question 4 views

Daily Challenge 2.2

(**Due: Wednesday 5/2 at 12:00 noon Eastern**)

For most people -- myself included -- writing a proof is a two-step process:

1. First there is an **exploration** phase where you play around with the statement, try examples, think about non-rigorous arguments, attempt to think of counterexamples and see whether they fail, and so on.

   At this point, you're not trying to be precise yet! You're just experimenting with the theorem to convince yourself that it's true.

   For me, the exploration phase usually involves scribbling down some thoughts on scratch paper.

2. Then comes the **argument**. After you've convinced yourself that the statement is true by experimenting, you attempt to convert your intuition into rigorous mathematics.

   This is the stage where you write the formal proof, beginning with your assumptions and reasoning logically toward the conclusion, but without including any of the non-rigorous arguments from your exploration.

The standard way of teaching mathematics (and what we've been doing) emphasizes argument over exploration: the student is expected to do the exploration on his own, and then "cover his tracks", so to speak, and present only the polished argument at the end.

But I think it might be useful to spend some time working on the "exploration" part of proofs, since this helps with the process of idea generation.

Review

I'll work through an example proof, including my thoughts in the exploration step and then the polished argument at the end.

Theorem. Prove that, if $x \geq 1$, then $x \geq \sqrt{x}$.

Exploration. I think this should be true, since taking the square root of a number greater than 1 will *reduce* that number. For instance, taking the square root of 9 reduces it to 3 and taking the square root of 49 reduces it to 7.

The only case, I think, where taking a square root *increases* a number is if the original number is smaller than one. For instance, the square root of $\frac{1}{9}$ is $\frac{1}{3}$, and $\frac{1}{3}$ is greater than $\frac{1}{9}$. But we don't have to worry about this case, since we are assuming that $x \geq 1$.

Now that I'm convinced that this is true, how should I prove it? I could try a direct proof, which would start with the assumption $x \geq 1$ and apply some algebraic manipulation to arrive at $x \geq \sqrt{x}$. But I don't think that would work, since taking the square root of both sides of $x \geq 1$ gives $\sqrt{x} \geq 1$, which is not what I want.

Ah, wait, but I can *square* both sides of an inequality, assuming both sides are positive! So perhaps I can prove this by going backwards. Here "backwards" means "by contradiction", so my strategy will be to assume the opposite is true and then show that it contradicts the assumption.

This seems likely to work: if our conclusion $x \geq \sqrt{x}$ is false, then it must be true that $x < \sqrt{x}$. Whenever we have two positive numbers a and b with $0 < a < b$, it's also true that $a^2 < b^2$ (I call this "squaring both sides of an inequality", though this is sloppy speech). Thus if $x < \sqrt{x}$, we have $x^2 < x$, which means $x < 1$, a contradiction.

Good, now I'm ready to go back and write up the argument in a precise form.

Argument. Suppose, by way of contradiction, that $x \geq 1$ but it is *not* true that $x \geq \sqrt{x}$. This means that $x < \sqrt{x}$.

Since both x and $\sqrt{x}$ are positive, the inequality $x < \sqrt{x}$ implies that $x^2 < x$. Again, since x is positive, I can divide both sides of the inequality by x, which gives $x < 1$.

But this statement $x < 1$ contradicts our assumption that $x \geq 1$! Therefore, we conclude that it must be true that $x \geq \sqrt{x}$. $\square$

Note that the exploration phase *is much longer and messier* than the argument itself! Writing a proof requires you to think hard, try a bunch of things, and then distill your thoughts into a short and clean argument at the end.

As an aside, this is also why I ask you never to answer "I don't know." Even if you don't have a proof of a statement, you can always include the things you tried in the exploration step (looking for counterexamples, thinking about special cases, etc.).

Problem

Attempt to prove the following, but include both a summary of your exploration phase (the things you thought about when trying to understand the theorem) and your final argument, if you find one.

Theorem. The number $\sqrt{3} - \sqrt{2}$ is irrational.

Hint 1: proceed by contradiction. You may assume that $\sqrt{3}$, $\sqrt{2}$, and $\sqrt{6}$ are irrational.

