Notes. This first section isn't a question; it's just a place for you to collect some notes on various derivative results that we discussed in the sessions.

Fill in each of the following with the corresponding result or formula.

Definition of the derivative.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{1}$$

Linearity.

$$\begin{aligned} (f+g)'(x) &= f'(x) + g'(x), \\ (cf)'(x) &= c \cdot f'(x). \end{aligned} \tag{2}$$

Product rule.

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x). \tag{3}$$

Quotient rule.

$$\left(\frac{f}{g}\right)'(x) = \frac{f'g - fg'}{g^2}. \tag{4}$$

Inverse function rule.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{5}$$

Power rule.

$$\frac{d}{dx}x^n = nx^{(n-1)} \tag{6}$$

Trig functions.

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \cos(x) \\ \frac{d}{dx}(\cos(x)) &= -\sin(x) \\ \frac{d}{dx}(\tan(x)) &= \sec^2(x) \\ \frac{d}{dx}(\sec(x)) &= \sec(x)\tan(x) \\ \frac{d}{dx}(\csc(x)) &= -\csc(x)\cot(x) \\ \frac{d}{dx}(\cot(x)) &= -\csc^2(x) \end{aligned} \tag{7}$$

Exponential and logarithm.

$$\begin{aligned} \frac{d}{dx}(e^x) &= e^x, \\ \frac{d}{dx}(\log(x)) &= \frac{1}{x}. \end{aligned} \tag{8}$$

Chain rule.

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x). \tag{9}$$

Question 1. *Quick calculations*
You should be able to compute each of the following derivatives quickly (in under one minute each). Practice until differentiation is as automatic as addition or multiplication.

(a) Differentiate $\sin^3\big(x^2+\log(x)\big)$.

(b) Differentiate $\exp\big((x+1)^2(x+2)\big)$.

Note that we will sometimes use the notation $\exp(x)$ instead of e^x; they mean the same thing. That is, this function can also be written as

$$\exp\big((x+1)^2(x+2)\big) \equiv e^{(x+1)^2(x+2)}. \tag{10}$$

(c) Differentiate $\tan^2\left(\frac{x}{x+1}\right)$.

(d) Differentiate $\left(e^{x^2}\right)^2$.

(e) Find the derivative of $\cos^{-1}(x)$ and simplify until there are no trigonometric functions remaining.

(f) Calculate $\frac{dy}{dx}$ for $x^{1/3}+y^{1/3}=1$ by implicit differentiation (that is, differentiate both sides with respect to x, then solve for $y'(x)$). Then calculate $\frac{dy}{dx}$ in a second way: solve for y explicitly and calculate y' using the chain rule. Confirm that your answers are the same.

(g) Suppose $PV^c = nRT$, and assume $P = P(V)$ with all other letters representing constants. Compute $\frac{dP}{dV}$.

(h) Suppose $F = \frac{mg}{(1+r^2)^{3/2}}$, and assume $F = F(r)$ with all other letters representing constants. Compute $\frac{dF}{dr}$.

Solution 1.
(a): We see first by the chain rule that

$$\begin{aligned} f'(x) &= 3\sin^2(x^2+\log(x))\cdot\big(\cos\big(x^2+\log(x)\big)\big)' && (11)\\ &= 3\sin^2(x^2+\log(x))\cdot -\sin\big(x^2+\log(x)\big)\cdot(2x+\frac{1}{x}) && (12)\end{aligned}$$

(b): We see by the chain rule that

$$\begin{aligned} f'(x) &= \exp\big((x+1)^2(x+2)\big)\cdot\big(((x+1)^2)'(x+2)+x(x+1)^2\big) && (13)\\ &= \exp\big((x+1)^2(x+2)\big)\cdot\big(2x(x+1)(x+2)+x(x^2+2x+1)\big) && (14)\\ &= \exp\big((x+1)^2(x+2)\big)\cdot\big(2x(x^2+3x+2)+(x^3+2x^2+x)\big) && (15)\\ &= \exp\big((x+1)^2(x+2)\big)\cdot\big((3x^3+8x^2+5x)\big) && (16)\end{aligned}$$

