British Tutorial: 2015 Integration Bee

Logan Pachulski

February 12, 2019

An introduction.

Today I am going to present my work for all questions presented in the 2015 Integration Bee, and the solutions I was able to derive.

Question 1.

Solve the following integral:

$$\int \cos^4(x) - \sin^4(x) dx \tag{1}$$

Solution 1.

We begin by noting the difference of squares to see that

$$\int \cos^4(x) - \sin^4(x)dx = \int (\cos^2(x) - \sin^2)(\cos^2(x) + \sin^2(x))dx \tag{2}$$
$$= \int (\cos^2(x) - \sin^2(x))dx. \tag{3}$$

Then recall the double angle identity

$$\cos(2x) = \cos^2(x) - \sin^2(x). \tag{4}$$

We see that

$$\int (\cos^2(x) - \sin^2(x))dx = \int \cos(2x)dx \tag{5}$$
$$= \frac{\sin(2x)}{2} \tag{6}$$

Question 2.

Solve the following integral:

$$\int \frac{dx}{\sqrt{2+4x}} \tag{7}$$

Solution 2.

We shall apply u-sub where $u = 2 + 4x \implies x = \frac{u-2}{4}$ and $du = 4dx$; thus

$$\int \frac{u-2}{16\sqrt{u}} du = \frac{1}{16} \int \frac{u-2}{\sqrt{u}} du \tag{8}$$

$$= \int \sqrt{u}\, du - 2 \int \frac{1}{\sqrt{u}} \tag{9}$$

Then, by the power rule,

$$\int \sqrt{u}\, du - 2 \int \frac{1}{\sqrt{u}} = \frac{2u^{3/2}}{3} - 2\sqrt{u} \tag{10}$$

Question 3.

Compute the following integral:

$$\int_0^8 \frac{\cos(\sqrt{x})}{\sqrt{x}} dx \tag{11}$$

Solution 3.

We start by applying u-sub where $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}} dx = \frac{1}{2u} dx$:

$$\int_0^8 \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx = \int_0^2 2\cos(u)\, du \tag{12}$$

$$= 2\Big[\sin(u)\Big]_0^{2\sqrt{2}} = 2\sin(2\sqrt{2}) \tag{13}$$

Question 4.

Solve the following integral:

$$\int \sec(x)\, dx \tag{14}$$

Solution 4.

This is literally the hardest integral on this paper, but also because it was the first hard one I tried. We start by recalling the definition of $\sec(x)$ and multiplying by 1 in a peculiar form:

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)} \cdot \frac{\cos(x)}{\cos(x)} \tag{15}$$

$$= \int \frac{\cos(x)}{1 - \sin^2(x)} \tag{16}$$

We u-sub $u = \sin(x)$ and $du = \cos(x)dx$:

$$\int \frac{\cos(x)}{1 - \sin^2}\,dx = \int \frac{du}{1 - u^2} = \int \frac{du}{(1+u)(1-u)} \tag{17}$$

Solution 4: Conclusion.

We now apply partial fractions to see that

$$\int \frac{du}{(1+u)(1-u)} = \frac{1}{2}\left(\int \frac{1}{1+u} + \frac{1}{1-u}\right) \tag{18}$$

$$= \frac{1}{2}(\log(1+\sin(x)) + \log(1-\sin(x))) \tag{19}$$

Question 5.

Solve the following integral:

$$\int_0^{\pi/2} \frac{e^{\sin(x)}}{\tan(x)\csc(x)}dx \tag{20}$$

Solution 5.

We begin by noting the definitions of the involved trig functions:

$$\cdots = \int_0^{\pi/2} e^{\sin(x)} \cos(x)\, dx \tag{21}$$

We then apply u-sub where we let $u = \sin(x)$ and $du = \cos(x)dx$, and change the bounds of integration respectively to see that

$$\begin{aligned} \int_0^{\pi/2} e^{\sin(x)} \cos(x)\, dx &= \int_0^1 e^u\, du & (22) \\ &= \Big[e^u\Big]_0^1 & (23) \\ &= e - 1 & (24) \end{aligned}$$

Question 6.

Solve the following integral:

$$\int_1^e x \log^2(x) dx \tag{25}$$

Solution 6.

