Question 1. *Polynomial roots.*

Suppose that f is a degree n polynomial with n distinct roots. Show that f' has exactly $n-1$ distinct roots.

Solution 1.

We are given that $p'(x)$ is an n degree polynomial with n distinct roots. Let a be the smallest of the distinct roots, and b be the largest of the distinct roots. We know that polynomials are continuous and differentiable at all points; We can then apply the mean value theorem to see that there exists some point c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. We see that since $f(b) = f(a) = 0$, then it must be true $f'(c) = 0$ and therefore c is a root of the derivative. We see that we can choose various pairs of the original distinct roots and find a point in that range where the derivative is zero. Intuitive idea follows: like a fork, there is one point in each "gap" between roots where the derivative equals zero, and as a result of that derivative being zero it is a distinct root of the derivative's graphed equation.

Question 2. *The function kitchen, redux.*
Cook up a function f with the following properties:

1. f is increasing on $(\infty, -2)$ and on $(3, \infty)$,
2. f is decreasing on $(-2, 3)$,
3. f has an inflection point at $x = 1$.

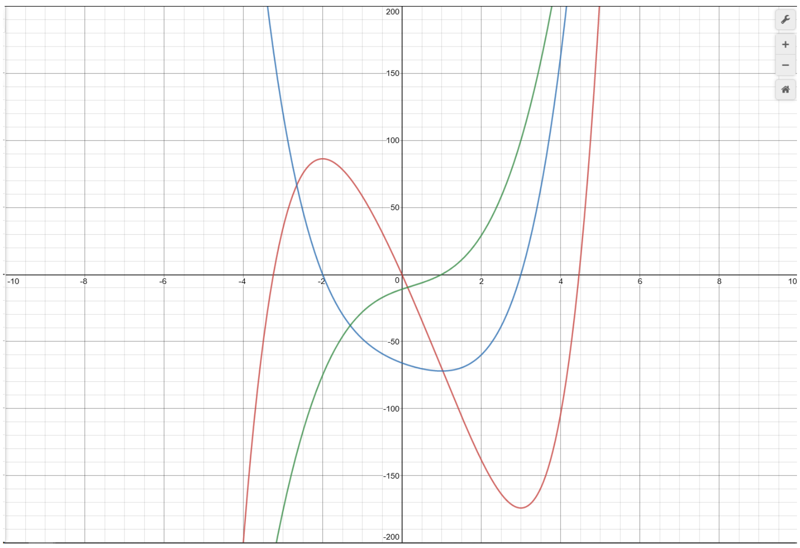
Once you have constructed your function, write down f, f', and f''. Confirm explicitly that it has the properties listed above (e.g. by checking the sign of the first derivative), and identify the intervals on which it is convex and concave. Finally, sketch the three graphs of f, f', and f'' together (you may either sketch all three on the same axes, or else stack the three graphs vertically, whichever you prefer).

Solution 2.
We shall attempt this time by taking the method that Christian suggests; finding some way of multiplying the first derivative in such a way that the second derivative is 0 when $x = 1$. We see that a polynomial that satisfies that traits looked for in the first two bullet points is $p(x) = (x-2)(x+3) = x^2 - x - 6$, so we want to find some $q(x)$ such that $f'(x) = (x^2 - x - 6)q$, and $f''(x) = (2x-1)q + (x^2 - x - 6)q'$ We see that we want some function that is always positive, as to not mess with the sign of our base polynomial; we allow this always positive function to be $x^2 + c$ where c is some value greater than 0. We can insert 1 into our equation and begin solving for c. $f''(x) = (2(1)-1)q(1) + (1-1-6)q'(1)$, or after simplification $q(1) + -6q'$; since this is a function of our choosing we can "demand" that $6g'(1) = g(1)$, in which case we see by inserting $x = 1$ that $12(1) = 1 + c$, or $c = 11$. We then work back; and see by substituting this $c = 11$ that

$$f''(x) = (2x-1)(x^2+11) + (x^2 - x - 6)(2x) \tag{1}$$

$$f'(x) = (x^2 - x - 6)(x^2 + 11) = (x^4 - x^3 - 6x^2) + (11x^2 - 11x - 66) = x^4 - x^3 + 5x^2 - 11x - 66 \tag{2}$$

$$f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{5}{3}x^3 - \frac{11}{2}x^2 - 66x \tag{3}$$



And finally, by referring to the green line representing $f''(x)$, f is concave when $x < 1$ and convex when $x > 1$.

