question 2 views

# Daily Challenge 6.3

(**Due: Thursday 5/31 at 12 noon Eastern**)

This is the last daily challenge which will introduce new material from chapter 1; after this, we can review and consolidate before beginning limits.

Review

In the past few challenges, we've proven several useful identities for the logarithm. If $a, b, x, y$ are positive real numbers, then

1. $\log_a(xy) = \log_a(x) + \log_a(y)$. This was proven using the exponential sum rule, $a^{x+y} = a^x a^y$.
2. $\log_a(x^y) = y \log_a(x)$, which was derived from the power-to-a-power rule $(a^x)^y = a^{xy}$.
3. The *change of base formula*, $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$. You proved this by exponentiating the defining relationship of $\log_b(x)$, then taking the log of that equation with base $a$.

The change of base formula is especially interesting, because it suggests that all logarithmic functions are somehow "the same", up to multiplication by a constant: that is, for any pair of bases $a$ and $b$, we have

$$\log_a(x) = (\text{constant}) \log_b(x),$$

so *all logarithmic functions are multiples of one another*.

One might ask: why don't we just fix a "standard" choice of the base of the logarithm, and stop writing the base altogether? If everyone agrees on a convention for the base, then we can simply write $\log$ from now on (with no base specified). Whenever we need a different base besides this standard choice, we can simply convert by multiplying by the appropriate constant.

**Definition**. The *natural logarithm*, which we will simply call the *logarithm*, is the logarithm with base $e$. (Recall that $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, which is roughly $2.78$. You may wish to re-read daily challenge 3.7 if this definition is unfamiliar.)

So, for instance, we have the results

$$\begin{aligned} \log(\sqrt{e}) &= \frac{1}{2}, \\ \log\left(\frac{1}{e^5}\right) &= -5, \\ &\text{etc.} \end{aligned}$$

> **Important notational comment**: from now on, whenever we write $\log(x)$ with no base, we will mean the natural logarithm $\log_e(x)$. This is the notation that mathematicians in academia use, and the one in the AoPS textbook. **Other textbooks and high school sources (including the AP Calculus Exam) use a different convention**: they write $\ln(x)$ for the natural logarithm, and they write $\log(x)$ with no base to mean $\log_{10}(x)$. For instance, to compute the logarithm with base $e$ on the TI-84 calculator, you press the $\ln$ button; the $\log$ button will give you $\log_{10}$.

Finally, you will reasonably wonder: what is so special about the number $e$ that we choose it as the base of our "standard" logarithm?

For now, the best I can say is that we saw in 3.7 that the number $e$ is somehow closely tied to continuous exponential growth: a colony of bacteria that is continuously dividing will roughly grow like $e^x$, whose inverse function is $\log(x)$. One might suspect that this makes the natural logarithm special, since it is somehow the inverse of the notion of continuous exponential growth.

The special-ness of $e$ and the natural logarithm will become much clearer when we develop calculus properly. Recall from the first meeting that the two central operations of calculus, the *derivative* (or *differential*) and *integral*, measure the slope of a line tangent to a curve, and the area under a curve between two endpoints, respectively. The notation for the derivative is $\frac{d}{dx}$ and the notation for the integral is $\int \, dx$.

The function $e^x$ has the very special properties

$$\frac{d}{dx}(e^x) = e^x \quad , \quad \int_{-\infty}^{x} e^y \, dy = e^x.$$

In other words, $e^x$ is the only function which is its own derivative and integral! This rightly makes $e^x$ the most important function in calculus, and singles out the natural logarithm $\log(x)$ (its inverse function) as special: in fact, the log satisfies the equation

$$\frac{d}{dx}(\log(x)) = \frac{1}{x},$$

which is somehow "dual" to the statement $\frac{d}{dx}(e^x) = e^x$.

All of this will seem very mysterious for now. We will first need to develop limits rigorously, since both derivatives and integrals are defined as limiting procedures, before we can understand all of the above at the level of non-bullshitters.

Problem

Answer the following questions about the (natural) logarithm.

