Question 1. *Proving that a few limits exist*

In the following questions, prove that the given limits exist directly from the definition (i.e. without resorting to other results that we've proven, like the squeeze theorem or continuity).

(a) Let $c \in \mathbb{R}$ and let $f(x) = c$ be the constant function with value c. Prove that $\lim_{x\to a} f(x) = c$ for all $a \in \mathbb{R}$.

(b) Let $f(x) = x^4$. Prove that $\lim_{x\to 2} f(x) = 16$.

(c) Let $f(x) = \frac{1}{x}$ and $a \in \mathbb{R}$ with $a \neq 0$. Prove that $\lim_{x\to a} f(x) = \frac{1}{a}$.

Solution 1.

(a) Proof: Let $\epsilon > 0$ be given, and define $\delta = \epsilon$. We then have that for $0 < |x - a| < \delta$, it is then true that since $f(x) = c$ then $|f(x) - c| = |c - c| = 0 < \epsilon$.

(b) Exploration: Let us begin by looking at the implication. Let $\epsilon > 0$ be given. our implication is along the lines of $0 < |x - 2| < \delta \implies |x^4 - 16| < \epsilon$. To find what we should set δ equal to, we have to factor the current right side; $|x^4 - 16| = |(x^2 - 4)(x^2 + 4)| = |(x - 2)(x + 2)(x^2 + 4)|$ We can restrict our foremost factor, ie $|x - 2| < 1$, and in turn restrict x, so $1 < x < 3$. We can then place 3 into our other two (more annoying to work with) factors and get that $65|x - 2| < \epsilon$, and in turn $|x - 2| < \frac{\epsilon}{65}$, allowing us to know that $\frac{\epsilon}{65}$ is a valid delta, and we can now continue to the proof.

Proof: Let $\epsilon > 0$ be given. We can then select $\delta = \min(1, \frac{\epsilon}{65})$. We then know that $0 < |x-2| < \delta \leq 1$, and this restricts x to less than or equal to 3. We have latter portion of the statement implied by the limit we are trying to prove; $|x^4 - 16| < \epsilon$. We factor: $|x^4 - 16| = |(x - 2)(x + 2)(x^2 + 4)| \leq |(x - 2)(3 + 2)(3^2 + 4)| = 65|x - 2|$. We then refer to our other option for selection of δ, $\frac{\epsilon}{65}$, and this entailing that $|x - 2| < \frac{\epsilon}{65}$. It is then true that $65|x - 2| < 65 \times \frac{\epsilon}{65} = \epsilon$, showing that thanks to our chosen δ, we have shown that $|x^4 - 16| < \epsilon$ when $0 < |x - 2| < \delta = \min(1, \frac{\epsilon}{65})$.

(c) Proof: Let $\epsilon > 0$ be given. Identical to all limit proofs, we must prove $\lim_{x\to a} f(x) = \frac{1}{a}$ by showing that $|\frac{1}{x} - \frac{1}{a}|$ can be bounded by restricting $|x - a|$. We can begin by stating that $\delta = \min\left(\frac{|a|}{2}, \frac{\epsilon|a|^2}{2}\right)$, but more relevant in the short term is $|x - a| < \frac{|a|}{2}$ since this makes it such that our domain restriction doesn't include 0.

To be more detailed, $|x-a| < \frac{|a|}{2}$ is equivalent to (by the reverse triangle inequality) $|a|-|x| < \frac{|a|}{2}$, and in turn $\frac{|a|}{2} < |x|$, and as Spivak requests, we find by placing each side to the negative first power that $\frac{1}{|x|} < \frac{2}{|a|}$.

