question 2 views

Daily Challenge 20.2

Let's do another Feynman trick, this time to compute

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx.$$

(a) Introduce a parameter b and define the new function

$$I(b) = \int_0^{\infty} \frac{\sin(x)}{x} e^{-bx}\,dx.$$

What is our original integral in terms of $I(b)$? (Be careful about the bounds of integration.)

(b) Differentiate under the integral sign. Evaluate the resulting integral by using integration by parts twice, which relates the integral to itself (the "recursive definition trick") and solve. Do this to show that

$$I'(b) = -\frac{1}{1+b^2}.$$

[Aside: this is technically a doubly-improper integral, since the integrand is undefined at the lower endpoint and the upper endpoint is infinity. However, the area is still well-defined because $\lim_{a\to 0}\lim_{b\to\infty}\int_a^b \frac{\sin(x)}{x}\,dx$ exists.]

(c) Anti-differentiate to find $I(b)$ in terms of an inverse trigonometric function and a constant C. Note that $\lim_{b\to\infty} I(b) = 0$; use this to figure out what C is.

(d) Finish the argument to conclude that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi.$$

daily_challenge

Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a): We see that the function $\frac{\sin(x)}{x}$ is symmetric for substitutions of $x = -x$, as the negatives cancel out. Thus,

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} = 2\int_0^{\infty} \frac{\sin(x)}{x} = f(x)$$

We see that the function defined as

$$I(b) = \int_0^{\infty} \frac{\sin(x)}{x} e^{-bx}\,dx \implies 2I(0) = f(x)$$

(b): We now would like to take the derivative of each side with respect to b.

$$I'(b) = -\int_0^{\infty} \sin(x)e^{-bx}\,dx$$

We then must integrate by parts; we let $a = \sin(x), a' = \cos(x), b' = e^{-bx}, b = \frac{e^{-bx}}{b}$; we then see that

$$I'(b) = \Big[\sin(x)\frac{e^{-bx}}{b}\Big]_0^{\infty} + \int_0^{\infty} -\cos(x)\frac{e^{-bx}}{b}$$

We then assume that $b > 0$, and see that as $x \to \infty$ that $\sin(x)\frac{e^{-bx}}{b} \to 0$ due to the e-power with a negative. We also see that as $x \to 0$ that $\sin(x)\frac{e^{-bx}}{b} \to 0$ due to $\sin(0) = 0$. Thus,

$$\begin{aligned} I'(b) &= \int_0^{\infty} -\cos(x)\frac{e^{-bx}}{b} \\ &= \frac{-1}{b}\int_0^{\infty} \cos(x)e^{-bx} \end{aligned}$$

We then integrate by parts again, this time letting (sidenote: honestly there are so few ways to type that you are applying IBP) $a = \cos(x), a' = -\sin(x), b' = e^{-bx}, b = \frac{e^{-bx}}{b}$. Recall

$$\begin{aligned} \int ab' &= ab - \int a'b \\ &= \frac{-1}{b}\left(\Big[\cos(x)\frac{e^{-bx}}{b}\Big]_0^{\infty} - \int -\sin(x)\frac{e^{-bx}}{b}\right) \\ &= \frac{-1}{b}\Big[\cos(x)\frac{e^{-bx}}{b}\Big]_0^{\infty} - \frac{1}{b}\int \sin(x)\frac{e^{-bx}}{b} \end{aligned}$$

We calculate the \Big[boi, once again assuming $b > 0$ and see that it equals -1. We also pull a factor of $\frac{1}{b}$ out of the integral, and note that a negative is donated by the definition of $I'(b)$, to see that

$$I'(b) = \frac{-1}{b}\left(\frac{1}{b} + \frac{1}{b}I(b)\right) \implies b^2 I'(b) = -1 - I'(b) \implies I'(b) = -\frac{1}{1+b^2}$$

(c): Of course, we immediately recognize the final formula because we've seen it at least twice in the past; we see that this implies
$I(b) = -\arctan(b) + C$
We see that $I(b) \to 0$ as $b \to \infty$ due to the presence of the e^{-bx}. We also see that as $b \to \infty$, $-\arctan(b) \to -\frac{\pi}{2}$ which then implies that $C = \frac{\pi}{2}$.

(d): We can now input $b = 0$ to calculate the case of interest:
$I(0) = \frac{\pi}{2}$
But we saw in the second sentence that $2I(0)$ equals our integral of interest, thus we conclude
$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi$

Updated 2 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) The function $\frac{\sin(x)}{x}$ is symmetric under the replacement $x \to -x$, since $\sin(-x) = -\sin(x)$. Thus

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = 2\int_{0}^{\infty} \frac{\sin(x)}{x}\, dx.$$

The original integral of interest, then, is equal to $2I(0)$, since

$$2I(0) = 2\int_{0}^{\infty} \frac{\sin(x)}{x} e^{-b\cdot 0}\, dx = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx.$$

(b) Differentiating with respect to b, one finds

$$I'(b) = -\int_{0}^{\infty} \sin(x) e^{-bx}\, dx.$$

Let $u = \sin(x)$, $du = \cos(x)\, dx$, $dv = e^{-bx}\, dx$, $v = -\frac{1}{b}e^{-bx}$. Integrating by parts, we see that

$$\begin{aligned} I'(b) &= -\int_{0}^{\infty} u\, dv \\ &= -\Big[uv\Big]_0^{\infty} + \int_0^{\infty} v\, du \\ &= -\left[-\frac{1}{b}\sin(x)\, e^{-bx}\right]_0^{\infty} - \frac{1}{b}\int_0^{\infty} \cos(x) e^{-bx}\, dx. \end{aligned}$$

Don't forget that we must evaluate the uv term at the endpoints when applying integration by parts to a definite integral. At the top bound, the expression in brackets vanishes because $\lim_{x\to\infty} e^{-bx} = 0$; at the bottom bound it vanishes because $\sin(0) = 0$.

Now integrate by parts again, this time letting $u = \cos(x)$, $du = -\sin(x)\, dx$, $dv = e^{-bx}\, dx$, $v = -\frac{1}{b}e^{-bx}$ in the second integral. This gives

$$\begin{aligned} I'(b) &= -\frac{1}{b}\int_0^{\infty} \cos(x) e^{-bx}\, dx \\ &= -\frac{1}{b}\int_0^{\infty} u\, dv \\ &= -\frac{1}{b}[uv]_0^{\infty} + \frac{1}{b}\int_0^{\infty} v\, du \\ &= -\frac{1}{b}\left[-\frac{1}{b}\cos(x) e^{-bx}\right]_0^{\infty} + \frac{1}{b^2}\int_0^{\infty} e^{-bx}\, \sin(x)\, dx. \end{aligned}$$

Note that the integral at the end of the final line is, by definition, $-I'(b)$ again. Evaluating the bracketed expression at the top and bottom endpoints, then multiplying both sides by b^2, we find

$$b^2 I'(b) = -1 - I'(b),$$

or solving,

$$I'(b) = -\frac{1}{1+b^2}.$$

(c) We have hopefully memorized that the derivative of the arctangent function is

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}.$$

Hence the statement that $I'(b) = -\frac{1}{1+b^2}$ means that

$$I(b) = -\arctan(b) + C$$

for some constant C.

As the hint suggests, $\lim_{b\to\infty} I(b) = 0$ because $\lim_{b\to\infty} e^{-bx} = 0$. On the other hand, $\lim_{b\to\infty} \arctan(b) = \frac{\pi}{2}$. This means $C = \frac{\pi}{2}$, and

$$I(b) = -\arctan(b) + \frac{\pi}{2}.$$

(d) Plugging in $b = 0$, we have

$$I(0) = -\arctan(0) + \frac{\pi}{2} = \frac{\pi}{2}.$$

But we argued in part (a) that the integral of interest is twice $I(0)$, so we conclude

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi.$$

Behold.

Updated 2 months ago by Christian Ferko

followup discussions *for lingering questions and comments*