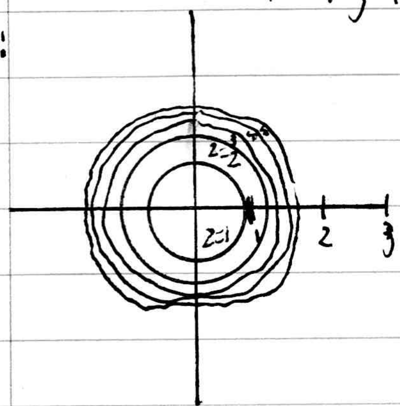
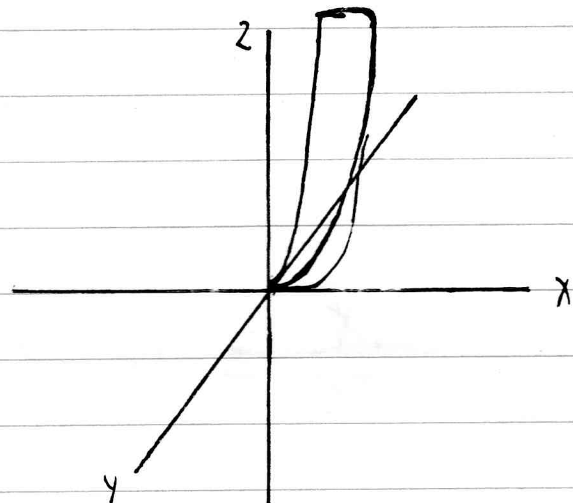
Part 1, Problem 2A-1c

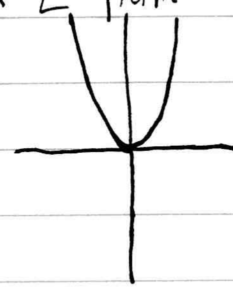
(a): The rings get denser (and harder to draw) as x and y go further from the origin.



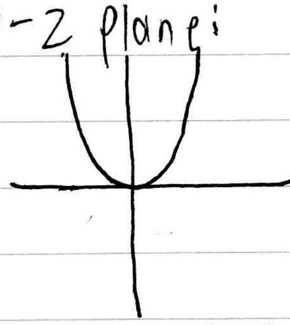
(b):



Trace on the x-z plane:



and on the ~~x~~y-z plane:



Both are parabolas of 2nd degree polynomial form.

Part 1, Problem 2A-2b e

(b): $\frac{\partial f}{\partial x} f(x,y) = \frac{\partial f}{\partial x} \frac{1}{y} x = \frac{1}{y}$

$\frac{\partial f}{\partial y} = x y^{-1} = -x y^{-2} = \frac{-x}{y^2}$

(e): $\frac{\partial f}{\partial x} = \log(2x+y) + \frac{x}{2x+y} \cdot 2$

$= \log(2x+y) + \frac{2x}{2x+y}$

$\frac{\partial f}{\partial y} = \frac{x}{2x+y} \cdot 1 = \frac{x}{2x+y}$

Part 1, Problem 2A-3b

~~Firmst~~ First find

$$\frac{\partial f}{\partial x}\left(\frac{x}{x+y}\right) = 1\cdot(x+y)^{-1} + x\left(\left((x+y)^{-1}\right)'\right)$$

$$+ -x(x+y)^{-2}$$

$$= (x+y)^{-1} + \frac{-x}{(x+y)^2}$$

Then take the partial derivative with respect to y.

$$\frac{\partial f}{\partial y}\left((x+y)^{-1} + -x(x+y)^{-2}\right)$$

$$= -(x+y)^{-2} + -x\left(-2(x+y)^{-3}\right)$$

$$= \frac{-1}{(x+y)^2}\cdot\frac{(x+y)}{(x+y)} + \frac{+2x}{(x+y)^3} = \frac{x-y}{(x+y)^3} = f_{xy}$$

Now we must find f_{yx}

$$\frac{\partial}{\partial y}\frac{x}{x+y} = -x(x+y)^{-2}$$

then

$$\frac{\partial}{\partial x}\left(-x(x+y)^{-2}\right) = -(x+y)^{-2} + -x\cdot -2(x+y)^{-3}$$

$$= \frac{-1}{(x+y)^2} + \frac{+2x}{(x+y)^3}$$

$$= \frac{(x-y)}{(x+y)^3} = f_{yx} = f_{xy} \quad \square$$

Part1, Problem 2A-5a

Evaluate $w_{xx} = (w_x)_x$

$$w_x = \frac{\partial}{\partial x} e^{ax} \sin(ay) = a e^{ax} \sin(ay)$$

$$(w_x)_x = \frac{\partial}{\partial x} a e^{ax} \sin(ay) = a^2 e^{ax} \sin(ay)$$

