question 3 views

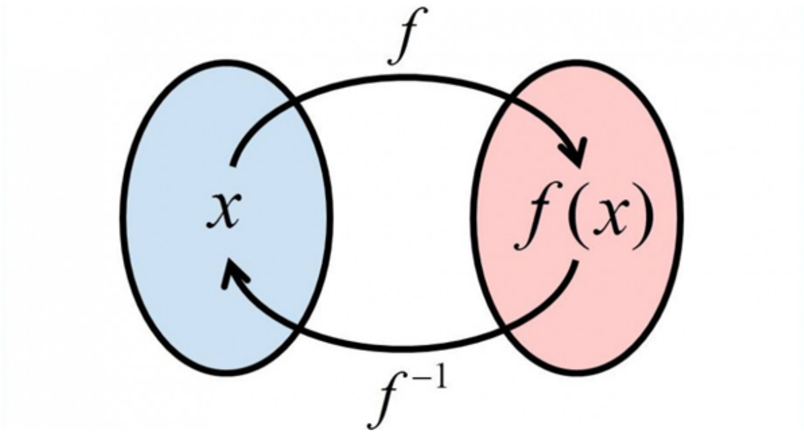
Daily Challenge 3.3

(**Due: Thursday 5/10 at 12:00 noon**)

Soon we will introduce *inverse* trigonometric functions, like arcsin, but first we will need a more nuanced understanding of what inverse functions are.

Review

Earlier, we defined a function $f : A \to B$ as a sort of machine that takes elements of A and outputs elements of B. The set A is called the **domain** of f and written $\mathrm{Dom}(f)$, the set B is called the **codomain** of f and written $\mathrm{Cod}(f)$, and the subset of values in the codomain actually output by the function is called the **range** and written $\mathrm{Rng}(f)$.



Definition. Let f be a real-valued function. A real-valued function g is called an **inverse** of f if $f(g(x)) = x$ for all $x \in \mathrm{Dom}(g)$ and $g(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$. We write $g(x) = f^{-1}(x)$.

The part of this definition which requires x to be in the domain of g or f, respectively, is important! For instance, we normally think of the two functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ as inverse functions. However, this cannot be true for all inputs x, since the parabola traced out by $f(x) = x^2$ fails the **horizontal line test**, so it cannot truly be invertible.

The resolution to this apparent problem is to restrict the domain to all *nonnegative* values of x. If we consider only the half of the parabola to the right of the y axis, then this curve *passes* the horizontal line test. More precisely, we are really modifying the function $f(x) = x^2$ to a new function f_+ as follows:

$$f_+ : [0, \infty) \to [0, \infty),$$
$$f_+(x) = x^2,$$

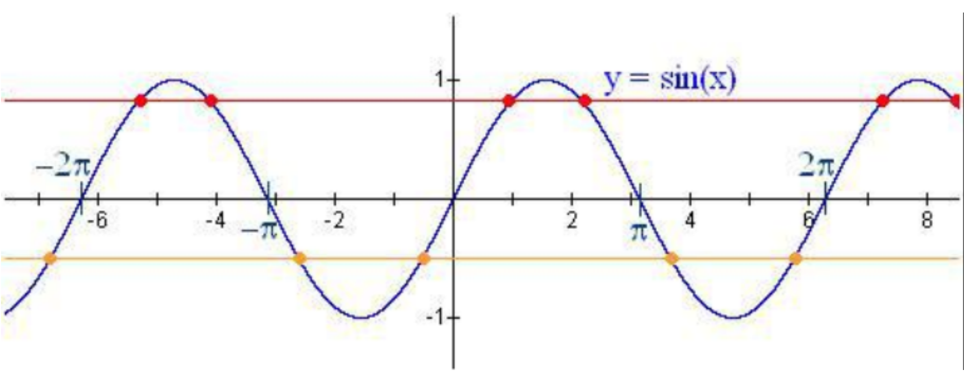
which agrees with the original function $f(x)$ but has a smaller domain. Then the function

$$g : [0, \infty) \to [0, \infty),$$
$$g(x) = \sqrt{x}$$

is indeed an inverse of f_+. To prove this, we check the definition: for all $x \in [0, \infty)$, it is true that $(\sqrt{x})^2 = x$ and that $\sqrt{x^2} = x$, where we take the positive square root.

These statements would not be true if we had included negative values of x. Surely if $x = -1$, it is not true that $\sqrt{x^2} = x$, since the left side is 1 but the right side is -1.

This idea of restricting the domain to get a good inverse function will be *critical* when we study inverse trigonometric functions. As you may have noticed, the sine function fails the horizontal line test pretty badly!



We will need to make a similar restriction to get an honest inverse function out of this beast.

Problem

Prove the following result, including both your exploration and your argument.