(c): First by the chain rule where the inner is $\tan\left(\frac{x}{x+1}\right)$ and the outer is x^2, then

$$\begin{aligned} f'(x) &= 2\tan\left(\frac{x}{x+1}\right)\cdot(\tan\left(\frac{x}{x+1}\right))' && (17)\\ &= 2\tan\left(\frac{x}{x+1}\right)\cdot\sec^2(\frac{x}{x+1})\cdot(\frac{x}{x+1})' && (18)\\ &= 2\tan\left(\frac{x}{x+1}\right)\cdot\sec^2(\frac{x}{x+1})\cdot(\frac{1}{(x+1)^2}) && (19)\end{aligned}$$

d: :thinking:

$$\begin{aligned} f'(x) &= ((e^x)^4)' && (20)\\ &= (e^{x^2} \cdot e^{x^2})' && (21)\\ &= (e^{2x^2})' && (22)\\ &= (e^{2x^2}) \cdot 4x && (23)\\ &= 4xe^{2x^2} && (24) \end{aligned}$$

e: We see by the inverse function rule that

$$f'(x) = \frac{1}{\sin(\cos^{-1}(x))}.$$

By inserting this denominator into Wolfram Alpha (because I have no better ideas and don't recognize it) we see that $\sin\bigl(\cos^{-1}(\theta)\bigr) = \sqrt{1-x^2}$, therefore

$$f'(x) = \frac{1}{\sqrt{1-x^2}}.$$

f: Pasted for reference: $x^{1/3} + y^{1/3} = 1$ We see by implicit differentiation of each side that first $\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$, or after applying algebra (subtract from each side and divide) that $y' = \frac{x^{-2/3}}{y^{-2/3}} = \frac{x^{-2/3}}{y^{-2/3}}$. We can explicitly find the derivative of y by applying algebra:

$$\begin{aligned} x^{1/3} + y^{1/3} &= 1 && (25)\\ y^{1/3} &= 1 - x^{1/3} && (26)\\ (y)' &= \left(\left(1 - x^{1/3}\right)^3\right)' && (27)\\ y' &= 3\left(1 - x^{1/3}\right)^2 \cdot \left(-\left(x^{1/3}\right)'\right) && (28)\\ y' &= 3\left(1 - x^{1/3}\right)^2 \cdot \left(-\left(\frac{1}{3}x^{-2/3}\right)\right) && (29)\\ y' &= -\left(1 - x^{1/3}\right)^2 \cdot x^{-2/3} && (30) \end{aligned}$$

g: The algebraic steps follow; 31-32 is by the product and power rules, and 32-34 are algebraic manipulations.

$$\begin{aligned} (PV^c)' &= (nRT)' && (31)\\ P'V^c + PcV^{c-1} &= 0 && (32)\\ P' &= \frac{-PcV^{c-1}}{V^c} && (33)\\ P' &= \frac{-Pc}{V} && (34) \end{aligned}$$

h: We start by applying the quotient rule to see that

$$
\begin{align}
F'(r) &= \frac{(0-(1+r^2)^{3/2})-(mg((1+r^2)^{3/2})')}{\left((1+r^2)^{3/2}\right)^2}, \text{ then by the chain rule} \tag{35}\\
&= \frac{-(1+r^2)^{3/2}-(-2mgr((1+r^2)^{1/2}))}{\left((1+r^2)^{3/2}\right)^2} \tag{36}\\
&= \frac{-(1+r^2)^{3/2}+2mgr((1+r^2)^{1/2})}{(1+r^2)^3} \tag{37}\\
&= \frac{-(1+r^2)^{3/2}}{(1+r^2)^3}+\frac{2mgr((1+r^2)^{1/2})}{(1+r^2)^3} \tag{38}\\
&= \frac{-1}{(1+r^2)^{3/2}}+\frac{2mgr}{(1+r^2)^{5/2}} \text{ And we can mult. term 1 by 1} \tag{39}\\
&= \frac{-(1+r^2)}{(1+r^2)^{5/2}}+\frac{2mgr}{(1+r^2)^{5/2}} \tag{40}\\
&= \frac{-(1+r^2)+2mgr}{(1+r^2)^{5/2}} \tag{41}
\end{align}
$$

This form looks satisying enough for my tastes (I also can't see any further way to *simplify* it).

Question 2. *The definition of the derivative.*
In each of the following questions, establish the result using only the definition of the derivative and not any of the theorems we've proven.

(a) Prove, working directly from the definition, that if $f(x) = x^2$, then $f'(a) = 2x$.

(b) Prove, from the definition of derivative, that if $g(x) = cf(x)$ (where $c \in \mathbb{R}$ is some constant) then $g'(x) = cf'(x)$.

(c) Find $f'(x)$ in two cases: first, if $f(x) = g(t + x)$, and second, if $f(t) = g(t + x)$. The answers will *not* be the same.

Solution 2.
(a) First, recall by the definition of derivative that $f'(x) = \lim_{h\to 0} \frac{(a+h)^2-a^2}{h}$. We can then foil and see that $\lim_{h\to 0} \frac{(a+h)^2-a^2}{h} = \lim_{h\to 0} \frac{2ah+h^2}{h}$ (the a^2's cancel out). We can then divide by h as we do not have to account for $h = 0$, and get $\lim_{h\to 0} \frac{2ah+h^2}{h} = \lim_{h\to 0} 2a + h$, which in turn as the limit of a sum is the sum of the limits, we have that $\lim_{h\to 0} 2a + h = \lim_{h\to 0} 2a + \lim_{h\to 0} h = 2a$, as 2a does not contain the variable h and is not affected by the limit, meanwhile $\lim_{h\to 0} h = 0$, because this function is a polynomial and is therefore continuous and the limit equals the solution normally. We then conclude that $f'(x) = 2a$. □

(b) We are given the information that f is a function that is differentiable (and in turn continuous) everywhere. Given a constant $c \in \mathbb{R}$, we must somehow show that $(cf(x))' = c\,(f(x))'$, or otherwise $c \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{c(f(a+h))-c(f(a))}{h}$. We can factor out the c in the numerator of the right side to get that $\lim_{h\to 0} \frac{c(f(a+h))-c(f(a))}{h} = \lim_{h\to 0} \frac{c(f(a+h)-f(a))}{h}$. We can take the c out of the numerator and apply the product rule of limits and get $\lim_{h\to 0}(c) \times \lim_{h\to 0} \frac{(f(a+h)-f(a))}{h}$, and since the former limit is a constant not containing h then we have what we want to prove, that $c \times \lim_{h\to 0} \frac{(f(a+h)-f(a))}{h} = (cf(x))' = c\,(f(x))'$. □

(c) Exploration (potentially answer): We first consider the case that $f(x) = g(t + x)$; Consider a point $a \in \mathbb{R}$; we have that $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{a\to 0} \frac{g(t+a+h)-g(a)}{h} = g'(t + a)$. The third line is true both by the substitution of $f(x) = g(t+x)$ and as well the definition of derivative applied to $g'(t + x)$. Therefore $f'(x) = g'(t + x)$. We can then consider the case $f(t) = g(t + x)$. Consider a point $a \in \mathbb{R}$ as well. We have that $f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = lim_{h\to 0} \frac{g(a+h+x)-g(a+x)}{h}$, with the 2nd step in the prior equation the result of substituting $t = a + h$ into $f(t) = g(t + x)$. We see that $\lim_{h\to 0} \frac{g(a+h+x)-g(a+x)}{h} = g'(a + x)$, and we see by replacing the dummy variable a with x that $f'(x) = g'(2x)$.

Question 3. *Some absolute value bounds.*
(a) Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

(b) Let $\alpha > 1$. If f satisfies $|f(x)| \leq |x|^\alpha$, prove that f is differentiable at zero.

