question 2 views

Daily Challenge 21.6

(**Due: Sunday 2/17 at 12:00 noon Eastern**)

In the previous challenge, you saw how to compute volumes for solids of revolutions by slicing them into discs:

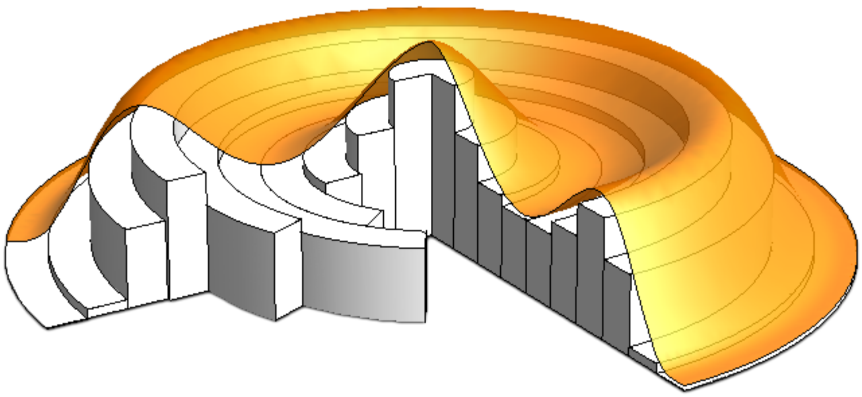
$$V = \int A_{\text{disc}}\,dx = \int \left(\pi r_{\text{disc}}^2\right)\,dx.$$

For other shapes, however, it is more convenient to slice the volume into cylindrical shells. You did this in the torus, for instance. Today we'll see how to do this in general, using

$$V = \int A_{\text{shell}}\,dx = \int \left(2\pi r_{\text{shell}} h_{\text{shell}}\right)\,dx.$$

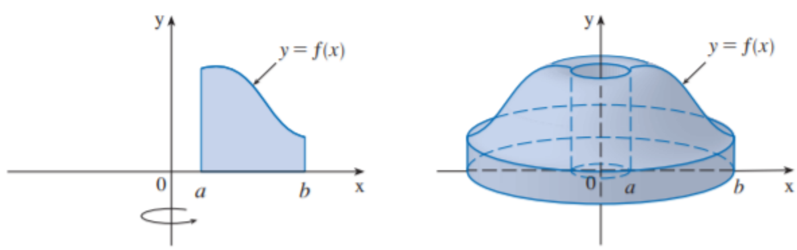
(1) Shell integration.

Recall that, to find the volume of the torus, you sliced it into thin cylindrical strips.

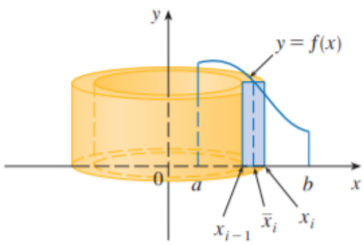


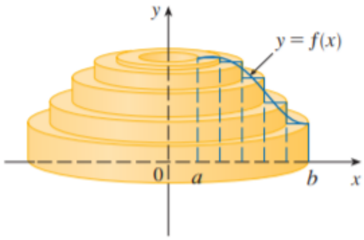
Another example: to find the volume under the orange surface in the figure above, we could work inward from the outer rim ("peeling an onion"), and chop it into shells of fixed height. Adding up the volumes of these shells as we move inward in radius, we get an approximation to the total volume.

Now let's be slightly more specific. Let $f(x)$ be an integrable function, and suppose we rotate the area under $f(x)$ on the interval $[a, b]$ around the y axis to create a solid of revolution.

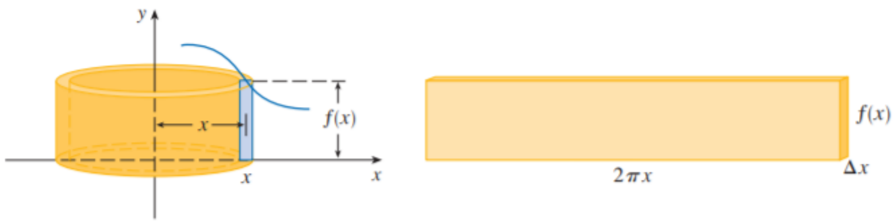


If we were finding the *area* under f, we would partition $[a, b]$ into rectangles $[x_{i-1}, x_i]$ and compute the area of each rectangle as $f(x_i)\,(x_i - x_{i-1})$. However, we now see that each rectangle will be rotated about the y axis into a shell:





Each shell is a cylindrical shell with radius x_i and height $f(x_i)$, which has volume $V_{\text{shell}} = (2\pi x_i)f(x_i)$.

