Question 1. *Quick computations*

It is sufficient to state your results without detailed proof.

(a) Evaluate $\sin\left(\pi \sin\left(\frac{\pi}{6}\right)\right)$, $\tan(21\pi)$, and $\sec\left(-\frac{7\pi}{2}\right)$.

(b) Find the value of $\log_5\left(\frac{(125)\cdot(625)}{(25)}\right)$.

(c) Suppose that $a < b < c < d$ are real numbers. Write the set $(a, c) \cap (b, d)$ as an interval.

(d) What are the domain and range of $f(x) = \sqrt{1-x}$?

Solution 1.

(a)

- The sine of $\frac{\pi}{6}$ is 0.5. so $\sin\left(\pi \sin\left(\frac{\pi}{6}\right)\right) = \sin\left(\frac{1}{2}\pi\right)$, and the sine of $\frac{\pi}{2}$ (or 90 degrees) is 1, so $\sin\left(\frac{\pi}{2}\right) = 1$.
- By the definition of tangent, $\tan(21\pi) = \frac{\sin(21\pi)}{\cos(21\pi)}$, and this simplifies to $\frac{0}{-1}$, so $\tan(21\pi) = 0$.
- By the definition of secant, $\sec\left(-\frac{7\pi}{2}\right) = \frac{1}{\cos\left(-\frac{7\pi}{2}\right)}$, and since cosine of $-\frac{7\pi}{2} = 0$, then $= \frac{1}{0}$, which is undefined.

(b) By the logarithm quotient rule, $\log_5(\frac{125\times625}{25}) = \log_5(125\times625) - \log_5(25)$, and by the logarithm product rule the first logarithm simplifies to create the expression $\log_5(125) + \log_5(625) - \log_5(25)$. Since $5^2 = 25$, $5^3 = 125$, and $5^4 = 625$, this can finally simplify to $3 + 4 - 2 = 5$.

(c) The intersection of these two intervals will have the left endpoint of the rightmost interval, and the right endpoint of the leftmost interval. Therefore, this interval is (b, c).

(d) The domain of this function is $(-\infty, 1]$ since the $f(x)$ enters the complex domain when $x > 1$. The range of this function begins at 0 since the minimum real $f(x)$ attainable is 0, and on up; $[0, +\infty)$.

Question 2. *Definitions and Concepts*

Here I ask you to state some definitions of important objects, or to explain ideas which we will need to understand well.

After we move on to chapter 2, you should be able to answer all of these without referring to notes or the textbook, but feel free to look them up for now if you've forgotten.

(a) Give the limit definition of the number e.

(b) Explain what is meant by the *completeness property* of the real numbers.

(c) What are the definitions of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$?

(d) What is the difference between the *codomain* of a function and the *range* of a function?

(e) Give the formal definition of the graph of a function. State the condition which this set must satisfy in order for f to pass the horizontal line test.

Solution 2.

(a) To simply copy and paste the definition, we have the limit $e \equiv \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$. As m approaches infinity it gets closer to the true value of e, which is along the lines of $2.71828\cdots$

(b) The set $\mathbb{R}$ is complete because for all $A \subset \mathbb{R}$, if A has an upper bound then $\sup A$ exists and $\sup A \in \mathbb{R}$.

(c)

- $\mathbb{N}$: The $\mathbb{N}$atural number set, generally known as the counting number set is all positive integers, and in some cases (but indecisively) includes zero; $1, 2, 3, 4, \cdots$
- $\mathbb{Z}$: The integer set, positive and negative "whole numbers" including zero; $\cdots, -1, 0, 1, \cdots$
- $\mathbb{Q}$: The rational set, all numbers that can be represented as a fraction of integers; $\frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \cdots$
- $\mathbb{R}$: The real number set, made up of all rational and irrational numbers; all the real numbers can be placed on a number line.
- $\mathbb{C}$: The $\mathbb{C}$omplex number set, all values here are generally represented on the complex plane and follow the form $a + bi$ where a and b are real and $i = \sqrt{-1}$.

(d) If $f : A \to B$ is a function, then B is called its codomain. The range is defined by $\text{Rng}(f) = \{x \in B \mid x = f(a) \text{ for some } a \in A\}$. In words, the range of a function is the set of possible outputs, while the codomain is a set which the range is a subset of.

(e) The definition of the graph of a function f is the set $\{(x, f(x)) \mid x \in \text{Dom}(f)\}$. For this set to pass the horizontal line test, then the graph cannot contain any two pairs (x_1, a) and (x_2, a) where $x_1, x_2, a \in \mathbb{R}$ and $x_1 \neq x_2$. In other words, none of the elements in this set can have a multiplicity.

Question 3. *Revising a solution (daily challenge 6.4)*
Please pick either

1. one daily challenge that you couldn't solve correctly or used a weekly skip on, or
2. one of the review problems with solutions that I posted

Try to choose a problem on a skill you haven't mastered yet (e.g. trig identities or images and inverse images).

In the space below, (re-)submit a solution to the problem you chose.

Solution 3.
I have chosen to revise my non-existent response to daily challenge 1.4, which I shall post here for reference:

Theorem. Let m be an integer. Then the number $n = (m-1) \times m \times (m+1)$ is divisible by 3

Proof: Assume m is an integer. For the suggested equation to be divisible by 3 (evenly), first it must be a multiple of 3. For this to be true, then one of three scenarios must be true:

A) $m - 1$ is a multiple of 3.

B) m is a multiple of 3.

or C) $m + 1$ is a multiple of 3.

Due to the logistics of this, one of these scenarios must be true for any given value m, and as such the result will be a multiple of 3. Since any integer that is a multiple of 3 is divisible by 3 due said operations being inverse, then n must be divisible by 3. $\square$

Question 4. *More practice with proofs*
(a) Prove from the definitions of "odd" and "even" that the sum of two odd integers is even.

(b) Suppose that the equation $ax^3 + bx^2 + cx + d = 0$ has only real roots, and that the four coefficients a, b, c, d are all positive. Prove that all of the roots are negative.

(c) Let A be any set. Prove that $\emptyset \subseteq A$.

(d) *(Daily challenge 7.1)* Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Prove that f passes the horizontal line test.

(e) *(Daily challenge 7.4)* If A and B are open intervals with $A \cap B \neq \emptyset$, $\sup(A) = \alpha$ and $\sup(B) = \beta$, show that $\sup(A \cap B) = \min(\alpha, \beta)$.

Solution 4.
(a) Recall the definitions of odd and even numbers: a number x is odd if $x = 2k + 1$ where $k \in \mathbb{Z}$. Similarly, x is even if $x = 2k$ where $k \in \mathbb{Z}$. Suppose x and y are each odd numbers. By the definition of odd, $x = 2k_1 + 1$ and $y = 2k_2 + 1$, therefore $x + y = 2k_1 + 2k_2 + 2 = 2(k_1 + k_2 + 1)$. Since $k_1 + k_2 + 1 \in \mathbb{Z}$, then $x + y$ is even. $\square$

(b) Suppose the coefficients of the polynomial f_1 of the form $f_1(x) = ax^3 + bx^2 + cx + d = 0$ are positive, ie $a, b, c, d > 0$ and all roots are real. Assume by way of contradiction, that at least one root of f_1 is non-negative (positive or zero). Therefore by the zero product property, the factorization of this polynomial of the third degree will follow the form $(fx - j)(gx + k)(hx + l)$, where $f, g, h > 0$ and $j \geq 0$. The non-negative root is $x = \frac{j}{f}$, which is non-negative since $j \geq 0$ and $f > 0$.

Multiplying this out results in the polynomial

$$(fx - j)(gx + k)(hx + l) = fghx^3 + (-ghj + fhk + fgl)x^2 + (-hjk - gjl + fkl)x - jkl$$

There is no need to pay attention to what occurs to the individual letters, but to observe that at least one of the coefficients in this polynomial has become negative. For the constant term, $-jkl$, to be positive, we must have $kl < 0$. For the coefficient of the linear term to be positive, we then need $fkl > hjk + gjl$, which means $hjk + gjl < 0$ since $f > 0$ and $kl < 0$. Then for the coefficient of the quadratic term to be positive, we need $fhk + fgl > ghj \geq 0$. So we need the two conditions

$$\begin{aligned} hjk + gjl &< 0, \\ hk + gl &\geq 0. \end{aligned}$$

If $j = 0$, the first line is false, and we have a contradiction. If $j > 0$, divide the first line by j, and then the first line contradicts the second line.

