question 2 views

Daily Challenge 10.3

(**Due: Tuesday 7/24 at 12:00 noon Eastern**)

Let's get back on track with daily challenges. It would be nice to finish limits by the end of July

(1) Counterexamples help you spot false claims.

As one learns a new field of mathematics, it is very useful to build a collection of *counterexamples*.

Good counterexamples help you to recognize when an otherwise reasonable-sounding claim is actually false. For instance, the following proposition seems plausible:

(False) proposition. Let $f : \mathbb{R} \to \mathbb{R}$. Then there exists at least one point a where f is continuous.

Counterexample. The claim is incorrect; consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

This f is discontinuous everywhere, since every open interval contains both rational and irrational points, and thus contains some points with $f(x) = 0$ and some points with $f(x) = 1$. Thus we cannot choose a δ to guarantee that $f(x)$ will be within say $\epsilon = \frac{1}{2}$ of any proposed limiting value. It follows that f is discontinuous everywhere. $\square$

Knowing this counterexample saved us a lot of time -- if we weren't aware of it, we might have instead spent a lot of effort trying in vain to prove that the proposition is *true*!

Entire books have been written which contain only good counterexamples. To a mathematician, they are indispensable.

(2) We can cook up functions with desired continuity properties.

To find counterexamples to claims about continuity, it is useful to be able to design functions with certain properties.

Example. Design a function which is discontinuous at every integer $m \in \mathbb{Z}$ but continuous at every non-integer point.

Solution. Let $f(x) = \lceil x \rceil$, where $\lceil \cdot \rceil$ is the ceiling function. Then $\lim_{x \to m} f(x)$ does not exist for each integer m, but f is constant on any open interval that does not contain an integer, so f is continuous away from the integers. $\square$

When we are making up a function which is discontinuous at an *infinite* number of points, one must be careful not to accidentally "break" continuity at another point.

Example. Design a function which is discontinuous at every point $\frac{1}{m}$, where m is a positive integer (that is, f is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \cdots$) but continuous elsewhere.

Solution. Our first guess might be to take a constant function with point discontinuities at each $\frac{1}{m}$, like

$$f \stackrel{?}{=} \begin{cases} 1 & \text{if } x = \frac{1}{m} \text{ for } \in \mathbb{N}_+ \\ 0 & \text{otherwise} \end{cases}$$

But this does *not* work because f is now discontinuous at 0. To see this, note that we can never find a δ so that $|f(x)| < \frac{1}{2}$ when $0 < |x| < \delta$. Indeed, for any δ, the interval $(-\delta, \delta)$ will contain some point $\frac{1}{N}$, since we can always find an integer N large enough that $\frac{1}{N} < \delta$.

Therefore, the function f defined above is discontinuous at each point $\frac{1}{m}$ for $m \in \mathbb{Z}$, and at $x = 0$, which is not what we wanted.

To fix this, we should re-define the function as

$$f = \begin{cases} \frac{1}{m} & \text{if } x = \frac{1}{m} \text{ for } \in \mathbb{N}_+ \\ 0 & \text{otherwise} \end{cases}.$$

Now f is continuous at 0 (do you see how to prove that?). Hence this f has the desired property of being discontinuous at each $\frac{1}{m}$ but continuous elsewhere.

(3) Problem: the function kitchen.

Note that this is question 3 on consolidation document 2.

For each question below, cook up an example function which has the stated properties.

(a) Find an example of a function $f(x)$ with domain $\mathbb{R}$ for which $\lim_{x\to 0^+} f(x)$ exists but $\lim_{x\to 0^-} f(x)$ does not exist. [Hint: oscillate wildly.]

(b) Cook up a function $f(x)$ with domain $\mathbb{R}$ that is continuous nowhere, but where $|f(x)|$ is continuous on all of $\mathbb{R}$. [Hint: flip between rationals and irrationals.]

(c) Find a function $f(x)$ defined on $\mathbb{R}$ which is continuous at the two points $x = -1$ and $x = 1$, but is discontinuous at every other point. [You actually already did this in DC 9.6(c), so it is fine to copy-paste the response.]

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: $f(x) = \begin{cases} 2 & \text{if } x > 0 \\ 1 & \text{if } x \in \mathbb{Q} \text{ and } x < 0 \\ 0 & \text{if } x \notin \mathbb{Q} \text{ and } x < 0 \end{cases}$

b: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$

c: $f(x) = \begin{cases} (x-1)(x+1) & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Nice, your solutions to 10.2 and 10.3 look good. Only small revisions (e.g. in (a) I think you want one of the conditions to be $x \leq 0$ so that the function is defined at zero), but we'll worry about those when we move them to the consolidation document.

Thanks for working on these!

Another interesting function which satisfies the properties of (a) is

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ \sin\left(\frac{1}{x}\right) & \text{if } x < 0 \end{cases}$$

but yours is nice as well.

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*