Application of the Schwarzian Derivative

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The Problem Statement

Show that

$$\mathscr{D}(f \circ g) = ((\mathscr{D}f) \circ g) \cdot (g')^2 + \mathscr{D}g, \tag{1}$$

where $\mathscr{D}$ represents taking the Schwarzian derivative of a function, equal to

$$\mathscr{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{2}$$

Assume f is C^3.

Intense Algebra

Our proof begins by finding the first, second, and third derivatives of our function, which I have calculated ahead of time and will not go over. Said algebra will be left *as an exercise.* Going down the line,

$$(f(g(x)))' = f'(g(x)) \cdot g'(x), \tag{3}$$

$$(f(g(x)))'' = f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x), \tag{4}$$

and

$$(f(g(x)))''' = f'''(g(x)) \cdot (g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x). \tag{5}$$

The longest equation (for now)

For an appropriate first step, we insert the calculated derivatives into the Schwarzian to get

$$\mathscr{D}(f(g(x)) = \frac{f'''(g(x)) \cdot (g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)}{f'(g(x)) \cdot g'(x)} - \frac{3}{2} \cdot \left(\frac{f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x)}{f'(g(x)) \cdot g'(x)} \right)^2 . \tag{6}$$

Or after applying FOIL to the rightmost element,

$$\mathscr{D}f(g(x)) = \frac{f'''(g(x)) \cdot (g'(x))^2}{f'(g(x))} + \frac{3f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} \tag{7}$$

$$- \frac{3}{2}\left(\left(\frac{f''(g(x)) \cdot g'(x)}{f'(g(x))} \right)^2 + 2\frac{f''(g(x))g''(x)}{f'(g(x))} + \left(\frac{g''(x)}{g'(x)} \right)^2 \right) \tag{8}$$

We can now distribute the $-\frac{3}{2}$ seen surrounding the 4th element and get that

$$\mathscr{D}(f\circ g) = \frac{f'''(g(x))(g'(x))^2}{f'(g(x))} + 3\frac{f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} \tag{9}$$

$$-\frac{3}{2}\left(\frac{f''(g(x))g'(x)}{f'(g(x))}\right)^2 - 3\frac{f''(g(x))g''(x)}{f'(g(x))} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2. \tag{10}$$

Or after the second and fourth terms coincidentally cancel,

$$\mathscr{D}(f\circ g) = \frac{f'''(g(x))(g'(x))^2}{f'(g(x))} + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{f''(g(x))g'(x)}{f'(g(x))}\right)^2 - \frac{3}{2}\left(\frac{g''(x)}{g'(x)} \tag{11}$$

We can now note that a $g'(x)^2$ is present in the the first and fourth terms; we group them and factor out said $g'(x)^2$ to see that

$$\mathscr{D}(f \circ g) = \left(\frac{f'''(g(x))}{f'(g(x))} - \frac{3}{2} \left(\frac{f''(g(x))}{f'(g(x))} \right)^2 \right) (g'(x))^2 + \frac{g'''(x)}{g'(x)} - \frac{3}{2} \left(\frac{g''(x)}{g'(x)} \right) \tag{12}$$

Oh, who could have known that this perfectly matches and proves that

$$\mathscr{D}(f \circ g) = ((\mathscr{D}f) \circ g) \cdot (g')^2 + \mathscr{D}g. \square \tag{13}$$

Show that, if $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$ then $\mathscr{D}f = 0$.

Once again, we shall find the first, second, and third derivatives;

$$f'(x) = \frac{ad - bc}{(cx + d)^2}, \tag{14}$$

$$f''(x) = \frac{-2c(ad - bc)}{(cx + d)^3}, \tag{15}$$

$$f'''(x) = \frac{6c^2(ad - bc)}{(cx + d)^4}. \tag{16}$$

We can then insert these calculated derivatives in the definition of the Schwarzian and see that

$$\mathscr{D}f(x) = \frac{\frac{6c^2(ad-bc)}{(cx+d)^4}}{\frac{ad-bc}{(cx+d)^2}} - \frac{3}{2}\left(\frac{\frac{-2c(ad-bc)}{(cx+d)^3}}{\frac{ad-bc}{(cx+d)^2}}\right)^2 \tag{17}$$

Or after placing the denominators of all these fractions in the "correct" places, and in turn simplifying them,

$$\mathscr{D}f(x) = \frac{6c^2(ad-bc)}{(cx+d)^4}\frac{(cx+d)^2}{ad-bc} - \frac{3}{2}\left(\frac{-2c(ad-bc)}{(cx+d)^3}\frac{(cx+d)^2}{ad-bc}\right)^2 \tag{18}$$

$$= \frac{6c^2}{(cx+d)^2} - \frac{3}{2}\left(\frac{-2c}{cx+d}\right)^2 \tag{19}$$

$$= 0.\square \tag{20}$$