Question 3. *Tuning a for a minimum.*
Determine a real number a with the property that the function $f(x) = x^4 - x^3 - x^2 + ax + 1$ has a local minimum at the point $x = a$.

(Source: Harvard-MIT math tournament)

Solution 3.
We know that a local minimum has a first derivative equal to zero when said minimum is inputted, and a positive second derivative; We shall go down the line of derivatives;

$$f(x) = x^4 - x^3 - x^2 + ax + 1 \tag{4}$$
$$f'(x) = 4x^3 - 3x^2 - 2x + a \tag{5}$$
$$f''(x) = 12x^2 - 6x - 2 \tag{6}$$

We can then look at the first derivative; substitute $x = a$, then $f'(a) = 4a^3 - 3a^2 - a = a(4a^2 - 3a - 1)$ and of course we can note that a is a dummy variable and replace all a with x. One potential root of the first derivative is 0, but inserting this into the second derivative yields the negative number -2, so my next idea is to apply the quadratic formula to the element in parentheses; We factor to see $4x^2 - 3x - 1 = (4x + 1)(x - 1)$ and with this info we can pass these root values onto the second derivative and check; We finally see that $x = \frac{-1}{4}$ results in a zero second derivative and a positive second derivative, so the solution to our problem is $a = \frac{-1}{4}$.

Question 4. *You can't be positive and concave everywhere.*
Suppose that f is a continuous differentiable function on $\mathbb{R}$. Prove that we cannot have $f(x) > 0$ and $f''(x) < 0$ for all x.

Solution 4.
Exploration: I can only think of one function that has this property, an even root set infinitely far to the left; like $\sqrt[n]{x + \inf}$ where $n = 2k$ and $k \in \mathbb{Z}$

Proof: Suppose by way of contradiction that such a function can exist; in English, we see that such a function is positive and concave everywhere. We see that for $f''(x) < 0$ then $f'(x)$ is strictly decreasing, and in turn we would like to prove that since $f'(x)$ is strictly decreasing, then there must be some point where $f'(x) < 0$. We can begin to prove this by referring to the mean value theorem, and see that there exists a point c on some open interval of interest $(a, b) \subseteq Dom(f')$ such that $f''(c) = \frac{f'(b)-f'(a)}{b-a}$. However we are told that $f''(x) < 0 \; \forall \; x$, therefore there is some point in this domain where $\frac{f'(b)-f'(a)}{b-a} < 0$. This then tells us that there is some point in the domain where the slope of the first derivative is negative. However, since we are also told that the first derivative is strictly decreasing, then by combining this knowledge we see that there is some points where $f'(x) < 0$, therefore $f(x)$ is strictly decreasing. We see that an asymptote is the only way a function could be strictly decreasing and never reach zero, but said asymptote (when positive) is also convex. Therefore there exists a point where $f(x) \leq 0$, contradicting our claim. Therefore it must be true that we cannot have a function where $f(x) > 0$ and $f''(x) < 0$. $\square$ (This needs to be more rigorous)

Question 5. *Some L'Hopital calculations.*
Find the following limits.

1. $\lim_{x\to1} \frac{x^2-1}{\log(x)}$
2. $\lim_{x\to0} \frac{\cos^2(x)-1}{x^2}$
3. $\lim_{t\to0} \frac{\sin(at)}{\sin(bt)}$, where a and b are nonzero real numbers.
4. Explain what is wrong with the following application of L'Hopital's rule:

$$\lim_{x\to1} \frac{x^3+x-2}{x^2-3x+2} = \lim_{x\to1} \frac{3x^2+1}{2x-3} = \lim_{x\to1} \frac{6x}{2} = 3. \tag{7}$$

   Then find the true value of the limit.

Solution 5.
A few basic calculations to get back in the swing of things (finally).

1. We can apply L'Hopital's rule here since the top and bottom go to zero as $x \to 1$; We see that

$$\lim_{x\to1} \frac{x^2-1}{\log(x)} = \lim_{x\to1} \frac{2x}{\frac{1}{x}} = \lim_{x\to1} 2x^2. \tag{8}$$

   This is a polynomial and is therefore continuous, and the value of this limit equals 2.

2. We can apply L'Hopital's rule here since the top and bottom go to zero as $x \to 0$; We take the derivative of the numerator and denominator of this quotient of interest and see by the chain rule that

$$\lim_{x\to0} \frac{\cos^2(x)-1}{x^2} = \lim_{x\to0} \frac{2\cos(x)\cdot -\sin(x)}{2x}; \tag{9}$$

   this also abides by the assumption of our lovely Frenchman, so we see

$$\lim_{x\to0} \frac{2\cos(x)\cdot -\sin(x)}{2x} = \lim_{x\to0} \frac{-2\sin(x)\cdot -\sin(x) - 2\cos^2(x)}{2}, \tag{10}$$

   and this final limit isn't discontinuous as $x \to 0$, so $L = -1$.

3. L'Hopital's rule is satisfied; we continue and can see that

$$\lim_{t\to0} \frac{\sin(at)}{\sin(bt)} = \lim_{t\to0} \frac{\cos(at)\times a}{\cos(bt)\times b}. \tag{11}$$

   We then see that this is equal to $\frac{a}{b}$.

4. This application of L'Hopital's rule is incorrect; we see in the second step that L'Hopital's rule is applied even though the top and bottom do not approach zero at the limit. I shall shortcut and note that neither approach zero, and neither are close to zero, so I simply insert the value and we see that the result matches the instructor value, -4.

Question 6. *Using the definition of convexity.*
We showed in a meeting that a function f is *convex* on an interval if, for any a, x, and b in the interval with $a < x < b$, we have

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}. \tag{12}$$

(a) Show that this definition can be restated in the following equivalent way: a function f is convex on an interval if any only if for all x and y in the interval we have

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y) \quad \text{for } 0 < t < 1. \tag{13}$$

(b) Prove that, if f and g are convex and f is increasing, then $f \circ g$ is convex.

(c) Give an example where f and g are convex and f is increasing but $g \circ f$ is *not* convex.

(d) Suppose that f and g are twice differentiable. Give another proof of the result of part (b) by considering second derivatives.

Solution 6.
Life things and social things got to me, and therefore all following unsolved problems will be revisited in the future.

Question 7. *The rule of 72.*

The *rule of 72* says that, if an amount of money is invested at $r\%$ interest per year, then it will take approximately $\frac{72}{r}$ years for the money to double.

Use a tangent line approximation to explain why this rule is a good estimate for small values of r.

Solution 7.

We begin by referring the compound interest formula, $B(t) = P \cdot (1 + \frac{r}{100})^{nt}$, and in turn we would like to solve the equation $2P = P \cdot (1 + \frac{r}{100})^{t}$ or $2 = (1 + \frac{r}{100})^{nt}$. We take the log of each side and see $\log(2) = t \cdot \log\big(1 + \frac{r}{100}\big)$, and the form of interest follows, $t = \frac{\log(2)}{\log\left(1+\frac{r}{100}\right)}$. We can then approximate the expression in the denominator for some small $x = \frac{r}{100}$ and see that $\log(1 + x) \approx \log(1) + x = x$, therefore $t = \frac{\log(2)}{\frac{r}{100}} = \frac{100\log(2)}{r} \approx \frac{72}{r}$.

Question 8. *Concentric circles.*
The area between two varying concentric circles is at all times 9π in^2. The rate of change of the area of the larger circle is $10\pi\frac{\text{in}^2}{s}$. How fast is the circumference of the smaller circle changing when it has area 16π in^2?

Solution 8.
We can let the plural radius of the larger and smaller circle be $R(t)$ and $r(t)$ (what an excellent usage of capitalization) respectively; we then see we can write the difference in area as $\pi R^2 - \pi r^2 = 9\pi in^2$. We then see by the chain rule that the derivative, with respect to time, of each side of this is $\pi 2RR' - \pi 2rr' = 0$. We then see that due to the structure of the former element, we can simply substitute into this our given rate of change for the area of the larger circle; then

$$10\pi\frac{\text{in}}{s^2} - \pi 2rr' = 0$$

or

$$10\frac{\text{in}}{s^2} = 2rr'.$$

We are also told that the area of the smaller circle at the time in questions is $16\pi\text{in}^2$ and therefore it has a radius of 4in. Therefore

$$10\frac{\text{in}}{s^2} = 2 \cdot 4r',$$

and in turn at the time in question $\frac{5}{4}\frac{\text{in}}{s^2} = r'$, and we see by taking the first derivative of the formula for circumference $C' = 2\pi r'$, then by inserting our changing radius in, the circumference is increasing by $\frac{5\pi}{2}\frac{\text{in}}{s^2}$.

Question 9. *Newton's method.*

Implement Newton's method in Python. Compare its performance to a bisection search for the root on an interval.

Solution 9.

Refer to the github repository at atomScratch/Python/Mathematics for the base of my Python Newton's Method.

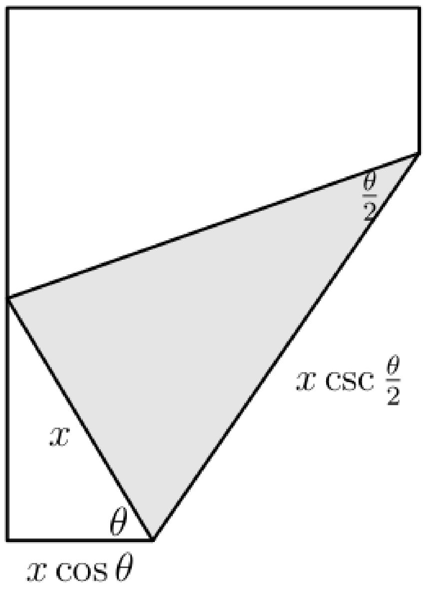
Question 10. *The mean value theorem and bounded derivatives.*
(a) Suppose f is a differentiable function and m is a real number such that $f'(x) \geq m$ for all $x \in [a, b]$. Prove that $f(b) \geq f(a) + m(b - a)$.

(b) Suppose that f is a differentiable function such that $|f'(x)| \leq 1$ for all $x \in [-1, 1]$. Write an inequality relating $f(1)$ and $f(-1)$.

Solution 10.
(a): Supposed by way of contradiction that said inequality of interest is not true. We see by the definition of the mean value theorem that there exists some point c in the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. We see by algebraically reorganizing the inequality we claim is false that $\frac{f(b)-f(a)}{b-a} \geq m$, and as we have shown by the MVT, $\frac{f(b)-f(a)}{b-a} = f'(c)$, and therefore $f'(c) \geq m$. We have known this to be true though, so our false inequality gave us something true, therefore said inequality must be true and we can conclude our proof.

(b): First a little basic thing, by the definition of absolute value, $|f'(x)| \leq 1 \implies -1 \geq f'(x) \leq 1$. We can then apply the fact we found in the previous problem, and see that for $a = -1$ and $b = 1$, then $(1) \geq f(-1) + -(1 - (-1))$, therefore $f(1) \geq f(-1) - 2$. Similarly, an alternative version of the above proves gives us the info that it is also true that $f(1) \leq f(-1) + (1 - (-1)) = f(-1) + 2$. Thus $f(-1) - 2 \leq f(1) \leq f(-1) + 2$.

Question 11. *Folding a piece of paper.*
A standard 8.5 inches by 11 inches piece of paper is folded so that one corner touches the opposite long side, as shown in the picture below. What is the minimum length of the crease?



[Hint: let θ be the angle between the fold and the bottom of the page, as in the picture. Write down a function $f(\theta)$ that gives the length of the crease as a function of θ. Then find the minimum of $f(\theta)$.]

