

# Tapering, DDE, and PSFs

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**Abstract.**

## 1. Introduction

## 2. Motivation

## 3. Matrix formulation of the problem - 1D interferometer

Here I intend to use convolution matrices properties to *qualitatively* study how “pseudo-PSF” vary as a function of source location. Here I limit myself to a 1-dimensional interferometer (scalar only), so that Convolution matrices are Toeplitz-symetric (see below). In a more general case, (along my intuition - but should be thought more carefully), convolution matrices should be block-Toeplitz (each block is a Toeplitz), while symetricity should still be true.

### 3.1. Remarks on the convolution and linear algebra

In functional form the convolution theorem can be written as follows:

$$\mathcal{F}\{a.b\} = \mathcal{F}\{a\} * \mathcal{F}\{b\} \quad (1)$$

Noting the convolution product is linear, we can re-express the convolution product and associated theorem using linear transformations:

$$\mathbf{F}\mathbf{A}\mathbf{b} = \mathbf{C}_\mathbf{A}\mathbf{F}\mathbf{b} \quad (2)$$

where  $\mathbf{F}$  is the Fourier operator of size  $n_{uv} \times n_{lm}$  ( $\mathbf{F}$  is unitary  $\mathbf{F}^H\mathbf{F} = \mathbf{1}$ ),  $\mathbf{b}$  is a vector with size  $n_{lm}$ . The matrix  $\mathbf{A}$  models the scalar multiplication of each point in  $\mathbf{b}$ , and is therefore diagonal of size  $n_{lm} \times n_{lm}$ , and  $\mathbf{C}_\mathbf{A}$  is the convolution matrix of size  $n_{uv} \times n_{uv}$ . There is a bijective relation

$$\mathbf{A} \longleftrightarrow \mathbf{C}_\mathbf{A} \quad (3)$$

in the sense that a scalar multiplication defines a convolution function and conversely. The matrices  $\mathbf{A}$  and  $\mathbf{C}_\mathbf{A}$  always have the following properties:

- $\mathbf{A}$  is diagonal
- In the 1D case
  - $\mathbf{C}_\mathbf{A}$  is Toeplitz
  - In addition, for radiointerferometry, because the uv plane is symetric,  $\mathbf{C}_\mathbf{A}$  is symetric

The matrix  $\mathbf{C}_\mathbf{A}$  being Toeplitz, each row  $[\mathbf{C}_\mathbf{A}]_l$  with sky coordinate  $l$  can be built using a rolling operator  $\Delta_l$  that shifts the first row (the PSF at the field center for example) to location of row  $l$ :

$$[\mathbf{C}_\mathbf{A}]_l = \Delta_l \{[\mathbf{C}_\mathbf{A}]_0\} \text{ and} \quad (4)$$

$$[\mathbf{C}_\mathbf{A}]_0 = \mathbf{F}^H \text{diag}(\mathbf{A}) \quad (5)$$

The rolling operator is essentially just a reindexing, and has the following properties:

$$\Delta_l \{a\mathbf{x}\} = a\Delta_l \{\mathbf{x}\} \quad (6)$$

$$\Delta_l \left\{ \sum_i \mathbf{x}_i \right\} = \sum_i \Delta_l \{\mathbf{x}_i\} \quad (7)$$

### 3.2. PSF behaviour

If  $\mathbf{X}$  is the true sky, then the dirty image  $\mathbf{X}_{ij}^D$  of baseline  $(ij)$  can be written as:

$$\mathbf{x}_{ij}^D = \mathbf{F}^H \mathbf{S}_{c,ij} \mathbf{C}_\mathbf{T} \mathbf{S}_{\square,ij} \mathbf{F} \mathbf{A} \mathbf{x} \quad (8)$$

where  $\mathbf{A}_{ij}$  models the DDE effects and is an  $n_{pix} \times n_{pix}$  diagonal matrix (taking polarisation into account it is an  $4n_{pix} \times 4n_{pix}$  block diagonal matrix),  $\mathbf{T}$  is the tapering/averaging function,  $\mathbf{S}_{\square}$  samples the region over which the tapering/averaging is made, and  $\mathbf{S}_{c,ij}$  selects the central point of the averaged/tapered visibility set. Using Eq. 4, we have:

