question 4 views

Daily Challenge 3.4

(**Due: Friday 5/11 at 12:00 noon Eastern**)

I think we're making good progress. Perhaps we can spend the rest of May on chapter 1 (proofs, trigonometry, exponentials, and logarithms), and then work on chapter 2 through June.

Review

In our study of trigonometry, we've been building up a toolkit of useful *identities*. A **trigonometric identity** is an equation which is true for *any* choice of angles appearing in the trig functions. For instance, we've seen that

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

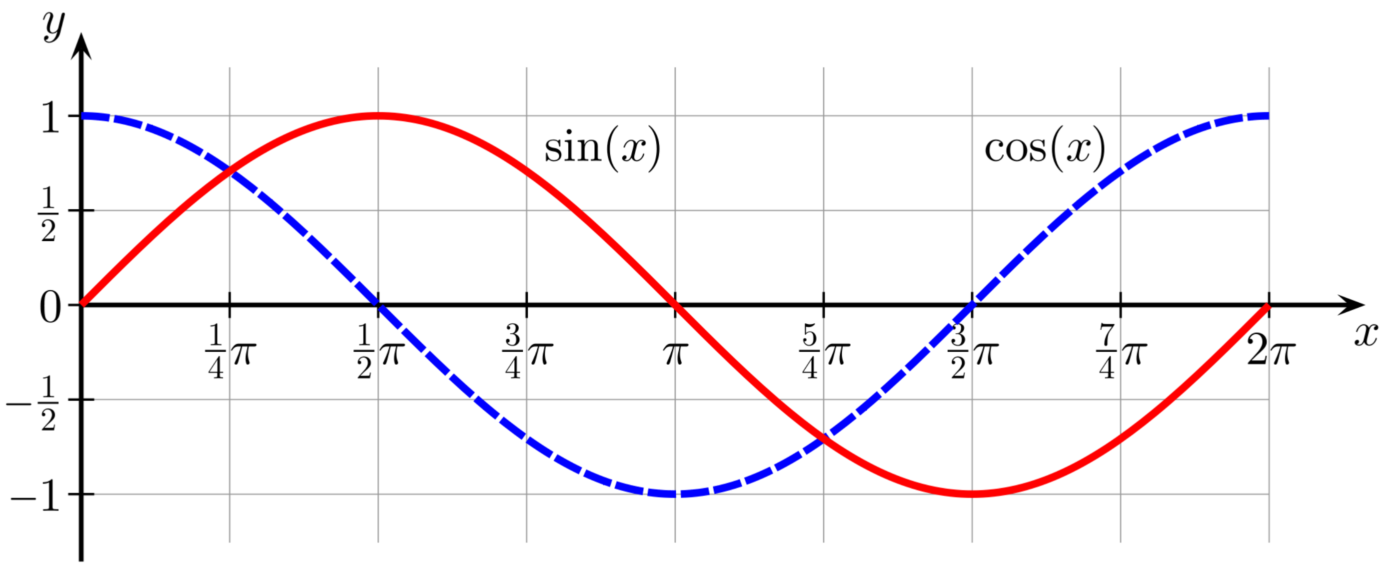
holds for all values of θ. This is the most important identity, sometimes called the Pythagorean identity. In our last meeting, we derived the so-called **angle addition formulas**:

$$\begin{aligned} \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta). \end{aligned}$$

These formulas are useful for getting other identities. Let's prove the following:

Theorem. For any $\theta \in \mathbb{R}$, it is true that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$, where the arguments of both functions are measured in radians.

Exploration. To get some intuition, look at the graphs of sine and cosine:



We see that the blue curve $\cos(x)$ can be obtained by shifting the red curve $\sin(x)$ to the left by $\frac{\pi}{2}$. And we remember from our study of functions that the graph of the function $f(x+a)$, for any function f, is the shift of the graph of $f(x)$ to the *left* by a (do you see why it is left and not right, for positive a?).

Here's another way to see it: we defined $\cos(\theta)$ as the x-coordinate of the point at angle θ on the unit circle, and we defined $\sin(\theta)$ as the y-coordinate. Adding an angle of $\frac{\pi}{2}$ is the same as *rotating* by 90 degrees counter-clockwise. But such a rotation sends the x axis to the y axis, so we expect it to relate the sine and cosine.

