question 3 views

Daily Challenge 3.2

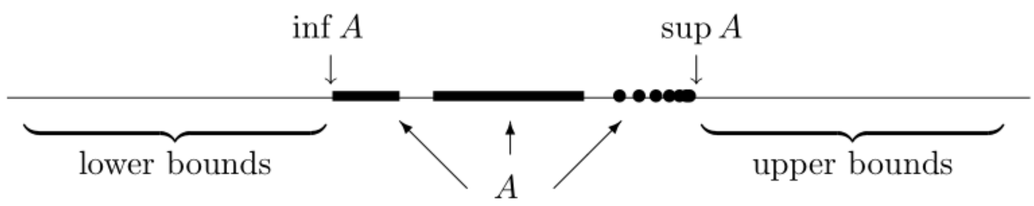
(**Due: Wednesday 5/9 at 12:00 noon Eastern**)

Let's take another crack at some more proofs from AoPS chapter 1. Since we've had a few chances to practice these, I'll start grading proofs with the expectation of correctness rather than just effort. We really need fluency in mathematical proof to have any chance of understanding chapters 2 and beyond.

Review

In AoPS chapter 1, we saw that $\mathbb{R}$ is "better" than $\mathbb{Q}$ because it has a **completeness** property: every subset of $\mathbb{R}$ with an *upper bound* also has a *least upper bound*, or *supremum*. In other words, $\mathbb{R}$ "has no holes." This is why we can do calculus in the reals but not in the rationals.

To understand this statement, we'll need a few notions of bounds. For example, consider the interval $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$. The number 2 is larger than every number in this interval, so we say that 2 is an *upper bound* for $(0,1)$. But 2 is not the only number with this property: 10, for instance, is also larger than every number in $(0,1)$, so it is also an upper bound for $(0,1)$. In fact, there are infinitely many upper bounds: any number greater than 1 is an upper bound for $(0,1)$.



The smallest number which is still an upper bound for $(0,1)$ is 1 itself. We call this the *least upper bound*, or the *supremum*, of the interval $(0,1)$.

In a similar way, the numbers -1, -2, and any other negative number, all have the property of being smaller than every number in $(0,1)$, so we call them *lower bounds*. The largest lower bound is 0, so we call this the *greatest lower bound* or *infimum*.

Pictures and intuition are great, but to prove things, we need precise definitions.

Definition. Let S be a subset of the real numbers.

- If a real number x satisfies $x \leq y$ for all $y \in S$, then x is a **lower bound** for S.
- If a real number x satisfies $x \geq y$ for all $y \in S$, then x is an **upper bound** for S.
- If x is an upper bound for S, and for every other upper bound z we have $x \leq z$, then x is the **supremum** of S.
- If x is a lower bound for S, and for every other lower bound z we have $x \geq z$, then x is the **infimum** of S.

Now let's prove our above claim that 1 is the supremum of the interval $(0,1)$. Actually, let's be slightly more general and consider any interval (a,b).

Theorem. Let a and b be real numbers with $a < b$. The supremum of the interval (a,b) is b.

Exploration. This certainly seems true. The interval (a,b) is defined as $\{x \in \mathbb{R} \mid a < x < b\}$, so every number in that interval is less than b by definition; this means that b is an upper bound. But we cannot have any smaller upper bound since there are no numbers "between" b and b, so b should also be the least upper bound. Let's see if we can make that intuition precise.

Argument. Let a, b be two real numbers with $a < b$. First we show that b is an upper bound for (a,b). The interval (a,b) is defined as

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$

so if $x \in (a,b)$, then $x < b$. By the definition of upper bound given above, this means that b is an upper bound for (a,b).

Now we show that b is the smallest upper bound. Suppose $a < z < b$ were any number less than b. Then z cannot be an upper bound for (a,b), since the number $\frac{z+b}{2}$ is in the interval (a,b) but $\frac{z+b}{2} > z$. Thus b is the smallest upper bound, so it is the supremum. $\square$

Problem

Try proving the following statement about suprema and infima, including your exploration and scratch work as well as the polished argument.

