# Final Exam

## Problem 1

(a): Let $L$ be the line connecting the value of the base line with dir vector $\langle -1, 2, -3\rangle$ at $t=0$ to $(-1, 1, 2)$;

$P_0 = L(0) = (1, 1, \cancel{2})$;

Then, the direction vector of ~~the~~ this line is

$$(-1, 1, 2) - (1, 1, 2) = \langle -2, 0, 0\rangle$$

and thus we see that

$$n = v \times \overrightarrow{P_0 Q} = \begin{vmatrix} i & j & k \\ -1 & 2 & 3 \\ -2 & 0 & 0 \end{vmatrix} = \langle 0, 6, 4\rangle$$

and thus the plane must satisfy

$$0(x-1) + 6(y-1) + 4(z-2) = 0$$

$$6y - 6 + 4z - 8 = 0$$

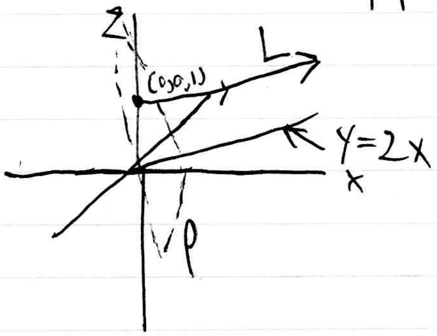
$$3y + 2y = 7$$

(b): Evaluate

$$\frac{\langle 2, 1, 1\rangle}{\sqrt{6}} \cdot \frac{\langle -1, 2, -3\rangle}{\sqrt{14}} = \frac{(-2+2-3)}{\sqrt{84}} = \frac{-3}{\sqrt{84}}$$

# Problem 2

(a):




$r = \langle x, y, z \rangle = \langle 0, 0, 1 \rangle + t\langle 1, 2, 0 \rangle$

~~implies~~ and/or

$x = t, \quad y = 2t, \quad z = 1$

(b): $x + 2y = 0$

(c): (i): $p = (t, 2t, 1)$

(ii): $p^* = (-t, -2t, 1)$

Problem 3

(a): Evaluate

$$\begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= \cancel{4 + 3 \cdot (-2+1)}$$

$$= 3 + 3(-2+1)$$

$$= 3 - 3 = 0.$$

(b): See that

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ -2 & 1 & -1 \end{vmatrix} = \langle -3, -5, 1 \rangle$$

Then ~~a potential~~ all potential solutions to

$Ax = 0$ are $x = t\begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}$

(c): We want that

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \cdot & \cdot & \cdot \\ -3 & p & 5 \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ -3 & p & 5 \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

implies

$$0 = -3 - 2p - 5$$

$$p = -4$$

# Problem 4

(a) First see that

$$r'(t) = \langle -\sin(e^t)e^t, +\cos(e^t)e^t, e^t \rangle$$

and that

$$|r'(t)| = \sqrt{\sin^2(e^t)e^{t2} + \cos^2(e^t)e^{t2} + e^{2t}}$$

$$= \sqrt{2}e^t,$$

thus

$$\frac{r'(t)}{|r'(t)|} = \frac{\langle -\sin(e^t)e^t, +\cos(e^t)e^t, e^t \rangle}{\sqrt{2}e^t}$$

$$= \frac{\langle -\sin(e^t), +\cos(e^t), 1 \rangle}{\sqrt{2}}$$

(b): $T'(t) = \frac{e^t}{\sqrt{2}} \langle -\cos(e^t), -\sin, 0 \rangle$

Problem 5

(a): Find $F_x, F_y, F_z$:

$$F_x = \frac{z}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{xz}{\sqrt{x^2+y^2}} \quad = \frac{1}{1}$$

$$F_y = \frac{z}{2\sqrt{x^2+y}} + \frac{2}{z} \quad = \frac{3}{2}$$

$$F_z = \frac{1}{\sqrt{x^2+y}} + -2\frac{y}{z^2} \quad = \frac{1}{2}$$

At $P_0$

implies

$$1(x-1) + \frac{3}{2}(y-3) + \frac{1}{2}(z-2) = 0$$

$$x-1 + \frac{3}{2}y - \frac{9}{2} + \frac{1}{2}z - 1 = 0$$

$$2x - 2 + 3y - 9 + z - 2 = 0$$

$$2x + 3y + z = 13.$$

(b): Changing y produces the greatest change; 0.15.

(c): Evaluate

$$\hat{u} \cdot \nabla F(P_0) = \pm \frac{1}{3}(-2 + 3 - 1/2) = \pm 1/6$$

$$0.1 = \frac{1}{6}\Delta s \Rightarrow \Delta s = 0.6.$$

Problem 6

(a): Find $f_x$ and $f_y$:

$f_x = 1 + -\frac{2}{x^2 y} \overset{!}{=} 0 \quad x^2 y = 2$

$f_y = 4 + -\frac{2}{x y^2} \overset{!}{=} 0 \quad x y^2 = \frac{1}{2}$

$\Rightarrow x = 4y \Rightarrow 4 y^3 = \frac{1}{2}$

and thus

$y = \frac{1}{2} \Rightarrow x = 2$, is the critical point.

