Exam 2, Problem 1

(a): Recall

$\nabla f = \langle f_x, f_y \rangle$,

then

$\nabla f = \langle 2xy^2 - 1, 2x^2 y \rangle$

and thus

$\nabla f(2,1) = \cancel{\langle 0, 4 \rangle = \langle 4, 8 \rangle} = \langle 3, 8 \rangle$

(b): We see, given the point $(2,1,2)$ that

$z - 2 = 3(x-2) + 8(y-1) \Rightarrow z = 3x + 8y - 12$

(c): We let $\Delta x = 1.9 - 2 = -0.1$, $\Delta y = 1.1 - 1 = 0.1$; then,

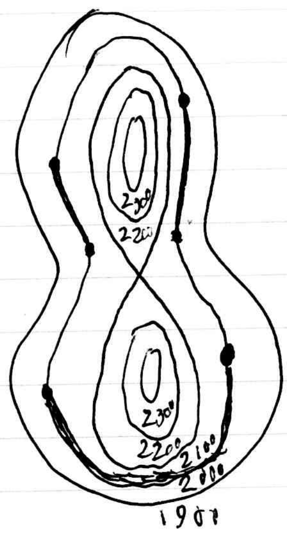
$z = 3 \cdot -0.1 + 8 \cdot 0.1 + f(2,1) = 2.5$

(d): $\cancel{\langle 2,1 \rangle \cdot \langle -1, 1 \rangle = -1}$

$\cancel{\langle 2,1 \rangle \cdot \frac{\langle -1,1 \rangle}{\sqrt{2}} = \frac{-1}{\sqrt{2}}}$ Let's do this using the right vector.

$\langle 3,8 \rangle \cdot \frac{\langle -1,1 \rangle}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

Exam 2, Problem 2



Exam 2, Problem 3

(a): To find the critical points, see that

$$w_x = -6x - 4y + 16 \stackrel{!}{=} 0 \qquad\qquad 6x + 4y = 16$$

$$\Longrightarrow$$

$$w_y = -4x - 2y - 12 \stackrel{!}{=} 0 \qquad\qquad 4x + 2y = -12$$

$\cancel{w_z}$ $\qquad\qquad -8x - 4y = 24$

$w_{xx} = -6$

$w_{xy} = -4$ $\qquad\qquad -2x = 40$

$w_{yy} = -2$ $\qquad\qquad x = -20$

$\qquad\qquad y = 34$

$$w_{xx}\, w_{yy} - w_{xy}^2 = (-6)(-2) - 16 = -4$$

Thus, the critical point is a saddle point.

(b): The critical point is not in the first quadrant, so to find the maximum we must consider the boundary and the points infinitely far away:

$x = 0,\ y \geq 0$: $w = -y^2 - 12y \Rightarrow$ vertex at $(-6, 36)$, thus max at $(0,0)$

$y = 0,\ x \geq 0$: $w = -3x^2 + 16x \Rightarrow$ vertex at $\left(\frac{8}{3}, \frac{64}{3}\right)$

Now for the points infinitely far away:

If $y \geq 0$ and $x \to \infty$, then we must only consider $w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to \infty$

If we let $C \geq x \geq 0$, and $y \to \infty$, then we must consider $w \leq -y^2 + 16C$, which again goes to $-\infty$ as $y \to \infty$

Exam 2, Problem 4

(a): $w_x = w_u u_x + w_v v_x = w_u \frac{-y}{x^2} + w_v 2x$

$w_y = w_u u_y + w_v v_y = w_u \frac{1}{x} + w_v 2y$

(b): By (a), see that

$$x w_x + y w_y = x\left(w_u \frac{-y}{x^2} + w_v 2x\right) + y\left(w_u \frac{1}{x} + w_v 2y\right)$$

$$= w_v(2x^2+2y^2) + w_u\left(\frac{-y}{x} + \frac{y}{x}\right)$$

$$= 2v\, w_v$$

(c): Given $w = v^5$, $w_v = 5v^4$ and then

$2v\, w_v = 10 v^5$

Exam 2, Problem 5

(a): We wish to find the formulas of

$\nabla f = \lambda \nabla g$

where $f(x,y,z) = x$ and $g(x,y,z) = x^4 + y^4 + z^4 + xy + yz + zx = 6$,

$f_x = 1 = \lambda(4x^3 + y + z)$

$f_y = 0 = \lambda(4y^3 + x + z)$

$f_z = 0 = \lambda(4z^3 + x + y)$

(b): If we have that at $P = (x_0, y_0, z_0)$, ~~and $\lambda \neq 0$, then~~

$1 = \lambda g_x$

$0 = \lambda g_y \implies \lambda \neq 0$ and then $\langle g_x, g_y, g_z \rangle = \langle 1/\lambda, 0, 0 \rangle$

$0 = \lambda g_z$

and in turn the tangent plane is $x = x_0$.

Exam 2, Problem 6

(a): $\cdot\ 2x\,dx + 3y^2\,dy - 4z^3\,dz = 0$ (1)

$\cdot\ (2+y)\,dx + x\,dy + (3z^2 + x)\,dz = 0$ (2)

(b): ~~We~~ see by (2) that

$$dy = \frac{-(3z^2 + x)\,dz + -(2+y)\,dx}{x}$$

then at $(1,1,1)$,

$dy = -(3+1)\,dz + -(1+1)\,dx$ $\cancel{\Rightarrow \frac{dy}{dz} = (-2) \text{ at } (1,1,1)}$

$dy = -4\,dz - 2\,dx$

and by (1) that since

$$dy = \frac{4z^3\,dz - 2x\,dx}{3y^2} \Rightarrow dy(1,1,1) = \frac{4}{3}\,dz$$

Insert $p = (1,1,1)$ into (1) and (2):

$$\begin{array}{r} 2\,dx + 3\,dy - 4\,dz = 0 \\ +\ 2\,dx + dy + 4\,dz = 0 \\ \hline 4\,dx + 4\,dy = 0 \end{array}$$

$dy = -dx$

$$\boxed{\frac{dy}{dx} = -1}$$