Some of the Fundamentals of Mathematics

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The Study of Perfect Knowledge

The Iterative Process

Mathematics is the study of formal systems. Given some system, structure, space, relation, or any other logically definable object, take certain axioms as truth and, through deduction, arrive at results trying to reach contradictions, paradoxes, or other things to refine the model of axioms given. This is very similar to the scientific process, but instead of observing something and making a hypothesis, the Tower of Babel for mathematics is simply declaration of truths.

Zermelo-Fraenkel Set Theory

An **axiom** is a proposition that is taken to be true. A **schema** is a set of rules that takes place for a possibly infinite number of axioms. Zermelo-Fraenkel Set Theory, initialized to ZF, is constructed of six axioms and two schemas. Zermelo-Fraeknel Set Theory with the Axiom of Choice is initialized as ZFC and consists of ZF along with the Axiom of Choice, initialized to AC. The formulation of any axiom or schema uses the reliance of logical notation. Sets and membership are not explicitely defined in ZFC, rather, they are recursively defined through context.

Axioms and Schemas

Axiom of Extensionality

If x and y contain the same elements, then they belong to the same sets.

$$\forall x \, \forall y \, [\forall z \, (z \in x \iff z \in y) \implies \forall w \, (x \in w \iff y \in w)]$$

Axiom Schema of Specification

If ϕ is a formula in the language of ZF given free variables $x, z, w_1, w_2, \cdots, w_n$, then:

$$\forall z \, \forall w_1 \, \forall w_2 \cdots \forall w_n \, \exists y \, \forall x \, [x \in y \iff x \in z \land \phi] \, .$$

This axiom holds true for any ϕ, thus, there are an infinite number of axioms for this schema.

The Empty Set Using the Axiom of Specification, define the empty set as:

$$\emptyset := \{u \in w \mid (u \in u) \land (u \notin u)\} \, .$$

Axiom of Regularity

Every non-empty set x contains an element y such that x and y are disjoint.

$$\forall x \, (x \neq \emptyset \implies \exists y \in x \, (x \cap y = \emptyset))$$

Axiom of Paring (Deprecated)

If x and y are sets, then there exists a set that contains both of them as elements.

$$\forall x \, \forall y \, \exists z \, (x \in z \wedge y \in z)$$

Axiom of Union

For any set of sets $\mathcal{F}$, there exists a set A containing every element that is some element in $\mathcal{F}$.

$$\forall \mathcal{F} \, \exists A \, \forall Y \, \forall x \, [(x \in Y \wedge Y \in \mathcal{F}) \implies x \in A]$$

Axiom Schema of Replacement

Let ϕ be a formula in ZF with free variables $x, y, A, w_1, w_2, \cdots, w_n$, then:

$$\forall A \, \forall w_1 \cdots \forall w_n \, [\forall x \, (x \in A \implies \exists! y \, \phi) \implies \exists B \, \forall x \, (x \in A \implies \exists y \, (y \in B \wedge \phi))]$$

Axiom of Infinity

There exists a set I that contains the empty set and for every element, contains the union of the element with its singleton.

$$\exists I \, (\emptyset \in I \wedge \forall x \in I \, (x \cup \{x\}) \in I) \, .$$

Axiom of Power Set

A set z is a subset of x if and only if every element of z is an element of x.

$$(z \subseteq x) \iff (\forall q \, (q \in z \implies q \in x))$$

This axiom states that for any set x, there exists a set y that contains every subset of x.

$$\forall x \, \exists y \, \forall z \, [z \subseteq x \implies z \in y]$$

Axiom of Choice

For any set X of non-empty sets, there exists a function f from X to $\bigcup X$ such that for every A in X, $f(A)$ is in A.

$$\forall X \left[\emptyset \notin X \implies \exists f : X \to \bigcup X \quad \forall A \in X \, (f(A) \in A) \right]$$

Implications

Deriving the Axiom of Pairing

The Axiom Schema of Replacement and an axiom that gauruntees the existence of two such as the Axiom of Infinity or iteration of the Axiom of Power Set imply the Axiom of Pairing. Using the Axiom of Infinity, one can construct a set like:

$$I = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \cdots\}.$$

By declaration of $0 = \emptyset$, this set constructs the naturals.

$$\{0, \{0\}, \{0, \{0\}\}, \cdots\} = \{0, 1, \{1\}, \cdots\} = \{0, 1, 2, \cdots\}$$

This set has order type ω. By construction, it's clear that $\omega = \beth_0$ by definition of ordinals. By Axiom of Power Set, one can show that the set $\{0, 1\}$ is a subset of such a set. Such a set has order type 2. Thus, 2 must exist. Suppose ϕ is the formula mapping two sets x and y to themselves; it holds by the Axiom Schema of Replacement that this mapping is contained in a set. By extension, there then exists a set z that contains both x and y as elements, showing the Axiom of Pairing. This is one reason why the Axiom of Pairing is sometimes not included in the standard ZFC as the Axiom Schema of Replacement and the Axiom of Infinity imply it. Thus, it is not really independent of the other axioms: not strictly needed to contruct ZFC.