Then find $w_{yy} = (w_y)_y$

$$w_y = \frac{\partial}{\partial y} e^{ax} \sin(ay) = a e^{ax} \cos(ay)$$

$$(w_y)_y = \frac{\partial}{\partial y} a e^{ax} \cos(ay) = -a^2 e^{ax} \sin(ay)$$

$$\Rightarrow$$

$$w_{xx} + w_{yy} = 0$$

Part 1, Problem 2B-16

Recall
$$z_0 + \text{ — } z_0 = a(x - x_0) + b(y - y_0)$$

then, since $P = (1,2,4)$ and $w = \frac{y^2}{x}$

$$4 = w_x(1)(x-1) + w_y(2)(y-2)$$

$$w_x = \frac{\partial}{\partial x}\left(\frac{y^2}{x}\right) = \frac{-y^2}{x^2} \Rightarrow w_x(1,2,4) = \frac{-2^2}{1^2} = -4$$

$$w_y = \frac{\partial}{\partial y}\left(\frac{y^2}{x}\right) = \frac{2y}{x} \Rightarrow w_y(1,2,4) = 4$$

$$\cancel{z = -4x + 4 + 4y - 8 + 4}$$

$$\cancel{z = -4x + 4y + 4}$$

$$\cancel{z = -x + y}$$

$$\text{tangent } w(1,2,4) = 4 + -4(x-1) + 4(y-2)$$
$$= 4 - 4x + 4 + 4y - 8$$

$$\text{tangent } w = -4x + 4y$$

Part 1, Problem 2B-6

~~Rec~~ Recall the volume formula of a cylinder;

$V_c(r,h) = \pi r^2 h$

Then,

$$\frac{\partial V_c}{\partial r} = 2\pi r h$$

$$\frac{\partial V_c}{\partial h} = \pi r^2$$

Then by ~~t~~ the approximation formula:

$$\Delta z \approx \cancel{V_x \Delta x + V_y} \approx V_h \Delta h + V_r \Delta r$$

$$\approx \frac{\partial V_c}{\partial r}(2,3)\Delta r + \frac{\partial V_c}{\partial h} \cancel{(3)}(2,3)\Delta h$$

$$\approx 12\pi \Delta r + 4\pi \Delta h$$

Then we'd like $|\Delta z| = |\Delta V|$, and assuming $|\Delta r| \le \epsilon$ and $|\Delta h| \le \epsilon$, then

$|\Delta V| \le 12\pi\epsilon + 4\pi\epsilon = 16\pi\epsilon.$

Then demand

$$|\Delta V| < 0.1,$$

which can be accomplished by ~~assum~~ asking

$$16\pi\epsilon < 0.1$$

$$\Longrightarrow$$

$$\boxed{\epsilon < \frac{1}{160\pi}}$$

Part 1, Problem 2B-9

(a): Given $w(x,y) = x^2(y+1)$, we find

$w_x = 2x(y+1) \Rightarrow w_x(1,0) = 2$

$w_y = x^2 \Rightarrow w_y(1,0) = 1$

Thus, this function is more sensitive to changes in x.

(b): See by the approximation formula that

$$\Delta w = 2\Delta x + \Delta y$$
$$0 \overset{!}{=} 2\Delta x + \Delta y$$
$$-\Delta y = 2\Delta x$$
$$-2^{-1} = \frac{\Delta \cancel{y}}{\Delta \cancel{x}}$$
$$-2 = \frac{\Delta x}{\Delta y}$$

Part 1, Problem 2F-1a

We define the distance-squared from the origin

$D = x^2 + y^2 + \frac{1}{xy}$,

since ~~$\left(z = \frac{1}{\sqrt{xy}}\right)^2 \Rightarrow z^2 = \frac{1}{xy}$~~ by the statement

$xyz^2 = 1 \Rightarrow z^2 = \frac{1}{xy}$.

