question 2 views

Daily Challenge 9.3

(**Due: Thursday 7/12 at 12:00 noon Eastern**)

Last time, we offered a definition of the symbols "$\lim_{x\to a} f(x)$", namely "the unique number L, if it exists, with the property that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$".

As we've emphasized, this is logically equivalent to our original definition for the entire sequence of symbols "$\lim_{x\to a} f(x) = L$", except now it allows us to speak about the object "$\lim_{x\to a} f(x)$" as a number rather than the boolean "$\lim_{x\to a} f(x) = L$" as something which is true or false. (Recall that two sentences p and q are said to be *logically equivalent* if p is true if and only if q is true.) This led us to ask what properties these numbers associated with limits satisfy.

Today we'll start to answer this question, investigating the properties of these numbers, which will eventually allow us to show that all polynomials are "continuous" (although we have not yet defined the word "continuous" so this sentence is officially meaningless for now).

(1) Limits can be multiplied by constants.

We actually proved this in session 13, but I think it's worth re-visiting the proof to get more experience with epsilonic arguments.

Proposition. Let f be a real-valued function and $a, c, L \in \mathbb{R}$. If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} (cf(x)) = cL$.

Proof. Let $\epsilon > 0$ be given. We must handle two cases separately, although one of them is a mere annoyance.

The first, annoying, case is if $c = 0$. In this case, $cf(x) = 0$ for all x and $cL = 0$. In this case, we may simply let $\delta = \epsilon$. It is automatically true that $0 < |x-a| < \delta$ implies $|f(x)| < \epsilon$, because $|f(x)| = 0 < \epsilon$ for all x.

The second, more interesting, case is $c \neq 0$. Since we have assumed $\lim_{x\to a} f(x) = L$, choose some δ with the property that $0 < |x-a| < \delta$ implies $|f(x) - L| < \frac{\epsilon}{|c|}$ (note that we have divided by $|c|$). But by multiplying each side of this inequality by the positive quantity $|c|$, we see that this condition is equivalent to

$$0 < |x-a| < \delta \implies |cf(x) - cL| < \epsilon.$$

This proves that $\lim_{x\to a} (cf(x)) = cL$, which was to be shown. $\square$

This strategy -- of using the assumption that a limit exists to bootstrap up the existence of another limit -- is quite common.

(2) Two limits can be added.

I will ask you to write a careful proof of this proposition in today's problem, but I'll begin by scaffolding the argument for you.

By "scaffold", I mean "write some of the non-rigorous ideas that one might use to brainstorm a proof in the exploration phase." The sort of words appearing in the scaffold are *not* legal to appear in a proof. Your job is to re-write the argument in a totally rigorous way.

Proposition. Suppose that $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$. Then $\lim_{x\to a} (f(x) + g(x)) = F + G$.

Scaffold. Begin by fixing some $\epsilon > 0$. Your goal is to prove that $|f(x) + g(x) - F - G|$ can be made smaller than this ϵ by choosing $0 < |x-a| < \delta$ for some δ.

1. First, pick two numbers δ_f and δ_g which guarantee that the outputs of f and g are within *half* of epsilon -- that is, $\frac{\epsilon}{2}$ -- of their limit values F and G.
2. Pick your δ to be the smaller of δ_f and δ_g.
3. Use the triangle inequality to bound $|f(x) + g(x) - F - G|$ by a sum of two terms, each of which is upper-bounded by $\frac{\epsilon}{2}$, which proves the claim.

In the problem, you'll make this scaffold precise.

(3) Two limits can be multiplied.

This is assigned as a problem in AoPS, but it's a bit difficult, so instead I will solve it for you.

Even though it is not a problem you're responsible for, you should still read the proof and try to understand the steps, since a similar strategy will be needed when you prove the squeeze theorem in consolidation document 2.

Proposition. Suppose that $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$. Then $\lim_{x\to a} (f(x)g(x)) = FG$.

Exploration. We want to bound the quantity

$$|f(x)g(x) - FG|$$

to make it smaller than some ϵ. We add and subtract the quantity $f(x)G$ and apply the triangle inequality to find

$$\begin{aligned}|f(x)g(x)-FG| &= |f(x)g(x)-f(x)G+f(x)G-FG| \\ &\leq |f(x)||g(x)-G|+|f(x)-F||G|.\end{aligned}$$

Now, since we have assumed $\lim_{x\to a} f(x)=F$, we may choose δ_1 such that

$$|f(x)-F| < \frac{\epsilon}{2(|G|+1)}$$

whenever $0<|x-a|<\delta_1$. Note that we have added 1 to $|G|$ in the denominator to avoid any problems when $|G|=0$, for which the right side would otherwise be undefined.

Next we need to bound $|f(x)|$, to which we can again apply the triangle inequality to find

$$|f(x)| \leq |f(x)-F|+|F|.$$

Choose δ_2 to make $|f(x)-F|<1$ when $0<|x-a|<\delta_2$. This guarantees that

$$|f(x)||g(x)-G| < (1+|F|)\,|g(x)-G|.$$

Finally, we need to bound $|g(x)-G|$. But since $\lim_{x\to a} g(x)=G$, we can choose δ_3 so that $0<|x-a|<\delta_3$ guarantees

$$|g(x)-G| < \frac{\epsilon}{2(1+|F|)}.$$

Then we should choose our δ to be the *smallest* of these, namely $\delta=\min(\delta_1,\delta_2,\delta_3)$.

