# PSET 9

## Part 1, Problem 4D-1c

- First we shall evaluate this using line integrals; let $C_1$ be the portion of the line where we follow $y = x^2$, $x = x$, $dx = dx$, $dy = 2x\,dx$ over the x-domain $[0,1]$:

$$\int_{C_1} F \cdot dr = \int_0^1 xy\,dx + y^2\,dy$$

$$= \int_0^1 xy + 2xy^2\,dx$$

$$= \int_0^1 x^3 + 2x^5\,dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^6}{6}\right]_0^1 = \frac{7}{12}$$

Now let $C_2$ be the portion returning to the origin; $x = x$, $y = x$, $dx = dx$, $dy = dx$, over the x-domain $[0,1]$; the order is reversed since we are returning from $(1,1)$ to $(0,0)$:

$$= \int_1^0 2x^2$$

$$= -\frac{2}{3}$$

Thus,

$$\int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = \frac{-1}{12}$$

- Now evaluate by Green's Theorem:

$$\int_C F \cdot dr = \iint \text{curl}(F)\,dA$$

$$= \int_0^1 \int_{x^2}^{x} -x\,dy\,dx = \int_0^1 \left[-xy\right]_{x^2}^{x} = \int_0^1 -x^2 + x^3$$

$$\cancel{= \int_0^1 \frac{x^2}{2} - \frac{x^4}{2}\,dx}$$

$$\cancel{= \left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1} =$$

$$= \left[\frac{-x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{-1}{12} \quad \square$$

Part 1, Problem 4D-2

- First, take the curl of

$\vec{F} = 4x^3 y \cancel{\partial x}\,\mathbf{i} + x^4\,\mathbf{j} = M\mathbf{i} + N\mathbf{j}$

The curl

$\text{curl}(\vec{F}) = \cancel{A} N_x - M_y$

$= 4x^3 - 4x^3$

$= 0$

- Thus, we can apply Green's Theorem and see that

$\int_C F \cdot dr = \iint_A \text{curl}(\vec{F}) \cdot dA$

$= \iint_A 0 \cdot dA$

$= 0 \quad \square$

# Part 1, Problem 40-3

We need some $\vec{F}$ such that curl $(F) = 1$, since

$$\int_C F \cdot dr = \iint_A \text{curl}(F)\, dA = \iint_A dA = \text{Area}(A)$$

Let that F be

$$\vec{F} = -y\,\mathbf{i} + x\,\mathbf{j} \text{; thus}$$

$$\int_C F \cdot dr = \int_C -y\,dx + x\,dy \text{;}$$

Then parametrize with

$x = \cos^3(\theta)$

$dx = -3\sin(\theta)\cos^2(\theta)\, d\theta$

$y = \sin^3(\theta)$

$dy = 3\sin^2(\theta)\cos(\theta)$,

$$\ldots = \int_C 3\sin^4(\theta)\cos^2(\theta)\, d\theta + 3\sin^2(\theta)\cos^4(\theta)$$

$$= 3\int_C (\sin^2(\theta)\cos^2(\theta))(\sin^2(\theta) + \cos^2(\theta))\, d\theta$$

$$= 3\int_C \sin^2(\theta)\cos^2(\theta)$$

Then since

$$\sin^2(\theta)\cos^2(\theta) = \frac{1}{8}\left(1 - \cos(4\theta)\right),$$

$$\ldots = \frac{3}{8}\left(\int_C 1\, d\theta - \int_C \cos(4\theta)\, d\theta\right.$$

$$= \frac{3}{8}\left(\left[\frac{\theta}{2} - \frac{\sin(4\theta)}{8}\right]_0^{2\pi}\right) = \frac{3\pi}{8}$$

Part 1, Problem 6D-4

- By Green's theorem, see that

$$\oint_C -y^3\, \partial x + x^3 = \iint_A \text{curl}(\vec{F})\, \partial A$$

where $\vec{F} = -y^3 \mathbf{i} + x^3 \mathbf{j}$ and thus

$$\text{curl}(F) = 3x^2 - (-3y^2) = 3x^2 + 3y^2$$

then since we are told that the curve C in question is positively oriented,

$$3x^2 + 3y^2 > 0 \Rightarrow \iint_A 3x^2 + 3y^2\, \partial A > 0. \quad \square$$

Part I, Problem 4E-1ac

(a): No material crosses the boundary, zero $= \emptyset$.