Solution 11.
We begin by defining some of the values seen in the image; We define θ to be the angle of the crease and x to be the length of the fold. We then see by the trigonometric identities that the length of the "shorter" leg of the bottom left triangle (as it is depicted in the image) can be represented as $cos(\theta) = \frac{l}{x}$ and solving for l gives us $x\cos(\theta)$. We then see by the laws of basic intuition that the width of the paper is comprised of $x + x\cos(\theta) = 8.5$, and in turn by algebra $x = \frac{8.5}{1+\cos(\theta)}$. We see by constructing an image of the changes from a non-folded paper that we get a quadrilateral who's angles we can solve for, allowing us to see that the upper right corer of the shaded triangle has an angle of $\frac{\theta}{2}$. We can then apply this knowledge to find the length of the crease is in the equation $\sin\left(\frac{\theta}{2}\right) = \frac{x}{y}$, or the length of the crease is $\frac{x}{\sin\left(\frac{\theta}{2}\right)}$ We want to minimize this length of the crease, so we begin by substituting into the value of x what we found earlier, so the length of the crease can be represented as $L(\theta) = \frac{\frac{8.5}{1+\cos(\theta)}}{\sin\left(\frac{\theta}{2}\right)} = \frac{8.5}{(1+\cos(\theta))\cdot\sin\left(\frac{\theta}{2}\right)} = \frac{8.5}{\sin\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\cos(\theta)}$ We then see that the minimum occurs at an endpoint, a critical point, or a where $L(\theta)$ is not defined; We begin by evaluating the foremost; the upper limit upon observation is $\frac{\pi}{2}$, while the lower limit is much harder to calculate... (wip :sad:)

Question 12. *The mean value theorem and constant functions.*
Suppose that f is differentiable and that

$$|f(x) - f(y)| \leq (x - y)^2 \tag{14}$$

for all $x, y \in \mathbb{R}$. Prove that f must be a constant function.

Solution 12.
We see by the definition of absolute value that $|f(x) - f(y)| \leq (x - y)^2 \implies -(x - y)^2 \leq f(x) - f(y) \leq (x - y)^2$. This turned out to be alot less useful than I expected. We continue by applying the mean value theorem (somehow) to our function of interest f for some interval $[y, x]$ where $y < x$, then there exists some point $c \in [y, x]$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. The numerator of this is very similar to something we have already defined, sans an absolute value sign; that is exactly what we add in on our own by taking the absolute value of each side, ie $|f'(c)| = \frac{|f(x)-f(y)|}{|x-y|}$. We then see by dividing each side of the information we are given by

$$|x - y|$$

that this is in turn less than or equal to $\frac{(x-y)^2}{|x-y|}$. But since we also required in the creation of the interval $[y, x]$ that $y < x$, then there also exist no points where $x - y$ is negative, and therefore the absolute value can be removed; then $\frac{(x-y)^2}{|x-y|} = \frac{(x-y)^2}{x-y} = x-y$, and we see overall that $f'(c) \leq (x-y)$. We can then let $x - y$ be equal to some infinitesimally small ϵ; thus $x - y \neq 0$. We then also see that as ϵ gets small, $-\epsilon \leq f'(c) \leq \epsilon$, and almost as if by the squeeze theorem we can conclude that $f'(c) = 0$.

Question 13. *Racetracks.*
The *racetrack theorem* states that, if f and g are differentiable functions with $f(a) = g(a)$ for some a and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$.

(a) Explain in words what this theorem is saying and why it is called the racetrack theorem. [Hint: think of f and g as representing the position of two cars on a racetrack as functions of time.]

(b) Prove the racetrack theorem.

(c) Use the racetrack theorem to prove the following result.

Proposition. Suppose f is a twice-differentiable function with $f(0) = 0$, $f(1) = 1$, and $f'(0) = f'(1) = 0$. Prove that $|f''(x)| \geq 4$ for some $x \in [0, 1]$.

[Hint: break it up into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Solution 13.
(a) This, intuitively, is the statement that if one car is always going faster than another, the the car always going faster will also always be further ahead.

