question 4 views

Daily Challenge 1.4

(**Due: Friday 4/27 at 12:00 noon Eastern**.)

Friendly reminder that reading assignment 3 and the nine associated reading questions (Corbin answers here, Logan answers here) are due by our meeting today.

Try not to leave this until the last minute!

Review

Some proofs require **casework**. This means that we don't know which of several logical possibilities are true, so we consider each possibility separately and prove the theorem in that case.

Perhaps an example would help.

Theorem. Let a be an integer. Then the integer $b = a(a+1)$ is even.

Proof. The issue is that we don't know whether a is even or odd. We handle each case separately.

1. Suppose a is even. Then $a = 2k$ for some integer k. The quantity $b = a(a+1)$ can therefore be written as

   $$b = 2k(2k+1) = 2\left(k(2k+1)\right).$$

   But $k(2k+1)$ is an integer, so b can be written as twice an integer and is therefore even.

2. Now suppose a is odd. Then $a = 2m+1$ for some integer m. In this case, b is

   $$b = (2m+1)(2m+2) = 2\left((2m+1)(m+1)\right).$$

   The quantity in parentheses, $(2m+1)(m+1)$ is an integer, so we have written b as twice an integer, and hence it is even.

Since a must be either even or odd (aside: actually we haven't proved this yet, but we will), and we have proved that $b = a(a+1)$ is even in either of these two logical cases, b must be even in general. $\square$

For another example of a proof involving casework, see part (3) of Daily Challenge 1.2, where we proved that an irrational number raised to an irrational power can be rational, by considering two separate cases.

Problem

Here's another example which requires cases.

Theorem. Let m be an integer. Then the number $n = (m-1) \times m \times (m+1)$is divisible by 3.

[Hint: Every integer either gives remainder 0, 1, or 2 when divided by 3. That means that every integer a can either be written as $a = 3k$, or $a = 3\ell + 1$, or $a = 3j + 2$, where k, ℓ, j are some integers. Consider the three cases separately.]

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Proof (Corbin) -
Looking at the hint there are only 3 possible cases for an integer

1. We get remainder 0 when $\div 3$. This would be for $m = 3x | x \in \mathbb{Z}$.
2. We get remainder 1 when $\div 3$. This would be for $m = 3x + 1 | x \in \mathbb{Z}$.
3. We get remainder 2 when $\div 3$. This would be for $m = 3x + 2 | x \in \mathbb{Z}$.

This also means that m is divisible by 3, $m-1$ is divisible by 3 and (I had to look at Christian's answer here.) $m+1$ is divisible by 3

We can then use this to determine that any of the numbers $m, m+1,$ or $m-1$ is divisible by 3. So $n = (m-1) \times (m) \times (m+1)$contains all 3 of our numbers that are divisible. This means that atleast one, most likely all 3, of the numbers are divisible by 3, and the product of these is n. Which means that n is divisible by 3.$\square$

Proof (Logan) - I activate my weekly skip.

Updated 10 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Proof (Christian). Following the hint, we note that, if m is an integer, one of the following is true:

1. m gives remainder 0 when divided by 3. This means $m = 3a$ for some integer a.

2. m gives remainder 1 when divided by 3, so $m = 3b + 1$ for some integer b.
3. m gives remainder 2 when divided by 3, so $m = 3c + 2$ for some integer c.

In case (1), m is divisible by 3. In case (2), the number $m - 1$ is divisible by 3 (since $m - 1 = 3b$). In case (3), the number $m + 1$ is divisible by 3 (since $m + 1 = 3c + 3 = 3(c + 1)$).

Thus, in any case, one of the numbers m, $m - 1$, or $m + 1$ must be divisible by 3. So the number

$$n = (m - 1) \times (m) \times (m + 1)$$

is a product of three numbers, one of which must be divisible by 3, which means that n is divisible by 3, as desired. $\square$

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*