Daily Challenge 21.1
Evaluate each of the following integrals using substitutions of the form $x = \sin(u)$, $x = \cos(u)$, etc. You might need to use that

$$\int \sec(x)\,dx = \log(\sec(x) + \tan(x)), \tag{1}$$

$$\int \csc(x)\,dx = -\log(\csc(x) + \cot(x)). \tag{2}$$

(a) $\int \frac{dx}{\sqrt{x^2-1}}$.

(b) $\int \frac{dx}{x\sqrt{x^2-1}}$.

(c) $\int \frac{dx}{x\sqrt{1+x^2}}$.

(d) $\int \sqrt{1+x^2}\,dx$.

Solution
(a): We let $x = \sec(\theta)$ and thus $dx = \sec\theta\tan\theta\, d\theta$ we then see that

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2(\theta)-1}}\,d\theta \tag{3}$$

$$= \int \frac{\sec\theta\tan\theta}{\tan(\theta)}\,d\theta \tag{4}$$

$$= \int \sec(\theta)\,d\theta \tag{5}$$

$$= \log(\sec(\theta) + \tan(\theta)). \tag{6}$$

We need to substitute something for theta, so we shall assume that there does somehow exist a function arcsec such that $y = \operatorname{arcsec}\sec(y + k2\pi)$ where $k \in \mathbb{Z}$. Then $\theta = \operatorname{arcsec}(x)$ and

$$\int \frac{dx}{\sqrt{x^2-1}} = \log(x + \tan(\operatorname{arcsec}(x))). \tag{7}$$

(b): Once again let $x = \sec(\theta)$ and thus $dx = \sec\theta\tan\theta\, d\theta$. Then we have (omitting the step involving substitution of $\sec(\theta)$ for x and simplifying)

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta}{\sec(\theta)\tan(\theta)}\,d\theta \tag{8}$$

$$= \int 1\,d\theta \tag{9}$$

$$= \theta \tag{10}$$

$$= \operatorname{arcsec}(x). \tag{11}$$

(c): Let $x = \tan(\theta)$ and thus $dx = \sec^2(\theta)\,d\theta$. We then see that

$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec^2(\theta)}{\tan(\theta)\sqrt{1+\tan(\theta)^2}}\,d\theta \tag{12}$$

$$= \int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta = \int \frac{1}{\cos(\theta)} \cdot \frac{\cos(\theta)}{sin(\theta)} \tag{13}$$

$$= \int \csc(\theta) = -\log(\csc(\theta) + \cot(\theta)) \tag{14}$$

We then see that we can substitute $\theta = \arctan(x)$ and conclude that

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\log(\csc(\arctan(x)) + 1/x) \tag{15}$$

(d): Let $x = \tan(\theta)$ and thus $dx = \sec^2(\theta)\ d\theta$. We then see that

$$\int \sqrt{1+x^2}\ dx = \int \sec(\theta)\ d\theta = \log(\sec(\arctan(x)) + x) \tag{16}$$

Daily Challenge 21.2

You will compute the volume of a unit n-ball for any n in Python using Monte Carlo integration.

Write a function which takes an integer n as input and returns the approximate volume of the unit n-ball by proceeding as follows. Pick a large number of sample points. For each sample point, generate n random numbers between -1 and 1, then put then into a list or array $(x_1, \cdots, x_n)$. Store your sample points in a list.

Count how many of the points in your list of samples satisfy $x_1^2 + \cdots + x_n^2 \leq 1$. For instance, you could create a counter, loop through the list, and increment the counter each time a point satisfies the condition. Call this final count something like 'n_inside'.

Return the approximate volume $\frac{n_{\text{inside}}}{n_{\text{samples}}} \cdot 2^n$, since we recall from above that 2^n is the volume of the unit n-cube.

When you have implemented your code, do the following:

1. Output the volumes of the unit n-balls for $n = 1$ up to $n = 12$. Compare to the numbers in this table on Wikipedia (set $R = 1$ in their formulas). Debug if they disagree.
2. For which n is the volume of the unit n-ball largest? You should find it is maximized for $n = 5$. All higher-dimensional beings for $n > 5$ have tiny balls.
3. Upload your code and any supporting documents (e.g. a Jupyter notebook where you perform the check against Wikipedia) to Github.

