question 2 views

Daily Challenge 20.7

Use the Weierstrass substitution $t = \tan\left(\frac{x}{2}\right)$ to compute

$$\int \frac{dx}{a\sin(x) + b\cos(x)}.$$

You may like to refer to the slides or Wikipedia.

daily_challenge

Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

We let $t = \tan(\frac{x}{2})$; we then see that we can substitute out our sine and cosine,

$$\begin{aligned}\int \frac{dx}{a\sin(x) + b\cos(x)} &= \int \frac{dx}{a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2}} \\ &= \int \frac{(1+t^2)\,dx}{a2t + b(1-t^2)}\end{aligned}$$

And then the $dx = \frac{2}{1+t^2}$:

$$\begin{aligned}\int \frac{(1+t^2)\,dx}{a2t + b(1-t^2)} &= 2\int \frac{dt}{a2t + b(1-t^2)} \\ &= 2\int \frac{dt}{-bt^2 + a2t + b}\end{aligned}$$

~~Lemme just start faffing about with certain chain rules of \log(-b t^2 + a2t + b) and see what we have to divide out.~~ Okay, that didn't work because of the presence of the t^2, so we have to get creative. There is no way to apply partial fractions to this, as intuitively the polynomial cannot be factored. Let's try writing this in a slightly more obvious way;

$$2\int \frac{dt}{-bt^2 + a2t + b} = 2\int (-bt^2 + a2t + b)^{-1}\,dt$$

How about we try a u-sub where $u = t^2$ and thus $\sqrt{u} = t$, then $du = 2tdt$. This complicates what we have going on by even more, where we can rewrite our problem as

$$2\int (-bt^2 + a2t + b)^{-1}\,dt = 2\int ((2t)\cdot(-bu + a\sqrt{u} + b))^{-1}\,dt$$

This didn't help and I still don't have any better ideas or alternate u-subs. ~~We cannot apply integration by parts here, as no two functions are multiplied.~~ False, it just doesn't help. Let's go back to trying to factor the denominator. We shall apply the wikipedia article on completing the square. We then see that:

$$2\int \frac{dt}{-bt^2 + a2t + b} = 2\int \frac{dt}{-b(t - \frac{a}{b})^2 + b + 2\frac{a}{b}}$$

We can now factor a $b + 2\frac{a}{b}$ out of the denominator and thus totally out to see that

$$2\int \frac{dt}{-b(t - \frac{a}{b})^2 + b + 2\frac{a}{b}} = \frac{2}{b + 2\frac{a}{b}}\int \frac{dt}{\frac{-b}{b+2\frac{a}{b}}(t - \frac{a}{b})^2 + 1}$$

And apply the hinted formula for $\int \frac{dx}{1+x^2} = \arctan(x)$

To conclude that

$$\frac{2}{b + 2\frac{a}{b}}\int \frac{dt}{\frac{-b}{b+2\frac{a}{b}}(t - \frac{a}{b})^2 + 1} = \frac{2}{\sqrt{-b^2 + 2a}}\cdot\arctan\left(\sqrt{\frac{-b}{b + 2\frac{a}{b}}}t\right) + C$$

where then $t = \tan\frac{x}{2}$ and we conclude that

$$\int \frac{dx}{a\sin(x) + b\cos(x)} = \frac{2}{\sqrt{-b^2 + 2a}}\cdot\arctan\left(\sqrt{\frac{-b}{b + 2\frac{a}{b}}}\tan\frac{x}{2}\right) + C$$

Updated 2 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Click to start off the wiki answer

followup discussions *for lingering questions and comments*

Resolved Unresolved

Christian Ferko 2 months ago

Hint: complete the square in the denominator and use that $\int \frac{dx}{1+x^2} = \arctan(x)$.

Logan Pachulski 2 months ago :logwow: