31.5

(c): For $n=-m$, we see that

$$\int_0^L \exp\left(\frac{2\pi i n x}{L}\right) \exp\left(\frac{2\pi i m x}{L}\right) dx = [\,1\,]_0^L = L$$

For $n \neq -m$, we have that

$$\int_0^L \exp\left(\frac{2\pi i m x}{L}\right) \exp\left(\frac{2\pi i n x}{L}\right) dx =$$

$$\int_0^L \exp\left(\left(\frac{2\pi i}{L}\right)(n+m)\, x\right) =$$

$$\left[\frac{\exp\left(\left(\frac{2\pi i}{L}\right)(n+m)\, x\right)}{(2\pi i/L)(n+m)}\right]_0^L = \frac{\exp\left(2\pi i\,(n+m)\right)}{(2\pi i/L)(n+m)} - \frac{1}{(2\pi i/L)(n+m)}$$

We chose these cases simply because we wish to consider; does the multiplication cancel or not?

~~Now recall the~~

$$\cancel{f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right)}$$

~~multiply eachside by~~

By euler,

$$e^{2\pi i(n+m)} = \cos(2\pi(n+m)) + i\sin(2\pi(n+m))$$

for integer n and m,

$$= 1 + 0$$

$$= 1$$

Then, $\frac{1-1}{(2\pi i/L)(n+m)} = 0$

~~Now, recall~~

$$\cancel{f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right)}$$

~~multiply eachside by $\exp\left(\frac{-2\pi i k}{L}x\right)$ and apply orthogonality~~

$$\cancel{\exp\left(\frac{-2\pi i k}{L}x\right) f(x) = c_n \cdot L}$$

~~since only at $n=-k$ does orthogonality return non-zero.~~

Recall the original Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp\left(\frac{2\pi i n x}{L}\right)$$

~~Inte~~ multiply each side by $\exp\left(\frac{-2\pi i k x}{L}\right)$

$$\exp\left(\frac{-2\pi i k x}{L}\right) f(x) = \sum_{n=-\infty}^{\infty} C_n \exp\left(\frac{2\pi i n x}{L}\right) \exp\left(\frac{-2\pi i k x}{L}\right)$$

And ~~evalu~~ integrate each side from 0 to L; to see that at $k = n$,

$$\int_0^L \exp\left(\frac{2\pi i n x}{L}\right) \exp\left(\frac{-2\pi i k x}{L}\right) dx = L$$

and is zero for $k \neq n$; thus

$$\int_0^L \left(\frac{-2\pi i k x}{L}\right) f(x) = C_n \cdot L$$

$$C_n = \frac{1}{L} \int_0^L \exp\left(\frac{-2\pi i n x}{L}\right) f(x)\, dx$$