(c): The $\vec{F} = -y\,i + x\,j$ and $\hat{n} = -j$, thus

$$\int_0^1 x \cdot -1 \, dx = \frac{-1}{2}. \square$$

Part 1, 4E-2

(a): Have the line segment start and end at:
~~$(0, \sqrt{2}) \to (\sqrt{2}, 0)$~~ $(\sqrt{2}, 0) \to (0, \sqrt{2})$

(b): ~~$(0,0) \to (\sqrt{2}, \sqrt{2})$~~ $(0, \sqrt{2}) \to (\sqrt{2}, 0)$

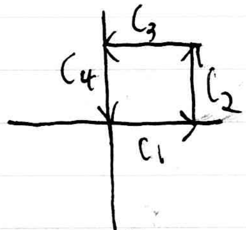
(c): $(0,0) \to (\sqrt{2}, \sqrt{2})$

(d): $(0,1) \to (1,1)$

(e): $\pm\sqrt{2}$

# Part 1, Problem 4E-4

Sketch the curve:




Then for $C = C_1 + C_2 + C_3 + C_4$,

$$\int_C \vec{F} \cdot n = \int_{C_1} \vec{F} \cdot n + \int_{C_2} \vec{F} \cdot n + \int_{C_3} \vec{F} \cdot n + \int_{C_4} \vec{F} \cdot n$$

$$= \int_1^0 dy + \int_0^1 -dx$$

$$= -2$$

# Part 1, Problem 4E-5

(a): Both F and n point outwards from the origin, so

$F \cdot n = |F| = r^m$;

then, ~~for positive~~ for positive y,

$\cancel{\int_0^{\pi/2} \cos(\theta)\, a \quad y = \sqrt{a^2 - x^2}}$

$\cancel{\int_0^{2\pi} a^m}$

we have

$\int_C a^m \, ds = \int_C ds \; a^m$

$= \text{circumferene}(C)\, a^m$

$= 2\pi a^{m+1}$

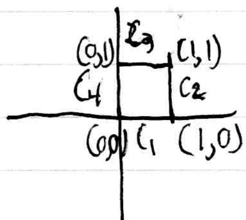
(b): $-1$

# Part 1, Problem 4F-4

Recall Green's Theorem:

$$\int_C F\cdot dr = \iint_R \text{curl}\, F \; dy \; dx$$

Sketch the path:




Break this into parametrizations:

$$\int_{C_1} F\cdot dr = \int_0^1 \cancel{x^2}\, 0 \, dx = \cancel{1/3} = 0$$

$$\int_{C_2} F\cdot dr = \int_0^1 \cancel{2x + y}\, 1 \, dy = \cancel{\tfrac{1}{2}} = 1$$

$$\int_{C_3} F\cdot dr = \int_1^0 \cancel{x^2}\, -x \, dx = \cancel{1/3} = \frac{-1}{2}$$

$$\int_{C_4} F\cdot dr = \int_1^0 0 \, dx = 0$$

Thus,

$$\int_C F\cdot dr = \frac{1}{2}$$

Now evaluate the right side:

$$\int_0^1\int_0^1 (y-0)\, dx\, dy = \int_0^1 y \, dy = \frac{1}{2}$$

# Part 2, Problem 1

(a): See by Green's Theorem that

$$\int_C F \cdot dr = \iint_R \text{curl}(F)\, dA$$
$$= \iint_R ((6-3x^2)-(3y^2-6))\, dA$$
$$= \iint_R 12-3x^2-3y^2$$

Then we want

$$12-3x^2-3y^2 \geq 0$$

for this to be maximized; thus we want

$$3x^2+3y^2 \leq 12$$
$$x^2+y^2 \leq 4$$

Thus, we want $C$ = circle at the origin of radius 2

(b): We see by substituting $dx\,dy = r\,dr\,d\theta$,

$$\iint_R F \cdot ds = \int_0^{2\pi}\int_0^2 (12-3r^2)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[ 12r - \frac{3r^4}{4} \right]_0^2$$
$$= \int_0^{2\pi} \cancel{2}\, 12\, d\theta$$
$$= 24\pi$$