Then,

$\frac{\partial D}{\partial x} = 2x + \cancel{\frac{-1}{(xy)^2}} \quad \frac{-1}{x^2 y}$

$\frac{\partial D}{\partial y} = 2y - \frac{1}{xy^2}$

Set each equal to zero

$2y - \frac{1}{xy^2} = 0 \Rightarrow 2y = \frac{1}{xy^2} \Rightarrow 2y^2 = \frac{1}{xy}$

$2x - \frac{1}{x^2 y} = 0 \Rightarrow 2x = \frac{1}{x^2 y} \Rightarrow 2x^2 = \frac{1}{xy}$

which then implies that $x^2 = y^2 \Rightarrow x = \pm y$; consider the former case; then

$x^2 = \frac{1}{2x^2} \Rightarrow x^4 = \frac{1}{2} = y^4 \Rightarrow x, y = 2^{-1/4}$

which by plugging into ~~D returns the~~ z gives the point $(2^{-1/4}, 2^{-1/4}, 2^{1/4})$; $y = -x$ returns no point since $x^4 = \frac{-1}{2}$ is undefined here.

Part 1, Problem 2F-2

Let x be the width of the box, y be the depth, and z be the height; then we can write a formula for the area of cardboard;

$$A(x,y,z) = 3xy + 4xz + 2yz$$

and the volume of the box

$$V(x,y,z) = xyz = 1.$$

Substitute $z = 1/xy$ into the area equation, then

$$A(x,y,z) = 3xy + \frac{4}{y} + \frac{2}{x}$$

$$\frac{\partial A}{\partial x} = 3y + \frac{-2}{x^2} \overset{!}{=} 0 \Rightarrow \frac{2}{x} = 3xy$$

$$\frac{\partial A}{\partial y} = 3x + \frac{-4}{y^2} \overset{!}{=} 0 \Rightarrow \frac{4}{y} = 3xy$$

then meaning

$$\frac{2}{x} = \frac{4}{y} \Rightarrow \frac{x}{2} = \frac{y}{4} \Rightarrow y = 2x$$

~~Recall $V = xyz = 1$;~~ Recall $3xy = \frac{2}{x}$

$\cancel{2x^2 \cdot \frac{1}{xy} =}$

$$3x^2 y = 2$$
$$6x^3 = 2$$
$$3x^3 = 1$$
$$x = \frac{1}{3}^{1/3} = 3^{-1/3}$$
$$\Rightarrow$$
$$y = 2 \cdot 3^{-1/3} \Rightarrow 1 : 2 : \frac{3}{2}$$
$$\Rightarrow$$
$$z = \frac{3}{2 \cdot 3^{1/3}}$$

Part 1, Problem 2G-1c

We see by solving the system of equations

$$\begin{cases} \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) a + n b = \sum_{i=1}^{n} y_i \end{cases}$$

that

$a = \frac{1}{2}$ and thus $y = \frac{1}{2} x + 1$

$b = 1$

~~Now we must minimize~~

$\cancel{D(\tfrac{1}{2}, 1)}$

Part 1, Problem 20-4

We let

$$D = \sum_{i=1}^{n} (a + b x_i + c y_i - z_i)^2$$

Then,

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(a + b x_i + c y_i - z_i) \overset{!}{=} 0$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2x_i(a + b x_i + c y_i - z_i) \overset{!}{=} 0$$

$$\frac{\partial D}{\partial c} = \sum_{i=1}^{n} 2y_i(a + b x_i + c y_i - z_i) \overset{!}{=} 0$$

Pull out all 2's by dividing each sides and separate the sums:

$$\sum a + b\sum x_i + c\sum y_i = \sum z_i$$

...

Let's use ~~produ~~ dot products to save writing time; $X = [x_1, .., x_n]$, $1 = [1, ..., 1]$, $\mathbf{y} = [y_1, ..., y_n]$, and $\mathbf{Z} = [z_1, ..., z_n]$, then our prior equations suggest

$$na + (X \cdot 1)b + (y \cdot 1)c = (Z \cdot 1)$$

$$(X \cdot 1)a + (X \cdot X)b + (X \cdot y)c = X \cdot Z$$

$$(y \cdot 1)a + (X \cdot y)b + (y \cdot y)c = y \cdot Z$$

Part 1, Problem 2H-1C

If

$f(x,y) = 2x^4 + y^2 - xy + 1$

then

$$\frac{\partial f}{\partial x} = 8x^3 - y \overset{!}{=} 0 \qquad y = 8x^3$$

$$\frac{\partial f}{\partial y} = 2y - x \overset{!}{=} 0 \Rightarrow y = \frac{1}{2}x$$

