Part I, Problem 6D-1a

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$, where C is the twisted cubic curve $x = t$, $y = t^2$, $z = t^3$ running from $(0, 0, 0)$ to $(1, 1, 1)$.

Solution

Part I, Problem 6D-2

Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any curve C lying on a sphere of radius a centered at the origin.

Solution

Part I, Problem 6D-4

(a) Let $f(x, y, z) = x^2 + y^2 + z^2$; calculate $\mathbf{F} = \nabla f$.

(b) Let C be the helix $x = \cos(t)$, $y = \sin(t)$, $z = t$, running from $t = 0$ to $t = 2n\pi$. Calculate the work done by $\mathbf{F}$ moving a unit point mass along C using three methods:

i directly

ii by using the path independence of the integral to replace C by a simpler path

iii by using the first fundamental theorem for line integrals.

Solution

Part I, Problem 6D-5

Let $\mathbf{F} = \nabla f$, where $f(x, y, z) = \sin(xyz)$. What is the maximum value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ over all possible paths C? Give a path C for which this maximum value is attained.

Solution

Part I, Problem 6E-3ab(ii)

The field $\mathbf{F} = (2xy + z)\mathbf{i} + x^2\mathbf{j} + x\mathbf{k}$ is defined for all (x, y, z).

(a) Show that $\text{curl}(\mathbf{F}) = 0$.

(b) Find a potential function $f(x, y, z)$ using both methods.

Solution

Part I, Problem 6E-5

For what values of a and b will $\mathbf{F} = yz^2\mathbf{i} + (xz^2 + ayz)\mathbf{j} + (bxyz + y^2)\mathbf{k}$ be a conservative field? Using these values, find the corresponding potential function $f(x, y, z)$ by one of the systematic methods.

Solution

Part I, Problem 6F-1b

Verify Stokes' theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e. calculate both integrals in the theorem and show they are equal. Do this for the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

Solution

Part I, Problem 6F-2

Verify Stokes' theorem if $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

Solution

Part I, Problem 6F-5

Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Find the flux of $\nabla \times \mathbf{F}$ through S

(a) directly, by calculating the two surface integrals; and

(b) by using Stokes' theorem.

Solution

Part II, Problem 1

Let $\mathbf{F}(x,y,z) = \left(\frac{-z}{x^2+z^2}\right)\mathbf{i} + y\mathbf{j} + \left(\frac{x}{x^2+z^2}\right)\mathbf{k}$ defined for all points (x,y,z) in 3-space not on the y-axis (that is, all points for which $x^2+z^2>0$).

(a) By direct computation, show that $\nabla \times \mathbf{F} = 0$ for all points not on the y-axis.

(b) By direct computation, show that $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$ where C_1 is the closed curve defined by $x^2+y^2=1$, $z=1$.

(c) Can you use Stokes' theorem and the fact (from part (a)) that $\nabla\times\mathbf{F}=0$ to conclude that $\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}=0$ when C_2 is the closed curve defined by $x^2+z^2=1$, $y=0$? Why or why not?

(d) Compute $\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}$ and see what happens.

Solution

Part II, Problem 2

Suppose the field $\mathbf{F}$ in the previous problem is replaced by the field

$$\mathbf{G}(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}\right)\mathbf{i} + \left(\frac{y}{x^2 + y^2 + z^2}\right)\mathbf{j} + \left(\frac{z}{x^2 + y^2 + z^2}\right)\mathbf{k}, \tag{1}$$

defined for all points $(x, y, z) \neq (0, 0, 0)$.

(a) Show that $\nabla \times \mathbf{G} = 0$ for all points $(x, y, z) \neq (0, 0, 0)$.

(b) Can you use Stokes' theorem in this case and the fact that $\nabla \times \mathbf{G} = 0$ for all points $(x, y, z) \neq (0, 0, 0)$ to conclude that $\oint_C \mathbf{G} \cdot d\mathbf{r} = 0$ for all simple closed curves C which do not pass through the origin?

(c) Explain the difference between these two cases in terms of the connectedness type of the domains of definition of the two fields.

Solution

In problems 3 and 4, we have a non-steady flow $\mathbf{F}(x, y, z, t) = \rho(x, y, z, t)\mathbf{v}(x, y, z, t)$ given by

$$\mathbf{F}(x, y, z, t) = (z\sin(t))\mathbf{i} + (-z\cos(t))\mathbf{j} + (-x\sin(t) + y\cos(t))\mathbf{k}. \tag{2}$$

We'll take the case where the density $\rho(x, y, z, t) = 1$, constant, so that $\mathbf{F}(x, y, z, t) = \mathbf{v}(x, y, z, t)$ as vectors (although in different physical units).

In general for non-steady flows, the divergence $\nabla \cdot \mathbf{F}$ of the field $\mathbf{F}(x, y, z, t)$ is defined using differentiation $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ with respect to the *space* variables only. For this reason, at any fixed time t the divergence theorem holds, and the physical interpretations of the divergence $\nabla \cdot \mathbf{F}$ of the field and of the flux of $\mathbf{F}$ through a surface are the same as the ones given in the Notes V10 for the case of steady flows. By the same token, Stokes' theorem holds at each fixed time t, and hence the physical interpretation of the curl $\nabla \times \mathbf{F}$ is the same as that given in the Notes V13 (pages 3-4) for the case of steady flows.

Part II, Problem 3

(a) Show that this flow $\mathbf{F}$ in equation (2) satisfies the *equation of continuity* $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F} = 0$.

(b) Referring to the facts about non-steady flows given above: what is the net outward flux of $\mathbf{F}$ through any simple closed surface, and why?

Solution

Part II, Problem 4

(a) Compute the curl $\nabla \times \mathbf{F}$ of $\mathbf{F}$ in equation (2).

(b) Referring to the facts about non-steady flows given above: at any fixed time t and position (x, y, z), what is the axial direction perpendicular to the plane in which the flow produces the maximum angular velocity, and what is the maximum angular velocity? Does it vary at different points (x, y, z) in space?

(c) Show that, at any time t:

i the velocity vectors $\mathbf{v}(x, y, z, t)$ of the flow lie in the plane through the origin with normal $\mathbf{n}_t = \langle \cos(t), \sin(t), 0\rangle$, which we'll call $\mathcal{P}_t$; and

ii the velocity vectors $\mathbf{v}(x, y, z, t)$ are perpendicular to the radial vectors $\mathbf{r} = \langle x, y, z\rangle$.

Combining the results of part (b) with (i): what is the plane of greatest spin for this flow?

(d) Use the results of part (c) to give a sketch of some representative velocity vectors in the plane $\mathcal{P}_t$ at a fixed time t. (Take t to be about $\frac{\pi}{4}$, and first sketch $\mathcal{P}_t$ in 3-space; then for the purposes of sketching the velocity field, you can take $\mathcal{P}_t$ to be the plane of the paper.)

(e) Putting together the results above, describe in general terms the pattern of the motion of the flow in 3-space over time.

Solution