question 2 views

Daily Challenge 5.2

(**Due: Wednesday 5/23 at 12:00 noon Eastern**)

Let's keep at it -- I'd like to get us to the point where we are able to correctly solve proof-based exercises, like those in AoPS chapter 1, by the end of this week.

Review

I've recorded two more videos to practice proof-reading. The first video solves this problem from the textbook:

Exercise 1.1.1. Show that, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq (B \cap C)$.

The second video walks through 1.2.2:

Exercise 1.2.2. Prove that, for any $a \leq b$, the open interval (a, b) is indeed an interval (using our original definition of interval).

For convenience, I repeat the definitions of "interval" and "open interval" here:

Definition. A subset I of $\mathbb{R}$ is called an *interval* if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, we have $x \in I$.

Definition. If $a \leq b$ are real numbers, the *open interval* (a, b) is defined as the set $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

Problem

Consider the following questions, and answer the ones that I have not answered for you.

1. In exercise 1.1.1, we began by assuming $x \in A$ and then showed that it is also true that $x \in (B \cap C)$. But x was just *one* element of A, and the definition of subset requires that *every* element $a \in A$ also belongs to $B \cap C$. Explain why our proof is still valid.

2. The statement of exercise 1.2.2 assumes $a \leq b$ and then considers the open interval (a, b). But if $a = b$, the open interval (a, b) is simply the empty set $\emptyset$. Does the proof still apply in that case?

3. In exercise 1.2.2, we wanted to show that (a, b) satisfies the definition of interval, so we started by assuming c and d belong to (a, b). But the definition of interval was

   "A subset I of $\mathbb{R}$ is called an *interval* if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, we have $x \in I$."

   If the above definition talks about a and b, why does our proof consider points c and d?

   Answer. The definition states a condition which must hold for any pair of points in the interval, but the specific names a and b used in the definition are just *dummy variables* -- we can call them whatever we like. Since we have already used the names a and b in our definition of the interval $I = (a, b)$, we instead choose to call the two points c, d in our proof.

4. Read the following proof of exercise 1.2.3 part (a) in AoPS.

   Exercise. Show that the intersection of any two intervals is an interval.

   Proof. Suppose I and J are intervals and let $K = I \cap J$. Let $a, b \in K$ and $x \in \mathbb{R}$ with $a \leq x \leq b$. To prove that K is an interval, we must show that $x \in K$. Since $a, b \in I$ and I is an interval, we have that $x \in I$. Likewise, since $a, b \in J$ and J is an interval, it follows that $x \in J$. Since $x \in I$ and $x \in J$, we see that $x \in K$, so we conclude that K is an interval. $\square$.

   Explain the step "since $a, b \in J$ and J is an interval, it follows that $x \in J$". Why is it true that a and b must belong to J, given what we have assumed above? Why does it follow that $x \in J$?

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

1. I I don't believe I have any better way to word it than the following: For every element of A, x_1 through to x_n, the statement will be proven true in the same way, ie we are accounting for the whole set A but using x to reach our end goal.
2. Yes, the proof still applies because it is already accounting for an equal a and b, denoted the less than or *equal to*, $\leq$.
3. The points there are simply there to make a point, they can easily be replaced with other variables and have the same meaning, and therefore they hold the status of *dummy variables*.
4. This step, in layman's terms, states that x is an element of the interval J since for it to contain both a and b, it must also contain the values in between a and b, and in this case x is one of those values. Therefore, x is in the interval J.

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My answers follow.

1. In exercise 1.1.1, we began by assuming $x \in A$ and then showed that it is also true that $x \in (B \cap C)$. But x was just *one* element of A, and the definition of subset requires that *every* element $a \in A$ also belongs to $B \cap C$. Explain why our proof is still valid.

   Answer. When we declared the variable x, the only assumption we made about x is that it belongs to A. Therefore, all of the deductions about x which follow apply equally well to *any* element of A. Thus all elements x must satisfy the conclusion $x \in B \cap C$, not just some single x which we have chosen.

   For instance, suppose $A = \{1, 2\}$. When we say "suppose $x \in A$", we could have $x = 1$ or $x = 2$, but we're not saying which. All of the logical deductions about x apply regardless of whether x is 1 or 2. Thus one could imagine applying these arguments twice, one when $x = 1$ and one when $x = 2$, to show that the conclusion is true in both cases. Similar reasoning applies when A has more than two elements.

2. The statement of exercise 1.2.2 assumes $a \leq b$ and then considers the open interval (a, b). But if $a = b$, the open interval (a, b) is simply the empty set $\emptyset$. Does the proof still apply in that case?

   Answer. Yes, the proof is still valid if $a = b$. In this case, we claimed that the empty set $\emptyset$ is an interval. But the definition of an interval is a condition which must be satisfied by any pair of points in the interval. Since the empty set contains no points, this condition is vacuously true, and hence the empty set is indeed an interval.

3. In exercise 1.2.2, we wanted to show that (a, b) satisfies the definition of interval, so we started by assuming c and d belong to (a, b). But the definition of interval was

   "A subset I of $\mathbb{R}$ is called an *interval* if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, we have $x \in I$."

   If the above definition talks about a and b, why does our proof consider points c and d?

   Answer. The definition states a condition which must hold for any pair of points in the interval, but the specific names a and b used in the definition are just *dummy variables* -- we can call them whatever we like. Since we have already used the names a and b in our definition of the interval $I = (a, b)$, we instead choose to call the two points c, d in our proof.

4. Read the following proof of exercise 1.2.3 part (a) in AoPS.

   Exercise. Show that the intersection of any two intervals is an interval.

   Proof. Suppose I and J are intervals and let $K = I \cap J$. Let $a, b \in K$ and $x \in \mathbb{R}$ with $a \leq x \leq b$. To prove that K is an interval, we must show that $x \in K$. Since $a, b \in I$ and I is an interval, we have that $x \in I$. Likewise, since $a, b \in J$ and J is an interval, it follows that $x \in J$. Since $x \in I$ and $x \in J$, we see that $x \in K$, so we conclude that K is an interval. $\square$.

   Explain the step "since $a, b \in J$ and J is an interval, it follows that $x \in J$". Why is it true that a and b must belong to J, given what we have assumed above? Why does it follow that $x \in J$?

   Answer. We have defined $K = I \cap J = \{x \in \mathbb{R} \mid x \in I \text{ and } x \in J\}$. Thus, if a and b belong to K, then by the definition of intersection, a and b must also belong to J.

   The deduction that $x \in J$ follows from the definition of interval: since J is an interval, for any $c, d \in J$ and y such that $c \leq y \leq d$, we have that $y \in J$. In particular, this statement holds when $c = a$, $d = a$, and $y = x$, which means that $x \in J$.

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*