question 2 views

Daily Challenge 11.7

(**Due: Saturday 8/18 at 12:00 noon Eastern**)

I still owe you a proof of the chain rule; I will present a written version in today's challenge, and then discuss the proof more carefully in our Saturday 8/18 meeting.

(1) The chain rule gives $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

We have already seen several cases where it is useful to differentiate a composition of two functions, like $\sin(x^2)$, and that we can do this using the chain rule.

However, I have not yet proven the chain rule for you, a deficiency which I will now rectify.

Theorem. If g is differentiable at a and f is differentiable at $g(a)$, then their composition $f \circ g$ is differentiable at a, and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

Proof. We aim to show that the limit $\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h}$ exists.

We will use a strategy common in mathematics: when you are trying to prove something hard, define a new function and translate the claim you want to prove into a statement about the new function.

Define a new function ϕ by

$$\phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0, \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0 \end{cases}.$$

The first step of our proof will be to show that ϕ is continuous at 0. Let $\epsilon > 0$ be given. By assumption, we have that f is differentiable at $g(a)$, which means that the limit $\lim_{k \to 0} \frac{f(g(a)+k) - f(g(a))}{k}$ exists. This means that, with our ϵ given above, we may find a number δ_1 such that

$$(*) \;\; 0 < |k| < \delta_1 \implies \left| \frac{f(g(a)+k) - f(g(a))}{k} - f'(g(a)) \right| < \epsilon.$$

Next, note that we have also assumed that g is differentiable at a, and hence it is continuous at a. Using continuity, we can find a $\delta_2 > 0$ such that

$$(**) \;\; |h| < \delta_2 \implies |g(a+h) - g(a)| < \delta_1.$$

(Note that we have applied the definition of continuity using δ_1 as the value of the "epsilon" in this case.)

Now suppose $0 < |h| < \delta_2$; we will prove that $|\phi(h) - \phi(0)| < \epsilon$. There are two cases to consider.

1. Case 1: $k = g(a+h) - g(a) \neq 0$

   In this case, we have

   $$\phi(h) = \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+k) - f(g(a))}{k}.$$

   But by (**), we have $k < \delta_1$, and thus (*) guarantees that $|\phi(h) - f'(g(a))| < \epsilon$.

2. Case 2: $g(a+h) - g(a) = 0$.

   In this case, by definition $\phi(h) = f'(g(a))$, so it is automatically true that $|\phi(h) - f'(g(a))| < \epsilon$.

In either case, we have shown that the given choice of δ_2 has the property that $|h| < \delta_2$ implies $|\phi(h) - \phi(0)| < \epsilon$, so by definition ϕ is continuous at zero:

$$\lim_{h \to 0} \phi(h) = f'(g(a)).$$

Now we complete the proof. If $h \neq 0$, we have

$$\frac{f(g(a+h)) - f(g(a))}{h} = \phi(h) \frac{g(a+h) - g(a)}{h}$$

by the definition of ϕ; note that this equation even holds if $g(a+h) - g(a) = 0$, because then both sides are zero. But this implies that

$$\begin{aligned} (f \circ g)'(a) &= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} \\ &= \left(\lim_{h \to 0} \phi(h) \right) \left(\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \right) \\ &= f'(g(a)) \cdot g'(a). \end{aligned}$$

This is what was to be shown. ☐

(2) Problem: some chain rule calculations.

Answer the following questions using the chain rule. You may assume the power rule and linearity of the derivative.

(a) If $g(x) = (f(x))^2$, find a formula for the derivative $g'(x)$ (your formula will involve $f'(x)$).

(b) If $g(x) = (f'(x))^2$, find a formula for the derivative $g'(x)$ (which will involve the *second derivative* $f''(x)$).

(c) Suppose that the function f satisfies $f(x) > 0$ for all x, and in addition, that

$$(f'(x))^2 = f(x) + \frac{1}{(f(x))^3}$$

for all x. Find a formula for $f''(x)$ in terms of $f(x)$. (There is one step where you will need to be careful.)

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Let $f(x)$ be the inner equation and x^2 be the outer equation. We then have that $g'(x) = 2f(x) \times f'(x)$.

b: Once again, let $f'(x)$ (this time around it is already a prime :eyes:) be the outside equation, and x^2 be the outer equation; we then have that $g'(x) = 2f'(x) \times f''(x)$.

c: To find an equation for $f''(x)$, we can begin with the equation we are given; $(f'(x))^2 = f(x) + \frac{1}{(f(x))^3}$. We begin by differentiating each side of this equation; we get on the left side that $((f'(x))^2)' = 2f'(x) \times f''(x)$. We have on the right side via the linearity of derivatives that $\left(f(x) + \frac{1}{(f(x))^3}\right)' = f'(x)$ plus a lovely composition of functions, with the outer being x^{-3} and inner being $(f(x))$; we then have that $\left(f(x) + \frac{1}{(f(x))^3}\right)' = f'(x) + -3(f(x))^{-4} \cdot f'(x)$. We can now set these sides equal to eachother, ie $f'(x) - 3(f(x))^{-4} \cdot f'(x) = 2f'(x) \times f''(x)$. We would like to divide each side by $2f$ " (x), but first we must show that $f'(x) \neq 0$. Suppose by way of contradiction that there exists x where $f'(x) = 0$. We then have by referring to the original equation that $0 = f(x) + (f(x))^{-3}$, or by multiplying each side by $(f(x))^3$, $0 = (f(x))^4 + 1$. There is no real number where $(f(x))^4 = -1$, therefore it must be true that $f'(x) \neq 0$. We receive through this division that $\frac{f'(x) - 3(f(x))^{-4} \cdot f'(x)}{2f'(x)} = f''(x)$. We can turn this into an addition of quotients to make it easier to understand; $\frac{f'(x)}{2f'(x)} - \frac{3(f(x))^{-4} \cdot f'(x)}{2f'(x)} = \frac{1}{2} - \frac{3(f(x))^{-4}}{2}$. We conclude that

$$f''(x) = \frac{1}{2} - \frac{3}{2}(f(x))^{-4}.$$

Updated 7 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

(a) We differentiate both sides of the equation $g(x) = (f(x))^2$. By the chain rule, we have $g'(x) = 2f(x)f'(x)$.

(b) Again by the chain rule, one has $g'(x) = 2f'(x)f''(x)$.

(c) Differentiate both sides of the equation $(f'(x))^2 = f(x) + \frac{1}{(f(x))^3}$ to find

$$2f'(x)f''(x) = f'(x) - 3(f(x))^{-4}f'(x).$$

We are asked to find a formula for $f''(x)$ in terms of $f(x)$, which means that we must eliminate $f'(x)$ from the equation. We would like to divide both sides by $f'(x)$, but to do so, we must check that $f'(x)$ is nonzero.

Return to the defining equation $(f'(x))^2 = f(x) + (f(x))^{-3}$, and suppose by way of contradiction that $f'(x) = 0$. This implies that $(f(x))^4 = -1$, which cannot hold for any real number $f(x)$, so we conclude that $f'(x) \neq 0$.

Since $f'(x) \neq 0$, we may now divide our equation through by $f'(x)$ and then by 2 to arrive at

$$f''(x) = \frac{1}{2} - \frac{3}{2}(f(x))^{-4},$$

which is the desired result.

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*