question 2 views

Daily Challenge 9.6

(**Due: Sunday 7/15 at 12:00 noon Eastern**)

Today we'll finish the Spivak reading.

(1) Problem: Spivak reading on limits.

Continue reading this excerpt from Spivak and answer the following reading questions.

(a) In his discussion of $f(x)=\sqrt{|x|}\sin\left(\frac{1}{x}\right)$, Spivak claims that we can impose $|x|<\epsilon^2$ and $x\neq 0$ to ensure that $|f(x)|<\epsilon$, but says "the algebra is left to you." Fill in the necessary algebra to explain why this statement is true.

(b) To show that $\sin\left(\frac{1}{x}\right)$ does not approach any value near $x=0$, Spivak makes the following argument. For any open interval A containing 0, we may find large enough n and m so that there are two numbers

$$x_1=\frac{1}{\frac{\pi}{2}+2\pi n} \quad , \quad x_2=\frac{1}{\frac{3\pi}{2}+2m\pi}$$

in the open interval A such that $f(x_1)=1$ and $f(x_2)=-1$.

Fill in the missing steps in this argument. Why can we always find n and m large enough so that x_1, x_2 lie in the open interval A containing zero? Why is it true that $f(x_1)=1$ and $f(x_2)=-1$?

(c) Spivak presents two of our amusing examples of functions which take different values on the rationals and irrationals, namely

$$f_1(x)=\begin{cases}0 & \text{if } x \text{ is irrational}\\ 1 & \text{if } x \text{ is rational}\end{cases},$$

and

$$f_2(x)=\begin{cases}x & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}.$$

The function f_1 does not approach any limit at any point; the function f_2 approaches 0 near $x=0$, but does not approach any limit at any point $a\neq 0$.

Following this strategy, define a similar function which approaches 0 near $x=1$ and $x=-1$, but does not approach any limit at any other points besides $x=\pm 1$.

(Aside: this is problem 3(c) on consolidation document 2.)

(d) Summarize and explain the discussion leading up to the introduction of the $\epsilon-\delta$ definition (from "The time has come..." to "This definition is so important..."). What is being clarified in each of these steps?

daily_challenge

Updated 9 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: The statement $|x|<\epsilon$ is true since if $|x|<\epsilon^2$, then one can take the square root of both sides and get $\sqrt{|x|}<\epsilon$. The latter half $x\neq 0$ is true since $|\sin(\frac{1}{x})|$ is undefined for $x=0$, and as well since the range of the sine function is $[-1,1]$, then $|\sin(\frac{1}{x})|\leq 1$. Multiplying these two results in $\sqrt{|x|}\sin\left(\frac{1}{x}\right)<\epsilon$.

b: We have to show that it is possible to choose values n and m large enough that x_1 and x_2 lie in an open interval A containing 0. We begin by referring to the definition of open interval; it is true our open interval A contains zero, then it is also true that since $\epsilon>0$, then $(-\epsilon,\epsilon)\subset A$. We can define a number $z=\frac{1}{\epsilon}$, and define an integer n where $n=\lceil z\rceil$. Therefore, $n>z$, and placing each side to the negative first power results in $\frac{1}{n}<\frac{1}{z}$, and in turn $\frac{1}{z}=\epsilon$. Therefore, $\frac{1}{n}<\epsilon$ and this implies that $\frac{1}{n}\in A$. Adding or multiplying by values greater than 1 within a denominator will only make the resulting fraction smaller, so we can confidently say $\frac{1}{2\pi n+\frac{\pi}{2}}\in A$. Therefore it is entirely possible to find a sufficiently large integer n such that $\frac{1}{2\pi n+\frac{\pi}{2}}=x_1\in A$. An nearly identical with the exception of substituting m for n shows that $\frac{1}{2\pi m+\frac{\pi}{2}}=x_2\in A$. Now we must show that $f(x_1)=1$ and $f(x_2)=-1$. Begin by substituting x_1 into our original function $\sin(\frac{1}{x})$. Therefore, $\sin(\frac{1}{x})=\sin(2\pi n+\frac{\pi}{2})$, and since n is an integer multiplied by the period of sine, then $\sin(\frac{\pi}{2})=1$, therefore $\sin(\frac{1}{x_1}=1$, what we wanted to prove. Identically, we begin by substituting x_2 into $\sin(\frac{1}{x})$. Therefore, $\sin(\frac{1}{x})=\sin(2\pi m+\frac{3\pi}{2})$, and once again since m is an integer multiplied by the period of sine, it is eliminated and then $\sin(\frac{3\pi}{2})=-1$, therefore $\sin(\frac{1}{x_2}=-1$. Proving the "second half" of this man Spivak's claim.

c: I shall simply take a hint from the instructor response and define a function that only has limits at ± 1: $f(x)=\begin{cases}(x-1)(x+1) & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$.

d: In general, we are building up a more concrete definition of a limit as we go further, but going step by step: We begin with the "plushy" definition of a limit given to us by Spivak:

> "The function f approaches the limit L near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal to a."