thus,

$8x^3 = \frac{1}{2}x$

$8x^2 = \frac{1}{2}$

$16x^2 = 1$

$x^2 = \frac{1}{16}$

$x = \pm\frac{1}{4} \Rightarrow y = \frac{1}{8}$ & $x = -\frac{1}{4} \Rightarrow y = \frac{-1}{8}$

and $(0,0)$; we must find the second derivative criterion of

$(0,0)$

$(\frac{-1}{4}, \frac{-1}{8})$

$(\frac{1}{4}, 1/8)$

Then find

$$\frac{\partial^2 f}{\partial x^2} = 24x^2 \qquad \Delta(0,0) = -1 \quad \text{Saddle}$$

$$\frac{\partial^2 f}{\partial xy} = f_{xy} = -1 \Rightarrow \Delta\left(\frac{1}{4}, \frac{1}{8}\right) = 2 \quad \text{Minimum}$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \qquad \Delta\left(\frac{-1}{4}, \frac{1}{4}\right) = 2 \quad \text{Minimum}$$

Part 1, Problem 2H-3

We want to find all critical points (x, y) such that $y > -x$;

$f(x,y) = x^2 + y^2 + 2x + 4y - 1$

$f_x = 2x + 2 \overset{!}{=} 0 \Rightarrow x = -1$

$f_y = 2y + 4 \overset{!}{=} 0 \Rightarrow y = -2$

this point does not satisfy the $y > -x$ bound, thus the critical point is on the bound $y = -x$.

~~Substitute f(x, -x)~~

$f(x, -x) = 2x^2 + 2x - 4x - 1$

$$\frac{\partial f}{\partial x} = 4x - 2 \overset{!}{=} 0$$

$$x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$$

and away from this line $y = -x$ the solutions go to infinity; this fundtion has maximum $f(x, -x)$ as $x \to \infty$ and minimum at $(1/2, -1/2)$.

Part 1, Problem 2H-4

(a): Given $f(x,y) = xy - x - y + 2$,

$$\frac{\partial f}{\partial x} = y - 1 \stackrel{!}{=} 0 \Rightarrow y = 1$$

$$\frac{\partial f}{\partial y} = x - 1 \stackrel{!}{=} 0 \Rightarrow x = 1$$

$$\begin{aligned} f_{xx} &= 0 \\ f_{xy} &= 1 \\ f_{yy} &= 0 \end{aligned} \Rightarrow (1,1) \text{ is a saddle point}$$

We also see that as $x, y \to \infty$, $f(x,y) \to \infty$ since xy outpaces the subtractions, and that

$x = 0,\ y \to \infty : f(x,y) \to -\infty$

$x \to \infty,\ y = 0 : f(x,y) \to -\infty$

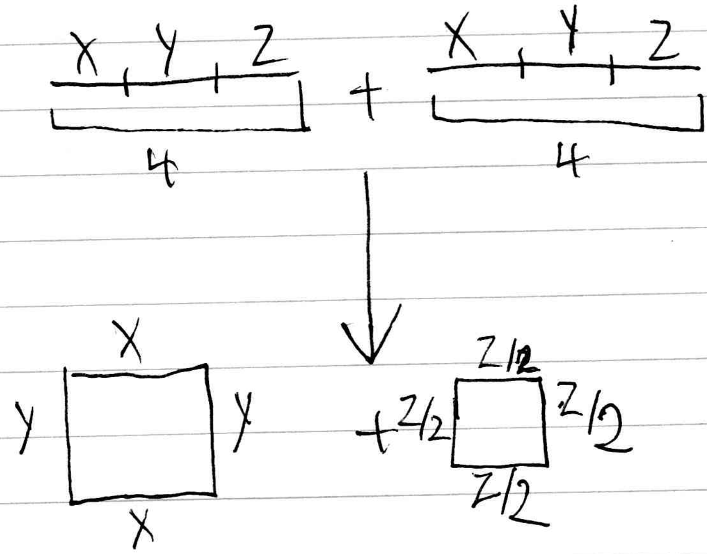
thus there is no minimum.

