question 2 views

Daily Challenge 20.1

Today we'll see another example of the Feynman trick for evaluating integrals by differentiation with respect to a parameter.

(1) Problem: turning the exponent into a parameter.

Suppose we would like to evaluate the integral

$$\int_0^1 \frac{x^2-1}{\log(x)}\,dx.$$

Do this using the Feynman trick using the following steps.

(a) Replace the integral by a new function

$$I(b) = \int_0^1 \frac{x^b-1}{\log(x)}\,dx.$$

This is the "introduce a parameter" step of the Feynman trick. Note that $I(2)$ is the quantity we want.

(b) Take the derivative $\frac{dI}{db}$ and apply our theorem to move the derivative under the integral sign.

(c) The resulting integral for $\frac{dI}{db}$ you find in part (b) is easy to evaluate; do so. [Hint: you should get $I'(b) = \frac{1}{b+1}$].

(d) Now integrate both sides of the equation $I'(b) = \frac{1}{b+1}$ to find $I(b) = \log(b+1) + C$, where C is a constant. Figure out what C needs to be. [Hint: what is $I(0)$?]

(e) Conclude that $I(2) = \int_0^1 \frac{x^2-1}{\log(x)}\,dx = \log(3)$, solving the original problem.

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Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

We shall begin operating on the given integral:

$$I(b) = \int \frac{x^b-1}{\log(x)}\,dx = \int \frac{e^{b\log(x)}-1}{\log(x)}\,dx$$

We can now take the derivative and see that

$$I'(b) = \int \frac{dI}{db}\,\frac{e^{b\log(x)}-1}{\log(x)}\,dx$$
$$= \int \frac{dI}{db}\,\frac{e^{b\log(x)}-1}{\log(x)}\,dx$$

We shall treat x as a constant and see that

$$\int \frac{d}{db}\,\frac{e^{b\log(x)}-1}{\log(x)}\,dx = \int e^{b\log(x)}\,dx = \int x^b\,dx$$

We then see by the power rule that

$$\int x^b\,dx = \frac{x^{b+1}}{b+1}$$

We evaluate this at $x = 0$ and $x = 1$ by the FTC and see

$$\frac{1^{b+1}}{b+1} - \frac{0^{b+1}}{b+1} = \frac{1}{b+1} = I'(b)$$

We see by anti-differentiating I' here that

$$I(b) = \log(b+1) + C \implies I(0) = 0 + C$$

We see by inserting $b = 1$ into our foremost equation,

$$I(0) = \int_0^1 \frac{e^{0\cdot\log(x)}-1}{\log(x)}\,dx, = \int_0^1 0 = 1 - 1 = 0.$$

Then, $I(0) = 0 = \log(0+1) + C \implies C = 0$, and thus we can insert $b = 2$ into our equation with a now known constant to conclude that $I(2) = \log(3)$

Updated 2 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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followup discussions *for lingering questions and comments*

Resolved Unresolved

Christian Ferko 2 months ago
Hint: look back at the definition of $I(b)$ in part (a) to figure out what $I(0)$ must be.

Logan Pachulski 2 months ago :logwow:

Logan Pachulski 2 months ago ~~If I'm not mistaken, that introduces another C though.~~ I am mistaken, this is a definite integral.