$$\mathbf{x}_{ij}^D = \mathbf{C}_{\mathbf{S}_{c,ij}} \mathbf{T} \mathbf{C}_{\mathbf{S}_{\square,ij}} \mathbf{F}^H \mathbf{F} \mathbf{A}_{ij} \mathbf{x} \quad (9)$$

$$= \mathbf{C}_{\mathbf{S}_{c,ij}} \mathbf{T} \mathbf{C}_{\mathbf{S}_{\square,ij}} \mathbf{A}_{ij} \mathbf{x} \quad (10)$$

$$\sim \mathbf{C}_{\mathbf{S}_{c,ij}} \mathbf{T} \mathbf{A}_{ij} \mathbf{x} \quad (11)$$

where Eq. 11 is true when the support of the function  $T$  is smaller than the sampling domain of  $\mathbf{S}_{\square}$ .

Averaged over all baselines, the dirty image becomes:

$$\mathbf{x}^D = \mathbf{C}_{STA} \mathbf{x} \quad (12)$$

$$\text{with } \mathbf{C}_{STA} = \sum_{ij} \mathbf{C}_{S_{c,ij}} \mathbf{T} \mathbf{A}_{ij} \quad (13)$$

### 3.3. Deriving the Pseudo-PSF

#### 3.3.1. PSF and Pseudo-PSF

We can already see that  $\mathbf{C}_{S_{c,ij}} \mathbf{T} \mathbf{A}_{ij}$  in Eq. 11 is NOT Toeplitz anymore because each column is multiplied by a different value (DDE multiplied by the tapering function). The dirty sky is therefore not anymore the convolution of the true sky by the psf *ie* the PSF varies across the field of view.

#### 3.3.2. Slow way

Calculate the psf estimating  $\mathcal{C}$  from direct calculation. Eventually at discrete locations on a grid.

#### 3.3.3. Quickly deriving the Pseudo-PSF

This is tricky part. The problem amount to finding any column  $l$  of  $\mathbf{C}$  on demand. For notation convenience, we merge  $\mathbf{T}$  and  $\mathbf{A}_{ij}$  together in  $\mathbf{A}_{ij}$ . Operator  $[\mathbf{M}]_l$  extracts column  $l$  from matrix  $\mathbf{M}$ , and using Eq. 6, 7 and 12:

$$[\mathbf{C}]_l = \left[ \sum_{ij} \mathbf{C}_{S_{c,ij}} \mathbf{A}_{ij} \right]_l \quad (14)$$

$$= \sum_{ij} a_{ij}^l [\mathbf{C}_{S_{c,ij}}]_l \quad (15)$$

$$\text{with } a_{ij}^l = \mathbf{A}_{ij}(l) \quad (16)$$

$$= \sum_{ij} \Delta_l \{ a_{ij}^l [\mathbf{C}_{S_{c,ij}}]_0 \} \quad (17)$$

$$= \sum_{ij} \Delta_l \{ \mathbf{F}^H a_{ij}^l \text{diag}(\mathbf{S}_{c,ij}) \} \quad (18)$$

If we now assume that at any given location  $l$ , the scalar  $a_{ij}^l$  can be described by a smooth *function* of the uv coordinates ( $(ij)$ -indices), then we can write:

$$[\mathbf{C}]_l = \sum_{ij} \Delta_l \{ \mathbf{F}^H \mathbf{A}^l \text{diag}(\mathbf{S}_{c,ij}) \} \quad (19)$$

$$= \sum_{ij} \Delta_l \{ \mathbf{C}_{A^l} \mathbf{F}^H \text{diag}(\mathbf{S}_{c,ij}) \} \quad (20)$$

$$= \sum_{ij} \Delta_l \{ \mathbf{C}_{A^l} [\mathbf{C}_{S_{c,ij}}]_0 \} \quad (21)$$

$$= \Delta_l \left\{ \mathbf{C}_{A^l} \sum_{ij} [\mathbf{C}_{S_{c,ij}}]_0 \right\} \quad (22)$$

$$= \Delta_l \{ \mathbf{C}_{A^l} [\mathbf{C}_{S_c}]_0 \} \quad (23)$$

$$(24)$$

The approximate observed Pseudo-PSF is the convolution of the PSF at the phase center ( $[\mathbf{C}_{S_c}]_0$ ) and the fourier transform of the uv-dependent tapering function at given  $lm$  ( $\mathbf{C}_{A^l}$ ).