We can now go on to the what we aim to prove, $|f(x)-f(a)| < \epsilon$, in turn $|\frac{1}{x} - \frac{1}{a}| < \epsilon$. Algebraically we know that $|\frac{1}{x} - \frac{1}{a}| = |\frac{x-a}{ax}|$, and in turn it is true algebraically that $|\frac{1}{x} - \frac{1}{a}| = |x - a| \times \frac{1}{|a|} \times \frac{1}{|x|}$. We know thanks to earlier preparations that $|x - a| < \frac{\epsilon|a|^2}{2}$ and that $\frac{1}{|x|} < \frac{2}{|a|}$, so we can conclude $|x - a| \times \frac{1}{|a|} \times \frac{1}{|x|} < \frac{\epsilon|a|^2}{2} \times \frac{1}{|a|} \times \frac{2}{|a|} < \epsilon$. $\square$

Question 2. *Some properties of limits*
(a) Let $f(x)$ and $g(x)$ be two real-valued functions such that $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$. Prove that

$$\lim_{x\to a} (f(x) + g(x)) = F + G. \tag{1}$$

(b) Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$. Prove that $L_1 = L_2$.

(c) Let f and g be real-valued functions and $a, b, c \in \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = c$. Show that it is *not* necessarily true that $\lim_{x\to a} (g(f(x))) = c$, by constructing a counter-example

(d) Same assumptions as in part (c). Prove that the conclusion $\lim_{x\to a} (g(f(x))) = c$ *is* true if we also assume $g(b) = c$.

Solution 2.
(a) Suppose that $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$. We aim to prove that $\lim_{x\to a} (f(x) + g(x)) = F + G$.

Like most limit proofs, we must show that this limit exists. Let $\epsilon > 0$ be given. By the definition of limit applied to our $f(x)$, there exists a $\delta_f > 0$ such that $0 < |x - a| < \delta_f \implies |f(x) - F| < \frac{\epsilon}{2}$ Similarly for $g(x)$, there exists a $\delta_g > 0$ such that $0 < |x - a| < \delta_g \implies |g(x) - G| < \frac{\epsilon}{2}$

Now we define $\delta = \min(\delta_f, \delta_g)$. Suppose that $0 < |x-a| < \delta$ and now consider $|f(x)+g(x)-F-G|$ which in turn by the triangle inequality $|f(x) + g(x) - F - G| \leq |f(x) - F| + |g(x) - G|$ Now since we have previously restricted these two absolute values individually sufficiently, then $|f(x) - F| + |g(x) - G| < (\frac{\epsilon}{2} + \frac{\epsilon}{2}) = \epsilon$ We have now successfully shown that $|f(x) + g(x) - F - G| < \epsilon$ and our proof is complete. $\square$

(b) Suppose that that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$, but by way of contradiction suppose $L' \neq L$. We can treat the limits of these functions with the same ϵ, in which case I shall subsequently define $\epsilon = \frac{|L'-L|}{2}$.

By the definition of limit, there exists a δ_1 such that $0 < |x - a| < \delta_1 \implies |f(x) - L| < \epsilon$. Similarly, there exists a δ_2 such that $0 < |x - a| < \delta_2 \implies |f(x) - L'| < \epsilon$. We can now unite these implications by defining that $\delta = \min(\delta_1, \delta_2)$ and supposing there exists a value x_0 such that $0 < |x_0 - a| < \delta$ is true.

We aim to show that the value x_0, while satisfying the previously described condition, does not satisfy the one it implies. Recall $\epsilon = \frac{|L'-L|}{2}$, and therefore $2\epsilon = |L' - L|$. We can now apply the first of the most important rules in algebra, adding zero. We can both add and subtract $f(x_0)$ within this absolute value to receive $2\epsilon = |(L' + f(x_0)) - (L + f(x_0))|$ By the triangle inequality, $|(L' + f(x_0)) + -(L + f(x_0))| \leq |f(x_0) + L'| + |f(x_0) + L|$ We can now use the implications made by the original limits to see that $|L' + f(x_0)| < \epsilon$ and likewise $|L + f(x_0)| < \epsilon$, therefore these are collectively less that 2ϵ, ie $|L' + f(x_0)| + |L + f(x_0)| < 2\epsilon$ Comparing the start and end of this "chain" shows that $2\epsilon < 2\epsilon$. This is false for obvious reasons, therefore L must equal L'. $\square$

(c) It is not necessarily true that $\lim_{x\to a} (g(f(x))) = c$ since we can suppose $f(x) = 0$ and $g(x)$ is the piecewise function $g(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, and in this case $\lim_{x\to a} (g(f(x))) \neq \lim_{x\to f(a)} g(x)$.

(d) Proof: Suppose $g(b) = c$ and $\lim_{x\to b} g(x) = c$ and therefore the function g is continuous at the point b. Given $\epsilon > 0$ we can then find a δ_g such that $0 \leq |x-b| < \delta_f \implies |g(x)-c| < \epsilon$. Next we can suppose $\lim_{x\to a} f(x) = b$ and in turn we can find a δ_f such that $0 < |x-a| < \delta_f \implies |f(x)-b| < \delta_g$. This proves that $\lim_{x\to b} g(f(x)) = c$ as we have shown $f(x)$ can be "fed into" $g(x)$ while remaining true. $\square$

Question 3. *The function kitchen*

For each question below, cook up an example function which has the stated properties.

(a) Find an example of a function $f(x)$ with domain $\mathbb{R}$ for which $\lim_{x\to 0^+} f(x)$ exists but $\lim_{x\to 0^-} f(x)$ does not exist.

(b) Cook up a function $f(x)$ with domain $\mathbb{R}$ that is continuous nowhere, but where $|f(x)|$ is continuous on all of $\mathbb{R}$.

(c) Find a function $f(x)$ defined on $\mathbb{R}$ which is continuous at the two points $x = -1$ and $x = 1$, but is discontinuous at every other point.

Solution 3.

a: $f(x) = \begin{cases} 2 & \text{if } x > 0 \\ 1 & \text{if } x \in \mathbb{Q} \text{ and } x < 0 \\ 0 & \text{if } x \notin \mathbb{Q} \text{ and } x < 0 \end{cases}$

b: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$

c: $f(x) = \begin{cases} (x-1)(x+1) & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Question 4. *The squeeze theorem*
In this problem, we will prove the squeeze theorem, which is copied below for reference.

Theorem. Let $a \in \mathbb{R}$ and let f, g, h be real-valued functions such that $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ in an open interval containing a (note that the inequality need not hold at a itself). If

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \tag{2}$$

then $\lim_{x \to a} g(x) = L$ as well.

I will scaffold the proof for you.

(a) First we will prove a useful inequality as an intermediate step. Suppose that the assumptions of the squeeze theorem hold. Then prove that, for all $x \neq a$ in the open interval containing a,

$$|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right). \tag{3}$$

(b) Now we can begin the proof itself. Let $\epsilon > 0$ be given. Since we have assumed $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, we can choose two numbers $\delta_f, \delta_h > 0$ such that

$$\begin{aligned} 0 < |x - a| < \delta_f &\implies |f(x) - L| < \epsilon, \\ 0 < |x - a| < \delta_h &\implies |h(x) - L| < \epsilon. \end{aligned} \tag{4}$$

Complete the proof.

(c) Use the squeeze theorem to prove that

$$\lim_{x \to 0} \left(\frac{1 - \cos(x)}{x} \right) = 0. \tag{5}$$

You may use the inequality we derived in session 14 to prove that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

Solution 4.
(a) Case one is simple enough, we already know that $g(x) \leq h(x)$ by the beginning of the theorem, and it follows algebraically that subtracting L from both sides is still true, therefore it is true that $g(x) - L \leq h(x) - L$.

Case two took a wee bit more creative thought; we can take the clearly true statement $L = L$ and subtract an inequality we know, in this case $f(x) \leq g(x)$. Via this we receive the inequality we are interested in, $L - g(x) \leq L - f(x)$.

Awesomely enough, we then know that we can compact these phrases into the following inequality:

$$|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right) \tag{6}$$

This is true since in the former case ($|g(x) - L| \leq |h(x) - L|$) as we know that when $g(x) \geq L$, also relatedly and importantly $g(x) - L \geq 0$ it is then true by the definition of absolute value that $|g(x) - L| \leq |h(x) - L|$. The latter case is also true by the definition of absolute value since, when $g(x) \leq L$ and $L - g(x) \leq L - f(x)$, then $|g(x) - L| \leq |f(x) - L|$. We know that $|g(x) - L| \leq |h(x), f(x) - L|$, and it is then true that

$$|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right) \tag{7}$$

(b) We have assumed in our hypothesis that the functions f and g have limits, therefore for all $\epsilon > 0$ we have that there exists δ_f and δ_g such that $0 < |x - a| < \delta_f \implies |f(x) - L| < \epsilon$ and as well $0 < |x - a| < \delta_h \implies |h(x) - L| < \epsilon$. We can then set δ_g (the delta we are using for the g function's limit) to be equal to the minimum of these two deltas, ie $\delta_g = \min(\delta_f, \delta_h)$. We can now show that $|g(x) - L| < \epsilon$. Suppose that $0 < |x - a| < \delta_g$. We refer back to our *useful* inequality and know that $|g(x) - L| \leq \max(|h(x) - L|, |f(x) - L|)$. However, since $\delta_g \leq \delta_f, \delta_h$, it is in then true that $|h(x) - L| < \epsilon$ and $|g(x) - L| < \epsilon$, allowing us to conclude that since $|g(x) - L| \leq \max(\epsilon, \epsilon)$, in turn $|g(x) - L| < \epsilon$ and therefore $\lim_{x \to a} f(x) = L$. $\square$

(c) Our end goal is to show that the

$$\lim_{x \to 0} \left(\frac{1 - \cos(x)}{x} \right) = 0.$$

Begin by beginning the function appearing in this limit as $g(x)$. We would like to prove said equation via the squeeze theorem, so we can start by finding one function that is greater than or equal to $g(x)$ and one less than to, ie we must find $f(x)$ and $h(x)$ such that

$$f(x) \leq g(x) \leq h(x)$$

We can multiply the numerator and denominator of $g(x)$ by $1 + \cos(x)$ to get that

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} \frac{1 - \cos^2(x)}{x\,(1 + \cos(x))}$$

We begin the search for suitable $f(x)$ and $h(x)$ by referring to the comparison of three triangles' areas, which stated

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1.$$

We know by Pythagoras that $\sin(x) = \sqrt{1 - \cos^2(x)}$, therefore

$$\cos(x) \leq \frac{\sqrt{1 - \cos^2(x)}}{x} \leq 1.$$

All quantities here are positive, therefore we can square all sides and receive

$$\cos^2(x) \leq \frac{1 - \cos^2(x)}{x} \leq 1.$$

Next up, we can multiply x through and divide by $1 + \cos(x)$ to finally receive the functions we are interested in,

$$\underbrace{\frac{x \cos^2(x)}{1 + \cos(x)}}_{f(x)} \leq \frac{1 - \cos^2(x)}{x(1 + \cos(x))} \leq \underbrace{\frac{x}{1 + \cos(x)}}_{h(x)}.$$

We can now see that both the left and right sides are quotients of functions we have shown in the past to be continuous, therefore the quotient is continuous; $\lim_{x \to 0} \frac{x \cos^2(x)}{1+\cos(x)} = 0$ and $\lim_{x \to 0} \frac{x}{1+\cos(x)} = 0$. By the squeeze theorem it is then true that $\lim_{x \to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = 0$. $\square$

Question 5. *Using the power of continuity*
Each of the following questions involves proving a result about continuous functions. You may use the intermediate value theorem, boundedness theorem, and extreme value theorem.

(a) Show that if f is continuous on $[a, b]$, then f attains a minimum on $[a, b]$.

(b) Suppose that f is continuous on $[a, b]$ and $f([a, b]) \subseteq \mathbb{Q}$ (that is, the image of $[a, b]$ under f lies in the rationals). What can we conclude about f?

(c) Let f be a continuous function with domain $[0, 1]$ and range $[0, 1]$. Prove that f must have a *fixed point*: that is, there exists some $a \in [0, 1]$ such that $f(a) = a$.

Solution 5.
(a) We must show that if f is continuous on $[a, b]$, then f attains a minimum on the closed interval $[a, b]$. We begin by defining that $g(x) = -f(x)$. It is then true by the extreme value theorem that within the interval $[a, b]$ the function g attains a maximum, and therefore f attains a minimum in this interval.

(b) We begin with the information that f is continuous on the interval $[a, b]$. Suppose by way of contradiction that f is not a constant function. We can then say that since this interval is not a single point, then there is some $y \in f[a, b]$ such that y is irrational. We then know, assuming $a \neq b$, there exists a value $c \in [a, b]$ such that $f(c) = y$. This contradicts the information that we are given that $f[a, b] \in \mathbb{Q}$. Therefore, f must be constant.

(c) f is a continuous function with the domain and range $[0, 1]$. First we consider the case where either $f(0) = 0$ or $f(1) = 1$. This then makes it automagically true that there exists $f(x) = x$ in this range. If said case is not is not true then we define a new function, $g(x) = f(x) - 1$ where $g(0) > 0$ and $g(1) < 1$. Then by the intermediate value theorem there exists a point $c \in [0, 1]$ such that $g(c) = 0$. This then implies that $f(c) = c$ as desired.

Question 6. *Stars over Babylon*
Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ , with } p, q \in \mathbb{Z} \text{ in lowest terms} \end{cases} . \tag{8}$$

This function is called the *stars over Babylon* or *Thomae's function.*

(a) Prove that $\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$.

(b) Using your result from (a), show that $f(x)$ is continuous at every irrational point and discontinuous at every rational point in $\mathbb{R}$.

Solution 6.
(a) Proof: We begin as the hint suggests; let $\epsilon > 0$ be given. We can then choose a number N such that $N \in \mathbb{Z}$ so that $\frac{1}{N} < \epsilon$. It is then true that the list of rational numbers with denominators up to N is finite. In turn we can choose a value $\frac{p}{q}$ in this set that is the closest to our target value a. We can then say $\delta = |\frac{p}{q} - a|$, and it is then true that if $0 < |x - a| < \delta$, then either x is irrational or $\frac{1}{M} < \frac{1}{N} < \epsilon$.

(b) We have shown in our previous proof that if x is irrational then $\lim_{x \to a} f(x) = 0 = f(x)$, therefore $f(x)$ is continuous as irrational x. Similarly, it is true that for rational x that $\lim_{x \to a} f(x) = 0 \neq f(x)$ since f doesn't output zero at any rationals.

Question 7. *Continuity of sine and cosine*
In this problem, we will prove that the sine and cosine functions are continuous everywhere.

(a) Use the fact that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ (which we proved in session 14 using the squeeze theorem) and the properties of limits of products (discussed in daily challenge 9.3) to show that $\sin(x)$ is continuous at $x = 0$.

(b) Using your result from (a), along with the identity $\sin^2(x) + \cos^2(x) = 1$, show that $\cos(x)$ is continuous at $x = 0$.

(c) Use the angle-addition formulas to prove that sine and cosine are continuous everywhere, by showing that

$$\begin{aligned}\lim_{h\to 0}\sin(x+h) &= \sin(x),\\ \lim_{h\to 0}\cos(x+h) &= \cos(x).\end{aligned} \tag{9}$$

Solution 7.
(a) Proof: We now it is true algebraically that $\lim_{x\to 0}\sin(x) = \lim_{x\to 0}(\frac{\sin(x)}{x}\times x)$ since we need not worry about $x = 0$, and we also know that $\lim_{x\to 0}(\frac{sin(x)}{x}\times x) = \lim_{x\to 0}(\frac{\sin(x)}{x})\times\lim_{x\to 0}(x)$ as we have also shown previously that the limit of a product is the product of the limits. In turn, we have proven that the former limit ($\lim_{x\to 0}(\frac{\sin(x)}{x})$) exists by the squeeze theorem, and that $\lim_{x\to 0} x$ exists since x is a polynomial. In turn, we can conclude that $\lim_{x\to 0}\sin(x) = 1\times 0 = 0$. □

(b) Our end goal is to show that $\cos(x)$ is continuous at 0, so we must prove that $\lim_{x\to 0}\cos(x) = \cos(0)$. We begin by noting that the hint suggests using the squeeze theorem to "trap" the $\cos(x)$ between $\cos^2(x)$ and 1, so I hop over to desmos to visualize a domain of x that will work for our purposes, including containing zero. The closed interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ works beautifully.

We can now claim that for the domain $[-\frac{\pi}{2},\frac{\pi}{2}]$, it is true that $\cos^2(x)\le\cos(x)\le 1$. We have previously shown that $\sin(x)$ is continuous at 0, and we know by Pythagoras that $\cos^2(x) = 1-\sin^2$, where in turn $\lim_{x\to 0}\cos^2(x) = 1-\lim_{x\to 0}\sin(x)$, and since it is true that the sine function is continuous at zero then we know thanks to this that $\lim_{x\to 0}\cos(x) = \cos(0) = 1$. It is also true that the zero degree polynomial $f(x) = 1$ is continuous at all points, since it is a polynomial. We can now conclude that since $1\le\lim_{x\to 0}\cos(x)\le 1$, it must be true that $\lim_{x\to 0}\cos(x) = 1$ by the squeeze theorem. □

c: We now by the angle addition formula of sine that $\lim_{h\to 0}\sin(x+h) = \sin(x)\lim_{h\to 0}\cos h + \cos(x)\lim_{h\to 0}\sin(h)$, and in turn thanks to previously proving that $\lim_{x\to 0}\sin(x) = \sin(0) = 0$, and $\lim_{x\to 0}\cos(x) = \cos(0) = 1$ it is true that $\sin(x)\lim_{h\to 0}\cos h + \cos(x)\lim_{h\to 0}\sin(h) = \sin(x)$, therefore $\lim_{h\to 0}\sin(x+h) = \sin(x)$ and proving our first claim. Almost identically, $\lim_{h\to 0}\cos(x+h) = \cos(x)\lim_{h\to 0}\cos(h) + \sin(x)\lim_{h\to 0}\sin(h) = \cos(x)$, allowing us to conclude that the sine and cosine functions are continuous everywhere. □

Question 8. *Discontinuity and the converse of IVT.*
(a) Define a function f by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0 \end{cases}. \tag{10}$$

Show that f is not continuous on $[-1, 1]$.

(b) Show that f satisfies the *conclusion* (not the hypotheses) of the intermediate value theorem on $[-1, 1]$. That is, show that if f takes on two values somewhere on $[-1, 1]$, then it takes on every value in between.

(c) Now consider some different function g. Suppose that g also satisfies the *conclusion* of the intermediate value theorem, and that g takes on each value *only once*. Prove that g is continuous.

Solution 8.
(a) Simply by looking at this piecewise function, we can see that we must show that f is discontinuous at 0 (just by the way f is enticingly constructed). To do so, we have to somehow show that $\lim_{x\to 0} f(x) \neq f(x)$. As well, by looking at Desmos one can see that the function f begins increasingly rapidly oscillating as $x \to 0$. Since this function has a range of $[1, -1]$ for the domain $[1, -1]$, we can simply set epsilon sufficiently small, after all by the definition of limit it only need be true that $\epsilon > 0$; we shall set $\epsilon = \frac{1}{2}$.

