question 3 views

Daily Challenge 5.1

(**Due: Tuesday 5/22 at 12:00 noon Eastern**)

Let's make another focused push this week to become comfortable reading and writing proofs.

Review

Every theorem and proof have the following form.

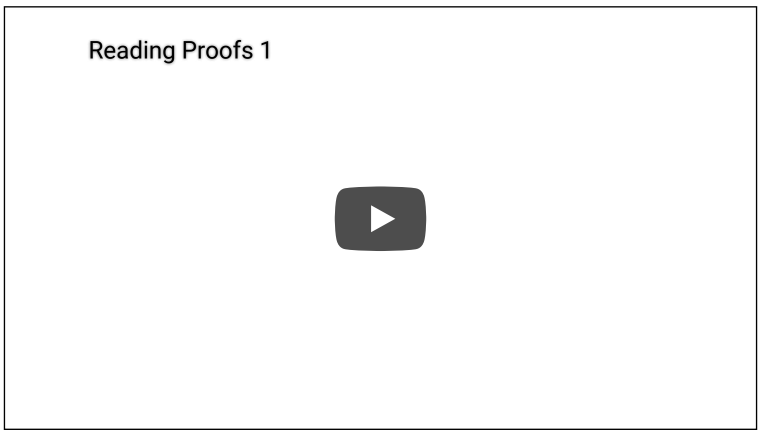
> **Theorem**. If (assumptions), then (conclusion).
>
> **Proof**. Suppose (assumptions). Then (logical deduction). Hence (deduction). $\cdots$ Therefore, (conclusion). $\square$

Before reading a proof, it's important to clearly understand the assumptions and conclusion, and to recall the precise definitions of all words and symbols in the statement.

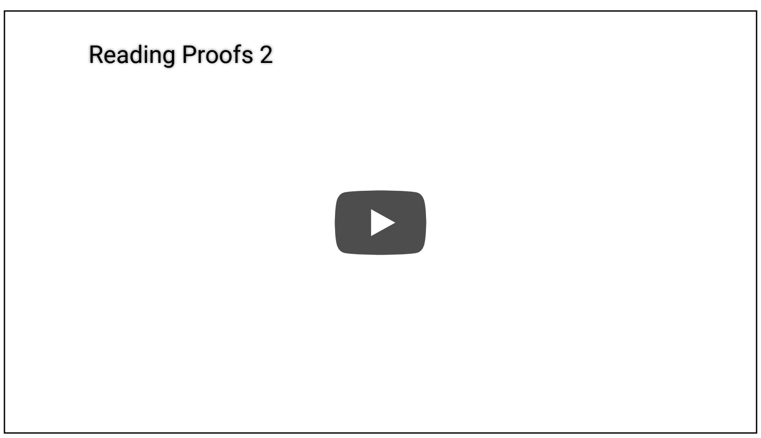
While reading a proof, one should stop after every sentence and ask: what was the goal of this sentence? Did it state an assumption or introduce a new variable? Was it a logical deduction, and if so, why is it true?

I've recorded two videos with more detail on proof-reading technique, including one which reads through an example proof. Please watch both.

Video one:



Video two:



Problem

Let's revisit the proof in problem 1.6 of AoPS, which we saw in reading assignment 1.

> **Theorem.** Suppose S is a subset of $\mathbb{R}$. If S has a least upper bound, then this least upper bound is unique.
>
> **Proof**. Suppose x and y are each least upper bounds of S. Then $x \leq y$ (since x is a least upper bound and y is an upper bound), but also $y \leq x$ (since y is a least upper bound and x is an upper bound). Since $x \leq y$ and $y \leq x$, we must have $x = y$. So the least upper bound is unique. $\square$

Consider the following questions, and answer the ones that I have not answered for you.

1. What are the assumptions and conclusion of this theorem?

2. Give the precise definition of "least upper bound".

3. What is the goal of the first sentence?

   Answer. The first sentence declares the variables x and y and assumes that they are least upper bounds of S, which means that they satisfy the definition you gave in question (2) above. We assume that there are *two* least upper bounds because this is a uniqueness proof, so we wish to show that x and y must be equal.

   Note that the subset S was not declared in the proof, but is still used in the first sentence. It is acceptable to do this if a variable was declared in the statement of the theorem, which is true in this case: the theorem said "Suppose S is a subset of $\mathbb{R}$", so we can treat this as a declaration of S without repeating it.

4. Explain why the deduction in the second sentence, $x \leq y$, is true. Give as much detail and precision as you can.

5. The parenthetical comment after the deduction $y \leq x$ says "y is a least upper bound and x is an upper bound", but we assumed that x was a *least* upper bound, not an upper bound. Why is it correct to say this? Justify your explanation using the definitions of upper bound and least upper bound.

