question 2 views

Daily Challenge 12.2

~~(Due: Tuesday 8/21 at 12:00 noon Eastern)~~
(**Due: Tuesday 8/28 at 11:59 pm Eastern**)

Today we'll discuss the second derivative a bit more.

(1) The derivative of the derivative is the second derivative, $f''(x)$.

We have seen how to differentiate a real-valued function, where by *function* we mean the formal definition: any set of ordered pairs (a, b) of real numbers such that no real number appears as the first element in two distinct ordered pairs (this is the *vertical line test*).

If $f(x)$ is a differentiable function, then we know how to compute its derivative $f'(a)$ at any real number a. Therefore, this defines a new function $f'(x)$ as the set of all ordered pairs $(a, f'(a))$ which map a point on the real line to the slope of the tangent line to f at that point.

Since $f'(x)$ is itself a function, we may ask whether it is differentiable.

Definition. Let $f(x)$ be a differentiable function and let $f'(x)$ be its derivative. We say that f is **twice differentiable at** a if the limit

$$\lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h}$$

exists. If the limit exists, we denote its value by $f''(a)$ or $\left.\frac{d^2f}{d^2x}\right|_a$ and call this value the **second derivative of** f **at** a.

Note in particular that a function may be differentiable, but not twice differentiable; we saw one example,

$$f(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x > 0 \end{cases},$$

which has a first derivative everywhere, namely

$$f'(x) = \begin{cases} 0 & x \le 0 \\ 2x & x > 0 \end{cases},$$

but its second derivative does not exist at $x = 0$:

$$f'(x) = \begin{cases} 0 & x < 0 \\ 2 & x > 0 \\ \text{undefined} & x = 0 \end{cases},$$

(2) We can iteratively take derivatives n times, if the limits exist.

We can generalize the above discussion -- obviously we can just as well consider the *second derivative* $f''(x)$ as a function in its own right, and ask whether the limit

$$f'''(a) = \lim_{h\to 0} \frac{f''(a+h)-f''(a)}{h}$$

exists, and so on. This leads us to the following definition.

Definition. Let f be an n-times differentiable function, let $f^{(1)}$ be its first derivative, $f^{(2)}$ be its second derivative, and so on, up to its n-th derivative $f^{(n)}$. We say that f is $(n+1)$-**times differentiable at** a if the limit

$$\lim_{h\to 0} \frac{f^{(n)}(a+h)-f^{(n)}(a)}{h}$$

exists. If it exists, we denote its value by $f^{(n+1)}(a)$ or $\left.\frac{d^{n+1}f}{dx^{n+1}}\right|_a$ and call it the $(n+1)$**-th derivative of** f **at** a.

As the discussion in section (1) should make clear, *it is not guaranteed that a function will be $(n+1)$-times differentiable just because it is n-times differentiable*. For instance, let

$$f(x) = \begin{cases} 0 & x \le 0 \\ x^{n+1} & x > 0 \end{cases},$$

Then $f(x)$ has n derivatives everywhere, but it is not $(n+1)$ times differentiable at zero. In particular, the function $f^{(n+1)}(x)$ is discontinuous.

Sometimes we would like to explicitly assume that a function has a certain number of derivatives, for instance in the assumptions of a proof. We use the following definition to do so.

Definition. We say that a function f is C^k if f has k continuous derivatives; that is, f is C^k if the function $f^{(k)}$ exists and is continuous everywhere.

For instance, C^1 is the space of differentiable functions whose derivative is continuous. By an abuse of notation, we often write C^0 to denote the space of continuous functions, with no restrictions on differentiability.

(3) Problem: the Leibniz rule.

In this problem, you will generalize the product rule $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$ to the n-th derivative. This is question 9 on CD 3, so **please copy over your solution to Overleaf when you're done.**

Suppose f and g are C^n functions.

(a) Prove that $(fg)'' = f''g + 2f'g' + fg''$.

(b) Generalize part (a) to show that, for any positive integer n,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

This result is called the *Leibniz rule*.

Here the symbol $f^{(n)}$ means to the n-th derivative, as defined in section (2) above. For instance, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$, and so on.

The notation $\binom{n}{k}$ means $\frac{n!}{k!(n-k)!}$. Here the *factorial* of an integer is defined by $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$with the exception of $0! = 1$. For instance, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: We begin by seeing what, by definition of derivative, $(fg)'' = ((fg)')' = ((f'g) + (fg'))'$, where he have proven that $((f'g) + (fg'))' = (f'g)' + (fg')'$. Then by the product rule we have that $(f'g)' + (fg')' = (f''g + f'g') + (f'g' + fg'') = f''g + 2(f'g') + fg''$. We can then conclude that $(fg)'' = f''g + 2(f'g') + fg''$.

b: How about a little bit of exploration :thinking: Suppose that one can take the third derivative of what we proved above in a, ie $(fg)'''$ exists, and by the definition of derivative we have that $((fg)'')' = (f''g + 2(f'g') + fg'')'$. The right side is going to get very ugly very quickly but we proceed; $(f''g)' = f'''g + f''g'''$, $(2(f'g'))' = 2(f'g')' = 2(f''g' + f'g'') = 2f''g' + 2f'g''$, and $(fg'')' = f'g'' + fg'''$. Appropriate prediction, this is *very ugly (algebraic)*. We then have that $(fg)'''$ =

I'm not gonna bother with the exploration, I'm just gonna read and make sure I understand every step of the instructor response.

