question 3 views

Daily Challenge 10.1

(~~**Due: Tuesday 7/17 at 12:00 noon Eastern**~~)
(~~**Skip: now due Wednesday 7/18 at 12:00 noon Eastern**~~)
(**Due Thursday 7/19 at 12:00 noon Eastern**)

Today, the hard work we put into proving limit properties will pay off and let us prove continuity for a large class of functions.

(1) Continuity means the limit exists and equals the function output.

We've seen many examples where a function's limiting value at a point is *not* equal to its output at that point, i.e. where $\lim_{x \to a} f(x) \neq f(a)$.

Recall that the symbols $\lim_{x \to a} f(x)$ mean "the unique number K, if it exists, such that for every $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |x-a| < \delta$ implies $|f(x) - K| < \epsilon$." Note in particular the "if it exists" clause. By a strict reading of this definition, the equation

$$\lim_{x \to a} f(x) = f(a)$$

will be false whenever the limit does not exist, since in such cases the left side of the equation is not a number (and an equation is true only when both sides of the equation are numbers and the two numbers are equal).

For instance, if $f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$, then it is not true that $\lim_{x \to 0} f(x) = f(0)$ because the limit does not exist.

A second way for the equation $\lim_{x \to a} f(x) = f(a)$ to fail is if both sides are defined -- the limit exists and the function is defined at $f(a)$ -- but the two are unequal. For instance, this is the case with $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Here we see $\lim_{x \to 0} f(x) = 0$ but $f(0) = 1$.

A third way for the equation to fail is if *both* the limit does not exist and the function is undefined. If $f(x) = \frac{1}{x}$, the equation $\lim_{x \to 0} f(x) = f(0)$ is "twice as false" since the left side is undefined (the limit does not exist) and the right side is undefined (we cannot evaluate $\frac{1}{0}$).

We would like to restrict our attention to functions which do not exhibit any of the three "bad behaviors" or "pathologies" above. This motivates the following

Definition. Let f be a real-valued function defined on an open interval containing a point $a \in \mathbb{R}$. We say that f is *continuous at a* if $\lim_{x \to a} f(x) = f(a)$. (This means *both* that the limit exists, and that it equals $f(a)$.) If f is continuous at each point in its domain, we say that f is *continuous*.

We can unpack the definition of continuity at a point a by saying

$$\text{for all } \epsilon > 0 \text{ there exists } \delta > 0 \text{ so that } |x-a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

The attentive reader will notice that I have deleted the $0 < |x-a|$ part of the inequality which usually appears. This is because, in contrast to the usual definition of a limit, in the case of continuity <u>we actually knows what happens at</u> $x = a$: we have assumed $f(x) = f(a)$, so the inequality must also hold when $|x-a| = 0$.

(2) Polynomials and rational functions are continuous.

Now we can use our powerful results for the sum, product, and quotient of limits to prove that a large class of functions is continuous.

Proposition. Let $f(x)$ be a polynomial. Then $f(x)$ is continuous.

Proof. Let $a \in \mathbb{R}$. We must show that $\lim_{x \to a} f(x)$ exists, and that it is equal to $f(a)$.

By definition, a polynomial $f(x)$ can be written as $f(x) = c_0 + c_1 x + \cdots + c_n x^n$ for some coefficients c_i and a positive integer n. But since we have proven that $\lim_{x \to a}(f+g) = (\lim_{x \to a} f) + (\lim_{x \to a} g)$, repeatedly applying this result gives

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} (c_0 + c_1 x + \cdots + c_n x^n) \\ &= \lim_{x \to a} (c_0) + \lim_{x \to a} (c_1 x) + \cdots + \lim_{x \to a} (c_n x^n). \end{aligned}$$

We have also proven that $\lim_{x \to a}(cf(x)) = c \lim_{x \to a} f(x)$, if c is a constant. But all of the c_i above are constants, so

$$\lim_{x \to a} (c_0) + \lim_{x \to a} (c_1 x) + \cdots + \lim_{x \to a} (c_n x^n) = c_0 + c_1 \lim_{x \to a} x + \cdots + c_n \lim_{x \to a} (x^n).$$

