Part I, Problem 3A-2c

Express each double integral over the given region R as an iterated integral, using the given order of integration. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

(c) R is the sector of the circle at the origin and radius 2 lying between the x-axis and the line $y = x$.

Express as an iterated integral:

(i) $\int\int_R dy\, dx$

(ii) $\int\int_R dx\, dy$

Solution

Part I, Problem 3A-3b

Evaluate each of the following double integrals over the indicated region R. Choose whichever order of integration seems easier – given the integrand, and the shape of R.

(b) $\int\int_R (2x+y^2)\,dA$; R is the finite region in the first quadrant bounded by the axes and $y^2 = 1-x$); ($dx\,dy$ is easier).

Solution

Part I, Problem 3A-4c

Find by double integration the volume of the finite solid lying underneath the graph of $x^2 - y^2$, above the xy-plane, and between the planes $x = 0$ and $x = 1$.

Solution

Part I, Problem 3A-5a

Evaluate the iterated integral

$$\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx \tag{1}$$

by changing the order of integration (begin by figuring out what the region R is, and sketch it).

Solution

In evaluating the integrals, the following definite integrals will be useful:

$$\int_0^{\pi/2} \sin^n(x)\,dx = \int_0^{\pi/2} \cos^n(x)\,dx = \begin{cases} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)}{2\cdot 4\cdot\cdots\cdot n}\frac{\pi}{2}, & \text{if } n \text{ is an even integer } \geq 2 \\ \frac{2\cdot 4\cdot\cdots\cdot(n-1)}{1\cdot 3\cdot\cdots\cdot n}, & \text{if } n \text{ is an odd integer } \geq 3 \end{cases}. \tag{2}$$

For example, $\int_0^{\pi/2} \sin^2(x)\,dx = \frac{\pi}{4}$, $\int_0^{\pi/2} \sin^3(x)\,dx = \frac{2}{3}$, $\int_0^{\pi/2} \sin^4(x)\,dx = \frac{3\pi}{16}$, and the same holds if $\cos(x)$ is substituted for $\sin(x)$.

Part I, Problem 3B-1a

Express each double integral over the given region R as an iterated integral in polar coordiantes. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into twwo parts. In each case, begin by sketching the region.

(a) The region lying inside the circle with center at the origin and radius 2, and to the left of the vertical line through $(-1, 0)$.

Solution

Part I, Problem 3B-2d

Evaluate by iteration the double integral

$$\int\int_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}}, \tag{3}$$

where R is the right half-disk of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$. Use polar coordinates.

Solution

Part I, Problem 3B-3ac

Find the volumes of the following domains by integrating in polar coordinates.

(a) a solid hemisphere of radius a (place it so its base lies over the circle $x^2 + y^2 = a^2$)

(c) the domain lying under the cone $z = \sqrt{x^2 + y^2}$ and over the circle of radius one and center at $(0, 1)$

Solution

If no coordinate system is specified for use, you can use either rectangular or polar coordinates, whichever is easier. In some of the problems, a good placement of the figure in the coordinate system simplifies the integration a lot.

Part I, Problem 3C-1

Let R be a right triangle, with legs both of length a, and density 1. Find the following ((b) and (c) can be deduced from (a) with no further calculation).

(a) its moment of inertia about a leg;

(b) its polar moment of inertia about the right-angle vertex;

(c) its moment of inertia about the hypotenuse.

Solution

Part I, Problem 3C-2a

Find the center of mass of the region inside one arch of $\sin x$ if $\delta = 1$.

Solution

Part I, Problem 3C-4

Find the center of gravity of a sector of a circular disc of radius a, whose vertex angle is 2α. Take $\delta = 1$.

Solution

Part I, Problem 3D-1

Evaluate $\int\int_R \frac{x-3y}{2x+y}\,dx\,dy$ where R is the parallelogram bounded on the sides by $y = -2x+1$ and $y = -2x+4$, and above and below by $y = \frac{x}{3}$ and $y = \frac{x-7}{3}$. Use a change of variables $u = x-3y$, $v = 2x+y$.

Solution

Part I, Problem 3D-2

Evaluate $\int\int_R \cos\left(\frac{x-y}{x+y}\right)\,dx\,dy$ by making the change of variables $u = x + y$, $v = x - y$; take as the region R the triangle with vertices at the origin, $(2, 0)$, and $(1, 1)$.

Solution

Part I, Problem 3D-4

Evaluate $\int\int_R (2x-3y)^2(x+y)^2\,dx\,dy$, where R is the triangle bounded by the positive x-axis, negative y-axis, and line $2x-3y=4$, by making a change of variable $u=x+y$, $v=2x-3y$.

Solution

Part II, Problem 1

(a) Sketch the solid in the first octant bounded by the xy, yz, and xz coordinate planes, the plane $x + y = 4$, and the surface $z = \sqrt{4 - x}$.

(b) Find the volume of the solid of part (a).

Solution

Part II, Problem 2

The equation of a surface of revolution obtained by spinning the $(1D)$ graph of the function $z = f(y)$ in the yz plane around the z axis is given in polar coordinates by $z = f(r, \theta) = f(r)$ (that is, there is no dependence on θ. Assume that $h \equiv f(0) > 0$.