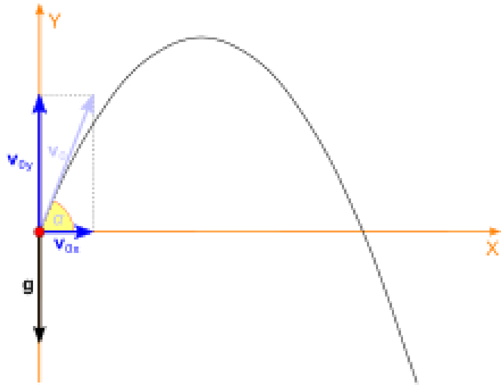
Question 14. *Estimating a root.*
Use linear approximation to estimate $\sqrt{1.1}$. Is your estimate too high or too low? Why?

Solution 14.
We see for $a = 1$ that when $f(x) = \sqrt{x}$ then $f(1.1) \approx 1 + \frac{0.5}{1} \cdot (0.1) \approx 1.05$, and this is just slightly too high, since one can see that the second derivative at 1 is concave and therefore the function itself is slightly lower than the approximation. A very accurate result nonetheless.

Question 15. *Minimizing a sum of areas.*
A wire of length 10 meters is to be cut into two pieces. The first piece will then be folded to form a square, while the second piece will be bent into a circle. What is the minimum total area that can be enclosed?

Solution 15.
We see that the amount remaining of wire after removing a length for the square can be represented as $10 - l$, which implies we have l length to work with for the perimeter of the square and $10 - l$ to work with for the circle. A square has 4 sides of equal length, therefore the area of our square can be represented $A_s = \left(\frac{l}{4}\right)^2$, and the area of a circle in terms of our circumference is $A_c = \pi(\frac{10-l}{2\pi})^2 = \pi(\frac{100+l^2-20l}{4\pi^2}) = \frac{100+l^2-20l}{4\pi}$ We add these to see that the cumulative area is $\frac{l^2}{16} + \frac{100+l^2-20l}{4\pi} = A = \frac{1}{16}l^2 + \frac{1}{4\pi}l^2 - \frac{5}{\pi}l + \frac{25}{\pi} = \frac{1+4\pi^{-1}}{16}l^2 - \frac{5}{\pi}l + \frac{25}{\pi}$. This is an upwards opening parabola, so we know that the vertex is where this function is at its minimum at $\frac{-b}{2a}$, therefore this function has its minimum at $\frac{40}{\pi+4}$, about 5.6 meters. :wow:

Question 16. *Firing a cannon.*
A cannon ball is shot from the ground with velocity v at an angle α, so that the vertical component of its velocity is initially $v\sin(\alpha)$ and the horizontal component is initially $v\cos(\alpha)$.



The ball obeys Newton's second law in the x and y directions, namely $F_x = m\frac{d^2x}{dt^2}$ and $F_y = m\frac{d^2y}{dt^2}$. Suppose that gravity acts downward in the y direction but there is no force in the x direction. Then

$$\begin{aligned} \frac{d^2x}{dt^2} &= 0, \\ \frac{d^2y}{dt^2} &= -g. \end{aligned} \tag{15}$$

Assume that the cannon ball begins at the origin at time $t = 0$. That is, $x(0) = 0$ and $y(0) = 0$.

(a) Write down the coordinates $x(t)$ and $y(t)$ of the cannon ball as functions of time.

[Hint: they should satisfy the two equations (15), and we know the initial velocities are $\dot{x}(0) = v\cos(\alpha)$ and $\dot{y}(0) = v\sin(\alpha)$, while the initial positions are $x(0) = 0$ and $y(0) = 0$.]

(b) Show that the trajectory of the cannon ball is parabolic. That is, show that the points $(x(t), y(t))$ lie on a parabola.

(c) Find the time t at which the parabola hits the ground ($y = 0$) and the horizontal distance it traveled.

(d) Find the angle α which maximizes the horizontal distance that the cannonball travels.