Solution

Answer on Github here.

Daily Challenge 21.3

In this problem, you will find the 4-volume of the four-dimensional ball

$$B^4 = \left\{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 \leq 1\right\} \tag{17}$$

and compare it to your Monte Carlo result in Python. I will scaffold the calculation for you.

(a) If we slice at a fixed value of x, the cross-section is

$$\left\{(y, z, w) \mid y^2 + z^2 + w^2 \leq 1 - x^2\right\}. \tag{18}$$

What three-dimensional shape in (y, z, w) is this (remember that we treat x as a constant, so this equation is of the form $y^2 + z^2 + w^2 \leq A$ for some constant A)? What is the volume of the three-dimensional cross-section?

(b) The four-volume is the integral of the three-volumes you found in part (a),

$$V^4 = \int_{-1}^{1} V_{\text{cross}}(x)\, dx. \tag{19}$$

Since this integral involves the quantity $\sqrt{1 - x^2}$, make the trig substitution $x = \sin(\theta)$ and $dx = \cos(\theta)\, d\theta$. Plug this in and simply. You should be able to write the integral as

$$V = \frac{4\pi}{3} \int_{-\pi/2}^{\pi/2} \cos^4(\theta)\, d\theta. \tag{20}$$

(c) Use the double-angle formula $\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$ and some algebra to prove that

$$\cos^4(\theta) = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta). \tag{21}$$

(d) Use your result from (c) to evaluate the integral, and therefore prove that the volume of the four-dimensional ball is

$$V^4 = \frac{1}{2}\pi^2 \approx 4.9348. \tag{22}$$

How close was your Python result yesterday?

Solution

(a): This is a 3-ball with radius $r = \sqrt{1 - x^2}$ and volume (3 dimensional cross section) $\frac{4}{3}\pi r^3$.

(b): We begin by writing a formula for $V(x)$; I believe we can find that $V(x) = \frac{4}{3}\pi r^3$. We can plug this into an integral for the volume of the 4-ball, where we see that

$$\int_{-1}^{1} \frac{4}{3}\pi(\sqrt{1 - x^2})^3\, dx \tag{23}$$

We shall apply the trig sub $x = \sin(\theta)$ and thus $dx = \cos(\theta)\, d\theta$ and see that

$$\cdots = \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} (\sqrt{1-\sin^2})^3 \cos\, d\theta \tag{24}$$

$$= \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \cos(\theta)^3 \cos(\theta) \tag{25}$$

Where in the second line we substituted $\sqrt{1-\sin^2(\theta)} = \cos(\theta)$; we conclude that

$$\int_{-1}^{1} V(x) = \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \cos^4(\theta) \tag{26}$$

(c): We see by foiling that

$$\cos^4(\theta) = (\cos^2(\theta))^2 \tag{27}$$

$$= \frac{1}{4}(1+\cos(2x))(1+\cos(2x)) \tag{28}$$

$$= \frac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right) \tag{29}$$

and substituting the $\cos^2(2x)$ term where by the cosine double angle that $\cos^2(2x) = \frac{1}{2}(1+\cos(4x))$, thus

$$\cdots = \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\right) \tag{30}$$

and by distributing,

$$\cos^4(\theta) = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta) \tag{31}$$

(d): We must evaluate

$$\frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\, d\theta = \frac{4}{3}\pi\Big[\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\Big]_{-\pi/2}^{\pi/2}. \tag{32}$$

Which is then equal to

$$\cdots = \frac{4}{3}\pi\left((\frac{3}{8}\cdot\frac{\pi}{2} + 0 + 0) - (\frac{3}{8}\cdot\frac{-\pi}{2} + 0 + 0)\right) = \frac{4}{3}\pi(\frac{3}{8}\pi) = 4.9348... \tag{33}$$

Daily Challenge 21.4

After the sphere, the torus (the shape of a bagel or donut) is perhaps the most important shape in string theory. Like the sphere, it can be generalized to higher-dimensional versions; these are useful in so-called string compactifications.

