# PSET 1

## Part 1, Problem 6~~D-1a~~A-3

(A): $\omega(-z\mathbf{j} + y\mathbf{k})$

Part 1, Problem 6B-1

(A): $\frac{4}{3} \pi a^3$

Part 1, Problem 6B-2

(A): $\langle 0,0,1 \rangle$ is parallel to the surface of the cylinder, so Flux $= 0$.

# Part 1, Problem 6B-3

See that

$n = \langle 1, 1, 1 \rangle \Rightarrow \hat{n} = \dfrac{\langle 1, 1, 1 \rangle}{\sqrt{3}};$

$\vec{F} \cdot \hat{n} = \dfrac{1}{\sqrt{3}}$

$\text{Flux} = \dfrac{\text{area}}{\sqrt{3}} = \dfrac{1}{2}.$

Part1, Problem 6B-6

$$\iint_S F\cdot d\vec{S} = \iint_S \langle x,y,z\rangle\cdot\langle 2x,2y,1\rangle\,dx\,dy$$
$$= \int_0^1\int_0^1 -2x^2 + -2y^2 + x^2+y^2\,dx\,dy$$
$$= \int_0^1 \left[\frac{x^4}{4} + y^2 x\right]_0^1 dy$$
$$= \int_0^1 \frac{3}{4} + 3y^2\,dy$$
$$= \frac{3}{4} + 1$$

Attempt No.2:

$d\vec{S} = \langle -2x,-2y,1\rangle$

$F = \langle x,y,z\rangle$

$$\iint_S F\cdot d\vec{S} = \iint_S -2x^2 - 2y^2 + z$$
$$= \iint_S -2x^2 - 2y^2 + x^2 + y^2$$
$$= -\iint_S x^2 + y^2$$

Make a change of coordinates to circular coordinates

$$= -\int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta$$
$$= -\int_0^{2\pi}\frac{1}{4}\,d\theta$$
$$= \frac{-\pi}{2}$$

Comparing to a cone

$z=\sqrt{x^2+y^2}$, this makes sense because f stays low longer than the field, and thus intersects at an appreciable angle opposite the normal.

# Part 1, Problem 6B-8

We have

$$\vec{F} = \langle 0, y, 0 \rangle$$

and

$$\hat{n} = \frac{\langle x, y, 0 \rangle}{\sqrt{x^2+y^2}} = \frac{\langle x, y, 0 \rangle}{a}$$

Thus

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \frac{y^2}{a}$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^h a^2 \sin^2\theta \, dz\, d\theta$$

$$= a^2 h \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2} a^2 h$$

# Part 1, Problem 6C-3

First consider the triple integral:

$$\iiint_D \operatorname{div}(\vec{F})\, dD = \iiint 3 \; dD = 3 \cdot \frac{2}{3}\pi a^3$$

$$= 2\pi a^3$$

Then consider the left side

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \langle x, y, z \rangle \cdot \frac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}} + \iint_{S_2} -z \, dz$$

$$= \iint_{S_1} x^2+y^2+z^2 \cdot \frac{1}{a}$$

$$= \iint_{S_1} a^2 \cdot \frac{1}{a} + \iint_{S_2} -0 \, dz$$

$$= a \cdot \iint_{S_1} dS$$

$$= 2\pi a^3$$

The integrals match and the divergence theorem is satisfied.

Part I, Problem 6C-5

By the divergence theorem,

$$\iint_S \vec{F} \cdot dS = \iiint_D \operatorname{div}(\vec{F}) \, dV$$

$$= \iiint_D (1 + z^2 + y^2$$

$$= 1 \cdot \iiint_D dV$$

$$= 1 \cdot \left(\frac{1}{3} \cdot \frac{1}{2} \cdot 1\right) = \frac{1}{6}$$

# Part 1, Problem 6C-7a

~~Fir~~ Recall the divergence theorem:

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_D \operatorname{div}(\vec{F}) \, dD$$

$$\cancel{= \iint_S \langle x^2 + xy \rangle \frac{\langle x^2 + y^2 \rangle}{\sqrt{x^2+y^2}} \, dS}$$

$$\cancel{= \iint_S}$$

$$= \iint_S \langle x^2, xy \rangle \langle x, y \rangle \, dS$$

$$= \iint_S x^3 + xy^2 \, dS$$

$$= \iint_S x(\cancel{x} - x^2) \, dS$$

$$= \iint_S x \, dS$$

$$= \int_0^{2\pi} \int_0^1 \cos\theta \, dz \, d\theta$$

$$= \int_0^{2\pi} \cos\theta \, d\theta$$

$$= 0$$

Now evaluate the right side!

$$\iiint_D \operatorname{div}(\vec{F}) \, dD$$

$$\iiint_D 3x \, dS$$

Since this shape is symmetric across the yz plane,

$$3 \iiint_D x \, dS = 3 \cdot 0 = 0.$$

Part 1, Problem 6C-8

(a) We see by the divergence theorem that for $S_2'$ is the surface with normal vector opposite $S_2$'s and on the same surface, that

$$\int_S \vec{F}\cdot d\vec{S} = \int_{S_1} \vec{F}\cdot d\vec{S} + \int_{S_2'} \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}(\vec{F})\, dD$$
$$= 0 \quad \text{since } \operatorname{div}(\vec{F}) = 0$$

implies

$$\int_{S_1} \vec{F}\cdot d\vec{S} = -\int_{S_2'} \vec{F}\cdot d\vec{S} = \int_{S_2} \vec{F}\cdot d\vec{S}$$

