PSET 10

Part 1, Problem 5A-2d

~~Begin by finding the formula of the shadow by setting the 2 functions equal:~~

$$\cancel{\sqrt{x^2+y^2} = \sqrt{x^2+y^2-2}}$$

Easier way by visualization; the radius of the circular shadow is

$$r = \sqrt{2}\cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 1$$

Write the triple integral:

$$\cancel{\int_0^{2\pi}\int_0^1\int_{\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2-2}} 1\, dz\, r\, dr\, d\theta}$$

$$\cancel{\ldots = \int_0^{2\pi}\int_0^1 \left(\sqrt{r-2} - \sqrt{r}\right) r\, dr\, d\theta}$$

$$(1) \quad \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta$$

$$\ldots = \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2} - r\right) r$$

and oh wait we solved the question asked in line (1).

Part 1, Problem 5A-3

By symmetry ~~about the x=y line~~ see that $\bar{x} = \bar{y} = \bar{z}$, then to continue we must find the mass of the tetrahedron; evaluate

$$\int_0^1\int_0^{1-x}\int_0^{1-(x+y)} dz\, dy\, dx$$

$$\int_0^1\int_0^{1-x} 1-x-y\, dy\, dx$$

$$\int_0^1 (1-x)-x(1-x)-\frac{1}{2}(1-x)^2$$

$$1-x+x^2-x-\frac{1}{2}(1+x^2-2x)$$

$$-\frac{1}{2}+-\frac{1}{2}x^2+x$$

$$\int_0^1 \frac{1}{2}-x+\frac{1}{2}x^2\, dx$$

$$\text{mass} = \frac{1}{2}-\frac{1}{2}+\frac{1}{6} = \frac{1}{6}$$

Then find the $\bar{x}$ by

$$\bar{x} = 6\int_0^1\int_0^{1-x}\int_0^{1-x-y} x\, dz\, dy\, dx$$

$$= 6\int_0^1\int_0^{1-x} x(1-x-y)\, dy\, dx$$

$$= 6\int_0^1\int_0^{1-x} x-x^2-y$$

$$= 6\int_0^1 (x-x^2)-x^2(1-x)-\frac{(1-x)^2}{2}$$

$$= \frac{1}{24}$$

Part 1, Problem 5A-4

(a): ~~Evaluate the integral~~ Assume the cone in question is a subset of the space above $z^2 = x^2 + y^2$; then evaluate the integral

$$\cancel{\int_0^h \int_0^{2\pi} \int_0^r r \cdot dz \, r \, dr \, d\theta}$$

$$\int_0^h \int_0^{2\pi} \int_0^z r^2 \cdot dr \cdot d\theta \, dz$$

$$\int_0^h \int_0^{2\pi} \frac{z^3}{3} \, d\theta \, dz$$

$$\int_0^h 2\pi \frac{z^3}{3} \, dz$$

$$\frac{1}{4} 2\pi \frac{h^4}{3} = \pi \frac{h^4}{6}$$

(b): Since we have central axis on the y-axis, $\bar{x} = \bar{y} = 0$; we just need to evaluate

$$\bar{z} = \frac{6}{\pi h^4} \int_0^h \int_0^{2\pi} \int_0^z r^2 z \, dr \, d\theta \, dz$$

$$= \quad | \quad | \quad \frac{z^4}{3} \, d\theta \, dz$$

$$= \frac{6}{\pi h^4} \int_0^h 2\pi \frac{z^4}{3}$$

$$= \frac{6}{\pi h^4} \cdot \left(\frac{1}{5} \cdot 2\pi \frac{h^5}{3} \right) = \frac{12}{15} \cdot \frac{h^5}{h^4} = \frac{4}{5} h$$

Part 1, Problem 6A-5

Sketch the shape

$$2^2 = 2a^2 \Rightarrow a = \sqrt{2}$$

Then, the integral representing the MOI is

$$\int_0^2 \int_{-1/2\sqrt{2}}^{1/2\sqrt{2}} \int_{-1/2\sqrt{2}}^{1/2\sqrt{2}} (x^2+y^2)\,dy\,dx\,dz.$$

Part1, Problem 5B-1bc

(b): $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} \partial \rho \; \partial \phi \; \partial \theta$

(c): $\int_0^{2\pi} \int_0^{\pi/4} \int_{\sec \phi}^{2\cos \phi} \partial \rho \; \partial \phi \; \partial \theta$

Part 1, Problem 5B-2

Place the hemisphere to have base in the x-y plane and central axis on the z axis; by symmetry $\bar{x} = \bar{y} = 0$, thus we only need to calculate

$$\bar{z} = \frac{1}{mass} \iiint_V z \, dV = \frac{1}{2/3 a^3} \iiint_V (\rho \cos\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Then notice that the integral has limits.