Theorem. Suppose that f is a real-valued function, and that g and h are both inverses of f. Then g and h are equal as functions (this means they have the same domain and range, and that $g(x) = h(x)$ for all x in their shared domain).

Some scaffolding:

- What does this statement mean, in ordinary language? Something like "inverse functions are unique"?
- Note that your proof will have two steps: you must
  1. show that g and h have the same domain and range, and
  2. show that $g(x) = h(x)$ for any x in their shared domain.
- To show (1) and (2), use the definition of the inverse function given above. If g and h are both inverses of f, this means four things:
  1. $g(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$, and
  2. $f(g(x)) = x$ for all $x \in \mathrm{Dom}(g)$, and
  3. $h(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$, and
  4. $f(h(x)) = x$ for all $x \in \mathrm{Dom}(h)$.
- What can you conclude from these four statements? Can you explain why these imply that $\mathrm{Dom}(g) = \mathrm{Rng}(f)$, for instance?

[Hint: this is problem 1.11 in AoPS chapter 1. Feel free to read the solution, as long as you rewrite it in your own words and understand the reasoning.]

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Exploration (Logan). Your thoughts and scratch work go here.

Activating my weekly skip. (But I do wanna work on this later.)

Argument (Logan). Your proof goes here.

Updated 11 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Exploration (Christian). The statement of this theorem means that a function can have *at most one* inverse function. This means that we can speak about "the" inverse, rather than "an" inverse.

Proofs of this form are among the most common in mathematics: they are called *uniqueness* results, since we are showing that the inverse function is unique.

Do I believe that this is true? It seems plausible! If $f : A \to B$ is a function, and g is an inverse, then the condition that $g(f(x)) = x$ seems to tightly constrain the possible outputs of g. In other words, it doesn't seem that we have any "wiggle room" to change the definition of the inverse, which suggests that there can be only one.

To show this precisely, I will use the general strategy of uniqueness proofs: assume that there are two of the object in question, and prove that they must be the same. Here "the same" means "equal as functions," which requires that they have the same domain and range and their values agree.

Argument (Christian). I'll prove this in two steps. The first step will be an intermediate result called a *lemma*.

Lemma. Let f and g be real-valued functions. If g is an inverse of f, then $\mathrm{Dom}(g) = \mathrm{Rng}(f)$ and $\mathrm{Rng}(g) = \mathrm{Dom}(f)$.

Proof. First we wish to prove that $\mathrm{Dom}(g) = \mathrm{Rng}(f)$. The left and right side are sets, and to prove that two sets are equal, we must prove that each set is a subset of the other.

Suppose $x \in \mathrm{Dom}(g)$. Then by the definition of inverse, we have that $f(g(x)) = x$. But this means that x is an output of f, so $x \in \mathrm{Rng}(f)$. Thus $\mathrm{Dom}(g) \subseteq \mathrm{Rng}(f)$.

Now suppose $y \in \mathrm{Rng}(f)$, which means that $y = f(x)$ for some $x \in \mathrm{Dom}(f)$. By the definition of inverse, then, $g(y) = x$. But this implies that y is in the domain of g, so $\mathrm{Rng}(f) \subseteq \mathrm{Dom}(g)$.

Since we have showed that $\mathrm{Dom}(g) \subseteq \mathrm{Rng}(f)$ and $\mathrm{Rng}(f) \subseteq \mathrm{Dom}(g)$, it follows that $\mathrm{Rng}(f) = \mathrm{Dom}(g)$.

Finally, notice that if g is an inverse of f, the definition requires that f is an inverse of g. Thus repeating the above argument with the letters f and g interchanged also proves that $\mathrm{Rng}(g) = \mathrm{Dom}(f)$. ■

Now let's return to the main proof.

Let f be a real-valued function and let g and h be inverses of f. By the lemma we proved above, we have

- $\mathrm{Dom}(g) = \mathrm{Rng}(f) = \mathrm{Dom}(h)$, and
- $\mathrm{Rng}(g) = \mathrm{Dom}(f) = \mathrm{Rng}(h)$.

Thus the two functions g and h have the same domain and range. Now let $y \in \mathrm{Rng}(f)$, which means y is in the shared domain of g and h. Since y is in the range of f, we can write $y = f(x)$ for some $x \in \mathrm{Dom}(f)$. By the definition of inverse, then, we see that

$$g(y) = g(f(x)) = x \text{ , and } h(y) = h(f(x)) = x.$$

This means that the two functions g and h agree on their shared domain, so they are equal as functions. Thus any two inverses of f are identical. □

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved

Christian Ferko 11 months ago

You may prefer to read the solutions to problems 1.10 and 1.11 in AoPS, which prove the same things as my solution above, but with fewer words.

Both their proof and mine are technically correct, but I've been trying to write longer proofs to explain more of the intermediate steps.