EE-305 Design Laboratory

Indian Institute of Technology Patna
Department of Electrical Engineering

Report on:
Control of Uncertain LTI Systems Based on an Uncertainty and Disturbance Estimator (UDE)

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Introduction:

About 15 years ago, Youcef-Toumi and Ito proposed a robust control method for systems with unknown dynamics. It is called time delay control (TDC). Based on the assumption that a continuous signal remains unchanged during a small enough period, the past observation of uncertainties and disturbances is used to modify the control action directly, rather than to adjust controller gains, such as gain scheduling or to identify system parameters, such as adaptive control. During the last decade, TDC has been widely studied and has been applied to a number of industrial applications, such as DC servo motors, four-wheel steering systems, hybrid position/force control of robots, brushless DC motors, overhead cranes, and robot trajectory control.

However, TDC inherently requires that all the states and their derivatives be available for feedback. This imposes very strict limitations on the application. Another inherent drawback is that oscillations always exist in the control signal. Moreover, as is well known in the control community, delay is not good for control in most cases. In particular, it brings difficulties into system analysis and tends to destabilize a system. It is still desirable to get rid of delays, if possible, so as to simplify system analysis.

The assumption used in TDC is a time-domain assumption. But here, we use an assumption in the frequency domain to propose an alternative control strategy to obtain similar performance to TDC. The major part of the controller is called an uncertainty and disturbance estimator (UDE). The two inherent drawbacks of TDC, namely, there is no need to measure the derivative of the states and there is no oscillation in the control signal, are addressed. Also, an additional benefit is that the delay is removed. The system stability is more easily analyzed. In addition to all these advantages, the system performance is very similar to that obtained using TDC.

Proposed Solution:

In this report, I have analyzed a robust control strategy for uncertain LTI systems. The strategy is based on an uncertainty and disturbance estimator (UDE). It brings similar performance as the time-delay control (TDC).
The advantages over TDC are:
(i) no delay is introduced into the system
(ii) there are no oscillations in the control signal

(iii) there is no need of measuring the derivatives of the state vector.

The robust stability of LTI-SISO systems is analyzed, and simulations are given to show the effectiveness of the UDE-based control with a comparison made with TDC.

Model Description (UDE-Based Control Law):

System Description:

The system to be considered is formulated as

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{F})\mathbf{x} + \mathbf{B}\mathbf{u}(t) + \mathbf{d}(t) \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the state, $\mathbf{u}(t) = [u_1(t), \ldots, u_r(t)]^T$ is the control input, $\mathbf{A}$ is the known state matrix, $\mathbf{F}$ is the unknown state matrix, $\mathbf{B}$ is the control matrix having full column rank, and $\mathbf{d}(t)$ is the unpredictable external disturbances.

Reference Model and Structural Constraint.

Assume that the desired specification can be described by the reference model

$$\dot{\mathbf{x}}_m = \mathbf{A}_m \mathbf{x}_m + \mathbf{B}_m \mathbf{c}(t) \tag{2}$$

where $\mathbf{c}(t) = [c_1(t), \ldots, c_r(t)]^T$ is the desired state. The control objective is to make the state error $\mathbf{e}$ between the system and the reference model

$$\mathbf{e} = (x_{m1} - x_1 \ \cdots \ x_{mn} - x_n)^T = \mathbf{x}_m - \mathbf{x} \tag{3}$$

converge to zero. In other words, the error dynamics

$$\dot{\mathbf{e}} = (\mathbf{A}_m + \mathbf{K})\mathbf{e} \tag{4}$$

is stable, where **K** is called the error feedback gain.

Combining Eqs. (1)–(4), we obtain

$$\mathbf{A}_m\mathbf{x}+\mathbf{B}_m\mathbf{c}(t)-\mathbf{Ax}-\mathbf{Fx}-\mathbf{Bu}(t)-\mathbf{d}(t)=\mathbf{Ke} \tag{5}$$

Then the control action $\mathbf{u}(t)$ is obtained as

$$\mathbf{u}(t)=\mathbf{B}^{+}[\mathbf{A}_m\mathbf{x}+\mathbf{B}_m\mathbf{c}(t)-\mathbf{Ax}-\mathbf{Fx}-\mathbf{d}(t)-\mathbf{Ke}] \tag{6}$$

where $\mathbf{B}^{+}=(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$ is the pseudoinverse of $\mathbf{B}$. Since $\mathbf{u}(t)$ given in (6) is only an approximate solution of (5) rather than the accurate solution, (4) and (5) will only be met fully under the following structural constraint:

$$[\mathbf{I}-\mathbf{BB}^{+}]\cdot[\mathbf{A}_m\mathbf{x}+\mathbf{B}_m\mathbf{c}(t)-\mathbf{Ax}-\mathbf{Fx}-\mathbf{d}(t)-\mathbf{Ke}]=0 \tag{7}$$

UDE and the Control Law.

The control law in (6) can be represented in the s-domain by using Laplace transforms (assuming zero-initial states) as

$$\begin{aligned}\mathbf{U}(s)=&\mathbf{B}^{+}[\mathbf{A}_m\mathbf{X}(s)+\mathbf{B}_m\mathbf{C}(s)-\mathbf{KE}(s)-\mathbf{AX}(s)]\\&+\mathbf{B}^{+}[-\mathbf{FX}(s)-\mathbf{D}(s)]\end{aligned} \tag{8}$$

The first part in (8) is known while the second part, denoted hereafter by $U_d(s)$, includes the uncertainties and the external disturbance. According to the system dynamics (1), $U_d(s)$ can be rewritten as:

$$\mathbf{U}_d(s)=\mathbf{B}^{+}[-\mathbf{FX}(s)-\mathbf{D}(s)]=\mathbf{B}^{+}[(\mathbf{A}-sI)\mathbf{X}(s)+\mathbf{BU}(s)] \tag{9}$$

In other words, the unknown dynamics and the disturbances can be observed by the system states and the control signal. However, it cannot be used in the control law directly. TDC adopts an estimation of this signal by using a small delay in the time domain. Here, we will use a different estimation strategy by looking at this problem in the frequency domain.

Assume that $G_f(s)$ is a strictly proper low-pass filter with unity steady-state gain and broad enough bandwidth, then $U_d(s)$ can be accurately approximated to UDE, where

$$\mathbf{UDE}=\mathbf{B}^{+}[(\mathbf{A}-sI)\mathbf{X}(s)+\mathbf{BU}(s)]G_f(s) \tag{10}$$

This is called the uncertainty and disturbance estimator (UDE). The UDE only uses the control signal and the states to observe the uncertainties and the unpredictable disturbances. Hence

$$\mathbf{U}(s) = \mathbf{B}^{+} \cdot [\mathbf{A}_m\mathbf{X} + \mathbf{B}_m\mathbf{C} - \mathbf{KE} - \mathbf{AX}] + \mathbf{UDE}$$
$$= \mathbf{B}^{+} \cdot [\mathbf{A}_m\mathbf{X} + \mathbf{B}_m\mathbf{C} - \mathbf{KE} - \mathbf{AX}(1 - G_f) - sG_f\mathbf{X} + G_f\mathbf{BU}]$$

The **UDE**-based control law is then derived as

$$\mathbf{U}(s) = (I - \mathbf{B}^{+}\mathbf{B}G_f)^{-1}\mathbf{B}^{+} \cdot [\mathbf{A}_m\mathbf{X} + \mathbf{B}_m\mathbf{C} - \mathbf{KE} - \mathbf{AX}(1 - G_f) - sG_f\mathbf{X}] \tag{11}$$

The control signal is formed by the state, the low-pass filter, the reference model, and the error feedback gain. It has nothing to do with the uncertainty and the disturbance. Since G_f is strictly proper, sG_f is physically implementable and there is no need for measuring the derivative of states.