Show that the formula which you get by using the double integral in polar coordinates for the volume under the graph of this surface of revolution and over the xy plane is the same as that given by the "shell method" in single-variable calculus.

Solution

Part II, Problem 3

(a) Find the formula for the centroid (center of mass assuming $\delta = 1$) of a uniform plane region in the shape of a circular sector of radius a and central angle θ (in terms of a and θ).

(b) Write down the formula for the centroid of a uniform plane region in the shape of an iscosceles triangle with height a and angle θ between the two equal sides. (You may use the formula from elementary geometry or compute it out using the integral.)

(c) If you align the two regions above using the same values for a and θ and with the central angle of the sector coinciding with the apex angle of the triangle, which centroid will be closer to the center (= the apex)? What do the math and the physics predict and why, and are they consistent?

Solution

Background for problems 4 and 5:

In fluid mechanics, the *fluid flow map* ϕ is defined as follows: if (x, y, z) is the position of a point mass in the flow at time $t = 0$, then $(X, Y, Z) = \phi(x, y, z, t)$ is the downstream position of that same point mass after an elapsed time t.

The standard assumptions on ϕ are that it is smooth and one-to-one.

We will call a flow *volume in-compressible* if for any bounded space region $\mathcal{R}$ in the flow, the volume of $\phi(\mathcal{R}, t)$ is the same as the volume of $\mathcal{R}$ for all t. In other words, if $\mathcal{R}_t = \phi(\mathcal{R}, t) = \{\phi(x, y, z, t) \mid (x, y, z) \in \mathcal{R}\}$ is the region formed by the points from $\mathcal{R}$ which have been carried downstream by the flow, then $\mathcal{R}_t$ can have a different shape but must have the same volume at all times, if the flow is 'v-i'.

In problems 4 and 5, we'll take the simpler case of a $2D$ flow (which could be e.g. a 2D section of a flow in 3D). Let $(X(x, y, t), Y(x, y, t)) = \phi(x, y, t)$. Note that then by definition, the velocity vectors $\mathbf{v}(x, y, t)$ of the flow are given by $\mathbf{v}(x, y, t) = \langle \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} \rangle$. A v-i flow in this case is one that preserves area, since area is the 2D version of volume.

For a fixed value of t, let $J(x, y, t) = \frac{\partial(X,Y)}{\partial(x,y)}$ be the Jacobian of the transformation $(x, y) \mapsto (X(x, y, t), Y(x, y, t))$. The general change-of-variables formula says that if a region $\mathcal{R}$ goes to a region $\mathcal{R}'$ by a transformation $(x, y) \mapsto (X, Y)$ with Jacobian $\frac{\partial(X,Y)}{\partial(x,y)}$, then the areas of $\mathcal{R}$ and $\mathcal{R}'$ are related by

$$A(\mathcal{R}') = \int \int_{\mathcal{R}} |J(x, y)| \, dA. \tag{4}$$

Here this gives that $A(\mathcal{R}_t) = \int \int_{\mathcal{R}} |J(x, y, t)| \, dA$, and therefore that a 2D flow is v-i if and only if $|J(x, y, t)| = 1$ for all (x, y, t).

In problems 4 and 5, we will look at three examples of 2D flows, v-i and non-vi, in order to illustrate this idea.

Example A: $\phi(x, y, t) = ((1 + t)x, (1 + t)y)$; $\mathcal{R}$ = the triangle with vertices at $(0, 0)$, $(1, 1)$, and $(1, -1)$.

Example B: $\phi(x, y, t) = (x \cos(t) - y \sin(t), x \sin(t) + y \cos(t))$; $\mathcal{R}$ = the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(2, 1)$.

Example C: $\phi(x, y, t) = \left((1 + t)x, \left(\frac{1}{1+t}\right) y\right)$; $\mathcal{R}$ = the rectangle with vertices at $(1, 1)$, $(1, 4)$, $(2, 1)$, and $(2, 4)$.

Part II, Problem 4

In each of the cases A, B, and C:

(i) compute the Jacobian $J(x, y, t)$,

(ii) compute the area $A(\mathcal{R}_t)$.

Solution

Part II, Problem 5

In each of the cases A, B, and C:

(i) Sketch the pattern of the flow paths over time, including some starting from points in $\mathcal{R}$.

(ii) Compute the velocity vectors of the flow and sketch in a few on the flow lines.

(iii) Sketch the regions $\mathcal{R}$ and $\mathcal{R}_t$ and check this against the areas calculated in problem 5 to see if it looks correct in each case. Use the following values for t:

A: $t = 2$, B : $t = \frac{\pi}{2}$, and $C : t = 3$.

Suggestion for sketching $\mathcal{R}_t$: see where the corners of $\mathcal{R}$ end up on $\mathcal{R}_t$.

For the flow lines in case C, also note that $X(x, y, t)Y(x, y, t) = xy$ for all values of t.

(iv) Identify which flows are v-i, using the computed results (and the sketches).

(v) In the cases A and B, describe what the flow is doing.

Note: after we learn Green's theorem in normal form in 2D, or equivalently the divergence theorem in 3D, we'll be able to prove that incompressability and (our ad-hoc terminology) "volume-incompressability" are in fact equivalent conditions.

Solution