Part 1, Problem 2C-1ad

(a) Recall the definition of differentials

$\partial f = \partial x\ f_x + \partial y\ f_y + \partial z\ f_z$

then since

$w = \log(xyz)$

$\frac{\partial w}{\partial x} = f_x = \frac{1}{xyz} \cdot yz = \frac{1}{x}$

$f_y = \frac{1}{y}$

$f_z = \frac{1}{z}$

thus

$\partial w = \frac{1}{x}\, \partial x + \frac{1}{y}\, \partial y + \frac{1}{z}\, \partial z$

(d) Recall

$\frac{d}{\partial x} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

then if

$w = \arcsin\left(\frac{u}{t}\right)$

$\frac{\partial w}{\partial u} = \frac{1}{\sqrt{1-\left(\frac{u}{t}\right)^2}} \cdot \frac{1}{t}$

$\frac{\partial w}{\partial t} = \frac{1}{\sqrt{1-\left(\frac{u}{t}\right)^2}} \cdot u$

then

$d w = \frac{1}{t\sqrt{1-\left(\frac{u}{t}\right)^2}}\, du + \frac{u}{\sqrt{1-\left(\frac{u}{t}\right)^2}}\, dt$

Part 1 Problem 2C-2

We see that the volume has the formula

$V = xyz$

Then,

$dV = V_x\, dx + V_y dy + V_z\, dz$

$V_x = yz$

$V_y = xz$

$V_z = xy$

thus,

$dV = yz\, dx + xz\, dy + xy\, dz$

and by inserting $x=5$, $y=10$, $z=20$,

$dV = 200\, dx + 100\, dy + 50\, dz$

then by letting each d-value $= \pm 0.1$, then

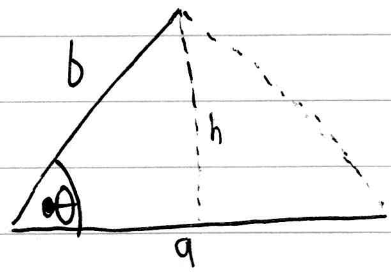
$dV = \pm 35$

Thus, the volume varies by

$(5 \cdot 10 \cdot 20) \pm 35 = 1000 \pm 35$

Part 1, Problem 2(-3

(a): We see that for the below triangle, $h = b\sin(\theta)$



Then since ~~for the~~ the formula

$$A = \frac{1}{2} b H \Rightarrow A = \frac{1}{2} ab\sin(\theta)$$

we then see that

$$A_a = \frac{1}{2} b \sin(\theta)$$

$$A_b = \frac{1}{2} a \sin(\theta)$$

$$A_\theta = \frac{1}{2} ab \cos(\theta)$$

and in turn

$$dA = \frac{1}{2}\left(b\sin(\theta)\,da + a\sin(\theta)\,db + ab\cos(\theta)\,d\theta\right)$$

(b): Insert $a = 1,\ b = 2,\ \theta = \frac{\pi}{6}$

$$dA = \cancel{0.5\,d\ 0.25\,da}$$

$$0.5\,da + 0.25\,db + \frac{\sqrt{3}}{2}\,d\theta$$

or 0.866

thus, our area is most sensitive to changes in θ and least sensitive to changes in b.

(c): Since

$$dA = 0.5\,da + 0.25\,db + 0.866\,d\theta$$

insert for d-values 0.02

$$dA = \pm 0.03$$

Part 1, Problem 2E-5ab

(a) We see that

$$\frac{1}{w} = \frac{1}{t} + \frac{1}{u} + \frac{1}{v} \Rightarrow w = \left(\frac{1}{t} + \frac{1}{u} + \frac{1}{v}\right)^{-1}$$

thus ~~if~~ since

$$W_t = (t^{-1} + u^{-1} + v^{-1}) \cdot -t^{-2}$$

$$W_u = w \cdot -u^{-2}$$

$$W_v = w \cdot -v^{-2}$$

$$dw = A - t^{2}\, dt + A - u^{2}\, du + A - v^{-2}\, dv$$

(b): We have that

$$u^2 + 2v^2 + 3w^2 = 10$$

thus,

$$2u\, du + 4v\, dv + 6w\, dw = 0$$

$$\Longrightarrow$$

$$dw = \frac{-(2u\, du + 4v\, dv)}{6w}$$

Part 1, Problem 2E-1c

(i): By the chain rule,

$\frac{\partial f}{\partial t}$ where the base function is dependent on $(w(t), v(t))$,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$= w\,2u \cdot -2\sin(t) + w\,2v \cdot 2\cos(t)$$

factor out w and $\frac{1}{8}$;

$$= w \cdot 8\left(u\,\frac{-1}{2}\sin(t) + \frac{1}{2}\,v\,\cos(t)\right)$$

and insert $u = 2\cos(t)$, $v = 2\sin(t)$

$$= w \cdot 8\left(\sin(t)\cos(t) - \sin(t)\cos(t)\right)$$

$$= w \cdot 8 \cdot 0$$

$$= 0$$

(ii): Write the function;

$$w = \log(4\cos^2(t) + 4\sin^2(t))$$

$$= \log(4)$$

thus,

$$\frac{\partial w}{\partial t} = 0$$

Part 1, Problem 2E-2bc

(b): We see by the chain rule that

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

by the given gradient

$$\nabla w = \langle y, x \rangle$$

then

$$\frac{dw}{dt} = y \cdot -\sin(t) + x \cdot \cos(t)$$

$$= x^2 - y^2 \text{ or } \cos^2(t) - \sin^2(t) = \cos(2t)$$

then, $\frac{dw}{dt} = 0$ when $\frac{1}{2}\pi$ n where $n \in \mathbb{Z}$

(c): Once again by the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$= 1 \cdot 1 + -1 \cdot 2t + 2 \cdot 3t^2$$

$$= 1 - 2t + 6t^2$$

then plug in $t=1$;

$$= 5$$

Part 1, Problem 2E-8a

(a): ~~If $f(u(x,y))$, then~~

$$\cancel{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}}$$

$$\cancel{= f'(u)\,(}$$

By the chain rule for functions of one variable

$$\frac{\partial f}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} \quad ; \quad \frac{\partial f}{\partial y} = f'(u)\cdot\frac{\partial u}{\partial y}$$

$$= f'(u)\cdot -\frac{y}{x^2} \quad ; \quad = f'(u)\cdot\frac{1}{x}$$

By plugging into the given formula:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f'u\cdot\frac{-y}{x} + f'(u)\cdot\frac{y}{x} = 0 \quad \square$$

Part 1, Problem 20-1a,e

(a) We are given

$f(x,y) = x^3 + 2y^3$

$P = (1,1)$

$A = \langle 1, -1 \rangle$

Recall the definition of gradient

$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2, 6y^2 \rangle$

then at point P

$\nabla f(P) = \langle 3, 6 \rangle$

Then since

$$\frac{df}{ds}\Big|_{\hat{A}} = \nabla f \cdot \hat{A} = \langle 3, 6 \rangle \cdot \sqrt{2}^{-1} \langle 1, -1 \rangle$$

$$= \frac{-3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-3\sqrt{2}}{2}$$

(e): We are given

$f(x,y,z) = (x+2y+3z)^2$

and thus

$\nabla f = \langle 2(x+2y+3z), 4(x+2y+3z), 6(x+2y+3z) \rangle$

at the point $P = (1,-1,1)$

$\nabla f(P) = \langle 4, 8, 12 \rangle$.

And if $A = -2i + 2j - k$, then

$\hat{A} = \frac{A}{3}$

thus

$$\frac{df}{ds}\Big|_{\hat{A}} = \nabla f \cdot \hat{A} = \langle 4, 8, 12 \rangle \cdot \frac{1}{3} \cdot \langle -2, 2, 1 \rangle$$

$$= (-8 + 16 + 12) \cdot \frac{1}{3} = 20/3$$

Part 1, Problem 20-26

We have that

$f(x,y,z) = xy + yz + xz$

and thus

$\nabla f = \langle y+z, x+z, y+x \rangle$.

We write the formula (where $P = (1,-1,2)$)

$$\frac{df}{ds}\Big|_{\hat{u}} = \nabla f(P) \cdot \hat{u} = \langle 1, 3, 0 \rangle \cdot \hat{u}$$

we see by the rules following the dot product that

$\langle 1, 3, 0 \rangle \cdot \hat{u} = \cos(\theta)\sqrt{10}$

thus, when the angle between the gradient and $\hat{u}$ is 0, $\frac{df}{ds}\Big|_{\hat{u}} = \sqrt{10}$,

while when $\theta = n\pi$ where $n \in \mathbb{Z}$,

$$\frac{df}{ds}\Big|_{\hat{u}} = -\sqrt{10}$$

and when $\theta = \frac{n\pi}{2}$ where $n \in \mathbb{Z}$ then

$$\frac{df}{ds}\Big|_{\hat{u}} = 0.$$

Part 1, Problem 20-3a

Recall the theorem stating that ~~if~~ $\nabla w \perp$ level surface of w when ∇w is constant; a normal vector to aforementioned surface is also perpendicular ~~and which~~.

~~$\nabla f =$~~

~~$z = \left(\frac{12}{xy^2}\right)^{1/3}$~~

$f(x,y,z) = x y^2 z^3 = 12$

then since $\nabla f = \langle f_x, f_y, f_z \rangle$

$\nabla f = \langle y^2 z^3, 2xyz^3, 3xy^2z^2 \rangle$

and plugging in our point $P = (3,2,1)$

$\nabla f(P) = \langle 4, 12, 36 \rangle$

Which implies by the theorem that the tangent plane

$= 4x + 12y + 36 = 72$

by plugging in P.

Part 1, Problem 20-8

We are given

$P = 30 + (x+1)(y+2)e^{z}$

We then want to travel opposite the gradient at the origin, since $P(0,0,0) = 32$.

We see that

$\nabla P = \langle P_x, P_y, P_z \rangle = \langle (y+2)e^{z}, (x+1)e^{z}, P \rangle$

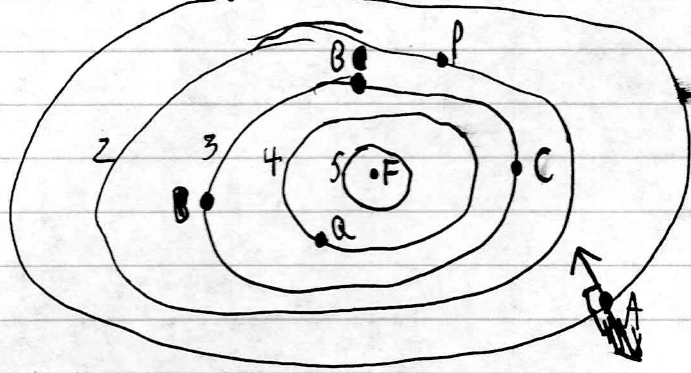
and thus

$\nabla P(0,0,0) = \langle 2, 1, 2 \rangle \Rightarrow |\nabla P(0,0,0)| = 3$

Then since $3 \cdot \Delta s = 0.9 \Rightarrow \Delta s = 0.3$, we should travel 0.3 units away from $\langle 2,1,2 \rangle$ and thus travel $0.1\langle -2,-1,-2 \rangle = \langle -0.2, -0.1, -0.2 \rangle$

Part 1, Problem 20-9

(a):
(b):
(c):



(d): $\cancel{\frac{\partial w}{\partial x} < 0}$ $\quad \frac{\partial w}{\partial x} \approx \frac{-1}{2} = -0.5$

$\cancel{\frac{\partial w}{\partial y} < 0}$ $\quad \frac{\partial w}{\partial y} \approx -1$

(e): $\left.\frac{\partial w}{\partial s}\right|_{i+j} \approx \frac{1}{\sqrt{2}}$

$\left.\frac{\partial w}{\partial s}\right|_{i-j} \approx \frac{-1}{\sqrt{2}}$

(f): at the peak, designated F on the above plot

Part 1, Problem 2E-7

We see by the chain rule that

$$f_u = f_x x_u + f_y y_u$$

$$f_v = f_x x_v + f_y y_v$$

thus by the ~~the~~ matrix multiplication operates;

$$\langle f_u, f_v \rangle = \langle f_x, f_y \rangle \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

Part 2 Problem 1

(a): Given

$$R = \cancel{k\frac{w}{r^4}} = k\frac{w}{r^4}$$

then

$$dR = R_w \, dw + R_r \, dr$$

$$dR = \frac{k}{r^4}\, dw + -4k\frac{w}{r^5}\, dr$$

(b): Divide each side of our (a) result to see

$$\frac{dR}{R} = \frac{1}{R}\frac{k}{r^4}\, dw + \frac{1}{R}\cdot -4k\frac{w}{r^5}\, dr$$

$$= \cancel{w^{-1}\, dw}\ \frac{dw}{w} + -4r^{-1}\, dr$$

(c): By (b) the function is more sensitive to changes in r, and signs should be opposite so that, due to the subtraction, the differences compound.

Part 2 Problem 2

We are given that

$$\frac{Df}{Dt} = \frac{d}{dt}\left(f(r(t), T)\right)$$

and thus

$$\ldots = \frac{d}{dt} f(x(t), y(t), z(t), t)$$

which by thee chain rule is equal to

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} + f_x \frac{\partial z}{\partial t} + f_t \frac{\partial t}{\partial t}$$

which by recalling the given ∇f, is equal to

$$\ldots = r'(t) \cdot \nabla f(r(t)) + \frac{\partial f}{\partial t} = v \cdot \nabla f + \frac{\partial f}{\partial t}$$

Part 2, Problem 3

(a): Recall that

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho$$

If $\rho = \rho(t)$, then

$\nabla \rho = \langle \rho_x, \rho_y, \rho_z \rangle = 0$, which means that

$\frac{D\rho}{Dt} = 0$ if $\frac{\partial \rho}{\partial t} = 0$

(b): If

$$\frac{\partial \rho}{\partial t} = 0$$

then

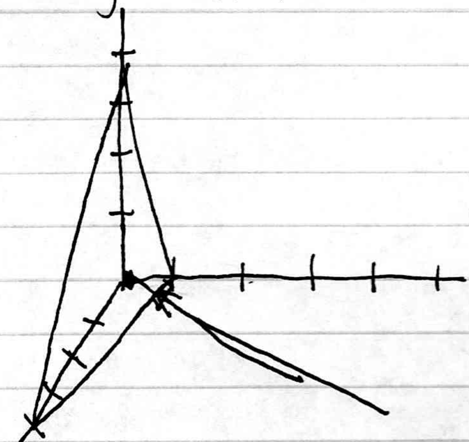
$$\frac{D\rho}{Dt} = 0$$

if $v \cdot \nabla \rho = 0$.

(c): If $\rho = \rho(y)$, then $\nabla \rho = \langle 0, \rho_y \rangle$ such that the gradien of the density $\parallel$ to j.

Part 2, Problem 4

(a):



(e):

(b): $\nabla f = \langle f_x, f_y \rangle = \langle -1, -4 \rangle$

(c): Demand $f(x,y) = 0$;

$0 = 4 - x - 4y \Rightarrow 4 = x + 4y$

then we want to find

$\langle x, y \rangle = s\langle -1, -4 \rangle$

and thus

$x = -s, \quad y = -4s = 4x$

then by plugging into

$4 = 16x + x \Rightarrow x = 4/17 \Rightarrow y = 16/17$

(d): The directional derivative

$$\nabla f(x,y) \cdot \frac{w}{|w|} = \langle -1, -4 \rangle \cdot \frac{\langle -2, -1 \rangle}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$