By this contradiction, we now know that f_1 has all negative roots. $\square$

(c) By the definition of subset, a set A is a subset of set B if for all $a \in A$ then $a \in B$. The null set is a subset of A since it doesn't have any elements x such that $x \notin A$. Therefore, $\emptyset \subseteq A$. $\square$

(d) First, I would like to give the definition of a strictly increasing: "A function f is strictly increasing if, whenever $x > y$ we have $f(x) > f(y)$." Fortunately, this definition gives us the base of our argument.

The horizontal line test checks whether any horizontal line $y = c$ intersects the graph of a function $f(x)$ at two different points, say $x = a$ and $x = b$. Thus a function fails the horizontal line test if there exist two distinct real numbers a and b such that $f(a) = c$ and $f(b) = c$.

If f is strictly increasing, then f can never fail the horizontal line test. To see why, consider two unequal real numbers a and b. Since $a \neq b$, it must either be true that $a > b$ or that $b > a$. If $a > b$, then $f(a) > f(b)$ (by the definition of strictly increasing), and if $b > a$, then $f(b) > f(a)$. Hence it cannot be true that $f(a) = f(b)$, so f cannot fail the horizontal line test. $\square$

(e) **Exploration**: I begin my exploration by recognizing that the ranges of the intervals A and B overlap by at least one element since their intersection isn't the null set. I do not find myself convinced of the truthfulness of this theorem upon reading it, so I decide to create an example. Say the interval A is $(0, 2)$ and B is the interval $(1, 3)$ The supremum of A is 2 and the supremum of B is 3. The supremum of the intersections of these sets is 2, and the minimum of the individual supremums is 2. I want to say I'm convinced but something still seems wrong, like a coincidence. After further thinking I find myself convinced. I believe I can proceed to this proof.

Proof: Suppose that A is the interval (a, b) and B is the interval (c, d). The theorem states that the intersection of A and B is not a null set, therefore they do overlap; assume without loss of generality that $a < c < b < d$. I shall define this intersection as C and the resulting interval is (c, b). We can now define the supremum of A as α and the supremum of B as β. We can now proceed. The supremum of the interval previously defined as C is b, and this is equal to $\min(\alpha, \beta)$ which is in turn b. We have at this point proven that $\sup(A \cap B) = \min(\alpha, \beta)$. $\square$

Question 5. *Giving precise definitions (daily challenge 6.5)*
Choose three learning goals from the list on Piazza which involve recalling or stating a definition which you cannot yet produce from memory. In the space below, write precise and mathematically rigorous definitions of those terms. You may need to refer to chapter 1 of the textbook.

"Precise" means to avoid using vague or colloquial language, name all mathematical objects using appropriate variables, and unambiguously define the term you're speaking about. For instance,

1. Bad (imprecise): The set difference of two sets is the set of all elements in one set but not in the other.
2. Good (precise): Let A and B be sets. The set difference of A and B, written $A \setminus B$, is defined by $A \setminus B = \{a \in A \mid a \notin B\}$.

Another example:

1. Bad (imprecise): The inverse image is the set of all elements that get mapped into the set under a function.
2. Good (precise): Let f be a function and suppose A is a subset of the codomain of f. Then the inverse image or preimage of A under f, denoted by $f^{-1}(A)$, is $f^{-1}(A) = \{x \in \mathrm{Dom}(f) \mid f(x) \in A\}$.

A bad definition fails to give names to the objects it speaks about (e.g. "a function" rather than f), uses words when symbols would be more appropriate, and conveys a shaky understanding at best. A good definition is crisp, clear, and is appropriate for use in a formal proof.

Solution 5.