In this problem, you will compute the volume of a torus with radius a and cross-sectional radius b:

This shape is obtained by rotating a circle of radius b around a line in the same plane as the circle, where a is the distance between the line and center of the circle.

(a) Note that a circle of radius b centered at $(a, 0)$ has the equation

$$(x-a)^2 + y^2 = b^2. \tag{34}$$

Solve this for y, including both the positive and negative signs on the square root; the positive sign gives the upper half of a cross-section of the torus (shown red below), and the negative sign gives the lower half.

(b) First consider the upper half of a cross-section (positive root in part (a)). Slice the torus into a cylindrical shell at a fixed value of x. You may want to draw a picture to help visualize this.

Write down an integral which adds up the volume contributions from these cylindrical shells. You may use that the area of a cylinder of radius r and height h is $2\pi rh$. (Hint: your integral should run from $x = a - b$ to $x = a + b$).

(c) Double the integral you wrote down in (b) to account for the lower half of the cross-section. Make the substitution $u = x - a$ and evaluate the resulting integral (one of the terms vanishes by symmetry; if you can explain why, you need not compute it!).

You should find that the volume of the torus is $2\pi^2 ab^2$.

Solution

Logan Pachuk:

(a): This is a simple bit of algebra;

$$(x-a)^2 + y^2 = b^2 \implies y = \pm\sqrt{b^2 - (x - a^2)^2} \tag{35}$$

(b): We would like to write down a formula to find the area of a cylindrical cross section of the top half of a torus a distance from the center; recall that the a cylinder has wall surface area $2\pi rh$, where we see by the equation we found in (a) that $h = y = \sqrt{b^2 - (x - a^2)^2}$ and simply we simply substitute $r = x$:

$$A(x) = 2\pi x\sqrt{b^2 - (x - a^2)^2} \tag{36}$$

We can now integrate over x from $a - b \to a + b$:

$$\frac{1}{2}V_t(a, b) = 2\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x-a)^2}\, dx \tag{37}$$

Where of course the leading $\frac{1}{2}$ is because this is the top half of the torus.

(c): We begin evaluating this integral by doubling each side of the equation we left off on in (b), since that only considered one half, the top half, of our torus.

$$V_t(a, b) = 4\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x-a)^2}\, dx \tag{38}$$

We now $u - sub$, where we let $u = x - a$ and we don't have to consider the du since it is 1.

$$V_t(a,b) = 4\pi \int_{-b}^{b} (u+a)\sqrt{b^2 - u^2}\, du \tag{39}$$

We now trig-sub where we let $u = b\sin(\theta)$ and thus $du = b\cos(\theta)\, d\theta$. Thus,

$$V_t(a,b) = 4\pi \int_{-\pi/2}^{\pi/2} (b\sin(\theta) + a) b\cos(\theta)\sqrt{b^2(1 - \sin^2)}\, d\theta \tag{40}$$

Recall that $1 - \sin^2(z) = \cos^2(z)$, and pass everything through the square root.

$$V_t(a,b) = 4\pi b^2 \int_{-\pi/2}^{\pi/2} (b\sin(\theta) + a)\cos^2(\theta)\, d\theta \tag{41}$$

We then distribute,

$$4\pi b^2 \int_{-\pi/2}^{\pi/2} b\sin(\theta)\cos^2(\theta)\, d\theta + 4\pi b^2 \int_{-\pi/2}^{\pi/2} a\cos^2(\theta)\, d\theta \tag{42}$$

We see by graphing the first term that it is odd for the range of interest, and thus is equal to zero. We just need to find

$$\cdots = 4\pi b^2 a \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta \tag{43}$$

We also recall the cosine double angle formula $\cos(2x) = 2\cos^2(x) - 1 \implies \cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$

$$\begin{aligned}
\cdots &= 2\pi b^2 a \int_{-\pi/2}^{\pi/2} \cos(2\theta) + 1\, d\theta && (44)\\
&= 2\pi b^2 a \Big[\frac{1}{2}\sin(2\theta) + \theta\Big]_{-\pi/2}^{\pi/2} && (45)\\
&= 2\pi b^2 a(\pi/2 - (-\pi/2)) && (46)\\
&= 2\pi^2 b^2 a && (47)
\end{aligned}$$

:clap:

Daily Challenge 21.5

Try these straightforward exercises.

