note 2 views

Daily Challenge 4.6

No problem today, but I will again post a couple of worked examples which are roughly at the level of difficulty of problems that I'll expect you to be able to solve before we can move on from chapter 1.

Example 1 (source: AoPS volume 1). Prove that, if $|x| + x > 0$, then x must be positive.

Solution 1. We instead choose to prove the contrapositive: if x is non-positive (that is, if $x \leq 0$), then $|x| + x \leq 0$.

So suppose $x \leq 0$. The definition of absolute value is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} .$$

Thus, by definition, if $x \leq 0$, we have that $|x| + x = -x + x = 0$. Thus we have proven the contrapositive, which is logically equivalent to the statement that if $|x| + x > 0$, x must be positive. $\square$

Example 2 (source: AoPS online calculus course, week 1 homework). A function f is called *injective* if $f(a) = f(b)$ implies that $a = b$. If f and g are functions such that $g \circ f$ is injective, can we conclude that f is injective or g is injective?

Solution 2. Suppose that f and g are functions and that $g \circ f$ is injective (recall that $(g \circ f)(x) = g(f(x))$).

This means that, whenever $g(f(a)) = g(f(b))$, we also have that $a = b$.

I claim that f must also be injective. We prove this by contradiction; suppose there were two numbers a, b so that $f(a) = f(b)$ but $a \neq b$. Then $g(f(a)) = g(f(b))$ but $a \neq b$, which means that $g \circ f$ is not injective, a contradiction.

However, g does *not* need to be injective. For a counterexample, let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$. Then the composition is $g \circ f : [0, \infty) \to \mathbb{R}$ with $(g \circ f)(x) = x$, which is injective. However, g is not injective since $g(1) = g(-1)$ but $1 \neq -1$. $\square$

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Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*