1. I have repeatedly demonstrated my inability to remember Euler's Formula, and as such shall provide the formula and a "text spoken" version to give this definition some substance: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, spoken "e to the power i theta equals cosine theta plus i sine theta." For some reason typing that makes me feel better, so I guess it works! :shrug:
2. I understand the inverse trigonometric functions $\sin^{-1}$ and $\cos^{-1}$. However, I do not know each of these functions' domain and range as well as I would like to. In this case as well as I would like to is quick, perhaps with a little thought. I now know to apply the idea of chopping the original function into a single piece to make inverting easier, and conveniently now understand that.
   (a) $\sin^{-1}$ Domain: $[-1, 1]$ Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
   (b) $\cos^{-1}$ Domain: $[-1, 1]$ Range: $[0, \pi]$
3. Let f be a real-valued function. A real-valued function g is called an *inverse* of f if $f(g(x)) = x$ for all $x \in \mathrm{Dom}(g)$ and $g(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$.

Question 6. *A few qualitative questions (daily challenge 6.7)*
Watch the YouTube video reviewing chapter 1 and previewing chapter 2, then answer the following questions.

(a) Explain what a “limit” is, as you presently understand it. (It’s okay to be non-rigorous for now.)

(b) In your own words, what is meant by a "pathology" in mathematics?

(c) Of all the things we’ve discussed and worked on in this group so far, which has been the most helpful to you?

Solution 6.
(a) As I understand it, a limit is the output of a given function for the value $x = \infty$.

(b) A pathology is the possible minor and major variations that affect the calculated limit of a function in undesirable ways, or ways that aren’t known whether they should be taken into account.

(c) The most helpful and probably most enjoyable concept I have learned more about thanks to the calculus group would likely be the trigonometric section as a whole. In particular, I’d like to see how the functions cotangent, secant, and cosecant are used since we never really “applied” them.

Question 7. *Proving Set Equality*
(a) Let A, B, C be sets. Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(b) Let f be a function with $A, B \subseteq \mathrm{Dom}(f)$. Is it true that $f(A \cap B) = f(A) \cap f(B)$? If so, prove it (stating the precise definition of image in your answer). If not, give a counterexample.

(c) Let f be a function with $C, D \subseteq \mathrm{Cod}(f)$. Is it true that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$? If so, prove it (stating the precise definition of inverse image in your answer). If not, give a counterexample.

Solution 7.
(a) To prove that one set expression is equal to another, I must first prove that one of the given expressions is a subset of the other and vice versa. In this case, I must prove 2 things: $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$ I shall begin my proof now:

Recall the definitions of union, intersection, and set subtraction. Respectively,

$$\begin{aligned} X \cup Y &= \{x \mid x \in X \text{ or } Y\}, \\ X \cap Y &= \{x \mid x \in X \text{ and } Y\}, \\ X \setminus Y &= \{x \mid x \in X \text{ but not in } Y\}. \end{aligned}$$

Suppose that $x \in A \setminus (B \cap C)$. First we must prove that $x \in (A \setminus B) \cup (A \setminus C)$. Due to presence of the set subtraction in our definition of the dummy variable x, we know $x \in A$ and likewise $x \notin B$ or $x \notin C$. There are two possible cases to handle here. To reiterate, $x \notin B$ or $x \notin C$. We shall begin by assuming the first statement is true, $x \in A$ and $x \notin B$. This now means that, by the definition of set subtraction, that $x \in (A \setminus B)$. It is true that $x \in (A \setminus B) \cup (A \setminus C)$, since by the definition of union, an element need be present in just one set to be present in the resulting set. The other case to account for would be that $x \in A$ and $x \notin C$. For similar reasons, by the definition of set subtraction then $x \in (A \setminus C)$. Now by the definition of union then $x \in (A \setminus B) \cup (A \setminus C)$. Since we have now proven that $x \in (A \setminus B) \cup (A \setminus C)$ in both cases, then $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Next we must prove that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$. Suppose that $y \in (A \setminus B) \cup (A \setminus C)$. By the definition of union (in particular its *logical* or), then is true that either $y \in (A \setminus B)$ or $y \in (A \setminus C)$. Assume that the former case is true, ie $y \in (A \setminus B)$. Since $y \notin B$, then y is not in the intersection of sets B and C. Therefore, $y \in A \setminus (B \cap C)$. We can now continue to next case, $y \in (A \setminus C)$. By the definition of set subtraction, $y \in A$ and $y \notin C$. Now since $y \notin C$ then $y \notin (B \cap C)$. Therefore, $y \in A \setminus (B \cap C)$, and in turn $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$. We have now proven that both these sets are subsets of each other, therefore $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. $\square$

