28.1

(a): Claim: If $\lim(a_n) = L$ and $\lim(b_n) = M$, then $\lim(a_n + b_n) = L + M$

Proof: Suppose $\lim(a_n) = L$, $\lim(b_n) = M$. Let $\epsilon > 0$ be given. ~~We fin~~ By the two supposed statements, applying the definition of a limit with $\epsilon' = \frac{\epsilon}{2}$, then

(1) $|a_n - L| < \frac{\epsilon}{2}$ for all $n > N_1$, and $|b_n - M| < \frac{\epsilon}{2}$ for all $n > N_2$

We set $N = \max(N_1, N_2)$. Then, for any $n > N$, we have

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \text{ by algebra}$$

$$\leq |a_n - L| + |b_n - M| \text{ by (1),}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon$$

Thus, $n > N$ implies $|(a_n + b_n) - (L + M)| < \epsilon$, proving $\lim(a_n + b_n) = L + M$

There also exist a few recurring tricks:

- The $\epsilon/2$ trick - when summing n limits, consider making the job easier by asking for each individual limit to be more precise.

- The $N = \max(N_1, N_2)$ trick - since the limit ~~as~~ just asks for a_n to be greater than some value, if n is greater than all values N, all limits work out.

- Triangle inequality - the sum of values in an absolute value is $\leq$ than the sum of the absolute values of all values.

(b): Idea: • Write down the definition of the given limit.

- multiply each side by c
- ~~Let the~~ Distribute into the absolute value.
- Let the new limit have $\epsilon' = \epsilon c$
- Conclude.

Authors ~~Idea~~ Proof:

Suppose $\lim(a_n) = L$, and let $c \in \mathbb{R}$ be given. Applying the definition of a limit with $\epsilon' = \epsilon/(|c|+1)$, we obtain an $N \in \mathbb{N}$ such that

$$(1) \qquad |a_n - L| < \epsilon' = \frac{\epsilon}{|c|+1}$$

Then see that

$$|c|\,|a_n - L| = |ca_n - cL| < |c|\frac{\epsilon}{|c|+1} \quad \text{by (1)}$$

and since

$$\frac{|c|}{|c|+1} < 1, \text{ then}$$

$$|c|\frac{\epsilon}{|c|+1} < \epsilon;$$

thus

$$|ca_n - cL| < \epsilon.$$

$n \geq N$ implies $|ca_n - cL| < \epsilon$, thus proving $\lim(ca_n) = cL$.

(c): Ideas:

· Write down the implied inequalites for the limits of a and b; fit b in some where there and find some where that the limit definition can be made to come up

Proof: We see that

$$|a_n - L| < \epsilon \text{ for } n > N_a,$$
$$|c_n - L| < \epsilon \text{ for } n > N_c$$

Or likewise,

$$-\epsilon < a_n - L < \epsilon$$
$$-\epsilon + L < a_n < \epsilon + L$$

and

$$-\epsilon + L < c_n < \epsilon + L.$$

By the given $a_n \leq b_n \leq c_n$, then

$$-\epsilon + L < a_n \leq b_n \leq c_n < \epsilon + L,$$

this line implies

$$|b_n - L| < \epsilon \text{ for some } n > N_b.$$

(d): Solving this problem is all about choosing an ~~at~~ apparently obscure choice of ϵ for the given limit and its definition. You can find or come up with ϵ's by writing the difference:

$$\frac{1}{a_n} - \frac{1}{L} = \frac{L - a_n}{a_n L}$$

Draw a picture to visualize how a_n is always "close" to L. Apply N maxing, recognize algebraic simplifications and substitutions, and you are done :clap:.