Solution 3.
(a): We must show that the function f where $|f(x)| \leq x^2$ for all x is differentiable at zero. First we recall the definition of a derivative, ie $f'(x) = \lim_{h\to 0} \frac{f(0+h)+f(0)}{h}$. We can begin by seeing that for $x = 0$ we have that $|f(0)| \leq 0^2$, therefore $f(0) = 0$. We can apply this to our limit and see that we must show that the limit $\lim_{h\to 0} \frac{f(h)}{h}$ exists at 0. Let $\epsilon > 0$ be given and let $\delta = \epsilon$. We then have that since $|f(x)| \leq x^2$, it is true that when $0 < |h| < \delta$, it is true that $|f(h)| \leq \delta^2$ where $\delta^2 = \epsilon^2$, and we can then see that $\left|\frac{f(h)}{h}\right| \leq \left|\frac{\epsilon^2}{\epsilon}\right| = \epsilon$ and the limit exists, therefore f is differentiable at 0.

(b): We see that the way this problem is worded suggests a nearly identical argument to the previous; simply instead of x^2 we have "parabolas" of all powers greater than 1. Once again, we have that $|f(x)| \leq |x|^a$ where $a > 1$. We then have that for $x = 0$ that $|f(0)| \leq |0|^a$ and therefore $f(0) = 0$. We can now refer to the derivative and see that $\lim_{h\to 0} \frac{f(h)}{h}$ for $x = 0$. We must show that this limit exists at zero. Let $\epsilon > 0$ be given. We have by assumption that $\left|\frac{f(h)}{h}\right| \leq \left|h^{a-1}\right|$. Let $\delta = \epsilon^{\frac{1}{a-1}}$. We then have that $\left|\frac{f(h)}{h}\right| \leq \left|\delta^{a-1}\right| = \left(\epsilon^{\frac{1}{a-1}}\right)^{a-1} = \epsilon$. Therefore the limit exists and f is differentiable at 0.

Question 4. *More proofs using the definition.*
(a) Suppose that f is differentiable at a. Prove that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}. \tag{42}$$

Hint: we can always add zero in the form $0 = a - a$ for some quantity a.

(b) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$ for all x, and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$.

Hint: begin with the definition of $g'(a)$.

Solution 4.
a: We begin by referring to the equation we would like to prove; We see that we can add 0 in the form of $(f(a) - f(a))$ to the numerator of this equation to see

$$f'(a) = \lim_{h\to 0} \left(\frac{f(a+h) - f(a-h) + f(a) - f(a)}{2h}\right) \tag{43}$$

$$f'(a) = \lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{2h} + \frac{-f(a-h) + f(a)}{2h}\right) \tag{44}$$

We factor out a $\frac{1}{2}$ from each of the elements in the derivative; We see that the first element is of the form of the generic definition of derivative, while the latter is of a different form that we have proven to be equal to $f'(a)$ previously. We then see that

$$f'(a) = \frac{1}{2}\left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{-f(a-h) + f(a)}{h}\right) \tag{45}$$

$$= \frac{1}{2}\left(f'(a) + f'(a)\right) \tag{46}$$

$$= f'(a) \tag{47}$$

We can then conclude that it is true, by performing all these steps in reverse, that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}. \ \square \tag{48}$$

b: We are given the hint to start with the derivative of g, so we see by the definition of derivative that

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}. \tag{49}$$

The problem is phrased in such a way that use of the squeeze theorem is implied; We are told that $f(x) < g(x) < h(x)\ \forall\ x$, and that $f(a) = g(a) = h(a)$, therefore we can write, by subtracting the latter inequality from the former and for $x = a + k$, then

$$f(a+k) - f(a) \leq g(a+k) - g(a) \leq h(a+k) - h(a). \tag{50}$$

We can divide this all by k and get closer to what we are interested in;

$$\frac{f(a+k) - f(a)}{k} \leq \frac{g(a+k) - g(a)}{k} \leq \frac{h(a+k) - h(a)}{k}. \tag{51}$$