Part 2, Problem 2

(a): We see that by integrating each side, that the given equation of continuity states that

$$\iint_R \frac{\partial \rho}{\partial t}\, dA + \iint_R \operatorname{div}(F)\, dA = 0$$

for all ~~regions R closed~~ simple bounded regions $R$. ~~To~~ Let some $M(R;t) = \iint_R \rho(x,y,t)$; then

$$\iint \frac{\partial \rho}{\partial t}\, dA = \frac{\partial}{\partial t} M(R;t)$$

We see by Green's theorem that

$$\iint_R \operatorname{div}(F)\, dA = \oint_C F(x,y,t) \cdot \hat{n}_{out}\, ds,$$

which is then the mass flux out of the region $R$ at time $t$. The mass is conserved if this mass flux is equal to

$$-\frac{\partial}{\partial t} M(R;t)$$

(b): $$\operatorname{div}(gG) = \frac{\partial (gM)}{\partial x} + \frac{\partial (gN)}{\partial y} = g_x M + g M_x + g_y N + g N_y$$

$$= g_x M + g_y N + g M_x + g N_y = G \cdot \nabla g + g \operatorname{div}(G)$$

(c) We see that

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(F) = \frac{\partial \rho}{\partial t} + V \cdot \nabla \rho + \rho \operatorname{div}(V) = \frac{D\rho}{Dt} + \rho \operatorname{div}(V)$$

first by (b) then by the chain rule for convective derivatives. Thus,

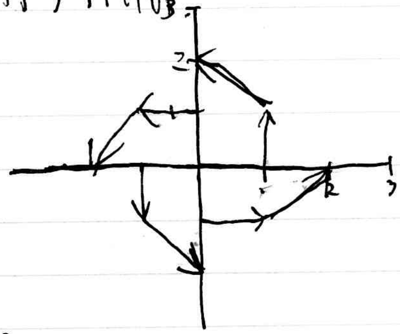
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(F) = 0 \text{ if } \frac{D\rho}{Dt} + \rho \operatorname{div}(V) = 0$$

$$\Downarrow$$

$$\frac{D\rho}{Dt} = 0 \text{ if } \operatorname{div}(V) = 0$$

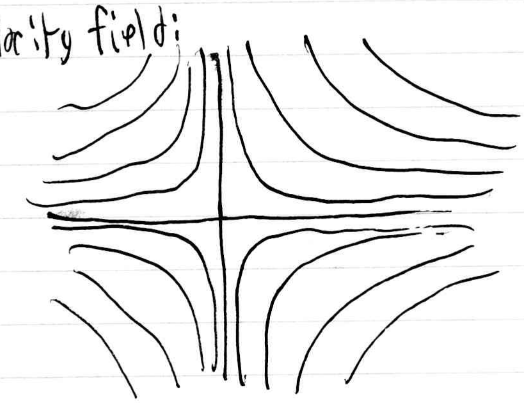
# Part 2, Problem 3

(a): Sketch the velocity field:



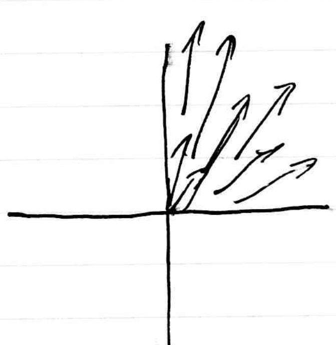

We see that $\frac{\partial \rho}{\partial t} = 0$, $V \cdot \nabla \rho = 0$, and $\operatorname{div}(V) = 0$. Thus, this flow is satisfies the continuity equation, is incompressible, and is stratified.

(b): Sketch the velocity field:



We see that $\partial \rho / \partial t = 0$, $V \cdot \nabla \rho = 0$, and $\operatorname{div}(V) = 0$

(c): Sketch the flow:



Then, $\frac{\partial \rho}{\partial t} = -2te^{-t^2}$, $\operatorname{div}(\rho(t)V) = \rho(t)\operatorname{div}(V) = e^{-t^2} 2t$, thus satisfying the continuity equation; $\operatorname{div}(V) = 2t \neq 0$, thus the flow is not incompressible.