question 3 views

Daily Challenge 3.5

(**Due: Saturday 5/12 at 12:00 noon Eastern**)

Review

We've seen that a function like $f(x)=x^2$ which fails the horizontal line test cannot have an honest-to-god inverse function.

One way to probe this failure of invertibility is by studying the *inverse image*. Recall that, if g is a function and A is a subset of the range of g, then $g^{-1}(A)$ is the set of all inputs which map to an element of A. For instance, if we return to the $f(x)=x^2$ example above, we see that

$$f^{-1}(\{1\})=\{1,-1\},$$

since both $x=1$ and $x=-1$ are mapped to $f(1)=f(-1)=1$ under $f(x)=x^2$.

This gives us another way to discuss invertibility: *a function has an inverse if and only if the inverse image of every one-element set also contains only one element.*

Trig functions fail this test pretty badly!

Example. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=\sin(x)$ and $g(x)=\cos(x)$. Find the inverse images $f^{-1}(\{1\})$ and $g^{-1}(\{1\})$.

Solution. By definition, $f^{-1}(\{1\})$ is the set of all angles which are mapped to 1 under the sine function; in other words, it is the set of all angles θ such that $\sin(\theta)=1$. Thus

$$\begin{aligned} f^{-1}(\{1\}) &= \left\{\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2},\cdots,-\frac{3\pi}{2},-\frac{7\pi}{2},\cdots\right\} \\ &= \left\{\frac{(1+4n)\pi}{2}\mid n\in\mathbb{Z}\right\}. \end{aligned}$$

Similarly, $g^{-1}(\{1\})$ is the set of all angles θ with $\cos(\theta)=1$, or

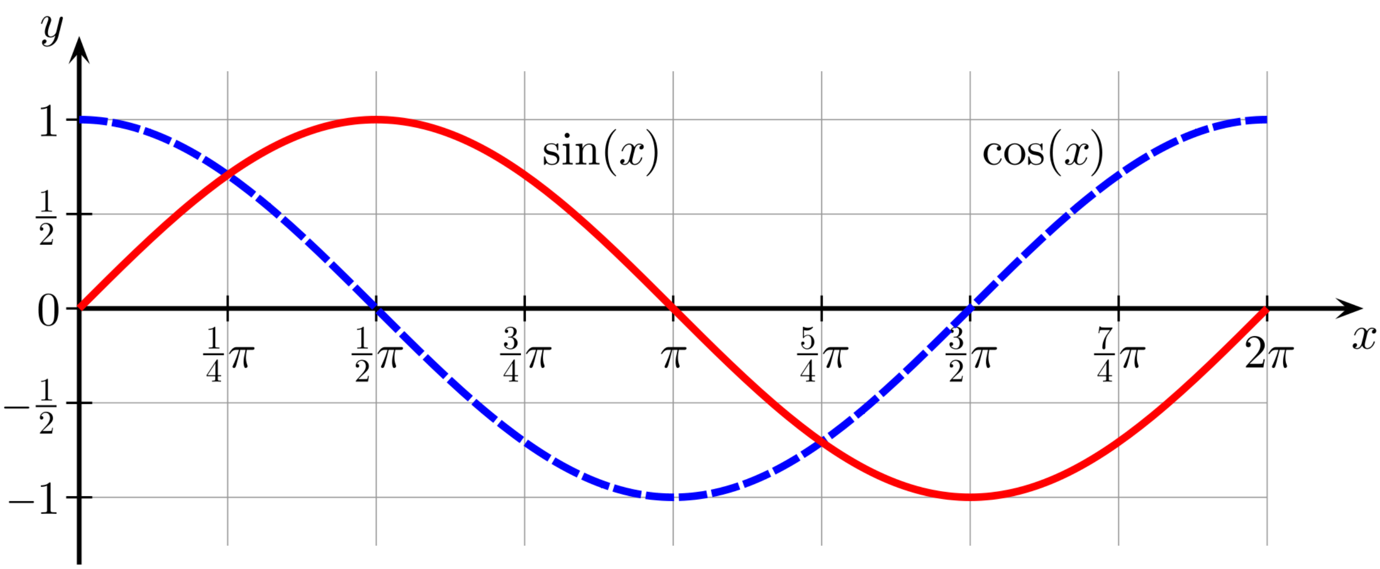
$$\begin{aligned} g^{-1}(\{1\}) &= \{0,\pm 2\pi,\pm 4\pi,\cdots\} \\ &= \{2n\pi\mid n\in\mathbb{Z}\}. \end{aligned}$$