Whew, that should do it! Let's begin the formal proof.

Proof. Let $\epsilon>0$ be given. Since $\lim_{x\to a} f(x)=F$ and $\lim_{x\to a} g(x)=G$, we may choose three numbers $\delta_1,\delta_2,\delta_3$ such that

$$\begin{aligned}0<|x-a|<\delta_1 &\implies |f(x)-F| < \frac{\epsilon}{2(|G|+1)}, \\ 0<|x-a|<\delta_2 &\implies |f(x)-F|<1, \\ 0<|x-a|<\delta_3 &\implies |g(x)-G| < \frac{\epsilon}{2(1+|F|)}.\end{aligned}$$

Now we pick $\delta=\min(\delta_1,\delta_2,\delta_3)$. We claim that this satisfies the definition of the desired limit. Indeed, if $0<|x-a|<\delta$, then by repeated application of the triangle inequality we have

$$\begin{aligned}|f(x)g(x)-FG| &= |f(x)g(x)-f(x)G+f(x)G-FG| \\ &\leq |f(x)||g(x)-G|+|f(x)-F||G| \\ &\leq (|f(x)-F|+|F|)\,|g(x)-G|+|f(x)-F||G| \\ &\leq (1+|F|)\,\frac{\epsilon}{2(1+|F|)}+\frac{|G|\epsilon}{2(|G|+1)}.\end{aligned}$$

But clearly $(1+|F|)\,\frac{\epsilon}{2(1+|F|)}=\frac{\epsilon}{2}$ and $\frac{|G|}{|G|+1}<1$ so the right side is strictly smaller than ϵ, which proves the claim. $\square$

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(4) Problem: finishing the addition proof.

In the student response below, convert the scaffold of the limit addition proposition above into a rigorous epsilonic argument.

I have repeated the statement and scaffold below for convenience.

Proposition. Suppose that $\lim_{x\to a} f(x)=F$ and $\lim_{x\to a} g(x)=G$. Then $\lim_{x\to a}(f(x)+g(x))=F+G$.

Scaffold. Begin by fixing some $\epsilon>0$. Your goal is to prove that $|f(x)+g(x)-F-G|$ can be made smaller than this ϵ by choosing $0<|x-a|<\delta$ for some δ.

1. First, pick two numbers δ_f and δ_g which guarantee that the outputs of f and g are within *half* of epsilon -- that is, $\frac{\epsilon}{2}$ -- of their limit values F and G.
2. Pick your δ to be the smaller of δ_f and δ_g.
3. Use the triangle inequality to bound $|f(x)+g(x)-F-G|$ by a sum of two terms, each of which is upper-bounded by $\frac{\epsilon}{2}$, which proves the claim.

daily_challenge

Updated 9 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(Began at 10:04)
Logan Pachulski:

Proof: Suppose that $\lim_{x \to a} f(x) = F$ and $\lim_{x \to a} g(x) = F$. We aim to prove that $\lim_{x \to a} (f(x) + g(x)) = F + G$. Like most limit proofs, we must show that this limit exists. Let $\epsilon > 0$ be given. By the definition of limit applied to our $f(x)$, for every $\epsilon > 0$ there exists a $\delta_f > 0$ such that $0 < |x - a| < delta_f \implies |f(x) - F| < \frac{\epsilon}{2}$. Similarly for $g(x)$, for every $\epsilon > 0$ there exists a $\delta_g > 0$ such that $0 < |x - a| < \delta_g \implies |g(x) - G| < \frac{\epsilon}{2}$. We can now claim there exists an ϵ such that $\delta = \min(\delta_f, \delta_g)$. Therefore, $\lim_{x \to a} (f(x) + g(x)) = F + G$ can be represented that $0 < |2x - 2a| < \delta \implies |f(x) + g(x) - F - G|$ which in turn by the triangle inequality $|f(x) + g(x) - F - G| \leq |f(x) - F| + |g(x) - G|$ Now since we have previously restricted these two absolute values individually sufficiently, then $|f(x) - F| + |g(x) - G| < (\frac{\epsilon}{2} + \frac{\epsilon}{2}) = \epsilon$ Therefore, $|f(x) + g(x) - F - G| < \epsilon$, and our proof is complete. $\Box$

Updated 9 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Proposition. Suppose that $\lim_{x \to a} f(x) = F$ and $\lim_{x \to a} g(x) = G$. Then $\lim_{x \to a} (f(x) + g(x)) = F + G$.

Proof (Christian). Let $\epsilon > 0$.

Since $\lim_{x \to a} f(x) = F$, there exists some $\delta_f > 0$ so that

$$0 < |x - a| < \delta_f \implies |f(x) - F| < \frac{\epsilon}{2}.$$

Likewise, because $\lim_{x \to a} g(x) = G$, there exists some $\delta_g > 0$ so that

$$0 < |x - a| < \delta_g \implies |g(x) - G| < \frac{\epsilon}{2}.$$

Now we choose $\delta = \min(\delta_f, \delta_g)$. We claim that this proves the claim about the limit of the sum. Indeed, if $0 < |x - a| < \delta$, then

$$\begin{aligned} |f(x) + g(x) - F - G| &\leq |f(x) - F| + |g(x) - G| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where we have applied the triangle inequality in the first line and our two limit assumptions in the second line. This proves that $\lim_{x \to a} (f(x) + g(x)) = F + G$. $\Box$

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*