question 2 views

Daily Challenge 12.4

(**Due: Thursday 8/23 at 12:00 noon eastern**)

(1) Implicit differentiation lets us find derivatives without solving an equation first.

As we saw in session 26, it is often unwieldy or impossible to solve an *implicit equation* which relates a function $f(x)$ to its argument x.

For instance, suppose we are interested in the slope of the tangent line to the hyperbola $x^2 - y^2 = 1$ at the point $(2, \sqrt{3})$. In principle, we could solve for the function $y = y(x)$ which represents the restriction of the graph of the hyperbola to the first quadrant, and then differentiate that expression.

However, it is much faster to simply differentiate both sides of the defining equation $x^2 - y^2 = 1$, keeping in mind that we are treating $y = y(x)$ as a function of x. This gives

$$2x - 2yy' = 0,$$

and hence

$$y'(x) = \frac{x}{y}.$$

At the point $(2, \sqrt{3})$, then, the tangent line has slope $\frac{2}{\sqrt{3}}$.

(2) Problem: implicit differentiation.

Use the strategy outlined above to find the slope of the tangent line to the curve $x\sin(x+y) = y\cos(x-y)$at the point $\left(0, \frac{\pi}{2}\right)$.

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Begin by taking the derivative of both sides;

$$((x \cdot \sin(x+y))' = (y \cdot \cos(x-y))'$$

We see on the left side, by the product rule and chain rule that

$$((x \cdot \sin(x+y))' = \sin(x+y) + \cos(x+y) \cdot (1 + y'.)$$

We then take the derivative of the right side, which by the product rule

$$(y \cdot \cos(x-y))' = y' \cdot \cos(x-y) - y \cdot \sin(x-y) \cdot (1 + y')$$

We then have that

$$\cos(x+y) + y'\cos(x+y) + \sin(x+y) = y'\cos(x-y) - y\sin(x-y) + yy'\sin(x-y),$$

and when all elements with a factor of y' are moved to the left side,

$$y'\cos(x+y) - yy'\sin(x-y) - y'\cos(x-y) = -\sin(x+y) - \cos(x+y) - y\sin(x-y)$$

Or when the y' is factored out,

$$y'\left(\cos(x+y) - y\sin(x-y) - \cos(x-y)\right) = -\sin(x+y) - \cos(x+y) - y\sin(x-y).$$

We can then solve for y', ie

$$y' = \frac{-(\sin(x+y) + \cos(x+y) + y\sin(x-y))}{(\cos(x+y) - y\sin(x-y) - \cos(x-y))}$$

We can then see that for $x = 0$ and $y = \frac{\pi}{2}$, then we have plugging these values all into the above beautiful equation that $y' = 1 - \frac{2}{\pi}$
:boi: :yungman:

Updated 7 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Using the product rule, the derivative of the left side is

$$\frac{d}{dx}(x\sin(x+y)) = \sin(x+y) + x\cos(x+y)\cdot(1+y')\,.$$

Meanwhile, the derivative of the right side is

$$y'\cos(x-y) - y\sin(x-y)\cdot(1-y')\,.$$

Setting these equal, one finds

$$\sin(x+y) + x\cos(x+y)\,(1+y') = y'\cos(x-y) - y\sin(x-y)\cdot(1-y')$$

Move everything with factors of y' to the left, and everything else to the right, to find

$$y'x\cos(x+y) - y'y\sin(x-y) - y'\cos(x-y) = -\sin(x+y) - x\cos(x+y) - y\sin(x-y),$$

which we can then factor and solve to obtain

$$y' = \frac{-\sin(x+y) - x\cos(x+y) - y\sin(x-y)}{x\cos(x+y) - y\sin(x-y) - \cos(x-y)}.$$

We plug in the point $x=0, y=\frac{\pi}{2}$, noting that $\sin(x+y) = \sin\left(\frac{\pi}{2}\right) = 1,\ \sin(x-y) = \sin\left(-\frac{\pi}{2}\right) = -1$, and $\cos(x\pm y) = \cos\left(\pm\frac{\pi}{2}\right) = 0$, so that

$$y'(0) = \frac{-1-0-\frac{\pi}{2}(-1)}{0-\frac{\pi}{2}(-1)-0} = \frac{-1+\frac{\pi}{2}}{\frac{\pi}{2}},$$

and simplifying yields

$$y'(0) = 1 - \frac{2}{\pi}.$$

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*