question **2** views

Daily Challenge 12.7

~~(**Due: Sunday 8/26 at 12:00 noon eastern**)~~
(**Due: Thursday 8/30 at 12:00 noon eastern**)

Apologies again for the second round of fairly routine questions, but one must really master this stuff.

(1) Problem: taking some derivatives.

Find the derivative of the specified function in each of the following cases. Again, this should not take longer than 20 minutes; practice until this becomes as automatic as addition.

After you have computed all of the derivatives, **choose one function to differentiate in Mathematica** and verify your result. The syntax in Mathematica is

```
D[f[x], x]
```

For instance, if you were checking your result for part (c), you would open Mathematica and type

```
D[Log[Sin[x] Cos[x]], x]
```

and then hit Shift+Enter.

(a) $f(x) = \frac{1}{(2x+3)^2}$

(b) $f(x) = \sqrt{x^3 + e^x}$

(c) $f(x) = \log(\sin(x)\cos(x))$

(d) $f(x) = \sqrt[3]{1 + \sqrt{x + \sqrt{e^x}}}$

(e) An attempt to liven things up: an electron in the $2s$ orbital of hydrogen has a wavefunction of the form

$$\psi(x) = c\,(2 - x)\,e^{-\frac{1}{2}x},$$

where c is a constant and x is a variable representing the distance between the electron and proton in some units. The momentum of the electron in this orbital is described by $\hat{p}\psi$, which is proportional to $\frac{d\psi}{dx}$.

Compute the latter; that is, find $\frac{d\psi}{dx}$.

(f) $f(x) = e^{-x^2}$

(g) $f(x) = \sin^{-1}(x^2)$. As usual, this is the *inverse sine function* or *arcsine*, not $\frac{1}{\sin}$.

(h) $f(x) = \sin^3(x)$

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

a: We see that by the chain rule and power rule that $f'(x) = \left(\frac{1}{(2x+3)^2}\right) = \left((2x+3)^{-2}\right)' = -2(2x+3)^{-3} \cdot 2x.$

b: Once again, by the chain rule and power rule
 $f'(x) = \left((x^3+e^x)^{0.5}\right)' = 0.5(x^3+e^x)^{-0.5} \cdot (3x^2+e^x)$

c: First, by the chain rule we see that $f'(x) = (\log(\sin(x)\cos(x)))' = \frac{1}{\sin(x)\cos(x)} \cdot (\sin(x)\cos(x))' = \frac{1}{\sin(x)\cos(x)} \cdot (cos^2(x) + -\sin^2(x)) = \frac{cos^2(x)+-\sin^2(x)}{\sin(x)\cos(x)}$

d: We see first by the application of the chain rule and product rule that

$$\begin{aligned}((1+\sqrt{x+\sqrt{e^x}})^{\frac{1}{3}})' &= \frac{1}{3}(1+\sqrt{x+\sqrt{e^x}})^{\frac{-2}{3}} \cdot (1+\sqrt{x+\sqrt{e^x}})' \\ &= \frac{1}{3}(1+\sqrt{x+\sqrt{e^x}})^{\frac{-2}{3}} \cdot (\sqrt{x+\sqrt{e^x}})'\end{aligned}$$

We then see by the chain rule and power rule applied to the derivative of interest that $(\sqrt{x+\sqrt{e^x}})' = 0.5(\sqrt[-2]{x+\sqrt{e^x}}) \cdot (1 + (\sqrt{e^x})') = 0.5(\sqrt[-2]{x+\sqrt{e^x}}) \cdot (1 + 0.5(\sqrt[-2]{e^x}) \cdot e^x)$ or after substituting into the original equation, $((1+\sqrt{x+\sqrt{e^x}})^{\frac{1}{3}})' = \frac{1}{3}(1+\sqrt{x+\sqrt{e^x}})^{\frac{-2}{3}} \cdot 0.5(x+\sqrt{e^x})^{\frac{-1}{2}} \cdot (1 + 0.5(\sqrt[-2]{e^x}) \cdot e^x)$ Simplifying this is gonna suck, so I'll do it later!

e: We see by the product rule that

$$\begin{aligned}\psi'(x) &= (c(2-x))' \cdot e^{-\frac{1}{2}x} + c(2-x) \cdot (e^{-\frac{1}{2}x})' \\ &= -c \cdot e^{-\frac{1}{2}x} + c(2-x) \cdot ((e^x)^{-\frac{1}{2}})' \\ &= -c \cdot e^{-\frac{1}{2}x} + c(2-x) \cdot \frac{-1}{2}((e^x)^{-\frac{3}{2}}) \cdot e^x\end{aligned}$$

f: We can break this into two functions, e^x and $-x^2$, and then chain rule where the latter is the inner function; $f'(x) = -2x \cdot e^{-x^2}$.
g: Chain rule this boitch; The derivative of arcsin is defined by the inverse function rule, where we see $(\sin^- 1(x))' = \frac{1}{\cos(\sin^{-1}(x))}$, therefore $f'(x) = \frac{1}{\cos(\sin(x^2))} \cdot 2x$
h: Apply the chain rule where the inner function is $\sin(x)$ and the outer is x^3, we then see $f'(x) = 3\sin^2(x) \cdot \cos(x)$

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) $f(x) = \frac{1}{(2x+3)^2}$

Solution. This can be written as $f(x) = (2x+3)^{-2}$, so by the chain rule and power rule, $f'(x) = -4(2x+3)^{-3}$.

(b) $f(x) = \sqrt{x^3 + e^x}$

Solution. We write this as $f(x) = \left(x^3 + e^x\right)^{\frac{1}{2}}$, so $f'(x) = \frac{1}{2}\left(x^3 + e^x\right)^{-\frac{1}{2}} \cdot \left(3x^2 + e^x\right)$, or

$$f'(x) = \frac{3x^2 + e^x}{2(x^3 + e^x)^{\frac{1}{2}}}.$$

(c) $f(x) = \log(\sin(x)\cos(x))$

Solution. Apply the chain rule and product rule to find $f'(x) = \frac{1}{\sin(x)\cos(x)} \cdot \left(\cos^2(x) - \sin^2(x)\right)$. If you like, we can re-express this using a Pythagorean identity as

$$f'(x) = \frac{2\cos^2(x) - 1}{\sin(x)\cos(x)}.$$

(d) $f(x) = \sqrt[3]{1 + \sqrt{x + \sqrt{e^x}}}$

Solution. This is $f(x) = \left(1 + \left(x + e^{x/2}\right)^{1/2}\right)^{1/3}$, so

$$\begin{aligned}f'(x) &= \frac{1}{3}\left(1 + \left(x + e^{x/2}\right)^{1/2}\right)^{-2/3} \cdot \frac{1}{2}\left(x + e^{x/2}\right)^{-1/2} \cdot \left(1 + \frac{1}{2}e^{x/2}\right) \\ &= \frac{1 + \frac{1}{2}e^{x/2}}{6\left(1 + \left(x + e^{x/2}\right)^{1/2}\right)^{2/3}\left(x + e^{x/2}\right)^{1/2}}.\end{aligned}$$

(e) An attempt to liven things up: an electron in the $2s$ orbital of hydrogen has a wavefunction of the form

$$\psi(x) = c\,(2-x)\,e^{-\frac{1}{2}x},$$

where c is a constant and x is a variable representing the distance between the electron and proton in some units. The momentum of the electron in this orbital is described by $\hat{p}\psi$, which is proportional to $\frac{d\psi}{dx}$.

Compute the latter; that is, find $\frac{d\psi}{dx}$.

Solution. By the product rule and chain rule,

$$\frac{d\psi}{dx} = -ce^{-\frac{1}{2}x} - \frac{c}{2}(2-x)e^{-\frac{1}{2}x}.$$

(f) $f(x) = e^{-x^2}$

Solution. We've done this in a meeting before; it is the Gaussian distribution. We see that $f'(x) = -2xe^{-x^2}$.

(g) $f(x) = \sin^{-1}\left(x^2\right)$. As usual, this is the *inverse sine function* or *arcsine*, not $\frac{1}{\sin}$.

Solution. Recall from the inverse function rule meeting that the derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. So by the chain rule,

$$f'(x) = \frac{2x}{\sqrt{1-x^4}}.$$

(h) $f(x) = \sin^3(x)$

Solution. By the chain rule, $f'(x) = 3\sin^2(x)\cos(x)$.

I have attached Mathematica verifications of these results below.

In[12]:= $\mathbf{D}\left[\dfrac{1}{(2x+3)^2}, x\right]$

Out[12]= $-\dfrac{4}{(3+2x)^3}$

In[13]:= $\mathbf{D}\left[\sqrt{x^3 + E^x}, x\right]$

Out[13]= $\dfrac{e^x + 3x^2}{2\sqrt{e^x + x^3}}$

In[14]:= `D[Log[Sin[x] Cos[x]], x]`

Out[14]= `Csc[x] Sec[x] (Cos[x]^2 - Sin[x]^2)`

In[15]:= $\mathbf{D}\left[\sqrt[3]{1+\sqrt{x+\sqrt{E^x}}}, x\right]$

Out[15]= $\dfrac{1+\dfrac{\sqrt{e^x}}{2}}{6\sqrt{\sqrt{e^x}+x}\left(1+\sqrt{\sqrt{e^x}+x}\right)^{2/3}}$

In[16]:= $\mathbf{D}\left[c\,(2-x)\,E^{\frac{-1}{2}x}, x\right]$

Out[16]= $-c\,e^{-x/2} - \dfrac{1}{2}c\,e^{-x/2}\,(2-x)$

In[17]:= $\mathbf{D}\left[E^{-x^2}, x\right]$

Out[17]= $-2\,e^{-x^2}\,x$

In[18]:= $\mathbf{D}\left[\mathrm{ArcSin}\left[x^2\right], x\right]$

Out[18]= $\dfrac{2x}{\sqrt{1-x^4}}$

In[19]:= $\mathbf{D}\left[(\mathrm{Sin}[x])^3, x\right]$

Out[19]= $3\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^2$

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*