(a) What is $\log(e^5)$?

(b) What is $\left(e^{-\log(4)}\right)^2$ ?

(c) Write $\log_3(7)$ in terms of the natural logarithm, using the change-of-base formula.

(d) Solve the equation $e^{2x} - 3e^x - 4 = 0$.

[Scaffold: define a new variable $y = e^x$. Replace all occurrences of $e^x$ in the original equation by $y$; the equation then becomes a quadratic in $y$. Solve for $y$ using your favorite method. Then use the natural logarithm to convert your solutions for $y$ into solutions for $x$).

daily_challenge

Updated 10 months ago by Christian Ferko

**the students' answer,** *where students collectively construct a single answer*

Logan Pachulski:
A: $\log(e^5)$ is simply 5, since $e^5 = e^5$.

B: I begin by simply squaring the given statement, therefore $(e^{-\log(4)})(e^{-\log(4)})$ and by the product rule, this equals $(e^{-2\log(4)})$
In turn by the log-of-a-power rule, $(e^{-2\log(4)}) = (e^{\log(\frac{1}{16})})$ Finally due to the inverse occurences in this statement, it simplifies to $\frac{1}{16}$.

C: Applying the change of base formula to $\log_3(7)$ results in $\log_3(7) = \frac{\log(7)}{\log(3)}$.

D: I apply the usage of an alternate variable suggested by the scaffold and define a variable $y$ for $y = e^x$. I can now substitute the occurences of $e^x$ for $y$ in the original equation to result in the equation that is much easier to handle equation $y^2 - 3y - 4 = 0$ I can now factor, and this results in $(y-4)(y+1) = 0$ By the rational zero theorem, $y = 4, -1$ and therefore$e^x = 4, -1$ However, no value of x in $e^x$ results in negative, therefore the only thing we have to solve for is $e^x = 4$, and finally we get our answer $x = log_e(4)$ by the definition of logarithm.

Updated 10 months ago by Logan Pachulski

**the instructors' answer,** *where instructors collectively construct a single answer*

My responses follow.

(a) What is $\log\left(e^5\right)$?

**Solution**. By definition, $\log\left(e^5\right) = 5$.

(b) What is $\left(e^{-\log(4)}\right)^2$ ?

**Solution**. We first simplify using the log-of-a-power rule, writing $-\log(4) = \log\left(4^{-1}\right) = \log\left(\frac{1}{4}\right)$. Then the quantity inside the parentheses is

$$e^{\log\left(\frac{1}{4}\right)} = \frac{1}{4},$$

and squaring the whole quantity gives the result

$$\left(e^{-\log(4)}\right)^2 = \frac{1}{16}.$$

(c) Write $\log_3(7)$ in terms of the natural logarithm, using the change-of-base formula.

**Solution**. In general, one has $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$. Applying the formula when $b = 3$, $x = 7$, and $a = e$, we see that

$$\log_3(7) = \frac{\log(7)}{\log(3)}.$$

Here, as always, the instances of $\log$ on the right side mean $\log_e$.

(d) Solve the equation $e^{2x} - 3e^x - 4 = 0$.

[Scaffold: define a new variable $y = e^x$. Replace all occurrences of $e^x$ in the original equation by $y$; the equation then becomes a quadratic in $y$. Solve for $y$ using your favorite method. Then use the natural logarithm to convert your solutions for $y$ into solutions for $x$).

**Solution**. Following the scaffold, let $y = e^x$. Then $e^{2x} = y^2$, so the equation becomes

$$e^{2x} - 3e^x - 4 = y^2 - 3y - 4 = 0.$$

This factors as

$$(y-4)(y+1) = 0,$$

which has solutions $y = 4$ and $y = -1$. But $y = e^x$ and the exponential function only outputs positive numbers, so the solution $y = -1$ is spurious and should be discarded. The only true solution (in the reals, at least) is furnished by $y = 4 = e^x$, which means that the value of $x$ which solves the original equation is

$$x = \log(4).$$

Updated 10 months ago by Christian Ferko

**followup discussions** *for lingering questions and comments*