Part I, Problem 6A-3

Write down the velocity field **F** representing a rotation about the x-axis in the direction given by the right-hand rule (thumb pointing in positive x-direction), and having constant angular velocity ω.

Solution

Part I, Problem 6B-1

Without calculating, find the flux of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the sphere of radius a and center at the origin. Take $\mathbf{n}$ pointing outward.

Solution

Part I, Problem 6B-2

Without calculation, find the flux of $\mathbf{k}$ through the infinite cylinder $x^2 + y^2 = 1$. Take $\mathbf{n}$ pointing outward.

Solution

Part I, Problem 6B-3

Without calculation, find the flux of $\mathbf{i}$ through that portion of the plane $x + y + z = 1$ lying in the first octant. Take $\mathbf{n}$ pointed away from the origin.

Solution

Part I, Problem 6B-6

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and S is the part of the paraboloid $z = x^2 + y^2$ lying underneath the plane $z = 1$, with $\mathbf{n}$ pointing generally upwards. Explain geometrically why your answer is negative.

Solution

Part I, Problem 6B-8

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{j}$ and S is that portion of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = h$, and to the right of the xz-plane; $\mathbf{n}$ points outwards.

Solution

Part I, Problem 6C-3

Verify the divergence theorem when $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface composed of the upper half of the sphere of radius a and center at the origin, together with the circular disc in the xy-plane centered at the origin and of radius a.

Solution

Part I, Problem 6C-5

By using the divergence theorem, evaluate the surface integral giving the flux of $\mathbf{F} = x\mathbf{i} + z^2\mathbf{j} + y^2\mathbf{k}$ over the tetrahedron with vertices at the origin and the three points on the positive coordinate axes at distance 1 from the origin.

Solution

Part I, Problem 6C-7a

Verify the divergence theorem when S is the closed surface having for its sides a portion of the cylinder $x^2 + y^2 = 1$ and for its top and bottom circular portions of the planes $z = 1$ and $z = 0$. Take $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$.

Solution

Part I, Problem 6C-8

Suppose $\operatorname{div}(\mathbf{F}) = 0$, and S_1 and S_2 are the upper and lower hemispheres of the unit sphere centered at the origin. Direct both hemispheres so that the unit normal is "up", i.e. has positive $\mathbf{k}$-component.

(a) Show that $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, and interpret this physically in terms of flux.

(b) State a generalization to an arbitrary closed surface S and a field $\mathbf{F}$ such that $\operatorname{div}(\mathbf{F}) = 0$.

Solution

Part II, Problem 1

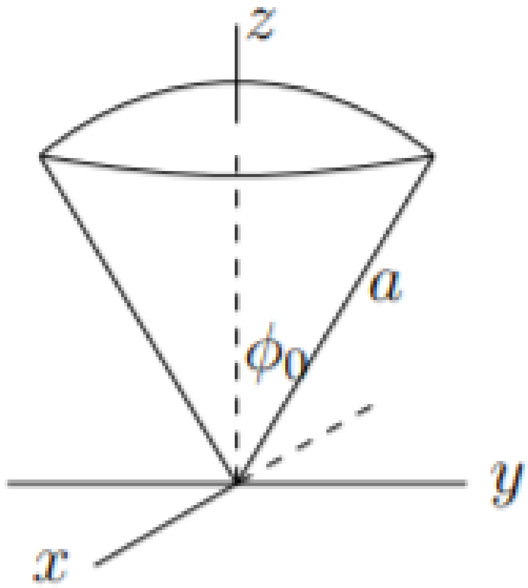
Show that the average straight-line distance to a fixed point on the surface of a sphere of radius a is $\frac{4a}{3}$.

Suggestion: take the sphere centered at the origin, and choose the "north pole" as the fixed point. Then compute the average straight-line distance to the north pole over all points on the surface of the sphere.

Solution

Part II, Problem 2

Consider a solid in the shape of an ice-cream cone. It's bounded above by (part of) a sphere of radius a centered at the origin. It's bounded below by the cone with vertex at the origin, vertex angle $2\phi_0$, and slant height a.



Find the gravitational force on a unit test mass placed at the origin. Assume density = 1.

Solution

Part II, Problem 3

Continuing with the solid in problem 2, take $a = \sqrt{2}$ and the vertex angle to be $\frac{\pi}{2}$ (so $\phi_0 = \frac{\pi}{4}$). Let $\mathbf{F} = z\mathbf{k}$.

Warning: the calculations are a bit messy.

(a) Let T be the horizontal disk whose boundary is the intersection of the sphere and the cone. Compute directly the upward flux of $\mathbf{F}$ through T.

(b) Let U be the boundary of the conical lower surface, and let S be the upper spherical cap of the solid. Use the divergence theorem and part (a) to compute the upward flux of $\mathbf{F}$ through U and S. (You will need to be careful with signs.)

(c) Set up, but don't compute, the integral for the flux of $\mathbf{F}$ through U. Write the integral in cylindrical coordinates.

Solution

Part II, Problem 4

Let $f(x, y, z) = \frac{1}{\rho}$, where $\rho = \sqrt{x^2 + y^2 + z^2}$ is the radial coordinate in spherical coordinates.

(a) Compute $\mathbf{F} = \nabla f$ and show $\operatorname{div}(\mathbf{F}) = 0$.

(b) Find the outward flux of $\mathbf{F}$ through the sphere of radius a centered at the origin. Why does this not contradict the divergence theorem?

Solution

Part II, Problem 5

The Laplacian of a function of three variables is defined by

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}. \tag{1}$$

Suppose that the simple closed surface S is the iso-surface of some smooth function $f(x, y, z)$, that is, the set of points in 3-space satisfying $f(x, y, z) = c$ for some constant c. Use the divergence theorem to show that, if G is the interior of S, then

$$\iint_S |\nabla f| \, dS = \pm \iiint_G \nabla^2 f \, dV. \tag{2}$$

Solution