question 3 views

# Daily Challenge 7.7

(**Due: Wednesday 6/13 at 11:59 pm Eastern**. Note that it is due in the *evening*; hopefully this allows enough time for finals studying on the preceding days, but if not, let me know and we can push to Thursday.)

This is problem 10 in the consolidation document, but let's try to post a first attempt here on Piazza by Wednesday in case we need to make revisions before copying the polished final version over.

Problem

A number $x$ is said to be *algebraic* if there exist integers $a_0, \cdots, a_n$, not all zero, such that

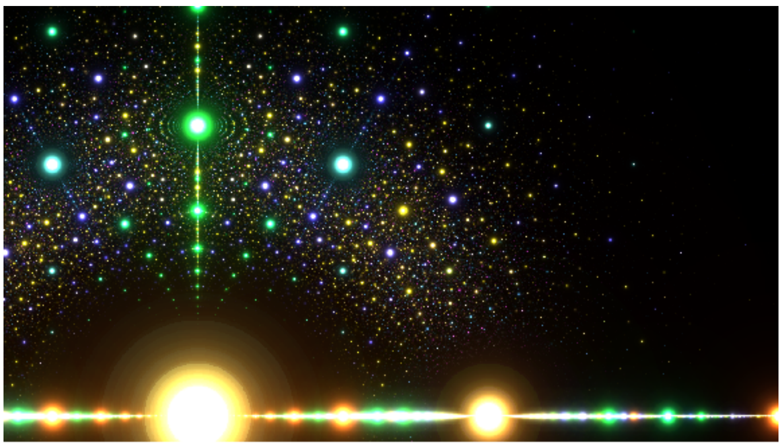
$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

In other words, a number $x$ is algebraic if it is the root of some polynomial equation with integer coefficients. We will denote the set of all algebraic numbers by $\mathbb{A}$.

If $x \in \mathbb{A}$, the smallest $n$ for which there exists a polynomial of the above form is called the *degree* of $x$.

For instance, the number $1$ is algebraic because it is a root of the polynomial $a_1 x + a_0 = 0$ with $a_1 = 1$ and $a_0 = -1$ (since this simply reduces to $x - 1 = 0$). This is a polynomial of order $n = 1$. But $1$ is also a root of the polynomial $a_2 x^2 + a_1 x + a_0 = 0$ if $a_2 = 1$, $a_1 = 1$, and $a_0 = -2$, and this is a polynomial with order $n = 2$. We can always cook up *higher*-order polynomials (larger $n$) for which $x = 1$ is a root. Since $n = 1$ is the *smallest* integer such that we can find a polynomial for which $1$ is a root, we say that the degree of the algebraic number $x = 1$ is $\deg(1) = 1$.

[Comment: There is some *deep* magic here. Take a look at this image which colors algebraic numbers by their degree.



Yellow is $n = 1$, green is $n = 2$, blue is $n = 3$, red is $n = 4$. The size of the points indicates the size of the integer coefficients $a_n$ of the polynomial, with smaller coefficients being marked with larger points. The large "sun" at the bottom is $0$, which is a root with $a_1 = 1$ and $a_0 = 0$; the next large yellow point to the right is $1$, which is a root with $a_1 = 1$ and $a_2 = -1$, and so on. The vertical line with many green dots is the imaginary axis; note that we're looking at the complex plane.

It's breathtaking.]

(a) Is $\sqrt{2}$ an element of $\mathbb{A}$? Is it true that $i \in \mathbb{A}$? Why or why not?

(b) Prove that every rational number is algebraic, i.e. show that $\mathbb{Q} \subseteq \mathbb{A}$. If $x \in \mathbb{Q}$, what is the degree of $x$?

(c) Suppose $x \in \mathbb{A}$ and $k \in \mathbb{Z}$. Prove that $kx \in \mathbb{A}$.

(Bonus) Prove that, if a complex number $z = a + bi$ is algebraic, then the complex conjugate $\bar{z} = a - bi$ is also algebraic with the same degree. Use this to explain why the image above is symmetric about the vertical line of green dots (that is, every point on the right side of this vertical line has a "mirror image" on the left side of the line).

