Algebraic Topology

Zack Walker

Vector Spaces

A good place to start studying spaces is vector spaces. Most math-derived areas of study use them in some capacity. We can start by declaring the idea of a **vector** as an object that is able to scale and add to other vectors in special ways. In the bigger picture of space, it is more useful to examine what constitutes a vector space rather than an individual vector. In the most general sense, a vector is a memeber of a vector space. Before looking at formal definitions of vector spaces, it's important to look at the definition of a field, first.

Definition of a Field. A **field** is a set F combined with field addition $+_F : F \times F \to F$ and field multiplication $\cdot_F : F \times F \to F$ that meets the seven axioms of a field:

1. $+_F$ is commutative over F; $\forall a, b \in F, +_F(a, b) = +_F(b, a)$
2. $+_F$ is associative over F; $\forall a, b, c \in F, +_F(a, +_F(b, c)) = +_F(+_F(a, b), c)$
3. There exists an additive identity for $+_F$ in F; $\exists i \in F : \forall a \in F, +_F(a, i) = a$
4. There exists an additive inverse for $+_F$ for every element in F; $\forall a \in F, \exists -a \in F : +_F(a, -a) = i$
5. There exists a multiplicative identity for $\cdot_F$ in F; $\exists j \in F : \forall a \in F, \cdot_F(a, j) = a$
6. There exists a multiplicative inverse for $\cdot_F$ for every element in F; $\forall a \in F, \exists a^{-1} \in F : \cdot_F\left(a, a^{-1}\right) = j$
7. Field multiplication is distributive over field addition; $\forall a, b, c \in F, \cdot_F(a, +_F(b, c)) = +_F(\cdot_F(a, b), \cdot_F(a, c))$

One example of a field is the rational numbers $\mathbb{Q}$ over standard multiplication and addition. It's a good thought exercise to go through each axiom and see why this is. Some may call fields spaces in their own right. In general, a **space** is just some kind of set combined with a structure over that set. So, the full description of a field is the set and two operations. In the definition above, I would say "a field F and field operations $+_F$ and $\cdot_F$," or simply "a field $(F, +_F, \cdot_F)$." With the structure of fields in place, vector spaces can now be formally constructed.

Definition of a Vector Space. A **vector space** is a set V over a field $(K, +_K, \cdot_K)$ combined with vector addition $+_V : V \times V \to V$ and scalar multiplication $\cdot_V : K \times V \to V$ that satisfies the eight axioms of a vector space:

1. $+_V$ is commutative over V; $\forall u, v \in V, +_V(u, v) = +_V(v, u)$
2. $+_V$ is associated over V; $\forall u, v, w \in V, +_V(u, +_V(v, w)) = +_V(+_V(u, v), w)$
3. There exists an additive identity for $+_V$ in V; $\exists I \in V : \forall v \in V, +_V(v, I) = v$
4. There exists an additive inverse for $+_V$ for every element in V; $\forall v \in V, \exists -v \in V : +_V(v, -v) = I$
5. There exists a multiplicative identity element in K for $\cdot_V$; $\exists J \in K : \forall v \in V, \cdot_V(J, v) = v$
6. Scalar multiplication and field multiplication are compatible; $\forall a, b \in K, \forall v \in V, \cdot_V(a, \cdot_V(b, v)) = \cdot_V(\cdot_K(a, b), v)$
7. Scalar multiplcation is distributive over scalar addition; $\forall a, b \in K, \forall v \in V, \cdot_V(+_K(a, b), v) = +_V(\cdot_V(a, v), \cdot_V(b, v))$
8. Scalar multiplication is distributive over vector addition; $\forall a \in K, \forall v, w \in V, \cdot_V(a, +_V(v, w)) = +_V(\cdot_V(a, v), \cdot_V(a, w))$

In order to make the notation easier to read, $+_V(u, v)$ is often written as $u +_V v$. If it is implied that the things you are adding are vectors in that specific space, $u + v$ is also sometimes used. If it's not implied through previous context that the things being added are vectors, sometimes notation such as $\vec{u} + \vec{v}$ is used. However, I think it is best to explicitely state which objects operations are being acted upon and which addition we're talking about, so I will avoid this notation. Likewise, $\cdot_V(a, v)$ is often written as $a \cdot_V v$. Similar notation is used for field addition and multiplication. Also similarly to fields, a complete description of a vector space is often denoted as $(V, K, +_V, \cdot_V)$. A really complete description of a vector space may even look something like $(V, (K, +_K, \cdot_K), +_V, \cdot_V)$.

