Part I, Problem 1F-5b

Find A^2, A^3, A^n if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution

Part I, Problem 1F-8a

If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$, and $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is the 3×3 matrix A?

Solution

Part I, Problem 1G-3

If $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, solve $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

Solution

Part I, Problem 1G-4

Referring to problem 1G-3 above, solve the system

$$x_1 - x_2 + x_3 = y_1 \qquad x_2 + x_3 = y_2 \qquad -x_1 - x_2 + 2x_3 = y_3 \tag{1}$$

for the x_i as a function of the y_i.

Solution

Part I, Problem 1G-5

Show that $(AB)^{-1} = B^{-1}A^{-1}$ by using the definition of inverse matrix.

Solution

Part I, Problem 1H-3abc

(a) For what c-value(s) will

$$\begin{aligned} x_1 - x_2 + x_3 &= 0, \\ 2x_1 + x_2 + x_3 &= 0, \\ -x_1 + cx_2 + 2x_3 &= 0, \end{aligned} \tag{2}$$

have a non-trivial solution?

(b) For what c-value(s) will $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix}$ have a non-trivial solution? (Write it as a system of homogeneous equations.)

(c) For each value of c in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to the three given vectors; find it by using the cross product.)

Solution

Part I, Problem 1E-1cd

Find the equations of the following planes:

(c) through $(1, 0, 1)$, $(2, -1, 2)$, and $(-1, 3, 2)$

(d) through the points on the x, y, and z-axes where $x = a, y = b, z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)

Solution

Part I, Problem 1E-2

Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.

Solution

Part I, Problem 1E-6

Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula

$$D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{3}$$

Solution

Part II, Problem 1

Suppose we know that when the three planes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ in $\mathbb{R}^3$ intersect in pairs, we get three lines L_1, L_2, L_3 which are *distinct* and *parallel*.

(a) Sketch a picture of this situation.

(b) Show that the three normals to $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ all lie in one plane, using a geometric argument.

(c) Show that the three normals to $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ all lie in one plane, using an algebraic argument. (Note that the three planes clearly do *not* all intersect at one point.)

Solution

Part II, Problem 2

A manufacturing process mixes three raw materials M_1, M_2, and M_3 to produce three products P_1, P_2, and P_3. The ratios of the amounts of raw materials (in the order M_1, M_2, M_3) which are used to make up each of the three products are as follows: For P_1 the ratio is $1 : 2 : 3$, for P_2 the ratio is $1 : 3 : 5$, and for P_3 the ratio is $3 : 5 : 8$.

In a certain production run, 137 units of M_1, 279 units of M_2, and 448 units of M_3 were used. The problem is to determine how many units of each of the products P_1, P_2, and P_3 were produced in that run.

(a) Set this problem up in matrix form. Use the letter A for the matrix, and write down the (one-line) formula for the solution in matrix form.

(b) Compute the inverse matrix of A and use it to solve for the production vector P.

(c) Find a choice for the ratios for the third product (in lowest form), different from the other two ratios, and for which the system has non-unique solutions.

Solution

Part II, Problem 3

For any plane $\mathcal{P}$ which is not parallel to the (x, y) plane, define the *steepest direction* on $\mathcal{P}$ to be the direction of any vector which lies in $\mathcal{P}$ and which makes the *largest* (acute) angle with the (x, y) plane.

(a) Let $\mathcal{P}$ be the plane through the origin with normal vector $\mathbf{n}$. Derive a formula, in terms of $\mathbf{n}$, for a vector $\mathbf{w}$ which points in the steepest direction on $\mathcal{P}$.

(b) Now let $\mathcal{P}$ be the plane through the origin which contains two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ do not both lie in the (x, y) plane. Derive a formula, in terms of $\mathbf{u}$ and $\mathbf{v}$, for a vector $\mathbf{w}$ which points in the steepest direction on $\mathcal{P}$.

Solution