daily_challenge

Updated 10 months ago by Christian Ferko

**the students' answer,** *where students collectively construct a single answer*

Logan Pachulski

1. Answers for a:
   1. $\sqrt{2}$ is an algebraic number, since it is a root of the polynomial $(x - \sqrt{2})(x + \sqrt{2}) = 0$, which by FOIL equals $x^2 - 2 = 0$ which has wholly integer coefficients.
   2. $i$ is a root of the polynomial $(x - i)(x + i) = x^2 - 1$ is also composed entirely of integer coefficients and therefore is algebraic.
2. Answer for b:
   1. Begin by defining the rational numbers: all rational numbers can be/are represented by fractions. For a value to be algebraic, there must exist a polynomial function where said value is a root. For all rational numbers, there exists a function to "convert" said value to zero and thereby show that it is algebraic. This equation can be represented as the following: Given a rational number of the form $\frac{a}{b}$, the polynomial of the first degree $(b \times x) - a = 0$ demonstrating the quote unquote undoing of a rational number, and thereby the fraction $\frac{a}{b} \in \mathbb{Q}$. Since all rational numbers follow the form $\frac{a}{b}$ by definition, it is true that $\mathbb{Q} \subset \mathbb{A}$. $\square$
3. Answer for c:

1. Proof: Recall the definition of an algebraic number: A number $x$ is algebraic if for a given set of integers $a_n, \cdots, a_0$, then $x$ is a root of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ Suppose $x \in \mathbb{A}$ and $k \in \mathbb{Z}$.

2. **Comment**: Your proof for (b) is fine, but when you write your solution to (c), could you try choosing explicit symbols for all mathematical objects that you introduce? For instance, rather than saying "For a value to be algebraic, there must exist a polynomial function where said value is a root", instead say something like "For a number $x$ to be algebraic, there must exist integers $a_0, \cdots, a_n$ such that $a_0 + \cdots + a_n x^n = 0$." The point is to give names to the value $x$ and a concrete form for the polynomial rather than using referents like "a value" or "a polynomial function."

   This goes back to my feedback at the end of daily challenge 6.5 about writing precise versus imprecise definitions (where "precise" means "choosing explicit names for every object" and "imprecise" means "using ordinary nouns like 'the function' rather than explicit names like $f$.")

Updated 9 months ago by Logan Pachulski and Christian Ferko

**the instructors' answer,** *where instructors collectively construct a single answer*

Some more feedback: I think you need to focus on using the formal definitions of terms (in terms of symbols) rather than the “intuitive” definitions in terms of words.

The definition of an algebraic number $x$ says that $x$ is a root of a polynomial $a_n x^n + \cdots + a_1 x + a_0 = 0$ where all of the $a_i$ are integers. In particular, the constant term $a_0$ must be an integer.

You claimed that the polynomial $x - \sqrt{2} = 0$ has integer coefficients, but here the constant term $a_0$ is $\sqrt{2}$ which is obviously not an integer.

This leads me to believe that you read the “in words” definition, which said that an algebraic number is a root of a polynomial with integer coefficients, but didn’t understand the rigorous definition with symbols.

That won’t work moving forward. An intuitive definition with words is good, but you always need to back up your intuition with formal mathematics.

**Edit**: Here is some copypasta from DC 6.5 which is also relevant in this problem.
"Precise" means to avoid using vague or colloquial language, name all mathematical objects using appropriate variables, and unambiguously define the term you're speaking about. For instance,

- Bad (imprecise): The set difference of two sets is the set of all elements in one set but not in the other.
- Good (precise): Let $A$ and $B$ be sets. The *set difference* of $A$ and $B$, written $A \setminus B$, is defined by $A \setminus B = \{a \in A \mid a \notin B\}$.

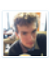
Another example:

- Bad (imprecise): The inverse image is the set of all elements that get mapped into the set under a function.
- Good (precise): Let $f$ be a function and suppose $A$ is a subset of the codomain of $f$. Then the *inverse image* or *preimage* of $A$ under $f$, denoted by $f^{-1}(A)$, is $f^{-1}(A) = \{x \in \mathrm{Dom}(f) \mid f(x) \in A\}$.

Updated 10 months ago by Christian Ferko

**followup discussions** *for lingering questions and comments*

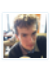
Resolved Unresolved



**Christian Ferko** 10 months ago
Comment: you might be tempted to think that *all* real numbers are algebraic. This is not true; for example, $\pi$ and $e$ are not algebraic. Real numbers which are not algebraic are called *transcendental*.

In a certain precise sense, "most" of the real numbers are transcendental.

Resolved Unresolved



**Christian Ferko** 10 months ago
Another comment: Mathematica knows how to check whether a number is algebraic, and if so, to find a polynomial with integer coefficients for which the given number is a root.

```
In[14]:= √5 ∈ Algebraics
Out[14]= True

In[15]:= MinimalPolynomial[√5, x]
Out[15]= -5 + x^2

In[16]:= π ∈ Algebraics
Out[16]= False

In[20]:= E ∈ Algebraics
Out[20]= False

In[18]:= (1 + I) ∈ Algebraics
Out[18]= True

In[19]:= MinimalPolynomial[1 + I, x]
Out[19]= 2 - 2 x + x^2
```