Part I, Problem 1E-3c

Find in parametric form the equations for all lines passing through $(1, 1, 1)$ and lying in the plane $x + 2y - z = 2$.

Solution

Part I, Problem 1E-4
Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?

Solution

Part I, Problem 1I-3ab

Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.

(a) $\mathbf{r} = 2\cos^2(t)\mathbf{i} + \sin^2(t)\mathbf{j}$

(b) $\mathbf{r} = \cos(2t)\mathbf{i} + \cos(t)\mathbf{j}$

Solution

Part I, Problem 1I-5

A string is wound clockwise around the circle of radius a centered at the origin O; the initial position of the end P of the string is $(a, 0)$. Unwing the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P.

(Use vectors; express the position vector OP as a vector function of one variable.)

Solution

Part I, Problem 1J-2
Let

$$OP = \frac{1}{1+t^2}\mathbf{i} + \frac{t}{1+t^2}\mathbf{j} \tag{1}$$

be the position vector for a motion.

(a) Calculate $\mathbf{v}$, $|\frac{ds}{dt}|$, and $\mathbf{T}$.

(b) At what point is the speed greatest? Smallest?

(c) Find the xy-equation of the curve along which the point P is moving, and describe it geometrically.

Solution

Part I, Problem 1J-4

Suppose a point P moves on the surface of a sphere with center at the origin; let

$$OP = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \tag{2}$$

Show that the velocity vector $\mathbf{v}$ is always perpendicular to $\mathbf{r}$ two different ways:

(a) using the x, y, z-coordinates;

(b) without coordinates (use the formula in problem 1J-3, which is valid also in space); and

(c) prove the converse: if $\mathbf{r}$ and $\mathbf{v}$ are perpendicular, then the motion of P is on the surface of a sphere centered at the origin.

Solution

Part I, Problem 1J-6

For the helical motion $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + bt\mathbf{k}$,

(a) calculate $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, and $|\frac{ds}{dt}|$; and

(b) show that $\mathbf{v}$ and $\mathbf{a}$ are perpendicular; explain using 1J-5.

Solution

Part I, Problem 1J-9

A point P is moving in space, with position vector

$$\mathbf{r} = OP = 3\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j} + 4\cos(t)\mathbf{k}. \tag{3}$$

(a) Show that it moves on the surface of a sphere.

(b) Show that its speed is constant.

(c) Show the acceleration is directed toward the origin.

(d) Show it moves in a plane through the origin.

(e) Describe the path of the point.

Solution

Part I, Problem 1K-3

In our proof that Kepler's second law is equivalent to the force being central, we used the following steps to show the second law implies a central force. Kepler's second law says the motion is in a plane and

$$2\frac{dA}{dt} = |\mathbf{r} \times \mathbf{v}| \text{ is constant.} \tag{4}$$

This implies $\mathbf{r} \times \mathbf{v}$ is constant. So

$$0 = \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{a}. \tag{5}$$

This implies $\mathbf{a}$ and $\mathbf{r}$ are parallel, i.e. the force is central.

Reverse these steps to prove the converse: for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate. You will need the statement in exercise 1K-2.

Solution

Part II, Problem 1

A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by P. Suppose that the disk is tangent to the x-axis with the center initially at $(0, 2)$ and P initially at $(0, 1)$, and that it starts to roll to the right on the x-axis at unit speed. Let C be the curve traced out by the point P.

(a) Make a sketch of what you think the curve C will look like.

(b) Use vectors to find the parametric equations for $\overrightarrow{OP}$ as a function of time t.

(c) Open the 'Mathlet' **Wheel** (with link on course webpage) and set the parameters to view an animation of this particular motion problem. Then activate the 'Trace' function to see a graph of the curve C. If this graph is substantially different from your hand sketch, sketch it also and then describe what led you to produce your first idea of the graph. (The mathlet is right, by the way; and *No Fair Working Backwards* from the mathlet – the object of the exercise is to give it a try first.)

Solution

Part II, Problem 2

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be two non-parallel unit vectors with $\mathbf{u} \perp \mathbf{v}$, and let $\mathbf{r}(t) = \mathbf{u}\cos(t) + \mathbf{v}\sin(t)$.

Show that the curve $\mathbf{r}(t)$ sweeps out the unit circle centered at O in the plane $\mathcal{P}$ defined by $\mathbf{u}$ and $\mathbf{v}$ (i.e. the plane through the origin which contains $\mathbf{u}$ and $\mathbf{v}$).

(b) Use the result of part (a) to find the parametric equations of $C = \{$the circle of radius 1 centered at the origin which lies in the plane $\mathcal{P} : x + 2y + z = 0\}$.

(c) Give a rough sketch of C lying in $\mathcal{P}$.

Solution

Part II, Problem 3

(a) Find the equations of all the lines which pass through the origin and which lie in the plane $\mathcal{P}$ given by the equation $x + 2y + z = 0$.

(b) Illustrate with a (*rough*) sketch, and then give a geometric description of this family of lines.

(Suggestion: what is the smallest number of variables needed to describe this family, and how does that fact relate to what you see on the sketch?

Solution

Part II, Problem 4 *A model for a photo enlarger.*

A simple mathematical model of a way to enlarge a plane figure is to put the transparent plane containing the figure in a horizontal position, place a point light source at some distance above the plane, and then project the figure – i.e. its shadow – onto a parallel plane at some distance on the other side from the light. In this problem, we'll use vector methods to compute the distortion created when the two planes are slightly out of parallel, for the case of a simple figure.

Suppose that the light source is at the point $(0, 0, 4)$ and that the figure to be projected is the circle $\mathcal{C} = \{x = \cos(t), y = \sin(t), z = 2\}$ in the horizontal plane $z = 2$. The imaging plane is meant to be the plane $z = 0$, in order to produce an enlarged circle. Suppose instead, however, that the bottom plane is slightly tilted. We'll take this tilted plane $\mathcal{P}_\alpha$ to be given by the equation $my + z = 0$ with $m = \tan\alpha$. Check first that this is the plane with normal $\langle 0, \sin\alpha, \cos\alpha\rangle$, so that $\mathcal{P}_\alpha$ contains the x-axis and is tilted to the $x - y$ plane with angle $-\alpha$ (if $\alpha > 0$). The horizontal plane $z = 2$ and $\mathcal{P}_\alpha$ are thus slightly out of parallel if $\alpha \approx 0$.

(a) Make a sketch showing the situation described above.

(b) Show that the equation of the curve $\mathcal{C}_\alpha = \{$ the shadow (or projection) of the curve $\mathcal{C}$ in the plane $\mathcal{P}_\alpha\}$ is given in vector-parametric form by

$$\mathbf{r}_\alpha(t) = \left(\frac{4\cos t}{2 - m\sin t}\right)\mathbf{i} + \left(\frac{4\sin t}{2 - m\sin t}\right)\mathbf{j} + \left(\frac{-4m\sin t}{2 - m\sin t}\right)\mathbf{k}. \tag{6}$$

(c) Check that when $\alpha = 0$ the curve $\mathbf{r}_0(t)$ is the enlarged circle $\mathcal{C}_0$ in the $x - y$ plane. Then use the following 'quick-and-dirty' method to estimate the distortion in $\mathcal{C}_\alpha$ from the circle $\mathcal{C}_0$ caused by the tilt: compute the distance $|\mathbf{r}_\alpha(t) - \mathbf{r}_0(t)|$ between the two curves at the four 'corner' points corresponding to $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and just take the largest value from these four.

Solution