question 4 views

Daily Challenge 2.1

(**Due: Tuesday 5/1 at 12:00 noon Eastern**)

A new week begins!

Review

We've seen that a **subset** is a smaller set which sits inside some larger set. For instance, if $A = \{6, 36\}$ and $B = \{6, 36, 216\}$, we say that A is a subset of B and write $A \subseteq B$.

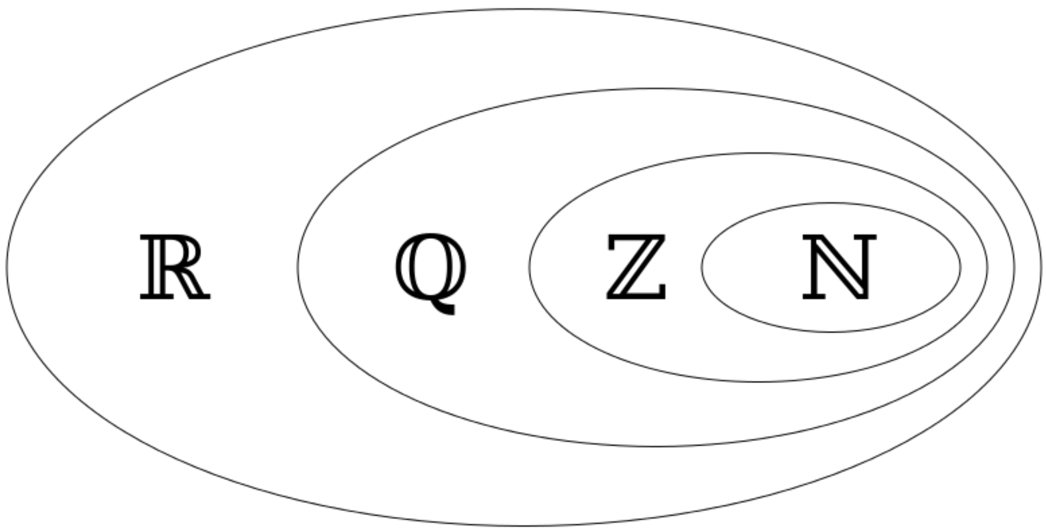
Here's a more formal definition. A set A is called a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B. That is,

$$A \subseteq B \iff \text{for each element } a \in A\ ,\ \text{ it is also true that } a \in B.$$

From the formal definition, we see that I was being sloppy when I spoke about a "smaller set" inside a "larger set" above, since the "larger set" can actually be the same size. Any set is a subset of itself: for any set A, we have $A \subseteq A$ (Proof: Every element of A is an element of A.)

For *finite* sets where we can list the elements, one can check whether some set is a subset of another by looking at each element, as we did above.

For *infinite* sets, we usually need to look at the condition defining the set instead. For instance, let's first recall the definitions of our number systems:



Working outward, these are:

- The **natural numbers** $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$are the "whole numbers" or "counting numbers". Some people also include 0 in their definition and write $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$. I will try to use the definition *without* zero.
- The **integers** $\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, \cdots\}$include the natural numbers, the number 0, and the negatives of all natural numbers.
- The **rationals** $\mathbb{Q} = \left\{\frac{n}{m} \mid n, m \in \mathbb{Z}, m \neq 0\right\}$ are all ratios of integers with non-zero denominator.
- The **reals** $\mathbb{R}$ include the entire number line, $(-\infty, \infty)$.

For example, how would we show that the integers are a subset of the rationals?

Theorem. $\mathbb{Z} \subseteq \mathbb{Q}$.

Proof. We must show that, for every $a \in \mathbb{Z}$, it is also true that $a \in \mathbb{Q}$. But given any integer a, we can also write a as the ratio $\frac{a}{1}$. By definition, the set of rationals

$$\mathbb{Q} = \left\{\frac{n}{m} \mid n, m \in \mathbb{Z}, m \neq 0\right\}$$

contains all ratios of integers with non-zero denominator, and $\frac{a}{1}$ is such a ratio, so we have $a \in \mathbb{Q}$. But this means $\mathbb{Z} \subseteq \mathbb{Q}$, as desired.

Problem

Try your hand at these questions involving subsets.

(a) The empty set $\emptyset$ has 0 elements and 1 subset, namely itself ($\emptyset \subseteq \emptyset$). The set $\{1\}$ has 1 element and 2 subsets, namely $\emptyset$ and $\{1\}$, and so on, as shown in the following table:

Set	# Elements	Subsets	# Subsets
$\emptyset$	0	$\emptyset$	1
$\{1\}$	1	$\emptyset, \{1\}$	2
$\{1,2\}$	2	$\emptyset, \{1\}, \{2\}, \{1,2\}$	4
$\{1,2,3\}$	3	$\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$	8

Guess the pattern: how many subsets do you think the set $\{1, 2, 3, \cdots, n\}$ has?

(b) Is it true that $\mathbb{N} \subseteq \mathbb{Q}$? (No proof required.)

(c) Is it true that $\mathbb{Q} \subseteq \mathbb{Z}$? (No proof required.)

(d) Is it true that $\mathbb{N} \subseteq \mathbb{R}$? (No proof required.)

(e) Is it true that $\mathbb{R} \subseteq \mathbb{Q}$? (No proof required.)

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Answers (Corbin). Your responses go here.
1. $\{1, 2, 3, \cdots, n\}$ has 2^n subsets
2. Yes
3. No
4. Yes
5. Yes

Answers (Logan).
1. The set $\{1, 2, 3, \cdots, n\}$ has 2^n subsets.
2. Yes.
3. No.
4. Yes.
5. Yes.

Updated 9 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Answers (Christian).

(a) The set $\{1, 2, \cdots, n\}$ has 2^n distinct subsets. In fact, we could even prove this as follows: let A be any subset of $\{1, 2, \cdots, n\}$. We construct a binary string by putting a 0 in the k-th place if the number k is *not* in A, and put a 1 in the k-th place if the number k *is* in A.

For instance, if $n = 5$, the subset $A = \{1, 3, 4\}$ of $\{1, 2, 3, 4, 5\}$ has the binary representation 1011, since

$$A = \{1, 3, 4\} \implies \underbrace{1}_{1 \in A} \underbrace{0}_{2 \notin A} \underbrace{1}_{3 \in A} \underbrace{1}_{4 \in A} \underbrace{0}_{5 \notin A}.$$

That is, just put ones where the element is present and zeros where it's absent.

Each such binary string defines a different subset. To count the number of such strings, note that we have n independent choices (we can choose 0 or 1 for each spot), so the number of such strings is $\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$.

(b) Yes; every natural number $n \in \mathbb{N}$ can also be written as $\frac{n}{1}$, so it's in $\mathbb{Q}$.

(c) No; a counterexample is $\frac{1}{2} \in \mathbb{Q}$, but $\frac{1}{2} \notin \mathbb{Z}$.

(d) Yes; every natural number sits on the number line $(-\infty, \infty)$.

(e) No; the reals also contain irrational numbers, like $\sqrt{2}$, which are not in $\mathbb{Q}$.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*