PSET 12

Part 1, Problem 60-1a

Evaluate

$$\int_C y\,dx + z\,dy - x\,dz$$

By parametrization,

$x = t$ $\quad \cancel{dx = dt}$ $\quad dx = dt$

$y = t^2$ $\quad \cancel{dy = 2t\,dt}$ $\quad dy = 2t\,dt$

$z = t^3$ $\quad \cancel{dz = 3t^2\,dt}$ $\quad dz = 3t^2\,dt$

$$\cdots = \cancel{\int_C t^3 + 2t^4 - 3t^3\,dt}$$

$$\cancel{= \left[\frac{2t^5}{5} - \frac{2t^4}{4}\right] = \frac{2}{5} - \frac{2}{4}}$$

$$= \int_0^1 t^2 + 2t^4 - 3t^3\,dt$$

$$= \left[\frac{t^3}{3} + \frac{2t^5}{5} - \frac{3t^4}{4}\right]_0^1 = \frac{-1}{60}$$

Part 1, Problem 60-2

The force is always perpendicular to the path on the sphere's surf-face; $\vec{F} \cdot d\vec{r} = 0$ then,

Part 1, Problem 60-4

(a): $F = \langle 2x, 2y, 2z \rangle$

(b): • Parametrize with
$x = \cos(t) \Rightarrow dx = -\sin(t)$
$y = \sin(t) \quad '' \quad dy = \cos(t)$
$z = t \quad '' \quad dz = dt$
over $[0, 2\pi n]$;

$$\int_C 2x\,dx + 2y\,dy + 2z\,dz = \int_0^{2\pi n} 2t\,dt = (2\pi n)^2$$

• Choosing the vertical path, we have an identical

$$\int_0^{2\pi n} 2t\,dt = (2\pi n)^2$$

• By FTLI;

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 0, 2\pi n) - f(1, 0, 0) = 1 + (2\pi n)^2 - 1 = (2\pi n)^2$$

Part 1, Problem 6D-5

By FTIL the ~~maximum~~ work over a curve connecting points R and S can be written

~~$f(x,y,z$~~

$$\sin(xyz)\big|_S - \sin(xyz)\big|_R$$

Since sin has range $[-1, 1]$, the maximum difference is 2.

Part 1, Problem 6E-3 ab

(a): Recall the tool for remembering curl:

$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

Thus,

$$\text{curl}(F) = \langle (0-0), (1-1), (2x-2x) \rangle = 0.$$

(b): $f_x = 2xy + z \Rightarrow f = x^2 y + xz + g(y,z)$

$f_y = x^2 \Rightarrow f = x^2 y + \cancel{h(z)}\ g_y \Rightarrow g_y = 0 \Rightarrow g = h(z)$

$f_z = x + h'(z) = x \Rightarrow h'(z) = 0 \Rightarrow h = C$; thus

$\vec{F} = x^2 y + xz + C.$

Part 1, Problem 6E-5

Recall

$\text{curl}(\vec{F}) = \langle (R_y - Q_z), (P_z - R_x), (Q_x - P_y) \rangle \stackrel{?}{=} 0$

$= \langle \cancel{(bxy \ldots - (bxz + 2y}$

$= \langle (bxz + 2y - (2xz + ay)), \ldots, \ldots \rangle$

$b = 2$

$a = 2$, Thus

$F = \langle yz^2, (xz^2 + 2yz), (2xyz + y^2) \rangle$

Then,

$P = f_x \qquad f = xyz^2 + g(y, z)$

$Q = f_y \Rightarrow \cancel{f_y = xyz^2 + h(x, z)}$

$R = f_z \qquad = xz^2 + 2yz \Rightarrow g = y^2 z + h(z)$

$f_z = 2xyz + y^2 + h(z) \Rightarrow h = C$

Thus,

$f(x, y, z) = xyz^2 + y^2 + C$

Part 1, Problem 6F-16

Recall Stoke's theorem

$$\text{work} = \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \ \hat{n} \ dS$$

First,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C y \, dx + z \, dy + x \, dz$$

Since $z = 0$ and thus $dz = 0$,

$$\ldots = \int_C y \, dx$$

Then since $x = \cos(\theta)$, $y = \sin(\theta)$, $dx = -\sin(\theta) \, d\theta$

$$\ldots = -\int_0^{2\pi} \sin^2 \theta \, d\theta = -\int_0^{2\pi} \frac{1 - \cos(\theta)}{2} \, d\theta = -\pi$$

Second,

$$\iint_S \text{curl}(\vec{F}) \ \hat{n} \ dS$$

$\text{curl}(\vec{F}) = -\langle 1, 1, 1 \rangle$,

$\hat{n} = \langle x, y, z \rangle$, thus

$$\ldots = -\int_S x + y + z \, dS$$

make the change of variables $x = r \cos\theta$, $y = r \sin\theta$, $z = \rho \cos\phi$, then

$$\ldots = -\int_0^{2\pi} \int_0^{\pi/2} (\sin\phi \, (\cos\theta + \sin\theta) + \cos(\phi)) \sin\phi \, d\phi \, d\theta$$

By algebra equals $-\pi$.

