question 2 views

Daily Challenge 11.6

(**Due: Friday 8/17 at 12:00 noon Eastern**)

The main goal of today's challenge is to prove the power rule for differentiation in two ways.

First let's review the proof of the product rule, which is used in the inductive proof of the power rule.

(1) The product rule gives $(fg)' = f'g + fg'$.

As in the case of limits, where we built up a "toolkit" of useful results like the fact that $\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$ assuming that both limits exist, we will establish a collection of rules that will allow us to differentiate a large class of functions.

Besides the linearity of the derivative, the first such result that we proved was the product rule. I will repeat the proof here.

Theorem. Suppose that f and g are differentiable at a. Then the product fg is also differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof. We begin with the definition,

$$(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

Now we add zero in the numerator, by both adding and subtracting the term $f(a+h)g(a)$, and split the result into two fractions.

$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(a+h)\,(g(a+h) - g(a))}{h} + \lim_{h\to 0} \frac{(f(a+h) - f(a))\,g(a)}{h} \right) \\ &= \left(\lim_{h\to 0} f(a+h)\right)\left(\lim_{h\to 0} \frac{g(a+h) - g(a)}{h}\right) + \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\right)\left(\lim_{h\to 0} g(a)\right).\end{aligned}$$

The last step requires some justification: we may only split limits like this if all limits exist. The first one, $\lim_{h\to 0} f(a+h)$, exists because f is continuous (because it is differentiable), and therefore the value of this limit is $f(a)$. The two limits involving the difference quotients of f and g exist by assumption, because we have said that f and g are differentiable. And of course the limit $\lim_{h\to 0} g(a)$ exists and equals $g(a)$, as it is the limit of a constant.

Thus we conclude that

$$f'(a) = f(a)g'(a) + f'(a)g(a),$$

which is what we set out to prove. $\square$.

(2) Two proofs of the power rule.

In this problem, you will prove the following theorem in two different ways.

Theorem. Let $f(x) = x^n$ for some $n \in \mathbb{N}$ and let $a \in \mathbb{R}$. Then $f'(a) = na^{n-1}$.

(a) Prove this by induction on n. Begin by proving the base case, i.e. that the claim is true when $n = 1$. Then prove that, if the theorem is true for some positive integer $n = k$, it is also true for the integer $n = k + 1$ due to the product rule.

[Hint: see the session 23 slides if you get stuck.]

(b) Prove this by explicit writing out the definition $\lim_{h\to 0} \frac{(a+h)^n - a^n}{h}$. Evaluate $(a+h)^n$ using the binomial theorem and then find the value of the limit.

[Hint: Note that you only need to compute the coefficient for one of the terms; the rest either cancel or do not affect the calculation (be sure to explain why!).]

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: We shall prove the power rule via induction, ie we are going to prove that it is true first when $n = 1$ and second, if the claim is true when $n = k$, then it is true for $n = k + 1$ where $k \in \mathbb{Z}$. First we have that $f(x) = x^1$, therefore this is a linear line with slope 1. We then have that $f'(a) = 1 = na^{n-1}$ where of course $n = 1$.

Now we must show that the product rule holds for $n + 1$. We have algebraically that $x^{n+1} = x^n \times x$. We can then write this as a multiplication of functions; let $g(x) = x^n$ and $h(x) = x$. We can then define $k(x) = g(x) \times h(x)$. We can then take the derivative of each side and see $k'(a) = (g \times h)'a$. By the product rule Christian has lovingly proved above we have that $k'(a) = g'(a)h(a) \times g(a)h'(a)$. We have shown previously that $h'(a) = 1$ since $h(x) = x$. We also have applied the assumption that $g'(a) = na^{n-1}$, and we have $k'(a) = na^{n-1} \cdot a + a^n = na^n + a^n$ We finally have that this finally simplifies to $(n + 1)a^n$, this claim then holds for all n. (jesus chrost feynman)

b: In this version of the proof of the power rule derivative, we will go about it by taking the derivative of the function $f(x) = x^n$. We see by the definition of derivative that $f'(x) = \lim_{h \to 0} \frac{(a+h)^n - (a)^n}{h}$. We have via the binomial theorem that $(a + h)^n$ equals $c_0 a^n + c_1 a^{n-1} h + c_2 a^{n-2} h^2 + \cdots + c_n h^n$, so $f'(x) = \lim_{h \to 0} \frac{(a^n + c_1 a^{n-1} h + c_2 a^{n-2} h^2 + \cdots + c_n h^n) - (a)^n}{h}$. We see that a^n zeroes out, all the while the h's present in this lovely polynomial cancel out, so we then have that $f'(x) = \lim_{h \to 0} c_1 a^{n-1} + c_2 a^{n-2} h + \cdots + c_n h^n$. This current equation follows the formula and therefore laws of polynomials, the law we are most interested in being the continuity of polynomials. We can then calculate for $h = 0$, giving us $f'(x) = c_1 a^{n-1}$. $c_1 = \binom{n}{1} = n$, therefore we can conclude $f'(x) = na^{n-1}$. $\square$

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) In the base case $n = 1$, we must prove that $\frac{d}{dx}(x) = \left(nx^{n-1}\right)|_{n=1} = 1$, but this is true because we have already proven that $\frac{d}{dx}(mx + b) = m$ and this is simply the case $m = 1$.

Now we induct. Suppose that the claim is true when $n = k$: that is, assume it is true that $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$. We must prove that the claim holds when $n = k + 1$, i.e., we must prove that $\frac{d}{dx}\left(x^{k+1}\right) = (k + 1)x^k$.

To prove this, write $x^{k+1} = x^k x$ and apply the product rule:

$$\begin{aligned}\frac{d}{dx}\left(x^{k+1}\right) &= \left(\frac{d}{dx}x^k\right)x + x^k\left(\frac{d}{dx}x\right) \\ &= kx^{k-1}x + x^k \\ &= (k+1)x^k.\end{aligned}$$

Note that, in the first term of the second line, we have used our inductive hypothesis: we have assumed that $\frac{d}{dx}x^k = kx^{k-1}$ because the claim holds for $n = k$.

Now that we have shown the power rule holds for $n = 1$, and that whenever it holds for $n = k$ it also holds for $n = k + 1$, it follows that the power rule is true for all nonnegative integers.

(b) Let $f(x) = x^n$. By the definition of the derivative,

$$f'(a) = \lim_{h \to 0} \frac{(a + h)^n - a^n}{h}.$$

By the binomial theorem, we have $(a + h)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + \binom{n}{n-1}h^{n-1}a + \binom{n}{n}h^n$. Thus

$$\begin{aligned}f'(a) &= \lim_{h \to 0} \frac{a^n + \binom{n}{1}a^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + \binom{n}{n-1}h^{n-1}a + \binom{n}{n}h^n - a^n}{h} \\ &= \lim_{h \to 0}\left(\binom{n}{1}a^{n-1} + \binom{n}{2}a^{n-2}h + \cdots + \binom{n}{n-1}h^{n-2}a + \binom{n}{n}h^{n-1}\right) \\ &= \binom{n}{1}a^{n-1}\end{aligned}$$

Here we have canceled one factor of h in the numerator and denominator, then applied continuity of polynomials in h to evaluate the limit by plugging in $h = 0$. But we have found in an earlier DC that $\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$, so we conclude

$$f'(a) = na^{n-1},$$

which is what was to be shown. $\square$

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*