question 2 views

Daily Challenge 10.2

(~~**Due: Wednesday 7/18 at 12:00 noon Eastern**~~)
(~~**Skip: now due Thursday 7/19 at 12:00 noon Eastern**~~)
(**Due Friday 7/20 at 12:00 noon Eastern**)

We've seen that all polynomial functions are continuous, and our powerful limit theorems immediately imply that the sum, difference, product, quotient, or composite of continuous functions is continuous.

You may wonder whether the so-called "transcendental" functions, like the exponential or trigonometric functions, are continuous; these cannot be built out of polynomials, so our theorems do not apply.

(1) Exponential functions are continuous.

Recall that we defined exponential functions in chapter 1 as follows.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We say that f is *exponential* if it

1. is either strictly increasing, strictly decreasing, or constant,
2. has domain $\mathbb{R}$ and range $(0, \infty)$, and
3. satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Almost all of the power of exponential functions comes from property (3). Indeed, we will now check that any function satisfying (3), if it is continuous at the origin, must be continuous everywhere.

Lemma. Let f be an exponential function and suppose that f is continuous at 0. Then f is continuous everywhere.

Proof. We must show that $\lim_{x\to a} f(x) = f(a)$ for any a.

Recall that our definition of "exponential function" requires that f satisfy the multiplicative property:

$$f(x+y) = f(x)f(y).$$

Also recall that, in general,

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h),$$

which we proved in DC 9.1. But by the multiplicative property, $f(a+h) = f(a)f(h)$. Thus

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{h\to 0} f(a)f(h)\\ &= f(a)\lim_{h\to 0} f(h),\end{aligned}$$

We have assumed that f is continuous at zero, so $\lim_{h\to 0} f(h) = f(0) = 1$, and we conclude that

$$\lim_{x\to a} f(x) = f(a)$$

for any other $a \neq 0$. $\square$

By the way, this is the last "theorem classification" word we will need:

- A *proposition* is a minor result.
- A *theorem* is a major result.
- A *corollary* is a conclusion which follows immediately from something we already showed.
- A *lemma* is an intermediate step we use to prove a larger proposition or theorem, not unlike a helper function.

Now we use this lemma to prove that functions of the form $f(x) = a^x$ for $a > 0$ (i.e. how one might have thought about exponential functions before we made the above definition) are continuous.

Proposition. Let $a > 0$ and $f(x) = a^x$. Then $f(x)$ is continuous.

Proof. Any such function is either strictly increasing (if $a > 1$), strictly decreasing (if $a < 1$), or constant (if $a = 1$), outputs only positive reals, and satisfies $f(x+y) = f(x)f(y)$, so $f(x)$ is exponential. By our lemma, then, we only need to prove that $f(x)$ is continuous at zero; after we do so, it follows immediately that f is continuous everywhere.

Since $f(0) = 1$, we must show that $\lim_{x\to 0} f(x) = 1$.We will assume $a > 1$; the proof for $a < 1$ is similar.

Let $\epsilon > 0$ be given and define $\delta = \log_a(1+\epsilon)$. Then if $|x| < \delta$, we have

$$-\log_a(1+\epsilon) < x < \log_a(1+\epsilon).$$

We may exponentiate this inequality with base a since $m < n < p$ implies $a^m < a^n < a^p$ if $a > 1$ (this follows from the definition of strictly increasing). Thus

$$a^{-\log_a(1+\epsilon)} < a^x < a^{\log_a(1+\epsilon)}$$

or, using the logarithm rules,

$$\frac{1}{1+\epsilon} < a^x < 1+\epsilon.$$

Subtracting 1 from each piece of the inequality, and simplifying $\frac{1}{1+\epsilon} - 1 = -\frac{\epsilon}{1+\epsilon} > -\epsilon$, we have $-\epsilon < a^x - 1 < \epsilon,$ or $|a^x - 1| < \epsilon$. This proves the claim. $\square$

---

(2) Problem: trigonometric functions are continuous.

Complete problem 7 in consolidation document 2. I have reproduced it below for convenience.

7. In this problem, we will prove that the sine and cosine functions are continuous everywhere.

(a) Use the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (which we proved in session 14 using the squeeze theorem) and the properties of limits of products (discussed in daily challenge 9.3) to show that $\sin(x)$ is continuous at $x = 0$.

(b) Using your result from (a), along with the identity $\sin^2(x) + \cos^2(x) = 1$, show that $\cos(x)$ is continuous at $x = 0$. [Hint: use the squeeze theorem, trapping $\cos(x)$ between $\cos^2(x)$ and 1.]

(c) Use the angle-addition formulas to prove that sine and cosine are continuous everywhere, by showing that

$$\lim_{h\to 0} \sin(x+h) = \sin(x),$$
$$\lim_{h\to 0} \cos(x+h) = \cos(x).$$

Note that this is sufficient to prove continuity since we showed in DC 9.1 that $\lim_{x\to a} f(x) = \lim_{h\to 0} f(x+a)$.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: Proof: We now it is true algebraically that $\lim_{x\to 0} sin(x) = \lim_{x\to 0}(\frac{sin(x)}{x} \times x)$ since we need not worry about $x = 0$, and we also know that $\lim_{x\to 0}(\frac{sin(x)}{x} \times x) = \lim_{x\to 0}(\frac{\sin(x)}{x}) \times \lim_{x\to 0}(x)$ as we have also shown previously that the limit of a product is the product of the limits. In turn, we have proven that the former limit ($\lim_{x\to 0}(\frac{\sin(x)}{x})$) exists by the squeeze theorem, and that $\lim_{x\to 0} x$ exists since x is a polynomial. In turn, we can conclude that $\lim_{x\to 0} sin(x) = 1 \times 0 = 0$. $\square$