(b) This equation is false. Suppose $f(x) = x^2$. We begin by defining set A as the interval $(1, 4)$ and set B as the interval $(-4, -1)$. In this case, $f(A \cap B) = \emptyset$ since the intersection of these sets is null. However, $f(A) \cap f(B)$ is the interval $(1, 16)$ since $f(A) = (1, 16)$ and $f(B) = (1, 16)$. $(1, 16) \neq \emptyset$, therefore $f(A \cap B) \neq f(A) \cap f(B)$

(c) This equation is true. Suppose x is element of the left side, ie $x \in f^{-1}(C \cap D)$. Now we define a value y for $y = f(x)$, therefore $y \in (C \cap D)$, and by the definition of intersection this means that $y \in C$ and D. This now means that $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. We can now conclude that $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$. To prove the converse, we must show that if $x \in f^{-1}(C) \cap f^{-1}(D)$, then $x \in f^{-1}(C \cap D)$. We begin by defining $y = f(x)$ for $y \in C \cap D$. By this definition, we know that $y \in C$ and D. Since $y \in C$ and D, we know that $x \in f^{-1}(C \cap D)$, therefore $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$. We have now proven that each side of the equation we must prove are subsets of eachother, therefore $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Question 8. *Through the complex domain (daily challenge 7.6)*
I am rather fond of the following quote:

> "The shortest path between two truths in the real domain passes through the complex domain." - Jacques Hadamard

We've seen Hadamard's comment apply to our derivation of the angle-addition formulas, which are purely statements about real numbers but which we proved by using Euler's identity (involving complex numbers).

In this problem, you will repeat this derivation and apply it to answer some related questions.

(a) Prove the angle-addition formulas $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$ and $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$.

(b) Using your result from (a), derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

(c) Using your result from (b), find the value of $\tan\left(\frac{5\pi}{12}\right)$.

Solution 8.
A:

Exploration: I begin my exploration by referring to Euler's formula as the scaffold suggests. For later reference I shall place it here:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

I also have a faint memory of applying the Pythagorean theorem in proving this, and as such I'll put it here for later reference as well:

$\sin^2(\theta) + \cos^2(\theta) = 1$.

Since I have seen this before, I need not convince myself of the truth of this equation. I can now begin my proof.

Proof:

We begin by defining $\theta = (\alpha+\beta)$. I believe that I can now substitute this value for θ in Euler's formula, and receive the equation

$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$.

However, we can improve this by applying the product of exponents rule, in turn providing me with $e^{i(\alpha)} \times e^{i(\beta)}$ which can now evolve into a much more useful equation by now applying Euler's formula to each of the expressions, and get

$e^{i(\alpha)} \times e^{i(\beta)} = (\cos(\alpha) + i\sin(\alpha)) \times (\cos(\beta) + i\sin(\beta))$

I can now apply foil and get

$e^{i(\alpha)} \times e^{i(\beta)} = (\cos(\alpha)\cos(\beta)) + (cos(\alpha)i\sin(\beta)) + (\cos(\beta)i\sin(\alpha)) - (\sin(\alpha)\sin(\beta))$

Our next step is to group what we received from foiling.

$\Big(\cos(\alpha)\cos(\beta)) - (\sin(\alpha)\sin(\beta)\Big) + i\Big((\cos(\alpha)\sin(\beta)) + (\cos(\beta)\sin(\alpha))\Big)$

I now take Christian's advice and set the equation I originally received and set it equal to the one I have just found.

