question 2 views

Daily Challenge 14.4

(~~**Due: Sunday 9/16 at 12:00 noon eastern**~~)
(**Due: Monday 9/17 at 12:00 noon eastern**)

Another hard problem today -- it's problem 4 on CD 4 so feel free to work directly there if you prefer.

(1) Problem: you can't be positive and concave everywhere.

Suppose that f is a continuous differentiable function on $\mathbb{R}$. Prove that we cannot have $f(x) > 0$ and $f''(x) < 0$ for all x.

Remark. This is pretty hard to prove rigorously. See if you can at least do some exploration and come up with a reasonable intuitive argument. One way to do the proof: assume by way of contradiction that such a function exists, then use the mean value theorem to show that something breaks.

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
Exploration: I can only think of one function that has this property, an even root set infinitely far to the left; like $\sqrt[n]{x+\inf} \text{ where } n = 2k \text{ and } k \in \mathbb{Z}$

Proof: Suppose by way of contradiction that such a function can exist; in English, we see that such a function is positive and concave everywhere. We see that for $f''(x) < 0$ then $f'(x)$ is strictly decreasing, and in turn we would like to prove that since $f'(x)$ is strictly decreasing, then there must be some point where $f'(x) < 0$. We can begin to prove this by referring to the mean value theorem, and see that there exists a point c on some open interval of interest $(a, b) \subseteq Dom(f')$ such that $f''(c) = \frac{f'(b)-f'(a)}{b-a}$. However we are told that $f''(x) < 0 \, \forall \, x$, therefore there is some point in this domain where $\frac{f'(b)-f'(a)}{b-a} < 0$. This then tells us that there is some point in the domain where the slope of the first derivative is negative. However, since we are also told that the first derivative is strictly decreasing, then by combining this knowledge we see that there is some points where $f'(x) < 0$, therefore $f(x)$ is strictly decreasing. We see that an asymptote is the only way a function could be strictly decreasing and never reach zero, but said asymptote (when positive) is also convex. Therefore there exists a point where $f(x) \leq 0$, contradicting our claim. Therefore it must be true that we cannot have a function where $f(x) > 0$ and $f''(x) < 0$. $\square$ (This needs to be more rigorous lol)

Updated 6 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Exploration. I initially didn't think the claim was true, and I spent longer than I care to admit looking for a counter-example.

It's certainly true for obvious examples, like a downward-opening parabola, since these eventually become negative. But I thought that you could somehow have a function which is downward-opening everywhere, but in such a way that it asymptotes to zero and never actually goes negative; to get this, I imagine that the second derivative would also approach zero from below.

First I thought that something like e^{-x^2} would do the trick, but this actually becomes convex ("upward-opening") at large positive x and large negative x.

Then I thought we could try a function whose *derivative* is $-\tanh(x) + 1$, since this is always decreasing so that $f''(x)$ would be always negative, but then I realized that the function itself would have to become negative at some points.

Finally I realized that it's important that the function is defined on all of $\mathbb{R}$. If we're allowed to "cheat" and define the function on only a subset of $\mathbb{R}$, like $[1, \infty)$, the claim fails. For instance, the function whose derivative is $-\tanh(x) + 1$ is positive on $[1, \infty)$ and has negative second derivative on $[1, \infty)$, but it goes negative when x becomes negative. So we really need the domain to be the whole real line.

With this realization in hand, I am ready to write the argument.

Proof. Suppose by way of contradiction that there were a function f defined on all of $\mathbb{R}$ with $f(x) > 0$ and $f''(x) < 0$ everywhere.

First I will argue that, without loss of generality, we can assume that $f'(x)$ becomes negative somewhere. If it didn't, so that $f'(x) > 0$ for all x, then define a new function $g(x) = f(-x)$. Then $g(x)$ is still positive everywhere, while by the chain rule, $g'(x) = -f'(x)$ and $g''(x) = f''(x)$, so it is still true that $g''(x) < 0$ everywhere. However, $g'(x)$ now becomes negative.

In short: if $f'(x)$ becomes negative, then apply the following argument to f. If not, apply it to g, which *does* have a negative derivative and which still satisfies the other hypotheses.

Now on to the second part of the argument. Find a point a where $f'(a) < 0$, which we can do by the preceding discussion. For any point b with $b > a$, we can apply the mean value theorem to $[a, b]$ to find that there exists some $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

On the other hand, since $f''(x) < 0$ everywhere, we know that f' is strictly decreasing, so $f'(c) < f'(a)$ since $c > a$. This means

$$f'(c) = \frac{f(b) - f(a)}{b - a} < f'(a).$$

By algebra, this means that

$$f(b) < f'(a) \cdot (b - a) + f(a).$$

Now for the kicker: the above applies for *any* point $b > a$. In particular, we can make b huge, so that the right side becomes negative. Indeed, let's pick b big enough so that

$$b > a + \frac{-f(a)}{f'(a)}.$$

Note that the second term is positive since $f(a) > 0$ by assumption (f is positive everywhere) but $f'(a) < 0$.

If b is that big, then $f'(a) \cdot (b - a) + f(a) < 0$. But this means $f(b) < 0$, so f becomes negative, contradicting that $f(x) > 0$ everywhere. $\square$

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*