question 2 views

Daily Challenge 11.4

~~(**Due: Tuesday 8/14 at 12:00 noon Eastern**)~~
(**Due: Wednesday 8/15 at 12:00 noon Eastern**)

(1) We can differentiate a function with a non-standard argument, but be careful.

If f is a real-valued function and we write the symbols $f(x)$ to emphasize that the function depends on a dummy variable x, this symbol x is called the *argument* of f.

Typically we write the argument of a function as a single letter, like x or t in $f(x)$ or $f(t)$. But often in scientific applications, it is necessary to evaluate a function on a more complicated argument, like $f(x+t)$ or $f(x^2 t)$.

The meaning of these symbols is, I hope, clear: if one has a formula or table of values defining f for various inputs, then to evaluate $f(x+t)$, one first adds x and t to find the appropriate input, and then looks up or computes the value of f for this input.

Technically, we can evaluate derivatives of these functions with non-standard argument using the *chain rule*, since they can be seen as composite functions. But since we have not proven the chain rule yet nor introduced it in daily challenges, we will work directly from the definition for today.

First consider a function whose composite argument is simply a re-scaling of a variable by a constant.

Theorem. If $g(x) = f(cx)$, $c \neq 0$ is a constant, and f is differentiable, then $g'(x) = cf'(x)$.

Proof. Using the definition of the deriative,

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(cx+ch) - f(cx)}{h}. \end{aligned}$$

Since c is a constant, we can multiply by 1 in the form $\frac{c}{c}$ to find

$$\begin{aligned} g'(x) &= \left(\lim_{h\to 0} \frac{f(cx+ch) - f(cx)}{h}\frac{c}{c}\right) \\ &= c\left(\lim_{h\to 0} \frac{f(cx+ch) - f(cx)}{ch}\right) \\ &\overset{!}{=} cf'(cx). \end{aligned}$$

First we have used that c is a constant, so we may pull it out of the limit due to our results from chapter 2. The last step requires some explanation; consider the limit $\lim_{h\to 0} \frac{f(cx+ch)-f(cx)}{ch}$.

We must refer to a result proved in chapter 2: if we have two functions v, w such that v is continuous at b, $\lim_{t\to b} v(t) = V$, and $\lim_{t\to a} w(t) = b$, then

$$\lim_{t\to a} v(w(t)) = V.$$

In this case, let $v(t) = \frac{f(cx+t)-f(t)}{t}$ and $w(t) = ct$. Then our theorem guarantees that

$$\begin{aligned} c\lim_{h\to 0} \frac{f(cx+ch) - f(cx)}{ch} &= c\lim_{t\to 0} \frac{f(cx+t) - f(cx)}{t} \\ &= cf'(cx). \end{aligned}$$

In conclusion, we have proven that if $g(x) = f(cx)$, then $g'(x) = cf'(cx)$. $\square$

Notice that we must be careful with these composite arguments! One **cannot** simply "put primes on both sides of the equation" to find

$$\textbf{Wrong: } g(x) = f(cx) \implies g'(x) = f'(cx).$$

(2) Problem: two functions with sum arguments.

Find $f'(x)$ in two cases: first, if $f(x) = g(t+x)$, and second, if $f(t) = g(t+x)$. The answers will *not* be the same.

(Note that this is problem 1(c) on CD 3. Please copy over your solution when you're done.)

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
Exploration (potentially answer): We first consider the case that $f(x) = g(t+x)$; Consider a point $a \in \mathbb{R}$; we have that $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{a\to 0} \frac{g(t+a+h)-g(a)}{h} = g'(t+a)$. The third line is true both by the substitution of $f(x) = g(t+x)$ and as well the definition of derivative applied to $g'(t+x)$. Therefore $f'(x) = g'(t+x)$. We can then consider the case $f(t) = g(t+x)$. Consider a point $a \in \mathbb{R}$ as well. We have that $f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = lim_{h\to 0} \frac{g(a+h+x)-g(a+x)}{h}$, with the 2nd step in the prior equation the result of substituting $t = a+h$ into $f(t) = g(t+x)$. We see that $\lim_{h\to 0} \frac{g(a+h+x)-g(a+x)}{h} = g'(a+x)$, and we see by replacing the dummy variable a with x that $f'(x) = g'(2x)$.

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

In the first case, $f(x) = g(t+x)$, we have $f'(x) = g'(t+x)$.

In the second case, $f(t) = g(t+x)$, we have $f'(x) = g'(2x)$.

To see why, one should return to the definition. The derivative of the first function, $f(x) = g(t+x)$, at a point $a \in \mathbb{R}$ is defined by

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{g(t+a+h)-g(t+a)}{h}, \end{aligned}$$

where in the second step we have plugged in using the definition $f(a) = g(t+a)$. But the second limit is, by definition, $g'(t+a)$. Replacing the dummy variable a with x, we find $f'(x) = g'(t+x)$ in this case.

Now consider the second function, $f(t) = g(t+x)$. Again by the definition of derivative,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{g(a+h+x)-g(a+x)}{h}, \end{aligned}$$

where in the second step we have substituted $a+h$ for t in $f(t) = g(t+x)$ and likewise in the second term. But this expression is precisely $g'(a+x)$. Replacing the dummy variable a by x, we see that $f'(x) = g'(2x)$ in this case.

The punchline is that the argument of the function matters: $f(x) = g(t+x)$ is different from $f(t) = g(t+x)$. In the first case, x is a dummy and t is some constant; in the second case, t is a dummy and x is some constant.

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*