Part I, Problem 1A-6

A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the **i**, **j**-components).

Solution

Our plane wishes to fly with the vector $0\mathbf{i} + 200\mathbf{j}$, which can be attained by addition of v, the velocity vector of the plane we are solving for, and w, the velocity vector of the wind. We first must solve for the wind vector. We let $w = c(-\mathbf{i} - \mathbf{j})$ where we will be solving for c shortly, and by Pythagoras let $|w|^2 = 50^2$. We begin by noting that

$$|w|^2 = w \cdot w, \tag{1}$$

since normally $A \cdot B = |A||B|\cos(\theta)$, but since this is the angle between it and itself, $\cos(\theta) = 0$. Then

$$|w|^2 = w \cdot w = 2c^2 \tag{2}$$

by the dot product applied to w and itself, with the scalar c multiplied in. We then see that

$$|w|^2 = 2c^2 = 50^2 \implies |w| = \sqrt{2}c = 50, \tag{3}$$

where we applied the Pythagoras ruling in the court of hardcore math. Thus, $c = \frac{50}{\sqrt{2}}$ and we see that

$$w = \frac{-50}{\sqrt{2}}\mathbf{i} - \frac{50}{\sqrt{2}}\mathbf{j}, \tag{4}$$

We then write that (where z is the airplane vector)

$$z = 0\mathbf{i} + 200\mathbf{j} = \left(\frac{-50}{\sqrt{2}}\mathbf{i} - \frac{50}{\sqrt{2}}\mathbf{j}\right) + v, \tag{5}$$

We then solve by splitting the **i**'s and the **j**'s that

$$v = \frac{50\mathbf{i}}{\sqrt{2}} + (200 + \frac{50}{\sqrt{2}})\mathbf{j}. \tag{6}$$

Part I, Problem 1A-7

Let $\mathbf{A} = a\mathbf{i} + b\mathbf{j}$ be a plane vector; find in terms of a and b the vectors $\mathbf{A}'$ and $\mathbf{A}''$ resulting from rotating A by $90°$ a) clockwise, and b) counterclockwise.

(Hint: make $\mathbf{A}$ the diagonal of a rectangle with sides on the x and y-axes, and rotate the whole rectangle.)

(c) Let $\mathbf{i}' = \frac{3\mathbf{i}+4\mathbf{j}}{5}$. Show that $\mathbf{i}'$ is a unit vector, and use the first part of the exercise to find a vector $\mathbf{j}'$ such that $\mathbf{i}', \mathbf{j}'$ forms a right-handed coordinate system.

Solution

(a) We see that rotating by 90 degrees clockwise we send ai to $-aj$ and bj to bi, thus $A' = bi - aj$

(b) We see that rotating by 90 degrees counter-clockwise we send ai to aj and bj to $-bi$, thus $A' = -bi + aj$

(c) We see that (From this point on all solutions can be found in the scanned work.

Part I, Problem 1A-9

Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides **A** and **B**.)

Solution

Part I, Problem 1B-2

Tell for what values of c the vectors $c\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ will a) be orthogonal, b) form an acute angle.

Solution

Part I, Problem 1B-5b

Find the component of the force $\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in the direction of the vector $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

Solution

Part I, Problem 1B-12

Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

Solution

Part I, Problem 1B-13

Prove the trigonometric formula: $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

(Hint: consider two unit vectors making angles θ_1 and θ_2 with the positive x-axis.)

Solution

Part I, Problem 1C-2

Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of: (a) the first row, (b) the first column.

Solution

Part I, Problem 1C-5a

Show that the value of a 2×2 determinant is unchanged if you add to the second row a scalar multiple of the first row.

Solution

Part I, Problem 1D-2

Find the area of the triangle in space having its vertices at the points $P : (2, 0, 1)$, $Q : (3, 1, 0)$, $R : (-1, 1, -1)$.

Solution

Part I, Problem 1D-5

What can you conclude about $\mathbf{A}$ and $\mathbf{B}$

a) if $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$;

b) if $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$.