(a) Find the volume of the region enclosed by the surface resulting when the curve $y = x^3$ on $[0, 2]$ is rotated about the x-axis. (Check that your answer is $\frac{128\pi}{7}$).

(b) Find the volume of the region enclosed by the surface resulting when the curve $y = \cos(x)$ on $[0, \pi/2]$ is rotated about the x axis. (Make sure you get $\frac{\pi^2}{4}$.

(c) You can plot and visualize surfaces of revolution in Wolfram Alpha. Think up another surface of revolution and plot it to get some practice with visualization.

Solution

(a): The surface area of the circle enclosed is

$$A(x) = \pi x^6. \tag{48}$$

Thus,

$V = pi \int_0^2 x^6$

$V = pi(\frac{2^7}{7}) = \frac{128\pi}{7}$

:thumbsup:

(b): We see that the surface area a distance from the origin is

$A(x) = \pi \cos^2(x)\, dx$, thus

$V(x) = \pi \int_0^{\pi/2} \cos^2(x)\, dx$

We then see that since $\cos(2a) = 2\cos^2(a) - 1$, then

$$\cdots = \pi/2 \left(\int_0^{\pi/2} \cos(2x) + \int_0^{\pi/2} 1 \right) \tag{49}$$

$$= \pi/2(1/2(\sin(\pi) - 1) + \pi/2) \tag{50}$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4} \tag{51}$$

(c):

rotate y=cos(x), 0<x<4pi about the y-axis

Browse Examples Surprise Me

Input interpretation:

surface of revolution	$y = \cos(x)$	$x = 0$ to 4π	about the y-axis

Parametric representation of surface:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \cos(\theta) \\ \cos(x_0) \\ x_0 \sin(\theta) \end{pmatrix} \text{ for } 0 < x_0 < 4\pi \text{ and } 0 < \theta < 2\pi$$

Open code

Implicit representation of surface:

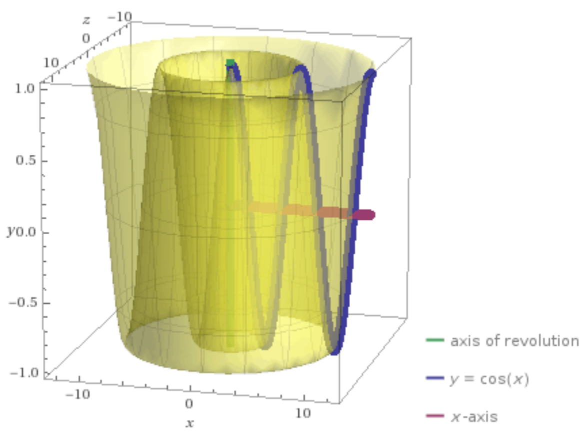
$$y = \cos\left(\sqrt{x^2 + z^2}\right) \text{ for } x^2 + z^2 \leq 16\pi^2$$

Area of surface:

$$\int_0^{4\pi} 2\pi |x| \sqrt{1 + \sin^2(x)}\, dx = 603.261$$

$|z|$ is the absolute value of z

Plot: Show solid



Daily Challenge 21.6

Do the following exercises.

(a) Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and the line $y = 0$ about the y axis (see figure below). Check that you get $\frac{16}{5}\pi$.

(b) Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$. (See figure below). Make sure you get $V = \frac{\pi}{2}$.

