note 3 views

Daily Challenge 3.7

Perhaps, given the trauma of reinstalling Windows and 300 GB of games (**edit**: maybe not the latter, thanks to D:\), we should take a make-up day to catch up on problems.

Thus I propose:

- No problem today, but I will write a quick note about the number e. Please read it!
- Let's focus on submitting **DC 2.4, DC 3.5,** and **DC 3.6**. It's fine to read my solutions and re-write them in your own words; it's more important to *understand* the results than to be able to figure them out yourself on the first try.

"Review"

When we discussed Euler's formula, I asked you to accept my use of the number e:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

I claimed that e was just some irrational number like π, with a decimal expansion $e \approx 2.718\cdots$, but this is really not doing it justice. This e is the most important number in *analysis*, which is the branch of mathematics of which calculus is a part.

Let me describe one way to define e (we will see others). Suppose I am a biologist modeling the growth of a colony of bacteria. I begin with N bacteria, and I know that each bacterial cell divides into two daughter cells once per day. My first attempt to model the population, then, might say that I have N bacteria until the end of the day, when they all divide and then I end up with $2N$, which all divide so that I get $4N$ at the end of the next day, and so on.

$$N_{\text{bacteria}}(t) = \begin{cases} N & 0 \text{ days } \le t < 1 \text{ day} \\ 2N & 1 \text{ day } \le t < 2 \text{ days} \\ 4N & 2 \text{ days } \le t < 3 \text{ days} \\ \vdots & \end{cases}.$$

This is a "jumpy" model -- my equation predicts that the population is constant all day, then "jumps" to twice its value at the end of the day.

How can we make the model better? Of course, not all bacteria will divide at the same time at the end of the day; they all have different "ages", so they will divide continuously throughout the day.

As a first step towards this, let's assume that *half* of these bacteria divide at noon. Then at the end of the day, we again suppose that *half* of them divide.

Then at noon, our population changes as

$$N \longrightarrow \underbrace{\frac{N}{2}}_{\text{half don't divide}} + \underbrace{2 \times \left(\frac{N}{2}\right)}_{\text{half do divide}} = \frac{3N}{2}.$$

At the end of the day, half of these $\frac{3N}{2}$ bacteria divide, so

$$\frac{3N}{2} \longrightarrow \underbrace{\frac{3N}{4}}_{\text{half don't divide}} + \underbrace{2 \times \frac{3N}{4}}_{\text{half do divide}} = \frac{9N}{4}.$$

So in the new model, there are *more bacteria at the end of the day,* $\frac{9N}{4} = 2.25 \times N$ rather than $2N$, than in the old model.

We've seen this story before: we have some approximation, and a method for making it better, so we should take a limit to get an exact answer. This is the whole idea of calculus.

Instead of breaking the bacteria into two groups, let's break them into m groups, where m is some integer. Then there will be m separate divisions throughout the day, and $\frac{1}{m} \times N$ bacteria will divide each time. After the first division, we have

$$\text{first division: } N \longrightarrow \underbrace{\frac{m-1}{m}N}_{\text{don't divide}} + \underbrace{2 \times \frac{1}{m}N}_{\text{do divide}} = N\left(1 + \frac{1}{m}\right).$$

After the second division, this happens again, which has the effect of multiplying by $\left(1 + \frac{1}{m}\right)$ a second time:

$$\text{second division: } N\left(1 + \frac{1}{m}\right) \longrightarrow N\left(1 + \frac{1}{m}\right)^2.$$

At the end of the day, we've multiplied by this factor m times:

$$\text{end of day: } N \to N\left(1 + \frac{1}{m}\right)^m.$$

Now for the coup de grâce. **Take the limit as m goes to infinity**. This should give us the *exact* result for the end-of-day population for a colony of bacteria that divides continuously. That is,

$$\text{Exact result:} \quad N_{\text{end-of-day}} = N_{\text{start-of-day}} \times \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m.$$

I haven't told you what a limit means, precisely (I will in chapter 2!), but for now just think of it as "let m get bigger and bigger, and see whether the result approaches some constant value." We define that constant value as the mathematical constant e:

$$\boxed{e \equiv \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.}$$

Savor this moment. The number e contains some deep magic.

What does this number look like, explicitly? We can ask Python to compute a few rounds of iteration:

In [3]:
```
import numpy as np, matplotlib.pyplot as plt

approx_e_list = [np.power(1+1/n, n) for n in range(1,20)]
approx_e_list
```

Out[3]:
```
[2.0,
 2.25,
 2.37037037037037,
 2.44140625,
 2.4883199999999994,
 2.5216263717421135,
 2.546499697040712,
 2.565784513950348,
 2.5811747917131984,
 2.5937424601000023,
 2.6041990118975287,
 2.613035290224676,
 2.6206008878857308,
 2.6271515563008685,
 2.6328787177279187,
 2.6379284973666,
 2.64241437518311,
 2.6464258210976865,
 2.650034326640442]
```