(b): By (a) this function is maximized at $(0,0)$ and $(2,2)$ and minimized at ~~(-2,0) and (0,~~ $(2,0)$ and $(0,2)$.

(c): Refer to (a)

Part 1, Problem 2H-6

Begin by writing a formula, with an image first for intuition:



Thus,

$$A_{tot}(x,y,z) = x \cdot y + z^2/4$$

where $x+y+z=4$

$$A_{tot} = x \cdot y + \frac{(4-x-y)^2}{4}$$

$$\frac{\partial A}{\partial x} = y + \frac{2(4-x-y)}{4} \cdot -1$$

$$= y + \frac{-1}{2}(4-x-y)$$

$$= \frac{3}{2}y + \frac{1}{2}x - 2 \overset{!}{=} 0$$

$$\frac{\partial A}{\partial y} = x + \frac{-1}{2}(4-x-y)$$

$$= \frac{3}{2}x + \frac{1}{2}y - 2 \overset{!}{=} 0$$

$$\rightarrow \frac{3}{2}y + \frac{1}{2}x = \frac{3}{2}x + \frac{1}{2}y$$

$$x = y$$

And thus we can ~~rewrite~~ recall

$\cancel{A_{tot} = -x^2 + 4 =}$

$\cancel{\frac{\partial A}{\partial x} = 2x}$

$$\frac{\partial A}{\partial x} = y + \frac{-1}{2}(4 - x - y) = x + \frac{-1}{2}(4-2x)$$

$$= x - 2 + x$$

$$0 \stackrel{!}{=} 2x - 2$$

$$x = 1$$

$$y = 1$$

And thus a critical point is ~~(0,~~ $(1, 1, 2)$; at this point $A = 1 + 1 = 2$; ~~Let is a~~ note that area is minimized when any ~~point = 4 to~~ single input $= 4$, and that $A(2,2,0) = A(0,0,4) = 4$; thus the area is maximized at $z = 4$ or $x = y = 2$, and minimized when $x, y,$ or $z = 4$.

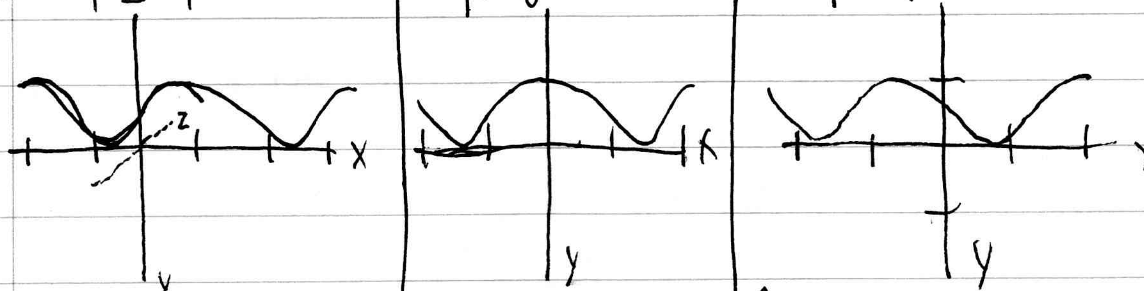
(6): $A_{xx} = 1/2$

$A_{xy} = 3/2$ Thus, $AC - B^2 = -2 < 0 \Rightarrow$ saddle point

$A_{yy} = 1/2$

Part 2, Problem 1

(a): $t = -1$ | $t = 0$ | $t = 1$



a normal cos wave, shifted left 2. | a normal cos wave. | A normal cos wave, shifted right 2.

(b): This describes a wave moving along a surface at 2 units per unit time.

Part 2, Problem 2

(a): The intersection of these surffaces can be described by setting the formulae of the surface equal:

$$x^2 - y^2 = 2 + (x-y)^2$$
$$= 2 + x^2 + y^2 \cancel{+} -2xy$$
$$\cancel{-2y^2 = 2 + 2xy}$$
$$\cancel{-y^2 = 1 + xy}$$
$$\cancel{-y^2 - 1 = xy}$$
$$-2y^2 = 2 + -2xy$$
$$-y^2 = 1 + -xy$$
$$-y^2 - 1 = -xy$$
$$y^2 + 1 = xy$$
$$x = y + \frac{1}{y}$$