Assume that the frequency range of the system dynamics and the external disturbance is limited by $\boldsymbol{\omega}_f$, then the ideal low-pass filter G_f has a low-frequency gain ($\boldsymbol{\omega} < \boldsymbol{\omega}_f$) of 1 and a high-frequency gain ($\boldsymbol{\omega} > \boldsymbol{\omega}_f$) of zero. Hence, the estimation error of the uncertainty and the disturbance =

$$\mathbf{B}^{+}[(\mathbf{A} - sI)\mathbf{X}(s) + \mathbf{BU}(s)](1 - G_f)$$

is approximately zero for all the frequency range because $1 - G_f = 0$ for the low-frequency range and $(A - sI)\, X(s) + BU(s) = 0$ for the high-frequency range. A practical low-pass filter is

$$G_f(s) = \frac{1}{Ts + 1} \tag{12}$$

where $T = 1 / \omega_f$. Although this will cause some error in the estimation, the steady-state estimation error is always zero because $G_f(0) = 1$ (the initial value of $G_f(s)$ can always be chosen to be zero).

Special Case (LTI - SISO Systems):

In this section, we consider LTI-SISO systems with uncertainties and disturbances described in the canonical form, i.e., the corresponding matrices in (1) have the following partitions:

$$\mathbf{A}=\begin{pmatrix} 0 & \mathbf{I}_{n-1} \\ & \mathbf{A}_1 \end{pmatrix};\quad \mathbf{F}=\begin{pmatrix} 0 \\ \mathbf{F}_1 \end{pmatrix};\quad \mathbf{B}=\begin{pmatrix} 0 \\ b \end{pmatrix};\quad \mathbf{d}(t)=\begin{pmatrix} 0 \\ d(t) \end{pmatrix} \tag{13}$$

where $\mathbf{A}_1=(-a_1,-a_2,\ \ldots\ ,-a_n)$ and $\mathbf{F}_1=(-f_1,-f_2,\ \ldots\ ,-f_n)$ are $1\times n$ row vectors, and $\underline{f}_i\leqslant f_i\leqslant\bar{f}_i$ $(i=1,\cdots,n)$ are uncertain parameters. $\mathbf{I}_{n-1}$ is the $(n-1)\times(n-1)$ identity matrix and $b\neq 0$. The output of the system is assumed to be $y=x_1$.

Control Scheme.

The reference model and the error feedback gain matrix are partitioned as

$$\mathbf{A}_m=\begin{pmatrix} 0 & \mathbf{I}_{n-1} \\ & \mathbf{A}_{m1} \end{pmatrix};\quad \mathbf{B}_m=\begin{pmatrix} 0 \\ b_m \end{pmatrix};\quad \mathbf{K}=\begin{pmatrix} 0 \\ \mathbf{K}_1 \end{pmatrix} \tag{14}$$

where $\mathbf{A}_{m1}=(-a_{m1},-a_{m2},\ \ldots\ ,-a_{mn})$ and $\mathbf{K}_1=(-k_1,-k_2,\ \ldots\ ,-k_n)$ are $1\times n$ row vectors.

Substituting the matrices into (11), the control law is

$$U(s)=\frac{1}{b[1-G_f(s)]}\left(-\sum_{i=1}^{n} a_{mi}X_i+b_mC+\sum_{i=1}^{n} k_iE_i-sG_fX_n\right)+\frac{1}{b}\sum_{i=1}^{n} a_iX_i \tag{15}$$

It can be further simplified as follows:

$$=\frac{P_m(s)+P_k(s)}{b[1-G_f(s)]}\left[\frac{b_m}{P_m(s)}C-Y\right]+\frac{P_a(s)}{b}Y \tag{16}$$

where $G_m(s)= b_m/P_m(s) = b_m/(s^n+a_{mn}s^{n-1}+\cdots+a_{m1})$ is the transfer function of the reference model, $P_a(s)=s^n+\Sigma_{i=1}^{n}a_is^{i-1}$ is the characteristic polynomial of the known system dynamics, and $P_k(s)=k_ns^{n-1}+\cdots+k_1$ is the error feedback polynomial.

Equivalent structure of UDE-based control system.

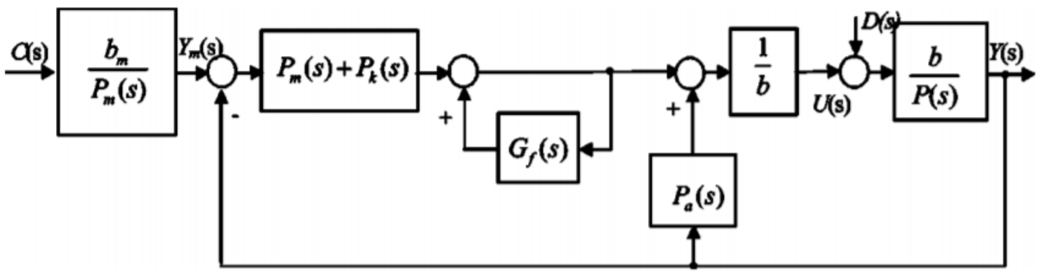


Fig. 1 The equivalent structure of UDE-based LTI-SISO control systems

The equivalent structure of the UDE-based control scheme is shown in Fig. 1. The UDE controller can be divided into four parts:
i) the reference model $b_m/P_m(s)$ to generate the desired trajectory
ii) a local positive-feedback loop via the low-pass filter (behaving like a PI controller)
iii) the polynomial $P_m(s) + P_k(s)$ for the tracking error and polynomial $P_a(s)$ for the output
iv) The last two perform the derivative effect.

Obviously, the UDE controller uses more derivative information than the common PID controller, and, hence, UDE-based control has the potential to obtain better performances than the common PID controller. It is worth noting that this structure is the equivalent of a UDE-based control, but not the structure for implementation. The control law should be implemented according to (15) under the assumption that all the states are available for feedback.

Stability Analysis.

