Daily Challenge 5.6

(**Due: Sunday 5/27 at 12:00 noon**)

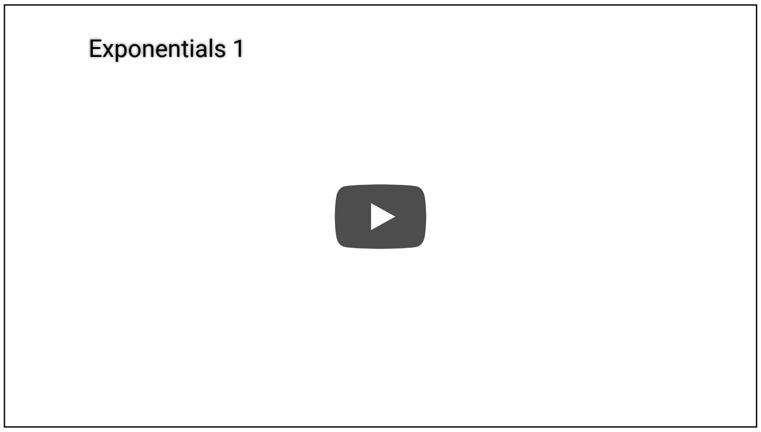
Let's start our discussion of exponential functions!

"Review"

First, our goal will be to rigorously define what we mean by "exponential function". Roughly, we want this definition to describe functions like $f(x) = 2^x$ or $g(x) = \left(\frac{1}{2}\right)^x$ which raise some positive number to a power x.

I've made a few videos walking through the logic of constructing functions like this, starting from our intuition about raising a positive number to a positive *integer* power.

The first video explains why we're doing this (we aspire to be "non-bullshitters", as I explain in the video, which means we want to understand *why* things are true):



The second video explains how we choose the definitions of numbers like 2^0, or 2^{-1} or $2^{2/3}$:

The third video presents a few ways to extend our definition to *irrational* numbers, which means we will have defined 2^x for all real numbers x.

For convenience, I repeat the definition of exponential function given in the video.

Definition. Let f be a function with domain $\mathbb{R}$ that only outputs positive numbers. We say that f is **exponential** if it satisfies the following two properties:

1. f is either strictly increasing, strictly decreasing or constant; and
2. f satisfies the condition $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$.

Problem

Let f be an exponential function.

(a) Show that $f(0)=1$, using the properties given in the definition.

[Aside: it is not enough to claim that any nonzero number raised to the zero-th power equals one -- we must prove it from the definition!]

(b) Suppose f is strictly increasing. Explain why f passes the horizontal line test; give as much detail about your reasoning as you can.

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

1. Using the definition we were given, we can show that it follows the second property of an exponetial function, $f(0)=1$ because $f(0+0)=f(0)\times f(0)$; Since this has only two options to output, 0 and 1, it must output the latter since 0 is not positive.
2. The function f passes the horizontal line test because one can solve for every x value of the graph given a corresponding y value, with no duplicate y values.

(Neither of these answers are very much into detail, but this is how I understand it.)

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My answers follow.

(a) Since f is exponential, we have that $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$. Letting $x=y=0$, this implies that

$$f(0)=f(0)^2.$$

The only solutions to the above equation are $f(0)=0$ and $f(0)=1$. (To see this, think of the number $f(0)$ as a variable, like x. Then the above equation says $x=x^2$, or $x^2-x=0$. Factoring the quadratic, we see that $x(x-1)=0$, so the possible roots are $x=0$ and $x=1$).

However, the definition of exponential function says that f only outputs *positive* values, so $f(0)$ cannot equal 0 since 0 is not positive. Thus we conclude that $f(0)=1$, as claimed. $\square$

(b) The horizontal line test checks whether any horizontal line $y=c$ intersects the graph of a function $f(x)$ at two different points, say $x=a$ and $x=b$. Thus a function fails the horizontal line test if and only if there exist two distinct real numbers a and b such that $f(a)=c$ and $f(b)=c$.

We claim that, if f is strictly increasing, then f can *never* fail the horizontal line test. Indeed, consider two unequal real numbers a and b. Since $a\neq b$, it must either be true that $a>b$ or that $b>a$. If $a>b$, then $f(a)>f(b)$ (by the definition of strictly increasing), and if $b>a$, then $f(b)>f(a)$. Hence it cannot be true that $f(a)=f(b)$, so f cannot fail the horizontal line test. $\square$

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*