question 2 views

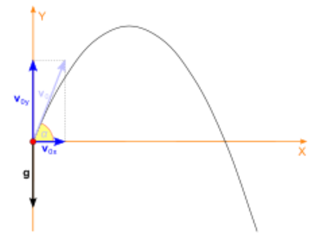
Daily Challenge 15.7

(**Due: by session 40 at the very latest)**

This is problem 16 on CD 4; feel free to work there directly. Here you'll reproduce some of our results on projectile motion from the meetings.

(1) Problem: Firing a cannon.

A cannon ball is shot from the ground with velocity v at an angle α, so that the vertical component of its velocity is initially $v\sin(\alpha)$ and the horizontal component is initially $v\cos(\alpha)$.



The ball obeys Newton's second law in the x and y directions, namely $F_x = m\frac{d^2x}{dt^2}$ and $F_y = m\frac{d^2y}{dt^2}$. Suppose that gravity acts downward in the y direction but there is no force in the x direction. Then

$$\frac{d^2x}{dt^2} = 0,$$
$$\frac{d^2y}{dt^2} = -g.$$

Assume that the cannon ball begins at the origin at time $t = 0$. That is, $x(0) = 0$ and $y(0) = 0$.

(a) Write down the coordinates $x(t)$ and $y(t)$ of the cannon ball as functions of time.

[Hint: they should satisfy the two Newton's law equations above, and we know the initial velocities are $\dot{x}(0) = v\cos(\alpha)$ and $\dot{y}(0) = v\sin(\alpha)$, while the initial positions are $x(0) = 0$ and $y(0) = 0$.]

(b) Show that the trajectory of the cannon ball is parabolic. That is, show that the points $(x(t), y(t))$ lie on a parabola.

(c) Find the time t at which the parabola hits the ground $(y = 0)$ and the horizontal distance it traveled.

(d) Find the angle α which maximizes the horizontal distance that the cannonball travels.

daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a): We begin by seeing that we can represent the x-coordinate as a function of time as the problem statement requests; we undifferentiate with the knowledge that $x(0) = 0$ and thus we know that $x'(t) = v\cos(\alpha) \implies x(t) = v\cos(\alpha)t$. Similarly, we see that $y''(t) = -g$ and $y'(0) = v\sin(\alpha)$, therefore by undifferentiating the former twice and the latter once, we see that $y(t) = -\frac{1}{2}gt^2 + v\sin(\alpha)t$ giving us positions on 2 axes in terms of time.

(b): To show that this is function is a parabola, we must find some way to express y as a function of x rather than t, simplifying this to two dimensions and allowing us to continue. We see that $x = v\cos(\alpha)t \implies t = \frac{x}{v\cos(\alpha)}$. *That Was Easy.* We then see that

$$y(x) = -\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 + v\sin(\alpha)\frac{x}{v\cos(\alpha)},$$

and by placing the top and bottom of the former element to the second power as suggested and separating, we see that

$$-\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 = \left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2 < br/ >< br/ > < br/ >$$

meanwhile by basic algebra

$$v\sin(\alpha)\left(\frac{x}{v\cos(\alpha)}\right) = \tan(\alpha)x\,.$$

We conclude that

$$y(x) = \left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2 + (\tan(\alpha))\,x\,,$$

which is a happy little parabola of the form $y(x) = ax^2 + bx + c$, where a, b are the respective previous alignments sans-x and $c = 0$.

(c): We see that a parabola is symmetric! Then the y position will equal zero at two times the x-value of the vertex, which can be found as the point where the $y'(t) = 0$. We see that $y'(t) = -gt + v\sin(\alpha)$, and by solving when this equals zero, we see that $t = \frac{v\sin(\alpha)}{g}$, therefore the horizontal distance can be found by inserting double this into $x(t)$, then the distance can be represented as $\frac{2v^2\sin(\alpha)\cos(\alpha)}{g}$. Lovely.

(d): We refer to the range function in terms of some angle α we previously defined, $R(\alpha) = \frac{2v^2\sin(\alpha)\cos(\alpha)}{g}$. $\frac{2v^2}{g}$ is given to us positive, so we simply must maximize $\sin(\alpha)\cos(\alpha)$; We find the derivative of this via the product rule to be $\cos^2(\alpha) - \sin^2(\alpha)$. We then solve for zero, and find by intuition that $\alpha = \frac{\pi}{4}$ is a potential answer; our range of interest is $\alpha \in [0, \frac{\pi}{2}]$, where at the endpoints $R = 0$, and there are no undefined points. Then launching an object at 45 degrees is the most efficient, who knew! Aerodynamics should make this *real fun*.

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) We have $\ddot{x} = 0$, $\dot{x}(0) = v\cos(\alpha)$. Anti-differentiating twice and using the initial conditions, we find $x(t) = v\cos(\alpha)t$.

Likewise, in the y direction one sees $\ddot{y} = -g$ while $\dot{y}(0) = v\sin(\alpha)$, so anti-differentiation yields $y(t) = -\frac{1}{2}gt^2 + v\sin(\alpha)t$.

(b) We wish to show that the function $y(x)$ associated with points $(x(t), y(t))$ on the cannon ball's trajectory is parabolic. To do this, we must eliminate the time variable t to express y as a function of x.

Solving our equation for $x(t)$ in part (a) gives $t = \frac{x}{v\cos(\alpha)}$. Plugging this into the equation for $y(t)$, we have

$$\begin{aligned} y(x) &= -\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 + v\sin(\alpha)\left(\frac{x}{v\cos(\alpha)}\right) \\ &= \left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2 + (\tan(\alpha))\,x + 0. \end{aligned}$$

This is indeed a parabola of the form $y = ax^2 + bx + c$with $a = -\frac{g}{2v^2\cos^2(\alpha)}$, $b = \tan(\alpha)$, $c = 0$.

(c) The time t at which the cannon ball hits the ground is, by symmetry, twice the time $t_{1/2}$ that it takes to reach the apex of its flight. The apex occurs where $\dot{y} = 0$, or where $-gt + v\sin(\alpha) = 0$, i.e. one has $t_{\text{apex}} = \frac{v\sin(\alpha)}{g}$. Thus the time at which it hits the ground is

$$t = \frac{2v\sin(\alpha)}{g}.$$

The horizontal distance it traveled is given by plugging this time into $x(t)$, which gives

$$\begin{aligned} \text{range} &= (v\cos(\alpha)) \cdot \frac{2v\sin(\alpha)}{g} \\ &= \frac{2v^2\sin(\alpha)\cos(\alpha)}{g}. \end{aligned}$$

(d) We wish to maximize the function $\text{range}(\alpha) = \frac{2v^2\sin(\alpha)\cos(\alpha)}{g}$ over the angle α. Since the constant $\frac{2v^2}{g}$ is positive, it suffices to maximize $\sin(\alpha)\cos(\alpha)$.

The derivative is

$$\frac{d}{d\alpha}(\sin(\alpha)\cos(\alpha)) = \cos^2(\alpha) - \sin^2(\alpha),$$

which vanishes if $\cos(\alpha) = \pm\sin(\alpha)$. We restrict to launching angles $\alpha \in \left(0, \frac{\pi}{2}\right)$, so the only sensible solution is $\alpha = \frac{\pi}{4}$.

To be thorough, we should check the endpoints and places where the derivative is undefined. But at both endpoints, the range is zero (clearly less than the point $\alpha = \frac{\pi}{4}$ we found), and there are no places where the derivative is undefined.

Thus launching a projectile at angle $\alpha = \frac{\pi}{4} = 45^\circ$ maximizes the range.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*