5/2/2019

28.3

(a): Recall the definition of Cauchy sequence:
"A sequence is Cauchy if $\forall \epsilon$, $\exists N \in \mathbb{N}$ such that $\forall$ $m,n > N$, $|x_m - x_n| < \epsilon$."

Now negate by pushing the "not" past everything:
"A sequence is not Cauchy if there exists ϵ such that, for all $N \in \mathbb{N}$, there exists $m,n > N$ such that $|x_m - x_n| \geq \epsilon$

(b): ~~By our definition of a non-cauchy sequence, let $\epsilon = 3$; then a_m, a_n can only be in two states: $a_m = a_n$, $a_m \neq a_n$~~
~~Pick any value N and let $m,n > N$; if $a_m = a_n$, then $|a_m - a_n| = 0$~~
~~If $a_m \neq a_n$, since $a_{m,n} = -1, 1$, then $|a_m - a_n| = |-(a_m - a_n)| = 2 < \epsilon = 3$.~~

(b): We must choose an ϵ and show that said choice satisfies the non-cauchy definition seen in (a); let $\epsilon = 1$, and let some N be given. Define
$m = N+1$
$n = N+2$
thus satisfying $m,n > N$; then a_m and a_n are some non-equal combination of $1, -1$; thus

$$|a_m - a_n| = 2 > \epsilon = 1$$

(c): Given some $\epsilon > 0$, we must find an $N \in \mathbb{N}$ such that

$$m,n > N \Rightarrow \big| |a_m - b_m| - |a_n - b_n| \big| < \epsilon$$

We are given that the sequences a_n and b_n are Cauchy; thus there exists N_a such that

$|a_m - a_n| < \frac{\epsilon}{2}$ for $n, m > N_a$,
and there exists N_b such that
$|b_m - b_n| < \frac{\epsilon}{2}$ for $n, m > N_b$

Now let our overall $N = \max(N_a, N_b)$.
We see that by demanding $m, n > N$, then $m, n > N_a, N_b$ is satisfied;
thus work through absolute values and see that

$$\begin{aligned}\big||a_m - b_m| - |a_n - b_n|\big| &\leq |(a_m - b_m) + (a_n - b_n)| \\ &\leq |a_m - a_n| + |b_m - b_n| \\ &< \frac{2\epsilon}{2} = \epsilon \quad \square\end{aligned}$$

We conclude that c_n is Cauchy.