PSET 8

Part 1, Problem 4A-1d

(d): This is a clockwise spiral about the origin at unit ~~speed~~ magnitude/r.
Rephrased: ~~of~~ At θ the vector in question is perpendicular ~~and~~ clock-wise, with magnitude of unit length.

Part 1, Problem 4A-2Bc

(b): Assuming $r = \sqrt{x^2+y^2}$, then

$$w(r) = \log(\sqrt{x^2+y^2})$$

$$\cancel{w'(r) = 1}$$

$$w_x = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \cancel{\frac{x}{\sqrt{x^2+y^2}}} = \frac{x}{x^2+y^2}$$

$$w_y = \cancel{\frac{y}{\sqrt{x^2+y^2}}} = \frac{y}{x^2+y^2}$$

Thus,

$$\nabla w = \cancel{\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle = \frac{\langle x, y \rangle}{r^2}$$

d: $$w_x = f'(r) \cdot \frac{1}{2r} \cdot 2x = \frac{2x\, f'(r)}{r}$$

$$w_y = \frac{y\, f'(r)}{r}$$

$$\nabla w = \frac{f'(r)}{r} \langle x, y \rangle$$

Part 1, Problem 4A-3bd

(b): $r^3\langle -x, -y\rangle$

(d): For some function $f(x,y)$ that returns the magnitude,
$f(x,y)\langle 1, 1\rangle$

Part 1, Problem 4B-1ab

(a): We see that

$$\int_C \vec{F} \cdot dr = \int_C (x^2-y)\,dx + 2x\,dy$$

(i): By the given line C_1, $x=x$, $y=0$ and thus $dx=1$, $dy=0$

$$=\int_{-1}^{1} x^2\,dx$$

$$=\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}-\left(\frac{-1}{3}\right) = \frac{2}{3}$$

(ii): Or by the given line C_2, $\cancel{y=y,\ x=\sqrt{1-y},\ dy=1,\ dx=\frac{1}{2\sqrt{1-y}}}$ $y=1-x^2$, $dy=-2x\,dx$

$$=\int_{-1}^{1} (x^2+x^2-1)\,dx + 2x\cdot -2x\,dx$$

$$=\int_{-1}^{1} -\cancel{4}2x^2-1\,dx$$

$$=\left[\frac{\cancel{4}2}{3}x^3+x\right]_{-1}^{1} = \cancel{\frac{-1}{3}-\left(\frac{4}{3}+\frac{1}{1}\right)} =$$

$$=\cancel{\left(\frac{-2}{3}-1\right)-\left(\frac{2}{3}+1\right)}$$

$$=-\left(\left(\frac{2}{3}+1\right)-\left(\frac{-2}{3}-1\right)\right)$$

$$=-\frac{4}{3}-2=\frac{-10}{3}$$

(b): We see that

$$\int_C xy\,dx + -x^2\,dy$$

We parametrize using the given C that $y=\sqrt{1-x^2}$, $x=x$, $dy=\frac{-2x}{\sqrt{1-x^2}}$

$$\dots = \int_0^1 x\sqrt{1-x^2}\,\cancel{dt} + \int_0^1 -x^2\,dx$$

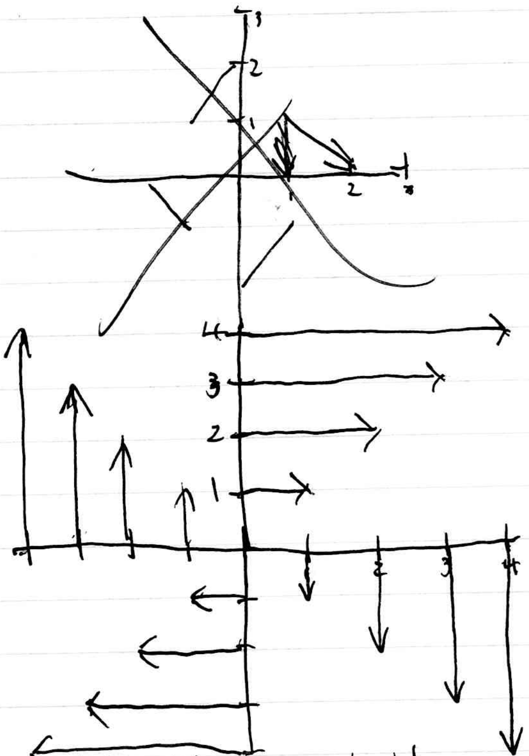
Hm; this is hard to solve. Let's try some other parametrization. Let $x=\cos t$, $y=\sin t \Rightarrow dx=-\sin(t)\,dt$, $dy=\cos(t)\,dt$

