question 2 views

Daily Challenge 15.2

(**Due: Saturday 9/22 at 12:00 noon eastern**)

Although most high school calculus classes say very little about convexity (just that it is related to the sign of the second derivative), I think it's a good exercise to get your hands dirty with at least one problem that uses the definition.

This is problem 6 in CD 4, so feel free to work there if you like.

(1) Problem: using the definition of convexity.

We showed in a meeting that a function f is *convex* on an interval if, for any a, x, and b in the interval with $a < x < b$, we have

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a}.$$

(a) Show that this definition can be restated in the following equivalent way: a function f is convex on an interval if any only if for all x and y in the interval we have

$$f(tx+(1-t)y) < tf(x)+(1-t)f(y) \quad \text{for } 0<t<1.$$

(b) Prove that, if f and g are convex and f is increasing, then $f \circ g$ is convex.

(c) Give an example where f and g are convex and f is increasing but $g \circ f$ is *not* convex.

(d) Suppose that f and g are twice differentiable. Give another proof of the result of part (b) by considering second derivatives.

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Updated 6 months ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

(a) First we claim that, if $x \in [a, b]$, then $x = (1-t)a + tb$ for some $t \in [0, 1]$. Indeed, if $x \in [a, b]$, then by definition $a \le x \le b$, which means

$$0 \le x - a \le b - a.$$

Define $t = \frac{x-a}{b-a}$; by the above inequality, we have that $0 \le t \le 1$, and

$$x = a + t(b-a) = (1-t)a + tb\,.$$

This establishes what we claimed above.

Now we return to the main problem. By the usual definition of convexity, f is convex on an interval if and only if, for all $x < z < y$ in the interval, we have

$$\frac{f(z)-f(x)}{z-x} < \frac{f(y)-f(x)}{y-x}$$

On the other hand, the points $z \in (x, y)$ are in one-to-one correspondence with the points $t \in (0, 1)$ via the map $z = tx + (1-t)y$ described above. Replacing z using this formula, the definition of convexity becomes

$$\frac{f(tx+(1-t)y)-f(x)}{tx+(1-t)y-x} < \frac{f(y)-f(x)}{y-x}.$$

Thus f is convex on an interval if and only if, for all x, y in the interval, we have

$$f(tx+(1-t)y) < tf(x)+(1-t)f(y).$$

(b) Suppose that f and g are convex and f is increasing. Since g is convex, by part (a) we have

$$g(tx+(1-t)y) < tg(x)+(1-t)g(y)$$

for all x, y in the interval under consideration. Then since f is increasing, applying f to both sides of the above inequality gives

$$f(g(tx+(1-t)y)) < f(tg(x)+(1-t)g(y)),$$

f is itself convex, so another application of part (a) yields

$$f(tg(x)+(1-t)g(y)) < tf(g(x))+(1-t)f(g(y)).$$

Putting together the pieces, we've proven that

$$f(g(tx+(1-t)y)) < tf(g(x)) + (1-t)f(g(y)),$$

which means that $f \circ g$ is convex.

(c) For instance, let $f(x) = 1 + x^2$ and $g(x) = \frac{1}{x}$ and consider any interval on the half-line $x > 0$. Then $f'' = 2$ and $g'' = \frac{2}{x^3}$, so both functions are convex, while f is increasing since $f'(x) = 2x > 0$ if $x > 0$. However, the function $g \circ f$ is

$$g \circ f = \frac{1}{1+x^2}$$

which is not convex everywhere on $x > 0$. Indeed, its derivatives are

$$\begin{aligned}(g \circ f)' &= \frac{-2x}{(1+x^2)^2}, \\ (g \circ f)'' &= \frac{-2(1+x^2)^2 - 2(1+x^2) \cdot 2x \cdot (-2x)}{(1+x^2)^4} \\ &= \frac{6x^2-2}{(1+x^2)^3},\end{aligned}$$

and this is negative for $x < \frac{1}{\sqrt{3}}$, where $g \circ f$ fails to be convex.

(d) By the chain rule,

$$\begin{aligned}(f(g(x)))' &= f'(g(x)) \cdot g'(x), \\ (f(g(x)))'' &= f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).\end{aligned}$$

Since f and g are convex, we have $f'' > 0$ and $g'' > 0$. Further, we have assumed that f is increasing, so $f' > 0$. Then all of the terms in the above expression for $(f(g(x)))''$ are positive, so $f \circ g$ is also convex.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*