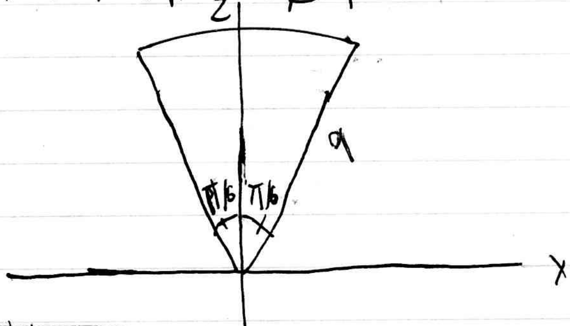
$$\cdots = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \cdot \frac{1}{mas}$$

$$= 2\pi \cdot \frac{1}{4} a^4 \cdot \frac{1}{2} = \frac{\pi a^4}{4} \cdot mass^{-1}$$

$$= \frac{3}{8} a$$

Part 1, Problem 5B-3

Sketch the outline in the x-z plane



$$\frac{3\pi}{6} - \frac{\pi}{6} = \frac{\pi}{3}$$

~~Begin by finding the mass~~

$$\cancel{\int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a} \rho\cos\phi \, d\rho \, d\phi \, d\theta}$$

Write the integral

$$MOI = \iiint_V r^2 \cdot z \, dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{a} (\rho\sin\phi)^2(\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6}\int_0^{a} \rho^5 \sin^3\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{\pi a^6}{196}$$

Part 1, Problem 5B-4b

We must evaluate the integral

$$\iiint_V r \, \partial V = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho \sin\phi \; \rho^2 \sin\phi \, \partial\rho \, \partial\phi \, \partial\theta$$

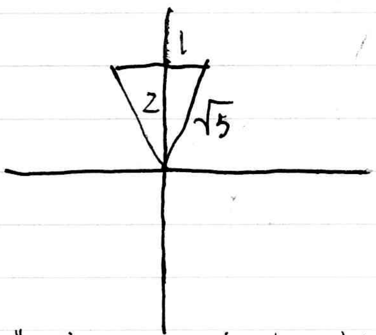
$$= \frac{\pi^2 a^4}{4}$$

The average is the MOI / mass, so

$$(\pi^2 a^4/4) / ((4/3)\pi a^3) = \frac{3\pi a}{16}$$

Part 1, Problem 5C-2

Sketch the region:



Then ~~er~~ we wish to evaluate the integral

$$G\int_0^{2\pi}\int_0^{\arccos(2/\sqrt{5})}\int_0^{2/\cos\phi} \sin\phi\,\cos\phi \; d\rho\, d\phi\, d\theta$$

$$\ldots = 4\pi G\left(1-\frac{2}{\sqrt{5}}\right)$$

Part 2, Problem 1

(a): Evaluate the integral

$$V = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r} dz \, r\,dr \, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} r^2 \, dr \, d\theta$$

$$= \int_0^{\pi} \frac{8}{3} \sin^3\theta \, d\theta$$

$$= \frac{16}{3} \int_0^{\pi/2} \sin^3\theta \, d\theta$$

$$= \frac{32}{9}.$$

(b): Evaluate the integral

$$\overline{z} = \frac{1}{V} \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r} z \, dz \, r\,dr \, d\theta$$

$$= \frac{1}{V} \int_0^{\pi} \int_0^{2\sin\theta} \frac{r^3}{2} \, dr \, d\theta$$

$$= \frac{1}{V} \int_0^{\pi} \frac{16}{8} \sin^4(\theta)$$

$$= \frac{4}{V} \int_0^{\pi/2} \sin^4\theta$$

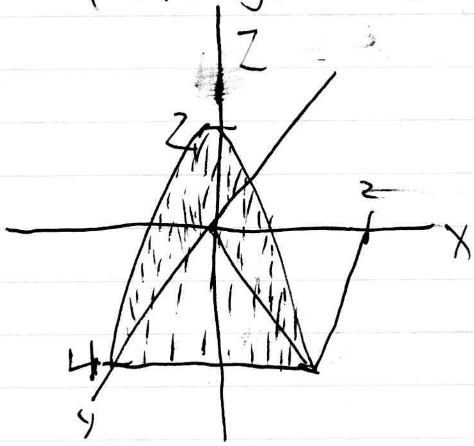
$$= \frac{3\pi}{4V}$$

Plug in $V = 32/9$,

$$\ldots \approx \frac{2}{3}$$

Part 2, Problem 2

(a):



(b): $\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{2x} f(x,y,z)\ dy\ dx\ dz$

(c): $\int_0^4 \int_0^{\sqrt{4-(y/2)^2}} \int_{y/2}^{\sqrt{4-z^2}} f(x,y,z)\ dx\ dy\ dz$

(d): First evaluate

$\int_0^2 \int_0^{2x} \int_0^{\sqrt{4-x^2}} dz\ dy\ dx$

$\dots = \int_0^2 \int_0^{2x} \sqrt{4-x^2}\ dy\ dx$

$= \int_0^2 2x\sqrt{4-x^2}$

$= \frac{-2}{3}(4-2^2)^{3/2} = \frac{16}{3}$

Then evaluate

$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{2x} dy\ dx\ dz$

$\dots = \int_0^2 (4-z^2)\ dz$

$= \frac{16}{3}$

Part 2, Problem 3

Evaluate the integral

$$\int_0^{100} \int_{-9}^{9} \int_{\frac{1}{81}x^4}^{81} 10^3 g\, z \, dz\, dx\, dy = 5.14 \cdot 10^{10} \text{ Joules}$$