Part I, Problem 2A-1c

Sketch five level curves for the function

$$f(x, y) = x^2 + y^2. \tag{1}$$

Also sketch the portion of the graph of the function lying in the first octant; include in your sketch the traces of the graph in the three coordinate planes, if possible.

Solution

Part I, Problem 2A-2be

Calculate the first partial derivatives of each of the following functions:

(b) $z = \frac{x}{y}$

(e) $z = x \log(2x + y)$

Solution

Part I, Problem 2A-3b

Verify that $f_{xy} = f_{yx}$ for the function

$$f(x, y) = \frac{x}{x+y}. \tag{2}$$

Solution

Part I, Problem 2A-5a

Show that the function

$$w = e^{ax}\sin(ay), \tag{3}$$

where a is a constant, satisfies the equation $w_{xx} + w_{yy} = 0$ (called the two-dimensional *Laplace equation*).

Solution

Part I, Problem 2B-1b

Give the equation of the tangent plane to the surface

$$w = \frac{y^2}{x} \tag{4}$$

at the point $(1, 2, 4)$.

Solution

Part I, Problem 2B-6

To determine the volume of a cylinder of radius 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed 0.1?

Solution

Part I, Problem 2B-9

(a) Around the point $(1, 0)$, is $w = x^2(y + 1)$ more sensitive to changes in x or y?

(b) What should the ratio of Δx to Δy be in order that small changes with this ratio produce *no* change in w, i.e., no first-order change – of course w will change a little, but like $(\Delta x)^2$, not like Δx.

Solution

Part I, Problem 2F-1a

Find the point(s) on the surface

$$xyz^2 = 1 \tag{5}$$

which is closest to the origin.

(Hint: it's easier to minimize the square of the distance, rather than the distance itself.)

Solution

Part I, Problem 2F-2

A rectangular produce box is to be made of cardboard; the sides of single thickness, the front and back of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what proportions for the sides will use the least cardboard?

Solution

Part I, Problem 2G-1c

Find by the method of least squares the line which best fits the three data points given. Do it from scratch, using the formula

$$D = \sum_{i=1}^{n} \left(y_i - (ax_i + b)\right)^2 , \tag{6}$$

which was in the reading on least squares, and differentiation (use the chain rule). Sketch the line and three points as a check.

Points: $(1, 1)$, $(2, 3)$, $(3, 2)$.

Solution

Part I, Problem 2G-4

What linear equations in a, b, c does the method of least squares lead to, when you use it to fit a linear function $z = a + bx + cy$ to a set of data points (x_i, y_i, z_i), $i = 1, \cdots, n$?

Solution

Part I, Problem 2H-1c
For the function

$$f(x, y) = 2x^4 + y^2 - xy + 1, \tag{7}$$

find the critical points and classify them using the second derivative criterion.

Solution

Part I, Problem 2H-3

Find the maximum and minimum of the function $f(x, y) = x^2 + y^2 + 2x + 4y - 1$ in the right half-plane R bounded by the diagonal line $y = -x$.

Solution

Part I, Problem 2H-4

Find the maximum and minimum points of the function $xy - x - y + 2$ on

(a) the first quadrant;

(b) the square $R : 0 \leq x \leq 2, 0 \leq y \leq 2$ (study its values at the unique critical point and on the boundary lines); and

(c) use the data to guess the critical point type, and confirm it by the second derivative test.

Solution

Part I, Problem 2H-6

(a) Two wires of length 4 are cut in the same way into three pieces, of length x, y, and z; the four x, y pieces are used as the four sides of a rectangle; the two z pieces are bent at the middle and joined at the ends to make a square of side length $\frac{z}{2}$. Find the rectangle and square made this way which together have the largest and smallest total area.

Using the answer, tell what type the critical point is.

(b) Confirm the critical point type by using the second derivative test.

Solution

Part II, Problem 1

In this problem we'll use the 'Mathlet' **Functions of Two Variables** (with link on the course webpage).

Let $f(u)$ be a function of one variable and let $F_c(x,t) = f(x - ct)$.

(a) Use the plotting feature of the mathlet to examine the graphs of F_c in the cases $c = -1, 1, 2$ for $f(u) = (\cos(u))^2$.

Note: you will need to use y for t when using *Input* to type the functions into the applet, since that's the only other letter allowed.

Then for $c = 2$ make a sketch by hand of the graphs of the single-variable function $x \to F_2(x,t)$ for the values $t = -1, 0, 1$.

