Exam 1, Problem 1

(a): $P = (1, 0, 0)$

$Q = (0, 2, 0)$

$R = (0, 0, 3)$

Then,

$\overrightarrow{QP} = \langle 1, -2, 0 \rangle$

$\overrightarrow{QR} = \langle 0, -2, 3 \rangle$

(b): $\overrightarrow{QP} \cdot \overrightarrow{QR} = |\overrightarrow{QP}|\ |\overrightarrow{QR}| \cos(\theta)$

$4 = \sqrt{5}\ \sqrt{13}\ \cos(\theta)$

$\implies$

$\cos(\theta) = \dfrac{4}{\sqrt{65}}$

Problem 2

(a): <u>Let</u>

$\overrightarrow{PQ} = \langle -1, 2, 0 \rangle$

$\overrightarrow{PR} = \langle -1, 0, 3 \rangle$;

then

$$A(PQR) = |\overrightarrow{PQ} \times \overrightarrow{PR}| = \left| \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} \right|$$

$$= \left| 6i + -3j + 2k \right| \qquad \text{(a)}$$

$$= \sqrt{39 + 9 + 4}$$

$$= \sqrt{52} \text{ units}^2$$

(b): By line (a), we can ~~w~~rite this as

$$6x + -3y + 2z = d$$

we plug in point P to find $d = 5$, thus

$$6x - 3y + 2z = 5$$

(c): The vector ~~dra~~ p between these points is

$$p = \langle 1, 0, -3 \rangle$$

if this is orthogonal to the normal vector

$\overrightarrow{PQ} \times \overrightarrow{PR} = 6i + -3j + 2k$, then it is parallel to the plane.

$$\langle 6, -3, 2 \rangle \cdot \langle 1, 0, -3 \rangle \overset{!}{=} \sqrt{52} \cdot \sqrt{10} \cos(\theta)$$

$$\cos(\theta) = 0$$

$$\Rightarrow$$

$\theta = 90°$, thus the line is parallel.

Problem 3

(a): We write the vector composition

$$\overrightarrow{OL} = \overrightarrow{OV} + \overrightarrow{VL}$$

to represent the connection of origin to car, and car to ladybug.

$$\overrightarrow{OL} = \langle 10t, 0 \rangle + \langle \cos(t), \sin(t) \rangle$$
$$= \langle 10t + \cos(t), \sin(t) \rangle$$

and thus

$$x(t) = 10t + \cos(t)$$
$$y(t) = \sin(t)$$

(b): Then, the bug has speed

$$\left| \frac{d}{dt} \overrightarrow{OL} \right| = \left| \langle 10 + -\sin(t), \cos(t) \rangle \right|$$
$$= \sqrt{(10-\sin(t))^2 + \cos^2(t)}$$

Square each side for ease of solution;

$$\left| \frac{d}{dt} \overrightarrow{OL} \right|^2 = (10 - \sin(t))^2 + \cos^2(t)$$

$$= 100 - 20\sin(t) + \sin^2(t) + \cos(t)$$

$$= 101 - 20\sin(t) \Rightarrow \left| \frac{d}{dt} \overrightarrow{OL} \right| = \sqrt{101 - 20\sin(t)}$$

Then, the speed is minimized when $\sin(t)$ is maximized, and maximized when $\sin(t)$ is maximized; the minimum occurs at $\pi/2$, and is maximized at $\emptyset$.

Problem 4

(a): $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}$

$= -1 + 2(-3-2) + 3(-3-4)$

$= -1 - 10 - 21$

$= -32$

(b): $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1/12 & 1/12 & 1/3 \\ a/12 & 7/12 & -2/3 \\ b/12 & -5/12 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Longrightarrow$

$\left(\frac{1}{12} + a/6 + b/4 = 1\right) \cdot 12 = 1 + 2a + 3b = 12$

$\left(\frac{1}{4} + a/6 + b/12 = 0\right) \cdot 12 = -(3 + 2a + b = 0)$

$-2 + 2b = 12$

$b = 7$

$a = -5$

(c): Write the equation

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 3 \end{bmatrix}$

Then,

$\frac{1}{12} \begin{bmatrix} 1 & 1 & 4 \\ -5 & 7 & -8 \\ 7 & -5 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \vec{r} = \left\langle \frac{1}{12}(13 + t), \frac{1}{12}(7t - 24), \frac{1}{12}(12 - 5t) \right\rangle$

$= \left\langle \frac{13}{12} + \frac{t}{12}, \frac{7t}{12} - 2, 1 - \frac{5t}{12} \right\rangle$

Problem 5

(a): This plane has normal vector $\vec{N} = \langle 4, -3, -2 \rangle$; since $\vec{r}(t)$ is on the plane, ~~Then if~~ Suppose we have both $P(t)$ and some point P_0 on the plane; let its position vector $\vec{P_0} = \langle 0, 0, -3 \rangle$. The vector given by $P(t) - \vec{P_0}$ is on the plane; thus

$$\vec{N} \cdot (P(t) - \vec{P_0}) = 0$$

$$\Rightarrow$$

$$\vec{N} \cdot P(t) = \langle 4, -3, -2 \rangle \cdot \langle 0, 0, -3 \rangle = 6. \quad (a)$$

(b): Differentiate (a);

$$\frac{d}{dt}\left(\vec{N} \cdot P(t)\right) = \vec{0} \cdot P(t) + \vec{N} \cdot \frac{d}{dt} P(t) = 0$$

$$= N \cdot \frac{d\,P(t)}{dt} = 0$$

showing that the normal vector and the derivative of the position vector are orthogonal