We see that the above inequality is true for $k > 0$, otherwise for $k < 0$ then

$$\frac{f(a+k) - f(a)}{k} \geq \frac{g(a+k) - g(a)}{k} \geq \frac{h(a+k) - h(a)}{k}. \tag{52}$$

We can now take the limits of all of these, as we have proven that taking limits does not change inequalities. We can now apply the squeeze theorem to see that since both

$$f'(a) \leq \lim_{k\to 0^+} \frac{g(a+k) - g(a)}{k} \leq h'(a) \tag{53}$$

$$\text{and} \tag{54}$$

$$f'(a) \geq \lim_{k\to 0^-} \frac{g(a+k) - g(a)}{k} \geq h'(a), \tag{55}$$

then it must be true that $g'(a)$ exists. Since $f'(a) = h'(a)$, then $g'(a) = f'(a) = h'(a)$. $\square$

Question 5. *Splitting the identity.*
Prove that it is impossible to write $x = f(x)g(x)$ for two functions f and g which are differentiable and satisfy $f(0) = g(0) = 0$.

Hint: Differentiate.

Solution 5.
We take the derivative of each side of the first given equation, with respect to x. Suppose by way of contradiction that the statement $x = f(x) \cdot g(x)$ is true. We see that $(x)' = 1 = f'g + fg' = (f \cdot g)'$. We can insert into this $x = 0$ and see that $1 = f'(0) \cdot (0) + f(0) \cdot g'(0) = 0$. The only assumption we have made here is that $x = f(x)g(x)\ \forall\ x$, therefore it must be false and impossible for $x = f(x)g(x)$ to be true for these functions.

Question 6. *The Schwarzian.*
Let f be a C^3 function with $f'(x) \neq 0$. We may define the *Schwarzian derivative* of f at x to be

$$\mathscr{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{56}$$

(a) Compute the Schwarzian derivative of $f(x) = e^{ax}$, where $a \neq 0$ is some constant.

(b) Show that

$$\mathscr{D}(f \circ g) = ((\mathscr{D}f) \circ g) \cdot (g')^2 + \mathscr{D}g. \tag{57}$$

(c) Show that, if $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$ then $\mathscr{D}f = 0$.

Solution 6.
(a): First we must find the first, second, and third derivatives of the function we are operating on; We see by the chain rule that the first derivative of $f(x) = e^{ax}$ where our outer function is e^x and outer is ax, then $f'(x) = e^{ax} \cdot a$. For identical reasons, we have that since a is a constant, then $f''(x) = a^2e^{ax}$, and once again for identical reasons, $f'''(x) = a^3e^{ax}$, and so on. We can then insert these functions we have found into the definition of Schwarzian derivative: $\mathscr{D}f(x) = \frac{a^3e^{ax}}{ae^{ax}} - \frac{3}{2}\left(\frac{a^2e^{ax}}{ae^{ax}}\right)^2$. We see in the problem statement that $f'(x) \neq 0$, therefore we can simplify this equation; $\frac{a^3e^{ax}}{ae^{ax}} - \frac{3}{2}\left(\frac{a^2e^{ax}}{ae^{ax}}\right)^2 = a^2 - \frac{3}{2}a^2 = \frac{-1a}{2}$, and we then have the Schwarzian derivative of our e^{ax}.

(b): Some crazy algebra is likely gonna be going on here; first we can see what Schwarzian derivative of this is by definition;

$$\mathscr{D}(f(g(x)) = \frac{(f(g(x)))'''}{(f(g(x)))'} - \frac{3}{2} \cdot \left(\frac{(f(g(x)))''}{(f(g(x)))'}\right)^2. \tag{58}$$

We are gonna be doing alot of taking derivatives. First we see by the chain rule the first derivative of this composition; $(f(g(x)))' = f'(g(x)) \cdot g'(x)$. To take the derivative of this we first apply the product rule; therefore $(f(g(x)))'' = (f'(g(x)))' \cdot g'(x) + f'(g(x)) \cdot g''(x)$, then by the chain rule applied to the foremost quantity

$$(f'(g(x)))' \cdot g'(x) + f'(g(x)) \cdot g''(x) = f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x). \tag{59}$$

