Part I, Problem 2I-1

A rectangular box is placed in the first octant so that one corner Q is at the origin and the three sides adjaent to Q lie in the coordinate planes. The corner P diagonally opposite Q lies on the surface $f(x, y, z) = c$. Using Lagrange multipliers, tell for which point P the box will have the largest volume, and tell how you know it gives a maximum point, if the surface is

(a) the plane $x + 2y + 3z = 18$,

(b) the ellipsoid $x^2 + 2y^2 + 4z^2 = 12$.

Solution

Part I, Problem 2I-3

(Repeat of 2F-2, but this time use Lagrange multipliers.) A rectangular produce box is to be made of cardboard; the sides of single thickness, the ends of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what should be its proportions in order to use the least cardboard?

Solution

Part I, Problem 2J-1

In Example 1 of notes N, we define

$$w = x^2 + y^2 + z^2, \tag{1}$$
$$z = x^2 + y^2. \tag{2}$$

Calculate by direction substitution:

(a) $\left(\frac{\partial w}{\partial y}\right)_z$,

(b) $\left(\frac{\partial w}{\partial z}\right)_y$.

Solution

Part I, Problem 2J-2

Calculate the two derivatives from problem 2J-1 by using

(i) the chain rule (differnetiate $z = x^2 + y^2$ implicitly), and

(ii) differentials.

Solution

Part I, Problem 2J-3a
Example 2 of Notes N defines

$$w = x^3y - z^2t, \tag{3}$$
$$xy = zt. \tag{4}$$

Use the chain rule to calculate $\left(\frac{\partial w}{\partial t}\right)_{x,z}$ in terms of x, y, z, t.

Solution

Part I, Problem 2J-4b

Repeat 2J-3, doing the calculation using differentials.

Solution

Part I, Problem 2J-5a

Let $S = S(p, v, T)$ be the entropy of a gas, assumed to obey the ideal gas law

$$pv = nRT. \tag{5}$$

Give an expression for $\left(\frac{\partial S}{\partial p}\right)_v$ in terms of the formal partial derivatives S_p, S_v, and S_T.

Solution

Part I, Problem 2J-7

Let P be the point $(1,-1,1)$ and assume $z = x^2 + y + 1$, and that $f(x,y,z)$ is a differentiable function for which $\nabla f(x,y,z) = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ at P.

Let $g(x,z) = f(x, y(x,z), z)$. Find ∇g at the point $(1,1)$, i.e. $x = 1, z = 1$.

Solution

Part II, Problem 1

Go to the 'Mathlet' **Lagrange Multipliers** (with link on the course webpage), and choose

$$f(x, y) = x^2 - y^2, \tag{6}$$

$$g(x, y) = x^2 + y^2. \tag{7}$$

(a) Solve by hand to find the two values of λ and the possibilities for the corresponding points (x, y) at which the gradients are proportional. Then check these possibilities on the applet and verify the predicted proportionality on the graph.

(b) Now take $b = 3$ and finish the solution of part (a) by hand to find the possible points which may give a relative extremum of f. Then return to the applet, set $b = 3$, move the f-levels until they make contact with the $g = 3$ constraint curve, and read the values of f at the points of contact. Compare with the results found by hand; how close could you get?

(c) What do the two values of λ correspond to in terms of the pairs of solution points? Do the gradients of f and g point in the same or the opposite direction at the contact points in the two different cases, and is this consistent with the signs of λ?

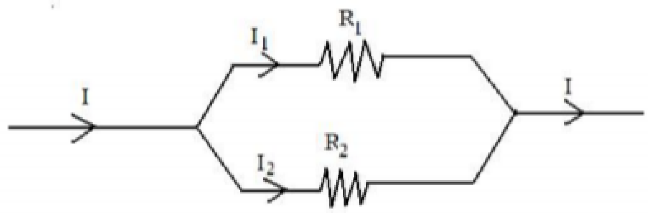
Solution

Part II, Problem 2

In this problem, we examine how electricity flows through circuits to minimize energy.

A current I flowing over a resistor R results in an energy loss (in the form of heat/light) equal to I^2R per second. It turns out that, in a sense, "electricity prefers to flow in the way that minimizes energy loss to resistance." For example, when an electric current comes to a fork, it will divide itself up in such a way that a large portion of the current flows where the resistance is low, and a small portion flows where the resistance is high (you might think all the electricity would flow where the resistance is low, but the energy loss is proportional to I^2, so it is better to spread the current around).

Suppose we have the following situation where a current I comes to a pair of resistors in parallel:



(a) Determine what choice of I_1 and I_2 will minimize energy loss and hence determine what the currents will be along the two paths. (Alternatively, if you already are familiar with resistors in parallel and circuit flows, verify that the currents I_1 and I_2 do in fact minimize energy loss.)

(b) Suppose instead we had three resistors in parallel. In terms of R_1, R_2, and R_3, determine the values of I_1, I_2, and I_3 which minimize energy loss.

Solution

Part II, Problem 3

Using the usual rectangular and polar coordinates, let w be the area of the right triangle in the first quadrant having its vertices at $(0,0)$, $(x,0)$, and (x,y). Using the equation expressing w in terms of x and y and the equations expressing y in terms of x and θ, calculate the two partial derivatives $\left(\frac{\partial w}{\partial x}\right)_\theta$ and $\left(\frac{\partial w}{\partial \theta}\right)_x$ in three different ways.

(a) Directly, by first expressing w in terms of the independent variables x and θ.

(b) By using the chain rule – for example, $\left(\frac{\partial w}{\partial x}\right)_\theta = w_x \left(\frac{\partial x}{\partial x}\right)_\theta + w_y \left(\frac{\partial y}{\partial x}\right)_\theta$, where w_x and w_y are the formal partial derivatives.

(c) By using differentials.

Solution