Both of these inverse images contain infinitely many angles, so they cannot have inverses. □

However, as we saw in the $f(x)=x^2$ example, we can get an inverse by *restricting the domain of the function*. If we define a new function $f_+:[0,\infty)\to\mathbb{R}$ by $f_+(x)=x^2$ which is defined only for nonnegative inputs, then this restricted function $f_+(x)$ has a good inverse $f_+^{-1}(x)=\sqrt{x}$.

We can do the same thing to define inverses for sine and cosine, but importantly, **the two inverse functions require different restrictions of the domain**.

To see why, let's look again at the graphs of sine and cosine:



The range of both sine and cosine is $[-1,1]$, so we would like the inverse functions $\sin^{-1}$ and $\cos^{-1}$ to have $[-1,1]$ as a domain.

From looking at the plots above, we see that the cosine function hits all outputs in $[-1,1]$ as its input ranges from 0 (since $\cos(0)=1$) to π (since $\cos(\pi)=-1$). Thus we should restrict the domain of cosine to $[0,\pi]$ to define an inverse.

However, the sine function does *not* achieve all values in $[-1,1]$ over this range: we see that $\sin(\theta)$ hits only positive values when the inputs lie in $[0,\pi]$. We need a different restriction of the domain for sine. The simplest choice is to consider inputs in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, since $\sin\left(-\frac{\pi}{2}\right)=-1$ and $\sin\left(\frac{\pi}{2}\right)=1$.

These restrictions will give us the desired inverse functions!

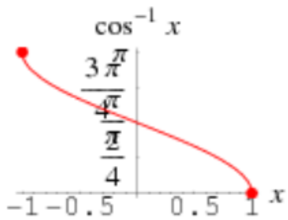
Definition. The *inverse sine function*, written $\sin^{-1}(x)$ or $\arcsin(x)$, has domain $[-1,1]$ and range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, and satisfies

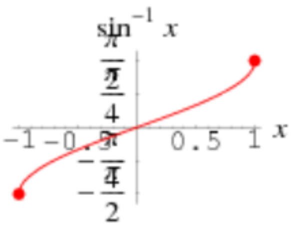
$$\sin^{-1}(\sin(x)) = x \text{ for all } x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$
$$\sin\left(\sin^{-1}(x)\right) = x \text{ for all } x \in [-1,1].$$

The *inverse cosine function*, written $\cos^{-1}(x)$ or $\arccos(x)$, has domain $[-1,1]$ and range $[0,\pi]$, and satisfies

$$\cos^{-1}(\cos(x)) = x \text{ for all } x \in [0,\pi],$$
$$\cos\left(\cos^{-1}(x)\right) = x \text{ for all } x \in [-1,1].$$

Their graphs are shown below:





Note in particular that, as we have emphasized above, the ranges differ.

Problem

Try the following exercises involving inverse trigonometric functions.

(a) Compute $\cos^{-1}(0)$, $\sin^{-1}\left(\frac{1}{2}\right)$, and $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

(b) Find an angle θ such that $\arccos(\cos(\theta)) = \theta$ but $\arcsin(\sin(\theta)) \neq \theta$. Explain why this is not a contradiction.

(c) Let $f(x) = \sin(x)$ and $g(x) = \arcsin(x)$. Compute the set $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ (i.e. the pre-image of $\left\{\frac{1}{2}\right\}$ under f) and the set $g\left(\left\{\frac{1}{2}\right\}\right)$ (i.e. the image of $\left\{\frac{1}{2}\right\}$ under g). Explain why the two sets are not equal.

[Comment: This is confusing because both $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ and $g\left(\left\{\frac{1}{2}\right\}\right)$ can be "written" as $\sin^{-1}\left(\left\{\frac{1}{2}\right\}\right)$, but they are not the same! Our notation for inverse images and inverse functions is a bit ambiguous.]