In other words, to compute the PSF at a given location ( $lm$ ):

- Find  $\mathbf{A}$ :
  - Compute weight  $w_{ij}$  for each baseline ( $ij$ )
  - Fit the uv-dependent weight by (for example), a Gaussian function  $w_{ij} \sim w(u, v) = \mathcal{G}(u, v)$
- Compute the  $PSF_{lm}$  at ( $lm$ ) from the PSF at the phase center  $PSF_0$  as  $PSF_{lm} = \mathcal{F}^{-1}(w) * PSF_0$

For example if the long baselines are more tapered, they are "attenuated". The effective PSF on the edge of the field will get larger by the convolution... Something like that...

## 4. Numerical Experiments

We demonstrate the computational complexity of the quick, the slow derived PSF as a function of sky coordinates and perform a direct numerical results.

### 4.1. Slow derivation and computation cost

### 4.2. Quick derivation and computation cost

We will now show how to derived a pseudo PSF which is base and resolves on the nominal PSF but labeled by a discrete set of integration.

#### 4.2.1. Averaging case

The approach is base on the discrete evaluation of the infinite integral:

$$s(x_0) = \int_{-\infty}^{+\infty} \exp(jx) dx \quad (25)$$

We are now in a position to discuss the discrete interpretation of the previous integral. If the integration interval is

kept small enough, then Eq25 can be rewritten as follows:

$$s(x_0) = \frac{1}{\Delta x_0} \int_{x_c - \Delta x_0}^{x_c + \Delta x_0} \exp(jx) dx \quad (26)$$

From Eq.26 we see that the integral is the Fourier transform of a top-hat function. Therefore,

$$s(x_0) = \text{sinc} \frac{\Delta x_0}{2} \exp(jx_c) \quad (27)$$

In a two dimensional case, the previous becomes:

$$s(x_0, y_0) = \text{sinc} \frac{\Delta y_0}{2} \text{sinc} \frac{\Delta x_0}{2} \exp(j(x_c, y_c)) \quad (28)$$

where  $x_c$  and  $y_c$  are the centre of  $\Delta x_0$  and  $\Delta y_0$  respectively.

In this section, knowing the response of an array to a source at the phase centre  $(l_0, m_0)$ , we want to measure the array response at a given location  $(l, m) \neq (l_0, m_0)$ . A pair wise element  $(p, q)$  of the array measures the quantity at  $(l, m)$ :

$$V_{pq}(t, \nu) = \exp \left\{ 2j\pi(u_{pq}l + v_{pq}m + w_{pq}n) \right\}, \quad (29)$$

where,

$$u_{pq} = u_{pq}(t, \nu) \quad (30)$$

$$v_{pq} = v_{pq}(t, \nu) \quad (31)$$

$$w_{pq} = w_{pq}(t, \nu) \quad (32)$$

The Earth rotation causes  $V_{pq}(t, \nu)$  to variate in time and frequency. Taking this effect into account, Eq. 29 is rewritten as an integration over narrower time and frequency band. From the above derivation,  $x_0 = x_0(t)$  and  $y_0 = y_0(\nu)$ , we then have (coming directly from RIME1, Oleg):

$$V_{pq}^{avg}(t_c, \nu_c) \simeq \text{sinc} \frac{\Delta \Psi}{2} \text{sinc} \frac{\Delta \Phi}{2} V_{pq}(t_c, \nu_c), \quad (33)$$

where  $t_c = \frac{t_s + t_e}{2}$ ,  $\nu_c = \frac{\nu_s + \nu_e}{2}$  and

$$\Delta \Phi = \arg V_{pq}(t_c, \nu_e) - \arg V_{pq}(t_c, \nu_s) \quad (34)$$

$$\Delta \Psi = \arg V_{pq}(t_e, \nu_c) - \arg V_{pq}(t_s, \nu_c) \quad (35)$$

The total array response :

$$V(t, \nu) \simeq \sum_{pq} V_{pq}^{avg}(t_c, \nu_c) \quad (36)$$

$$\simeq \sum_{pq} \text{sinc} \frac{\Delta \Psi}{2} \text{sinc} \frac{\Delta \Phi}{2} V_{pq}(t_{mid}, \nu_{mid}) \quad (37)$$

