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Fast Scheme to Approximate an Offset Point Spread Function Response

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Abstract.

1. INTRODUCTION AND MOTIVATION

2. PROBLEM STATEMENT and PSF SLOW DERIVATION

We demonstrate the computational complexity of the quick, the slow derived PSF as a function of sky coordinates and perform a direct numerical results.

2.1. interferometric visibility

It is worth noting that in the Radio Astronomy community the cross-correlator output of two elements interferometer in response to a source with spectral brightness distribution $I_\nu(\mathbf{s})$ as a function of the pointing direction $\mathbf{s}$ is the visibility function defined in Eq.1 and obtained by integrating over the solid angle $d\Omega$ (Thompson et al. 2008; Taylor et al. 1999)

$$V(\mathbf{b}) = \int_\Omega I_\nu(\mathbf{s}) e^{-2\pi i \mathbf{bs}} d\Omega, \tag{1}$$

where $\mathbf{b} = \mathbf{b}(t,\nu) = (u_{t\nu}, v_{t\nu}, w_{t\nu})$ is the so-called "*baseline vector*" in wavelength and with modulus the distance between the two elements interferometer. The measurement in Eq.1 is over the entire surface of the celestial sphere, practically the measurement is generally taken over a finite surface area of the celestial sphere due to the finite nature of the tracking source and other effects. Furthermore, the result is averaged over finite time/frequency bins. Suppose Δt centered at t_0 and $\Delta\nu$ centered at ν_0 the time and frequency sampling intervals respectively. Assuming that Δt and $\Delta\nu$ are small enough so that $I_\nu(\mathbf{s})$ remains constant while the complex phase, $-2\pi i\mathbf{bs}$ varies linearly. Eq. 1 becomes:

$$V(t_0,\nu_0) = \frac{1}{\Delta t \Delta\nu} \iint\limits_{\Delta t \Delta\nu} \left[\int\limits_\Omega I_\nu(\mathbf{s}) e^{-2\pi i \mathbf{bs}} d\Omega \right] dt d\nu. \tag{2}$$

However, performing the integration over a surface of size, $\Delta t \times \Delta\nu$ yields a nonzero value only in this surface area, which is recognized as integrating of a windowing function $\Pi_{t_0,\nu_0}(t,\nu)$ of size $\Delta t \times \Delta\nu$. Let $\mathbf{b}_0 = \mathbf{b}(t_0,\nu_0) = (u_{t_0\nu_0}, v_{t_0\nu_0}, w_{t_0\nu_0})$ be the baseline vector at the centre of the sampling intervals and $W_{t_0,\nu_0}(t,\nu)$ a weighting sampling function. A generalized sampling kernel for a given (u,v) track, $W_{\mathbf{b}_0}(\mathbf{b})$ associated with the windowing function can be defined as

$$W_{\mathbf{b}_0}(\mathbf{b}) = \frac{\Pi_{t_0,\nu_0}(t,\nu)}{\Delta t \Delta\nu} W_{t_0,\nu_0}(t,\nu) \tag{3}$$

$$= \frac{\Pi(\mathbf{b}-\mathbf{b}_0)}{\Delta u_{t\nu} \Delta v_{t\nu}} W(\mathbf{b}-\mathbf{b}_0). \tag{4}$$

where, $\frac{dt d\nu}{du_{t\nu} dv_{t\nu}} = \frac{\Delta t \Delta\nu}{\Delta u_{t\nu} \Delta v_{t\nu}}$. Similary, Eq.2 can be written in terms of $\mathbf{b}_0$ and can thus be expressed as an integration over uv-bins. We can write

$$V_{\mathbf{b}_0}(\mathbf{x}) = \iint\limits_{u_{t\nu} v_{t\nu}} W_{\mathbf{b}_0}(\mathbf{b}) \left[\int\limits_\Omega I_\nu(\mathbf{s}) e^{-2\pi i \mathbf{bs}} d\Omega \right] du_{t\nu} dv_{t\nu}, \tag{5}$$

where, $\mathbf{x} = (u,v,w)$ the resulting average baseline vector. As previously mentioned, we are interested in PSF response. Therefore, Eq.5 is restricted to the complex visibility measured by the two-element interferometer for a point source located towards the direction $\mathbf{s}$ with unit brightness. That said, $I_\nu(\mathbf{s}) = \delta(\mathbf{b}_0)$ with δ the Dirac delta. It then follows that Eq5 can be written as

$$V_{\mathbf{b}_0}(\mathbf{x}) = \iint\limits_{u_{t\nu} v_{t\nu}} W_{\mathbf{b}_0}(\mathbf{b}) e^{-2\pi i (u_{t\nu} l + v_{t\nu} m + w_{t\nu}(n-1))} du_{t\nu} dv_{t\nu}. \tag{6}$$