This is the second derivative of our composition, and taking the derivative of this is going to get even more ugly.

$$(f''(g(x)) \cdot g'(x) \cdot g'(x) + f'(g(x)) \cdot g''(x))' = \left(f''(g(x)) \cdot g'(x)^2\right)' + (f'(g(x)) \cdot g''(x))'. \tag{60}$$

So after half of a paper filled with algebra or whatevs,

$$(f(g(x)))''' = f'''(g(x)) \cdot (g'(x))^3 + f''(g(x)) \cdot 2g'(x)g''(x) + f''(g(x)) \cdot g'(x) \cdot g''(x) + f'(g(x)) \cdot g'''(x), \tag{61}$$

or after adding the middle terms, $(f(g(x)))''' = f'''(g(x)) \cdot (g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)$. That was pretty intense, but now we can go on and insert this into the definition of the Schwarzian derivative.

$$\begin{aligned}\mathscr{D}(f(g(x)) = &\frac{f'''(g(x)) \cdot (g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)}{f'(g(x)) \cdot g'(x)} \\ &- \frac{3}{2} \cdot \left(\frac{f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x)}{f'(g(x)) \cdot g'(x)}\right)^2. \end{aligned} \tag{62}$$

We now must break this entire thing into fractions. After *alot* of paper and suffering, we can see that eventually we have that

$$\mathscr{D}f(g(x)) = \frac{f'''(g(x)) \cdot (g'(x))^2}{f'(g(x))} + \frac{3f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} \tag{63}$$

$$- \frac{3}{2}\left(\left(\frac{f''(g(x)) \cdot g'(x)}{f'(g(x))}\right)^2 + 2\frac{f''(g(x))g''(x)}{f'(g(x))} + \left(\frac{g''(x)}{g'(x)}\right)^2\right) \tag{64}$$

We can then distribute the $\frac{-3}{2}$ seen as the rightmost element here to get that

$$\begin{aligned}\mathscr{D}(f\circ g) &= \frac{f'''(g(x))(g'(x))^2}{f'(g(x))} + 3\frac{f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} \qquad (65)\\ &- \frac{3}{2}\left(\frac{f''(g(x))g'(x)}{f'(g(x))}\right)^2 - 3\frac{f''(g(x))g''(x)}{f'(g(x))} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2. \qquad (66)\end{aligned}$$

We see that the second and fifth terms cancel, beautifully aligned for clarity; we then have that

$$\mathscr{D}(f\circ g) = \frac{f'''(g(x))(g'(x))^2}{f'(g(x))} + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{f''(g(x))g'(x)}{f'(g(x))}\right)^2 - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2. \qquad (67)$$

And as a result of power rules and basic algebra, we see that we can factor a $g'(x)^2$ out of the first and fourth terms and get that

$$\mathscr{D}(f\circ g) = \left(\frac{f'''(g(x))}{f'(g(x))} - \frac{3}{2}\left(\frac{f''(g(x))}{f'(g(x))}\right)^2\right)(g'(x))^2 + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2. \qquad (68)$$

And by construction, we see that this is identical to the thing that we wanted to prove, that

$$\mathscr{D}\left(f\circ g\right) = \left(\left(\mathscr{D}f\right)\circ g\right)\cdot\left(g'\right)^2 + \mathscr{D}g.\square \qquad (69)$$

(c): After about 1 sheet of paper (woah), I have been able to calculate the first, second, and third derivatives of the function of interest;

$$\begin{aligned}f'(x) &= \frac{ad-bc}{(cx+d)^2}, \qquad (70)\\ f''(x) &= \frac{-2c(ad-bc)}{(cx+d)^3}, \qquad (71)\\ f'''(x) &= \frac{6c^2(ad-bc)}{(cx+d)^4}. \qquad (72)\end{aligned}$$

