5/21/2019

30.7

(a): First consider the base case $n=1$; for $x \neq 0$

$$f(x) = \exp(-1/x^2) \Rightarrow f'(x) = (-x^{-2})' \cdot \exp(-1/x^2)$$
$$= 2x^{-3} \exp(-1/x^2)$$

What p_1 returns $2/x^3$ given $1/x$?

$p_1 = 2x^3$.

~~Now we must show that the theorem~~

~~Lets~~ We must now consider for $x=0$; the hint suggests we look at the limit definition of the derivative; recall

$$f'(x) = \lim_{k \to 0} \left(\frac{-f(x) + f(x+k)}{k} \right)$$

$$f'(0) = \lim_{k \to 0} \left(\frac{f(k)}{k} \right) = \lim_{k \to 0} \left(\frac{e^{(-1/k^2)}}{k} \right)$$

~~Both the top and bottom go to zero; apply L'Hopital:~~

$$\cancel{\lim_{k \to 0} \left(\frac{e^{(-1/k^2)}}{k} \right) = (-1k^{-2})' e^{(-1/k^2)}}$$

$$= \lim_{k \to 0}$$

~~Then by L'Hopital~~

$$\cancel{\lim_{k \to 0} \left(\frac{\exp(-1/k^2)}{k} \right) = \frac{(-k^{-2})' \exp(-1/k^2)}{1}}$$

$$\cancel{= 2k^{-3} \exp(-1/k^2)}$$

Christ I cannot get my mind made up. For the n-time, and this time making a change of variables $h = \frac{1}{k}$,

$$\lim_{k \to 0} \left(\frac{e^{-1/k^2}}{k} \right) = \lim_{h \to \infty} \left(h e^{-h^2} \right) = 0.$$

Now for the inductive step;

By induction, assume the theorem is true for $n=1$; then (by assumption)

$$f^n(x) = \begin{cases} P_n(\frac{1}{x}) \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 \text{ if } & x=0 \end{cases}$$

For $x \neq 0$, see that

$$f^n(x) = P_n\left(\frac{1}{x}\right) \exp\left(-1/x^2\right)$$

$$\Downarrow$$

$$f^{n+1}(x) = \left(P_n\left(\frac{1}{x}\right)\right)' \exp\left(-1/x^2\right) + P_n\left(\frac{1}{x}\right) \cdot \frac{2}{x^3} \exp\left(-1/x^2\right)$$

$$= \frac{-1}{x^2} P_n'\left(\frac{1}{x}\right) \exp\left(-1/x^2\right) + P_n\left(\frac{1}{x}\right) \cdot \frac{2}{x^3} \exp\left(-1/x^2\right)$$

Instructor response then suggests we let

$$\cancel{P_{n+1}\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 P_n'\left(\frac{1}{x}\right)}$$

$$P_{n+1}(Z) = -Z^2 P_n'(Z) + P_n(Z)(Z)^3$$

Thus,

$$f^{n+1}(x) = P_{n+1} \exp\left(-1/x^2\right).$$

We have proven the inductive case for $x \neq 0$, now we must consider $x=0$;

$$f^{n+1}(x) = \lim_{k \to 0} \left(\frac{f^n(k)}{k}\right) \text{ Since } f^n(0) = 0;$$

$$= \lim_{k \to 0} \left(\frac{P_n\left(\frac{1}{k}\right) \exp\left(-1/k^2\right)}{k}\right)$$

Substitute $t = 1/k$; then

$$\ldots = \lim_{t \to \infty} \left(t \, P_n(t) \exp(-t^2)\right)$$

Both the numerator and denominator go to infinity; repeatedly apply L'hopital until e^{t^2} in the denominator can dominate ct where c represents the coefficient of the final derivative of $P_n(t)$.

~~Eventually te~~ Then,

$$\lim_{t \to \infty} T\, p_n(t)\, e^{-t^2} = 0.$$

Proving the $x = 0$ and thus the inductive step of this proof, and thus the theorem is true for all n.

(b): Recall the definition of a Taylor series at 0:

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

We also see that $x \neq 0 \Rightarrow f(x \neq 0$; since $f(x)$ is not a polynomial in this scenario then the series does not converge to f.