question 2 views

Daily Challenge 9.5

(**Due: Saturday 7/14 at 12:00 noon Eastern**)

Today we continue the Spivak reading.

(1) Problem: Spivak reading on limits.

Continue reading this excerpt from Spivak which we started yesterday, and answer the following reading questions.

(a) Spivak scaffolds the proof that $f(x) = \frac{1}{x}$ approaches $\frac{1}{a}$ as $x \to a$, assuming $a \neq 0$. After giving some hints, he says

> "You should be able to check first that $x \neq 0$ and that $\frac{1}{|x|} < \frac{2}{|a|}$, and then work out the rest of the argument."

Do so; check the two given conditions and write out the complete argument that $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ for any $a \neq 0$.

(Aside: this is problem 1(c) on consolidation document 2.)

(b) To prove that $\lim_{x\to 0} \left(x \sin\left(\frac{1}{x}\right)\right) = 0$, Spivak first claims that

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$$

for all $x \neq 0$, and then says it follows that

$$\left|x\sin\left(\frac{1}{x}\right)\right| \leq |x|$$

for all $x \neq 0$.

Explain these two steps: why is the first line true? Why does the second follow?

(c) Unpack the meaning of the statement

> "We could do even better, and allow $|x| < \frac{1}{\sqrt{10}}$ and $x \neq 0$, but there is no particular virtue in being as economical as possible."

In particular, what is Spivak referring to in the phrase "being as economical as possible," and why is "being economical" unnecessary in order to prove that the given limit exists?

daily_challenge

Updated 9 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Proof: The end goal of proving $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ except for $a = 0$ is to show that $\frac{1}{x}$ is continuous with an exception at zero. First by the the definition of limit, $|x - a| < \delta$ and in this case we would like to make a delta that changes with our target value a and isn't zero, so we say $|x - a| < \frac{|a|}{2}$. This implies that $-\frac{|a|}{2} + a < x < \frac{|a|}{2} + a$ and $x \neq 0$ when $a \neq 0$ as the proposition tells. Now we can go on and show that $|f(x) - L|$ is less than some ϵ, in this case $|\frac{1}{x} - \frac{1}{a}| < \epsilon$. Then $|\frac{1}{x} - \frac{1}{a}| = |\frac{a-x}{ax}| = \frac{1}{a} \times \frac{1}{|x|} \times |x - a|$ We have previously restricted $|x - a|$ so that $x \neq 0$, so $\frac{1}{a} \times \frac{1}{|x|} \times |x - a| < \frac{1}{a} \times \frac{1}{|x|} \times \frac{|a|}{2}$. Fortunately everything is in line here because we have already shown that our δ does not permit $x = 0$. We now have that $|\frac{1}{x} - \frac{1}{a}| < \frac{1}{a} \times \frac{1}{|x|} \times \frac{|a|}{2}$ for $|x - a| < \frac{|a|}{2}$. $\square$

b: The first line is true since the range of $\sin$ is $[-1, 1]$, and because of the absolute value this domain is then $[0, 1]$. By intuition the second line is true simply because both sides of the above inequality have been multiplied by x.

c: He is referring to that $\frac{1}{\sqrt{10}}$ works perfectly to limit x since $\left(\frac{1}{\sqrt{10}}\right)^2 = \frac{1}{10}$.

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) For completeness, we review the beginning of Spivak's argument: we are attempting to bound $\left|\frac{1}{x}-\frac{1}{a}\right|$ by making $|x-a|$ small.

We begin by imposing $0<|x-a|<\frac{|a|}{2}$. With this assumption, x certainly cannot be zero. To see this, we can use the "reverse triangle inequality" $|u|-|v|\leq|u-v|$ with $u=a$ and $v=x$ to find

$$|a|-|x|\leq|x-a|<\frac{|a|}{2},$$

or after adding $|x|$ and subtracting $\frac{|a|}{2}$ from the left-most and right-most sides,

$$|x|>\frac{|a|}{2}.$$

This proves the first part of Spivak's request ("You should be able to check first that $x\neq 0$...").

Next we are asked to show that $\frac{1}{|x|}<\frac{2}{|a|}$. But we may simply take the inequality above, $|x|>\frac{|a|}{2}$, and invert both sides (which reverses the direction of the inequality) to find

$$\frac{1}{|x|}<\frac{2}{|a|},$$

which is exactly what we wanted to prove.