And upon inserting this into the Schwarzian derivative, we see that

$$\mathscr{D}f(x) = \frac{\frac{6c^2(ad-bc)}{(cx+d)^4}}{\frac{ad-bc}{(cx+d)^2}} - \frac{3}{2}\left(\frac{\frac{-2c(ad-bc)}{(cx+d)^3}}{\frac{ad-bc}{(cx+d)^2}}\right)^2 \qquad (73)$$

Or after placing everything in algebraically correct places, and multiple stages of simplifying,

$$\begin{aligned}\mathscr{D}f(x) &= \frac{6c^2(ad-bc)}{(cx+d)^4}\frac{(cx+d)^2}{ad-bc} - \frac{3}{2}\left(\frac{-2c(ad-bc)}{(cx+d)^3}\frac{(cx+d)^2}{ad-bc}\right)^2 \qquad (74)\\ &= \frac{6c^2}{(cx+d)^2} - \frac{3}{2}\left(\frac{-2c}{cx+d}\right)^2 \qquad (75)\\ &= 0. \qquad (76)\end{aligned}$$

Allowing us to conclude that $\mathcal{D}f(x) = 0$.

Question 7. *Tangent lines.*
(a) Find the equation of the tangent line to the graph $y = (x-1)^3 + 2$ at the point $(3, 10)$.

(b) Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose that the tangent lines to the graph of f at $x = a$ and at $x = b$ are parallel. What is $f(1)$?

Solution 7.
a: We calculate the derivative;

$$f'(x) = ((x-1)^3)' + (2)') \tag{77}$$
$$f'(x) = 3(x-1)^2 \cdot x \tag{78}$$

Then by calculating for $x = 3$ the slope is 36, or for that to intersect the point of interest our tangent line is $y = 36x - 98$

b: We begin by taking the derivative of this function as it is with respect to x; Then (and first by the power rule)

$$f'(x) = 3x^2 + (ax)' + (b)'. \tag{79}$$

We then see, by applying the product rule to $(ax)'$ and since the derivative of a constant is zero,

$$f'(x) = 3x^2 + (a'x + ax') + 0 \tag{80}$$
$$f'(x) = 3x^2 + a \tag{81}$$

We see that at the point a this function has the tangent line slope $3a^2 + a$ and at the point b this has the slope $3b^2 + a$; I shall keep going down this chain even if it gets me nowhere; since we are told the slopes at a and b are equal as given, then $3a^2 + a = 3b^2 + a$ or as a result of simple algebraic manipulation, $a = \pm b$. Since we are told in the problem statement that $a \neq b$, then we can conclude $a = -b$. Whether or not this helps us to finding out what $f(1)$ is is to be seen. I shall make a first attempt by substituting the b in the given cubic for $-a$, therefore $f(x) = x^3 + ax - a$. We input $x = 1$ and see that $f(1) = 1$.

Question 8. *The Leibniz rule.*
Suppose f and g are differentiable functions.

(a) Prove that $(fg)'' = f''g + 2f'g' + fg''$.

(b) Generalize part (a) to show that, for any positive integer n,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}. \tag{82}$$

Here the symbol $f^{(n)}$ mean to take n derivatives of f. For instance, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$, and so on.

The notation $\binom{n}{k}$ means $\frac{n!}{k!(n-k)!}$. Here the *factorial* of an integer is defined by $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$. For instance, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Solution 8.
(a): We begin by seeing what, by definition of derivative, $(fg)'' = ((fg)')' = ((f'g)+(fg'))'$, where he have proven that $((f'g) + (fg'))' = (f'g)' + (fg')'$. Then by the product rule we have that $(f'g)' + (fg')' = (f''g + f'g') + (f'g' + fg'') = f''g + 2(f'g') + fg''$. We can then conclude that $(fg)'' = f''g + 2(f'g') + fg''$.