Solution

(a) We shall write a formula for the area of the outer area of the cylinder:

$$A(x) = 2\pi x(2x^2 - x^3) \tag{52}$$

Thus, we can write the integral

$$\begin{aligned} V(x) &= 2\pi \int_0^2 2x^3 - x^4 \, dx && (53)\\ &= 2\pi(\Big[\frac{x^4}{2} - \frac{x^5}{5}\Big]_0^2) && (54)\\ &= 2\pi(\frac{40}{5} - \frac{32}{5}) && (55)\\ &= \frac{16\pi}{5} && (56) \end{aligned}$$

(b): We write another formula for the area of the face of a cylinder:

Everything past here is incorrect.

$$A(x) = 2\pi x(x - x^2) \tag{57}$$

We then make the integral

$$\begin{aligned} V(x) &= 2\pi \int_3^4 (x^2 - x^3) \, dx && (58)\\ &= 2\pi\Big[(\frac{x^3}{3} - \frac{x^4}{4})\Big]_3^4 && (59) \end{aligned}$$

Woops, $[3, 4]$ gave a negative result, let's try again with $[0, 1]$ because this is actually the area we are rotating through three dimensional space.

$$2\pi\Big[(\frac{x^3}{3} - \frac{x^4}{4})\Big]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6} \tag{60}$$

Well that isn't the right answer either, so let's look at our work.

Everything up to here has been incorrect, so let's do this with the right radius this time

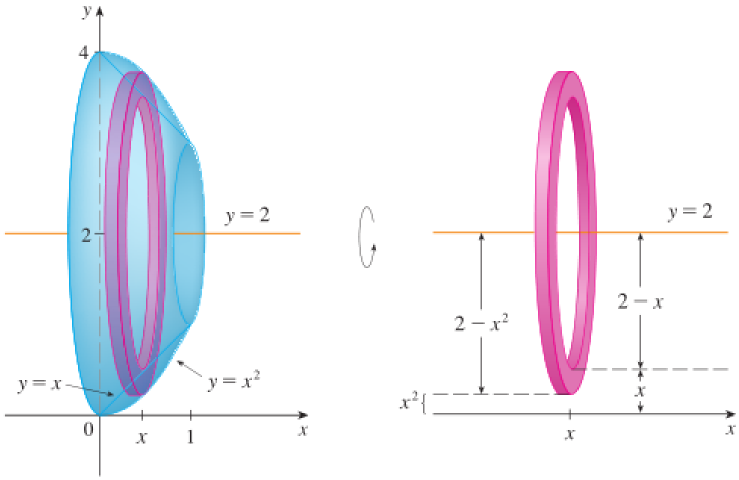
$$A(x) = 2\pi(2 - x)(x - x^2) \tag{61}$$

We then write out an integral:

$$\begin{aligned}
V(x) &= 2\pi \int_0^1 (2-x)(x-x^2) && (62)\\
&= 2\pi \int_0^1 (2-x)(x-x^2) && (63)\\
&= 2\pi \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 && (64)\\
&= 2\pi \frac{1}{4} && (65)\\
&= \frac{\pi}{2} && (66)
\end{aligned}$$

Daily Challenge 21.7

Consider the area between $y = x$ and $y = x^2$. Rotate the area about the line $y = 2$.



Find the volume of the resulting solid. Make sure your computation yields $\frac{8\pi}{15}$.

Solution

We recall the formula for an annulus:

$$A(x) = \pi(R^2 - r^2), \tag{67}$$

where R is the outer radius and r is the inner radius. We see visually that we shall let $R = 2 - x^2$ and $r = 2 - x$ and the intersection of interest is on the interval $[0, 1]$. Thus we will be integrating over

$$\begin{aligned}
V(x) &= \pi \int_0^1 ((2 - x^2)^2 - (2 - x)^2)\, dx && (68)\\
&= \pi \int_0^1 \left(-5x^2 + 4x + x^4\right)\, dx && (69)\\
&= \pi \left[\frac{-5x^3}{3} + 2x^2 + \frac{x^5}{5}\right]_0^1 && (70)\\
&= \pi \left(\frac{-5}{3} + 2 + \frac{1}{5}\right) && (71)\\
&= \pi \left(\frac{-25}{15} + \frac{30}{15} + \frac{3}{15}\right) && (72)\\
&= \frac{8\pi}{15}. && (73)
\end{aligned}$$