We begin to toughen this up by turning the variables of this sentence into more recognizable mathematical statements, for example:

> "The function f approaches the limit L near a, if we can make $|f(x) - L|$ as small as we'd like by requiring that $|x - a|$ be sufficiently small, but $x \neq a$."

Next we can give the former absolute value statement something solid to be less than, meaning:

> "The function f approaches the limit L near a, if for every $\epsilon > 0$, we can make $|f(x) - L| < \epsilon$ by requiring that $|x - a|$ be sufficiently small, but $x \neq a$."

We found that we also had to limit our input x values by a certain number δ, and at the same time compress our definition further to get:

> "The function f approaches the limit L near a, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that we can make $|f(x) - L| < \epsilon$ by requiring that $0 < |x - a| < \delta$."

We now have a concrete and uncomfortable to hug definition of a limit.

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) If $|x| < \epsilon^2$, then taking the square root (which preserves the inequality) tells us that $\sqrt{|x|} < \epsilon$, since $\epsilon > 0$. Further, we have that $|\sin(\frac{1}{x})| \leq 1$ for all $x \neq 0$ since the sine function takes values in $[-1, 1]$. Thus, if $|x| < \epsilon^2$, one has

$$\sqrt{|x|}\left|\sin\left(\frac{1}{x}\right)\right| \leq \epsilon \cdot 1 = \epsilon,$$

as desired.

(b) First we justify the claim that we can always find n and m large enough so that x_1 and x_2 lie in an open interval A containing 0.

If A is an open interval containing zero, then A also contains $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ (we showed this in a meeting; it follows from the definition of *open*).

Now consider the number $z = \frac{1}{\epsilon}$, which is just some real number. In particular, we can always find an integer greater than z; for instance, one could apply the ceiling function to z. Call this integer n. (Aside: the ability to find an integer greater than any real number is sometimes called the *Archimedean property*.)

Now, since $n > z$, we have $\frac{1}{n} < \frac{1}{z} = \epsilon$, so $\frac{1}{n}$ is smaller than ϵ and thus $\frac{1}{n} \in A$. Adding a positive quantity to the denominator will only make the result smaller, so it is also true that $\frac{1}{n+\frac{\pi}{2}} \in A$ and likewise that $\frac{1}{2\pi n+\frac{\pi}{2}} \in A$.

Thus we can always find a large integer n so that $x_1 = \frac{1}{2\pi n+\frac{\pi}{2}}$ belongs to any open interval A containing the origin. A similar argument shows that we can always find an m so that $x_2 = \frac{1}{2\pi m+\frac{3\pi}{2}}$ belongs to A.

Now we need only show $f(x) = \sin(\frac{1}{x})$ satisfies $f(x_1) = 1$ and $f(x_2) = -1$. But this follows from the usual properties of the sine function: we know that

$$\begin{aligned}\sin\left(\frac{1}{x_1}\right) &= \sin\left(2\pi n + \frac{\pi}{2}\right) \\ &= \sin\left(\frac{\pi}{2}\right) \\ &= 1,\end{aligned}$$

and likewise that

$$\begin{aligned}\sin\left(\frac{1}{x_2}\right) &= \sin\left(2\pi m + \frac{3\pi}{2}\right) \\ &= \sin\left(\frac{3\pi}{2}\right) \\ &= -1,\end{aligned}$$

This proves the second half of Spivak's claim.

(c) We generalize the above examples: define

$$f(x) = \begin{cases} (x-1)(x+1) & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Then $\lim_{x\to\pm 1} f(x) = 0$ but f has no limit at any other point.

(d) The purpose of this discussion is to move from the "provisional definition" of the limit to one which is mathematically rigorous.

The provisional definition was "The function f approaches the limit ℓ near a, if we can make $f(x)$ as close as we like to ℓ by requiring that x be sufficiently close to a."

This definition is officially meaningless, since the terms "as close as we like" and "sufficiently close" are undefined. This is what Spivak alludes to when he says that he "certainly hopes" you have "criticisms of our definition."