Note that Eq.6 is obtained after a delay correction of $\mathbf{bs_0}$ is applied to the signals from the antennas array to steer towards the direction $\mathbf{s}_0$ and $\mathbf{b}(\mathbf{s}-\mathbf{s_0}) = u_{t\nu} l + v_{t\nu} m + w_{t\nu}(n-1)$ describes the time difference between the two incoming signals. The three direction cosines l, m and n are components of $\mathbf{s}-\mathbf{s_0}$ in radians with $n=\sqrt{1-l^2-m^2}$. For an extensive discussion, see (Thompson et al. 2008; Taylor et al. 1999; Leshem & van der Veen 2000). Let $G_{\mathbf{b}}(l,m)=e^{-2\pi i[w_{t\nu}(n-1)]}$ the fringe from the w-term. The w-term depends on the variation of $u_{t\nu}$ and $v_{t\nu}$. In other words, $w_{t\nu}=(\gamma+\alpha du_{t\nu}+\beta dv_{t\nu})$; γ,α,β are reals numbers. In this paper, we consider the w-term in the general case where the array is coplanar ($w_{t\nu}=0$) or not ($w_{t\nu}\neq 0$) and/or when the array is tracking a small field ($n\approx 1$) or a wide field ($n\ll 1$). The fringe from the w-term is separated; the measurement in Eq6 becomes a two dimensional Fourier transform. Making use of the convolution theorem which states that the Fourier transform of the product of two functions result in a convolution and vice-versa [1], we have

$$V_{\mathbf{b}_0}(\mathbf{x})=\widetilde{G}_{\mathbf{b}}(l,m)\circ\iint\limits_{u_{t\nu}v_{t\nu}}W_{\mathbf{b}_0}(\mathbf{b})e^{-2\pi i(u_{t\nu}l+v_{t\nu}m)}du_{t\nu}dv_{t\nu}. \tag{7}$$

Also, $W_{\mathbf{b}_0}(\mathbf{b})$ is defined as a product of two functions (refer to Eq. 4), making use of the shifted Fourier transform properties and the convolution theorem, the following expansion is valid

$$V_{\mathbf{b}_0}(\mathbf{x})=\left(\widetilde{G}_{\mathbf{b}}\circ\left[C_{\mathbf{b}_0}\widetilde{\Pi}_{\mathbf{b}}\right]\circ\left[C_{\mathbf{b}_0}\widetilde{W}_{\mathbf{b}}\right]\right)(l,m) \tag{8}$$

The symbols $\circ$ and $\tilde{}$ denote the convolution operator and the Fourier transform respectively. The phase gradient, $C_{\mathbf{b}_0}(l,m)$ is defined as

$$C_{\mathbf{b}_0}(l,m)=e^{-2\pi i(u_{t_0\nu_0}l+v_{t_0\nu_0}m)} \tag{9}$$

2.1.1. Computational cost

3. PSF QUICK DERIVATION

In order to further optimize the slow derivation of the PSF described above to particularly reduce its computational cost, we will need to understand the concept and theory of signal correlation in aperture synthesis. If it is assumed that $\Pi(\mathbf{b})$ is the top-hat windowing function, then it can be demonstrated that its Fourier transform is (see Appendix A)

$$\widetilde{\Pi}(\mathbf{b})=sinc\Big(\frac{-2\pi t\Delta\nu}{2}\Big)sinc\Big(\frac{-2\pi\nu\Delta t}{2}\Big) \tag{10}$$