We shall apply integration by parts where we are moving the derivative onto the log:

$$\int_1^e x\log^2(x)dx = \left[\frac{x^2}{2}\cdot log^2(x)\right]_1^e - \int_1^e x\log(x)\,dx \tag{26}$$

We then apply integration by parts onto the remaining integral term, once again letting log eat the $\frac{d}{dx}$ to see that

$$\begin{aligned}\int_1^e x\log(x)\,dx &= \left[\frac{x^2}{2}\cdot\log(x)\right]_1^e - \int_1^e \frac{x}{2}dx &(27)\\ &= \left[\frac{x^2}{2}\cdot\log(x)\right]_1^e - \left[\frac{x}{2}\right]_1^e &(28)\end{aligned}$$

Then, by plugging all these values in and subtracting,

$$\int_1^e x \log^2(x) dx = \left[\frac{x^2}{2} \cdot log^2(x)\right]_1^e - \left(\left[\frac{x^2}{2} \cdot \log(x)\right]_1^e - \left[\frac{x}{2}\right]_1^e\right) \tag{29}$$

$$= \frac{e^2}{4} - \frac{1}{4} \tag{30}$$

Question 7.

Solve the following integral:

$$\int \frac{1}{x + 4\sqrt{x} + 5} dx \tag{31}$$

Solution 7.

We begin by applying u-sub where $u = \sqrt{x}$ and $du = \frac{1}{\sqrt{x}}$:

$$\int \frac{1}{x + 4\sqrt{x} + 5} dx = \int \frac{2u}{u^2 + 4u + 5} dx \tag{32}$$

Then, add zero in the form ± 4:

$$\begin{aligned} \int \frac{2u + 4 - 4}{u^2 + 4u + 5} dx &= \int \frac{2u + 4}{u^2 + 4u + 5} dx - 4 \int \frac{1}{u^2 + 4u + 5} dx && (33) \\ &= \log(u^2 + 4u + 5) - 4 \int \frac{1}{(u + 2)^2 + 1} && (34) \\ &= \log(x + 4\sqrt{x} + 5) - 4 \arctan(\sqrt{x} + 2) && (35) \end{aligned}$$

Question 8.

Solve the following integral:

$$\int 2015^x dx \tag{36}$$

Solution 8.

We start by placing this function into an $e^log(z)$ which by definition is equivalent to z;

$$\int 2015^x dx = \int e^{\log(2015^x)} dx = \int e^{x \log(2015)} \tag{37}$$

$$= \frac{e^{x \log(2015)}}{\log(2015)} \tag{38}$$

$$= \frac{2015^x}{\log(2015)} \tag{39}$$

Question 9.

Solve the following integral:

$$\int_0^2 \frac{x}{(x-3)(x+5)^2}\,dx \tag{40}$$

Solution 9

We apply partial fractions to find that

$$\int_0^2 \frac{x}{(x-3)(x+5)^2}dx = \int_0^2 \frac{\frac{3}{64}}{x-3} + \frac{\frac{-3}{64}}{x+5} + \frac{\frac{5}{8}}{(x+5)^2} \tag{41}$$

We then factor out a $\frac{3}{64}$ out of the former two terms and note that these two can just be evaluated with a log.

$$\cdots = \Big[\log(x-3)\log(x+5)\Big]_0^2 + \frac{5}{8}\int \frac{1}{(x+5)^2} \tag{42}$$

$$= \Big[\log(x-3)\log(x+5)\Big]_0^2 - \frac{5}{8}\left[\frac{1}{(x+5)}\right]_0^2 \tag{43}$$

$$= -\log(-3)\log(5) - \frac{5}{8}\left(\frac{1}{7} - \frac{1}{5}\right) \tag{44}$$

Question 10.

Solve the following integral:

$$\int \frac{\log(1 + \log(x))}{x} dx \tag{45}$$

Solution 10.

We begin with a u-sub. Again. $u = \log(x) + 1$ and $du = \frac{1}{x}dx$.

$$\cdots = \int \log(u) = u\log(u) - u \tag{46}$$

$$= (\log(x)+1)\log(\log(x)+1) - (\log(x)+1) \tag{47}$$

Question 11.

Solve the following integral:

$$\int \sqrt{\csc(x) - \sin(x)}dx \tag{48}$$

Solution 11

We begin by multiplying the integral by 1 in the form of $\frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}}$:

$$\frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}}\int\sqrt{\csc(x)-\sin(x)}dx = \frac{\sqrt{1-\sin^2(x)}}{\sqrt{sin(x)}} \tag{49}$$

$$= \frac{\cos(x)}{\sqrt{\sin(x)}} \tag{50}$$

We then apply u-sub, letting $u = \sin(x)$ and thus $du = \cos(x)dx$:

$$\int\frac{du}{\sqrt{u}} = 2\sqrt{\sin(x)} \tag{51}$$

Question 12.

Solve the following integral:

$$\int \frac{1}{\sqrt{x^2+25}}\,dx \tag{52}$$

Solution 12.