   Answer. By definition, a variable z is an upper bound of a subset $S \subseteq \mathbb{R}$ if $z \geq s$ for all $s \in \mathbb{R}$. But we see that the definition of least upper bound which you gave in question 2 assumes that this is true. Thus every least upper bound is also an upper bound.

   Since we assumed that x was a least upper bound, then, it is also true that x is an upper bound, and the parenthetical comment is correct.

6. How could we modify this proof to show that, if $S \subset \mathbb{R}$ has a *greatest lower bound*, then this greatest lower bound is unique?

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

1. The assumption of this theorem would be the statement: "Suppose S is a subset of $\mathbb{R}$"", and the conclusion would be "If S has a least upper bound, then this least upper bound is unique."
2. The proper definition of "least upper bound" would be a value that is greater than all the elements of a set but also such that there are no lower values that are an upper bound.
3. The goal of the first sentence is to declare variables for later usage.
4. This is a correct deduction because, as is described in the defining of x and y, they both have the status of least upper bound and therefore simply being an upper bound, hence one must be greater than or equal to the other.
5. This statement is true because the status of being a least upper bound is simply a special case of an upper bound; for example, If a is a least upper bound, then a is an upper bound.
6. Surprisingly, we could modify this proof to prove that S has a greatest lower bound by simply replacing all occurrences of "least upper bound" with "greatest lower bound." Quite amusing.

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

1. What are the assumptions and conclusion of this theorem?

   Answer. The assumptions are that $S \subset \mathbb{R}$ and that S has a least upper bound. The conclusion is that there is only *one* real number which satisfies the definition of the least upper bound of S. In other words, the conclusion is that, if x and y are any two numbers satisfying the definition of the least upper bound of S, then we must have $x = y$.

2. Give the precise definition of "least upper bound".

   Answer. If $S \subset \mathbb{R}$, a number x is called the *least upper bound* of S if x is an upper bound for S (which means that $x \geq s$ for all $s \in S$) and if, for every $z \in \mathbb{R}$ which is an upper bound of S, we have that $x \leq z$.

3. What is the goal of the first sentence?

   Answer. The first sentence declares the variables x and y and assumes that they are least upper bounds of S, which means that they satisfy the definition you gave in question (2) above. We assume that there are *two* least upper bounds because this is a uniqueness proof, so we wish to show that x and y must be equal.

   Note that the subset S was not declared in the proof, but is still used in the first sentence. It is acceptable to do this if a variable was declared in the statement of the theorem, which is true in this case: the theorem said "Suppose S is a subset of $\mathbb{R}$", so we can treat this as a declaration of S without repeating it.

4. Explain why the deduction in the second sentence, $x \leq y$, is true. Give as much detail and precision as you can.

   Answer. We have assumed that x is a least upper bound and y is a least upper bound. From the definition above, we see that every least upper bound is also an upper bound, so y is also an upper bound.

   Next, the definition of least upper bound requires that, for every upper bound z of S, we must have $x \leq z$. Since y is an upper bound, then, this sentence must be true when z is replaced by y. Thus we conclude that $x \leq y$.

5. The parenthetical comment after the deduction $y \leq x$ says "y is a least upper bound and x is an upper bound", but we assumed that x was a *least* upper bound, not an upper bound. Why is it correct to say this? Justify your explanation using the definitions of upper bound and least upper bound.

   Answer. By definition, a variable z is an upper bound of a subset $S \subseteq \mathbb{R}$ if $z \geq s$ for all $s \in \mathbb{R}$. But we see that the definition of least upper bound given in question 2 assumes that this is true. Thus every least upper bound is also an upper bound.

   Since we assumed that x was a least upper bound, then, it is also true that x is an upper bound, and the parenthetical comment is correct.

6. How could we modify this proof to show that, if $S \subset \mathbb{R}$ has a *greatest lower bound*, then this greatest lower bound is unique?

   Answer. The proof goes through almost identically, except with every instance of "least" replaced with "greatest", "upper" replaced with "lower", and every $\leq$ replaced by $\geq$.

   More precisely, here is the modified proof:

   Theorem. Suppose S is a subset of $\mathbb{R}$. If S has a greatest lower bound, then this greatest lower bound is unique.

   Proof. Suppose x and y are each greatest lower bounds of S. Then $x \geq y$ (since x is a greatest lower bound and y is a lower bound), but also $y \geq x$ (since y is a greatest lower bound and x is a lower bound). Since $x \geq y$ and $y \geq x$, we must have $x = y$. So the greatest lower bound is unique. $\square$

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*