$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \Big(\cos(\alpha)\cos(\beta)) - (\sin(\alpha)\sin(\beta)\Big) + i\Big((\cos(\alpha)\sin(\beta)) + (\cos(\beta)\sin(\alpha))\Big)$

From this, we can now conclude the following:

$\cos(\alpha+\beta) = \Big(\cos(\alpha)\cos(\beta)) - (\sin(\alpha)\sin(\beta)\Big)$

$i\sin(\alpha+\beta) = i\Big((\cos(\alpha)\sin(\beta)) + (\cos(\beta)\sin(\alpha))\Big)$

which then simplifies to

$\sin(\alpha+\beta) = \Big((\cos(\alpha)\sin(\beta)) + (\cos(\beta)\sin(\alpha))\Big)$

B: (Ah damnit, it has taken me 2.5 hours to do A, LOL)

Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, I believe that I can simply divide the angle addition formula of sine by the angle addition formula of cosine, therefore

$\tan(\alpha + \beta) = \frac{\left(\cos(\alpha)\sin(\beta)+\cos(\beta)\sin(\alpha)\right)}{\left(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\right)}$

I rearrange this to make my intention more clear:

$\tan(\alpha + \beta) = \frac{\left(\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\right)}{\left(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\right)}$

Once again, I take hint from Chris and apply divide every one of these value by $\cos(\alpha)\cos(\beta)$ to get the stupendously large equation:

$\tan(\alpha + \beta) = \frac{\left(\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\right)}{\left(\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\right)}$

This equation can now simplify to $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$.

C: $\tan\left(\frac{5\pi}{12}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$, which thanks to the formula I just derived equals:

$$\tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}.$$

I can now define that $\tan(\alpha) = 1$ and $\tan(\beta) = \frac{1}{\sqrt{(3)}}$

Therefore $\tan(\alpha + \beta) = \frac{1+\frac{1}{\sqrt{(3)}}}{1-\frac{1}{\sqrt{(3)}}}$.

I don't know if this can be simplified further but it'd probably be an abomination.

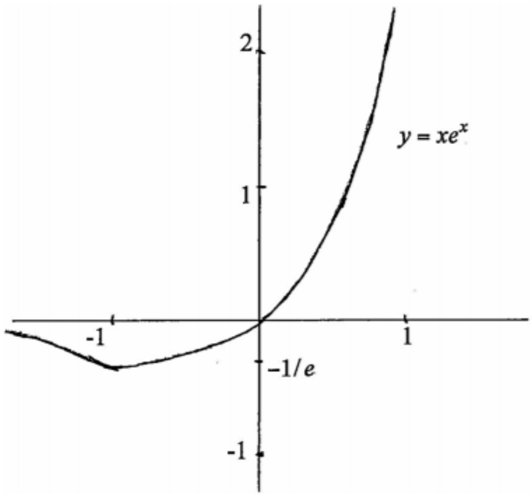
Question 9. *The Product Logarithm*
We introduced the natural logarithm because we had equations like $e^x = 5$ that we didn't know how to solve. With an inverse function for e^x in our toolkits, we simply apply log to both sides to find $x = \log(5)$.

When I was in high school, I was fascinated by equations like $x^x = 5$, or related equations that seem to admit no solution in terms of "common" functions. Could we invent a function like log, which could be applied to both sides of this equation to find a solution, in the same way we did in the previous paragraph?

My teachers told me that solving such equations was impossible, which only made me more interested.

In this problem, we will see that they were wrong.

(a) Let's define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = xe^x$. The graph of $f(x)$ looks like



Explain why $f(x)$ cannot have an inverse function.

(b) To remedy this problem, we will restrict the domain: define a new function $f_+ : [0, \infty) \to \mathbb{R}$ which is given by the same $f(x)$ above, but is defined only for non-negative values of x.

Prove that this restricted function $f_+(x)$ has an inverse function.

(c) Define a function $W(x)$ which is the inverse function of $f_+(x)$, since you have proved that such an inverse exists. This is called the *product logarithm*, also known as *Lambert's W function.*[1]

Consider the two equations

$$W\left(xe^x\right) = x,$$
$$W(x)e^{W(x)} = x. \tag{1}$$

Explain why (and whether) each of these equations is true, and for which values of x the equation holds.