Solution

Part I, Problem 1D-7

Find the volume of the tetrahedron having vertices at the four points $P : (1, 0, 1), Q : (-1, 1, 2), R : (0, 0, 2), S : (3, 1, -1)$.

Hint: volume of tetrahedron $= \frac{1}{6}$ (volume of parallelepiped with same 3 coterminous edges).

Solution

Part II, Problem 1

Find the dihedral angle between two faces of a regular tetrahedron.

Solution

Part II, Problem 2

a) Show that the "polarization identity"

$$\frac{1}{4}\left(|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}-\mathbf{v}|^2\right) = \mathbf{u}\cdot\mathbf{v} \tag{7}$$

holds for any two n-vectors $\mathbf{u}$ and $\mathbf{v}$. (Use vector algebra, not components.)

b) Given two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, give the formula for the unit vector which bisects the (smaller) angle between $\mathbf{u}$ and $\mathbf{v}$. (Use the notation $\hat{\mathbf{u}}$ for the unit vector in the $\mathbf{u}$-direction.)

Solution

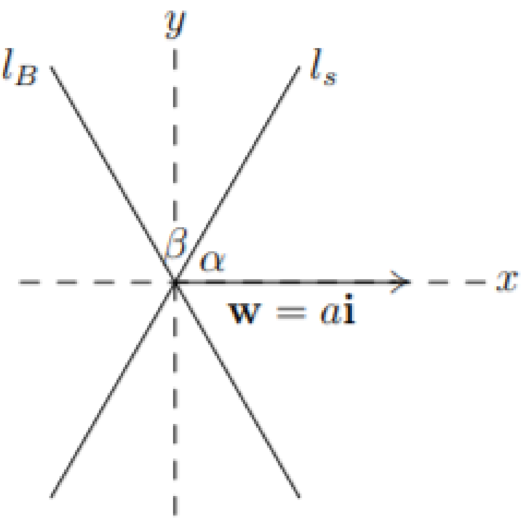
Part II, Problem 3

In this problem we examine tacking, which is the process sailboats use to travel against the wind. Sails are a familiar tool to harness the energy of the wind for transportation over the sea. Early ships had large fixed sails which would capture the wind blowing from behind to propel the ship forward. Even if the wind is blowing from behind at an (acute) angle, the component of the wind vector perpendicular to the sail will push on the sail and hence on the boat. However, these early fixed sail ships had no way to go against the wind and had to rely on oarsmen if the wind was blowing in the wrong direction.

A great advance that allowed boats to sail against the wind was the invention of movable sails in combination with a rudder and a keel. By carefully positioning the sail the boat can be made to sail into the wind – this process is called *tacking*.

As noted before, the component of the wind perpendicular to the sail pushes on the sail and, through it, the boat. The keel only allows the boat to move along its axis. (The rudder is used to turn the boat.) That is, for any force on the boat, only the component along the boat's axis actually pushes the boat.

Described mathematically, the wind vector is first projected on the perpendicular to the sail to get the direction of the force on the sail. This resultant force is projected on the axis of the boat to find the direction the boat is being pushed. By orienting the sail correctly this double projection can result in a vector with a component pointing into the wind.



In the picture $\mathbf{w} = a\mathbf{i}$ is the wind direction. The line ℓ_s is perpendicular to the sail (with $0 \leq \alpha \leq \frac{\pi}{2}$). And the line ℓ_B is along the the boat's axis (with $0 \leq \beta \leq \frac{\pi}{2}$).

a) Let $\mathbf{w}_1$ be the projection of $\mathbf{w}$ onto the line ℓ_s. Show that $\mathbf{w}_1$ does not have a nonzero component in the direction opposite $\mathbf{w}$. (It is sufficient to show the projections on the sketch.)

b) Find the projection of $\mathbf{w}_1$ onto ℓ_B. (Give an explicit formula in terms of α and β.) What is the condition on α and β that this projection has a component in the $-\mathbf{i}$ direction? (For a warm-up, you might try the specific case $\alpha = \frac{\pi}{3} = \beta$.)

Solution