(b) The statement applys for any closed surface, as long as it can be broken into 2 parts; $S_1$ and $S_2$

# Part 2, Problem 1

As the hint suggests, let a "north pole" = (0,0,a) and consider a sphere of radius a ~~at~~ centered at the origin; then the average distance from the north pole ~~can~~ be found by evaluating

$$\bar{D} = \frac{1}{4\pi a^2}\int_0^{2\pi}\int_0^{\pi} a^3\sqrt{2}\,(1-\cos\phi)^{1/2}\sin\phi \; d\phi \; d\theta$$

$$\cancel{= \frac{a\sqrt{2}}{4\pi}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^{1/2}\sin\phi\, d\phi\, d\theta}$$

$$\cancel{= \frac{a\sqrt{2}}{4\pi}}$$

$$= \frac{1}{4\pi a^2}\cdot 2\pi\sqrt{2}a^3\int_0^{\pi}(1-\cos\phi)^{1/2}\sin\phi\, d\phi$$

$$= \frac{1}{4\pi a^2}\cdot 2\pi\sqrt{2}\,a^3\,\frac{2}{3}\Big[(1-\cos\phi)^{3/2}\Big]_0^{\pi}$$

$$= \frac{1}{4\pi a^2}\cdot\frac{16\pi a^3}{3} = \frac{4}{3}a$$

Part 2, Problem 2

For $dm$ at $(x, y, z)$,

$$dF = G \frac{\langle x, y, z \rangle}{\rho^3} \, dm = G \frac{\langle x, y, z \rangle}{\rho^3} \, dV$$

Then consider the total force

$$F = \langle a, b, c \rangle$$

~~By symm~~ By symmetry $a = b = 0$, and see that

$$c = G \int_0^{2\pi} \int_0^{\phi_0} \int_0^{a} \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= G \pi a \sin^2(\phi_0)$$

Thus,

$$F = \langle 0, 0, G \pi a \sin^2(\phi_0) \rangle$$

# Part 2, Problem 3

(a): The surface $T$ is a circle of radius 1 at $z = 1$, ~~with~~ and $n = k$; thus

$F \cdot n = 1 \Rightarrow \text{flux} = \iint_T F \cdot n \, dS = (\text{area } t) = \pi$.

(b): We see that

$$\text{Vol}(D_1) = \text{Vol}(D_1 + D_2) - \text{Vol}(D_2)$$

$$\text{Vol}(D_1 + D_2) = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{4\pi\sqrt{2}}{2} - \frac{4\pi}{3}$$

Meanwhile,

$\text{Vol}(D_2) = \pi/3$, thus

$$\text{Vol}(D_1) = \frac{4\pi\sqrt{2}}{3} - \frac{5\pi}{3}.$$

(c): See that

$n\,dS = \langle -z_x, -z_y, 1 \rangle$ for $z = \sqrt{x^2 + y^2}$; then

$F \cdot d\vec{S} = z \, dx \, dy$

Then,

$$\iint_R z \, dx \, dy = \int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta = \frac{2\pi}{3}.$$

# Part 2, Problem 4

(a): Use that

$$\frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \quad \frac{\partial \rho}{\partial z} = \frac{z}{\rho}$$

to see that

$$F = -\left\langle \frac{x}{\rho^3}, \frac{y}{\rho^3}, \frac{z}{\rho^3} \right\rangle$$

then since

$$\frac{\partial}{\partial \lambda} \lambda \rho^{3} = \rho^{3} - 3\lambda^{2} \rho^{5} \text{ for } \lambda = x, y, z, \text{ then}$$

$$\operatorname{div} F = \left(-\rho^3 + \frac{\partial}{\partial x} x \rho^3\right) + \left(-\rho^3 + \frac{\partial}{\partial y} y \rho^3\right) + \left(-\rho^3 + \frac{\partial}{\partial z} z \rho^3\right)$$

$$= 0$$

(b): On $S$, $\hat{n} = \langle x, y, z \rangle / a$, $F = -\frac{\langle x, y, z \rangle}{a^3}$

Thus,

$$F \cdot n = \frac{-1}{a^2} \Rightarrow \text{flux} = \frac{-1}{a^2} \cdot \text{area} = -4\pi$$

Part 2, Problem 5

Use that $\nabla f \perp$ the surface $f = c$; then

$\nabla f \cdot n = \pm|\nabla f|$ assuming is pointing outwards; then

$$\iint_S \vec{\nabla} f \cdot n \, dS = \iiint_D \vec{\nabla}(\vec{\nabla} f) \, dD$$

Implies

$$\pm\iint |\nabla f| \, dS = \iint_S \nabla f \cdot n \, dS = \iiint_D \nabla^2 f \, dD. \quad \square$$