(d) Let a be a positive real number. Solve the equation $x^x = a$ for x.

Solution 9.
(a) $f(x)$ as it is can not have an inverse function because it does not pass the horizontal line test, since if left be its inverse wouldn't pass the vertical line test.

(b) We aim to prove that the function $f_+ = xe^x$ with the domain $[0, \infty)$ has an inverse. Since restricting the domain has changed this to a strictly increasing function (as both x and e^x are strictly increasing for positive x, so their product is as well), we can now apply the definition

[1]This function crops up in various scientific applications. For instance, there is a change of coordinates useful when studying black holes in Einstein's theory of general relativity which is written in terms of the product log. It also appears when studying enzyme kinetics in biochemistry.

of said function. The definition states that the outputs of this function follow the form that if $0 \leq x_1 < x_2$, then $0 \leq f_+(x_1) < f_+(x_2)$.To make this easier to digest, I'll rename $f_+(x_1)$ to y_1, ie $0 \leq y_1 < y_2$. Based off of what we have concluded, these points pass the horizontal line test because all points on this line do not share the same y-value unless x_1 and x_2 are the same. Since this strictly increasing function does pass the horizontal line test, then f_+ is invertible.

(c) The former equation is true because for non-negative values of x, $W(x)$ is the inverse function for the restricted function $f_+(x) = xe^x$.The latter equation is true for similar reasons since it is the Lambert W function substituted into f_+.

(d) Begin by assuming that $a > 0$. We begin by applying the definition of logarithm to our given equation, ie $x^x = a$. Since $e^{\log(x)} = x$, we can substitute this into our equation to receive $\left(e^{\log x}\right)^{e^{\log x}} = a$. We can now take the natural log of each side to receive $\log(x)e^{\log x} = \log(a)$. We can now take the Lambert W function of both sides and receive $W\left(\log(x)e^{\log x}\right) = W(\log(a))$. By the equations we were previously given then $\log(x) = W(\log(a))$. We can set both sides to a power of e and simplify to receive $x = e^{W(\log(a))}$.

Question 10. *Algebraic numbers*
A number x is said to be algebraic if there exist integers $a_0, \cdots, a_n$, not all zero, such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0. \qquad (2)$$

In other words, a number x is algebraic if it is the root of some polynomial equation with integer coefficients. We will denote the set of all algebraic numbers by $\mathbb{A}$.

If $x \in \mathbb{A}$, the smallest n for which there exists a polynomial of the above form is called the degree of x.[2]

For instance, the number 1 is algebraic because it is a root of the polynomial $a_1 x + a_0 = 0$ with $a_1 = 1$ and $a_0 = -1$ (since this simply reduces to $x - 1 = 0$). This is a polynomial of order $n = 1$. But 1 is also a root of the polynomial $a_2 x^2 + a_1 x + a_0 = 0$ if $a_2 = 1$, $a_1 = 1$, and $a_0 = -2$, and this is a polynomial with order $n = 2$. We can always cook up higher-order polynomials (larger n) for which $x = 1$ is a root. Since $n = 1$ is the smallest integer such that we can find a polynomial for which 1 is a root, we say that the degree of the algebraic number $x = 1$ is $\deg(1) = 1$.

(a) Is $\sqrt{2}$ an element of $\mathbb{A}$? Is it true that $i \in \mathbb{A}$? Why or why not?

(b) Prove that every rational number is algebraic, i.e. show that $\mathbb{Q} \subseteq \mathbb{A}$. If $x \in \mathbb{Q}$, what is the degree of x?

(c) Suppose $x \in \mathbb{A}$ and $k \in \mathbb{Z}$. Prove that $kx \in \mathbb{A}$.

(Bonus) Prove that, if a complex number $z = a + bi$ is algebraic, then the complex conjugate $\bar{z} = a - bi$ is also algebraic with the same degree. Use this to explain why the image above is symmetric about the vertical line of green dots (that is, every point on the right side of this vertical line has a "mirror image" on the left side of the line).

