Useful formula: $\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right)$.

Problem 1

(a) Find the equation in the form $Ax + By + Cz = D$ of the plane $\mathcal{P}$ which contains the line L given by $x = 1 - t$, $y = 1 + 2t$, $z = 2 - 3t$, and the point $(-1, 1, 2)$.

(b) Let $\mathcal{Q}$ be the plane $2x + y + z = 4$. Find the component of a unit normal vector for $\mathcal{Q}$ projected on a unit direction vector for the line L of part (a).

Solution

Problem 2

Let L denote the line which passes through $(0,0,1)$ and is parallel to the line in the xy-plane given by $y = 2x$.

(a) Sketch L and give its equation in vector-parametric form.

(b) Let $\mathcal{P}$ be the plane which passes through $(0,0,1)$ and is *perpendicular* to the line L of part (a). Sketch in $\mathcal{P}$ (above) and give its equation in point-normal form.

(c) Suppose that the point $P \neq (0,0,1)$ lies on L. Write down the method or formula you would use to find the point P^* which is: (i) on L; (ii) the same distance away from the point $(0,0,1)$ as P; and is (iii) on the *other* side of $\mathcal{P}$ from P.

Solution

Problem 3

Given the 3×3 matrix $A_a = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & a \end{bmatrix}$:

(a) Let $a = 2$: show that $|A_2| = 0$.

(b) Find the line of solutions to $A_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) Suppose now that $a = 1$, and that $A_1^{-1} = \begin{bmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{bmatrix}$. Find p.

Solution

Problem 4

Let $\mathbf{r}(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$.

(a) Compute and simplify the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

(b) Compute $\mathbf{T}'(t)$.

Solution

Problem 5

Consider the function $F(x, y, z) = z\sqrt{x^2 + y} + 2\frac{y}{z}$.

(a) The point $P_0 : (1, 3, 2)$ lies on the surface $F(x, y, z) = 7$. Find the equation of the tangent plane to the surface $F = 7$ at P_0.

(b) If starting at P_0 a small change were to be made in only *one* of the variables, which one would produce the largest change (in absolute value) of F? If the change in this variable was of size 0.1, approximately how large would the change in F be?

(c) What distance from P_0 in the direction $\pm\langle -2, 2, -1 \rangle$ will produce an approximate change in F of size 0.1 units, according to the (already computed) linearization of F?

Solution

Problem 6

Let $f(x, y) = x + 4y + \frac{2}{xy}$.

(a) Find the critical point(s) of $f(x, y)$.

(b) Use the second-derivative test to test the critical point(s) found in part (a).

Solution

Problem 7

Let $\mathcal{P}$ be the plane with equation $Ax + By + Cz = D$ and $P_0 = (x_0, y_0, z_0)$ be a point which is *not* on $\mathcal{P}$.

Use the Lagrange multiplier method to *set up* the equations satisfied by the point (x, y, z) on $\mathcal{P}$ which is *closest* to P_0. Do not solve.

Solution

Problem 8

(a) Let $F(x, y, z)$ be a smooth function of three variables for which $\nabla F(1, -1, \sqrt{2}) = \langle 1, 2, -2 \rangle$.

Use the chain rule to evaluate $\frac{\partial F}{\partial \phi}$ at $(\rho, \phi, \theta) = \left(2, \frac{\pi}{4}, -\frac{\pi}{4}\right)$.

(Use $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$.)

(b) Suppose $f(x, y)$ is a smooth, non-constant function. Is it possible that, for *all* points (x, y), the gradient of f at the point (x, y) is equal to the vector $\langle -y, x \rangle$? Justify (briefly).

Solution

Problem 9

Consider $\int\int_R f\, dA = \int_0^2 \int_{x^2}^{2\sqrt{2x}} f(x,y)\, dy\, dx$.

(a) Sketch the region R.

(b) Rewrite the double integral as an iterated integral with the order interchanged.

Solution

Problem 10

Set up the integral $\int \int_R f(x,y)\,dA$ where R is the region bounded by the four curves $x^2 y = 4$, $x^2 y = 9$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 2$, as a double integral in the variables $u = x^2 y$ and $v = \frac{y}{x}$. (Your answer should be completely ready to integrate, once the function f is given.)

Note: the inverse transformation is given by $x = u^{1/3} v^{-1/3}$, $y = u^{1/3} v^{2/3}$.

Solution

Problem 11

Let $\mathbf{F}(x, y) = x(\mathbf{i} + \mathbf{j})$, and let C be the closed curve in the xy plane formed by the triangle with vertices at the origin and the points $(1, 0)$ and $(0, 1)$.

(a) Give a rough sketch of the field $\mathbf{F}$ in the first quadrant, and use it to predict whether the net flux out of the region $R = \{$the interior of $C\}$ will be positive or negative.

(b) Compute the flux integral $\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds$ directly. Specify which orientation you are using for C.

(c) Compute the flux integral $\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds$ using the appropriate *double* integral. (Setting up, and then using a shortcut, is okay.)

Solution

Problem 12

Let G be the solid $3D$ cone bounded by the lateral surface given by $z = 2\sqrt{x^2 + y^2}$ and by the plane $z = 2$. The problem is to compute $\overline{z}$, the z-coordinate of the center of mass of G, in the case where the density is equal to the height above the xy-plane.

(a) Find the mass of G using cylindrical coordinates.

(b) Set up the calculation for $\overline{z}$ using cylindrical coordinates. (Answers should be ready to integrate out, but do not evaluate.)

(c) Set up the calculation for $\overline{z}$ using spherical coordinates. (Answers should be ready to integrate out, but do not evaluate.)

Solution

Problem 13

Let $\mathbf{F}(x, y, z) = (y + y^2 z)\mathbf{i} + (x - z + 2xyz)\mathbf{j} + (-y + xy^2)\mathbf{k}$.

(a) Show that $\mathbf{F}(x, y, z)$ is a gradient field using the derivative conditions.

(b) Find a potential function $f(x, y, z)$ for $\mathbf{F}(x, y, z)$ using any *systematic* method. Show the method used and all work clearly.

(c) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the straight line joining the points $(2, 2, 1)$ and $(1, -1, 2)$ (in that order), using as little computation as possible.

Solution

Problem 14

In this problem, S is the surface given by the quarter of the right-circular cylinder centered on the z-axis, of radius 2 and height 4, which lies in the first octant. The field $\mathbf{F}(x, y, z) = x\mathbf{i}$.

(a) Sketch the surface S and the field $\mathbf{F}$.

(Suggestion: use a coordinate system with y pointing out of the paper.)

(b) Compute the flux integral $\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$.

(Use the normal which points "outward" from S, i.e. on the side away from the z-axis.)

(c) Let G be the $3D$ solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $\mathbf{F} = x\mathbf{i}$ out of the region G.

(d) The boundary surface of G is comprised of S together with four other faces. What is the flux outward through these four faces, and why? Use the answers to parts (b) and (c), and also verify using the sketch of part (a).

Solution

Problem 15

Let $\mathbf{F}(x, y, z) = (yz)\mathbf{i} + (-xz)\mathbf{j} + \mathbf{k}$. Let S be the portion of the surface of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2, and C_3 are the three curves formed by intersecting S with the xy, yz, and xz planes, respectively (so that C is the boundary of S). Orient C so that it is traversed counter-clockwise when seen from above in the first octant.

(a) Use Stokes' theorem to compute the loop integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by using the *surface integral* over the capping surface S.

(b) Set up and evaluate the loop integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parameterizing each piece of the curve C and then adding up the three line integrals.

Solution