(b): We begin by establishing that the claim $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$ is true when $n = 1$ by the product rule and $n = 2$ as we have demonstrated in (a). We can then suppose that our claim is valid for some $n = k$, where through the method we want to prove this by we must demonstrate that our claim is true for $n = k + 1$. First we see what we have already shown; $(fg)^{(k)} = \left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}\right)$. Then we see that when $n = k + 1$, we have that $((fg)^{k+1} = ((fg)^{k})' = \left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}\right)'$. We can then apply the product rule to the right side to see that $\left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}\right)' = \left(\sum_{i=0}^{k} \binom{k}{i} f^{(i+1)} g^{(k-i)} + f^{(i)} g^{(k-i+1)}\right)$. Now how do we handle this $i+1$? :thinking: We can split this into two separate sums since addition is commutative, therefore

$$\left(\sum_{i=0}^{k} \binom{k}{i} f^{(i+1)} g^{(k-i)} + f^{(i)} g^{(k-i+1)}\right) = \left(\sum_{i=0}^{k} \binom{k}{i} f^{(i+1)} g^{(k-i)}\right) + \left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i+1)}\right) \tag{83}$$

We can then modify the indexes of the left sum to get it to something we can add to the right side. We can define a new variable $j = i + 1$, and substitute our new value for i in our sum to see that and change bounds appropriately to see

$$\left(\sum_{i=0}^{k} \binom{k}{i} f^{(i+1)} g^{(k-i)}\right) = \left(\sum_{j=1}^{k+1} \binom{k}{j-1} f^{(j)} g^{(k-j+1)}\right) \tag{84}$$

Of course we can be cheeky young men, and note that j is a dummy variable, and it is perfectly acceptable to replace all occurrences of j here for i, giving us

$$\left(\sum_{i=1}^{k+1} \binom{k}{i-1} f^{(i)} g^{(k-i+1)}\right) \tag{85}$$

and we can then substitute this sum in place of what we were originally operating on to see

$$\left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)}\right)' = \left(\sum_{i=1}^{k+1} \binom{k}{i-1} \left(f^{(i)} g^{(k-i+1)}\right)\right) + \left(\sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i+1)}\right). \tag{86}$$

We see that the terms are the same, but we have to handle the difference in binomial coefficients to progress. So we break each of these down by their definition to see

$$\binom{k}{i-1} + \binom{k}{i} = \frac{k!}{(i-1)!(k-i+1)!} + \frac{k!}{i!(k-i)!} \tag{87}$$

We can then multiply the first fraction by i and the second by $(k-i+1)$ and see that

$$\begin{aligned}\frac{k!}{(i-1)!(k-i+1)!}+\frac{k!}{i!(k-i)!} &= \frac{k!i}{i!(k-i+1)!}+\frac{k!(k-i+1)!}{i!(k-i+1)!} && (88)\\ &= \frac{k!i+k!(k-i+1)}{i!(k-i+1)!} && (89)\\ &= \frac{k!i+k!k-k!i+k!}{i!(k-i+1)!} && (90)\\ &= \frac{k!k+k!}{i!(k-i+1)!} && (91)\\ &= \frac{k!(k+1)}{i!(k-i+1)!} && (92)\\ &= \binom{k+1}{i} && (93)\end{aligned}$$

We then see, now that we have added the binomial coefficients, then we have that

$$\left(\sum_{i=1}^{k+1}\binom{k}{i-1}\left(f^{(i)}g^{(k-i+1)}\right)\right)+\left(\sum_{i=0}^{k}\binom{k}{i}f^{(i)}g^{(k-i+1)}\right)=\sum_{i=0}^{k}\binom{k+1}{i}\left(f^{(i)}g^{(k-i+1)}\right)+fg^{(k+1)}+gf^{(k+1)} \tag{94}$$

Or by reindexing to make the elements to the right succesfully add,

$$\sum_{i=0}^{k+1}\binom{k+1}{i}\left(f^{(i)}g^{(k-i+1)}\right) \tag{95}$$

And as a result we have successfully proven the inductive portion of the Liebniz rule.