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Tapering, DDE, and PSFs

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Abstract.

1. Introduction

2. Motivation

3. Matrix formulation of the problem - 1D interferometer

Here I intend to use convolution matrices properties to *qualitatively* study how "pseudo-PSF" vary as a function of source location. Here I limit myselves to a 1-dimensional interferometer (scalar only), so that Convolution matrices are Toeplitz-symetyric (see bellow). In a more general case, (along my intuition - but should be thought more carefully), convolution matrices should be block-Toeplitz (each block is a Toeplitz), while symetricity should still be true.

3.1. Remarks on the convolution and linear algebra

In functional form the convolution theorem can be written as follows:

$$\mathcal{F}\left\{a.b\right\} = \mathcal{F}\left\{a\right\} * \mathcal{F}\left\{b\right\} \tag{1}$$

Noting the convolution product is linear, we can reexpress the convolution product and associated theorem asing linear transformations:

$$\boldsymbol{F}\boldsymbol{A}\boldsymbol{b} = \boldsymbol{\mathcal{C}}_{\boldsymbol{A}}\boldsymbol{F}\boldsymbol{b} \tag{2}$$

where $\boldsymbol{F}$ is the Fourier operator of size $n_{uv} \times n_{lm}$ ($\boldsymbol{F}$ is unitary $\boldsymbol{F}^H\boldsymbol{F} = \boldsymbol{1}$), $\boldsymbol{b}$ is a vector with size n_{lm}. The matrix $\boldsymbol{A}$ models the scalar multiplication of each point in $\boldsymbol{b}$, and is therefore diagonal of size $n_{lm} \times n_{lm}$, and $\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}$ is the convolution matrix of size $n_{uv} \times n_{uv}$. There is a bijective relation

$$\boldsymbol{A} \longleftrightarrow \boldsymbol{\mathcal{C}}_{\boldsymbol{A}} \tag{3}$$

in the sense that a scalar multiplucation defines a convolution function and conversely. The matrices $\boldsymbol{A}$ and $\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}$ always have the following properties:

- $\boldsymbol{A}$ is diagonal
- In the 1D case
  - $\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}$ is Toeplitz
  - In addition, for radiointerferometry, because the uv plane is symetric, $\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}$ is symetric

The matrix $\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}$ being Toeplitz, each row $\left[\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}\right]_l$ with sky coordinate l can be built using a rolling operator Δ_l that shifts the first row (the PSF at the field center for example) to location of row l:

$$\left[\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}\right]_l = \Delta_l\left\{\left[\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}\right]_0\right\} and \tag{4}$$

$$\left[\boldsymbol{\mathcal{C}}_{\boldsymbol{A}}\right]_0 = \boldsymbol{F}^H \ \mathrm{diag}(\boldsymbol{A}) \tag{5}$$

The rolling operator is essentially just a reindexing, and has the following properties:

$$\Delta_l\left\{a\boldsymbol{x}\right\} = a\Delta_l\left\{\boldsymbol{x}\right\} \tag{6}$$

$$\Delta_l\left\{\sum_i \boldsymbol{x}_i\right\} = \sum_i \Delta_l\left\{\boldsymbol{x}_i\right\} \tag{7}$$

3.2. PSF behaviour

If $\boldsymbol{X}$ is the true sky, then the dirty image $\boldsymbol{X}^D_{ij}$ of baseline (ij) can be written as:

$$\boldsymbol{x}^D_{ij} = \boldsymbol{F}^H\boldsymbol{S}_{c,ij}\boldsymbol{\mathcal{C}}_{\boldsymbol{T}}\boldsymbol{S}_{\Box,ij}\boldsymbol{F}\boldsymbol{A}\boldsymbol{x} \tag{8}$$

where $\boldsymbol{A}_{ij}$ models the DDE effets and is an $n_{pix} \times n_{pix}$ diagonal matrix (taking polarisation into account it is an $4n_{pix} \times 4n_{pix}$ block diagonal matrix), $\boldsymbol{T}$ is the tapering/averaging function, $\boldsymbol{S}_{\Box}$ samples the region over which the tapering/averaging is made, and $\boldsymbol{S}_{c,ij}$ selects the central point of the averaged/tapered visibility set. Using Eq. 4, we have:

$$\boldsymbol{x}^D_{ij} = \boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\boldsymbol{T}\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{\Box,ij}}\boldsymbol{F}^H\boldsymbol{F}\boldsymbol{A}_{ij}\boldsymbol{x} \tag{9}$$

$$= \boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\boldsymbol{T}\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{\Box,ij}}\boldsymbol{A}_{ij}\boldsymbol{x} \tag{10}$$

$$\sim \boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\boldsymbol{T}\boldsymbol{A}_{ij}\boldsymbol{x} \tag{11}$$

where Eq. 11 is true when the support of the function T is smaller than the sampling domain of $\boldsymbol{S}_{\square}$.

Averaged over all baselines, the dirty image becomes:

$$\boldsymbol{x}^D = \boldsymbol{\mathcal{C}}_{STA}\boldsymbol{x} \tag{12}$$

$$\text{with } \boldsymbol{\mathcal{C}}_{STA} = \sum_{ij} \boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}} \boldsymbol{T}\boldsymbol{A}_{ij} \tag{13}$$

3.3. Deriving the Pseudo-PSF

3.3.1. PSF and Pseudo-PSF

We can already see that $\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\boldsymbol{T}\boldsymbol{A}_{ij}$ in Eq. 11 is NOT Toeplitz anymore because each colunm is multiplied by a different value (DDE muliplied by the tapering function). The dirty sky is therefore not anymore the convolution of the true sky by the psf *ie* the PSF varies accross the field of view.

3.3.2. Slow way

Calculate the psf estimating $\mathcal{C}$ from direct calculation. Eventually at discrete locations on a grid.

3.3.3. Quickly deriving the Pseudo-PSF

This is tricky part. The problem amount to finding any column l of $\boldsymbol{\mathcal{C}}$ on demand. For notation convenience, we merge $\boldsymbol{T}$ and $\boldsymbol{A}_{ij}$ together in $\boldsymbol{A}_{ij}$. Operator $[\boldsymbol{M}]_l$ extracts column l from matrix $\boldsymbol{M}$, and using Eq. 6, 7 and 12:

$$[\boldsymbol{\mathcal{C}}]_l = \left[\sum_{ij} \boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}} \boldsymbol{A}_{ij}\right]_l \tag{14}$$

$$= \sum_{ij} a^l_{ij} \left[\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\right]_l \tag{15}$$

$$\text{with } a^l_{ij} = \boldsymbol{A}_{ij}(l) \tag{16}$$

$$= \sum_{ij} \Delta_l \left\{a^l_{ij} \left[\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\right]_0\right\} \tag{17}$$

$$= \sum_{ij} \Delta_l \left\{\boldsymbol{F}^H a^l_{ij} \,\mathrm{diag}\left(\boldsymbol{S}_{c,ij}\right)\right\} \tag{18}$$

If we now assume that at any given location l, the scalar a^l_{ij} can be described by a smooth *function* of the uv coordinates ((ij)-indices), then we can write:

$$[\boldsymbol{\mathcal{C}}]_l = \sum_{ij} \Delta_l \left\{\boldsymbol{F}^H \boldsymbol{A}^l \,\mathrm{diag}\left(\boldsymbol{S}_{c,ij}\right)\right\} \tag{19}$$

$$= \sum_{ij} \Delta_l \left\{\boldsymbol{\mathcal{C}}_{\boldsymbol{A}^l}\ \boldsymbol{F}^H \mathrm{diag}\left(\boldsymbol{S}_{c,ij}\right)\right\} \tag{20}$$

$$= \sum_{ij} \Delta_l \left\{\boldsymbol{\mathcal{C}}_{\boldsymbol{A}^l}\ \left[\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\right]_0\right\} \tag{21}$$