Axiom of Choice

The Axiom of Choice is a historically controversial axiom as this axiom seems to imply certain paradoxes. The first is the construction of sets without size under the Lebesgue Measure. Consider the interval over the reals from 0 to 1. Make a collection for every pair of numbers, x and y, whose difference is rational. The union of all of these sets would contain the interval. Further more, they're all disjoint. Using the Axiom of Choice, build a set S that contains one and only one member from each of these sets. Next, list all of the rationals from negative one to one. Suppose this set is $r_1, r_2, \cdots$. One then constructs sets $S_1 = \{s_1 + r_1, s_2 + r_1, s_3 + r_1, \cdots\}$, $S_2 = \{s_1 + r_2, s_2 + r_2, s_2 + r_2, \cdots\}$, etc. These sets are disjoint and contain all of the members from zero to one. Since they are all disjoint, and by translation invariance, it holds that the measure is the sum of the measures of all of the S sets of the same size. Since a set contains all from zero to one, the measure must be bigger than three. Moreover, since the interval is contained within the range negative one to two, the measure must be less than three. However, there is no number that, when added to itself infinitely many times, falls in the range between one and three. Thus, such a set S has no size. The Axiom of Choice can construct an infinite number of such sets.

The Naturals

Peano Axioms

The Peano axioms define the naturals. The first axiom declares that 0 is a natural. The next axioms describe a successor function, $S : \mathbb{N} \to \mathbb{N}$. The first axiom of the successor functions declares that for $n \in \mathbb{N}, S(n) = 0$ is false. The second axiom of the successor function states that S is an injection; $\forall n, m \in \mathbb{N}, S(m) = S(n) \implies n = m$. The third axiom of the successor function states that S is closed over the naturals; $n \in \mathbb{N} \implies S(n) \in \mathbb{N}$. The final Peano axiom states that if a set K contains 0 and for any natural n, $n \in K \implies S(n) \in K$, then K contains all of the naturals. One can show that the Von Neumann ordinals satisfy the Peano axioms.

Von Neumann Ordinals

The Von Neumann Ordinals start by delcaring that $0 = \emptyset$. There is then a successor function defined as $S(n) := n \cup \{n\}$. By the Axiom of Infinity, one can show that there exists a set X that contains zero and contains every successor from thereon.

$$\exists X \left(\emptyset \in X \land \forall n \in X, S(n) \in X \right)$$

By declaration of the Axiom of Infinity, it's clear that such a set would satisfy the last Peano axiom. Moreover, since zero is declared as the empty set, it's seen that zero would be contained in such a set. Using this successor, one can show that $S(m) = S(n) \implies m = n$ for such a set. First, for two given elements, m and n, assume that $m \cup \{m\} = n \cup \{n\}$. Then is holds that $\{a \mid a \in m \lor a \in \{m\}\} = \{a \mid a \in n \lor a \in \{n\}\}$. By the Axiom of Extensionality, these belong to the same set. Thus, $m = n$. Next, show that there is no set whose successor is zero, or the empty set. Suppose there exists a set q such that $q \cup \{q\} = \emptyset$. It would then follow by definition of union that $\emptyset = \{t \mid t \in q \lor t \in \{q\}\}$. If this were true, that would imply that the empty set has elements, which is contradictory to its definition by the Axiom Schema of Specification. Finally, sets that are constructed by the Axiom of Infinity, call them inductive sets. The intersection of all inductive sets are the naturals; $\bigcap X = \mathbb{N}$. Show that $n \in \mathbb{N} \implies S(n) \in \mathbb{N}$. Since n would be a set, by the Axiom Schema of Replacement, it would hold that the mapping of n would belong to a set. By declaration of the inductive set, it holds that its successor would also be contained in the set. This is the modern definition of the natural numbers, one of the most fundamental sets in mathematics that can be used to build the integers, rationals, algebraic numbers, real numbers, complex numbers, etc.

Relations

A relation is a way of relating two sets. For a formal definition of relations, a definition of the Cartesian product of sets would be handy. Given two sets A

and B, the Cartesian product of A and B is defined as:

$$A \times B := \{(a, b) \mid a \in A \wedge b \in B\} .$$

In short, a Cartesian product between two sets generates a set of pairs of the two sets. Now, given two sets A and B, a relation between them, R, is a subset of their Cartesian product; $R \subseteq A \times B$. There are potentialy many different relations on sets. Taking a relation over a set A means a subset of $A \times A$. For example, take $R = \{(1, 1), (1, 2), (3, 5)\}$. This is a relation over the natural numbers $\mathbb{N}$. In that example, one could say that 3 relates 5, denoted as $3 \sim 5$ or $3 \; R \; 5$. Order matters in relations, so, 5 does *not* relate 3; $5 \not\sim 3$, or $5 \not R \; 3$.

Constructing Relations

Consider an $R = \{(0, 0), (1, 2), (2, 4)\}$ over the naturals $\mathbb{N}$. Denote elements of the left side of pairs a and right side b. From this set, it's seen that $a \sim b \iff b = 2a$. a relating b and b being twice a is equivalent. Another interpretation is that $a \sim b \iff b = a + a$. Thus, $b = 2a \iff b = a + a$. One may go the other way with constructing relations. Suppose one defines a relation over $\mathbb{N}$ as $x \sim y \iff y = x^2$. Then, the relation, R, is exactly $R = \{(0, 0), (1, 1), (2, 4), (3, 9), \cdots\}$.

Reflexive Relations

A relation R over a set X is reflexive if for every $x \in X$, $x \sim x$. An example is the equals relation; $x \sim y \iff x = y$. Every element in X relates itself. Thus, $=$ is a reflexive relation.

Symmetric Relation

A relation R over a set X is symmstric if for every $x, y \in X, x \sim y \implies y \sim x$. One example is $x \sim y \iff x$ and y have the same parity. That is, if they're both even or both odd.

Transitive Relation

A relation R over a set X is transitive if for every $x, y, z \in X$,

$$x \sim y \wedge y \sim z \implies x \sim z.$$

An example is $x \sim y \iff x \leq y$. That is, if x is less than or equal to y, and y is less than or equal to z, then x is less than or equal to z.

Equivalence Relation

A relation is an equivalnce relation if it is reflexive, symmetric, and transitive.