question 2 views

Daily Challenge 10.7

(~~Due: Saturday 7/28 at 12:00 noon Eastern~~)
(**Due: Sunday 7/29 at 12:00 noon Eastern**)

Short challenge today! All that's left is two small limit existence proofs in CD 2 problem 1. Then we can do revisions and the British-style tutorial, and onward to derivatives and glory.

(1) Problem: proving two limits exist.

In the following questions, prove that the given limits exist directly from the definition (i.e. without resorting to other results that we've proven, like the squeeze theorem or continuity).

(a) Let $c \in \mathbb{R}$ and let $f(x) = c$ be the constant function with value c. Prove that $\lim_{x\to a} f(x) = c$ for all $a \in \mathbb{R}$.

(b) Let $f(x) = x^4$. Prove that $\lim_{x\to 2} f(x) = 16$.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Proof: Let $\epsilon > 0$ be given, and define $\delta = \epsilon$. We then have that for $0 < |x-a| < \delta$, it is then true that since $f(x) = c$ then $|f(x) - c| = |c - c| = 0 < \epsilon$.

b: Exploration: Let us begin by looking at the implication. Let $\epsilon > 0$ be given. our implication is along the lines of $0 < |x-2| < \delta \implies |x^4 - 16| < \epsilon$ To find what we should set δ equal to, we have to factor the current right side; $|x^4 - 16| = |(x^2-4)(x^2+4)| = |(x-2)(x+2)(x^2+4)|$We can restrict our foremost factor, ie $|x-2| < 1$, and in turn restrict x, so $1 < x < 3$ and x infinitesimally close to 3. We can then place 3 into our other two (more annoying to work with) factors and get that $65|x-2| < \epsilon$, and in turn $|x-2| < \frac{\epsilon}{65}$, allowing us to know that $\frac{\epsilon}{65}$ is a valid delta, and we can now continue to the proof.

Proof: Let $\epsilon > 0 begiven$. We can then select $\delta = \min(1, \frac{\epsilon}{65})$. We then know that $0 < |x-2| < \delta \leq 1$, and this restricts x to less than or equal to 3. We have latter portion of the statement implied by the limit we are trying to prove; $|x^4 - 16| < \epsilon$. We factor: $|x^4 - 16| = |(x-2)(x+2)(x^2+4)| \leq |(x-2)(3+2)(3^2+4)| = 65|x-2|$We then refer to our other option for selection of δ, $\frac{\epsilon}{65}$, and this entailing that $|x-2| < \frac{\epsilon}{65}$. It is then true that $65|x-2| < 65 \times \frac{\epsilon}{65} = \epsilon$ showing that thanks to our chosen δ, we have shown that $|x^4 - 16| < \epsilon$ when $0 < |x-2| < \delta = \min(1, \frac{\epsilon}{65})$.

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) Let $\epsilon > 0$ be given and choose any δ whatsoever (for concreteness, say $\delta = \epsilon$).

Whenever $0 < |x-a| < \delta$, it is automatically true that $|f(x) - c| < \epsilon$, since $f(x) - c = c - c$is identically zero.

(b) Fix $\epsilon > 0$ and let $\delta = \min\left(1, \frac{\epsilon}{65}\right)$. We will prove that, whenever $0 < |x-2| < \delta$, then $|x^4 - 16| < \epsilon$.

If $0 < |x-2| < \delta \leq 1$, then certainly $x \leq 3$. Therefore

$$|x^4 - 16| = \left|(x-2)(x+2)(x^2+4)\right| \leq \left|(x-2)(3+2)(3^2+4)\right| = 65\,|(x-2)|\,.$$

But we have also assumed that $|x-2| < \delta \leq \frac{\epsilon}{65}$, so

$$|x^4 - 16| \leq 65\,|(x-2)| < 65 \cdot \frac{\epsilon}{65} = \epsilon,$$

which shows that $|x^4 - 16| < \epsilon$ whenever $0 < |x-2| < \delta$ with our chosen δ. $\square$

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*