question 2 views

Daily Challenge 9.4

(**Due: Friday 7/13 at 12:00 noon Eastern**)

Because limits are so important, I think it's useful for you to read several explanations written by different people to get a broader perspective.

Thus we'll switch gears and start reading an excerpt from Spivak's chapter on limits, which will be broken up over three daily challenges whose problems take the form of reading questions.

First, I'd like to make some comments on what constitutes good mathematical writing.

(1) Good explanations are usually not simple.

Beginning mathematics students sometimes believe that a "good" explanation is one which is simple, or easy to read; that is, it allows the reader to understand the topic of conversation with minimal effort.

Not only is this untrue, but it is a harmful misconception. A good explanation of almost anything interesting enough to be worth learning will *not* be easy to follow.

In mathematics, the ideas we discuss are so precise and abstract that a simple explanation cannot tell you very much about them at all. At best, a simple explanation will *fool* you into thinking that you understand the topic. Later, you realize that you have been bamboozled when you cannot solve problems or prove any new properties of the objects in question.

Rather, a good explanation is thought-provoking and rewards effort with insight. It forces you to connect the dots and construct the necessary mathematical knowledge for yourself.

For example, the gold standard for good introductory mathematical writing is Spivak's Calculus textbook.

Spivak's discussion of limits is the single best introduction to the subject I've seen, but it is not simple to read. This is entirely necessary, since limits are a complicated thing to understand. What makes his writing good is that it repays the work you invest in reading it by giving you a rich view of the epsilon-delta definition.

Almost every other mathematician with whom I have discussed introductory calculus books agrees that Spivak's is the best (except for the occasional rebel who prefers Apostol). For instance, the AoPS book list claims "Top students swear by this book," the UChicago math bibliography says "Of course, as we all know, the One True Calculus Book is Spivak, Calculus. This is a book everyone should read. If you don't know calculus and have the time, read it and do all the exercises," and the PROMYS reading list says "Spivak takes the honors as the best introductory calculus text."

(2) Problem: Spivak reading on limits.

Open this excerpt from Spivak's chapter on limits. Read the first four pages and answer the following reading questions.

(a) Explain what is meant by

> $h(a)$ is defined "the wrong way"

What would the "right" definition of $h(a)$ be?

(b) Consider figure 2(b). Explain why the arrows from the lower line to the upper line tend to "bunch up" near zero but "spread out" for values outside of $[-1, 1]$.

(c) Spivak generalizes the argument about $f(x) = 3x$ by saying

> "Naturally, the same sort of argument works for the function $f(x) = 3,000,000\,x$. We just have to be $1,000,000$ times as careful, choosing $|x - a| < \frac{\epsilon}{3{,}000{,}000}$ in order to ensure that $|f(x) - a| < \epsilon$."

However, there is a mistake in this paragraph; it is incorrect as written. Where is the error?

(d) In the discussion of $f(x) = x^2$, Spivak says

> "To make sure you understand the reasoning in the previous paragraph, it is a good exercise to figure out how the argument would read if we chose $|x - 3| < 10$."

Do this exercise; that is, rewrite the argument using the alternate initial assumption $|x - 3| < 10$.

daily_challenge

Updated 9 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski: Reading completed and work begun at 9:21
a: By Spivak saying that the function $h(a)$ is defined the "wrong" way, he means that it is counter-intuitive for this (apparently) strictly increasing function to become piecewise for the x-value a.

b: These arrows are seen bunching up since multiplying fraction less than by itself results in an even smaller fraction, for example $(\frac{1}{2})^2 = \frac{1}{4}$, or in this scenario since fractions are multiplied by themselves twice, $(\frac{1}{2})^3 = \frac{1}{8}$. This results from the numerator being 1, where being placed to any power does not change it, but since the denominator is being placed to the third power, the fraction become much smaller.

c: The mistake in this line seems to be that $|f(x) - a| < \epsilon$ should be $|f(x) - L| < \epsilon$.

d: We need to show that the function $f(x) = x^2$ approaches 9 near 3, so we begin by "translating" this limit to what it implies, so $|f(x) - 9| < \epsilon$, and in turn this can be factored to $|x-3| \times |x+3|$. We begin differing from the Spivak book here, where we choose to restrict $|x-3| < 10$. This then means that $-10 < x-3 < 10$, which then is $-7 < x < 13$. We can now modify this by adding 3 to all sides and get $-4 < x+3 < 16$, which then guarantees to us that $|x+3| < 16$. We now have $|x^2-9| = |x-3| \cdot |x+3| < 16|x-3|$. This then shows we have $|x^2 - 9| < \epsilon$ for $|x-3| < \frac{\epsilon}{16}$. We require that $|x-3| < \min(\frac{\epsilon}{16}, 10)$, and therefore there exists a ϵ and δ for this limit.

Updated 9 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) We say that $h(a)$ is defined the "wrong way" because $\lim_{x \to a} h(a) = L$ but $h(a)$ is itself not equal to L; the actual output of the function at a is shifted downward from its limiting value.

It is implied that the "right way" would be to define $h(a) = L$.

(b) The arrows describe the mapping $x \mapsto x^3$. For $|x| < 1$, we have $|x^3| < |x|$, so values on the lower line whose absolute value is less than 1 will be pushed "inward" to an output with smaller absolute value. Likewise, if $|x| > 1$, we have $|x^3| > |x|$, so inputs with absolute value above 1 are pushed "outward" to outputs with increasing norms.

(c) The first statement is that

$$|x-a| < \frac{\epsilon}{3,000,000},$$

and multiplying both sides by $3,000,000$ gives

$$|3,000,000\,x - 3,000,000\,a| < \epsilon.$$

Recall that the function is $f(x) = 3,000,000\,a$. Thus this statement can be written as

$$|f(x) - 3,000,000\,a| < \epsilon.$$

This is the correct statement. Spivak dropped the factor of $3,000,000$, instead writing

$$|f(x) - a| \overset{?!?}{<} \epsilon.$$

(d) We aim to show that, given $\epsilon > 0$, we can find $\delta > 0$ so that $0 < |x-3| < \delta$ implies $|x^2 - 9| < \epsilon$. First factor the latter to obtain the equivalent condition

$$|x-3||x+3| < \epsilon.$$

Now assume $|x-3| < 10$. This means that x is upper-bounded by 13, so $|x+3|$ is upper bounded by 16. Thus we choose

$$\delta = \min\left(10, \frac{\epsilon}{16}\right).$$

Then if $0 < |x-3| < \delta$, we have $|x-3||x+3| < 16\delta \leq \epsilon$, which proves that $\lim_{x\to 3} x^2 = 9$.

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*