Theorem. Let S be a subset of $\mathbb{R}$ which is bounded above and below. If $\sup(S) = \alpha$ and $\inf(S) = \alpha$ for some $\alpha \in \mathbb{R}$, then the set S must consist of a single element: $S = \{\alpha\}$.

I will scaffold the exploration and argument for you.

Exploration. Here are some questions you might consider while exploring the claim.

- Can you re-phrase the statement of the theorem in simple language?
- Do you remember the precise definitions of supremum and infimum?
- Can you draw a picture to understand the claim? Perhaps draw a picture where the supremum and infimum are very close, but not quite equal, and then think about what happens as they get closer and closer.

Argument. Here is an outline of the proof; use this to get the general idea, but your final argument should be re-written in your own words and with the missing details filled in.

1. We wish to show that the set S contains a single element, so begin your proof with the sentence "Let $x \in S$." If we can prove that $x = \alpha$, we've established the claim.
2. Use the definition of supremum to show that $x \leq \alpha$.
3. Use the definition of infimum to argue that $x \geq \alpha$.

4. Reason that the two inequalities $x \leq \alpha$ and $x \geq \alpha$ can both be true only if $x = \alpha$.
5. Explain why, since we started by assuming that x was *any old element* of S and then proved that $x = \alpha$, it follows that S can contain only the single element α.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Exploration (Logan).
I copy and paste the statement here just to avoid constant scrolling: Let S be a subset of R which is bounded above and below. If sup(S)=α and inf(S)=α for some α∈R, then the set S must consist of a single element: S={α}.

First, I can make this more readable by converting the phrase to its more easily understandable definitions, something along the lines of: "S is a subset of the real numbers. Its greatest lower bound is a and its lowest upper bound is also a. I decide to phrase this pseudo-algebraically as $a \leq S \leq a$. At the moment visualizing is enough for a "simple" problem like this, so I believe I can proceed to the proof.

Argument (Logan). I am given the statement: "If $\sup(S) = a$ and $\inf(S) = a$ for some $a \in R$, then the set S must consist of a single element: $S = a$." We are operating on the real numbers. I must prove that set S has only one element, of which I will define as x. First, I define the supremum of the set algebraically. By the definition of supremum, it is the lowest number that still remains an upper bound to the set, therefore $x \leq \sup(x)$. Similarly, by the definition of infimum, the infimum of a set is the greatest number that is still a lower bound to the set, therefore $\inf(x) \leq x$. I can assemble this into the statement $\inf(x) \leq x \leq \sup(x)$ due to the transitive property. However, as given in the original problem $\sup(s) = a$ and $\inf(x) = a$ therefore, $\sup(s) = \inf(x)$ Because of this, the only situation that the statement $\inf(x) \leq x \leq \sup(x)$ is true in is that $\inf(x) = x = \sup(x)$. From this statement, one can conclude that because the set S has an equal infimum and supremum, then the set S only contains one element. ☐

