34.6

(a): As said in the problem statement, make the ansatz

$$y(t) = Ce^{i\omega t}$$

and note corresponding derivatives

$$y'(t) = Ci\omega e^{i\omega t}$$

$$y''(t) = -C\omega^2 e^{i\omega t}.$$

Plug y, y', and y'' into

$$y'' + \gamma y' + \omega_0^2 y = Fe^{i\omega t},$$

factoring out $Ce^{i\omega t}$,

$$Ce^{i\omega t}(\omega_0^2 + \gamma i\omega - \omega^2) = Fe^{i\omega t}$$

Since $e^{i\omega t}$ cannot equal zero, divide it from each side. Then,

$$C = \frac{F}{\omega_0^2 + \gamma i\omega - \omega^2}$$

(b): We want to demonstrate that the real part of our $y(t)$ matches our result in session for $x(t)$; begin with

$$x(t) = Re(y(t)) = Re(Ce^{i\omega t})$$

$$= Re\left(\frac{F}{\omega_0^2 + \gamma i\omega - \omega^2} e^{i\omega t}\right)$$

Multiply the inside by 1 in the form of the complex conjugate:

$$Re(y(t)) = Re\left(\frac{F(\omega_0^2 - \gamma i\omega - \omega^2)}{(\omega_0^2 + \gamma i\omega - \omega^2)(\omega_0^2 - \gamma i\omega - \omega^2)} e^{i\omega t}\right)$$

$$= Re\left(\frac{F((\omega_0^2 - \omega^2) - \gamma i\omega)}{\omega_0^4 - \omega^2\omega_0^2 - \omega^2\omega_0^2 + \omega^4 + \gamma\omega} e^{i\omega t}\right)$$

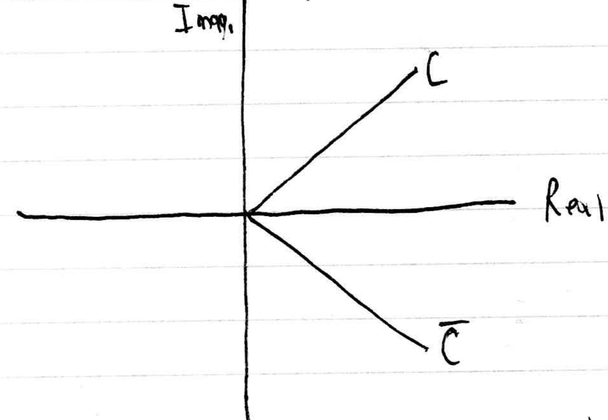
Then factoring and applying Euler's formula,

$$Re(y(t)) = Re\left(\frac{11}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}(\cos(\omega t) + i\sin(\omega t))\right)$$

And finally, by ~~only consitering where~~ passing the Re through and ~~Re(y(t)~~ ignoring terms where i's don't multiply,

$$Re(y(t)) = \frac{F(w_0^2 - w^2)}{(w_0^2 - w^2) + \gamma^2 w^2}\cos(wt) + \frac{F\gamma w}{(w_0^2 - w^2) + \gamma^2 w^2}\sin(wt)$$

(d): The hint suggest we draw a picture:



We see by complex number multiplication (add angles, square root of multiple lengths) that $-\theta + \theta = 0$, and $\sqrt{len(C) \cdot len(\bar{C})} = len(C)$, thus in polar coordinates

$$r = \sqrt{C\bar{C}},$$

and visually

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{Im(C)}{Re(C)}.$$

Recall

$$C = \frac{F((w_0^2 - w^2) - \gamma i w)}{(w_0^2 - w^2)^2 + \gamma^2 w^2}$$

Then,

$$\tan(\theta) = \frac{-\gamma w}{w_0^2 - w^2}$$

since F and denominators cancel, and

$$r = \frac{F\sqrt{((w_0^2 - w^2) - \gamma i w)((w_0^2 - w^2) + \gamma i w)}}{(w_0^2 - w^2)^2 + \gamma^2 w^2}$$

$$r = \frac{F\sqrt{(w_0^2 - w^2)^2 + \gamma^2 w^2}}{(w_0^2 - w^2) + \gamma^2 w^2} = \frac{F}{\sqrt{(w_0^2 - w^2) + \gamma^2 w^2}}$$

and thus

$$x(t) = \frac{F}{\sqrt{(\omega_0^2 - \omega^2) + \gamma^2 \omega^2}} \cos\left(\omega t + \arctan\left(\frac{-\gamma \omega}{\omega_0^2 - \omega^2}\right)\right).$$