The transfer function of the LTI - SISO plant given by (13) is

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b}{s^n+(a_n+f_n)s^{n-1}+\cdots+(a_1+f_1)}=\frac{b}{P(s)} \tag{17}$$

From (16), the closed-loop transfer function of the system is derived to be

$$\frac{Y(s)}{C(s)} = \frac{b_m}{P_m(s)} \frac{P_m + P_k}{P_m + P_k + (1 - G_f)(P - P_a)} \tag{18}$$

Since the reference model is always chosen to be stable, only the stability of the second part in the above equation needs to be discussed hereafter. If the low-pass filter is chosen as given in (12), then the characteristic polynomial of the closed-loop system is

$$\begin{aligned} P_c(s) &= (Ts+1)(P_m + P_k) + Ts(P - P_a) \\ &= (P_m + P_k) + Ts(P_m + P_k + P - P_a) \end{aligned} \tag{19}$$

The system stability depends on the reference model, the error feedback gain, the time constant of the filter, and, of course, the plant itself. The reference model is chosen to obtain the desired specifications, and the time constant of the low-pass filter is chosen to have a broad enough bandwidth. Hence, the error feedback gain **K** can be used to guarantee the robust stability of the closed-loop system. Intuitively, if T is small enough, the system is always stable. However, this might be difficult to implement, and, hence, the value of T has to be a compromise. The robust stability of the polynomial (19) can be guaranteed by the following theorem, which is obtained directly using the box theorem (a generalization of the well-known Kharitonov's theorem) for parametric uncertain LTI systems.

Robust stability.

Theorem (Robust stability). The closed-loop system is stable if and only if the four polynomials $P_m(s) + P_k(s) + TsP_{pq}(s)$ $(p,q = 1,2)$ are stable, in which $P_{pq}(s) = P_p(s) + Q_q(s)$ $(p,q = 1,2)$ are the Kharitonov polynomials of the uncertain polynomial $P_m + P_k + P - P_a = s^n + \Sigma_{i=1}^{n}(a_{mi} + k_i + f_i)s^{i-1}$ with

$$P_1(s)=(\beta_1+\underline{f}_1)+(\beta_3+\overline{f}_3)s^2+(\beta_5+\underline{f}_5)s^4+\cdots$$
$$=\sum_{i=0,even}^{n}(\beta_{1+i}+j^i\cdot\min\{j^i\underline{f}_{1+i},j^i\overline{f}_{1+i}\})\cdot s^i$$
$$P_2(s)=(\beta_1+\overline{f}_1)+(\beta_3+\underline{f}_3)s^2+(\beta_5+\overline{f}_5)s^4+\cdots$$
$$=\sum_{i=0,even}^{n}(\beta_{1+i}+j^i\cdot\max\{j^i\underline{f}_{1+i},j^i\overline{f}_{1+i}\})\cdot s^i$$

$$Q_1(s)=(\beta_2+\underline{f}_2)s+(\beta_4+\overline{f}_4)s^3+(\beta_6+\underline{f}_6)s^5+\cdots$$
$$=\sum_{i=1,odd}^{n}(\beta_{1+i}+j^{i-1}\cdot\min\{j^{i-1}\underline{f}_{1+i},j^{i-1}\overline{f}_{1+i}\})\cdot s^i$$
$$Q_2(s)=(\beta_2+\overline{f}_2)s+(\beta_4+\underline{f}_4)s^3+(\beta_6+\overline{f}_6)s^5+\cdots$$
$$=\sum_{i=1,odd}^{n}(\beta_{1+i}+j^{i-1}\cdot\max\{j^{i-1}\underline{f}_{1+i},j^{i-1}\overline{f}_{1+i}\})\cdot s^i$$

where $\beta_i=a_{mi}+k_i$ $(i=1,\cdots,n)$, $\beta_{n+1}=1$, $\underline{f}_{n+1}=\overline{f}_{n+1}=0$ and $j=\sqrt{-1}$.

Some Examples:

1) A First-Order Uncertain LTI System

Consider the following first-order LTI system:

$$\dot{x}=fx+u+d$$

where $0.5 \leqslant f \leqslant 2$ is an uncertain parameter with the nominal value $f = 1$, and d is a unit-step disturbance, acting at 1 s.

The reference model is chosen as $\dot{x}_m = -x_m + c$, and the error feedback gain is chosen as $k = 0$. The low-pass filter is chosen as $G_f(s) = 1/(Ts+1)$ with $T = 0.01$ s. According to (11), the control law is

$$U(s) = \frac{1}{1 - G_f(s)}[-X(s) + C(s) - sG_f(s)X(s)]$$

The measurement of ẋ is not needed. In the simulations, c(t) is chosen to be the step signal 1(t).

The nominal response is shown in Fig. 2.

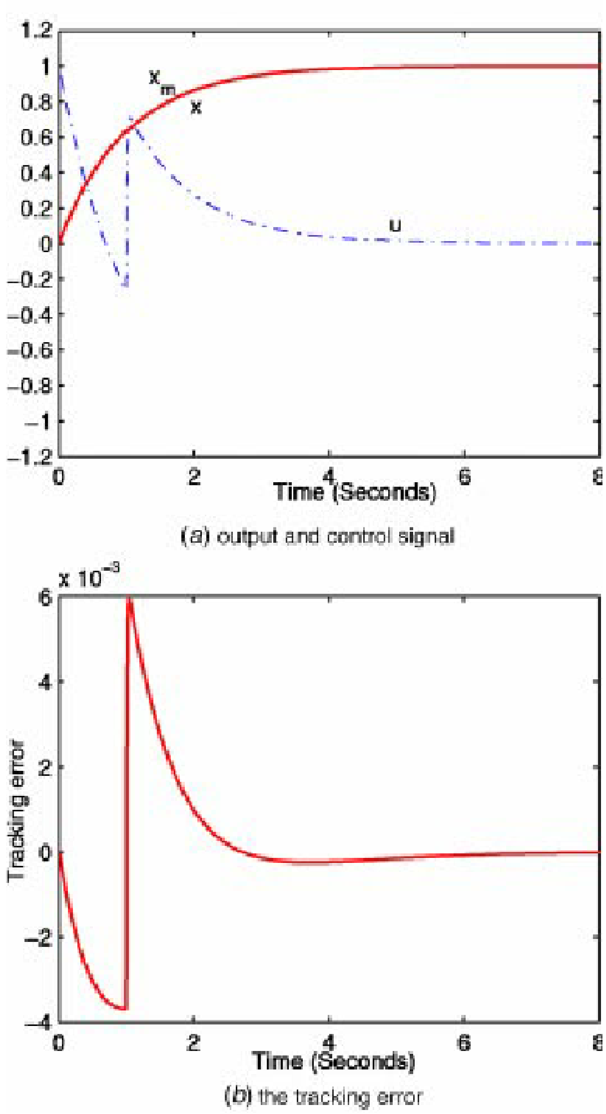


Fig. 2 Example 1: Nominal response

The response when f = 2 is shown in Fig. 3.

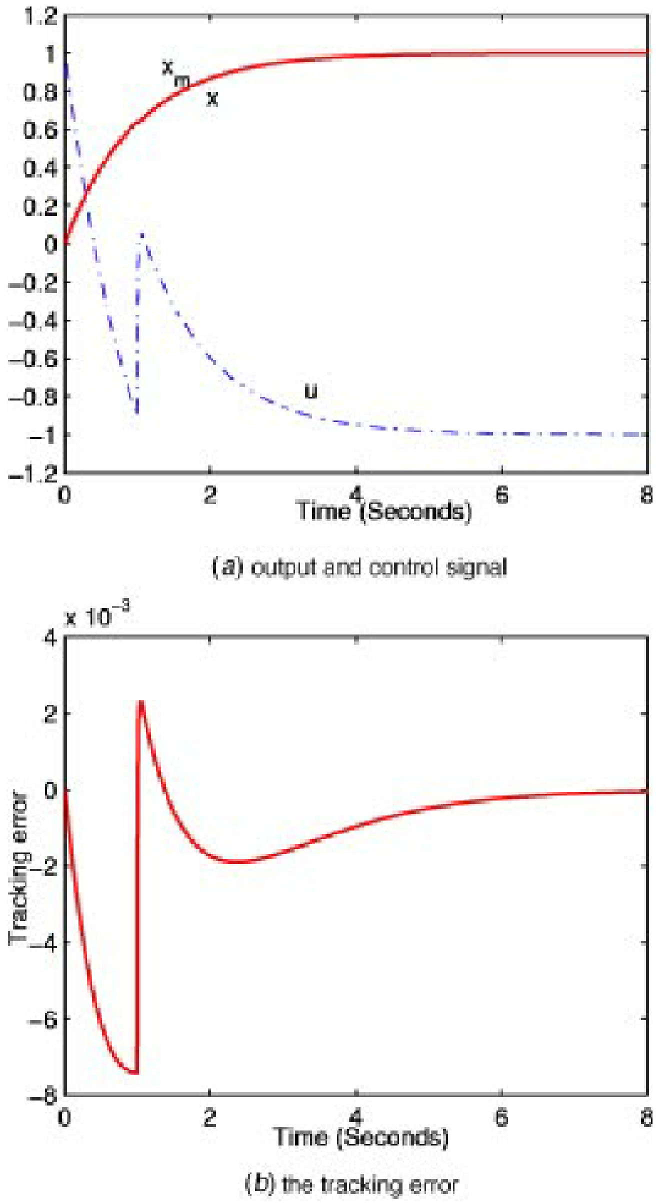


Fig. 3 Example 1: Response when f = 2

The system state x tracks the desired state x_m very well. When the uncertain parameter f is changed to f = 2, the performance does not degrade significantly because the controller can well estimate the uncertainty. It can also be seen from the control signal u (there is a jump in u when the disturbance starts at t=1s) that the controller estimates the disturbance very quickly.

The bandwidth (time constant) of the low-pass filter is very important for the system performance. The responses of three cases with different time constants are shown in Fig. 4.

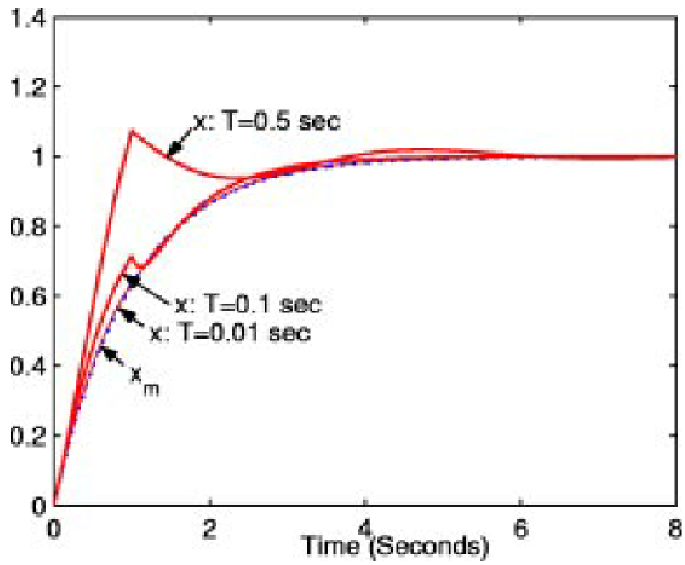


Fig. 4 Example 1: Response when f = 2 for different t

The smaller the time constant (the broader the bandwidth), the better the tracking (i.e., the better the performance). However, in practice, the time constant might be limited by the computation capability and the measurement noise.

2) A Servomotor Positioning System

In order to make a clear comparison with TDC, we take the servo motor positioning system as the second example.
The inherent unknown dynamics in this system are the viscous and dry frictions. Additional uncertainty is simulated by an elastic spring that is physically attached to the motor load. The dynamic equation of the system is

$$\ddot{\theta} = -\frac{k_s}{J}\theta - \frac{b_s}{J}\dot{\theta} - d_s\,\mathrm{sgn}(\dot{\theta}) + \frac{1}{J}\tau, \qquad (20)$$

where θ is the motor rotation angle, t is the motor torque, k_s is the unknown spring constant, b_s is the unknown viscous friction coefficient, d_s is the unknown dry friction coefficient, and J = 4.0 x 10^-4 kg m^2 is the total inertia.

The reference model is chosen as the following second-order system:

$$\begin{pmatrix} \dot{\theta}_m \\ \ddot{\theta}_m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega \end{pmatrix} \begin{pmatrix} \theta_m \\ \dot{\theta}_m \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_n^2 \end{pmatrix} c \tag{21}$$

where $\boldsymbol{\omega_n} = 5$ rad/s and $\boldsymbol{\zeta} = 1$ and the command c = 1 rad. The error feedback gain is chosen to be 0.

The UDE-based control law is

$$\tau(s) = \frac{1}{1 - G_f(s)} J[-25\theta(s) - 10\dot{\theta}(s) + 25C(s) - sG_f(s)\dot{\theta}(s)] \tag{22}$$

Nominal Performance:

When the frictions are omitted and the spring is not attached to the motor load, ks = 0, bs = 0, ds = 0, the responses of u under the UDE-based control and the time delay control are shown in Fig. 5

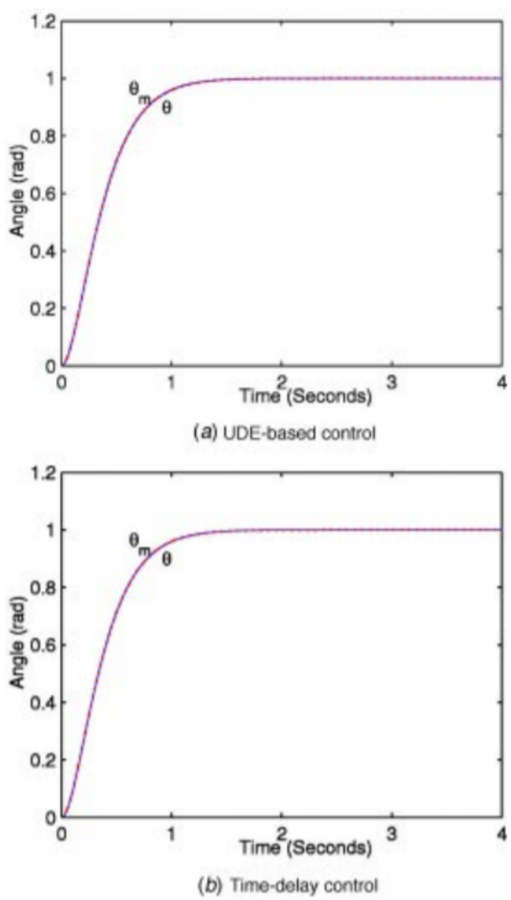


Fig. 5 Example 2: System responses in the nominal case

The control signals of both cases are shown in Fig. 6.

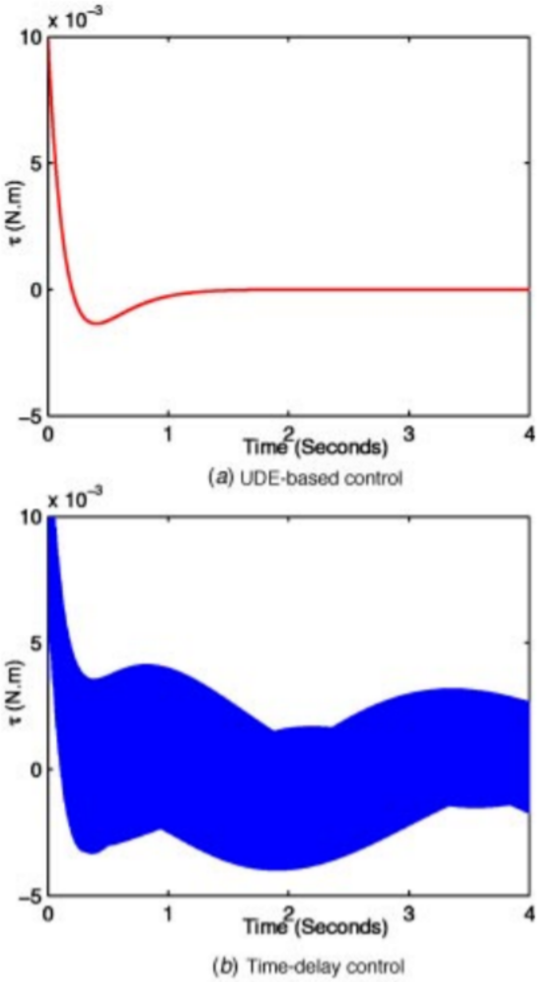


Fig. 6 Example 2: Control signals in the nominal case

Robust Performance

In this case, we assume that ks = 0.17 N•m/rad, bs = 0.01 N•m•s/rad and ds = 0.1 s^-1. The responses are shown in Fig. 7

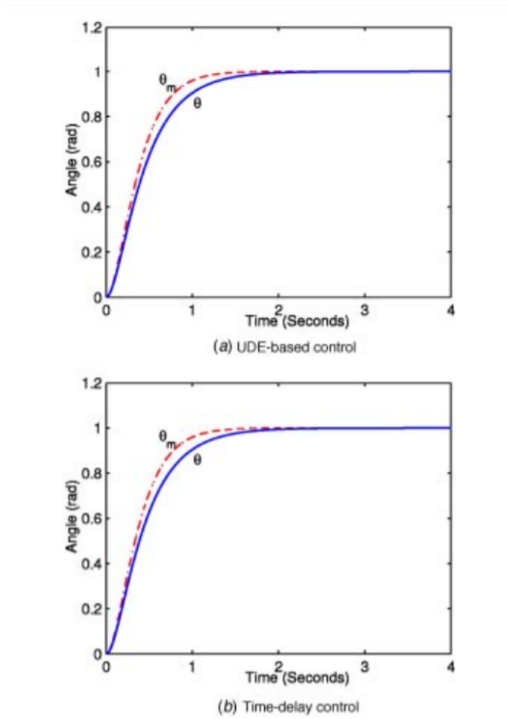


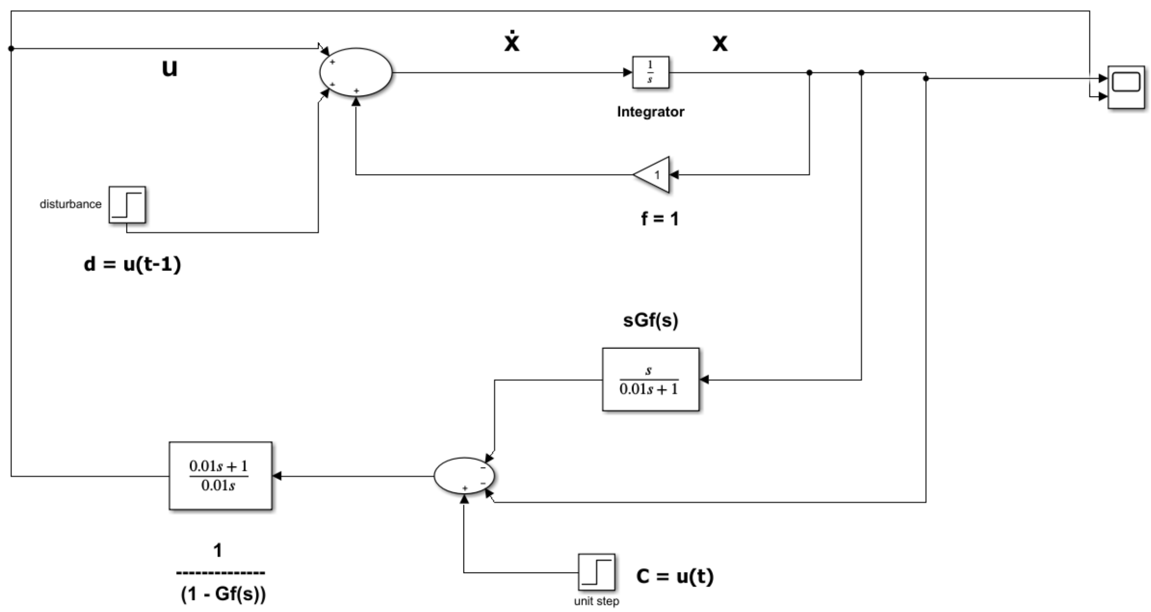
Fig. 7 Example 2: Robust performance ($L = 5$ ms or $T = 5$ ms)

Simulation and results in MATLAB (Simulink) of a 1st-Order Uncertain LTI System (Example 1)

<u>Block diagram:</u>

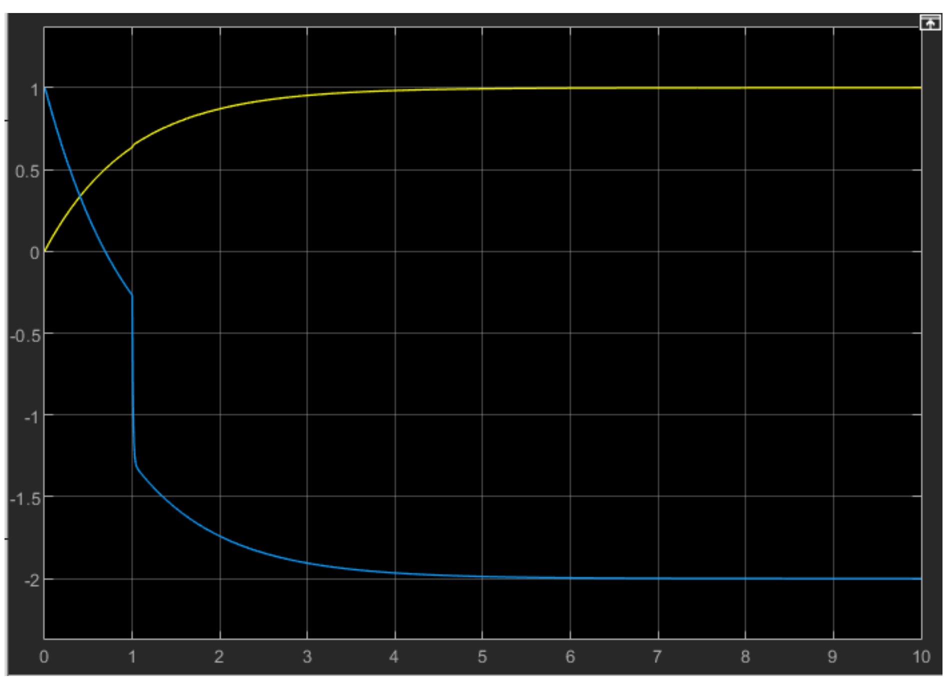
Where U(s) is calculated as -

$$U(s) = \frac{1}{1 - G_f(s)}[-X(s) + C(s) - sG_f(s)X(s)]$$

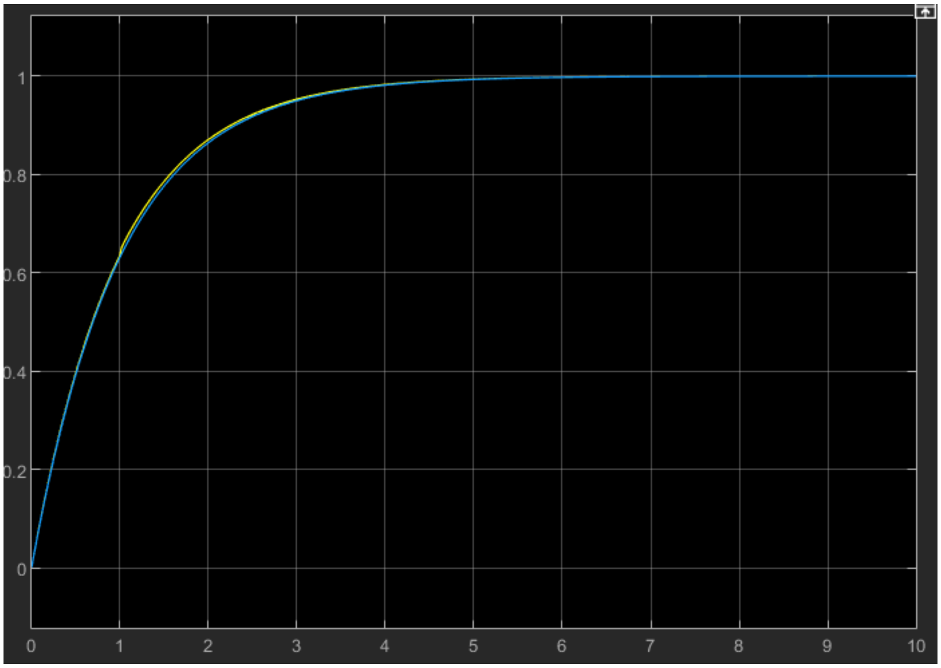


Nominal response (f = 1)

Yellow curve = x(t)
Blue Curve = u(t)

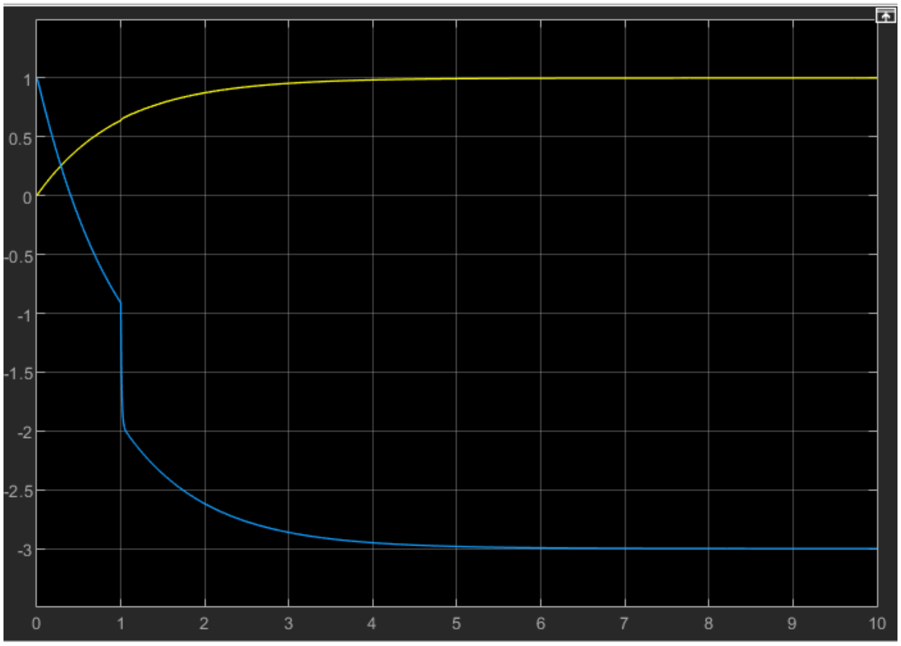


Yellow curve = x(t)
Blue Curve = x_m(t)

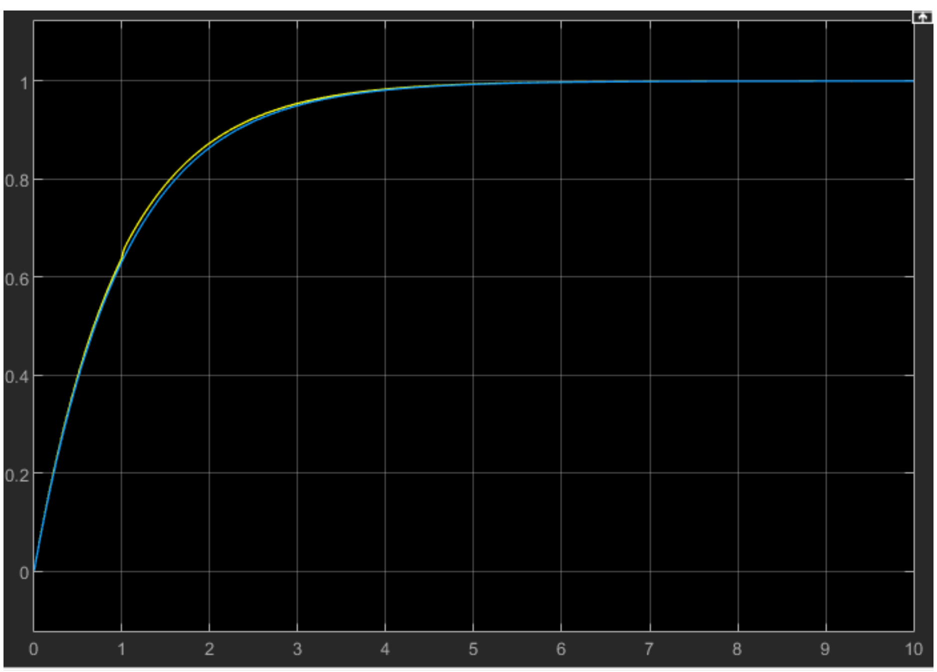


Response when f = 2

Yellow curve = x(t)
Blue Curve = u(t)