Hint 2: Why is the following "fake proof" not correct? "We have shown that the difference of two irrational numbers cannot be rational. We know that $\sqrt{3}$ and $\sqrt{2}$ are irrational. Therefore, the difference $\sqrt{3} - \sqrt{2}$ is also irrational."

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Exploration (Corbin) -
First I'm gonna start but contradiction, saying that $\sqrt{3}-\sqrt{2}=\frac{a}{b}$ such that $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. But because these are square roots we can square both sides to eliminate them and then manipulate the equation.

Argument (Corbin) -
Assume by way of contradiction that $\sqrt{3}-\sqrt{2}$ is rational, meaning the equation is $\sqrt{3}-\sqrt{2}=\frac{a}{b}$. This is because all numbers in $\mathbb{Q}$ can be created by the form $\frac{a}{b}$ for some integers a and b. We can start by manipulating the equation to square both sides, unraveling the squares on the left side. $\left(\sqrt{3}-\sqrt{2}\right)^2=\left(\frac{a}{b}\right)^2$ When we FOIL this becomes $3+2-2\sqrt{6}=\frac{a^2}{b^2}$. We can then rearrange this to get $-2\sqrt{6}=5+\frac{a^2}{b^2}$. Next we will divide both sides by -2 to isolate $\sqrt{6}$. This becomes $\sqrt{6}=-\frac{5}{2}-\frac{a^2}{-2b^2}$. The right side is a fraction of non-zero integers, showing that the right side is in $\mathbb{Q}$, but we know that $\sqrt{6}$ is irrational, therefore $\sqrt{3}-\sqrt{2}$ is irrational by contradiction. □

Exploration (Logan). *Activate weekly skip*
Argument (Logan).

Updated 9 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Exploration (Christian). I know that the difference of two irrational numbers can sometimes be rational, like $\sqrt{2}-\sqrt{2}=0$, so it's not immediately obvious why this difference should be irrational.

Roughly speaking, I know that irrational numbers have decimal expansions that never repeat. So if $\sqrt{3}$ and $\sqrt{2}$ each have non-repeating decimal expansions, it seems unlikely that subtracting the two of them would give something which repeats. It's *possible*, of course -- I could cook up some decimal expansion which, when subtracted from that of $\sqrt{3}$, would give something repeating like $0.\overline{123}$ -- but I would be very surprised if $\sqrt{2}$ happened to be that number.

Okay, so I'm not entirely sure why this is true, but I at least have a hunch that it is more likely true that false. How should I show it? The statement I'm trying to prove is of the form "some number cannot be written as a ratio $\frac{a}{b}$ where a and b are integers." To show something cannot be done, it's usually easier to proceed by contradiction.

My plan, then, will be to assume that $\sqrt{3}-\sqrt{2}=\frac{a}{b}$ for some integers a and b, and try to get a contradiction. Since the left side involves square roots, which are undone by squaring, it might be useful to square both sides of this equation. Then I'll try to apply some algebra and obtain a false result. Let's try it!

Argument (Christian). Suppose by way of contradiction that $\sqrt{3}-\sqrt{2}$ is rational. This means that

$$\sqrt{3}-\sqrt{2}=\frac{a}{b}$$

for integers a and b (and $b \neq 0$). If this were true, we could square both sides to find

$$3-2\sqrt{6}+2=\frac{a^2}{b^2},$$

where I have used FOIL on the left side. Re-arranging this equation, we find

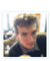
$$\sqrt{6}=\frac{5}{2}-\frac{a^2}{2b^2}=\frac{5b^2-a^2}{2b^2}.$$

The number on the right side is a ratio of integers with non-zero denominator (since we know $b \neq 0$), so the right side is rational. However, the number on the left side is $\sqrt{6}$, which we know is *irrational*. Thus we have arrived at a contradiction! We conclude that $\sqrt{3}-\sqrt{2}$ is irrational, as desired. □

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 11 months ago
Side note: the "exploration" phase is most of what I do all day long.