$$\dots = -\int_0^{\pi/2} \cos(t)\sin^2(t)\,dt + -\cos^3(t)\,dt$$

$$=-\int_0^{2\pi} -\cos(t)\,dt = \left[-\sin t\right]_{\pi/2}^{0} \cancel{=-(0-0)=0-(1)} = 1$$

Part 1, Problem 4B-2b

(b): Sketch the field:



Thus, by inspection, F is the rotating ~~count~~ clockwise vortex force field F imparts force $F = -a\,t$ on the particle, and ~~the particle~~ thus

$\int_C F\,dr = \int_C F \cdot t\,ds = -\int_C a\,ds = -2\pi a^2$

Part 1, Problem 4B-3

(a): In line with the field: $C = \frac{\langle 1, 1 \rangle}{\sqrt{2}}$

(b): Directly opposite the direction of the field:

$C = \langle -1, -1 \rangle / -\sqrt{2}$

(c): Perpendicular to the field: $C = \pm \frac{\langle 1, -1 \rangle}{\sqrt{2}}$

(d): By noting the max magnitude seen in (a) and (b); $max = \sqrt{2}$
$min = \sqrt{2}$

Part 1, Problem 4C-1

(a) We see that

$f_x = 3x^2 y$

$f_y = x^3 + 3y^2$,

thus

$\nabla f = \langle 3x^2 y, x^3 + 3y^2 \rangle$

(B): (i): $\int_{C_1} F \cdot dr = \int_{C_1} 3x^2 y \, dx + (x^3 + 3y^2)\, dy$

By the parameter $x = y^2$, and thus $dx = \cancel{dy}\ 2y\, dy$, then

$$= \int_{-1}^{1} 3y^5\, 2y\, dy + (y^6 + 3y^2)\, dy$$

$$= \int_{-1}^{1} 7y^6 + 3y^2 \, dy$$

$$= \left[y^5 + y^3 \right]_{-1}^{1} = 2 - -2 = 4$$

(ii): Replace the path with the simpler $x = 1$, $y = y$

$$\int_{C_2} F \cdot dr = \int_{-1}^{1} 1 + 3y^2 \, dy = \left[y + y^3 \right]_{-1}^{1} = 2 - (-2) = 4$$

(iii): By ~~the gradient,~~ $W = f(P_1) - f(P_0)$

$$= 2 - (-2) = 4$$

Part 1, Problem 4C-3

(a) We see that

$$f_x = \cos(x)\cos(y)$$

$$f_y = -\sin(x)\sin(y),$$

thus

$$\nabla f = \left\langle \cos(x)\cos(y), -\sin(x)\sin(y) \right\rangle$$

(b): We see by ~~the~~ path-independence ~~fo~~ and the fundamental theorem for line integrals that the maximum is where $f(P_1) - f(P_0)$ is maximized. Follow the line from $(2\pi, 0)$ to $(2\pi, \cancel{2\pi}\ \pi)$, then $W = 2$

Part 6, Problem 4C-5~~a~~b

This is only a gradient if $M_y = N_x$ in the format

$F = M\,dx + N\,dy$;

see that

$\cancel{M_x = 2}$ $\quad M_y = 2y$

$\quad\Rightarrow\quad a = 2$;