To go about an inductive proof, we have to show that for some $n = 1$ the proof is true at, it is also true at all subsequent integers. (reading)

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) We begin by taking the first derivative using the ordinary product rule,

$$(fg)' = f'g + g'f.$$

Now we differentiate again, using linearity to split into two terms and applying the product rule again to each term individually.

$$\begin{aligned}(fg)'' &= (f'g + g'f)' \\ &= (f'g)' + (g'f)' \\ &= f''g + f'g' + g''f + g'f' \\ &= f''g + 2f'g' + g''f.\end{aligned}$$

This is the desired result.

(b) We will prove this by induction. Clearly it is true for $n = 2$ by our result in part (a). Thus assume the claim holds for $n = k$; we will attempt to prove that it also holds for $n = k + 1$.

We have that $(fg)^{(k+1)} = \left((fg)^{(k)}\right)'$, but by our induction hypothesis, this is $\left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}\right)'$. We again apply linearity and differentiate each term using the product rule:

$$\left(\sum_{i=0}^{k}\binom{k}{i}f^{(i)}g^{(k-i)}\right)' = \sum_{i=0}^{k}\binom{k}{i}\left(f^{(i+1)}g^{(k-i)} + f^{(i)}g^{(k-i+1)}\right).$$

Now we need some re-indexing trickery. Consider the first term,

$$\sum_{i=0}^{k}\binom{k}{i}\left(f^{(i+1)}g^{(k-i)}\right).$$

To handle the undesirable $(i+1)$ index, we define a new dummy variable $j = i+1$. To express the sum in terms of j, we must do two things: (1) replace each i with $j-1$ in the summand, and (2) change the bounds on the summation accordingly; at the lower bound $i=0$, we have $j=1$, and at the upper bound $i=k$, we have $j=k+1$. This gives

$$\sum_{j=1}^{k+1}\binom{k}{j-1}\left(f^{(j)}g^{(k-j+1)}\right).$$

Since j is just a dummy variable, we may as well change its name back to i and write this term as

$$\sum_{i=1}^{k+1}\binom{k}{i-1}\left(f^{(i)}g^{(k-i+1)}\right).$$

Now we can combine the sums:

$$\begin{aligned}\left(\sum_{i=0}^{k}\binom{k}{i}f^{(i)}g^{(k-i)}\right)' &= \left(\sum_{i=1}^{k+1}\binom{k}{i-1}\left(f^{(i)}g^{(k-i+1)}\right)\right) + \left(\sum_{i=0}^{k}\binom{k}{i}f^{(i)}g^{(k-i+1)}\right)\\ &= \sum_{i=1}^{k}\left(\binom{k}{i-1}+\binom{k}{i}\right)\left(f^{(i)}g^{(k-i+1)}\right) + fg^{(k+1)} + gf^{(k+1)}.\end{aligned}$$

In the second step, I have split off the first and last terms in the sum (i.e. $i=0$ and $i=k+1$). We will consider the other terms, from $i=1$ to k, separately.

We simplify the sum of two binomial coefficients:

$$\begin{aligned}\binom{k}{i-1}+\binom{k}{i} &= \frac{k!}{(i-1)!(k-i+1)!} + \frac{k!}{i!(k-i)!}\\ &= \frac{k!i + k!(k-i+1)}{i!(k-i+1)!}\\ &= \frac{(k+1)!}{i!(k+1-i)!}\\ &= \binom{k+1}{i}.\end{aligned}$$

So all in all, we have

$$\begin{aligned}\left(\sum_{i=0}^{k}\binom{k}{i}f^{(i)}g^{(k-i)}\right)' &= \sum_{i=1}^{k}\binom{k+1}{i}\left(f^{(i)}g^{(k-i+1)}\right) + fg^{(k+1)} + gf^{(k+1)}\\ &= \sum_{i=0}^{k+1}\binom{k+1}{i}\left(f^{(i)}g^{(k+1-i)}\right).\end{aligned}$$

Now we have folded the first and last terms back into the sum. Note that the binomial coefficient is correct for these terms as well, since $\binom{k+1}{0}=1$ and $\binom{k+1}{k+1}=1$.

This proves the inductive step, so the claim holds for all n by induction. $\square$

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*