question 2 views

Daily Challenge 17.7

(**Due: Sunday 11/11 at 12:00 noon eastern**)

Another problem that takes 5-10 minutes. You should be able to finish this on time despite social obligations on Sunday.

(1) Problem: an integral equation.

Find all continuous functions f satisfying

$$\int_0^x f = (f(x))^2 + C$$

where C is some constant.

daily_challenge

Updated 5 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

I believe we can take the derivative of each side and see that $f = 2f(x) \cdot f'(x)$ which means $\frac{1}{2} = f'(x)$ where $x \neq 0$ due to our division; We see that this implies f is a line with slope $\frac{1}{2}$ and some constant; ie $f(x) = \frac{1x}{2} + C$, but I doubt this is a valid argument at all :shrug:

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Suppose there were such a continuous function f. Differentiating, we would have $f(x) = 2f(x)f'(x)$. This means that either $f(x) = 0$ or $f'(x) = \frac{1}{2}$. Since f is continuous, it cannot alternate between these possibilities; either $f(x) = 0$ for all x or $f(x) = \frac{x}{2} + b$ for all x.

We should check that both of these possibilities work.

1. If $f(x) = 0$ identically, then it is true that $\int_0^x f(t)\,dt = (f(x))^2 + C$ for $C = 0$.
2. On the other hand, say $f(x) = \frac{x}{2} + b$. Then the left side is

   $$\int_0^x \left(\frac{t}{2} + b\right)\,dt = \left[\frac{t^2}{4} + bt\right]_0^x = \frac{x^2}{4} + bx,$$

   while the right side is

   $$\left(\frac{x}{2} + b\right)^2 + C = \frac{x^2}{4} + bx + b^2 + C.$$

   The two are equal only if $b^2 + C = 0$.

Thus the complete answer is that all such functions are either of the form (1) $f(x) = 0$ and $C = 0$, or (2) $f(x) = \frac{x}{2} \pm \sqrt{-C}$ where C is negative.

Updated 4 months ago by Christian Ferko

followup discussions *for lingering questions and comments*