question 2 views

Daily Challenge 13.6

(**Due: Tuesday 9/11 at 12:00 noon eastern**)

Chapter 4 begins!

(1) The derivative controls the shape of a function's graph.

We have seen in the past several meetings that one can extract useful data about the graph of a function from the behavior of its first and second derivatives.

For instance, the sign of the derivative controls whether a function is increasing or decreasing:

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b).
- If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).
- If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).
- If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

We have also seen that local maxima and minima often occur at *critical points* where the first derivative vanishes, as encoded in the following theorem.

Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then f achieves its maximum and minimum value on $[a, b]$, and both occur at points x which satisfy one of the following conditions:

- $f'(x) = 0$,
- $x = a$ or $x = b$,
- the derivative of f at x does not exist.

Finally, as we will prove in the next meeting, the second derivative controls the *concavity* of the function: if $f''(x) > 0$ then the function is convex (or "concave up", i.e. it opens upward) and if $f''(x) < 0$ the function is concave (or "concave down", i.e. it opens downward).

A point where $f''(x) = 0$ is called an *inflection point*; this is a place at which the function's concavity changes direction.

(2) Problem: sketching a graph.

Sketch the graph of the function $f(x) = xe^{-x}$. Find all intervals where f is increasing and decreasing, where it is concave up and concave down, and any local maxima or minima. Include a picture or scan of your physical sketch in your response.

daily_challenge

Updated 7 months ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

The original function $f(x) = xe^{-x}$ has derivatives

$$f'(x) = e^{-x} - e^{-x}x,$$
$$f''(x) = -2e^{-x} + e^{-x}x.$$

First we find where f is increasing or decreasing, which is controlled by the sign of the first derivative. We see $f'(x) = 0$ where $e^{-x} = xe^{-x}$, which occurs at $x = 1$. On the other hand, $f'(x)$ is negative for large positive x and negative for large negative x, so we conclude that the original function f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

The second derivative vanishes where $2e^{-x} = e^{-x}x$, i.e. at $x = 2$, and is negative to the left of 2 and positive to the right of 2. Thus we see that the original function f is concave on $(-\infty, 2)$ and convex on $(2, \infty)$.

We also see that f has a local maximum at $x = 1$ since here the first derivative vanishes and the second derivative is negative.

Finally, the plots of f and its derivatives are below.

```
In[41]:= f[x_] := x E^-x;

In[42]:= Plot[{f[x], f'[x], f''[x]}, {x, -0.5, 3}, PlotLegends → "Expressions"]
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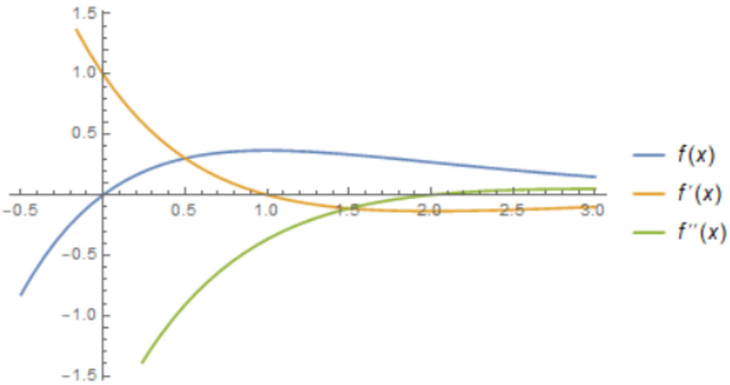
Out[42]=



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followup discussions *for lingering questions and comments*