Example of a Vector Space

Consider a set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ over the field of $\mathbb{R}$ with standard addition and multiplication. Define vector addition as the standard sum of their components and scalar multiplication as a the standard multiplication between the scalar and each component. Then, vector addition,$+_{\mathbb{R}^2} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, looks like this:

$$\begin{bmatrix} a \\ b \end{bmatrix} +_{\mathbb{R}^2} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a +_{\mathbb{R}} c \\ b +_{\mathbb{R}} d \end{bmatrix},$$

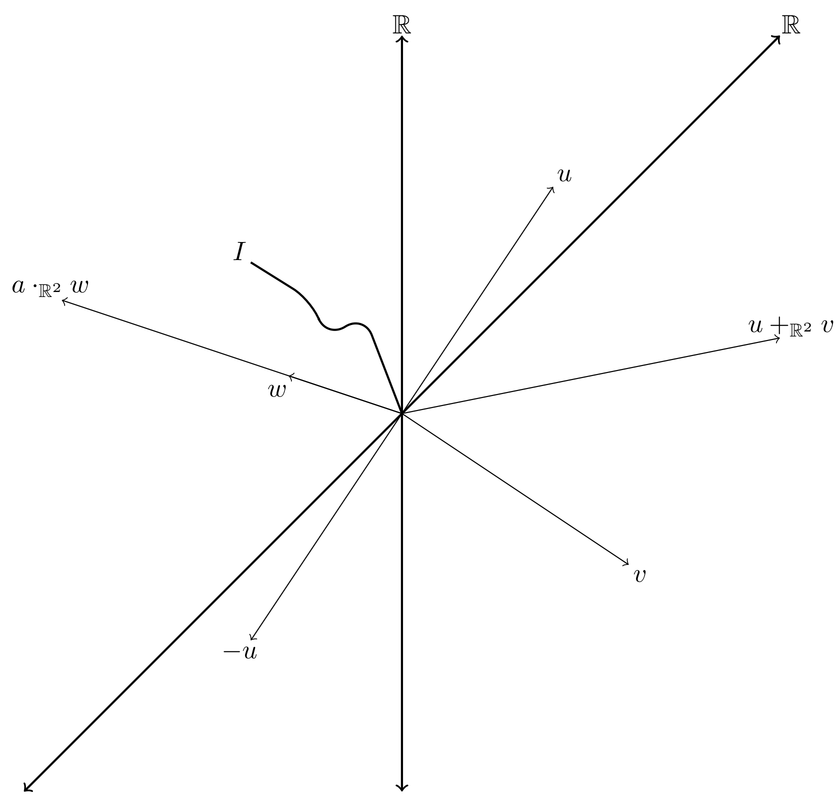
where $+_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ represents standard addition, the field addition. This kind of addition is clearly commutative and associative. The additive identity vector is $I = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Furthermore, one can construct an additive inverse for any element in this set via:

$$\forall v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2, -v = \begin{bmatrix} -a \\ -b \end{bmatrix}.$$

Scalar multiplcation, $\cdot_{\mathbb{R}^2} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, looks like this:

$$k \cdot_{\mathbb{R}^2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} k \cdot_{\mathbb{R}} a \\ k \cdot_{\mathbb{R}} b \end{bmatrix},$$

where $\cdot_{\mathbb{R}}$ represents standard multiplication, the field multiplication. One can check that the rest of the axioms are satisfied with these definitions of vector addition and scalar multiplication. This example of vectors is an example of the more general **Euclidean vector**, which is a specific type of vector that is an element of a set that is constructed by $\prod_i \mathbb{R}_i$. If there are n copies of $\mathbb{R}$, we call it an "n-touple." The useful things about these 2-touple vectors we described is that they can be easily visualized for an intuition of what these operations actually do. Let's consider a 2-touple graph:



One can see that these kinds of vectors can be represented as arrows stemming from the center, where the additive identity vector is. The actual components of the vectors are arbitrary, and thus so are the axes used to draw them; they don't *have* to be perpindicular. The actual conversions of these different components based off of different choices of axes is a major area of linear algebra and tensor algebra. As long as the axes aren't parallel to each other, one can make any 2-touple with any choice of axes. Formally, we say that as long as your basis vectors as linearly independent, but this visual representation captures the spirit of it for 2-touples.

Vector Subspace

In some sense, there is a way for one vector space to be "contained" inside of another. Formally, we say W is a vector subspace of V over field K if W is a subset of V and if, under the operations of V, $+_V$ and $\cdot_V$, W is a vector space over K. While there is no standard notation, $W \leq V$ is often used to denote that W is a vector subspace of V over field K. Likewise, if it is shown that W is not equal to V, one can denote W as a proper vector subspace of V via $W < V$. I will use this notation. For an example, consider the 2-touple space mentioned earlier. Next, consider all pairings $(a, 0)$ such that a is a real number, call this set $\mathbb{R}_0$. One can show that this is a proper subspace of $\mathbb{R}^2$ combined with its operations as decribed earlier. Thus, one can write $\mathbb{R}_0 < \mathbb{R}^2$.