(b): $f_{xx} = \frac{4}{x^3 y}$

$f_{xy} = \frac{2}{x^2 y^2}$

$f_{yy} = \frac{4}{x y^3}$

and thus

$f_{xx}(P_0) \cdot f_{yy}(P) - f_{xy}(P) = 12 > 0 \Rightarrow$ relative minimum

Problem 7

Consider some

$$f(x,y,z) = dist^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2.$$

Then,

$$\nabla f = 2\langle x-x_0, y-y_0, z-z_0\rangle$$

and the plane $G(x,y,z)$ $Ax + By + Cz = D$ and thus

$$\nabla G = \langle A, B, C\rangle$$

then

$$\nabla f = \lambda \nabla g \Rightarrow \begin{aligned} 2(x-x_0) &= \lambda A \\ 2(y-y_0) &= \lambda B \\ 2(z-z_0) &= \lambda C \\ Ax + By + Cz &= D \end{aligned}$$

Problem 8

(a): See that at $P$:

$$x_\phi = \rho\cos(\phi)\sin(\theta) \quad = 1$$

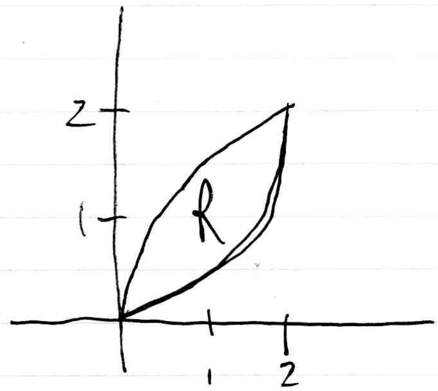
$$y_\phi = \rho\cos(\phi)\sin(\theta) \quad = -1$$

$$z_\phi = -\rho\sin(\phi) \quad = -\sqrt{2}$$

(b): No, for $\langle -y, x\rangle = \langle P, Q\rangle$, $P_y = -1 \neq 1 = Q_x$.

Problem 2

(a):




(b): $\int_0^2 \int_{\frac{1}{8}y^2}^{\sqrt{x}}$

$y = 2\sqrt{2x} \Rightarrow \cancel{x=} \quad \frac{1}{4}y^2 = 2x$

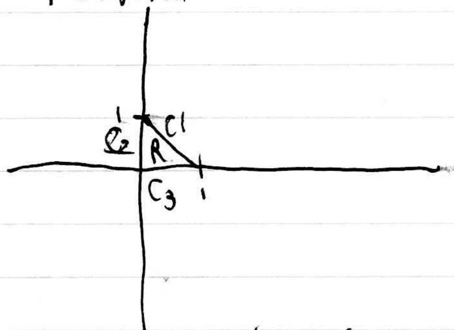
Problem 10

We integrate over u-range [4, 9] and v-range [1, 2]; then Jacobian $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \cdots = \frac{1}{3} u^{-1/3} v^{-2/3}$ implies the integral in question is

$$\int_1^2 \int_4^9 \frac{1}{3 \, u^3 \sqrt[3]{v}} .$$

## Problem 11

(a):




It looks like the flux out will be positive

(b): $\int_C F\cdot\hat{n}\, ds = \int_C -x\, dx + x\, dy$

On $C_1$: $x = 1-t$, $y = t$, $dx = -dt$, $dy = dt$

$\dots = \int_0^1 -(t-1) + 1 - t \quad dt = 1$

On $C_2$: $x = 0$,

$\int_{C_2} = 0$

On $C_3$: $y = 0$, $dy = 0$

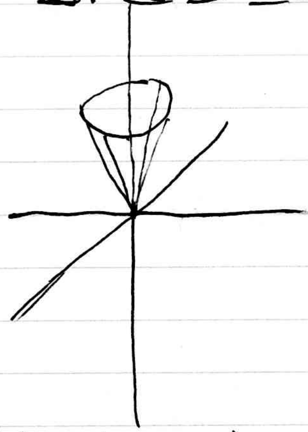
$+ \int_0^1 -x\, dx = \left[\frac{-x^2}{2}\right]_0^1 = \frac{-1}{2}$, thus

$\int_{C_1+C_2+C_3} = \frac{1}{2}$.

(c): $\operatorname{div}(F) = 1 \Rightarrow \iint_R \operatorname{div}(F)\, dA = \operatorname{area}(R) = \frac{1}{2}$

Problem 12

(a): The Limits on $\sigma$ are $2r \le z \le 2$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$



In cylindrical coordinates, where $dV = dz \cdot r\, dr\, d\theta$

$$m = \iiint_\sigma z\, dV = \int_0^{2\pi}\int_0^1\int_2^{2r} z\, dz\, r\, dr\, d\theta = \pi$$

(b): $$\bar{z} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_{2r}^{\cancel{m}2} z^2\, dz\, r\, dr\, d\theta$$

(c): In spherical, $dV = \rho^2 \sin\phi\, d\rho\, d\theta$ and $z = \rho\cos\phi$; thus

$$\bar{z} = \frac{1}{\cancel{m}\pi}\int_0^{2\pi}\int_0^{\tan^{-1}(1/2)}\int_0^{2\sec\phi} (\rho\cos\phi)^2\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Problem 13

(a): For

$$F = \langle P, Q, R \rangle = \langle y + y^2 z, x - z + 2xyz, -y + xy^2 \rangle$$

we see that

$$P_z = y^2 = R_x, \quad Q_z = -1 + 2xy = R_y, \text{ and } P_y = 1 + 2yz = Q_x$$

and thus F is a gradient field.

(b): $$f(x_1, y_1, z_1) = \int_0^{x_1} P(x, 0, 0)\, dx + \int_0^{y_1} P(x_1, y, 0)\, dy + \int_0^{z_1} P(x_1, y_1, z)\, dz$$
$$= 0 + \int_0^{y_1} x_1\, dy + \int_0^{z_1} -y_1 + x_1 y_1^2\, dz$$

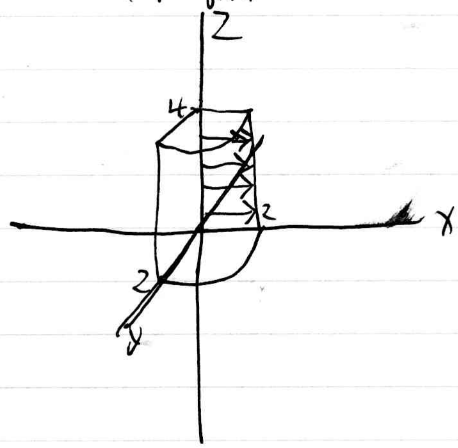
Implies

$$f(x,y,z) = xy - yz + xy^2 z + C.$$

(c): $-7$

Problem 14

(a):




(b): $F = \langle x, 0, 0 \rangle$

$\hat{n} = \frac{1}{2}\langle x, y, 0 \rangle$

$F \cdot \hat{n} = \frac{1}{2}x^2$, and with cylindrical coordinates where $dS = 2\,dz\,d\theta$:

$$\int_S F \cdot \hat{n}\, dS = \frac{1}{2}\int_0^{\pi/2} \int_0^4 (2\cos\theta)^2\, dz\, 2\, d\theta = 4\pi$$

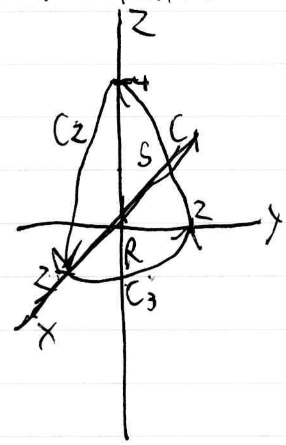
(c): $\mathrm{div}(F) = 1$, thus

$$\iiint_G \nabla \cdot F\, dV = \mathrm{vol}(G) = 4\pi.$$

(d): All 4 flat faces are either parallel to the ~~flux~~ force or where the movement is zero.

(a):

Problem 15




$$\nabla \times F = \langle x, y, -2z \rangle$$

and

$$\hat{n} \cdot dS = \langle 2x, 2y, 1 \rangle \, dA,$$

and the limits of integration are

$$0 \le r \le 2, \quad 0 \le \theta \le \pi/2; \text{ then}$$

$$\iint_S (\nabla \times F) \cdot \hat{n} \, dS = 4 \int_0^{\pi/2} \int_0^2 (r^2 - 2) r \, dr \, d\theta = 0.$$

(b): Break into the 3 parts seen above:

For $C_1$, $x = 0 \Rightarrow dx = 0$, $y = t$, $z = 4 - t^2$, $dz = -2t\,dt$; t from 2→

$$\int_{C_1} F \cdot dr = \int_2^0 -2t \, dt = 4$$

For $C_2$, $y = 0$, $dy = 0$, $x = t$, $z = 4 - t^2$, $dz = -2t\,dt$:

$$\int_{C_2} F \cdot dr = \int_0^2 -2t \, dt = -4$$

For $C_3$, $z = 0$, $dz = 0$

$$\int_{C_3} F \cdot dr = \int 0 \cdots = 0.$$

Thus,

$$\int_C F \cdot dr = 0 = -4 + 4 + 0.$$