30.4

(a): Recall that sine has taylor series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

then by substituting x^2 for x,

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

We see by Leibniz that if we can show the "non-alternating" series converges, and thus the terms go to zero, then aforementioned sum is convergent and has $R=\infty$; consider then the sum

$$\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}$$

apply the ratio test and evaluate the limit

$$\lim_{n\to\infty} \frac{\frac{x^{4n+6}}{(2n+3)!}}{\frac{x^{4n+2}}{(2n+1)!}} = \lim_{n\to\infty} \frac{x^4}{(2n+3)(2n+2)} = 0$$

thus all sums in question converge and.

$R = \infty$.

(b): Recall that e^x has Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

thus by substituting $-x^2$ for x and multiplying each side by x,

$$xe^{-x^2} = x + -x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(n)!}$$

thus by Leibniz and the ratio test evaluate

$$\lim_{n\to\infty} \frac{\frac{x^{2n+3}}{(n+1)!}}{\frac{x^{2n+1}}{(n)!}} = \lim_{n\to\infty} \frac{x^2}{(n+1)} = 0.$$

Thus ~~for~~ $R = \infty$

(c): This is the sum of

$\sum_{n=0}^{\infty} \cancel{\;} (x^2)^n$, which is convergent for $x^2 < 1 \Rightarrow |x| < 1$

Thus $R = 1$

(d): Let a_n be the Fibonacci at position n; then by the nondecreasing nature see that

$a_{n+1} = a_{n-1} + a_n \leq 2a_n$

Then apply the ratio test and evaluate

$\lim_{n\to\infty} \left| \frac{a_{n+1}\, x^{n+1}}{a_n\, x^n} \right| \leq \left| \frac{2a_n}{a_n} x \right| = |2x|$

Then we see by the results of the ratio test that if

$|2x| < 1$ then the sum converges; ie $|x| < \frac{1}{2} \Rightarrow R \geq \frac{1}{2}$

(e): We see that

$f(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$ (1)

$x f(x) = x + x^2 + 2x^3 + 3x^4 + \cdots$ (2)

$x^2 f(x) = x^2 + x^3 + 2x^4 + 3x^5 + \cdots$ (3)

We then see by the statement that $a_n = a_{n-1} + a_{n-2}$ that by subtracting (2) and (3) from (1), that

$(1 - x - x^2) f(x) = 1$

$\Downarrow$

$f(x) = \frac{1}{1-x-x^2}$

Thus we have a function for the sum that applies for $|x| < 1/2$.