Dual Space

The dual space of a vector space V over a field K, denoted V^*, is the collection of all linear mappings $\varphi : V \to K$. A mapping $\varphi : V \to K$ is considered linear if:

1. $\forall u, v \in V, \varphi(u +_V v) = \varphi(u) +_K \varphi(v)$
2. $\forall a \in K, \forall v \in V, \varphi(a \cdot_V v) = a \cdot_K \varphi(v)$

Example of Such a Linear Map

From our earlier 2-touple space example, consider a mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}$ that is defined as $\varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a +_{\mathbb{R}} b$. Let's prove that this is a linear mapping. First, show that it's linearly additive:

$$\varphi\left(\begin{bmatrix} a \\ b \end{bmatrix} +_{\mathbb{R}^2} \begin{bmatrix} c \\ d \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} a +_{\mathbb{R}} c \\ b +_{\mathbb{R}} d \end{bmatrix}\right) = a +_{\mathbb{R}} c +_{\mathbb{R}} b +_{\mathbb{R}} d = a +_{\mathbb{R}} b +_{\mathbb{R}} c +_{\mathbb{R}} d = \varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) +_{\mathbb{R}} \varphi\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$$

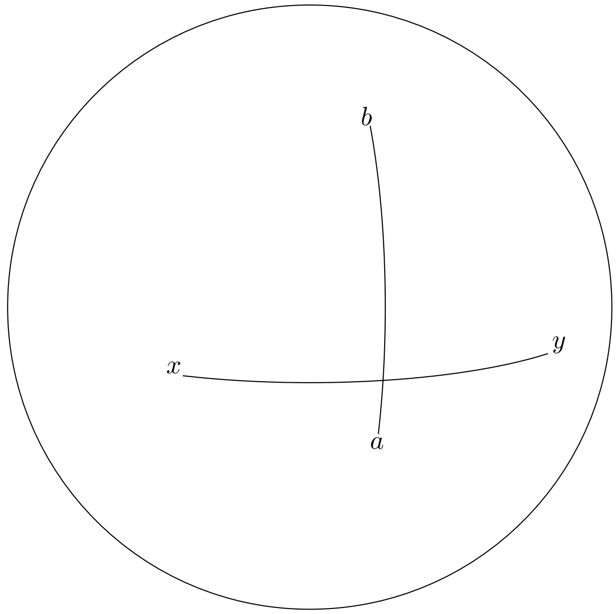
Next, one can show that this function linearly scales:

$$\varphi\left(k \cdot_{\mathbb{R}^2} \begin{bmatrix} a \\ b \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} k \cdot_{\mathbb{R}} a \\ k \cdot_{\mathbb{R}} b \end{bmatrix}\right) = k \cdot_{\mathbb{R}} a +_{\mathbb{R}} k \cdot_{\mathbb{R}} b = k \cdot_{\mathbb{R}} (a +_{\mathbb{R}} b) = k \cdot_{\mathbb{R}} \varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$$

Thus, we have proven that $\varphi \in V^*$. V^* is simply the collection of all such linear mappings from the vector space to the field.

Metric Spaces

A metric space can be thought of as a generalization of vector spaces; a metric space is a set that is paired with a structure that captures certain properties of distance called a metric. Let's start with an example by looking at points on a sphere.



There is some sense of a "distance" between x and y on the surface on the sphere. The same goes for a and b. It will actually end up that this holds for *any* two distint points on the sphere. Consider the metric the distance one would walk from two points in a straight line in terms of the surface of the sphere. Now that there is an intuition for metrics, one may now look at a formal definition.

Definition of a Metric Space. A **metric space** is a set X combined with a metric $d : X \times X \to \mathbb{R}^{\geq 0}$ that meets three axioms:

1. The metric is symmetric; $\forall x, y \in X, d(x, y) = d(y, x)$
2. The metric is always positive for different points; $\forall x, y \in X, d(x, y) = 0 \iff x = y$
3. The metric is triangular under addition; $\forall x, y, z \in X, d(x, y) + d(y, z) \geq d(x, z)$

where $\mathbb{R}^{\geq 0}$ is short hand for "all real numbers that are at least zero."

Traditionally, a fourth axiom is stated for a metric, namely

$$\forall x, y \in X, d(x, y) \geq 0.$$

However, the other three axioms imply this one. Thus, it is not independent of the other axioms. One can prove it.

Proposition. *The three standard axioms of a metric space imply the traditional fourth axiom.*

Proof. According to axiom 3, it holds that $d(x, y) + d(y, x) \geq d(x, x)$ if it stands that $x = z$. Next, according to axiom 2, it holds that $d(x, x) = 0$ since $x = x$. Thus, $d(x, y) + d(y, x) \geq 0$ must also hold. Since the metric is symmetric according to axiom 1, so it also holds that

$$\begin{aligned} d(x, y) + d(y, x) &\geq 0 \\ d(x, y) + d(x, y) &\geq 0 \\ 2d(x, y) &\geq 0 \\ d(x, y) &\geq 0. \end{aligned}$$

$\square$