Part 1, Problem 8F-2

First consider the left part of Stoke's theorem;

$$\int_C \vec{F} \cdot d\vec{r} = \int_C y\,dx + z\,dy + x\,dz$$

Then, let

$x = \cos(\theta) \Rightarrow dx = -\sin\theta$

$y = \sin(\theta) \Rightarrow dy = \cos\theta$

$z = -(\cos(\theta) + \sin(\theta)) \Rightarrow dz = -(-\sin\theta + \cos\theta)\,d\theta$

Thus,

$$\ldots = \int_0^{2\pi} \left(-\sin^2(\theta) - (\cos^2(\theta) + \sin\theta\cos\theta) + \cos\theta(\sin\theta - \cos\theta)\right) d\theta$$

$$= -3\pi$$

Now consider the right side; evaluate

$\operatorname{curl}(\vec{F}) = -\langle 1, 1, 1 \rangle$

and $n\,dS = \langle 1, 1, 1 \rangle\,dx\,dy$, thus

$$\iint_S \operatorname{curl} F \cdot n \cdot dS = -\iint_{S'} 3\,dA = -3\pi.$$

Part 1, Problem 6F-5

(a) The top surface has

$$\iint_{S_1} \text{curl}(\vec{F}) \cdot \hat{n} \cdot dS = 2\iint_{S_1} dS = 2\pi a^2,$$

while the ~~bottom surface~~ sides have

$$n = \frac{\langle x, y, 0\rangle}{a},$$ thus

$$\iint_{S_2} \text{curl}(F) \cdot \hat{n} \cdot dS = \int_0^{2\pi} \int_0^h \frac{-2xy}{a} \, dz \, d\theta$$

$$= \left[-h a^2 \sin^2\theta\right]_0^{2\pi} = 0$$

Thus,

work $(S_1 + S_2) = 2\pi a^2$

Part 2, Problem 1

(a): $\operatorname{curl}(\vec{F}) = \left\langle (0-0), \left(\frac{-1}{x^2+z^2} + \frac{2x^2}{(x^2+z^2)^2} - \frac{1}{(x^2+z^2)} + \frac{2z^2}{(x^2+z^2)^2} \right), (0-0) \right\rangle$

$= 0$ for $x^2+z^2>0$.

(b): We see that

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\left(\frac{-1}{1+\cos^2(t)} \right)(-\sin(t)) + \sin t \cos t \right) dt$

by ~~variable~~ variable manipulation.

In turn,

$\dots = 0.$

(c): No, any capping surface must pass through point(s) where $\operatorname{curl}(\vec{F})$ is undefined.

(d): $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta = 2\pi \neq 0.$

Part 2, Problem 2

(a): $\operatorname{curl}(\vec{F}) = \left\langle \frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right), -\left(\frac{\partial}{\partial x}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{x}{r^2}\right)\right), \frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^2}\right) \right\rangle$

$= 0.$

(b): Yes; there exists a capping surface that does not intersect the origin.

(c): $\mathbb{R}^3 - \{y\text{-axis}\}$ is not simply connected, $\mathbb{R}^3 - \{0\}$ is.

Part 2, Problem 3

(a): See that

$$\cancel{f}\,\mathrm{curl}(\vec{F}) = \frac{\partial}{\partial x}(z\sin\theta) + \frac{\partial}{\partial y}(-z\cos\theta) + \frac{\partial}{\partial z}(-x\sin\theta + y\cos\theta)$$

$$= 0.$$

(b): By the divergence theorem, flux is zero.

Part 2, Problem 4

(a): $\text{Curl}(\vec{F}) = 2\langle \cos\theta, \sin\theta, 0\rangle$

(b): $w_{max} = 1$

(c): $n_+ \cdot v = 0$, thus P_i is the plane of fastest spin.

(d): Pure rotational flow in P_+.

(e): The fluid rotates in P_+ while the P_+ themself revolve around the z-axis; Some swirling is occuring.