Part I, Problem 4D-1c

For the field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ and closed positively oriented curve $C : y = x^2$ and $y = x$, $0 \leq x \leq 1$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ both directly, as a line integral, and also by applying Green's theorem and calculating a double integral.

Solution

Part I, Problem 4D-2
Show that $\oint_C 4x^3y\,dx + x^4\,dy = 0$ for all closed curves C.

Solution

Part I, Problem 4D-3

Find the area inside the hypocycloid $x^{2/3} + y^{2/3} = 1$, by using Green's theorem. (This curve can be parameterized by $x = \cos^3(\theta)$, $y = \sin^3(\theta)$, between suitable limits on θ.)

Solution

Part I, Problem 4D-4

Show that the value of $\oint_C -y^3\,dx + x^3\,dy$ around any positively oriented simple closed curve C is always positive.

Solution

Part I, Problem 4E-1ac

Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Recalling the interpretation of this field as the velocity field of a rotating fluid or just by remembering how it looks geometrically, evaluate with little or no calculation the flux integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$, where

(a) C is a circle of radius a centered at $(0,0)$, directed counter-clockwise

(c) C is the line running from $(0,0)$ to $(1,0)$.

Solution

Part I, Problem 4E-2

Let **F** be the constant vector field $\mathbf{i}+\mathbf{j}$. Where would you place a directed line segment C of length one in the plane so that the flux across C would be

(a) maximal,

(b) minimal,

(c) zero,

(d) -1, and

(e) what would the maximal and minimal values be?

Solution

Part I, Problem 4E-4

Take C to be the square of side length 1 with opposite vertices at $(0, 0)$ and $(1, 1)$, directed clockwise. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$; find the flux across C.

Solution

Part I, Problem 4E-5

Let $\mathbf{F}$ be defined everywhere except at the origin by the description: dir($\mathbf{F}$) is radially outward, $|\mathbf{F}| = r^m$ where m is an integer.

(a) Evaluate the flux of $\mathbf{F}$ across a circle of radius a and center at the origin, directed counter-clockwise.

(b) For which value(s) of m will the flux be independent of a?

Solution

Part I, Problem 4F-4

Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$, and C is the square with opposite vertices at $(0,0)$ and $(1,1)$.

Solution

Part II, Problem 1

Let $\mathbf{F}(x, y) = (y^3 - 6y)\mathbf{i} + (6x - x^3)\mathbf{j}$.

(a) Using Green's theorem, find the simple closed curve C for which the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ (with positive orientation) will have the largest possible value.

(b) Compute this largest possible value.

Solution

Part II, Problem 2

In the reading V4.2 (pages 1-3), it is shown that in the context of $2D$ fluid flows, Green's theorem in normal form combined with the principle of conservation of mass imply that $\text{div}(\mathbf{F})$, the divergence of the flow field $\mathbf{F}(x, y)$, represents the (signed) rate of mass per unit time per unit area which originates at the point (x, y), or the source or sink rate for short.

This extends to non-steady flows $\mathbf{F}(x, y, t)$, and leads directly to the *equation of continuity* for fluid flows, which is the statement of conservation of mass and hence one of the basic physical principles of fluid dynamics. We'll continue to use ρ for the density (instead of δ used in the notes).

The divergence of a vector field $\mathbf{F}(x, y, t)$ in this context is defined with respect to the space variables only, that is, if $\mathbf{F}(x, y, t) = \langle M(x, y, t), N(x, y, t)\rangle$ is a smooth vector field, then $\text{div}(\mathbf{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

Then for the case of a flow field $\mathbf{F}(x, y, t) = \rho(x, y, t)\mathbf{v}(x, y, t)$ with density $\rho(x, y, t)$ and velocity $\mathbf{v}(x, y, t)$, the equation of continuity reads

$$\frac{\partial \rho}{\partial t} + \text{div}(\mathbf{F}) = 0. \tag{1}$$

Note that for *steady* flows, which by definition means $\rho = \rho(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$, the equation of continuity holds if and only if $\text{div}(\mathbf{F}) = 0$. Thus conservation of mass for steady flows is equivalent to the absence of any sources or sinks, which makes sense.

(a) For non-steady flows, assuming that the physical interpretation of $\text{div}(\mathbf{F})$ is the same as in the case of steady flows (at each time t), explain why the equation of continuity is in fact the statement of conservation of mass.

Hint: take an arbitrary bounded region $\mathcal{R}$ and integrate both terms of the continuity equation over $\mathcal{R}$. Then use Green's theorem in normal form.

(b) Let $g(x, y, t)$ be a smooth scalar function, and again define the gradient of $g(x, y, t)$ in this case to be with respect to just the space variables, $\nabla g = \langle g_x, g_y\rangle$. Then if $\mathbf{G}(x, y, t) = \langle M(x, y, t), N(x, y, t)\rangle$ is a smooth vector field, use the product rule to show that

$$\text{div}\,(g\mathbf{G}) = g\text{div}(\mathbf{G}) + \mathbf{G} \cdot \nabla g. \tag{2}$$

(c) Refer to the definition of the convective derivative $\frac{Df}{dt}$ given in pset 5 problem 2, and the definition of incompressibility for flows $\frac{D\rho}{Dt} = 0$, as given in pset 5 problem 3.

Combine the equation of continuity, the result of part (b) above, and the result of pset 5 problem 2 to show that the flow $\mathbf{F}(x, y, t) = \rho(x, y, t)\mathbf{v}(x, y, t)$ is incompressible if and only if

$$\text{div}(\mathbf{v}) = 0. \tag{3}$$

This is thus an equivalent condition for the incompressibility of a flow.

Solution

Part II, Problem 3

Sketch each of the following non-steady flows. Verify that it satisfies the equation of continuity.

(*Suggestion*: Use the expanded form of the equation of continuity found in problem 2(c) above.)

Then test it to determine whether it is incompressible, and if so, whether it is also stratified (see pset 5, problem 3(b)).

(a) $\mathbf{v}(x, y, t) = t\langle -y, x\rangle$ and $\rho(x, y, t) = \sqrt{x^2 + y^2}$.

(b) $\mathbf{v}(x, y, t) = \frac{1}{1+t}\langle x, -y\rangle$ and $\rho(x, y, t) = xy$.

(c) $\mathbf{v}(x, y, t) = t\langle x, y\rangle$ and $\rho(x, y, t) = e^{-t^2}$.

Solution