We begin by noting a potential trig sub:

$$x = 5\tan(\theta) \implies \theta = \arctan(\frac{x}{5}) \tag{53}$$

$$dx = 5sec^2(\frac{x}{5})\,d\theta \tag{54}$$

allows us to see that

$$\int \frac{1}{\sqrt{x^2+25}}\,dx = \int \frac{5sec^2(\theta)}{\sqrt{25(\tan^2(\theta)+1)}}\,d\theta \tag{55}$$

$$= \int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta \tag{56}$$

where in the line above we noted that $\tan^2(z) + 1 = \sec^2(z)$.

Solution 12: Conclusion.

Now refer to Solution 4 and see that:

$$\int \frac{\sec^2(\theta)}{\sec(\theta)}\, d\theta = \int \sec(\theta)\, d\theta = \operatorname{sech}^{-1}(\cos(\arctan(\frac{x}{5}))) \tag{57}$$

Question 13.

Solve the following integral:

$$\int_2^e \frac{\log^2(x) - 1}{x\log^2(x)}dx \tag{58}$$

Solution 13.

We begin with a u-sub: $u = \log(x)$ and thus $du = \frac{1}{x}$:

$$\int_2^e \frac{\log^2(x) - 1}{x \log^2(x)}\, dx = \int_{\log(2)}^1 \frac{u^2 - 1}{u^2}\, du \tag{59}$$

$$= \int_{\log(2)}^1 1\, du - \int_{\log(2)}^1 \frac{1}{u^2}\, du \tag{60}$$

$$= \Big[u\Big]_{\log(2)}^1 + \Big[\frac{1}{u}\Big]_{\log(2)}^1 \tag{61}$$

$$= 2 - \log(2) + \frac{1}{\log(2)} \tag{62}$$

Question 14.

Solve the following integral:

$$\int e^{3x} \arctan(e^x) dx \tag{63}$$

Solution 14.

Begin by applying the u-sub $u = e^x$ and thus $du = e^x dx$; then,

$$\int e^{3x} \arctan(e^x) dx = \int u^2 \arctan(u) dx \tag{64}$$

We now integrate by parts, moving the derivative onto the $\arctan(u)$ (surprisingly):

$$\int u^2 \arctan(u) dx = \frac{u^3}{3} \arctan(u) - \frac{1}{3} \int \frac{u^3}{(u^2 + 1)} \tag{65}$$

We then apply polynomial long division to see that

$$\int \frac{u^3}{(u^2 + 1)} = \int u + \frac{-u}{u^2 + 1} \tag{66}$$

Solution 14: Conclusion

We now apply the power rule to the former term and an intuitive log to the second to see that

$$\int u + \frac{-u}{u^2+1} = \frac{u^2}{2} - \frac{1}{2}\log(u^2+1) \tag{67}$$

We re-insert this into what we found before:

$$\int u^2 \arctan(u) dx = \frac{u^3}{3}\arctan(e^x) - \frac{1}{3}\left(\frac{e^{2x}}{2} - \frac{1}{2}\log(e^{2x}+1)\right) \tag{68}$$

Question 15.

Solve the following integral:

$$\int \frac{|x-1|}{|x-2|+|x-3|}\,dx \tag{69}$$

Solution 15.

We begin by breaking the integral into parts, where we con confidently replace the absolute values with parantheses and some $\pm$. We see that $[0,1],\cdots,[3,4]$ works. We see that

$$\int_0^1 \frac{|x-1|}{|x-2|+|x-3|}\,dx = \int \frac{-x+1}{-2x+5}\,dx \tag{70}$$

$$\int_1^2 \frac{|x-1|}{|x-2|+|x-3|}\,dx = \int \frac{x-1}{-2x+5} \tag{71}$$

$$\int_2^3 \frac{|x-1|}{|x-2|+|x-3|}\,dx = \int \frac{x-1}{5} \tag{72}$$

$$\int_3^4 \frac{|x-1|}{|x-2|+|x-3|}\,dx = \int \frac{x-1}{2x-5} \tag{73}$$

$$\tag{74}$$

Solution 15: Conclusion.

We see that the the first two sum to zero, thus we must find

$$\int \frac{x-1}{5} + \int \frac{x-1}{2x-5} \tag{75}$$

The former term

$$\int \frac{x-1}{5} dx = \frac{1}{5} \int x - 1 dx = \frac{1}{5} \left(\frac{x^2}{2} - x\right) \tag{76}$$

Question 16.

Solve the following integral:

$$\int_0^{2\pi} \frac{1}{\sin^4(x) + \cos^4(x)}\,dx \tag{77}$$

Solution 16.

