question 4 views

Daily Challenge 1.5

(**Due: Saturday 4/28 at 12:00 noon Eastern**)

Review

In reading assignment 3, we met the ideas of inverse, converse, and contrapositive. You may have seen these in the travesty that is a high school geometry class. In short, for an original statement of the form "p implies q" (written $p \implies q$, for short),

- The **inverse** is the statement "(not p) implies (not q)", or symbolically, $(\neg p) \implies (\neg q)$.
- The **converse** is the statement "q implies p", or $q \implies p$.
- The **contrapositive** is the statement "(not q) implies (not p)", or symbolically, $(\neg q) \implies (\neg p)$.

For example, suppose I make the statement "when I'm doing math, I'm happy." This is of the form "p implies q" where p is the condition "I'm doing math" and q is the condition "I'm happy".

1. The inverse is "when I'm not doing math, I'm not happy."
2. The converse is "when I'm happy, I'm doing math."
3. The contrapositive is "when I'm not happy, I'm not doing math."

Note that the inverse and converse need not be true when the original statement is true!

I am also happy when I'm doing physics, so statement (1) is false. When I'm doing physics, I'm not doing math but I am happy. This contradicts the claim of (1) that whenever I'm not doing math, I'm not happy.

Statement (2) is also false. If I'm doing physics, I am happy, so the hypothesis ("I'm happy") of statement (2) is true, and yet its conclusion ("I'm doing math") is not true.

Statement (3) is true. We know that I'm happy whenever I'm doing math. If you observe me and I'm not happy, then, it must follow that I'm not doing math.

A statement and its contrapositive are logically equivalent: if one is true, the other is true. We can use this fact in writing proofs. If you want to prove some statement, you can always choose to prove its contrapositive instead, since this is equivalent to proving the original claim.

Theorem. Let m be an integer. If m^2 is odd, then m is odd.

Proof. We seek to prove a statement of the form $p \implies q$, where p is the statement "m^2 is odd" and q is the statement "m is odd". Instead, we will prove the contrapositive, $(\neg q) \implies (\neg p)$. In words, the contrapositive is: "if m is not odd, then m^2 is not odd."

To prove the contrapositive, we begin by assuming that m is not odd. An integer must be either odd or even, so this means m is even. If m is even, then $m = 2k$ for some integer k. It follows that $m^2 = 4k^2$, but this can be re-written as $m^2 = 2(2k^2)$. The quantity inside parentheses, $2k^2$, is an integer, so we have written m^2 as twice an integer (specifically, $m^2 = 2a$ where $a = 2k^2$ is an integer), and hence m^2 is even. This means that m^2 is not odd.

Since we have proved that the contrapositive of the theorem is true, the original theorem is also true. $\square$

Problem

Consider the following statement.

Theorem. If n is an integer satisfying $n^4 + 4n^3 + 3n^2 + n + 4000 = 0$ then n is even.

It is easier to prove the contrapositive of this theorem. Do so.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Proof (Corbin). Your proof goes here.

We must show that if n is an integer satisfying $n^4 + 4n^3 + 3n^2 + n + 4000 = 0$then n is even. First we must define an even number as any number that is $n = 2k \mid k \in \mathbb{Z}$. I believe I can reference to the fact that if we add an odd and an even number we get an odd number. This shows us that our sequence of coefficients is $odd + even + odd + odd + even$ This comes out as an odd number, from here we can state that this odd number multiplied against an even n will always result in an even number. This shows that n will be even to result in zero because $odd \times even$ is even and zero is an even number. $\square$

Proof (Logan) - Began at 11:30 AM, 4/28 (Late ik.)

I must prove that: "If n is an integer satisfying $n^4 + 4n^3 + 3n^2 + n + 4000 = 0$ then n is even." I start by taking the contrapositive of this statement: "If n is an integer that does not satisfy the statement $n^4 + 4n^3 + 3n^2 + n + 4000 = 0$ then n is not even." This is the statement I must prove. Since n is an integer and integers must be even or odd, n must be odd. This can be written mathematically by the equation $n = 2k + 1$ such that $k \in \mathbb{Z}$. I will attempt to then substitute n in the given equation using synthetic substitution. Therefore,

	1	4	3	1	4000
n		n	$n^2 + 4n$	$n^3 + 4n^2 + 3n$	$n^4 + 4n^3 + 3n^2 + n$
	1	n+4	$n^2 + 4n + 3$	$n^3 + 4n^2 + 3n + 1$	$n^4 + 4n^3 + 3n^2 + n + 4000$

I don't believe I have achieved anything by doing this, so how about I try something different. n^4 given an odd number must be odd, and $4n^3$ given an odd number must be even, so n^4+4n^3 must be odd. In a similar manner, $3n^2+n$ must result in an even, once again assuming n is odd. And finally 4000 is even. These 3 statements taken individually add up to conclude that if n is odd, then the result must be odd, and therefore non-zero. Since an non-even n results in a non-zero output, then an even n can result in a zero output. ☐

Updated 11 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Proof (Christian). We want to prove the original statement, which is of the form

$$\text{if } \underbrace{\left(n \text{ is an integer and } n^4+4n^3+3n^2+n+4000=0\right)}_{p},\ \text{then } \underbrace{\left(n \text{ is even }\right)}_{q}.$$

It is easier to prove the contrapositive, so we switch the order of the implication and insert not in front of each:

$$\text{if }\left(\text{ not } \underbrace{\left(n \text{ is even }\right)}_{q}\right),\ \text{then }\left(\text{ not } \underbrace{\left(n \text{ is an integer and } n^4+4n^3+3n^2+n+4000=0\right)}_{p}\right).$$

So begin by assuming that n is not even, which means that n is odd. Let's consider the five terms in the sum

$$n^4+4n^3+3n^2+n+4000$$

separately.

- Since n is odd, n^4 is odd (one could prove this -- if n^4 were even, then 2 divides n^4, which means 2 divides n -- but perhaps it is obvious enough to simply state the result).
- The number $4n^3$ is even, since it is $2\left(2n^3\right)$ and $2n^3$ is an integer.
- The term $3n^2$ is odd (because n^2 is odd, and the product of odd integers is odd).
- The term n is odd, by assumption.
- The constant 4000 is even.

Thus the left side is a sum $(\text{odd})+(\text{even})+(\text{odd})+(\text{odd})+(\text{even})$ which must be odd. This means that the sum can never be 0, since 0 is even.

We have therefore shown that, if n is an odd integer, the equation $n^4+4n^3+3n^2+n+4000=0$can never hold. It follows by contraposition that, if $n^4+4n^3+3n^2+n+4000=0$*is* true, then n must be even. ☐

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*