question 2 views

Daily Challenge 17.2

(**Due: Monday 11/5 at 12:00 noon Eastern**)

(1) Problem: intuition, integrals, and area.

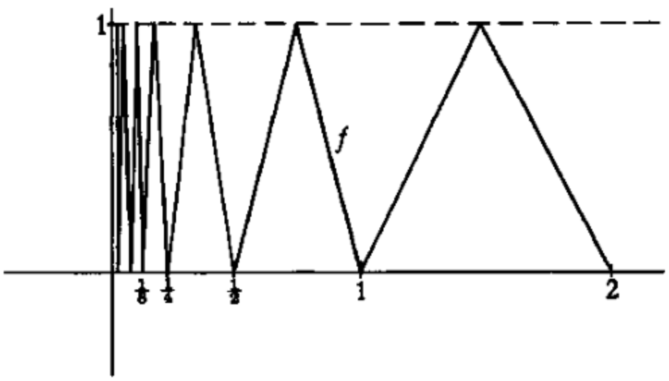
Try to come up with plausible arguments for the following two questions. You don't need to be totally rigorous and justify your answers by explicitly building partitions (although you can if you want). For this challenge, it's enough to use semi-rigorous geometrical reasoning about areas.

(a) Explain why $\int_0^x \frac{\sin(t)}{t+1}\,dt > 0$ for all x.

In other words: show that, if you find the signed area under $\frac{\sin(t)}{1+t}$ between $t=0$ and any positive point $t=x$, you will always get a positive result (even though the graph itself is sometimes negative).

[Hint: the graph switches between positive and negative values, but the amplitude dies down because of the $\frac{1}{t+1}$ suppressing factor.]

(b) Consider the function f shown below.



In words: f is made of infinitely many triangles with height 1 and bases $[1,2]$, $[\frac{1}{2},1]$, $[\frac{1}{4},\frac{1}{2}]$, $[\frac{1}{8},\frac{1}{4}], \cdots, [\frac{1}{2^{n+1}},\frac{1}{2^n}], \cdots$; all "stitched together."

Do you think f is integrable? Why or why not?

If you think it's integrable, what is the area? (No partition or epsilon proof needed; just write down the number and explain why it should be the answer.)

daily_challenge

Updated 5 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a): I believe that because of the initial area being positive, each "hill" will consistently always be larger than the following valley, and due to the high initial area eventually the area of the hill and valley will be pseudo equal, but overall the area is positive due to the infinitely high initial height.

(b): I do believe that this is integrable, as at every point expect for $x=0$ the area can be well defined with incredibly large partitions, and at $x=0$, the area will be zero; I believe that the area is going to be equal to the sum of all triangles prior to the point of interest, plus the shape making up the area between where the counted triangles stop and x. We see by semi-basic geometric intuition that the area will be $\frac{1}{2} * \text{ triangle endpoint } + \text{ current polygon}.$

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) First note that $\sin(t) > 0$ on $(0, \frac{\pi}{2})$. Thus

$$\int_0^x \frac{\sin(t)}{t+1}\,dt > 0$$

for $x \in (0, \frac{\pi}{2})$ since the integrand is positive here.

Next, note that the integral $\int_{\pi/2}^{\pi} \sin(t)\, dt$ is exactly the negative of $\int_0^{\pi/2} \sin(t)\, dt$ owing to the symmetries of the sine function, while the suppressing factor $\frac{1}{t+1}$ is smaller on $(\frac{\pi}{2}, \pi)$ than on $(0, \frac{\pi}{2})$, so the entire integral

$$\int_{\pi/2}^{\pi} \frac{\sin(t)}{t+1}\, dt$$

is smaller in absolute value than

$$\int_0^{\pi/2} \frac{\sin(t)}{t+1}\, dt$$

This means that the integral must be positive for any $x \in (\frac{\pi}{2}, \pi)$. An analogous argument holds on $(\pi, \frac{3\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$, and so on. The integral is therefore positive for any $x > 0$.

(b) The function certainly seems integrable. Its area should be the sum of infinitely many triangles, all with height 1 but with bases $1, \frac{1}{2}, \frac{1}{4}, \cdots$. But we know a trick for adding up infinitely many terms of this form. Let the area be

$$\begin{aligned} A &= \sum_{i=0}^{\infty} \left(\frac{1}{2} b_i h \right) \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right). \end{aligned}$$

Factor out a $\frac{1}{2}$ inside the parens to find

$$\begin{aligned} A &= \frac{1}{2} \left(1 + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \right) \\ &= \frac{1}{2} (1 + A). \end{aligned}$$

Solving for A, one finds $2A = 1 + A$ and thus $A = 1$.

Updated 5 months ago by Christian Ferko

followup discussions *for lingering questions and comments*