We begin by noting that the bottom can be solved via completing the square:

$$\int_0^{2\pi} \frac{1}{\sin^4(x) + \cos^4(x)}\, dx = \int_0^{2\pi} \frac{1}{1 - 2\sin^2(x)\cos^2(x)}\, dx \tag{78}$$

We have praised Pythagoras to handle the left square term, and recall the sine double angle formula and see that:

$$\int_0^{2\pi} \frac{1}{1 - 2\sin^2(x)\cos^2(x)}\, dx = \int_0^{2\pi} \frac{1}{1 - \frac{1}{2}\sin^2(2x)}\, dx \tag{79}$$

Solution 16: Conclusion.

We can now multiply the top and bottom by $\sec^2(2x)$ to see that

$$\int_0^{2\pi} \frac{1}{1 - \frac{1}{2}\sin^2(2x)}\, dx = \int_0^{2\pi} \frac{\sec^2(2x)}{\sec^2(2x) - \frac{1}{2}\tan^2(2x)}\, dx \tag{80}$$

We recall that $\sec^2 = \tan^2 + 1$ and multiply the top and bottom by 2:

$$\cdots = \int_0^{2\pi} \frac{2\sec^2(2x)}{2 + \tan^2(2x)}\, dx \tag{81}$$

We now u-sub where $u = \tan(2x) \implies du = 2\sec(2x)dx$:

$$\cdots = \frac{1}{2}\int_0^{2\pi} \frac{2\sec^2(2x)}{2 + \tan^2(2x)}\, dx \tag{82}$$

Question 17.

Solve the following integral:

$$\int \frac{1+e^x}{1-e^x}\,dx \tag{83}$$

Solution 17.

We begin by splitting this integral into 2:

$$\int \frac{1+e^x}{1-e^x}\,dx = \int \frac{1}{1-e^x}\,dx + \int \frac{e^x}{1-e^x}\,dx \tag{84}$$

$$= -\log(1-e^x) + \int \frac{1}{1-e^x}\,dx \tag{85}$$

We then must apply a u-sub to solve for the latter term; let $u = e^x$ and thus $du = e^x dx$. Then,

$$\int \frac{1}{1-e^x}\,dx = \int \frac{1}{u(1-u)}\,dx \tag{86}$$

We now apply partial fractions to the latter term and find that

$$\int \frac{1}{u(1-u)}\,dx = \int \frac{1}{u} + \frac{1}{(1-u)}\,du \tag{87}$$

$$= x - \log(1-e^{x}) \tag{88}$$

Question 18.

Solve the following integral:

$$\int \tan^4(x)\,dx \tag{89}$$

Solution 18.

We begin by recalling that $\sec^2(x) = \tan^2(x) + 1$; substitute one of the tan^2 as implied and see that the given integral equals

$$\int \tan^2(x)(\sec^2(x) - 1)dx = \int \tan^2(x)\sec^2(x)\,dx - \int \tan^2(x)\,dx \quad (90)$$

We apply u-sub to the left term where $u = \tan(x)$ and $du = \sec^2(x)$, meanwhile the right gets split by the $\sec^2(x) = \tan^2(x) + 1$ once again.

$$\cdots = \int u^2\,du - \left(\int \sec^2(x)\,dx - \int 1\,dx\right) \quad (91)$$

$$= \frac{u^3}{3} - \tan(x) + x \quad (92)$$

Question 19.

Solve the following integral:

$$\int \sin(x)\tan^2(x)dx \tag{93}$$

Solution 19

We begin by recalling that $\tan^2(x) = \sec^2(x) - 1$, thus allowing us to to see that

$$\int \sin(x)\tan^2(x)\,dx = \int \frac{\tan(x)}{\cos(x)}\,dx - \int \sin(x)\,dx \tag{94}$$

We then u-sub where $u = sec(x)$ and $du = \sec(x)\tan(x)$ and see that

$$\int \frac{\tan(x)}{\cos(x)}\,dx = \int 1\,du \tag{95}$$

We then evaluate the right term of the upper line and the right term of the lower and see that

$$\int \sin(x)\tan^2(x)\,dx = \sec(x) + \cos(x) \tag{96}$$

Question 20.

Solve the following integral:

$$\int \frac{x+1}{x^2+2x+3}\,dx \tag{97}$$

Solution 20.

This is an incredibly intuitive integral, but I may as well use an actual technique instead of approximate and refine. We u-sub where $u = x^2 + 2x + 3$ and thus $du = 2x + 2dx$. Thus,

$$\int \frac{x+1}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log(u) = \frac{1}{2}\log(x^2+2x+3) \quad (98)$$