question 2 views

Daily Challenge 26.4

(**Due: Tuesday 4/9 at 12:00 noon Eastern**)

In session 59, I told you that the Fourier transform of a convolution equals the product of the Fourier transform, i.e. $\mathcal{F}\left[f * g\right] = \mathcal{F}[f] \cdot \mathcal{F}[g]$. That is, the Fourier transform "turns convolution into multiplication of functions."

The proof was

$$\begin{aligned}\mathcal{F}\left[f * g\right] &= \int_{-\infty}^{\infty} e^{-ikx}\left[\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right] \\ &= \int_{\mathbb{R}^2} e^{-ik(x-y)} f(x-y) e^{-iky} g(y)\,dx\,dy \\ &= \left(\int_{-\infty}^{\infty} e^{-iku} f(u)\,du\right)\left(\int_{-\infty}^{\infty} e^{-iky} g(y)\,dy\right) \\ &= \mathcal{F}[f] \cdot \mathcal{F}[g].\end{aligned}$$

Today we will test this claim by computing the Fourier transform of a specific convolution, and comparing it to the product of the Fourier transforms.

Consider the square pulse

$$\mathrm{sq}(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

We showed on slide 24 of session 58 that the Fourier transform of the square pulse is

$$\mathcal{F}[\mathrm{sq}] = \int_{-\infty}^{\infty} e^{-ikx} \cdot \mathrm{sq}(x)\,dx = \frac{\sin\left(\frac{k}{2}\right)}{\left(\frac{k}{2}\right)} = \mathrm{sinc}\left(\frac{k}{2}\right).$$

On the other hand, we showed in session 56 that the convolution of two square pulses gives a triangle pulse:

$$(\mathrm{sq} * \mathrm{sq})(x) = \begin{cases} x+1 & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

That is, the convolution of two boxes gives a triangle. If our Fourier convolution theorem is correct, we *should* have

$$\mathcal{F}\left[\mathrm{sq} * \mathrm{sq}\right] \overset{\text{want}}{=} \mathcal{F}[\mathrm{sq}] \cdot \mathcal{F}[\mathrm{sq}] = \frac{\sin^2\left(\frac{k}{2}\right)}{\left(\frac{k}{2}\right)^2} = \mathrm{sinc}^2\left(\frac{k}{2}\right).$$

Check explicitly that this formula is correct by computing the Fourier transform of the left side. That is, compute the Fourier transform of the triangle function

$$\mathrm{tri}(x) = \begin{cases} x+1 & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

using the conventions

$$\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx,$$

and show that the result is equal to $\frac{4\sin^2(k/2)}{k^2}$, which is $\mathrm{sinc}^2\left(\frac{k}{2}\right)$.

Hint/Solution. Split the integral into two regions, one from $-1 \leq x \leq 0$ and one from $0 \leq x \leq 1$. In each region, $\mathrm{tri}(x)$ is just $1 \pm x$; handle the $\int 1 \cdot e^{-ikx}\,dx$ piece directly, since

$$\int_a^b e^{-ikx}\,dx = \left[\frac{e^{-ikx}}{-ik}\right]_a^b,$$

and handle the $\int x e^{-ikx}\,dx$ piece through integration by parts to move a derivative onto the x.

Combine the four terms you get from the resulting integrals and get a common denominator. Your numerator will involve $e^{ik} + e^{-ik}$, which you can simplify using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ to show

$$e^{ik} + e^{-ik} = 2\cos(k).$$

Then use the cosine double-angle formula $\cos(2x) = 1 - 2\sin^2(x)$ to simplify the result. You should find

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-ikx}\,\mathrm{tri}(x)\,dx &= \frac{2-\left(e^{ik}+e^{-ik}\right)}{k^2} \\
&= \frac{2-2\cos(k)}{k^2} \\
&= \frac{2-\left(2-4\sin^2\left(\frac{k}{2}\right)\right)}{k^2} \\
&= \frac{4\sin^2\left(\frac{k}{2}\right)}{k^2} \\
&= \mathrm{sinc}^2\left(\frac{k}{2}\right).
\end{aligned}
$$

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Updated 4 days ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

keet

Updated 2 days ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

blonde boi

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followup discussions *for lingering questions and comments*