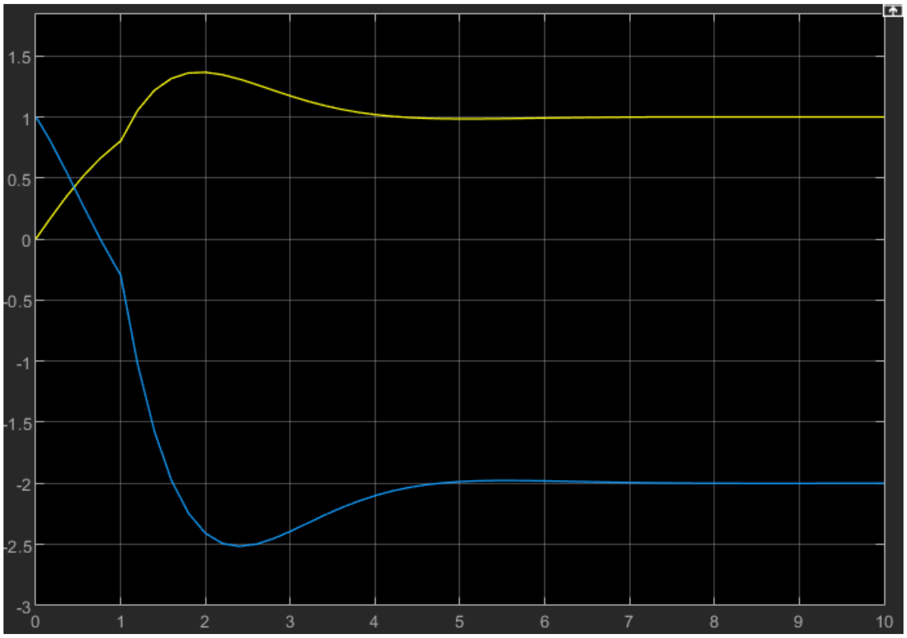


Yellow curve = $x(t)$
Blue Curve = $x_m(t)$

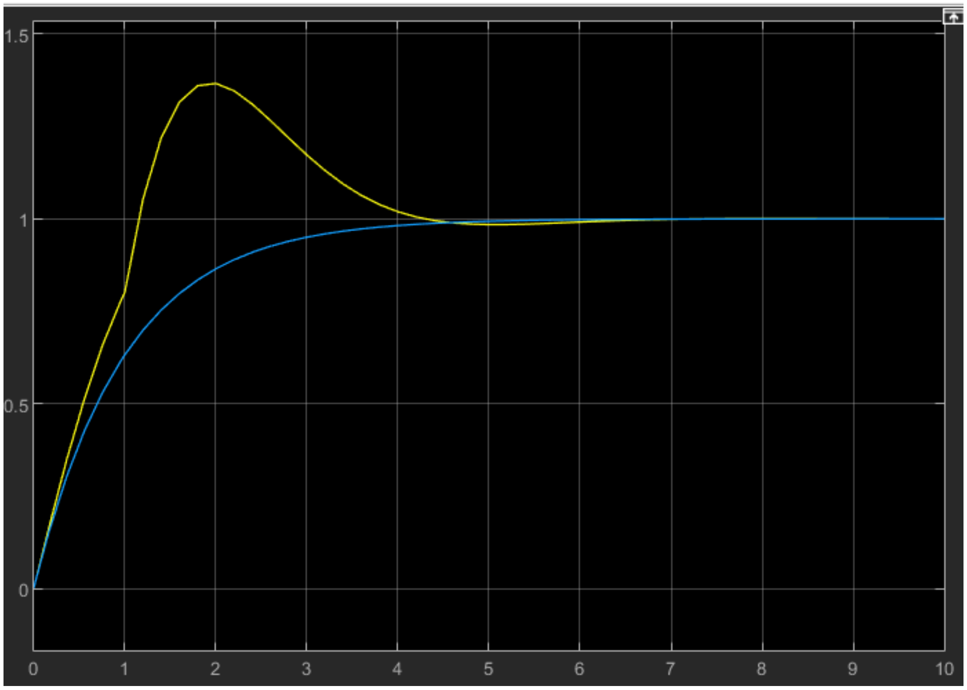


Nominal response (f = 1) when T (tau) = 0.5

Yellow curve = x(t)
Blue Curve = u(t)

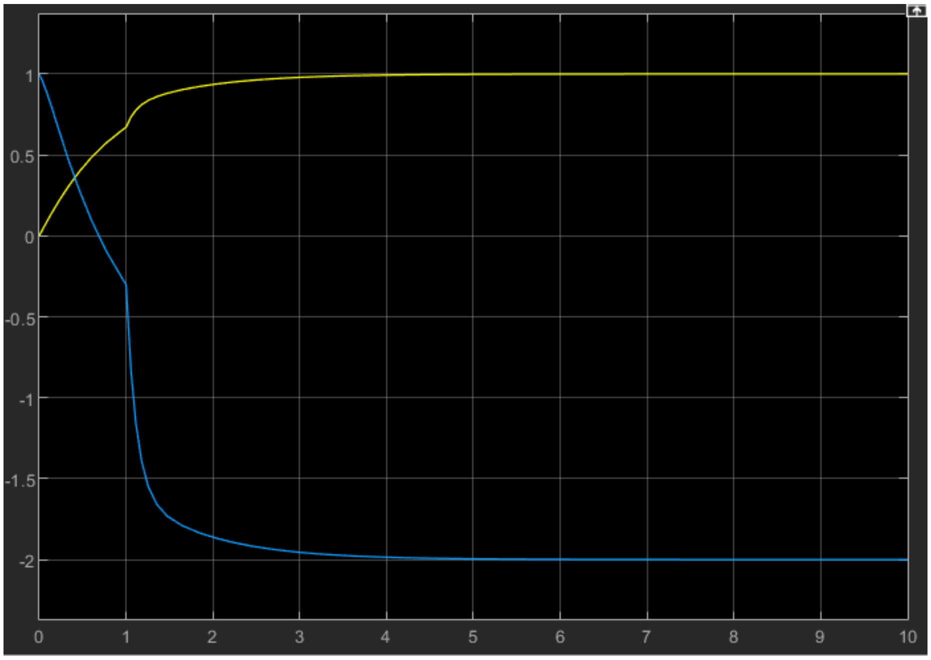


Yellow curve = x(t)
Blue Curve = x_m(t)

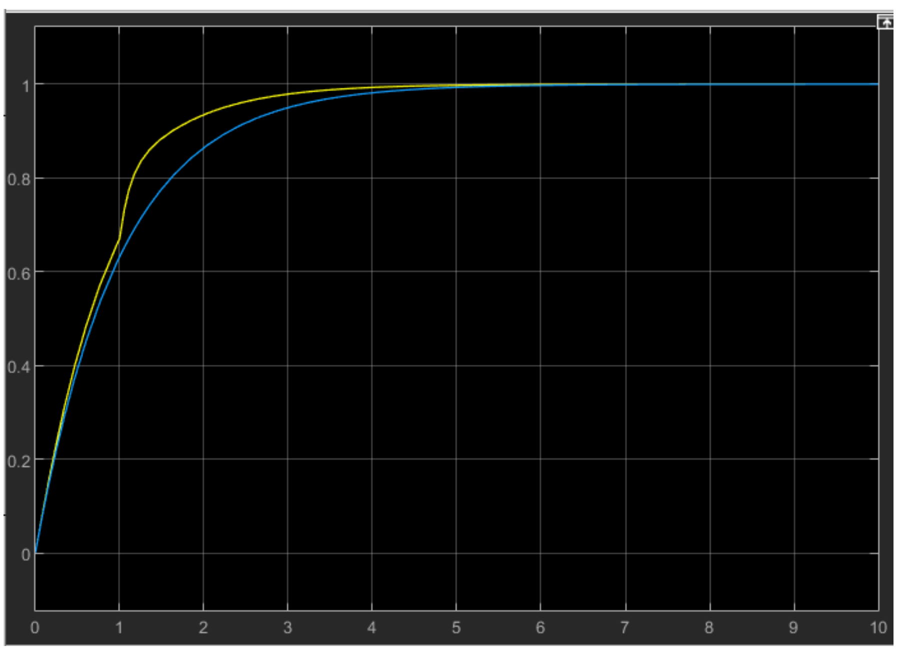


Nominal response (f = 1) when T (tau) = 0.1

Yellow curve = x(t)
Blue Curve = u(t)

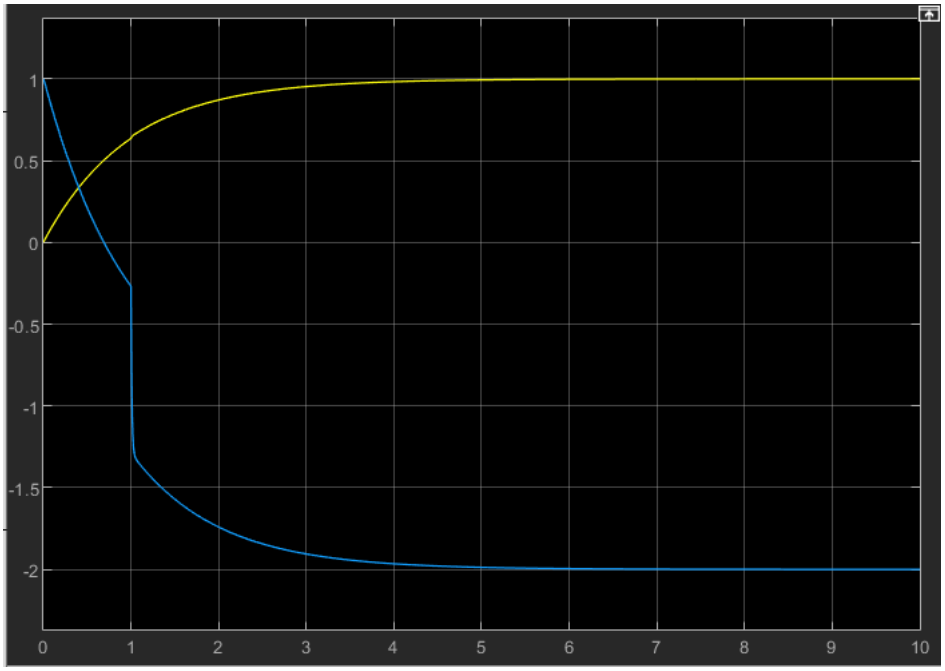


Yellow curve = $x(t)$
Blue Curve = $x_m(t)$

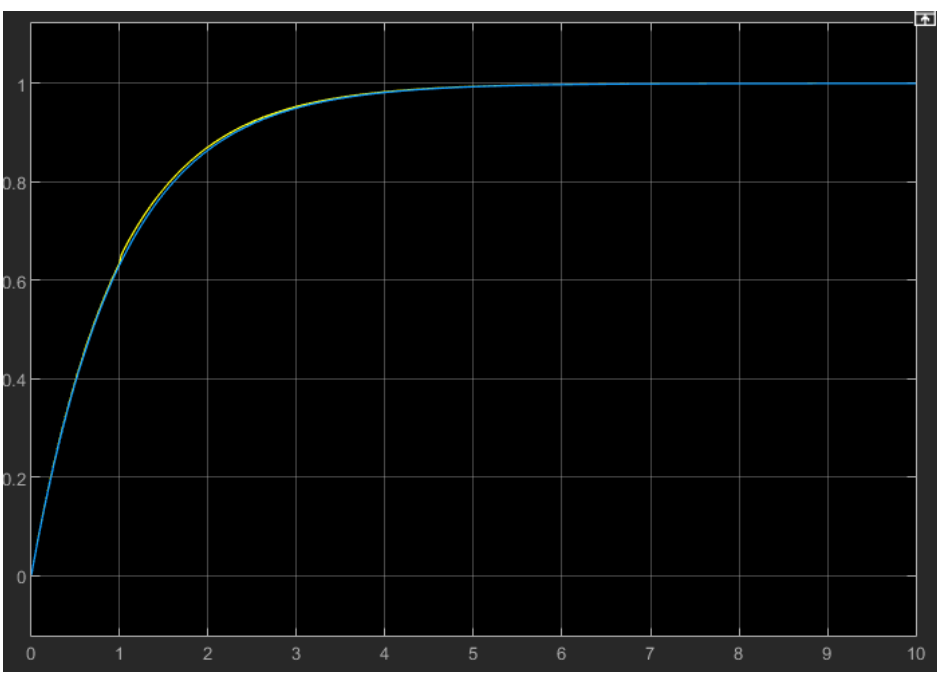


Nominal response (f = 1) when T (tau) = 0.01

Yellow curve = x(t)
Blue Curve = u(t)



Yellow curve = x(t)
Blue Curve = $x_m(t)$



Conclusions:

I presented an uncertainty and disturbance estimator (UDE) for LTI systems with uncertainties and disturbances. Similar performance as obtained by using TDC are obtained without the use of a delay element. Moreover, the inherent drawbacks of the TDC have been eliminated. There is no need for measuring the derivative of the states and no oscillation in the control signal. It is also easier to analyze the stability of a UDE-based control system than to analyze the stability of a TDC control system.

References:

1) Youcef-Toumi K., and Ito, O., 1987, "Controller Design for Systems With Unknown Nonlinear Dynamics," Proc. of ACC, Minnesota, pp. 836–844. https://ieeexplore.ieee.org/document/4789429

2) Youcef-Toumi, K., and Ito, O., 1990, "A Time Delay Controller for Systems With Unknown Dynamics," ASME J. Dyn. Syst., Meas., Control, 112~1!, pp. 133–142. https://asmedigitalcollection.asme.org/dynamicsystems/article-abstract/112/1/133/427748/A-Time-Delay-Controller-for-Systems-With-Unknown?redirectedFrom=PDF