26.7 cde

(c): We have that

$$C = \frac{\phi' \sin^2(\theta)}{\sqrt{1+\sin^2(\theta)\,\phi'^2}} \Rightarrow C^2(1+\sin^2(\theta)\phi'^2) = \phi'^2 \sin^4(\theta)$$

$$C^2 + C^2\sin^2(\theta)\phi'^2 = \phi'^2 \sin^4(\theta)$$

$$C^2 = \phi'^2\left(-C^2\sin^2(\theta) + \sin^4(\theta)\right)$$

$$\frac{C}{\sin(\theta)\sqrt{\sin^2(\theta) - C^2}} = \phi'$$

We then integrate each side over $[\theta_1, \theta_2]$ with respect to θ:

$$\int_{\theta_1}^{\theta_2} \phi' \, d\theta = \int_{\theta_1}^{\theta_2} \frac{C}{\sin(\theta)\sqrt{\sin^2\theta - C^2}}$$

We see by u-sub where $u = \phi \Rightarrow du = \phi' \, d\theta$, and the bounds become $[\phi(\theta_1), \phi(\theta_2)]$, thus

$$\int_{\phi(\theta_1)}^{\phi(\theta_2)} du = \quad ''$$

$$\phi(\theta_2) - \phi(\theta_1) = \quad ''$$

(d): We wish to evaluate

$$\int_{\theta_1}^{\theta_2} \frac{C}{\sin(\theta)\sqrt{\sin^2(\theta) - C^2}} \, d\theta = \int_{\theta_1}^{\theta_2} \frac{C}{\sin^2(\theta)\sqrt{1 - C^2\csc^2(\theta)}}$$

then u-sub $u = \cot(\theta) \Rightarrow -\csc^2(\theta)\, d\theta = du$; thus since

$$\sin^2 + \cos^2 = 1 \Rightarrow \csc^2 = 1 + \cot^2 \qquad (1)$$

then

$$\ldots = \int_{\cot(\theta_1)}^{\cot(\theta_2)} \frac{-C}{\sqrt{1 - C^2(1+u^2)}} \, du$$

where the du absorbs the $\sin^2$ in the denominator and the $\csc^2 = 1 + u^2$ by (1). If we let $a = \sqrt{1-C^2}/C$, then we see that

$$\ldots = \int_{\cot(\theta_1)}^{\cot(\theta_2)} \frac{-c}{\sqrt{1-c^2-c^2u^2}}$$

$$= \int \frac{-1}{\frac{1}{c}\sqrt{1-c^2-c^2u^2}}$$

$$= \int \frac{-1}{\sqrt{\frac{1-c^2}{c^2}-u^2}}$$

$$= \int \frac{-1}{\sqrt{a^2-u^2}}$$

Then trig-sub $u = a\sin\theta$, $du = a\cos(\theta)\,d\theta$

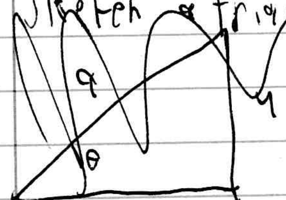
$$\cancel{= \frac{-1}{a}\int \frac{t}{\cos\theta}\,du} \quad = \frac{-1}{a}\int \frac{a\cos\theta\,d\theta}{\cos\theta}$$

$$= -\int d\theta$$

$$= -\theta$$

$$\phi(\theta_2) - \phi(\theta_1) \quad = -\left[\arcsin\left(\frac{u}{a}\right)\right]_{\cot(\theta_1)}^{\cot(\theta_2)}$$

~~Sketch a triangle:~~



~~Then~~

Suppressing bounds and letting the $\phi(\theta_1)$ absorb the differences, we see that

$$\phi(\theta_2) - \phi(\theta_1) = \arccos\left(\frac{u}{a}\right)$$

(e):

$$\phi(\theta_2) - \phi(\theta_1) = \arccos\left(\frac{\cot(\theta)}{a}\right)$$

$$a\cos(\phi(\theta_2) - \phi(\theta_1)) = \cot(\theta)$$