question 1 views

Daily Challenge 27.1

(**Due: Saturday 4/13 at 12:00 noon Eastern**)

There are other integral transforms besides Fourier. If $f(x)$ is any real-valued function, we can consider the expression

$$g(t) = \int_a^b f(x)K(x,t)\,dt,$$

where $K(x,t)$ is some function called the *kernel*. If the integral exists, we call it the integral transform of $f(x)$ with kernel $K(x,t)$.

Example 1. If $K(x,t) = e^{-itx}$, $a = -\infty$, and $b = +\infty$, this is the usual Fourier transform:

$$g(t) = \int_{-\infty}^{\infty} f(x)e^{-itx}\,dx = \mathcal{F}[f],$$

where we use the variable t rather than the usual k.

Example 2. If $K(x,t) = e^{-tx}$, $a = 0$, and $b = \infty$, this is called the Laplace transform:

$$g(t) = \int_0^{\infty} f(x)e^{-tx}\,dx = \mathcal{L}[f].$$

Although it looks very similar to Fourier except for a factor of i and lower bound of 0 rather than $-\infty$, the Laplace transform has different properties.

Example 3. If $K(x,t) = x^{t-1}$, $a = 0$, and $b = \infty$, this is the Mellin transform:

$$g(t) = \int_0^{\infty} x^{t-1} f(x)\,dx = \mathcal{M}[f].$$

The Mellin transform is less common than the Fourier and Laplace transform, but it comes up in algorithm analysis in theoretical computer science and in the AdS/CFT correspondence of string theory. Sav has some idea that Mellin transforms should be useful for studying Witten diagrams.

(**Part a**) What is the Mellin transform of $f(x) = e^{-x}$?

[Hint: write down the integral using the definition above, but don't try to evaluate it. Instead, recognize this integral as the definition of a special function we've seen before.]

(**Part b**) The *Heaviside step function*, usually written as $\theta(x)$ in theoretical physics, just returns 0 if the input is negative and 1 if the input is positive:

$$\theta(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Compute the Laplace transform $\mathcal{L}[\theta]$.

[Answer: $\mathcal{L}[\theta] = \frac{1}{t}$]

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Updated 3 days ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

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