It is demonstrated in (Smirnov 2011) that for natural weighting, Eq.7 can be approximated in term of the phase change in time, $\Delta\Psi$ and in frequency $\Delta\Phi$ for the case of smearing as:

$$V_{\mathbf{b}_0}(\mathbf{x})\simeq\Big[sinc\frac{\Delta\Psi}{2}sinc\frac{\Delta\Phi}{2}C_{\mathbf{b}_0}(l,m)\Big]\circ\left(\widetilde{G}_{\mathbf{b}_0}\circ\Big[C_{\mathbf{b}_0}\widetilde{W}_{\mathbf{b}_0}\Big]\right)(l,m) \tag{11}$$

where $\Delta\Psi$ and $\Delta\Phi$ are defined as

$$\begin{aligned}\Delta\Psi=&2\pi\Big[(u_{t_s\nu_0}-u_{t_e\nu_0})l+(v_{t_s\nu_0}-v_{t_e\nu_0})m\\&+(w_{t_s\nu_0}-w_{t_e\nu_0})(n-1)\Big],\end{aligned} \tag{12}$$

$$\begin{aligned}\Delta\Phi=&2\pi\Big[(u_{t_0\nu_s}-u_{t_0\nu_e})l+(v_{t_0\nu_s}-v_{t_0\nu_e})m\\&+(w_{t_0\nu_s}-w_{t_0\nu_e})(n-1)\Big],\end{aligned} \tag{13}$$

where t_s, t_e, ν_s and ν_e are the starting time, ending time, starting frequency and ending frequency respectively of the sampling intervals.

We then generalized the approximation of smearing where $\Pi(\mathbf{b})$ is a random windowing function as follows:

$$V_{\mathbf{b}_0}(\mathbf{x})\simeq\left(\widetilde{G}_{\mathbf{b}_0}\circ\Big[C_{\mathbf{b}_0}\widetilde{\Pi}_{\mathbf{b}_0}\Big]\circ\Big[C_{\mathbf{b}_0}\widetilde{W}_{\mathbf{b}_0}\Big]\right)(l,m) \tag{14}$$

3.0.2. Computational cost

4. IMAGING

During conventional interferometric imaging, the baseline visibilities are mapped on a uv-plane and the result is inverse Fourier transformed

$$I^D(l_0,m_0)=\sum_{\mathbf{b}kj}P_{\mathbf{b}kj}(l_0,m_0)\circ I^D_{\mathbf{b}kj}(l_0,m_0) \tag{15}$$

with,

$$P_{\mathbf{b}kj}(l_0,m_0)=\mathcal{F}^{-1}\{S_{\mathbf{b}kj}\},\quad I^D_{\mathbf{b}kj}(l_0,m_0)=\mathcal{F}^{-1}\{V_{\mathbf{b}kj}\} \tag{16}$$

where $\mathcal{F}^{-1}$ represents the inverse Fourier transform and $S_{\mathbf{b}kj}$ is the sampling function at the space kj of the baseline pq. The sampling function is only nonzero in the neighborhood of the track described by $\mathbf{b}kj$ and account for the interpolation coefficients of the gridding process.

4.1. *The case of hight time/frequency resolution post-correlation without averaging*

$V_{\mathbf{b}kj}$ is a hight time and frequency resolution samples defined as:

$$V_{\mathbf{b}kj}=\delta_{\mathbf{b}kj}V_{\mathbf{b}kj},\quad V_{\mathbf{b}kj}=e^{-2i\pi\mathbf{bs}} \tag{17}$$

The definition leads Eq. 16 to:

$$I^D_{\mathbf{b}kj}(l_0,m_0)=\delta_{\mathbf{b}kj} \tag{18}$$

Hence, Eq15 becomes

$$\begin{aligned}I_D(l_0,m_0)&=\sum_{\mathbf{b}kj}P_{\mathbf{b}kj}(l_0,m_0) &(19)\\&=P(l_0,m_0), &(20)\end{aligned}$$

where, $P(l,m)$ is the array PSF.

