question 2 views

Daily Challenge 16.2

(**Due: by session 40 at the very latest)**

This is problem 12 on CD 4.

(1) Problem: The mean value theorem and constant functions.

Suppose that f is differentiable and that

$$|f(x)-f(y)| \leq (x-y)^2$$

for all $x, y \in \mathbb{R}$. Prove that f must be a constant function.

daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

We see by the definition of absolute value that $|f(x)-f(y)| \leq (x-y)^2 \implies -(x-y)^2 \leq f(x)-f(y) \leq (x-y)^2$. This turned out to be alot less useful than I expected. We continue by applying the mean value theorem (somehow) to our function of interest f for some interval $[y, x]$ where $y < x$, then there exists some point $c \in [y, x]$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. The numerator of this is very similar to something we have already defined, sans an absolute value sign; that is exactly what we add in on our own by taking the absolute value of each side, ie $|f'(c)| = \frac{|f(x)-f(y)|}{|x-y|}$. We then see by dividing each side of the information we are given by $|x-y|$ that this is in turn less than or equal to $\frac{(x-y)^2}{|x-y|}$. But since we also required in the creation of the interval $[y, x]$ that $y < x$, then there also exist no points where $x-y$ is negative, and therefore the absolute value can be removed; then $\frac{(x-y)^2}{|x-y|} = \frac{(x-y)^2}{x-y} = x-y$, and we see overall that $f'(c) \leq (x-y)$. We can then let $x-y$ be equal to some infinitesimally small ϵ; thus $x-y \neq 0$. We then also see that as ϵ gets small, $-\epsilon \leq f'(c) \leq \epsilon$, and almost as if by the squeeze theorem we can conclude that $f'(c) = 0$.

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Let y, x be two real numbers with $y < x$. Apply the mean value theorem to the closed interval $[y, x]$ to conclude that there exists a point $c \in [y, x]$ where

$$f'(c) = \frac{f(x)-f(y)}{x-y}.$$

But by taking the absolute value of both sides of this equation, we see that

$$|f'(c)| = \frac{|f(x)-f(y)|}{|x-y|} \leq \frac{(x-y)^2}{x-y} = x-y,$$

where we have used the assumption that $|f(x)-f(y)| \leq (x-y)^2$.

This conclusion must hold for any choice of x, y. In particular, we can choose $x = y+\epsilon$ for any small number ϵ that we like, which means that $x-y=\epsilon$ will be arbitrarily small. This means that $|f'(c)| < \epsilon$ for every real $\epsilon > 0$, which can be true only if $f'(c) = 0$. But this means that $f'(c) = 0$ for all c, and we have proven before that any function whose derivative vanishes everywhere must be a constant function. □

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*