Part I, Problem 2C-1ad

Find the differential (dw or dz). Make the answer look as neat as possible.

(a) $w = \log(xyz)$.

(d) $w = \sin^{-1}\left(\frac{u}{t}\right)$.

Solution

Part I, Problem 2C-2

The dimensions of a rectangular box are $5, 10$, and 20 cm, with a possible measurement error in each side of ± 0.1 cm. Use differentials to find what possible error should be attached to its volume.

Solution

Part I, Problem 2C-3

Two sides of a triangle have lengths respectively a and b, with θ the included angle. Let A be the area of the triangle.

(a) Express dA in terms of the variables and their differentials.

(b) If $a = 1, b = 2, \theta = \frac{\pi}{6}$, to which variable is A most sensitive? Least sensitive?

(c) Using the values in (b), if the possible error in each value is 0.02, what is the possible error in A, to two decimal places?

Solution

Part I, Problem 2C-5ab

The following equations define w implicitly as a function of the other variables. Find dw in terms of all the variables by taking the differential of both sides and solving algebraically for dw.

(a) $\frac{1}{w} = \frac{1}{t} + \frac{1}{u} + \frac{1}{v}$

(b) $u^2 + 2v^2 + 3w^2 = 10$

Solution

Part I, Problem 2E-1c

In the following, find $\frac{df}{dt}$ for the composite function

$$\begin{aligned} w(u(t), v(t)) &= \log\left((u(t))^2 + (v(t))^2\right), \qquad (1) \\ u &= 2\cos(t) \qquad (2) \\ v &= 2\sin(t). \qquad (3) \end{aligned}$$

in two ways:

(i) use the chain rule, then express your answer in terms of t by using $u = u(t)$, etc.;

(ii) express the composite function f in terms of t, and differentiate.

Solution

Part I, Problem 2E-2bc

In each of these, information about the gradient of an unknown function $f(x, y)$ is given; x and y are in turn functions of t. Use the chain rule to find out additional information about the composite function $w = f(x(t), y(t))$, without trying to determine f explicitly.

(b) $\nabla w = y\mathbf{i} + x\mathbf{j}$; $x = \cos(t), y = \sin(t)$. Find $\frac{dw}{dt}$ and tell for what t-values it is zero.

(c) $\nabla f = \langle 1, -1, 2 \rangle$ at $(1, 1, 1)$. Let $x = t$, $y = t^2$, $z = t^3$; find $\frac{df}{dt}$ at $t = 1$.

Solution

Part I, Problem 2E-8a

(a) Let $w = f\left(\frac{y}{x}\right)$; i.e. w is the composite of the functions $w = f(u)$ and $u = \frac{y}{x}$. Show that w satsifies the PDE (partial differential equation) $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$.

Solution

Part I, Problem 2D-1ae

In each of the following, a function f, a point P, and a vector $\mathbf{A}$ are given. Calculat the gradient of f at the point, and the directional derivative $\frac{df}{ds}\big|_{\mathbf{u}}$ at the point, in the direction $\mathbf{u}$ of the given vector $\mathbf{A}$.

(a) $x^3 + 2y^3$; $(1, 1)$, $\mathbf{i} - \mathbf{j}$.

(e) $f(u, v, w) = (u + 2v + 3w)^2$; $(1, -1, 1)$, $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution

Part I, Problem 2D-2b

For the function

$$w = xy + yz + xz \tag{4}$$

at the point $P = (1, -1, 2)$,

(i) find the maximum and minimum values of $\frac{df}{ds}\big|_{\mathbf{u}}$, as $\mathbf{u}$ varies;

(ii) tell for which directions the maximum and minimum occur;

(iii) find the direction(s) $\mathbf{u}$ for which $\frac{df}{ds}\big|_{\mathbf{u}} = 0$.

Solution

Part I, Problem 2D-3a

By viewing the following surface as a contour surface of a function $f(x, y, z)$, find its tangent plane at the given point P.

$$\begin{aligned} xy^2z^3 &= 12, \\ P &= (3, 2, 1). \end{aligned} \tag{5}$$

Solution

Part I, Problem 2D-8
The atmospheric pressure in a region of space near the origin is given by the formula

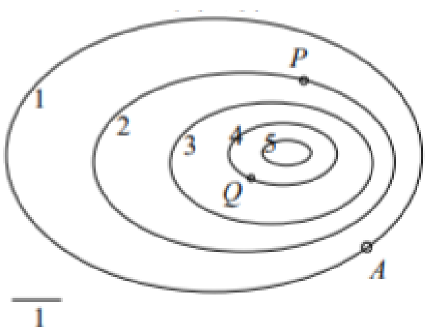
$$P = 30 + (x+1)(y+2)e^z. \tag{6}$$

Approximately where is the point closest to the origin at which the pressure is 31.1?

Solution

Part I, Problem 2D-9

The accompanying picture shows the level curves of a function $w = f(x, y)$. The value of w on each curve is marked. A unit distance is given.



(a) Draw in the gradient vector at A.

(b) Find a point B where $w = 3$ and $\frac{\partial w}{\partial x} = 0$.

(c) Find a point C where $w = 3$ and $\frac{\partial w}{\partial y} = 0$.

(d) At the point P, estimate the value of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

(e) At the point Q, estimate $\frac{dw}{ds}$ in the direction of $\mathbf{i} + \mathbf{j}$.

(f) At the point Q, estimate $\frac{dw}{ds}$ in the direction of $\mathbf{i} - \mathbf{j}$.