The first improvement is to replace the nonsense phrase "as close as we like" with the statement that $|f(x) - \ell|$ is small. Note that the word "small" is still meaningless.

The next step is to replace the word "small" with a quantitative statement, namely that $|f(x) - \ell| < \epsilon$ for any given real number ϵ.

The third change is to rigorize the phrase "requiring that x be sufficiently close to a" by replacing it with the well-defined statement that $|x - a| < \delta$ but $x \neq a$.

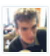
The final change is to make the two conditions "$|x - a| < \delta$ but $x \neq a$" more concise by re-stating them with the equivalent single condition $0 < |x - a| < \delta$.

The result, in its full glory, is the ϵ, δ definition of the limit which we know and love: we say that $\lim_{x\to a} f(x) = \ell$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - \ell| < \epsilon$.

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 8 months ago

Feedback:

> a: The statement $|x| < \epsilon$ is true since if $|x| < \epsilon^2$, then one can take the square root of both sides and get $\sqrt{|x|} < \epsilon$. The latter half $x \neq 0$ is true since $|\sin(\frac{1}{x})|$ is undefined for $x = 0$, and as well since the range of the sine function is $[-1, 1]$, then $|\sin(\frac{1}{x})| \leq 1$. Multiplying these two results in $\sqrt{|x|}\sin\left(\frac{1}{x}\right) < \epsilon$.

I think you meant "The statement $\sqrt{|x|} < \epsilon$ is true" in the first sentence, but otherwise good.

> b: We have to show that it is possible to choose values $n \text{ and } m$ large enough that $x_1 \text{ and } x_2$ lie in an open interval A containing 0. We begin by referring to the definition of open interval; it is true our open interval A contains zero, then it is also true that since $\epsilon > 0$, then $(-\epsilon, \epsilon) \subset A$.

Not "since $\epsilon > 0$"; the claim is that *there exists* some $\epsilon > 0$ so that $(-\epsilon, \epsilon) \subset A$.

> We can define a number $z = \frac{1}{\epsilon}$, and define an integer n where $n = \lceil z \rceil$. Therefore, $n > z$, and placing each side to the negative first power results in $\frac{1}{n} < \frac{1}{z}$, and in turn $\frac{1}{z} = \epsilon$. Therefore, $\frac{1}{n} < \epsilon$ and this implies that $\frac{1}{n} \in A$. Adding or multiplying by values greater than 1 within a denominator will only make the resulting fraction smaller, so we can confidently say $\frac{1}{2\pi n + \frac{\pi}{2}} \in A$. Therefore it is entirely possible to find a sufficiently large integer n such that $\frac{1}{2\pi n + \frac{\pi}{2}} = x_1 \in A$. An nearly identical with the exception of substituting m for n shows that $\frac{1}{2\pi m + \frac{\pi}{2}} = x_2 \in A$.

I think you meant $\frac{1}{2\pi m + \frac{3\pi}{2}} = x_2 \in A$, but otherwise okay.

> Now we must show that $f(x_1) = 1 \text{ and } f(x_2) = -1$. Begin by substituting x_1 into our original function $\sin(\frac{1}{x})$. Therefore, $\sin(\frac{1}{x}) = \sin(2\pi n + \frac{\pi}{2})$, and since n is an integer multiplied by the period of sine, then $\sin(\frac{\pi}{2}) = 1$, therefore $\sin(\frac{1}{x_1} = 1$, what we wanted to prove. Identically, we begin by substituting x_2 into $\sin(\frac{1}{x})$. Therefore, $\sin(\frac{1}{x}) = \sin(2\pi m + \frac{3\pi}{2})$, and once again since m is an integer multiplied by the period of sine, it is eliminated and then $\sin(\frac{3\pi}{2}) = -1$, therefore $\sin(\frac{1}{x_2} = -1$. Proving the "second half" of this man Spivak's claim.

Looks fine to me.

> c: I shall simply take a hint from the instructor response and define a function that only has limits at ± 1:
> $f(x) = \begin{cases} (x-1)(x+1) & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$

Sure.

> d: In general, we are building up a more concrete definition of a limit as we go further, but going step by step: We begin with the "plushy" definition of a limit given to us by Spivak:
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> We concrete and uncomfortable to hug definition of a limit.

The explanation is a bit sparse, but good enough for me.

Overall:

- (a) is 5/6; no revisions needed
- (b) is 5/6; no revisions needed
- (c) is 5/6; no revisions needed
- (d) is 4/6; no revisions needed