Solution 16.
(a): We begin by seeing that we can represent the x-coordinate as a function of time as the problem statement requests; we undifferentiate with the knowledge that $(0) = 0$ and thus we know that $x'(t) = v\cos(\alpha) \implies x(t) = v\cos(\alpha)t$. Similarly, we see that $y''(t) = -g$ and $y'(0) = v\sin(\alpha)$, therefore by undifferentiating the former twice and the latter once, we see that $y(t) = -\frac{1}{2}gt^2 + v\sin(\alpha)t$ giving us positions on 2 axes in terms of time.

(b): To show that this is function is a parabola, we must find some way to express y as a function of x rather than t, simplifying this to two dimensions and allowing us to continue. We see that $x = v\cos(\alpha)t \implies t = \frac{x}{v\cos(\alpha)}$. That was easy. We then see that

$$y(x) = -\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 + v\sin(\alpha)\frac{x}{v\cos(\alpha)}, \tag{16}$$

and by placing the top and bottom of the former element to the second power as suggested and separating, we see that

$$-\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 = \left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2 \tag{17}$$

meanwhile by basic algebra

$$v\sin(\alpha)\left(\frac{x}{v\cos(\alpha)}\right)=\tan(\alpha)x. \tag{18}$$

We conclude that

$$y(x)=\left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2+(\tan(\alpha))\,x, \tag{19}$$

which is a happy little parabola of the form $y(x)=ax^2+bx+c$, where a,b are the respective previous alignments sans-x and $c=0$.

(c): We see that a parabola is symmetric! Then the y position will equal zero at two times the x-value of the vertex, which can be found as the point where the $y'(t)=0$. We see that $y'(t)=-gt+v\sin(\alpha)$, and by solving when this equals zero, we see that $t=\frac{v\sin(\alpha)}{g}$, therefore the horizontal distance can be found by inserting double this into $x(t)$, then the distance can be represented as $\frac{2v^2\sin(\alpha)\cos(\alpha)}{g}$. Lovely.

(d): We refer to the range function in terms of some angle α we previously defined, $R(\alpha)=\frac{2v^2\sin(\alpha)\cos(\alpha)}{g}$. $\frac{2v^2}{g}$ is given to us positive, so we simply must maximize $\sin(\alpha)\cos(\alpha)$; We find the derivative of this via the product rule to be $\cos^2(\alpha)-\sin^2(\alpha)$. We then solve for zero, and find by intuition that $\alpha=\frac{\pi}{4}$ is a potential answer; our range of interest is $\alpha\in[0,\frac{\pi}{2}]$, where at the endpoints $R=0$, and there are no undefined points. Then launching an object at 45 degrees is the most efficient, who knew! Aerodynamics should make this real fun.

Question 17. *The ideal gas law.*

The ideal gas law states that a given mass of gas obeys the constraint $PV = kT$, where P is the pressure, V is the volume, T is the temperature, and k is a constant.

Suppose that at time $t = 0$, we have a gas at pressure 100 kilopascals (kPa), volume $1,000\ \mathrm{cm}^3$, and temperature 300 degrees Kelvin.

If the temperature increases by $1\frac{\mathrm{K}}{\mathrm{s}}$ and the pressure increases by $2\frac{\mathrm{kPa}}{\mathrm{s}}$, then after 60 seconds, what is the volume, and what is rate of change of the volume?

Solution 17.

We define P, V and T to be functions in terms of time; we then take the derivative of the given ideal gas law to see that $P'V + PV' = kT'$. We quickly calculate the value of our constant, given the initial values and applying them to the ideal gas law, we see by algebra that $k = \frac{1000\cdot\ \mathrm{kPa}\cdot\ \mathrm{cm}^3}{3K}$. We see that, for an increase of 60 Kelvin and 120 kilo-pascals over 60 seconds, the temperature at 60 seconds is 360 kelvin and the pressure is 220 kilo-pascals. We plug these into the original equation for the ideal gas law and solve for volume (after a few lines of maeth) to see that the volume equals $\frac{6000\mathrm{cm}^3}{11}$. We can then insert these values, including the just calculated volume at 60 seconds, into our differentiated gas law to see that the volume is decreasing by $\frac{1250\mathrm{cm}^3}{363\mathrm{s}}$. :wow: