question 4 views

Daily Challenge 1.1

(**Due: Tuesday 4/24 at 12:00 noon)**

Let's start daily problems! In the official AoPS online calculus course, they use the term "challenge sets" rather than "problem sets," to emphasize that all of the problems are challenging so you're only expected to attempt them, not necessarily solve them. In this spirit, I'll call these "Daily Challenges".

Each day, I'll write up a quick introduction, review, or example along with the challenge. Reading this should help you understand and solve it.

The game is:

- I post a new challenge every morning at breakfast, and it will be due the next day at noon.
- Enter your solutions directly into the student response, which I've separated into two sections for you. Usual rules apply (complete sentences, explain your reasoning, and give me any thoughts or ideas rather than an "I don't know").
- You're in good shape if you attempt at least 6 of the 7 challenges in a given week. (Of course, it's even better to try all 7!)
- **30 minute rule**: After you finish reading the review, note the time when you start working on the challenge. If you haven't solved it after 30 minutes, I suggest that you stop and write up your thoughts, ideas, and partial progress. I don't want to inflict too much agony by asking you to bang your head against a question for more than half of an hour.

Review

Yesterday, I presented a strategy for a certain type of argument called "direct proofs." These are proofs which mostly involve un-wrapping the definitions of terms and showing that they're equivalent. The steps were:

1. Replace all general statements with specific statements about named variables.
2. Un-pack the definitions of words and symbols in the **assumptions** by replacing them with simpler terms (you may have to look up the definitions).
3. Un-pack the definitions of words and symbols in the **conclusion**. This is what you want to show.
4. Construct a sequence of logical steps which shows that, if the un-packed statements in your assumptions are true, then the statements in your conclusion are also true.

We saw an example involving even numbers. An integer n is called **even** if we can write $n = 2k$ for some other integer k. In words, an integer is even if we can write it as two times some other integer.

Then we proved that the sum of two even numbers is even. Let's read this again carefully.

Theorem. The sum of two even integers is even.

Proof. We begin with step 1 of the strategy. The words "sum of two even numbers" are general, so we replace these with named variables. The theorem becomes: "if m is an even integer and n is an even integer, then $m + n$ is an even integer."

The assumptions are "m is an even integer and n is an even integer." We take this statement to be true. The conclusion is "$m + n$ is an even integer"; this is what we need to prove.

Next we un-pack the word "even" using the definition. If m and n are even, then this means we can write

$$m = 2k_1 \quad , \quad n = 2k_2 \quad ,$$

where k_1 and k_2 are both integers.

The statement which we need to prove, then, is that $m + n = 2k_3$ for some other integer k_3. How do we prove that this is true? Well, let's write out the sum $m + n$:

$$m + n = 2k_1 + 2k_2 = 2\underbrace{(k_1 + k_2)}_{\text{integer}}.$$

But we know that the sum of two integers is an integer, so $k_1 + k_2$ is an integer. Therefore, the sum $m + n = 2(k_1 + k_2)$ can be written as two times some integer! Specifically, we can write $m + n = 2k_3$, where $k_3 = k_1 + k_2$ is an integer. This completes the proof. $\square$

Problem

Make sure you understand the proof above, which shows that the sum of two even integers is even. Then prove the following:

Theorem. The square of an even integer is even.

This argument is very similar, so you can mimic the style of the previous proof and make the appropriate changes.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Proof (Corbin). Your proof goes here.

First let us define an even number as the variable n. Next n must be even so we say that $n = 2k$ where k is any realinteger. Next we must say that $n_1 \times n_1 =$ some even integer. Knowing that $n_1 = 2k$ I believe we can apply it so that our equation now becomes $n_1 \times n_1 = 2k_1 \times 2k_1$. This shows us also that $n_1 \times n_1 = 2k_1 \times 2k_1$ is equal to $n_1 \times n_1 = 4k_1$. And as we know adding even numbers together will always result in an even number so now we have the equivalent of adding k together 4 times. I think this shows how multiplication of 2 even numbers will always be even. I think another way to write this is that for the Mth power of an even integer we have $n^m = (m \times 2)k$ $\square$

Proof (Logan) -
First, I define the "original" even number as a variable, in this case I'll define it as m_1.
I must prove that the square of an even number is even, represented by $m_1 * m_1 = 2k$ such that $k \in \mathbb{R}$
m_1 is even, defined as such by the statement $m_1 = 2k$ once again such that $k \in \mathbb{R}$.
I can create the statement from my prior knowledge $m_1 * m_1 = 2k * 2k$.
I have at this point spent half an hour of active work on this problem, and I have once again hit this mental wall. I simply *don't know what I'm not understanding,* because this seems like it should be an incredibly easy problem. $\square$

Updated 11 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Proof (Christian). Let m be an even integer. Then we can write $m = 2a$, where a is also an integer.

The number m^2 is $m \cdot m = (2a) \cdot (2a) = 4a^2$. But we can write this as

$$m^2 = 2\underbrace{\left(2a^2\right)}_{\text{integer}}.$$

The number $2a^2$ is an integer, because it is a product of integers. Therefore, we have written m^2 as two times some integer -- in particular $m^2 = 2k$, where $k = 2a^2$ is an integer. This is the definition of "even", so we have shown that m^2 is an even integer. $\square$

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*