question 2 views

Daily Challenge 17.5

(**Due: Friday 11/9 at 12:00 noon Eastern**)

Let's revisit one of my favorite functions.

(1) Problem: Babylon's revenge.

We proved in a meeting that the function

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is not integrable on any closed interval $[a, b]$. Roughly, every lower sum is $0 \cdot (b - a)$ and every upper sum is $1 \cdot (b - a)$, so they never get close.

Now consider Stars over Babylon,

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases},$$

which one might think is also not integrable because it jumps around so much.

Prove that Stars over Babylon is integrable on $[0, 1]$ and $\int_a^b f = 0$.

[Hint: Every lower sum is zero, by an argument similar to the one we used for g. You must figure out how to make the upper sums small. It may be helpful to revisit the proof you wrote in chapter 2, which showed that f is continuous at irrational points.]

daily_challenge

Updated 5 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

To prove that the *Stars Over Babylon* function is integrable, we must first show that the upper and lower sums are equal. We let $\epsilon > 0$ be given, and we want to find that $U(f, P) - L(f, P) < \epsilon$. First we begin by proving the lower sum is less than epsilon; we see automatically that because there are irrational numbers between all rationals, that the lower sum $\sum_{i=1}^{n} \inf(f, [t_{i-1}, t_i]) * (t_i - t_{i-1}) = \sum_{i=1}^{n} 0 * (t_i - t_{i-1}) = 0$ since there are guaranteed to be irrationals between all rationals, and potentially irrationals between irrationals to handle a potentially irrationally ended interval. The upper sum is much more difficult to handle; We must show somehow create a partition such that $U(f, P) < \epsilon$ (because the lower sum is always zero). We begin by choosing some sufficiently large integer N such that $\frac{1}{N} < \frac{\epsilon}{2}$. We then create a list of the rationals in $(0, 1]$ with numerator less than N, ie

$$Q = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \cdots \frac{N-1}{N} \right\}$$

We then see that we can upper bound the amount of numbers in this set since there are less than N choices of integer for the numerator and N choices of integer for the denominator, therefore $\operatorname{len}(Q) < N^2$. We begin to assemble our partition, beginning with one made up of the endpoints a, b; we will append to this partition two values for each value in Q:

$$t_{i-1} = \frac{p_i}{q_i} - \frac{\delta_i}{2}$$
$$t_i = \frac{p_i}{q_i} + \frac{\delta_i}{2}$$

where $\frac{p_i}{q_i}$ corresponds to one of the fractions in Q, and δ_i is the minimum of half the distance between $\frac{p_i}{q_i}$ and some other faction in Q, or $\frac{\epsilon}{4N^2}$. We then must show that our upper sum with such a partition is less than ϵ, so first break the upper sum into two parts those with subintervals containing elements of Q, and those that do not.

For the first set of sums, consider the following: each supremum is $\frac{p_i}{q_i}$, and the width is $\frac{\epsilon}{2N^2}$. there are less than N^2 subintervals to be concerned about, and thus this first set is less than $\frac{\epsilon}{2}$.

For the second set of sums not containing values seen in Q, we see that the rationals here must have denominators greater than N, and therefore the supremum of all subintervals here is $\frac{1}{N+1}$. The total length of this interval is 1 and thus the maximum area of the addition of all these subintervals does not exceed $\frac{1}{N+1}$, and since we required above that $\frac{1}{N} < \frac{\epsilon}{2}$, then this second interval can not possibly exceed $\frac{\epsilon}{2}$.The sum of these intervals is thus less than ϵ, and therefore $U(f, P) - L(f, P) < \epsilon$ and THOMMAEEE is integrable on $[0, 1]$. We then see that since $L(f, P) = 0$ and $U(f, P)$ can be made less than epsilon, the integral $\int_a^b = 0$.

Updated 4 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Let $\epsilon > 0$ be given. We must construct a partition P such that $U(f, P) - L(f, P) < \epsilon$

If P is any partition, we have $L(f,P)=0$, since every interval contains an irrational number x at which $f(x)=0$. This is the infimum on every sub-interval, so all m_i are 0 and thus $L(f,P)=0$.

Thus all we need to do is build a partition P so that $U(f,P)<\epsilon$. Pick an integer N so that $\frac{1}{N}<\frac{\epsilon}{2}$. Now write down all of the fractions with denominators less than or equal to N (in lowest terms) in a list:

$$\frac{1}{1};\frac{1}{2};\frac{1}{3},\frac{2}{3};\frac{1}{4}\frac{3}{4};\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\cdots;\frac{1}{N},\ldots,\frac{N-1}{N}.$$

Note that there are at most N^2 numbers in that list (N different denominators, and at most N numerators per denominator).

We start with a partition $P=\{a,b\}$, and then for each number $\frac{p_i}{q_i}$ in the above list, we will add two numbers to our partition which "cuts out" a tiny interval around that fraction. More precisely, we add points

$$t_{i-1}=\frac{p_i}{q_i}-\frac{\delta_i}{2},$$
$$t_i=\frac{p_i}{q_i}+\frac{\delta_i}{2}$$

where δ_i are small widths chosen as follows: let δ_i be either half the minimum of the distances between $\frac{p_i}{q_i}$ and any other fraction in the above list, or else $\frac{\epsilon}{4N^2}$, whichever is smaller. (The first condition ensures that no two points in the list lie in the same interval; the second condition ensures that all sub-intervals have length at most $\frac{\epsilon}{2N^2}$.

We claim that the upper sum in the partition constructed in this way is less than ϵ. To see this, split the sum into two pieces: one piece includes all sub-intervals with a point in the above list, and one piece includes all other sub-intervals.

1. For the first sum, the supremum of the function on each sub-interval $\frac{p_i}{q_i}$ is $\frac{1}{q_i}$. This is multiplied by the length of the interval, which is $\frac{\epsilon}{2N^2}$. There can be at most N^2 such sub-intervals, as argued above, so this part of the sum is at most $\frac{\epsilon}{2}$.

2. Now the second sum. These pieces contain only sub-intervals with no points in the above list. But that list exhausts all denominators up to N, so these sub-intervals can therefore only contain rationals with denominators greater than N. This means that the supremum of the function on these intervals is at most $\frac{1}{N+1}$. The total length of these intervals is certainly at most 1, since the whole interval is $[0,1]$, so their contribution to $U(f,P)$ can be at most $\frac{1}{N+1}\cdot 1$. But we said above that $\frac{1}{N}<\frac{\epsilon}{2}$, so this contribution to the sum is again less than half of epsilon.

The two pieces (1) and (2) above combine to give the entire upper sum $U(f,P)$, but each piece is less than $\frac{\epsilon}{2}$, so we conclude that $U(f,P)-L(f,P)<\epsilon$and f is integrable on $[0,1]$.

Finally, since $0=L(f,P)\le\int_0^a f\le U(f,P)$in any partition, and the infimum of the upper sums equals 0, the value of the integral must be zero. $\square$

Updated 5 months ago by Christian Ferko

followup discussions *for lingering questions and comments*