30.3

- Define the $f_n(x) = \frac{1}{10^n}\{10^n x\}$ where $\{x\}$ is ~~the dist~~ $|$nearest integer to $x - x|$.

Notice that for $n=1$, distance is maximized at 0.05 + 0.1n for $n \in \mathbb{Z}$; we then notice that, since $\{10^n x\} \leq 1$, then

$$f_n(x) \leq \frac{1}{10^n}$$

but $\frac{1}{10^n}$ has a convergent sum, thus by Weierstrass $f_n(x)$ has a convergent sum. ~~Thus~~ I in turn see that

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \text{ is continuous,}$$

- We are asked to prove the following:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{10^n}\{10^n x\} \text{ is continuous everywhere and differentiable nowhere}$$

We have shown in the previous bullet that $f(x)$ is continous, thus we only need to prove this is ~~is~~ differentiable nowhere. Consider some a s.t. $0 < a \leq 1$, and ~~$a_n = 0.a, 10^{-n}$~~ a has decimal expansion

$$a = 0.a_1 a_2 a_3 \ldots$$

Then, let some $h_m = 10^{-m}$ if $a_m \neq 4, 9$, or if $a_m = 4, 9$, let $h_m = -10^{-m}$. Then see that the definition of differentiability

$$\frac{f(a+h_m) - f(a)}{h_m}$$

can be rewritten as

$$\sum_{n=1}^{\infty} \pm 10^{m-n}\left(\{10^n(a+h_m)\} - \{10^n a\}\right)$$

We see that, for $n > m$, the terms in the parentheses cancel since $10^n h_m$ will be an integer and the other 2 terms cancel.

In the case of $n < m$, we can see that

$$10^n a = \text{integer} + 0.a_{n+1} a_{n+2} a_{n+3} \cdots a_m \cdots$$

and

$$10^n(a + h_m) = \text{integer} + 0.a_{n+1} \cdots (a_m \pm 1) \cdots$$

by our choices for h_m. We now propose that

$$0.a_{n+1} a_{n+2} \cdots a_m \leq \frac{1}{2} \qquad (1)$$

and likewise

$$0.a_{n+1} \cdots (a_m \pm 1) \leq \frac{1}{2}$$

by our choice of h_m. This then allows us to see that

$$\{10^n(a + h_m)\} - \{10^n a\} = \pm 10^{n-m}$$

given (1); thus for $n > m$

$10^{m-n}\left(\{10^n(a+h_m)\} - \{10^n a\}\right) = \pm 1$; then we see that

$$\frac{f(a+h_m) - f(a)}{h_m}$$

is a sum of $m-1$ integers which are ± 1, which then allows us to see that the sequence of the derivatives at a cannot converge since it is built out of odd and even integers.