Multivariable Calculus: Tutorial 1

Logan Pachulski

February 25, 2018

Progress Update

Over the past week I have:

1. Learned about vectors.
2. Learned about dot products of vectors & applications of the dot product.
3. Used vectors to find areas, volumes, and relating those to determinants.
4. Learned about the cross product of vectors.

Vectors

We define a vector as a line with direction and magnitude (length). A vector A can take on a few different formats:

$$A = \langle 1, 2, 3 \rangle \quad (1)$$

$$= 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad (2)$$

Each of these vectors has the same associated change in position; 1 unit along the x-axis, 2 along the y-axis, and 3 along the y-axis.

A few functions of individual vectors

We can multiply a vector by a number/scalar.

$$cA = \langle ca_1, \cdots, ca_n \rangle \tag{3}$$

The scalar that is magnitude (length) of an individual vector A, by Pythagoras

$$\text{len}(A) = |A| = \sqrt{a_1^2 + \cdots + a_n^2} \tag{4}$$

Define the unit vector as any n-vector as a vector found on the unit n-sphere; then we can find the unit vector for any vector by letting

$$\text{dir}(A) = \frac{A}{|A|} \tag{5}$$

Dot product

The dot product of two vectors A and B is defined as

$$A \cdot B = a_1 * b_1 + \cdots + a_n * b_n \tag{6}$$

It is also true that

$$A \cdot B = |A||B|\cos(\theta) \tag{7}$$

where θ is the angle between the two vectors, assuming we place their base at the same point.

Determinant of vectors

The determinant of two 2-vectors represents the area of a parallelogram made up of those two, while the determinant of three 3-vectors represents the volume of a parallelepiped built out of those vectors. We shall define the determinant of the former:

$$\det(A, B) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 + a_2 b_1 \tag{8}$$

And of three 3-vectors:

$$\det(A, B, C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{9}$$

Where we evaluate the individual terms of (9) in the same way as (8).

Worked problem

Problem: Find the volume of the tetrahedron having vertices at the four points $P : (1, 0, 1), Q : (1, 1, 2), R : (0, 0, 2), S : (3, 1, 1)$.

Worked problem (solution)

Solution: We shall shift all of our points by the vector $-\mathbf{i} + 0\mathbf{j} - \mathbf{k}$, thus our set of points $P : (1, 0, 1), Q : (1, 1, 2), R : (0, 0, 2), S : (3, 1, 1)$ becomes $P' : (0, 0, 0), Q' : (2, 1, 1), R' : (-1, 0, 1), S' : (2, 1, 2)$.
We now let A be a vector connecting to Q', B to R', and C to S'. We then insert these values into a determinant and note that this tetrahedron has volume $1/6$ of the parallelepiped with these sides.

$$\det(A, B, C) = \begin{vmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & -2 \end{vmatrix} = \frac{1}{6}(1) = \frac{1}{6} \tag{10}$$

Cross Product

The cross product of two vectors returns a vector orthogonal to the plane drawn by two vectors by the right hand rule.

$$A \times B = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \tag{11}$$