(b) If the variable t represents time, and x the position on a one-dimensional string, and f measures the displacement of the string perpendicular to the x-axis, what phenomenon is being described by the function $F_2(x,t)$?

Solution

Part II, Problem 2

(a) Find the curve of intersection of the surfaces $z = x^2 - y^2$ and $z = 2 + (x - y)^2$ in parametric form.

(b) Find the angle of intersection of these two surfaces at the point $(2, 1, 3)$.

(The angle of intersection of two surfaces is defined to be the angle made by their tangent planes.)

(c) Check that the tangent vector to the curve of intersection found in part (a) at the point $(2, 1, 3)$ lies in (i.e. is parallel to) the tangent plane of each of the two surfaces.

Solution

Part II, Problem 3

Let $f(x,y) = \frac{4}{1+x^2+y^2}$ and let S be the surface given by the graph of $f(x,y)$.

(a) Make a sketch of the surface S in 3-space, and also a (separate) sketch of the contour plot of f.

(b) Let C_2 denote the curve in the xy-plane given by $\mathbf{r}_2(t) = \langle t, \frac{3}{2} - t^2 \rangle$; and let C denote the curve on the surface S which has C_2 as its shadow in the xy-plane. Find the parametric equations $\mathbf{r} = \mathbf{r}(t)$ for C.

(c) Sketch C on the picture of S of part (a), and also C_2 on the contour plot of part (a).

(d) Let $z(t)$ denote the z-component of the parametric equations $\mathbf{r} = \mathbf{r}(t)$ of C found in part (b). Find the points where $z(t)$ has its local maxima and minima, and add these in to the sketch of part (a).

(e) Set up the function $h(t)$ which gives the square of the distance from the origin to a variable point on the curve C_2, and then find the local maxima and minima of $h(t)$. How do these points relate to the maxima and minima found in part (d), and why?

Solution

Part II, Problem 4

Suppose that three non-negative numbers are restricted by the condition that the sum of their squares is equal to 27. Using critical point analysis, with second derivative and/or boundary tests as needed, find the maximum and minimum values of the sum of their cubes.

Solution

Part II, Problem 5

We return to problem 3 on pset 1, where we computed the component of the wind vector $\mathbf{w}$ which pointed *against* the wind after projecting $\mathbf{w}$ twice.

The problem now is to find the combination of projections of $\mathbf{w} = \langle 1, 0\rangle$ onto the vectors $\mathbf{w}_1$ and then onto $\mathbf{w}_2$ which yields the second projection which points *most strongly* into the wind, that is, the one with the most negative $\mathbf{i}$ component.

In pset 1, problem 3 we found the $\mathbf{i}$ component of $\mathbf{w}_2$ was

$$\cos(\alpha)\cos(\beta)\cos(\alpha+\beta). \tag{8}$$

(a) What choice(s) of α and β will give the most negative $\mathbf{i}$ component of $\mathbf{w}_2$?

(b) What fraction of the initial force of the wind is this?

This is the fraction of the force of the wind a sailboat can use to go against the wind with tacking.

Solution

Suggestion for Mathlet Experiments

Here are some suggested experiments to familiarize yourself with the *Level Curves and Partial Derivatives* selections in the applet **Functions of Two Variables**.

These are optional, but people often find them useful for strengthening 3D visualization. With the function $f(x, y) = y^2 - x^2$, start to familiarize yourself with the applet.

Try out different windows, in addition to $[-2, 2] \times [-2, 2]$, to get a better picture of the part of the graph around the saddle point. Read the directions given, and then try out the different input and mouse-controlled features to see how they work. (Note also that the surface graph may be rotated in different directions to get a better view of the features of interest.)

Also try these out on the given default example $f(x, y) = x(x-1)(x-2)+(y-1)(x-y)$ for more practice. Then: Using the *Partial Derivatives* applet for the function $f(x, y) = y^2 - x^2$, describe the behavior of f_x and f_y when you start at the saddle point $(0, 0)$ on the contour plot and then move the point in the following directions: E, NE, N, NW, W, SW, S, and SE (where e.g. SE means along the ray $\theta = -\frac{\pi}{4}$, etc.). Check the geometric pictures against the numerical results.

If an intrepid mountaineer were moving along this surface, what would the partial derivative information be telling you about the rate of climb/descent on the mountain in the $x-$ and $y-$ directions for the E, N, W, S routes on the mountain?