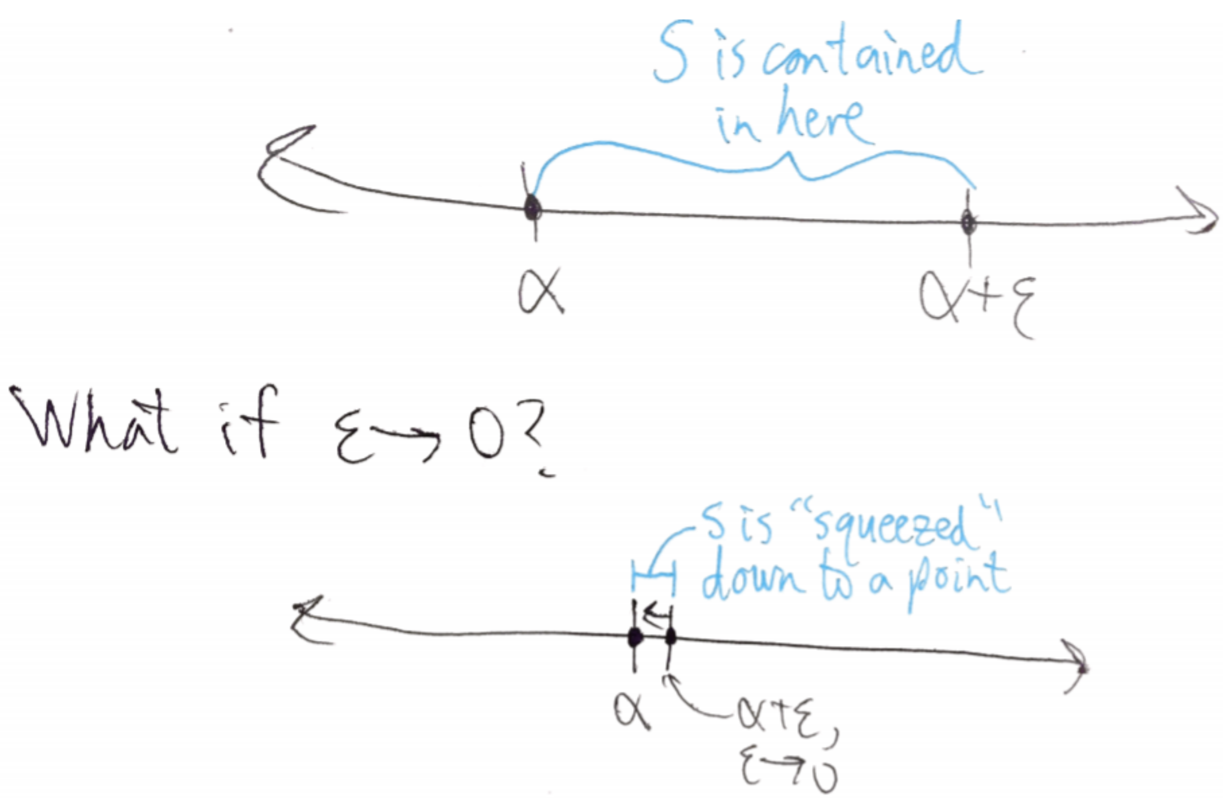
(Note: this is feeling rather poorly written but I don't know where and why)

Updated 11 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Exploration (Christian). If $\sup(S) = \alpha$, then α is an upper bound for S, and if $\inf(S) = \alpha$, then α is a lower bound for S. That means that every element of S lies between α and α. But the only number in this "interval" $[\alpha, \alpha]$ is the number α itself, so I think this implies that S must contain only a single element.

Another way to understand this is to change the statement slightly. Suppose $\sup(S) = \alpha$ and $\inf(S) = \alpha - \epsilon$, where ϵ is some small positive number. Then all the points of S lie in a tiny ϵ-interval:



As ϵ gets smaller and smaller, we see graphically that the "space" where S can live shrinks to a point.

Let's make this reasoning precise.

Argument (Christian). Let $x \in S$ and suppose that $\sup(S) = \alpha = \inf(S)$.

Since $\alpha = \sup(S)$, by the definition of supremum, every element $y \in S$ satisfies $y \leq \alpha$. In particular, x is an element of S, so $x \leq \alpha$.

Likewise, since $\alpha = \inf(S)$, every element of $y \in S$ satisfies $y \geq \alpha$, so $x \geq \alpha$.

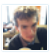
The only number x which obeys the compound inequality $\alpha \leq x \leq \alpha$ is $x = \alpha$ itself.

Thus we began by assuming that x is an arbitrary element of S and then proved that $x = \alpha$, which means that this must be true of *every* element in S. In particular, it follows that S contains only the single element α, so $S = \{\alpha\}$.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 11 months ago

Additional evidence for the importance of proof-writing: some problem sets from one of my freshman year math classes (typically the first course that math majors take). It's all proofs!