Suppose by way of contradiction that there exists δ such that $|f(x) - L| < \epsilon$ is true for the domain $[-\delta, \delta]$ that is implied in this limit. We can exploit the periodicity of the sine function and see that that we can choose integers m and n sufficiently large such that $x_1 = \frac{1}{2\pi n + \frac{\pi}{2}}$ and $x_2 = \frac{1}{2\pi m + \frac{3\pi}{2}}$ are in our domain $[-\delta, \delta]$. We can then see regardless of m and n then $f(x_1) = 1$ and $f(x_2) = -1$, contradicting and showing us that as $x \to 0$ the limit does not exist, and therefore f is not continuous on $[1, -1]$.

(b) To show that "if f takes on two values somewhere $[-1, 1]$, then it takes on every value in between," we must consider the potential placements of two values a and b.

First let $[a, b]$ be a non-null subset of $[-1, 1]$. First if $0 \notin [a, b]$, then it is automatically true that f is continuous for $[a, b]$ as f is only discontinuous at 0. Therefore by the intermediate value theorem, we can choose any number y between $f(a)$ and $f(b)$ and there exists some $c \in (a, b)$ such that $f(c) = y$, pretty generic for now.

In the case where $0 \in [a, b]$, we must show that for y in between $f(a)$ and $f(b)$, there exists a $c \in (a, b)$ such that $f(c) = y$. We must show that this c exists. We have $-1 \leq f(a), f(b) \leq 1$, then in turn $-1 \leq y \leq 1$. We can use the inverse sine function to do our bidding here as we are operating within its domain; apply $\sin^{-1}(y)$ to get that there exists some number c' such that $\sin(c') = y$. Once again we can exploit the periodicity of the sine function and let $c = \frac{1}{2\pi n + c'}$ where n is large enough such that $c \in [a, b]$. We then have that $f(c) = \sin(2\pi n + c') = \sin(c') = y$, and therefore $f(c) = y$ and the conclusion of the intermediate value theorem is true for this function f. $\square$

c: g is a function satisfying the conclusion of the intermediate value theorem, and takes on each value only once. We shall show that g is continuous by way of contradiction.

Suppose by way of contradiction that there exists a point a where $\lim_{x\to a} g(x) \neq g(x)$. We can take the negation of the definition of a limit at a continuous point to get what is meant by a limit not being continuous at a point: "There exists some $\epsilon > 0$ for which it is true that, no matter what $\delta > 0$ you pick, there will always be some values of x where $|x - a| < \delta$ but still $|g(x) - g(a)| > \epsilon$." We can then choose a value of epsilon such that the previous statement is true, and in turn it is true that regardless of how "close" we get to a, then our input x will either have that $g(a) + \epsilon < g(x)$ or $g(x) < g(a) - \epsilon$. Without loss of generality assume the former, and for a newly defined input x_1. Therefore $g(x_1) > g(a) + \epsilon$.

From this we then have that there exists a $c \in (a, x_1)$ so that $g(c) = g(a) + \frac{\epsilon}{2}$. We can once again refer to the fact that we have assumed g is discontinuous at some point a, and therefore there exists more values x such that $g(x) > g(a) + \epsilon$. We can then say x_2 is another value on this interval (a, c). We have now found that $a < x_2 < c < x_1$ and that $g(x_1), g(x_2) > \epsilon$, and that $g(c) = g(a) + \frac{\epsilon}{2}$.

By the IVT, we can see that this information contradicts our claim that g outputs each number only once, as we see that there exists $y_1 \in (x_2, c)$ where $g(y_1) = g(a) + \epsilon$ and another value $y_2 \in (c, x_1)$ where $g(y_2) = g(a) + \epsilon$. This contradiction then verifies our claim that g must be continuous. $\square$