question

Daily Challenge 19.2

(**Due: Wednesday 11/21 at 12:00 noon eastern**)

Let's transition back to problems rather than exercises, although today's is actually rather short.

(1) Problem: an amusing FTC application.

Suppose that f and g are differentiable functions satisfying

$$\int_0^{f(x)} fg = g(f(x))$$

and that $f(x_0) = 0$ at some point $x_0 > 0$. Prove that $g(0) = 0$.

daily_challenge

Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

logdog gets whipped:
We apply the FTC to the integral of the derivative of g alone, ie

$$\int_0^{f(x)} g'(t)\,dt = g(f(x)) - g(0)$$

and follow up with some algebra to separate out the $g(0)$:

$$g(0) = g(f(x)) - \int_0^{f(x)} g'(t)\,dt$$

We then see in the problem statement that $g(f(x)) = \int_0^{f(x)} f(t)g(t)\,dt$, and therefore

$$g(0) = \int_0^{f(x)} f(t)g(t) - g'(t)\,dt$$

We see that the left side is independent of the right side, and therefore we can let $x = x_0$; since the integral will be of zero width as a result, it will have zero area.

Updated 4 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

I'm very sorry, as I was going through my "proof" I realized that the claim (as originally stated) is false. Here is a counterexample showing why it could not have worked.

Let $f(x) = 1$ and $g(x) = 1$. Then

$$\int_0^{f(x)} fg = \int_0^1 1 = 1,$$

and

$$g(f(x)) = 1,$$

so the equation holds but $g(0) \neq 0$.

Modifying the problem.

We can make the problem solvable by assuming that $f(x)$ has a root somewhere (i.e. there exists some $x_0 > 0$ so that $f(x_0) = 0$), which I believe you suggested.

Begin by noting that the fundamental theorem of calculus tells us

$$\int_0^{f(x)} g'(t)\,dt = g(f(x)) - g(0),$$

and thus

$$g(0) = g(f(x)) - \int_0^{f(x)} g'(t)\,dt.$$

But by the equation given in the problem statement, $g(f(x)) = \int_0^{f(x)} fg$. Plugging this into the above, we find

$$g(0) = \int_0^{f(x)} (f(t)g(t) - g'(t))\ dt.$$

The right side must be a constant, independent of x, since the left side $g(0)$ is independent of x. Since it is constant, we can evaluate it at any point x and the result will be the same. Choose the point x to be a root of f, i.e. the point x_0 where $f(x_0) = 0$. Then the right side is $\int_0^0 (f(t)g(t) - g'(t))\, dt = 0$, so $g(0) = 0$. $\square$

Updated 4 months ago by Christian Ferko

followup discussions *for lingering questions and comments*