question 3 views

Daily Challenge 6.1

(**Due: Tuesday 5/29 at 12:00 noon Eastern**)

Excellent work -- thanks for the marathon session yesterday to catch up on the challenges. Now I think we're in good shape.

The last topic in chapter 1 is logarithms.

Review

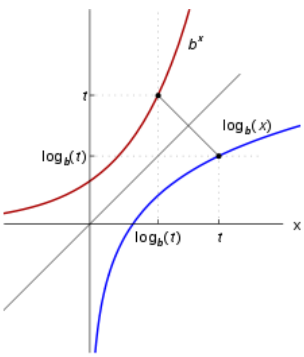
In daily challenge 5.7, we saw that any strictly increasing function passes the horizontal line test. The same is true of a strictly *decreasing* function. Since the horizontal line test determines whether a function is invertible, any strictly increasing or strictly decreasing function must have an inverse.

In particular, since exponential functions are either strictly increasing (if the base is greater than 1), strictly decreasing (if the base is less than 1), or constant (if the base is precisely 1), we see that an exponential function $f(x) = a^x$ will have an inverse function so long as $a \neq 1$.

The **logarithm with base a**, written as $\log_a$, is the inverse of the exponential function $f(x) = a^x$. Recall that an exponential function has domain $\mathbb{R}$ and only outputs positive values, so that its range is $(0, \infty)$. Thus the logarithm, as its inverse function, has the domain and range reversed:

$$\begin{aligned} \mathrm{Dom}(\log_a) &= (0, \infty), \\ \mathrm{Rng}(\log_a) &= \mathbb{R}, \\ \log_a(a^x) &= x \text{ for all } x \in \mathbb{R}, \\ a^{\log_a(x)} &= x \text{ for all } x \in (0, \infty). \end{aligned}$$

If you like to think of inverse functions as inverting a graph over the line $y = x$, then the following visualization may help:



Because it is the inverse function of the exponential, the quantity $\log_a(x)$ is the number to which we must raise the base a in order to get x. For instance

$$2^4 = 16 \iff \log_2(16) = 4.$$

Since we must raise 2 to the power 4 to achieve an output of 16, this means that the logarithm with base 2 of 16 is 4. A few more examples:

$$\begin{aligned} 3^{-1} = \frac{1}{3} &\iff \log_3\left(\frac{1}{3}\right) = -1, \\ 10^2 = 100 &\iff \log_{10}(100) = 2, \\ 5^0 = 1 &\iff \log_5(1) = 0, \end{aligned}$$

and so on.

Problem

Try the follow questions; the goal is for calculations like (a) - (c) to become almost automatic, like $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ or $2 + 2 = 4$.

(a) What is $\log_2(32)$?

(b) What is $\log_9(\sqrt{3})$?

(c) What is $\log_{\frac{1}{2}}(2)$?

(d) We know that exponential functions have the property that $a^{x+y} = a^x a^y$. Let's define $m = a^x$ and $n = a^y$. Then $\log_a(m) = x$ and $\log_a(n) = y$, and by the above property, we also have that $mn = a^{x+y}$. Take the logarithm with base a of both sides of this last equation and simplify it to obtain an equation which relates $\log_a(mn)$ to $\log_a(m)$ and $\log_a(n)$.

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

1. $\log_2(32) = 2^x = 32$, and $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$, therefore $\log_2(32) = 5$
2. Just like before, $\log_9(\sqrt{3}) = 9^x = \sqrt{3}$, and I know that $(\sqrt{3})^4 = 9$, so $9^(1/4) = \sqrt{3}$, therefore $\log_9(\sqrt{3}) = 1/4$.
3. I know that a number to the negative first power is it's reciprocal, so $\frac{1}{2}^- 1 = 2$, therefore $\log_{\frac{1}{2}}(2) = -1$
4. I don't totally understand what is being asked in this question, but my "answers" would be $y = \frac{y}{x}$, and $x = \frac{x}{y}$.

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My answers follow.

1. The equation $\log_2(32) = x$ can be re-written as $2^x = 32$, but we know that $2^5 = 32$, so we conclude that $\log_2(32) = 5$.

2. Similarly, we know that $9^{1/4} = 3^{1/2} = \sqrt{3}$, which means that $\log_9(\sqrt{3}) = \frac{1}{4}$.

3. We seek a number x with the property that $\log_{\frac{1}{2}}(2) = x$, or $\left(\frac{1}{2}\right)^x = 2$, which is satisfied for $x = -1$.

4. As suggested in the problem statement, let's define $x = \log_a(m)$ and $y = \log_a(n)$, which means that $a^x = m$ and $a^y = n$.

   On the other hand, by the sum rule for exponentials, we have $a^{x+y} = a^x a^y = mn$. We take the logarithm with base a of both sides of this equation:

   $$\log_a\left(a^{x+y}\right) = \log_a(mn).$$
   $$\implies x + y = \log_a(mn).$$

   Finally, we replace x and y on the left side using their definitions, so we conclude

   $$\log_a(mn) = \log_a(m) + \log_a(n).$$

   This is the desired result -- a logarithm of a product is the sum of the logs.

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*