question 2 views

Daily Challenge 13.2

(~~**Due: Tuesday 8/28 at 12:00 noon eastern**~~)
(**Due: Saturday 9/1 at 12:00 noon eastern**)

Here's another CD 3 problem that will give you more practice with proofs involving the definition of the derivative. As usual, please either work in Overleaf directly or copy over and re-format when you're done.

(1) Problem: more proofs using the definition of derivative.

(a) Suppose that f is differentiable at a. Prove that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

Hint: we can always add zero in the form $0 = a - a$ for some quantity a.

(b) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$ for all x, and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$.

Hint: begin with the definition of $g'(a)$.

daily_challenge

Updated 7 months ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

(a) We aim to show that the limit

$$L = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$$

exists and is equal to $f'(a)$. Following the hint, we add zero in the form $0 = -f(a) + f(a)$ in the numerator. This gives

$$\begin{aligned} L &= \lim_{h\to 0} \frac{f(a+h) - f(a) + f(a) - f(a-h)}{2h} \\ &= \lim_{h\to 0} \left(\frac{1}{2} \frac{f(a+h) - f(a)}{h} + \frac{1}{2} \frac{f(a) - f(a-h)}{h} \right). \end{aligned}$$

So far, we still don't know that the limit L exists. But the problem statement says that we may assume the ordinary derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. We have also proven that, if f is differentiable at a (hence continuous at a), then we may re-write the definition of the derivative as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h},$$

by applying our result about the limit of a composition of functions (here the inner function is the one which sends h to $-h$).

Therefore, the two limits inside the parentheses in our expression above separately exist and are equal to $f'(a)$, which means that we may split this into a sum of limits:

$$\begin{aligned} L &= \lim_{h\to 0} \left(\frac{1}{2} \frac{f(a+h) - f(a)}{h} + \frac{1}{2} \frac{f(a) - f(a-h)}{h} \right) \\ &= \frac{1}{2} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} \\ &= \frac{1}{2} f'(a) + \frac{1}{2} f'(a) \\ &= f'(a). \end{aligned}$$

Thus $L = f'(a)$, so we have proven that

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

if we assume that the limit on the right exists (i.e. if we assume that f is differentiable at a).

There are cases where the limit on the left exists but the limit on the right does not. For instance, let $f(x) = |x|$, which we know is not differentiable at zero. But the limit on the left is

$$\lim_{h \to 0} \frac{f(h) - f(-h)}{2h} = \lim_{h \to 0} \frac{|h| - |-h|}{2h} = 0.$$

So the "new definition" can disagree with the "usual definition" at places where the latter is undefined.

(b) We wish to show that the limit

$$\lim_{k \to 0} \frac{g(a+k) - g(a)}{k}$$

exists and is equal to $f'(a)$ and to $h'(a)$.

We will attempt to use the squeeze theorem. We know that $f(x) < g(x) < h(x)$ for all x and that $f(a) = g(a) = h(a)$, so

$$f(a+k) - f(a) \leq g(a+k) - g(a) \leq h(a+k) - h(a).$$

This means that, for $k > 0$, we can divide by k (and preserve the direction of inequality) to find

$$\frac{f(a+k) - f(a)}{k} \leq \frac{g(a+k) - g(a)}{k} \leq \frac{h(a+k) - h(a)}{k}, \ (k > 0)$$

while for $k < 0$ we reverse the direction to find

$$\frac{f(a+k) - f(a)}{k} \geq \frac{g(a+k) - g(a)}{k} \geq \frac{h(a+k) - h(a)}{k}, \ (k < 0)$$

We have proven that taking a limit preserves inequality, so the above results imply that

$$f'(a) \leq \lim_{k \to 0^+} \frac{g(a+k) - g(a)}{k} \leq h'(a) \text{ and } f'(a) \geq \lim_{k \to 0^-} \frac{g(a+k) - g(a)}{k} \geq h'(a).$$

But since $f'(a) = h'(a)$, by the squeeze theorem, this means that both the left and right limits of $\frac{g(a+k) - g(a)}{k}$ exist and are equal to $f'(a) = g'(a)$.

Thus we have shown that g is differentiable at a, and that $g'(a) = f'(a) = h'(a)$, as desired. $\square$

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*