Thus we expect that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$ should hold, based on graphical reasoning. Let's see if we can prove it from the angle addition formula.

Argument. We begin with the angle addition formula for sine, namely

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

This equation holds for any choice of α and β. In particular, we can let $\alpha = \theta$ and $\beta = \frac{\pi}{2}$, which gives

$$\sin\left(\theta + \frac{\pi}{2}\right) = \sin(\theta)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} + \cos(\theta)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}.$$

We recall that $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$, so we conclude that

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta),$$

which is what we wanted to show. □

Problem

In our last session, we used the angle-addition formulas and set $\alpha = \beta$ to obtain the **double-angle formula** for sine,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

(a) Derive the **half-angle** formulas for sine and cosine. That is, obtain equations which express $\sin\left(\frac{1}{2}\theta\right)$ and $\cos\left(\frac{1}{2}\theta\right)$ in terms of $\sin(\theta)$ and/or $\cos(\theta)$.

[Hint: begin by deriving the double-angle formula for cosine. You may need to use the Pythagorean identity to trade sines for cosines or vice-versa. Now replace $\theta \to \frac{\theta}{2}$ and solve for the desired quantity. Your results will involve square roots.]

(b) Using your results from (a), evaluate $\sin\left(\frac{\pi}{12}\right)$ and $\cos\left(\frac{\pi}{12}\right)$.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Solutions (Logan). Your reasoning and results go here.

(a) Of course I take the hint given, and I begin by pulling the angle addition formula for cosine, which is:
$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
Similar to how we found the double angle formula for sine, I will assign angles α and β the same value θ, therefore $\alpha, \beta = \theta$.
I substitute the occurrences of α and β in the cosine angle addition formula, which leads me to the equation $\cos(2\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)$. I *believe* I should simplify this to $\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$ As far as I can tell, there is nowhere to go with this. I don't know without looking at the instructor answer where to go with this, but I am liking it.

(b)

Updated 11 months ago by Logan Pachulski and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Solutions (Christian).

(a) Following the hint, we begin with the angle addition formula for cosine, namely

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

If we let $\alpha = \beta = \theta$, this gives the cosine double-angle identity,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

We have two separate calculations to do: one to obtain the half-angle formula for cosine, and one to obtain the half-angle formula for sine.

1. Half-angle formula for cosine.

   It's somewhat annoying to have two different trigonometric functions, $\cos^2$ and $\sin^2$, on the right side. Using the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, we are free to replace $\sin^2(\theta) = 1 - \cos^2(\theta)$. This gives

   $$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

   That's better. This equation holds for any value of θ. In particular, we can simply re-scale $\theta \to \frac{\theta}{2}$, which yields

   $$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1.$$

   Solving for $\cos\left(\frac{\theta}{2}\right)$ gives

   $$\boxed{\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}.}$$

   We should be a little careful about the plus-or-minus sign in front. This doesn't mean that you can simply choose whichever sign you like. We know that the cosine function is positive in the first and fourth quadrants, and negative in the second and third quadrants, so the formula above really means "look at the quadrant that $\frac{\theta}{2}$ is in, and choose the positive or negative root appropriately."

2. Half-angle formula for sine.

   This time, we use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, to replace $\cos^2(\theta) = 1 - \sin^2(\theta)$. This gives

   $$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right).$$

   Solving,

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}.}$$

Again, the $\pm$ sign means "choose the sign appropriate for the angle you're considering."

(b) To compute $\sin\left(\frac{\pi}{12}\right)$, let's use our brand-new identity

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}.$$

This holds for all values of θ. In particular, we can let $\theta = \frac{\pi}{6}$. We know that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, so the formula gives

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \\ &= \frac{\sqrt{2-\sqrt{3}}}{2}.\end{aligned}$$

In the last line, we have taken the positive square root (the angle $\frac{\pi}{12}$ is in the first quadrant!) and simplified the fraction under the square root. If you're really picky, you can simplify the expression further as

$$\frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

but I think that's really splitting hairs.

Likewise, to find $\cos\left(\frac{\pi}{12}\right)$, we use

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}.$$

Let $\theta = \frac{\pi}{6}$ to find

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \frac{\sqrt{2+\sqrt{3}}}{2}.\end{aligned}$$

Again we have taken the positive sign. If you really want to simplify this more, you can also write it as

$$\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}},$$

but again this is nit-picking.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*