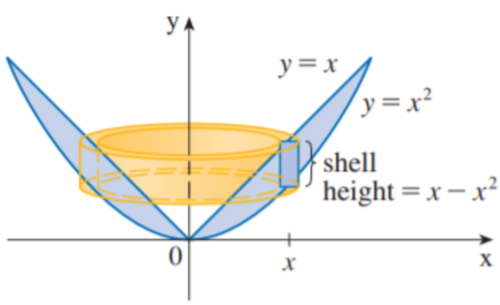


Thus the total volume of the region is obtained by adding the contributions from each shell. In the limit as Δx goes to zero, this becomes an integral:

$$V = \int_a^b (2\pi x)f(x)\,dx.$$

Again, the $(2\pi x)$ factor comes from the circumference of the cylinders, and $f(x)$ is the height.

Example. Suppose we want to find the volume obtained by rotating the area between the curves $y = x$ and $y = x^2$ about the y axis.



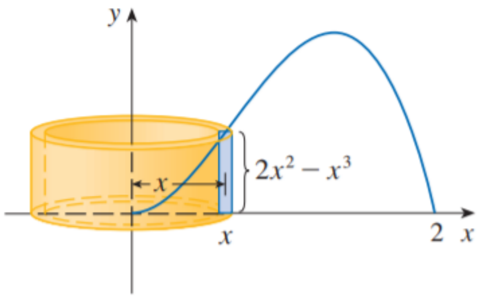
Each shell has radius x, thus circumference $2\pi x$, and height $x - x^2$ (the *difference* between the curves). We integrate from $x = 0$ to $x = 1$, the two intersection points of the curves. Thus the volume is

$$\begin{aligned} V &= \int_0^1 (2\pi x)(x - x^2)\,dx \\ &= 2\pi \int_0^1 (x^2 - x^3)\,dx \\ &= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= \frac{\pi}{6}. \end{aligned}$$

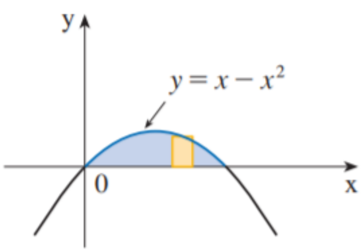
(2) Problem: shell practice.

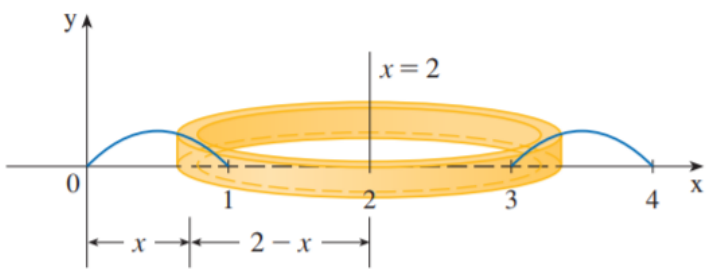
Do the following exercises.

(a) Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and the line $y = 0$ about the y axis (see figure below). Check that you get $\frac{16}{5}\pi$.



(b) Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$. (See figure below). Make sure you get $V = \frac{\pi}{2}$.





daily_challenge

Updated 1 month ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a) We shall write a formula for the area of the outer area of the cylinder:

$A(x) = 2\pi x(2x^2 - x^3)$

Thus, we can write the integral

$$
\begin{aligned}
V(x) &= 2\pi \int_0^2 2x^3 - x^4 \, dx \\
&= 2\pi(\Big[\frac{x^4}{2} - \frac{x^5}{5}\Big]_0^2) \\
&= 2\pi(\frac{40}{5} - \frac{32}{5}) \\
&= \frac{16\pi}{5}
\end{aligned}
$$

(b): We write another formula for the area of the face of a cylinder:

Everything past here is incorrect.

$A(x) = 2\pi x(x - x^2)$

We then make the integral

$$
\begin{aligned}
V(x) &= 2\pi \int_3^4 (x^2 - x^3) \, dx \\
&= 2\pi\Big[(\frac{x^3}{3} - \frac{x^4}{4})\Big]_3^4
\end{aligned}
$$

Woops, $[3, 4]$ gave a negative result, let's try again with $[0, 1]$ because this is actually the area we are rotating through three dimensional space.

$$2\pi\Big[(\frac{x^3}{3} - \frac{x^4}{4})\Big]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$

Well that isn't the right answer either, so let's look at our work.

Everything up to here has been incorrect, so let's do this with the right radius this time

$A(x) = 2\pi(2 - x)(x - x^2)$

We then write out an integral:

$$
\begin{aligned}
V(x) &= 2\pi \int_0^1 (2-x)(x-x^2) \\
&= 2\pi \int_0^1 (2-x)(x-x^2) \\
&= 2\pi \Big[\frac{x^4}{4} - x^3 + x^2\Big]_0^1 \\
&= 2\pi \frac{1}{4} \\
&= \frac{\pi}{2}
\end{aligned}
$$

Updated 1 month ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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followup discussions *for lingering questions and comments*