Finally, Spivak asks us to "work out the rest of the argument." We have shown that, if $0<|x-a|<\delta_1=\frac{|a|}{2}$, then we have the bound $\frac{1}{|x|}<\frac{2}{|a|}$. In this case, one has

$$\begin{aligned}\left|\frac{1}{x}-\frac{1}{a}\right|&=\frac{|x-a|}{|x||a|}\\&<\frac{2|x-a|}{|a|^2}.\end{aligned}$$

Now let $\epsilon>0$ be given. We see that, to guarantee $\left|\frac{1}{x}-\frac{1}{a}\right|<\epsilon$, we should impose both the previous assumption $0<|x-a|<\delta_1=\frac{|a|}{2}$ *and* the new assumption $0<|x-a|<\frac{\epsilon|a|^2}{2}$. In other words,

$$\begin{aligned}&0<|x-a|<\delta,\\&\delta=\min\left(\frac{|a|}{2},\frac{\epsilon|a|^2}{2}\right).\end{aligned}$$

With this choice of δ, we are guaranteed that $\left|\frac{1}{x}-\frac{1}{a}\right|<\epsilon$. This completes the argument.

(b) The first claim, $\left|\sin\left(\frac{1}{x}\right)\right|\leq 1$ for $x\neq 0$, is true simply because $-1\leq\sin(y)\leq 1$ for all y. The second inequality follows by multiplying both sides by $|x|$, which preserves the inequality since $|x|\geq 0$.

(c) By the phrase "being economical" Spivak means "choosing the weakest restriction on x necessary to give the desired result."

More specifically: we are trying to find a condition on x so that $|x|^2<\frac{1}{10}$. The weakest condition which guarantees this is $|x|<\frac{1}{\sqrt{10}}$. However, instead of using this condition, we impose the stronger condition $|x|<\frac{1}{10}$.

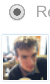
It may seem like we are being "uneconomical" by asking for more than we need; we only need $|x|$ to be smaller than $\frac{1}{\sqrt{10}}$, but we ask for the tighter restriction $|x|<\frac{1}{10}$.

The comment that "being economical" is unnecessary is justified in light of the definition of the limit. We need only prove that there exists *some* δ which guarantees that $f(x)$ is within distance ϵ of the limiting value. Nowhere in the definition does it say that we need to pick the "best" δ which just barely makes this result hold; we can always choose a smaller δ and the proof will still go through.

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 8 months ago

Feedback:

> a: Proof: The end goal of proving $\lim_{x\to a}\frac{1}{x}=\frac{1}{a}$ except for $a=0$ is to show that $\frac{1}{x}$ is continuous with an exception at zero. First by the the definition of limit, $|x-a|<\delta$

It doesn't make sense to assert that $|x-a|<\delta$ "by the definition of the limit", since the existence of the limit is what you're trying to prove!

> and in this case we would like to make a delta that changes with our target value a and isn't zero, so we say $|x-a| < \frac{|a|}{2}$. This implies that $-\frac{|a|}{2} + a < x < \frac{|a|}{2} + a$ and $x \neq 0$ when $a \neq 0$ as the proposition tells.

Good, so you're making an initial choice $\delta \leq \frac{|a|}{2}$.

> Now we can go on and show that $|f(x) - L|$ is less than some ϵ, in this case $|\frac{1}{x} - \frac{1}{a}| < \epsilon$. Then $|\frac{1}{x} - \frac{1}{a}| = |\frac{a-x}{ax}| = \frac{1}{a} \times \frac{1}{|x|} \times |x-a|$. We have previously restricted $|x-a|$ so that $x \neq 0$, so $\frac{1}{a} \times \frac{1}{|x|} \times |x-a| < \frac{1}{a} \times \frac{1}{|x|} \times \frac{|a|}{2}$. Fortunately everything is in line here because we have already shown that our δ does not permit $x = 0$. We now have that $|\frac{1}{x} - \frac{1}{a}| < \frac{1}{a} \times \frac{1}{|x|} \times \frac{|a|}{2}$ for $|x-a| < \frac{|a|}{2}$. $\square$

Okay, but I thought you said you were going to show that $|f(x) - L| < \epsilon$? The proof is incomplete.

> b: The first line is true since the range of $\sin$ is $[-1, 1]$, and because of the absolute value this domain is then $[0, 1]$. By intuition the second line is true simply because both sides of the above inequality have been multiplied by x.

No need to say "by intuition" here -- the operation of multiplying both sides of an inequality by $|x|$ is totally rigorous.

> c: He is referring to that $\frac{1}{\sqrt{10}}$ works perfectly to limit x since $\left(\frac{1}{\sqrt{10}}\right)^2 = \frac{1}{10}$.

In particular, although $\frac{1}{\sqrt{10}}$ "works perfectly", we are allowed to be "less economical" and use $\frac{1}{10}$ instead. The definition of the limit does not require us to find a δ which "works perfectly"; any smaller δ will do just fine.

Overall:

- (a) is 3/6, which should be revised to a 5/6 since it is a CD problem.
- (b) is 5/6 which is fine.
- (c) is 4/6 which is fine.