# 27.4

(a): Let's start by evaluating $f'(x), f''(x), f'''(x)$ given $f(x) = \arctan(x)$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$f''(x) = \quad = -2x(1+x^2)^{-2}$$

$$f'''(x) = -2(1+x^2)^{-2} + -2x(1+x^2)^{-3} \cdot -2 \cdot 2x$$

Thus, if

$$\begin{aligned} f(0) &= p(0) \\ f'(0) &= p'(0) \\ f''(0) &= p''(0) \\ f'''(0) &= p'''(0) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 0 &= a_0 \\ 1 &= a_1 \\ 0 &= 2a_2 \\ -2 &= 6a_3 \end{aligned}$$

and thus

$$\arctan(x) \approx x - \frac{1}{3}x^3$$

(b): Refer to soln_27.4-Arctan_Polynomial.png.

(c): We wish to find a Taylor series by looking at the derivative of the function in question. We see that

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

Then consider the sum

$$S = 1 + y + y^2 + \ldots$$

$$= 1 + y(1 + y + y^2 \ldots)$$

$$S = 1 + yS \Rightarrow (1-y)S = 1$$

$$S = \frac{1}{1-y}$$

If we let $y = x^2$, then

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \ldots$$

and by integrating each side:

~~WA~~ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(d): If we let $x=1$, then since $\arctan(1) = \frac{\pi}{4}$,

$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

~~<Aside: The square~~