$$= \Delta_l \left\{\boldsymbol{\mathcal{C}}_{\boldsymbol{A}^l}\ \sum_{ij} \left[\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_{c,ij}}\right]_0\right\} \tag{22}$$

$$= \Delta_l \left\{\boldsymbol{\mathcal{C}}_{\boldsymbol{A}^l}\ \left[\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_c}\right]_0\right\} \tag{23}$$

$$\tag{24}$$

The approximate observed Pseudo-PSF is the convolution of the PSF at the phase center ($[\boldsymbol{\mathcal{C}}_{\boldsymbol{S}_c}]_0$) and the fourier transform of the uv-dependent tapering function at given lm ($\boldsymbol{\mathcal{C}}_{\boldsymbol{A}^l}$).

In other words, to compute the PSF at a given location (lm):

- Find $\boldsymbol{A}$:
  - Compute weight w_{ij} for each baseline (ij)
  - Fit the uv-dependent weight by (for example), a Gaussian function $w_{ij} \sim w(u,v) = \boldsymbol{\mathcal{G}}(u,v)$
- Compute the PSF_{lm} at (lm) from the PSF at the phase center PSF_0 as $PSF_{lm} = \mathcal{F}^{-1}(w) * PSF_0$

For example if the long baselines are more tapered, they are "attenuated". The effective PSF on the edge of the field will get larger by the convolution... Something like that...

4. Numerical Experiments

We demonstrate the computational complexity of the quick, the slow derived PSF as a function of sky coordinates and perform a direct numerical results.

4.1. Slow derivation and computation cost

4.2. Quick derivation and computation cost

We will now show how to derived a pseudo PSF which is base and resolves on the nominal PSF but labeled by a discrete set of integration.

4.2.1. Averaging case

The approach is base on the evaluation of the band limited case of the following infinite signal:

$$\tilde{s}(\kappa) = \int_{-\infty}^{+\infty} exp(-j\kappa y)dy, \tag{25}$$

where κ is a real number and $\tilde{s}(\kappa)$ is the Fourier transform of a unitary function, $s(y) = 1$. We can also see $\tilde{s}(\kappa)$ as a delta function, $\delta(\kappa)$. If we band limited $\tilde{s}(\kappa)$ in a small finite phase range Δx_0 centered on the phase x_c, Eq.25 becomes:

$$\begin{aligned}(B_{\Delta x_0}s)(x) =&\frac{1}{\Delta x_0}\int_{x_c-\frac{\Delta x_0}{2}}^{x_c+\frac{\Delta x_0}{2}}\tilde{s}(\kappa)exp(jx\kappa)d\kappa\\ =&\frac{1}{\Delta x_0}\int_{x_c-\frac{\Delta x_0}{2}}^{x_c+\frac{\Delta x_0}{2}}\left(\int_{-\infty}^{+\infty}exp(-j\kappa y)dy\right)\\ &exp(jx\kappa)d\kappa\\ =&\int_{-\infty}^{+\infty}\left(\frac{1}{\Delta x_0}\int_{x_c-\frac{\Delta x_0}{2}}^{x_c+\frac{\Delta x_0}{2}}exp(j(x-y)\kappa)d\kappa\right)dy\\ =&\int_{-\infty}^{+\infty}K(x-y)dy\\ =&K(x)\circ s(x)\\ =&K(x)\end{aligned}$$

Where $s(x)=\mathcal{F}^{-1}\tilde{s}(\kappa)$ and,

$$\begin{aligned}K(x) =&\frac{1}{\Delta x_0}\int_{x_c-\frac{\Delta x_0}{2}}^{x_c+\frac{\Delta x_0}{2}}exp(jx\kappa)d\kappa\\ =&sinc\big(x\frac{\Delta x_0}{2}\big)exp(jxx_c)\end{aligned}$$

Then the result follows:

$$(B_{\Delta x_0}s)(x) =sinc\big(x\frac{\Delta x_0}{2}\big)exp(jxx_c)$$

In a two dimensional case, the previous becomes:

$$(B_{\Delta x_0\Delta e_0}s)(x,e) =sinc(x\frac{\Delta x_0}{2})sinc(e\frac{\Delta e_0}{2}) \tag{26}$$
$$exp\big(j(xx_c+ee_c)\big) \tag{27}$$

Prof:

$$\tilde{s}(\kappa,\gamma)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}exp\big(-j(\kappa y+\gamma h)\big)dydh, \tag{28}$$

where $\tilde{s}(\kappa,\gamma)=\delta(\kappa,\gamma)$ and if we band limited $\tilde{s}(\kappa,\gamma)$ in a small finite phase range Δx_0 centered on the phase x_c varieting in time and a small finite phase range Δe_0 centered on the phase e_c varieting in frequency, Eq.28 becomes:

$$\begin{aligned}(B_{\Delta x_0\Delta e_0}s)(x,e) =&\frac{1}{\Delta x_0\Delta e_0}\int_{x_c-\frac{\Delta x_0}{2}}^{x_c+\frac{\Delta x_0}{2}}\int_{e_c-\frac{\Delta e_0}{2}}^{e_c+\frac{\Delta e_0}{2}}\tilde{s}(\kappa,\gamma)\\ &exp(jx\kappa+je\gamma)d\kappa d\gamma\\ =&\frac{1}{\Delta x_0\Delta e_0}\int_{-\infty}^{+\infty}\int_{x_c-\frac{\Delta x_0}{2}}^{x_c+\frac{\Delta x_0}{2}}exp\big(j\kappa(x-y)\big)\\ &d\kappa dy\left[\int_{-\infty}^{+\infty}\int_{e_c-\frac{\Delta e_0}{2}}^{e_c+\frac{\Delta e_0}{2}}exp\big(j\gamma(e-h)\big)d\gamma dh\right]\\ =&K_1(e)K_2(x)\\ =&sinc(x\frac{\Delta x_0}{2})sinc(e\frac{\Delta e_0}{2})exp\big(j(xx_c+ee_c)\big)\end{aligned}$$

In this section, knowing the response of an array to a source at the phase centre (l_0,m_0), we want to measure the array response at a given location $(l,m)\neq(l_0,m_0)$. A pair wise element (p,q) of the array measures the quantity at (l,m):

$$V_{pq}(t,\nu)=\ exp\Big\{2j\pi\big(u_{pq}l+v_{pq}m+w_{pq}n\big)\Big\}, \tag{29}$$

where,

$$u_{pq}=u_{pq}(t,\nu) \tag{30}$$
$$v_{pq}=v_{pq}(t,\nu) \tag{31}$$
$$w_{pq}=w_{pq}(t,\nu) \tag{32}$$

The Earth rotation causes $V_{pq}(t,\nu)$ to variate in time and frequency. Taking this effect into account, Eq. 29 is rewritten as an integration over narrower time and frequency band. From the above derivation, $x_0=x_0(t)$ and $y_0=y_0(\nu)$, we then have (coming directly from RIME1, Oleg):

$$V_{pq}^{avg}(t_c,\nu_c)\simeq sinc\frac{\Delta\Psi}{2}sinc\frac{\Delta\Phi}{2}V_{pq}(t_c,\nu_c), \tag{33}$$

where $t_c=\frac{t_s+t_e}{2}$, $\nu_c=\frac{\nu_s+\nu_e}{2}$ and

$$\Delta\Phi=argV_{pq}(t_c,\nu_e)-argV_{pq}(t_c,\nu_s) \tag{34}$$
$$\Delta\Psi=argV_{pq}(t_e,\nu_c)-argV_{pq}(t_s,\nu_c) \tag{35}$$

The total array response :

$$V(t,\nu)\simeq\sum_{pq}V_{pq}^{avg}(t_c,\nu_c) \tag{36}$$
$$\simeq\sum_{pq}sinc\frac{\Delta\Psi}{2}sinc\frac{\Delta\Phi}{2}V_{pq}(t_{mid},\nu_{mid}) \tag{37}$$

For convenience let assume $V_{pq}(t_c,\nu_c)$ the visibility of a source at the phase centre (coordinates (l_0,m_0)) and $V_{pq}(t,\nu)$ the one of a source at $(l,m)\neq(l_0,m_o)$. That said:

$$V_{pq}(t_c,\nu_c)=\ exp\Big\{2j\pi\big(u_{pq}l_0+v_{pq}m_0+w_{pq}n_0\big)\Big\}, \tag{38}$$

then from the inverse Fourier transform and the convolution theorem we have:

$$PSF(l,m)\simeq\sum_{pq}\mathcal{F}^{-1}\Big\{sinc\frac{\Delta\Psi}{2}sinc\frac{\Delta\Phi}{2}\Big\}\circ PSF_{pq}(l_0,m_0) \tag{39}$$

Assuming that all the baselines are pointing at the same phase centre we have:

$$PSF(l,m)\simeq\Big(\sum_{pq}\mathcal{F}^{-1}\Big\{sinc\frac{\Delta\Psi}{2}sinc\frac{\Delta\Phi}{2}\Big\}\Big)\circ PSF_{pq}(l_0,m_0) \tag{40}$$
$$\simeq Tri(l,m)\circ PSF(l_0,m_0) \tag{41}$$

$$Tri(l,m)=\begin{cases}a_{lm}\neq1 & \text{Where there is a source}\\ 0 & \text{otherwise}\end{cases}$$

4.2.2. General case

From our previous work, we shows that averaging is similar to convolving the visibilities with a top-hat function, therefore a general case of equation Eq. 25 is derived as follows:

If we tape the visibilities with a window f_b then:

$$s(x_0) = \int_{-\infty}^{+\infty} f_b(x - x_c)exp(jx)dx \tag{42}$$

$$= exp(jx_c)\int_{-\infty}^{+\infty} f_b(u)exp(ju)du \tag{43}$$

$$= exp(jx_c)F_b(x_0) \tag{44}$$

(**Also see how you can use convolution to easily derive what you want: Eq44 is a convolution**)

For a narrower band limited, Eq.44 becomes (Yet to verify, not so sure):

$$s(x_0) = sinc\frac{\Delta x_0}{2} \circ F_b\left(\frac{\Delta x_0}{2}\right)exp(jx_c) \tag{45}$$

In a two dimensional case, the previous becomes:

$$s(x_0) = sinc\frac{\Delta x_0}{2} sinc\frac{\Delta y_0}{2} \tag{46}$$

$$\circ F_b\left(\frac{\Delta y_0}{2}\right)F_b\left(\frac{\Delta x_0}{2}\right)exp(j(x_c + y_c)) \tag{47}$$

Eq.33 can therefore be estimate in a general case as:

$$V_{pq}^{corr}(t_c, \nu_c) \simeq sinc\frac{\Delta\Psi}{2} sinc\frac{\Delta\Phi}{2} \tag{48}$$

$$\circ F_b\left(\frac{\Delta\Psi}{2}\right)F_b\left(\frac{\Delta\Phi}{2}\right)V_{pq}(t_c, \nu_c) \tag{49}$$

The function F_b depends on the nature of f_b. The total response in Eq.41 becomes:

$$PSF(l, m) \simeq \Big(Tri(l, m)c_b(l, m)\Big) \circ PSF(l_0, m_0) \tag{50}$$

We have to clarify well what we derived
Also verify if:

$$\Delta\Psi \simeq \Delta\Phi \tag{51}$$

5. Simulation and comparison

6. Discussion and conclusion

References

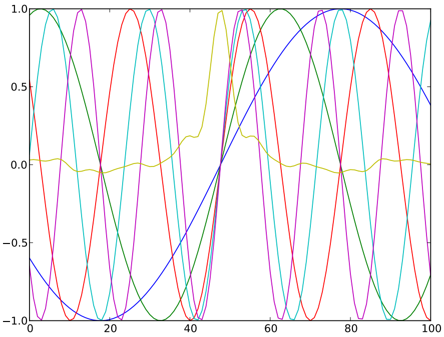

Fig. 1. averages sine functions vs. sinc function