Solution 10.
(a) $\sqrt{2}$ is an algebraic number, since it is a root of the polynomial $(x - \sqrt{2})(x + \sqrt{2}) = 0$, which by FOIL equals $x^2 - 2 = 0$ which has wholly integer coefficients. i is a root of the polynomial $(x - i)(x + i) = x^2 - 1$ is also composed entirely of integer coefficients and therefore is algebraic.

(b) Begin by defining the rational numbers: all rational numbers can be/are represented by fractions $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. For a number x to be algebraic, there must exist a polynomial function with integer coefficients so that x is a root.

For all rational numbers, there exists a function to "convert" said value to zero and thereby show that it is algebraic. This equation can be represented as the following: Given a rational number of the form $\frac{a}{b}$, the polynomial of the first degree $(b \times x) - a = 0$ demonstrating the quote unquote undoing of a rational number, and thereby the fraction $\frac{a}{b} \in \mathbb{Q}$. Since all rational numbers follow the form $\frac{a}{b}$ by definition, it is true that $\mathbb{Q} \subset \mathbb{A}$. $\square$

(c) Proof. Recall the definition of an algebraic number: A number x is algebraic if for a given set of integers $a_n, \cdots, a_0$, then x is a root of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$.

Suppose $x \in \mathbb{A}$ and $k \in \mathbb{Z}$. We begin by defining a new variable, $y = xk$ and in turn $x = \frac{y}{k}$. Since we already know that x is a root of a polynomial equation following the previously described form, this means that we can substitute $\frac{y}{k}$ into the place(s) of x, therefore

$$a_n(\frac{y}{k})^n + a_{n-1}(\frac{y}{k})^{n-1} + \cdots + a_1(\frac{y}{k}) + a_0 = 0.$$

We can now multiply this equation by k^n to remove all the seen fractions. This results in the equation

$$a_n y^n + k a_{n-1} y^{n-1} + \cdots + k^{n-1} a_1 y + k^n a_0 = 0.$$

[2] There is some deep magic here. Take a look at this image which colors algebraic numbers by their degree. Yellow is $n = 1$, green is $n = 2$, blue is $n = 3$, red is $n = 4$. The size of the points indicates the size of the integer coefficients a_n of the polynomial, with smaller coefficients being marked with larger points. The large "sun" at the bottom is 0, which is a root with $a_1 = 1$ and $a_0 = 0$; the next large yellow point to the right is 1, which is a root with $a_1 = 1$ and $a_2 = -1$, and so on. The vertical line with many green dots is the imaginary axis; note that we're looking at the complex plane. It's breathtaking.

But then y is a root of this newfound equation, which has integer coefficients since k is an integer and the product of integers is an integer, so we conclude that $y = kx$ is algebraic. $\square$

(Bonus) Suppose $z = a + bi$ is an algebraic number. Then there exist some integers $a_0, \cdots, a_n$ such that

$$a_0 + a_1 z + \cdots + a_n z^n = 0.$$

Now take the complex conjugate of both sides of the equation. Since the a_i are integers, they satisfy $\bar{a}_i = a_i$. Therefore

$$a_0 + a_1 \bar{z} + \cdots + a_n \bar{z}^n = 0.$$

So we see that the complex conjugate $\bar{z} = a - bi$ also satisfies a polynomial equation of degree n with the same coefficients a_i as that satisfied by z. It follows that $\bar{z}$ is an algebraic number with the same order.

Therefore, whenever $z \in \mathbb{A}$, we also have $\bar{z} \in \mathbb{A}$, as claimed. $\square$

This explains the symmetry about the vertical axis in the visualization of the algebraic numbers. If $z = a + bi$ is a complex number, then the complex conjugate $\bar{z} = a - bi$ has an imaginary part opposite that of z, which means that $\bar{z}$ is obtained by "flipping" z about the y axis. Thus the statement that $\bar{z}$ is an algebraic number whenever $z \in \mathbb{A}$ is the statement that the set of algebraic numbers has a symmetry under reflection about the $x = 0$ axis in the complex plane.