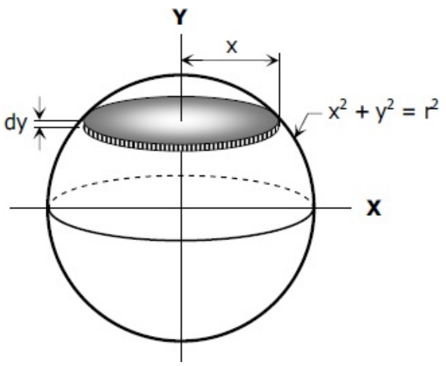
question 2 views

Daily Challenge 21.3

(**Due: Thursday 2/14 at 12:00 noon Eastern**)

(1) Volume of a sphere by slicing.

In the last meeting, we saw how to compute the volume of a sphere of radius r by slicing it into circular cross-sections and performing an integral.



At a distance y from the equator, the cross-section is a circle whose radius is $\sqrt{r^2 - y^2}$. Integrating these circular slices gives

$$\begin{aligned} V &= \int_{-r}^{r} A(y)\, dy \\ &= 2\int_0^r \left(\pi(r^2 - y^2)\right)\, dy \\ &= 2\pi\left[yr^3 - \frac{1}{3}y^3\right]_0^r \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

In the first step, we used that the cross-sectional area is $A(z) = \pi r_{\text{slice}}^2 = \pi\left(r^2 - z^2\right)$ and used symmetry to split the integral over $[-r, r]$ into two copies of the integral from $[0, r]$.

This technique is quite general: for any object with cross-sectional area $A(x)$ at a distance x along some slicing axis, one has

$$\text{volume} = \int_0^h A(x)\, dx.$$

This also generalizes to higher dimensions. An n-dimensional volume can be sliced into $(n-1)$-dimensional cross-sections:

$$V^n = \int_0^h V^{n-1}\, dx.$$

(2) Problem: Volume of 4-ball.

In this problem, you will find the 4-volume of the four-dimensional ball

$$B^4 = \left\{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 \le 1\right\}$$

and compare it to your Monte Carlo result in Python. I will scaffold the calculation for you.

(a) If we slice at a fixed value of x, the cross-section is

$$\left\{(y, z, w) \mid y^2 + z^2 + w^2 \le 1 - x^2\right\}.$$

What three-dimensional shape in (y, z, w) is this (remember that we treat x as a constant, so this equation is of the form $y^2 + z^2 + w^2 \le A$ for some constant A)? What is the volume of the three-dimensional cross-section?

(b) The four-volume is the integral of the three-volumes you found in part (a),

$$V^4 = \int_{-1}^{1} V_{\text{cross}}(x)\, dx.$$

Since this integral involves the quantity $\sqrt{1-x^2}$, make the trig substitution $x = \sin(\theta)$ and $dx = \cos(\theta)\, d\theta$. Plug this in and simply. You should be able to write the integral as

$$V = \frac{4\pi}{3} \int_{-\pi/2}^{\pi/2} \cos^4(\theta)\, d\theta.$$

(c) Use the double-angle formula $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ and some algebra to prove that

$$\cos^4(\theta) = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta).$$

(d) Use your result from (c) to evaluate the integral, and therefore prove that the volume of the four-dimensional ball is

$$V^4 = \frac{1}{2}\pi^2 \approx 4.9348.$$

How close was your Python result yesterday?

daily_challenge

Updated 2 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachinski:
(a): This is a 3-ball with radius $r = \sqrt{1-x^2}$ and volume (3 dimensional cross section) $\frac{4}{3}\pi r^3$.

(b): We begin by writing a formula for $V(x)$; I believe we can find that $V(x) = \frac{4}{3}\pi r^3$. We can plug this into an integral for the volume of the 4-ball, where we see that
$$\int_{-1}^{1} \frac{4}{3}\pi(\sqrt{1-x^2})^3\, dx$$
We shall apply the trig sub $x = \sin(\theta)$ and thus $dx = \cos(\theta)\, d\theta$ and see that
$$\begin{aligned} \cdots &= \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} (\sqrt{1-\sin^2})^3 \cos\ d\theta \\ &= \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \cos(\theta)^3 \cos(\theta) \end{aligned}$$
Where in the second line we substituted $\sqrt{1-\sin^2(\theta)} = \cos(\theta)$; we conclude that
$$\int_{-1}^{1} V(x) = \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \cos^4(\theta)$$

(c): We see by foiling that

$$\begin{aligned} \cos^4(\theta) &= (\cos^2(\theta))^2 \\ &= \frac{1}{4}(1+\cos(2x))(1+\cos(2x)) \\ &= \frac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right) \end{aligned}$$

and substituting the $\cos^2(2x)$ term where by the cosine double angle that $\cos^2(2x) = \frac{1}{2}(1+\cos(4x))$, thus

$$\cdots = \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\right)$$

and by distributing, $\cos^4(\theta) = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$

(d): We must evaluate
$$\frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\, d\theta = \frac{4}{3}\pi\Big[\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\Big]_{-\pi/2}^{\pi/2}.$$
Which is then equal to
$$\cdots = \frac{4}{3}\pi\left((\frac{3}{8}\cdot\frac{\pi}{2} + 0 + 0) - (\frac{3}{8}\cdot\frac{-\pi}{2} + 0 + 0)\right) = \frac{4}{3}\pi(\frac{3}{8}\pi) = 4.9348...$$

Updated 1 month ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

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followup discussions *for lingering questions and comments*