$\cancel{N_y = ax}$ $\quad N_x = ay$

thus,

$F = \langle y^2 + 2x, 2xy \rangle$

and

$f = xy^2 + x^2 + c$

Part 1, Problem 4C-5 6

~~We see that~~

$$\cancel{F = e^{x+y}((x+a)\hat{i} + x\hat{j})}$$

$$\cancel{\Rightarrow e^{x+y}((M\,dx + N\,dy)}$$

~~then by seeing~~

We have that

$M_y = e^{x+y}(x+a)$ $\Rightarrow$ $M_y = e^{x+y}(x+a)$

$N_x = xe^{x+y}$ $\quad$ $N_x = e^{(x+y)} + xe^{x+y}$

and thus

$$xe^{x+y} + ae^{x+y} = e^{(x+y)} + xe^{x+y}$$
$$ae^{(x+y)} = e^{(x+y)}$$
$$a = 1$$

Thus,

$M = xe^{x+y} + e^{x+y} = f_x$

$N = xe^{x+y} = f_y$

We should get that

$\int M\,dx = f + C,$

then evaluate

$$\int xe^{x+y} + e^{x+y}\,dx = \int xe^{x+y}\,dx + e^{x+y}$$

Integrate by parts;

$$\int x e^{x+y} = xe^{x+y} - e^{x+y}$$

And thus

$f = xe^{x+y} + C$

Part 1, Problem 4 C-6ab

(a): This is not exact since $y\,dx - x\,dy = M\,dx - N\,dy$ has that $M_y \neq N_x$ since $1 \neq -1$.

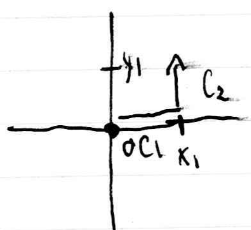
(b): $$\frac{\partial(2xy+y^2)}{\partial y} \stackrel{?}{=} \frac{\partial(2xy+x^2)}{\partial x}$$

$$2x+2y \stackrel{\checkmark}{=} 2y+2x$$

Yes, this is exact. We then see that

$$f(x_1,y_1) = \int_C F\,dr = \int_{0,0}^{x_1,y_1} 2xy+y^2\,dx + 2xy+x^2\,dy$$

Consider the path



Parametrize C_1 with $y=0$, $x=x$, $dy=0$; then the integral is 0.
Then parametrize with $y=y$, $x=x_1$, $dy=dy$, $dx=0$; thus

$$\cdots\int_{C1}\cdots = \int_0^{y_1} 2x_1 y + x_1^2\,dy$$

$$x_1 y_1^2 + x_1^2 y_1$$

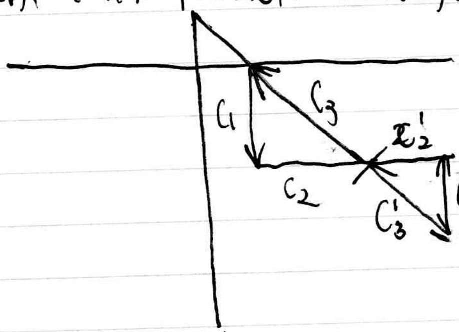
Thus,

$$f(x,y) = xy^2 + x^2 y$$

Part 2, Problem 1

(a): $F = 1 - x - y$; this sends any point up/down to be on the line.

(b): Sketch ~~a section~~ each possible triangle



C_3 and $C_3' = 0$, since they are not with/against any force.
C_1 and $C_1' > 0$, since they go against the force
C_2 and $C_2' = 0$, since they are perpendicular to the force.

$C_1 + C_2 + C_3 = C > 0$ by addition. Likewise,
$C_1' + C_2' + C_3' = C' > 0$ " " .