For convenience let assume  $V_{pq}(t_c, \nu_c)$  the visibility of a source at the phase centre (coordinates  $(l_0, m_0)$ ) and  $V_{pq}(t, \nu)$  the one of a source at  $(l, m) \neq (l_0, m_0)$ . That said:

$$V_{pq}(t_c, \nu_c) = \exp \left\{ 2j\pi(u_{pq}l_0 + v_{pq}m_0 + w_{pq}n_0) \right\}, \quad (38)$$

then from the inverse Fourier transform and the convolution theorem we have:

$$PSF(l, m) \simeq \sum_{pq} \mathcal{F}^{-1} \left\{ \text{sinc} \frac{\Delta \Psi}{2} \text{sinc} \frac{\Delta \Phi}{2} \right\} \circ PSF_{pq}(l_0, m_0) \quad (39)$$

Assuming that all the baselines are pointing at the same phase centre we have:

$$PSF(l, m) \simeq \left( \sum_{pq} \mathcal{F}^{-1} \left\{ \text{sinc} \frac{\Delta \Psi}{2} \text{sinc} \frac{\Delta \Phi}{2} \right\} \right) \circ PSF_{pq}(l_0, m_0) \quad (40)$$

$$\simeq Tri(l, m) \circ PSF(l_0, m_0) \quad (41)$$

$$Tri(l, m) = \begin{cases} a_{lm} \neq 1 & \text{if the source is in the map} \\ 0 & \text{otherwise} \end{cases}$$

#### 4.2.2. General case

From our previous work, we shows that averaging is similar to convolving the visibilities with a top-hat function, therefore a general case of equation Eq. 25 is derived as follows:

If we tape the visibilities with a window  $f_b$  then:

$$s(x_0) = \int_{-\infty}^{+\infty} f_b(x - x_c) \exp(jx) dx \quad (42)$$

$$= \exp(jx_c) \int_{-\infty}^{+\infty} f_b(u) \exp(ju) du \quad (43)$$

$$= \exp(jx_c) F_b(x_0) \quad (44)$$

**(Also see how you can use convolution to easily derive what you want: Eq44 is a convolution)**

For a narrower band limited, Eq.44 becomes (Yet to verify, not so sure):

$$s(x_0) = \text{sinc} \frac{\Delta x_0}{2} \circ F_b \left( \frac{\Delta x_0}{2} \right) \exp(jx_c) \quad (45)$$

In a two dimensional case, the previous becomes:

$$s(x_0) = \text{sinc} \frac{\Delta x_0}{2} \text{sinc} \frac{\Delta y_0}{2} \quad (46)$$

$$\circ F_b \left( \frac{\Delta y_0}{2} \right) F_b \left( \frac{\Delta x_0}{2} \right) \exp(j(x_c + y_c)) \quad (47)$$

Eq.33 can therefore be estimate in a general case as:

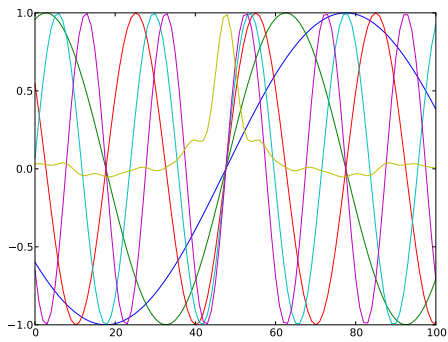
$$V_{pq}^{corr}(t_c, \nu_c) \simeq \text{sinc} \frac{\Delta \Psi}{2} \text{sinc} \frac{\Delta \Phi}{2} \quad (48)$$

$$\circ F_b \left( \frac{\Delta \Psi}{2} \right) F_b \left( \frac{\Delta \Phi}{2} \right) V_{pq}(t_c, \nu_c) \quad (49)$$

The function  $F_b$  depends on the nature of  $f_b$ . The total response in Eq.41 becomes:

$$PSF(l, m) \simeq \left( Tri(l, m) c_b(l, m) \right) \circ PSF(l_0, m_0) \quad (50)$$

We have to clarify well what we derived



**Fig. 1.** averages sine functions vs. sinc function

## 5. Simulation and comparison

## 6. Discussion and conclusion

## References