question 2 views

Daily Challenge 15.6

(**Due: Sunday 9/30 at 12:00 noon Eastern**)

Oftentimes in calculus problems, the static constraint which we differentiate in a "related rates" problem is geometrical, like Pythagoras or a volume formula. But in physics and chemistry, it is equally common to differentiate a non-geometrical relation between various quantities in a system, like the equation of state for a gas. In this problem, we'll see one example which involves time-differentiating the famous ideal gas law of chemistry.

Remark. This is another reason why I dislike the philosophy of "related rates" problems, which place undue emphasis on geometrical constraints and time derivatives. In fact, this technique is much more general; the variable with respect to which we differentiate need not be time. Whenever you have a one-parameter family of equations, it is sensible to differentiate with respect to that parameter, regardless of whether it has the interpretation of time in a dynamical system.

(1) Problem: the ideal gas law.

This is question 17 on CD 4.

The ideal gas law states that a given mass of gas obeys the constraint $PV = kT$, where P is the pressure, V is the volume, T is the temperature, and k is a constant.

Suppose that at time $t = 0$, we have a gas at pressure 100 kilopascals (kPa), volume $1,000\ \mathrm{cm}^3$, and temperature 300 degrees Kelvin.

If the temperature increases by $1\,\frac{\mathrm{K}}{\mathrm{s}}$ and the pressure increases by $2\,\frac{\mathrm{kPa}}{\mathrm{s}}$, then after 60 seconds, what is the volume, and what is rate of change of the volume?

daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

We define P, V and T to be functions in terms of time; we then take the derivative of the given ideal gas law to see that $P'V + PV' = kT'$. We quickly calculate the value of our constant, given the initial values and applying them to the ideal gas law, we see by algebra that $k = \frac{1000 \cdot \mathrm{kPa} \cdot \mathrm{cm}^3}{3K}$. We see that, for an increase of 60 Kelvin and 120 kilo-pascals over 60 seconds, the temperature at 60 seconds is 360 kelvin and the pressure is 220 kilo-pascals. We plug these into the original equation for the ideal gas law and solve for volume (after a few lines of maeth) to see that the volume equals $\frac{6000\mathrm{cm}^3}{11}$. We can then insert these values, including the just calculated volume at 60 seconds, into our differentiated gas law to see that the volume is decreasing by $\frac{1250\mathrm{cm}^3}{363\mathrm{s}}$. :wow:

Updated 6 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Differentiate the ideal gas law and apply the chain rule to find

$$\dot{P}V + P\dot{V} = k\dot{T}.$$

We need to pin down the constant k, but thankfully we are given the initial condition at time $t = 0$:

$$(100\ \mathrm{kPa})\left(1,000\ \mathrm{cm}^3\right) = k\,(300\ \mathrm{K}),$$

so solving for k yields

$$k = \frac{1000}{3}\ \frac{\mathrm{kPa} \cdot\ \mathrm{cm}^3}{K}.$$

We assume that the temperature and pressure increase linearly, so at the end of the 60 seconds, we have

$$P = 100\ \mathrm{kPa} + 2\frac{\mathrm{kPa}}{\mathrm{s}} \cdot 60\ \mathrm{s} = 220\ \mathrm{kPa},$$
$$T = 300\ \mathrm{K} + 1\frac{\mathrm{K}}{\mathrm{s}} \cdot 60\ \mathrm{s} = 360\ \mathrm{K}.$$

It seems that we don't know the volume at time $t = 60$ s, but we can plug the above data into the ideal gas law and solve for V, which yields

$$V\,(t = 60\ \mathrm{s}) = \frac{6000}{11}\ \mathrm{cm}^3.$$

Finally, plugging all of these numbers into our differentiated form of the ideal gas law and solving yields

$$\dot{V}\,(t = 60\ \mathrm{s}) = -\frac{1250}{363}\ \frac{\mathrm{cm}^3}{\mathrm{s}}.$$

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*