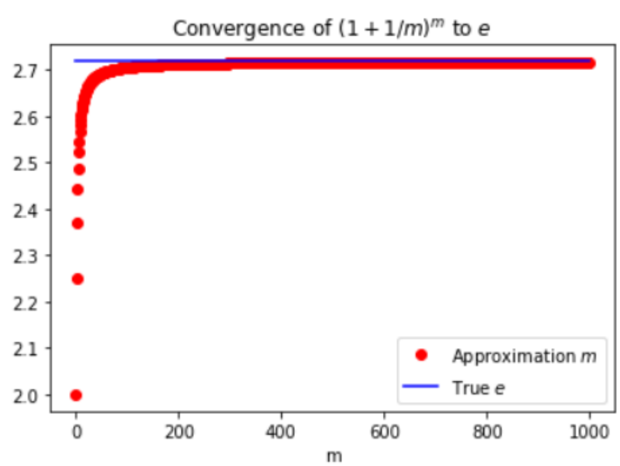
In [4]:
```
np.e
```

Out[4]:
```
2.718281828459045
```

At $m = 20$, Python gives us the approximation $e \approx 2.65\cdots$, which is getting closer to the true value $e \approx 2.718\cdots$ given on the next line. This definition converges rather slowly, so we need to go to higher m (say, 100 or 1000) to get better accuracy. Here's a plot up to $m = 1000$:

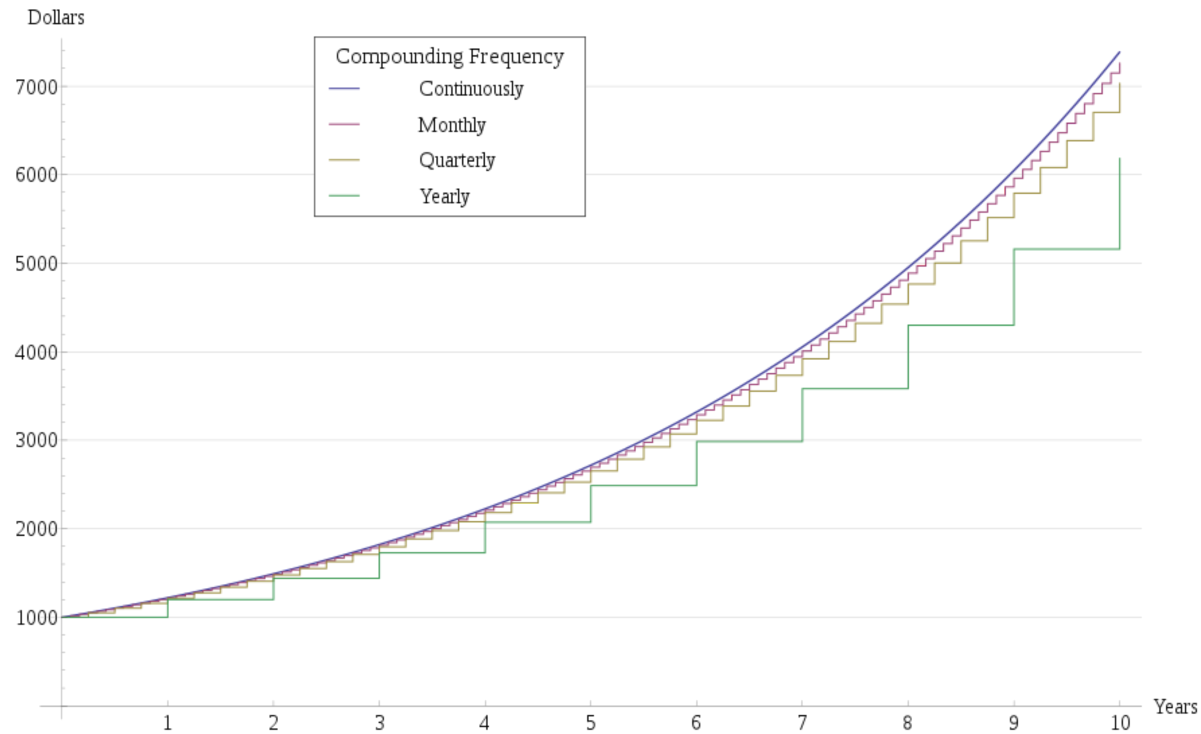
In [10]:
```
approx_e_list = [np.power(1+1/n, n) for n in range(1,1000)]
plt.plot(approx_e_list, 'ro')
plt.plot(np.full(np.shape(approx_e_list), np.e), 'b-')
plt.legend(["Approximation $m$", "True $e$"])
plt.xlabel("m")
plt.title("Convergence of $( 1 + 1/m )^m$ to $e$")
```

Out[10]:
```
Text(0.5,1,'Convergence of $( 1 + 1/m )^m$ to $e$')
```



One final comment: rather than bacteria, e is often introduced in the context of compound interest. Instead of dividing bacteria, you can think of investing $\$1000$ and having the interest compound annually, versus monthly, versus daily.

Here is a plot of the result:



You can think of the "jumpy" green curve, that stays mostly flat but jumps up at the end of each year, as our first model for the bacterial population. The quarterly and monthly curves are like our second model that jumps twice per day -- they are getting better, but still have sharp "corners" at each division time. Only in the limit as we take infinitely many divisions do we get the smooth continuous curve shown in purple above, which is in fact the plot of $f(x) = e^x$.

daily_challenge

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*