Thus, $W \neq 0$

Part 2, Problem 2

(a): We integrate

$$\int_C F\cdot dr = \int_C \frac{-x}{x^2+y^2}\,dx + \frac{-y}{x^2+y^2}\,dy$$

Then parametrized $y=1$, $dy=0$, $x=x$

$$\dots = \int_0^{\infty} \frac{-x}{x^2+1} = -\frac{1}{2}\Big[\log(x)\Big]_0^{\infty} = \boxed{-\infty}$$

(b): Parametrize with $x = a\cos t$, $y = a\sin(t)$, $dx = -a\sin(t)$, $dy = a\cos(t)$

$$\dots = \int_0^{2\pi} \frac{a^2\sin(t)\cos(t)}{a^2}\,dt + \frac{-a^2\sin(t)\cos(t)}{a^2}\,dt$$

$$= \int_0^{2\pi} 0\,dt = \boxed{0}$$

(c): Paramatrize with $x=t$, $y=1-t$, $dx=dt$, $dy=-dt$

$$\cancel{\dots = \int_0^1 \frac{-t}{2t^2}\,dt + \frac{t-1}{2t^2}\,dt}$$

$$\cancel{= \int_0^1 \frac{-1}{2t^2}\,dt = \frac{t^1}{2}\Big[t^{-1}\Big]_0^1 - }$$

$$= \int_0^1 \frac{-t}{t^2+1+t^2-2t}\,dt + \frac{\cancel{t-1}\ 1-t}{2t^2+1-2t}$$

$$= \int_0^1 \frac{-2t+1}{2t^2-2t+1} = \left[\frac{-1}{2}\log(2t^2-2t+1)\right]_0^1 = 0$$

Part 2, Problem 3

(a): Let $r = \sqrt{x^2+y^2}$; then $r_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$, $r_y = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r}$

and thus

$$-\nabla \ln(r) = -\left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

(b): $\displaystyle\int_{P_1}^{P_2} F \cdot dr = \left[-\ln(r)\right]_{P_1}^{P_2} = -\ln\left(\frac{r_2}{r_1}\right),$

Part 2, Problem 4

(a): See that

$$\int_C F \cdot dr = \int (2xy + 2y^2)\,dx + (x^2 + 4xy)\,dy$$

(b): Parametrize $x = \frac{1}{3}\cos(t)$, $y = \frac{1}{2}\sin(t)$ over $[0, \pi/2]$ and $dx = \frac{-1}{3}\sin(t)\,dt$, $dy = \frac{1}{2}\cos(t)$:

$$\dots = \int_0^{\pi/2} \left(\frac{1}{3}\cos(t)\sin(t) + \frac{1}{2}\sin^2 t\right)\frac{-1}{3}\sin(t)\,dt + \left(\frac{1}{9}\cos^2(t) + \frac{2}{3}\sin t\cos t\right)$$

$$x\,\frac{1}{2}\cos t$$

Absolute. Unit.

(c): Evaluate the difference in the gradient function:

$$\int_C F \cdot dr = \left[x^2y + 2xy^2\right]_{\cancel{0,1/3}\,(1/3,0)}^{(0,1/2)} = 0$$

(d): Pick the path along the axes; by path independence, the path will be the same at 0.

Part 2, Problem 5

(a): We see that, given

$M = xy$

$N = x^3$,

that $M_y - N_x = x - 3x^2 \neq 0$; thus F is not conservative

(b): We attempt method 1;

$\int_{P_1}^{P_2} xy\,dx + x^3\,dy$

Break into C_1 along the x-axis and C_2 along the y-axis:

Parametrize C_1 as $y = 0$, $dy = 0$, $x = x$, $dx = dx$

$\int_0^{x_1} 0\,dx = 0$

And parametrize as $y = y$, $x = x_1$, $dx = 0$, $dy = dy$:

$\int_0^{y_1} x_1^3\,dy = y_1 x_1^3 \Rightarrow f(x,y) = y x^3$

(c): We see that

$f_x = xy \Rightarrow f = \frac{x^2 y}{2} + g(y)$

and

$f_y = \frac{x^2}{2} + g'(y) \Rightarrow g'(y) = x^3 - \frac{x^2}{2}$

No such function g exists to satisfy this.