question 4 views

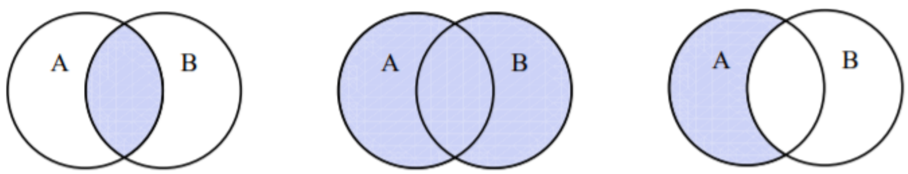
Daily Challenge 2.3

(**Due: Thursday 5/3 at 12:00 noon Eastern**)

Let's tackle some of the set theory proofs from chapter 1 of AoPS again.

Review

Our goal is to prove facts about the three basic set operations of **intersection**, **union**, and **set difference**. Their Venn diagrams are illustrated below:



$A \cap B$ $A \cup B$ $A \setminus B$

To write proofs about these operations, we will need precise definitions rather than pictures.

Definition. Let A and B be sets.

- The intersection $A \cap B$ is defined as $\{x \mid x \in A \text{ and } x \in B\}$. In words, this contains all of the elements that are in both A and B.
- The union $A \cup B$ is defined as $\{x \mid x \in A \text{ or } x \in B\}$. This contains all elements that are *either* in A, *or* in B (or possibly both).
- The set difference $A \setminus B$ is defined as $\{x \in A \mid x \notin B\}$. This is everything in A, but not in B.

Notice that these definitions rely on the logical operations of "and", "or", and "not". Most set-theoretic proofs reduce to properties of these three logical operations: for instance, if the statement "p and q" is true, we know that both p is true and q is true. Similarly, if we know that the statement "p or q" is true, then we know that at least one of the statements p and q is true.

Here's an example problem, where we reduce a set theory question to a logic question involving "or".

Theorem. If A and B are sets, then $A \subseteq A \cup B$.

Exploration. I think this should be true by looking at the Venn diagrams above. The diagram for union shows a shaded region which contains both A and B, so it should be true that A sits inside the union.

How will I prove this? To show that one set is a subset of another, we must show that all elements of the first set are also in the second set. Let's convert this general statement to a specific statement about a named variable: if a is an element of A, we must show that a is also an element of $A \cup B$. To do this, I must show that a satisfies the condition which appears after the "such that" bar when we write $A \cup B = \{x \in \text{ some set } \mid x \text{ satisfies some conditiion }\}$.

What's the definition of $A \cup B$ again? Ah, it's $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. The condition contains a logical "or", and I know that the statement "p or q" is true whenever p is true, or q is true, or both. So if the first statement, $a \in A$, is true, then the entire "or" statement is true.

Okay, I think I understand now. Let's go back and write the proof.

Argument. Suppose $a \in A$. The definition of $A \cup B$ is

$$\{x \mid x \in A \text{ or } x \in B\}.$$

Since we have assumed $a \in A$ is true, the statement $(a \in A \text{ or } a \in B)$ is also true. This means that a belongs to $A \cup B$. Since every $a \in A$ also satisfies $a \in A \cup B$, we conclude that $A \subseteq A \cup B$. $\square$

Problem

Prove the following, including *both* your thought process during **exploration** and your final polished **argument** (if you find one).

Theorem. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

To prove this, consider some element $x \in A \cup B$, and show that we also have $x \in C$ by using the two assumptions, the definition of union, and the properties of the logical "or" operator.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Exploration (Corbin) - I'm going to begin by creating some element x where $x \in A \cup B$. This means that x contains all the elements in A and B. And then from here I shall prove that $x \in C$. Although this problem makes intuitive sense this often means that it is very hard to prove the intuition, but I shall make an attempt.

Argument (Corbin) - First I shall define x where $x \in A$ and $x \in B$. This also means that $x \in A \cup B$ because x is in either A, B, or both. Therefore we have possible outcomes. If $x \in A$ and $A \subseteq C$ then $x \in C$. And if $x \in B$ and $B \subseteq C$ then $x \in C$. This proves that every element of $A \cup B$ is in C. This shows that $A \cup B \subseteq C$. $\Box$

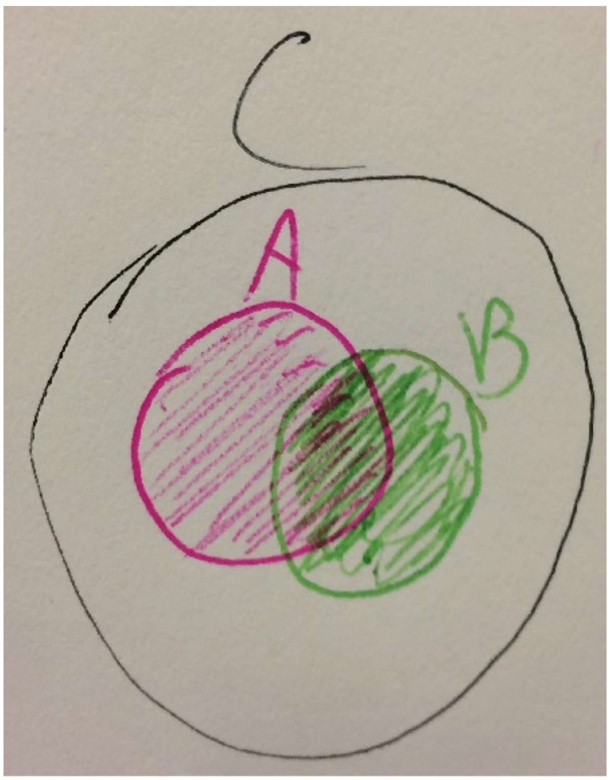
Exploration (Logan). Your thoughts go here.

Argument (Logan) - First, I can define the union of sets A and B as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ From this definition, I can conclude that x is either in set A, B, or both for it to appear in the union of said sets. Furthermore, taking into account our if statement, then all the elements of A are a subset of C, represented by $A \subseteq C$. In the same manner, all the elements of B are a subset of C, therefore $B \subseteq C$. The union of these two subsets can be represented by the statement $\{x \mid x \in A \text{ or } x \in B\} \subseteq C$, and this is the very thing we are trying to prove.

Updated 9 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Exploration (Christian). First I make sure that I understand the statement: if both A and B are contained in some larger set C, then their union $A \cup B$ should also be contained within C. I believe that this is true; to convince myself, I draw a Venn diagram:



It seems convincing that, if we combine the pink and green shaded regions to find $A \cup B$, the result must still be contained in C because all of its elements are in C.

How do I prove this? To show subset containment, I will consider some $x \in A \cup B$, then argue that $x \in C$. Now I start the proof.

Argument (Christian). Suppose $x \in A \cup B$. By the definition of union, this means that x belongs to either A, or B, or both. There are two cases:

1. If $x \in A$, then since $A \subseteq C$, it follows that $x \in C$.
2. If $x \in B$, we also have $B \subseteq C$, so it again follows that $x \in C$.

Thus it is generally true that $x \in C$, so every element of $A \cup B$ belongs to C and we conclude that $A \cup B \subseteq C$. $\Box$

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*