question 2 views

Daily Challenge 12.1

(**Due: Monday 8/20 at 12:00 noon Eastern**)

Today we will see some applications of the result $\frac{d}{dx}\log(x) = \frac{1}{x}$ which we derived last time using the inverse function rule.

(1) Sometimes it is easier to differentiate $\log(f)$ than f.

There is a trick for taking the derivative of complicated functions, especially those where a variable appears in the exponent.

For instance, consider the function $f(x) = a^x$. We cannot differentiate this using the power rule, since this result only applies to functions of the form x^n where x appears in the base but not in the exponent. Likewise, none of the other rules that we've developed -- the product rule, quotient rule, chain rule, etc. -- seem to apply clearly to a function like a^x.

We will need a new trick to handle functions like this. First we will develop the technique.

Exercise. Let $f(x)$ be a differentiable function and define $g(x) = \log(x)$. Find a formula for $g'(x)$ in terms of $f'(x)$.

Solution. By the chain rule,

$$\begin{aligned} g'(x) &= (\log)'(f(x)) \cdot f'(x) \\ &= \frac{f'(x)}{f(x)}, \end{aligned}$$

where we have used that the derivative of $\log(x)$ is $\frac{1}{x}$. We can re-write this result as

$$\frac{d}{dx}\log(f(x)) = \frac{f'(x)}{f(x)}.$$

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(2) A third proof of the power rule.

We can use the result above to give a new result of the power rule, *which holds for any real exponent rather than just integers*.

Let $f(x) = x^r$, where $r \in \mathbb{R}$. Take the logarithm of both sides of this equation to find

$$\log(f(x)) = \log(x^r) = r\log(x),$$

where we have used the log-of-a-power rule. Differentiate both sides. On the left, we have $\frac{d}{dx}\log(f(x)) = \frac{f'(x)}{f(x)}$, as we saw before. On the right, we have $r\frac{d}{dx}\log(x) = \frac{r}{x}$. Thus

$$\frac{f'(x)}{f(x)} = \frac{r}{x}.$$

Solving for the derivative, we find $f'(x) = \frac{rf(x)}{x}$, and plugging back in the definition $f(x) = x^r$, we conclude

$$f'(x) = rx^{r-1}.$$

Again I emphasize that r can be any real number, not just an integer. We have now proven that the power rule holds for *any* function of the form $f(x) = x^r$.

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(3) Problem: using the "logarithmic differentiation" trick.

Use the technique discussed above (take the logarithm of both sides, differentiate, and then solve for $f'(x)$) to compute the derivative of the following two functions.

(1) Let $f(x) = a^x$ for $a \in \mathbb{R}$; find $f'(x)$.

(2) Let $f(x) = x^x$. Find $f'(x)$. (Note that your result will **not** be the same as you would get by setting $a = x$ in your result for part (1); do you see why?)

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Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan:
a: We begin with an adorable small bit of information; $f(x) = a^x$ where $a \in \mathbb{R}$. We shall begin by taking the log (base e) of each side; We receive $\log(f(x)) = \log(a^x) = x\log(a)$. We can now take the derivative of each side. By the chain rule applied to the left side we have that $(\log(f(x))' = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$, and on the right side we have by the product rule that $(x\log(a))' = \log a + x \cdot \frac{1}{x} = \log(a)$. By setting each side equal to eachother we see that $\log(a) = \frac{f'(x)}{f(x)}$, and by multiplying each side by $f(x)$ that $a^x \log(a) = f'(x)$

b: Take the log of each side of the given equation; $log(f(x)) = log(x^x) = x\log(x)$. Then we shall take the derivative of each side and see on the left side via the chain rule, identically to last time, that $(log(f(x)))' = \frac{f'(x)}{f(x)}$. On the left side since both are variable, that by the product rule $(x\log(x))' = \log(x) + x \cdot \frac{1}{x} = \log(x) + 1$We then set each side equal to each-other; $\log(x) + 1 = \frac{f'(x)}{f(x)}$, and finally $f(x)(\log(x) + 1) = f(x)\log(x) + f(x) = f'(x)$. We conclude that $f'(x) = x^x \log(x) + x^x$.

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(1) Taking the logarithm of both sides gives $\log(f) = x\log(a)$, then differentiating yields

$$\frac{f'(x)}{f(x)} = \log(a).$$

Solving, we find $f'(x) = f(x)\log(a)$, and replacing the definition of $f(x)$, we conclude

$$f'(x) = a^x \log(a).$$

Note that, when $a = e$, this reduces to the usual rule $f'(x) = e^x \log(e) = e^x$ since $\log(e) = 1$.

(2) Take the log of both sides to find $\log(f) = x\log(x)$. Now differentiate and use the product rule

$$\frac{f'(x)}{f(x)} = \log(x) + 1.$$

This means $f'(x) = f(x)\left(\log(x) + 1\right)$, or

$$f'(x) = x^x \left(\log(x) + 1\right).$$

As expected, this is *not* the same as one would find by plugging $a = x$ into our result from (a); when a is not a constant, but is instead replaced by something which depends on x, then the effect of changing $f(x)$ to $f(x+h)$ in the definition of the derivative has *two* effects, one from changing the base and one from changing the exponent.

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*