[1] $http://mathworld.wolfram.com/ConvolutionTheorem.html$
$http://en.wikipedia.org/wiki/Convolution\_theorem$

4.2. The case of hight time/frequency resolution post-correlation with averaging

In the above section, an averaged visibility, $V_{\mathbf{b}_0}(\mathbf{x})$ given either by Eq. 8 (pseudo-averaging) or Eq. 13 (pseudo-averaging approximation) or Eq. 14 (baselines dependent windowing approximation) is predicted for each baseline corresponding to the averaged visibility. We can write:

$$V_{\mathbf{b}kj} = \delta_{\mathbf{b}kj} F_{\mathbf{b}kj}, \quad F_{\mathbf{b}kj} = V_{\mathbf{b}_0}(\mathbf{x}) \tag{21}$$

What leads Eq. 16 to the following:

$$I^D_{\mathbf{b}kj}(l_0, m_0) = \mathcal{F}^{-1}\{\delta_{\mathbf{b}kj} F_{\mathbf{b}kj}\} \tag{22}$$

$$\Rightarrow \quad I^D_{\mathbf{b}kj}(l_0, m_0) = \mathcal{F}^{-1}\{\delta_{\mathbf{b}kj} V_{\mathbf{b}_0}(\mathbf{x})\} \tag{23}$$

Hence, Eq15 becomes

$$I_D(l_0, m_0) = \sum_{\mathbf{b}kj} P_{\mathbf{b}kj}(l_0, m_0) \circ \mathcal{F}^{-1}\{\delta_{\mathbf{b}kj} V_{\mathbf{b}_0}(\mathbf{x})\} \tag{24}$$

$$\neq P(l_0, m_0) \tag{25}$$

4.3. The case of hight time/frequency resolution post-correlation with averaging: Image plane

$$I_D(l_0, m_0) = \sum_{\mathbf{b}kj} P_{\mathbf{b}kj}(l_0, m_0) \circ \mathcal{F}^{-1}\{\delta_{\mathbf{b}kj} V_{\mathbf{b}_0}(\mathbf{x})\} \tag{26}$$

$$= [\sum_{\mathbf{b}kj} P_{\mathbf{b}kj}(l_0, m_0)] \circ [\sum_{\mathbf{b}kj} \mathcal{F}^{-1}\{\delta_{\mathbf{b}kj} V_{\mathbf{b}_0}(\mathbf{x})\}] \tag{27}$$

$$= P(l_0, m_0) \circ F(l_0, m_0) \tag{28}$$

Algorithm.

Algorithm 1 Contruct Maxtrices, $\mathbf{\Delta\Psi}$ and $\mathbf{\Delta\Phi}$

procedure VARIATION IN TIME AND FREQUENCY
- $FoV = N_{pix}\Delta l \Delta m$ $//N_{pix}$ the number of pixels
- $\Delta u = \Delta v = 1/FoV$
- $v_0 = -(N_{pix} - 1)\Delta v/2$
- $h = (2v_0)/(N_{pix} - 1)$, $cst = \frac{-\pi}{432\times 10^2}$
- **for** i from 1 to N_{pix} **do**
  - $u_0 = -(N_{pix} - 1)\Delta u/2$
  - **for** j from 1 to N_{pix} **do**
    - $\theta = \arctan(u_0/v_0)$
    - $du_{i,j} = cst\sqrt{u_0^2 + v_0^2}\sin\theta$
    - $dv_{i,j} = cst\sqrt{u_0^2 + v_0^2}\cos\theta$
    - $\Delta\Psi_{i,j} = \pi(du_{i,j}l + dv_{i,j}m)\Delta t$
    - $\Delta\Phi_{i,j} = \pi\frac{\Delta\nu}{\nu}\sqrt{l^2 + m^2}\sqrt{u_0^2 + v_0^2}$
    - $u_0 = u_0 + h$
  - **end for**
  - $v_0 = v_0 + h$
- **end for**

end procedure

$$P = \mathcal{F}^{-1}\Big\{sinc(\Psi)sinc(\Phi)\mathcal{F}\{P_{l,m}\}\Big\} \tag{29}$$

PipLine

5. PSEUDO PSF's DECONVOLTION

5.1. Gridding and degridding

6. RESULTS

7. DISCUSSION AND CONCLUSION

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Appendix A: The Fourier Transform of a top-hat windowing function

A two dimensional top-hat window is defined as

$$\mathbf{\Pi}(\mathbf{b}) = \begin{cases} 1 & \text{for } t \times \nu \in [t_s, t_e] \times [\nu_s, \nu_e], \\ 0 & \text{for } t \times \nu \notin [t_s, t_e] \times [\nu_s, \nu_e] \end{cases} \tag{A.1}$$

where $[t_s, t_e] \times [\nu_s, \nu_e]$ is the passband region.

$$\widetilde{\Pi}(\mathbf{b}) = \frac{1}{\Delta t \Delta\nu}\int_{t_s}^{t_e}\int_{\nu_s}^{\nu_e} e^{-2\pi i t\nu} dt d\nu$$

Let suppose that

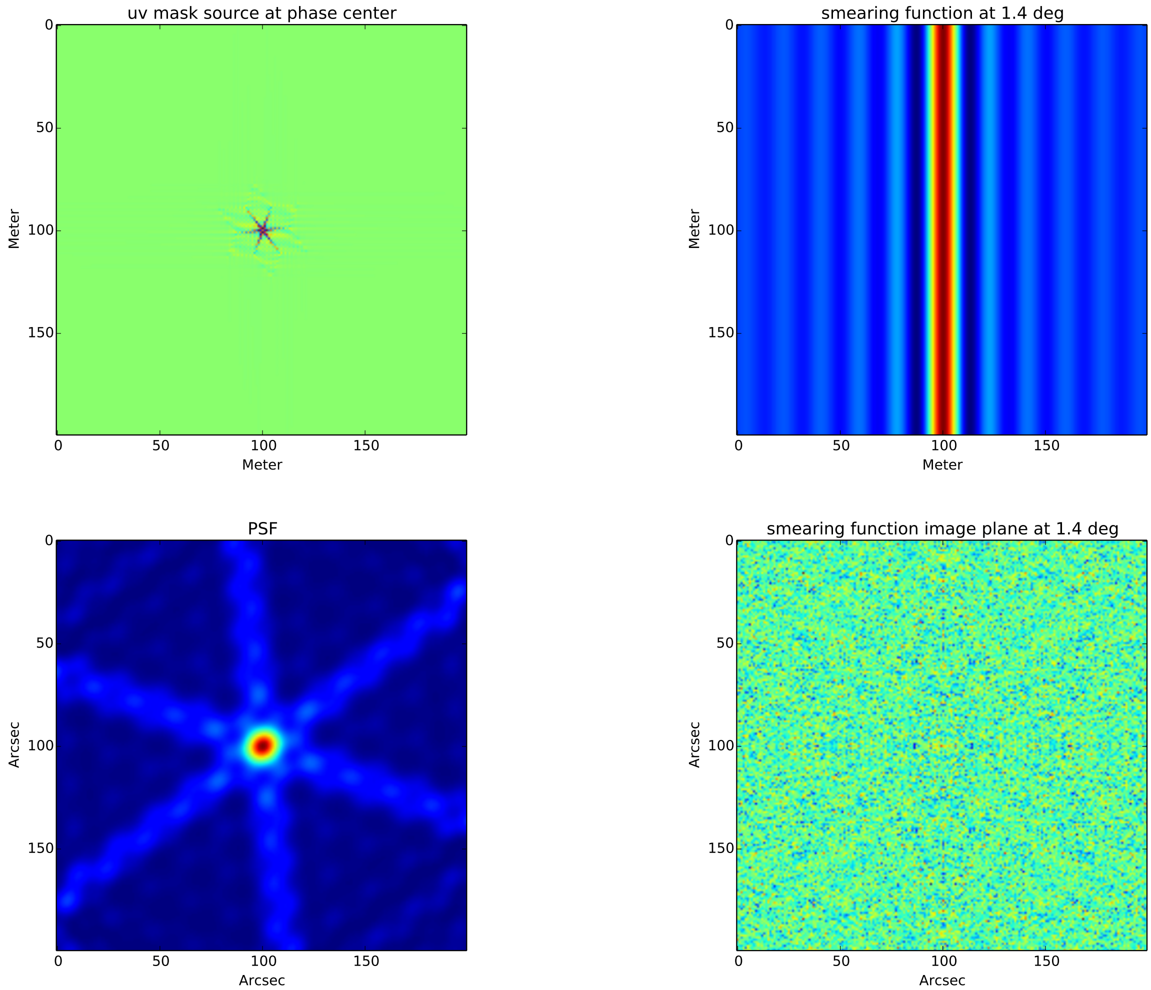

Fig. 1: Hanning-Bessel windowing function and its tapering response.

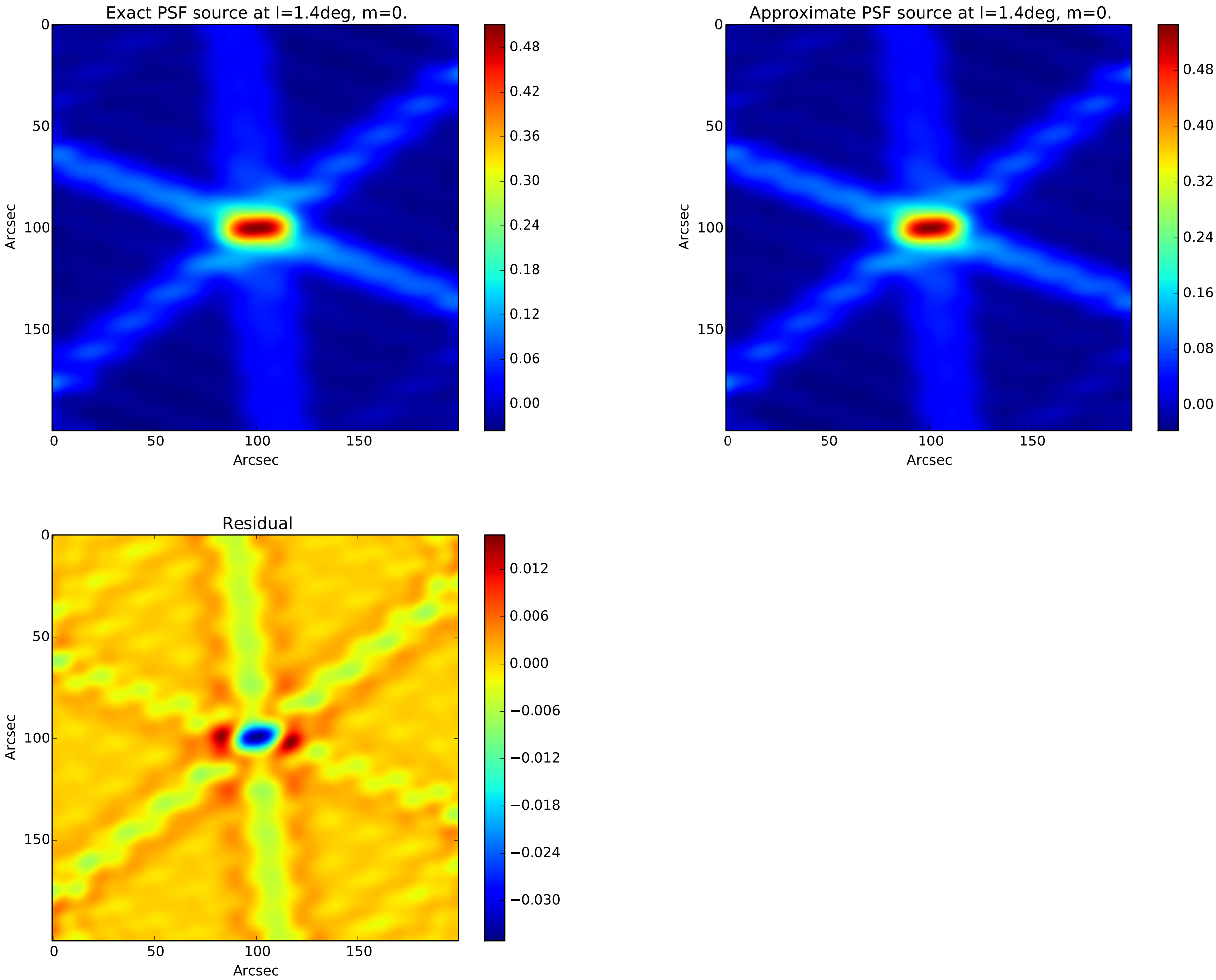

Fig. 2: Hanning-Bessel windowing function and its tapering response.