note 3 views

Daily Challenge 8.2

Again, two parts. Answer either in the student response or on Overleaf.

- **Due: Friday 6/15 at 12:00 noon Eastern.**

  Answer question 2 (definitions and concepts) on the consolidation document, copied below.

  (a) Give the limit definition of the number e.

  (b) Explain what is meant by the *completeness property* of the real numbers.

  (c) What are the definitions of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$?

  (d) What is the difference between the *codomain* of a function and the *range* of a function?

  (e) Give the formal definition of the graph of a function (page 17 of AoPS, which is the fifth page of RA2). State the condition which this set must satisfy in order for f to pass the horizontal line test.

  [Hint: The definition of a graph involves a set of ordered pairs. Your condition for the horizontal line test should be a statement about the ordered pairs (x, f(x)) appearing in the set, *not* a geometric statement about lines intersecting the graph.]

- **Due: Friday 6/15 at 11:59 pm Eastern.**

  Answer question 9 (the product logarithm) on the consolidation document. **This is the hardest question, so allow for some time to think about it.** The statement is reproduced below.

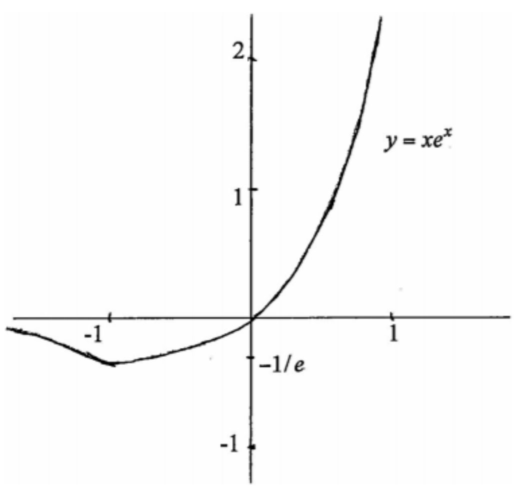
  We introduced the natural logarithm because we had equations like $e^x = 5$ that we didn't know how to solve. With an inverse function for e^x in our toolkits, we simply apply $\log$ to both sides to find $x = \log(5)$.

  When I was in high school, I was fascinated by equations like $x^x = 5$, or related equations that seem to admit no solution in terms of ``common'' functions. Could we invent a function like $\log$, which could be applied to both sides of this equation to find a solution, in the same way we did in the previous paragraph?

  My teachers told me that solving such equations was impossible, which only made me more interested.

  In this problem, we will see that they were wrong.

  (a) Let's define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = xe^x$. The graph of $f(x)$ looks like

  Explain why $f(x)$ cannot have an inverse function.

  (b) To remedy this problem, we will restrict the domain: define a new function $f_+ : [0, \infty) \to \mathbb{R}$ which is given by the same $f(x)$ above, but is defined only for non-negative values of x.

  Prove that this restricted function $f_+(x)$ has an inverse function.

  [Hint: the proof is very similar to the proof we used to show that the exponential function, e^x, has an inverse (namely the natural logarithm). In that case, we knew that e^x is strictly increasing and used this to show that it passes the horizontal line test. See if you can do something similar here.]

(c) Define a function $W(x)$ which is the inverse function of $f_+(x)$, since you have proved that such an inverse exists. This is called the *product logarithm*, also known as *Lambert's W function*.

(Comment: This function crops up in various scientific applications. For instance, there is a change of coordinates useful when studying black holes in Einstein's theory of general relativity which is written in terms of the product log. It also appears when studying enzyme kinetics in biochemistry.)

Consider the two equations

$$\begin{aligned} W\left(xe^x\right) &= x, \\ W(x)e^{W(x)} &= x. \end{aligned}$$

One of these equations is true for all real numbers x, and the other is true for some values of x but not for others. Explain which is which and why.

(d) Let a be a positive real number. Solve the equation $x^x = a$ for x.

[Scaffold:
(1) First notice that $x = e^{\log(x)}$ for all $x > 0$ (we will assume that the solution to $x^x = a$ is some positive x).
(2) Substitute this expression for x into both occurrences of x in the left side.
(3) Use the power-to-a-power rule to simplify the left side, writing it as $e^{\text{something}}$.
(4) Take the natural log of both sides of the equation. This should look like $(\text{something}) = \log(a)$.
(5) Apply the Lambert W function to both sides of your equation.
(6) Simplify the left side using one of the identities in (c).
(7) Exponentiate both sides to solve for x.
(8) Simplify the expression $e^{W(\text{something})}$ using the definition.]

daily_challenge

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*