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Solutions (Logan). Your reasoning and results go here.

(a)
$\cos^{-1}(0)$ has the solution $\frac{\pi}{2}$
$\sin^{-1}(\frac{1}{2})$ has the solution $\frac{\pi}{6}$
$\arccos\left(-\frac{\sqrt{3}}{2}\right)$ can be rewritten as $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$], and by the definition of inverse function $\cos(\theta) = -\frac{\sqrt{3}}{2}$. I know that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, and in my own special thinking the "mirror" of this will also be $\frac{\pi}{6}$ from the x-axis, and I can conclude that this mirror would be $\frac{5\pi}{6}$ giving me my answer $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

(b)

(c)

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the instructors' answer, *where instructors collectively construct a single answer*

Solutions (Christian).

(a) First we consider $\cos^{-1}(0)$. The range of the inverse cosine function is $[0, \pi]$, so we're looking for an angle $\theta \in [0, \pi]$ with the property that $\cos(\theta) = 0$. We see that $\theta = \frac{\pi}{2}$ does the trick, so

$$\boxed{\cos^{-1}(0) = \frac{\pi}{2}}.$$

Next we look at $\sin^{-1}\left(\frac{1}{2}\right)$. The range of inverse sine is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so we need some $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $\sin(\theta) = \frac{1}{2}$. Thankfully we've memorized that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so we can immediately write down

$$\boxed{\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}}.$$

Finally we turn to $\arccos\left(-\frac{\sqrt{3}}{2}\right)$. Of course, $\arccos$ is simply another name for $\cos^{-1}$, so we need a $\theta \in [0, \pi]$ with $\cos(\theta) = -\frac{\sqrt{3}}{2}$. This puts us in the second quadrant, where x is negative, and at an angle of $\frac{\pi}{6}$ away from the negative x axis. Thus the desired angle is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$, and hence

$$\boxed{\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}}.$$

(b) This boils down to the fact that $\sin^{-1}$ and $\cos^{-1}$ have different ranges: we saw $\operatorname{Rng}\left(\cos^{-1}\right) = [0, \pi]$ but $\operatorname{Rng}\left(\sin^{-1}\right) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, to find an angle θ such that

$$\cos^{-1}(\cos(\theta)) = \theta \text{ but } \sin^{-1}(\sin(\theta)) \neq \theta,$$

we should probably look for an angle in the range of $\cos^{-1}$ but *not* in the range of $\sin^{-1}$. Perhaps $\theta = \pi$ will do the trick. Let's check:

$$\cos^{-1}(\cos(\pi)) = \cos^{-1}(-1) = \pi,$$

but

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

Aha! So $\theta = \pi$ works. There is no contradiction because the inverse sine and inverse cosine arise from different restrictions on the domain of sine and cosine, respectively.

(c) The set $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ is the pre-image of $\frac{1}{2}$ under sine. We know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and that $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, but we can also go around the circle (a total of 2π radians) as many times as we like and get this same output, so

$$f^{-1}\left(\left\{\frac{1}{2}\right\}\right) = \left\{\frac{\pi}{6} + 2\pi n \mid n \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + 2\pi n \mid n \in \mathbb{Z}\right\}.$$

On the other hand, $g\left(\left\{\frac{1}{2}\right\}\right)$ is the image of the single-element set $\left\{\frac{1}{2}\right\}$ under the function $\arcsin$, so

$$g\left(\left\{\frac{1}{2}\right\}\right) = \left\{\frac{\pi}{6}\right\}.$$

Note that the two sets are not equal, but one is a subset of the other!

$$g\left(\left\{\frac{1}{2}\right\}\right) \subseteq f^{-1}\left(\left\{\frac{1}{2}\right\}\right).$$

This is as we expected. The set $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ is the inverse image of a function that fails the horizontal line test, so we expect it to have many elements. The set $g\left(\left\{\frac{1}{2}\right\}\right)$, on the other hand, is the image under an inverse function that came from restricting the domain of f so that it *passes* the horizontal line test.

In some sense, we've "thrown away" infinitely many elements of $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ and kept only $\frac{\pi}{6}$ to get a good inverse.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*