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(a): Make the exponential ansatz

$$y=e^{rx}$$

to see that

$$r^2e^{rx}+re^{rx}-6e^{rx}=0$$

$$r^2+r-6=0$$

then

$$r=\frac{-1\pm\sqrt{-4\cdot1\cdot-6}}{2}=\frac{-1\pm5}{2}=2,-3$$

and we have general solution

$$y=C_1e^{2x}+C_2e^{-3x}$$

(b): This time, make the ansatz

$$y=xe^{rx}$$

and ~~plug in~~ note

$$y'=e^{rx}+xre^{rx}$$

$$y''=re^{rx}+r(e^{rx}+xre^{rx})$$

$$=2re^{rx}+xr^2e^{rx}\text{, thus}$$

$$a2re^{rx}+axr^2e^{rx}+be^{rx}+bxe^{rx}+xe^{rx}=0$$

$$2ar+axr^2+b+bx+x=0$$

applying the quadratic formula to find r,

$$r=-2a\pm\sqrt{4a^2-4\cdot}$$

(b): Evaluate y' and y'':

$$y' = e^{rx} + rxe^{rx}$$

$$y'' = re^{rx} + re^{rx} + r^2xe^{rx},$$

thus

$$ay'' + by' + cy = 0$$

becomes

$$2are^{rx} + ar^2e^{rx} + be^{rx} + brxe^{rx} + cxe^{rx} = 0$$

~~but~~ dividing out all e^{rx} (since no x has $e^{rx} = 0$),

$$2ar + ar^2x + b + brx + cx = 0,$$

then plugging in $r = \frac{-b}{2a}$,

$$-b + \frac{b^2}{4a}x + b + \frac{-b^2}{2a}x + cx = 0$$

Refer to final page

b's cancel and ~~we can make the equation true by setting $x = a$,~~

Thus, we have general solution

$$y = C_1e^{rx} + C_2xe^{rx}.$$

(c): Given

$$y = C_1e^{(a+i\beta)x} + C_2e^{(a-i\beta)x},$$

separate powers ~~to factor~~ and factor to have

$$y = e^{ax}\left(C_1e^{i\beta x} + C_2e^{-i\beta x}\right).$$

Recall Euler's Formula,

$$e^{ix} = \cos(x) + i\sin(x),$$

Then

and distribute
$$y = e^{ax}\left(C_1\left(\cos(\beta x) + i\sin(\beta x)\right) + C_2\left(\cos(\beta x) - i\sin(\beta x)\right)\right)$$

Factor a lot of bois:

$$y = e^{ax}\left((C_1 + C_2)\cos(\beta x) + i(C_1 - C_2)\sin(\beta x)\right)$$

Or redefining

$$K_1 = C_1 + C_2$$

$$K_2 = i(C_1 - C_2),$$

$$y = e^{ax}\left(K_1\cos(\beta x) + K_2\sin(\beta x)\right).$$

Errata for (b): Consider

$$x\left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) = 0$$

$$x\left(c - \frac{b^2}{4a}\right) = 0$$

This can be true if

$$c - \frac{b^2}{4a} = 0,$$

so notice we can rearrange

$$c = \frac{b^2}{4a}$$

$$4ac = b^2$$

$$0 = b^2 - 4ac, \text{ a known true equation!}$$

Thus, refer to end of (b).