Then plug back into our first formula:

$$(y + y^{-1})^2 - y^2 = \cancel{2 + y^{-2}} = y^2 + 2 + y^{-2} - y^2$$
$$= 2 + y^{-2}$$
$$= ((y + y^{-1}) - y)^2 + 2$$

Thus we have common points ~~betwen~~ in the intersection. We parametrize our equations by letting $y = t$ and see that

$x = t + t^{-1}$

$y = t$

$z = 2 + t^{-2}$

~~(b): Then to find the normal, and in turn the plane,~~

~~$\langle f_x(2,1,3), f_y(2,1,3), -1\rangle = \langle 4, 2, -1\rangle$~~

~~$\Rightarrow$~~

~~$c = 4x + 2y - 1z$~~

(b): We define

$f(x,y) = x^2 - y^2$

$g(x,y) = 2 + (x-y)^2$

Then ~~to find the plane tangent to f~~ we find the normal to the planes tangent to f and g, we solve at point $P = (2,1,3)$

$n_f(2,1,3) = \langle f_x(2,1,3), f_y(2,1,3), -1 \rangle$

and

$n_g(2,1,3) = \langle g_x(P), g_y(P), -1 \rangle$

since

$f_x = 2x$

$f_y = -2y$

$g_x = 2(x-y)$

$g_y = -2(x-y)$

then

$n_f(P) = \langle 4, -2, -1 \rangle$

$n_g(P) = \langle 2, -2, -1 \rangle$

Then the angle between the normal vectors

$$\theta = \angle(n_g, n_f) = \angle(\text{Plane at } f(P), \text{Plane at } g(P))$$

$$\cos(\theta) = \frac{n_g \cdot n_f}{|n_g| \cdot |n_f|} = \frac{13}{\sqrt{9 \cdot 21}}$$

$\theta \approx 19$ degrees.

(c): $\vec{r}(t) = \langle t + t^{-1}, t, 2 + t^{-2} \rangle$

and thus

$\vec{r}\,'(t) = \langle 1 - t^{-2}, 1, 2t^{-3} \rangle$

At $t = 1$ the vector $\vec{r}(t)$ passes through P, Thus it has velocity $\vec{r}\,'(1) = \langle 0, 1, -2 \rangle$

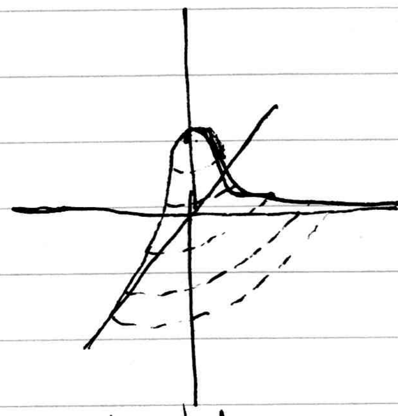
Then we must show the vector $\vec{r}\,'(1)$ is parallel to both planes:

$n_f(P) \cdot \vec{r}\,'(1) = \langle 4, -2, -1 \rangle \cdot \langle 0, 1, -2 \rangle = 0.$

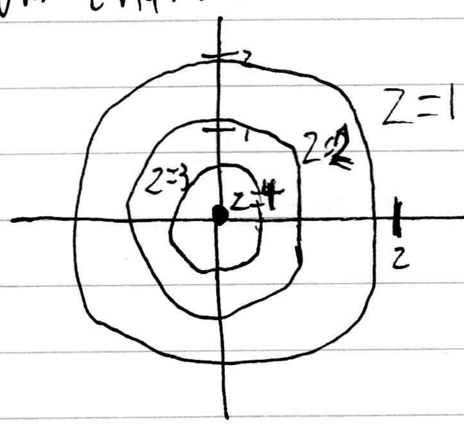
$n_g(P) \cdot \vec{r}\,'(1) = \langle 2, -2, -1 \rangle \cdot \langle 0, 1, -2 \rangle = 0.$

Part 2, Problem 3

(a): First, a graph in 3-space; this function is 4 when $x,y=0$ and goes to 0 as $x,y \to \infty, -\infty$ ie away from the origin:



And a contour chart:



$$3 = \frac{4}{1+x^2+y^2} \Rightarrow 3+3x^2+3y^2 = 4 \Rightarrow x^2+y^2 = \frac{1}{3}$$

$$2 = \quad '' \quad \Rightarrow 2+2x^2+2y^2 = 4 \Rightarrow x^2+y^2 = \frac{(4-2)}{2}$$

$$1 = \quad \dots \quad = \frac{(4-1)}{1} = 3$$

(b): We let $x = t$ and thus

$y = 1.5 - t^2$

and in turn

$$z = \frac{4}{1 + t^2 + (1.5 - t^2)^3}$$

(d): We see that by setting $\frac{dz}{dt}$ equal to zero that

$$4t(t^2 - 1) = 0$$

and as such critical points are $t = \{0, -1, 1\}$ with corresponding surface points $(0, \frac{3}{2}, \frac{16}{13})$, $(\pm 1, \frac{1}{2}, \frac{16}{9})$ where the first minimizes f and the latter two maximize f.

(e): Finding the max/min of $D^2 = t^2 + (\frac{3}{2} - t^2)^2$ is done by ~~differentiatiatt~~ differentiating and setting equal to 0, and gives us $4t(t^2 - 1) = 0$ and the same critical points as d.

Part 2, Problem 4

~~$27 = x^2 + y^2 + z^2$ such that $x, y, z \geq 0$~~

We are considering the sum

$f(x,y,z) = x^3 + y^3 + z^3$

where $x^2 + y^2 + z^2 = 27 \Rightarrow z = \sqrt{27 - x^2 - y^2}$

and thus

$f(x,y,z) = x^3 + y^3 + (27 - x^2 - y^2)^{3/2}$

Critical points occur where the partial derivatives are zero:

$$\frac{\partial f}{\partial x} = 3x^2 + \frac{3}{2}\sqrt{27 - x^2 - y^2} \cdot -2x$$

$$= 3x^2 - 3\sqrt{27 - x^2 - y^2}\ x$$

$$\frac{\partial f}{\partial y} = 3y^3 + \frac{3}{2}\sqrt{27 - x^2 - y^2} \cdot -2y$$

$$= 3y^3 + -3y\sqrt{27 - x^2 - y^2}$$

Then, these partial derivatives are zero if

$x = 0$

$y = 0$

$x = \sqrt{27 - x^2 - y^2}$

$y = \sqrt{27 - x^2 - y^2}$

which implies the points

$(0,0)$, $(0, \sqrt{27/2})$, $(\sqrt{27/2}, 0)$, $(3,3)$

maximum, saddle, saddle, minimum

$f(0,0,\sqrt{27})$
$= 81\sqrt{3}$

~~$f(3,3) = 81$~~

$f(3,3,3) = 81$

The maximum occurs at $(0, 0, \sqrt{3} \cdot 3)$ and the minim-um occurs at $(3, 3, 3)$.

Part 2, Problem 5

(a): We would like to minimize the equation

$$F_i(\alpha, \beta) = \cos(\alpha)\cos(\beta)\cos(\alpha+\beta)$$

$$\frac{\partial F}{\partial \alpha} = \left(-\sin(\alpha)\cos(\alpha+\beta) + -\cos(\alpha)\sin(\alpha+\beta)\right)\cos(\beta)$$
$$= -\cos(\beta)$$

$$\frac{\partial F}{\partial \beta} = \cos(\alpha)\left(+\sin(\beta)\cos(\alpha+\beta) + +\cos(\beta)\sin(\alpha+\beta)\right)$$

~~Then by setting each equal and dividing out encompassing terms,~~

then by the sine addition formula applied to each,

$$\frac{\partial F}{\partial \alpha} = -\cos(\beta)\sin(2\alpha+\beta) \stackrel{!}{=} 0$$

$$\frac{\partial F}{\partial \beta} = -\cos(\alpha)\sin(\alpha+2\beta) \stackrel{!}{=} 0$$

$\alpha = \beta = \pi/2$ is a valid solution (returning $F_i = 0$), but we can also consider where

$$2\alpha + \beta = \pi \quad \rightarrow \quad 2\alpha + \beta = \pi$$
$$(\alpha + 2\beta = \pi) \quad +(-2\alpha + -4\beta = -2\pi)$$
$$-3\beta = -\pi$$
$$\beta = \frac{\pi}{3}$$
$$\alpha = \frac{\pi}{3}$$

Which returns $F_i = \frac{-1}{8}$.

(b): Since the ~~magnitude of~~ magnitude of the wind vector is 1, the 1/8 of the velocity is captured.