H-ininfity equations and system description explanation

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1 Explanation

1.1 Linear system equations

Our nominal linear state space model has a form:

$$\begin{aligned} \dot{x} &= Ax + B_n n + B_w w + Bu = Ax + \begin{bmatrix} B_n & B_w & B \end{bmatrix} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \\ y &= Cx + D_n n + D_w w + Du = Cx + \begin{bmatrix} D_n & D_w & D \end{bmatrix} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \end{aligned} \tag{1}$$

where:

$$B_n = \mathbf{0}^{10x4}, \quad D_w = \mathbf{0}^{10x2} \quad and \quad D = \mathbf{0}^{10x3} \tag{2}$$

1.2 H-infinity representation

1.2.1 H-inifnity optimisation model

The H-infinity requires the system description to have form:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{3}$$

Where the system equations have form:

$$\begin{aligned} \dot{x} &= Ax + B_n n + B_w w + B_u u \\ z &= C_z x + D_{zn} n + D_{zw} w + D_{zu} u \\ y &= Cx + D_n n + D_w w + Du \end{aligned} \tag{4}$$

where signal $y \in R^{10}$ is the measured set of variables, $z \in R^{13}$ is the optimisation signal, in our case that is going to be normalised y with signals $u \in R^3$ included as outputs.

$$z = \begin{bmatrix} W_y y \\ W_u u \end{bmatrix} \tag{5}$$

Vectors $W_y \in R^{10}$ and $W_u \in R^3$ are weighting vectors based on the H_∞ norm of the system outputs.
Therefore the extended state space matrices of the optimisation system (the system used for H-inifnity optimisation) are:

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B_n & B_w & B \end{bmatrix} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} C_z \\ C \end{bmatrix} x + \begin{bmatrix} D_{zn} & D_{zw} & D_{zu} \\ D_n & D_w & D \end{bmatrix} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \end{aligned} \tag{6}$$

where:

$$B_n = \mathbf{0}^{10x4}, \quad D_w = \mathbf{0}^{10x2} \quad and \quad D = \mathbf{0}^{10x3} \tag{7}$$

Furthermore:

$$D_{zn} = \begin{bmatrix} D_n \\ \mathbf{0}^{3x3} \end{bmatrix}, \quad D_{zw} = \mathbf{0}^{13x2} \quad and \quad D_{zu} = \begin{bmatrix} \mathbf{0}^{10x3} \\ W_u \boldsymbol{I}^{3x3} \end{bmatrix} \tag{8}$$

And finally:

$$C_z = \begin{bmatrix} W_y \boldsymbol{I}^{10x10} \\ \mathbf{0}^{3x10} \end{bmatrix} \tag{9}$$

In the Matlab script I also used notation:

$$D_z = \begin{bmatrix} D_{zn} & D_{zw} & D_z u \end{bmatrix} \tag{10}$$

Final state space model of the system has matrices:

$$\begin{aligned} \boldsymbol{A} &= A \\ \boldsymbol{B} &= \begin{bmatrix} B_n & B_w & B \end{bmatrix} \\ \boldsymbol{C} &= \begin{bmatrix} C_z \\ C \end{bmatrix} \\ \boldsymbol{D} &= \begin{bmatrix} D_{zn} & D_{zw} & D_{zu} \\ D_n & D_w & D \end{bmatrix} \end{aligned} \tag{11}$$

and equations:

$$\begin{aligned} \dot{x} &= \boldsymbol{A}x + \boldsymbol{B} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix} &= \boldsymbol{C}x + \boldsymbol{D} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \end{aligned} \tag{12}$$

Making it the dynamical system with 10 states, 23 outputs and 9 inputs.

1.2.2 H-inifnity evaluation model

To test the model behaviour on our real model we need to test the outputs y not optimisation outputs z.

Therefore we construct new extended state space model where:

$$z = \begin{bmatrix} y \\ u \end{bmatrix} \tag{13}$$

So the state space model becomes:

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B_n & B_w & B \end{bmatrix} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} C_z \\ C \end{bmatrix} x + \begin{bmatrix} D_{zn} & D_{zw} & D_{zu} \\ D_n & D_w & D \end{bmatrix} \begin{bmatrix} n \\ w \\ u \end{bmatrix} \end{aligned} \tag{14}$$

where as before:

$$B_n = \mathbf{0}^{10x4}, \quad D_w = \mathbf{0}^{10x2}, \quad D = \mathbf{0}^{10x3}, \quad D_{zn} = \begin{bmatrix} D_n \\ \mathbf{0}^{3x3} \end{bmatrix} \quad and \quad D_{zw} = \mathbf{0}^{13x2} \tag{15}$$

But this time:

$$D_{zu} = \begin{bmatrix} \mathbf{0}^{10x3} \\ \boldsymbol{I}^{3x3} \end{bmatrix} \quad and \quad C_z = \begin{bmatrix} \boldsymbol{I}^{10x10} \\ \mathbf{0}^{3x10} \end{bmatrix} \tag{16}$$

In the Matlab for the case of (13) I used notation:

$$D_y = \begin{bmatrix} D_{zn} & D_{zw} & D_z u \end{bmatrix} \quad and \quad C_y = C_z \tag{17}$$

Final state space model of the system has matrices:

$$\begin{aligned} \boldsymbol{A} &= A \\ \boldsymbol{B} &= \begin{bmatrix} B_n & B_w & B \end{bmatrix} \\ \boldsymbol{C} &= \begin{bmatrix} C_z \\ C \end{bmatrix} \\ \boldsymbol{D} &= \begin{bmatrix} D_{zn} & D_{zw} & D_{zu} \\ D_n & D_w & D \end{bmatrix} \end{aligned} \tag{18}$$

1.3 Nomial and worst case model

Finally, to change between nominal and worst case models the only difference in equations are the linear model matrices based on the euqation (1):

$$A, \quad B_n, \quad B_w, \quad ,B, \quad C, \quad D_n, \quad D_w \quad and \quad D \tag{19}$$