(g) Approximately where is the gradient $\mathbf{0}$?

Solution

Part I, Problem 2E-7

The Jacobian matrix for the change of variables $x = x(u, v)$, $y = y(u, v)$ is defined to be

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}. \tag{7}$$

Let $\nabla f(x, y)$ be represented as the row vector $\langle f_x, f_y \rangle$. Show that

$$\nabla f(x(u, v), y(u, v)) = \nabla f(x, y) \cdot J, \tag{8}$$

where $\cdot$ means matrix multiplication.

Solution

Part II, Problem 1

In laminar flow in a cylinder (for example, blood flow in a vein or artery), the resistance R to the flow is related to the length w and radius r of the cylinder by the law of Poiseuille:

$$R = k\frac{w}{r^4} \tag{9}$$

for some constant k.

(a) Compute the linear approximation dR to the change in R, in terms of the changes in w and r.

(b) Compute the linear approximation $\frac{dR}{R}$ to the *relative* change in R in terms of $\frac{dw}{w}$ = the relative change in w and $\frac{dr}{r}$ = the relative change in r.

(c) For relative changes in w and r of about the same sizes, which variable contributes more to the relative change in R? Also, in order to produce the greatest relative change in R, should the changes in w and r both have the same sign or opposite signs (and why)?

Solution

Part II, Problem 2

Let $f(x, y, z, t)$ be a smooth function, and let $\nabla f = \langle f_x, f_y, f_z \rangle$ be the gradient in the *space* variables only. Let $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a smooth curve, and $\mathbf{v} = \mathbf{r}'(t)$; and usppose we use the notation

$$\frac{Df}{Dt} = \frac{d}{dt} f(\mathbf{r}(t), t). \tag{10}$$

Use the chain rule to show that $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$.

Background: The notation $\frac{D}{Dt}$ comes from the physics of fluid motion, where it is called the *convective derivative* (or material or substantial derivative, and by several other names), and means the rate of change along a moving path of some physical quantity (scalar or vector) which is being transported by fluid currents. In this macroscopic model, the fluid is pictured as a continuum of point masses rather than as individual molecules. At a location (x, y, z) in space and a time t, the point mass has a density $\rho = \rho(x, y, z, t)$ and a velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$. This means that the vector $\mathbf{v}(x, y, z, t)$ points in a direction tangent to the path of a particle at (x, y, z, t) in the flow, and has magnitude equal to the instantaneous speed of the particle located at that point and which is moving in the flow.

Now suppose that the curve $\mathbf{r} = \mathbf{r}(t)$ is a path of a point mass in the flow, so that (by definition) $\mathbf{r}'(t) = \mathbf{v}(\mathbf{r}(t), t)$. The convective derivative $\frac{Df}{Dt}$ of f along this path is the time rate of change of f using *only* the values of $f(x, y, z, t)$ for which the space variables (x, y, z) are *restricted to the path* $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ of a particle in the flow. For this reason, you will see the convective derivative described as the rate of change of the quantity f "moving along the flow" or "moving with an element of the fluid" (and other similar language).

Solution

Part II, Problem 3

Now take the case $f = \rho$, the density of the fluid. A fluid flow is called *incompressible* if

$$\frac{D\rho}{Dt} = 0. \tag{11}$$

As discussed above, this means that the mass density is constant along the paths of the flow. Any substance (like water, at moderate pressure) which has the property that its density is constant in all variables (x, y, z, t) will of course be incompressible, which is the usual way one pictures something which cannot be compressed. However, incompressability is in general a property of the *flow* rather than just the fluid itself, since it says only that the rate of change of the density moving along the flow is zero. The following examples illustrate this.

(a) Suppose that hte density function depends only on *time* t but is constant in the space variables (x, y, z), that is, $\rho = \rho(t)$. Then show that the flow is incompressible if and only if the density $\rho(t)$ is constant in all the variables (x, y, z, t) (that is, the constant-density case discussed above).

(b) Next suppose instead that the density depends only on the *space* variables (x, y, z) but not (explicitly) on t, so that $\rho = \rho(x, y, z)$. An incompressible flow in this case is called *stratified.*

Use the result of problem 2 to give the condition on ρ and $\mathbf{v}$ for stratified flow.

A flow is called *steady* if the density ρ and the velocity field $\mathbf{v}$ do not depend explicitly on the time t, i.e. $\rho = \rho(x, y, z)$ and $\mathbf{v} = \mathbf{v}(x, y, z)$. In this case, the term *streamlines* is used for the paths of the particles in the flow, since they keep their same shape over time.

(c) Suppose one has a $2D$ stratified steady flow, so that $\rho = \rho(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$, and suppose also that the density varies only with the height y. Draw a picture of the streamlines for such a flow. Then explain why they must follow this pattern, and why the term "stratified" fits in this case.

(This could be, for example, a cross-section of a very regular ocean current, if it is an incompressible steady flow whose density varies only with the depth.)

Solution

Part II, Problem 4

For the linear function $f(x, y) = 4 - x - 4y$,

(a) Sketch the portion of the graph in the first octant.

(b) Compute the gradient of f.

(c) Find the point on the level curve $f(x, y) = 0$ such that the line in the gradient direction passes through the origin, and gthen sketch in the gradient at that point.

(d) Compute the directional derivative of f in the direction $\mathbf{w} = -2\mathbf{i} - \mathbf{j}$.

(e) Sketch in the slope triangles for the rates of change of f in the gradient direction and in the direction of $\mathbf{w}$.

Solution