b: Our end goal is to show that $cos(x)$ is continuous at 0, so we must prove that $\lim_(x \to 0) \cos(x) = \cos(0)$. We begin by noting that the hint suggests using the squeeze theorem to "trap" the $\cos(x)$ between $\cos^2(x)$ and 1, so I hope over to desmos to visualize a domain of x that will work for our purposes, including containing zero. The closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ works beautifully. We can now claim that for the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is true that $\cos^2(x) \leq \cos(x) \leq 1$. We have previously shown that $\sin(x)$ is continuous at 0, and we know by Pythagoras that $\cos^2(x) = 1 - \sin^2$, where in turn $\lim_{x\to 0} \cos^2(x) = 1 - \lim_{x\to 0} \sin(x)$, and since it is true that the sine function is continuous at zero then we know thanks to this that $\lim_{x\to 0} \cos(x) = \cos(0) = 1$. It is also true that the zero degree polynomial $f(x) = 1$ is continuous at all points, since it is a polynomial. We can now conclude that since $1 \leq \lim_{x\to 0} \cos(x) \leq 1$, it must be true that $\lim_{x\to 0} \cos(x) = 1$ by the squeeze theorem. $\square$

c: We now by the angle addition formula of sine that $\lim_{h\to 0} \sin(x+h) = \sin(x) \lim_{h\to 0} \cos h + \cos(x) \lim_{h\to 0} \sin(h)$ and in turn thanks to previously proving that $\lim_{x\to 0} sin(x) = sin(0) = 0$, and $\lim_{x\to 0} cos(x) = cos(0) = 1$ it is true that $\sin(x) \lim_{h\to 0} \cos h + \cos(x) \lim_{h\to 0} \sin(h) = \sin(x)$ therefore $\lim_{h\to 0} \sin(x+h) = \sin(x)$ and proving our first claim. Almost identically, $\lim_{h\to 0} \cos(x+h) = \cos(x) \lim_{h\to 0} \cos(h) + \sin(x) \lim_{h\to 0} \sin(h) = \cos(x)$ allowing us to conclude that the sine and cosine functions are continuous everywhere. $\square$

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) We wish to prove that \sin(x) is continuous at 0, or that \lim_{x \to 0} \sin (x) = 0. But one has that

\begin{align} \lim_{x \to 0} \left(\sin (x) \right) = \lim_{x \to 0} \left(\frac{\sin (x) }{x} \cdot x \right) , \end{align}

where we have assumed x \neq 0 inside the limit (which is fine, since the value of the limit is insensitive to the point x = 0).

We proved that \lim_{x \to 0} \left(\frac{\sin (x) }{x} \right) exists by the squeeze theorem, and we have proven that \lim_{x \to 0} x exists since x is a polynomial. Thus the limit of the product exists and is equal to the product of the limits. This means

\begin{align} \lim_{x \to 0} \left(\sin (x) \right) &= \lim_{x \to 0} \left(\frac{\sin (x) }{x} \cdot x \right) \\ &= \left(\lim_{x \to 0} \frac{\sin(x)}{x} \right) \cdot \left(\lim_{x \to 0} x \right) \\ &= 1 \cdot 0 = 0 . \end{align}

Thus the sine function is continuous at zero, as claimed.

(b) We wish to show that \lim_{x \to 0} \cos (x) = 1. Naively we would like to argue that \cos (x) = \sqrt{ 1 - \sin^2 (x) }, and that the sine function is continuous at zero, so cosine must be continuous as well. Unfortunately this will not work since we haven't proved that the square root of a continuous function is continuous.

Here is a better way: we use the squeeze theorem. We know that \cos (x) \leq 1 so, in some open interval around x=0 -- for concreteness, let's say x \in \left(- \frac{\pi}{2} , \frac{\pi}{2} \right) -- we have

\begin{align} \cos^2 (x) \leq \cos (x) \leq 1 . \end{align}

We know that \cos^2 (x) = 1 - \sin^2 (x), and we have proven that \lim_{x \to 0} \sin (x) = 0 , so

\begin{align} \lim_{x \to 0} \cos^2 (x) = 1 - \left(\lim_{x \to 0} \sin (x) \right)^2 = 1 , \end{align}

while clearly \lim_{x \to 0} 1 = 1, so by the squeeze theorem we have \lim_{x \to 0} \cos (x) = 1. This proves that cosine is continuous at zero.

(c) We need to prove that \lim_{h \to 0} \sin (x + h) = \sin(x). By the angle addition formula,

\begin{align} \lim_{h \to 0} \sin (x + h) &= \lim_{h \to 0} \left(\sin (x) \cos (h) + \sin (h) \cos (x) \right) \\ &= \sin(x) \lim_{h \to 0} \left(\cos (h) \right) + \cos(x) \lim_{h \to 0} \left(\sin (h) \right) \\ &= \sin(x) , \end{align}

where we have used our results \lim_{h \to 0} \sin(h) = 0 and \lim_{h \to 0} \cos(h) = 1.

Likewise, using the cosine angle addition formula,

\begin{align} \lim_{h \to 0} \cos (x + h) &= \lim_{h \to 0} \left(\cos (x) \cos (h) - \sin (h) \sin (x) \right) \\ &= \cos(x) \lim_{h \to 0} \left(\cos (h) \right) - \sin(x) \lim_{h \to 0} \left(\sin (h) \right) \\ &= \cos (x) , \end{align}

where again we have used the results above.

Thus we see that the sine and cosine functions are continuous everywhere. \Box

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*