**Part I, Problem 4A-1d**

Describe geometrically how the vector fields determined by each of the following vector functions look. Tell for each what the largest region in which $\mathbf{F}$ is continuously differentiable is.

(d) $\frac{y\mathbf{i}-x\mathbf{j}}{r}$.

**Solution**

**Part I, Problem 4A-2bc**

Write down the gradient field $\nabla w$ for each of the following:

(b) $w = \log r$

(c) $w = f(r)$

**Solution**

**Part I, Problem 4A-3bd**

Write down an explicit expression for each of the following fields:

(b) The vector at $(x, y)$ is directed radially inward towards the origin, with magnitude $r^2$.

(d) Each vector is parallel to $\mathbf{i} + \mathbf{j}$, but the magnitude varies.

**Solution**

**Part I, Problem 4B-1ab**

For each of the fields $\mathbf{F}$ and corresponding curve $C$ or curves $C_i$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Use any convenient parameterization of $C$, unless one is specified. Begin by writing the integral in the differential form $\int_C M\,dx + N\,dy$.

(a) $\mathbf{F} = (x^2 - y)\mathbf{i} + 2x\mathbf{j}$. $C_1$ and $C_2$ both run from $(-1, 0)$ to $(1, 0)$. $C_1$ is the $x$-axis and $C_2$ is the parabola $y = 1 - x^2$.

(b) $\mathbf{F} = xy\mathbf{i} - x^2\mathbf{j}$. $C$ is the quarter of the unit circle running from $(0, 1)$ to $(1, 0)$.

**Solution**

**Part I, Problem 4B-2b**

For the following fields $\mathbf{F}$ and curves $C$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ without any formal calculation, appealing instead to the geometry of $\mathbf{F}$ and $C$.

(b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$. $C$ is the counter-clockwise circle, centered at $(0, 0)$, with radius $a$.

**Solution**

**Part I, Problem 4B-3**

Let $\mathbf{F} = \mathbf{i} + \mathbf{j}$. How would you place a directed line segment $C$ of length one so that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ would be

(a) a maximum,

(b) a minimum,

(c) zero, and

(d) what would the maximum and minimum values of the integral be?

**Solution**

**Part I, Problem 4C-1**

Let $f(x, y) = x^3y + y^3$, and let $C$ be $y^2 = x$ between $(1, -1)$ and $(1, 1)$, directed upwards.

(a) Calculate $F = \nabla f$.

(b) Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in three different ways:

(i) directly,

(ii) by using path independence to replace $C$ with a simpler path,

(iii) by using the fundamental theorem of calculus for line integrals.

**Solution**

**Part I, Problem 4C-3**

Let $f(x, y) = \sin(x)\cos(y)$.

(a) Calculate $\mathbf{F} = \nabla f$.

(b) What is the maximum value $\int_C \mathbf{F} \cdot d\mathbf{r}$ can have over all possible paths $C$ in the plane? Give a path $C$ for which this maximum value is attained.

**Solution**

**Part I, Problem 4C-5a**

Tell for what value of the constant $a$ the field $\mathbf{F} = (y^2 + 2x)\mathbf{i} + axy\mathbf{j}$ will be a gradient field, and for this value, find the corresponding (mathematical) potential function. Use V2 method 1.

**Solution**

**Part I, Problem 4C-5b**

Tell for what value of the constant $a$ the field $\mathbf{F} = e^{x+y}\left((x+a)\mathbf{i} + x\mathbf{j}\right)$ will be a gradient field, and for this value, find the corresponding (mathematical) potential function. Use V2 method 2.

**Solution**

**Part I, Problem 4C-6ab**

Decide which of the following differentials is exact. For each one that is exact, express it in the form $df$. Use both methods, V2 method 1 and V2 method 2.

(a) $y\,dx - x\,dy$

(b) $y(2x + y)\,dx + x(2y + x)\,dy$.

**Solution**

**Part II, Problem 1**

(a) Write down in $xy$-coordinates the vector field $\mathbf{F}$ whose vector at $P = (x, y)$ runs in the vertical direction from $P$ to the line $L : x + y = 1$.

(b) Show *without calculation* that for this field $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$, if $C$ is any positively oriented right triangle with legs parallel to the axes and hypotenuse on $L$.

**Solution**

**Part II, Problem 2**

Consider the vector field $\mathbf{F} = -c\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}$. (This is the gravitational field of an infinitely long uniform wire along the $z$-axis.) Take $c = 1$ and find by direct calculation (i.e. using a parameterization of $C$) the work done by the gravitational field in moving a unit mass along each of the following paths $C$ in the $xy$-plane.

(a) $C$ is the half-line $y = 1, x \geq 0$.

(b) $C$ is the circle of radius $a$ with center at the origin, traced counter-clockwise.

(c) $C$ is the line from $(0, 1)$ to $(1, 0)$.

**Solution**

**Part II, Problem 3**

(a) Let $\mathbf{F} = -\nabla(\log r)$. Show that $\mathbf{F}$ is the field in problem 2, when $c = 1$.

(b) Show that, for any path $C$ joining two points $P_1$ and $P_2$, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the ratio $\frac{r_2}{r_1}$, where $r_i$ is the distance of $P_i$ to the origin.

**Solution**

**Part II, Problem 4**

Let $\mathbf{F} = \nabla\left(x^2y + 2xy^2\right)$. Let $C$ be the quarter-ellipse $9x^2 + 4y^2 = 1$ running from the positive $x$-axis to the positive $y$-axis.

(a) Write the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in the form $\int_C M\,dx + N\,dy$.

(b) By parameterizing the curve $C$, write the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as an explicit definite integral in $t$. Do not evaluate.

(c) Use the fundamental theorem to (easily) compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(d) Use path independence to choose a different path and (again easily) compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

**Solution**

**Part II, Problem 5**
Let $\mathbf{F} = xy\mathbf{i} + x^3\mathbf{j}$.

(a) Show $\mathbf{F}$ is not conservative.

(b) Try to find a potential function for $\mathbf{F}$ using method 1 from the reading (section V2). What goes wrong?

(c) Answer the same question for method 2.

**Solution**