Finally, we have proven the "multiplication rule" $\lim_{x \to a} (fg) = (\lim_{x \to a} f)(\lim_{x \to a} g)$. This means, for instance, that

$$\lim_{x \to a}(x^2) = \left(\lim_{x \to a} x\right)\left(\lim_{x \to a} x\right) = a^2,$$

and

$$\lim_{x\to a}(x^3) = \left(\lim_{x\to a} x\right)\left(\lim_{x\to a} x\right)\left(\lim_{x\to a} x\right) = a^3,$$

and similarly that $\lim_{x\to a}(x^n) = a^n$ for any n. (Here we are also using that $\lim_{x\to a} x = a$, which we proved).

In conclusion,

$$\begin{aligned}\lim_{x\to a} f(x) &= c_0 + c_1 \lim_{x\to a} x + \cdots + c_n \lim_{x\to a}(x^n) \\ &= c_0 + c_1 a + \cdots + c_n a^n \\ &= f(a).\end{aligned}$$

Thus the polynomial $f(x)$ is continuous at a. But the same argument applies equally well to any $a \in \mathbb{R}$, so f is continuous everywhere. $\Box$

As an easy consequence of this, we can prove that a quotient of polynomials is continuous wherever it is defined.

Definition. If $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, we say that f is a *rational function*.

For example, the function $f(x) = \frac{x^3-2x+5}{x^2+1}$ is rational.

Corollary. Rational functions are continuous.

Proof. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function, so that $p(x)$ and $q(x)$ are polynomials.

Recall that "continuous" means "continuous at each point a on the function's domain." In particular, the domain of the function only includes points x where $q(x) \neq 0$, since otherwise we would be dividing by zero. Thus we may assume $q(x) \neq 0$.

Now let $a \in \mathrm{Dom}(f)$ and consider $\lim_{x\to a} f(x)$. By the limit quotient rule, this is $\frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)}$.

But p and q are each polynomials, which we have shown to be continuous, so $\lim_{x\to a} p(x) = p(a)$ and $\lim_{x\to a} q(x) = q(a)$.

Thus $\lim_{x\to a} f(x) = \frac{p(a)}{q(a)} = f(a)$, so f is continuous at a and hence on its entire domain. $\Box$

(3) Problem: an absolute value bound.

Suppose f is a real-valued function with the property that $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Prove that f is continuous at 0.

[Hint: the condition means that $f(0)$ can only be one thing. So you just need to prove that $\lim_{x\to 0} f(x)$ exists and is equal to $f(0)$.]

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski: Proof: Suppose $|f(x)| \leq |x|$. The only possible solution for $f(0)$ when taking aforementioned property into account means that $f(0) = 0$. To show that f is continuous is at zero we must show $\lim_{x\to 0} f(x) = 0$. Let $\epsilon > 0$ and choose simply that $\delta = \epsilon$, then we have that $0 < |x| < \delta \implies |f(x)| \leq |x| < \delta = \epsilon$, proving that f is continuous at 0. $\Box$

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Proposition. If $$f$$ is a real-valued function with $$|f(x)| \leq | x |$$ for all $$x \in \mathbb{R}$$, then $$f$$ is continuous at $$0$$.

Proof. Applying the bound $$|f(x)| \leq |x|$$ at $$x=0$$, we see that $$f(0) = 0$$. Thus we need only prove that $$\lim_{x \to 0} f(x) = 0$$.

Let $$\epsilon > 0$$ be given and choose $$\delta = \epsilon$$. Then whenever $$0 < | x | < \delta$$, we have

$$\begin{align} | f(x) | \leq | x | < \delta = \epsilon , \end{align}$$

where in the first step we have applied the assumption that $$|f(x)| \leq | x | $$ for all $$x$$. This proves that $$\lim_{x \to 0 } f(x) = 0 = f(0)$$, so $$f$$ is continuous at zero. $$\Box$$

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*