question 2 views

Daily Challenge 12.5

~~(**Due: Friday 8/24 at 12:00 noon eastern**)~~
(**Due: Tuesday 8/28 at 11:59 pm eastern**)

I will skip the content today to leave you time for one of the harder consolidation document 3 problems. This one involves the *Schwarzian derivative*, which appears in string theory when one considers the behavior of the stress tensor under a finite conformal transformation (e.g. see the discussion around equation (4.32) here).

Note that this is problem 6 in CD 3; please copy over your solution (and preferably change the formatting for Overleaf, e.g. replace double-dollars with dollars where appropriate) when you're done.

(1) Problem: the Schwarzian.

Let f be a C^3 function with $f'(x) \neq 0$. We may define the *Schwarzian derivative* of f at a point x to be

$$\mathscr{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

(a) Compute the Schwarzian derivative of $f(x) = e^{ax}$, where $a \in \mathbb{R}$ is some constant, $a \neq 0$.

(b) Show that

$$\mathscr{D}(f \circ g) = ((\mathscr{D}f) \circ g) \cdot (g')^2 + \mathscr{D}g.$$

(c) Show that, if $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$ then $\mathscr{D}f = 0$.

daily_challenge

Updated 7 months ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Let f be a C^3 function with $f'(x) \neq 0$. We may define the *Schwarzian derivative* of f at a point x to be

$$\mathscr{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

(a) Compute the Schwarzian derivative of $f(x) = e^{ax}$, where $a \in \mathbb{R}$ is some constant.

Solution. If $f(x) = e^{ax}$, then

$$\begin{aligned} f'(x) &= ae^{ax}, \\ f''(x) &= a^2 e^{ax} \\ f'''(x) &= a^3 e^{ax}. \end{aligned}$$

Thus the Schwarzian is

$$\begin{aligned} \mathscr{D}f(x) &= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \\ &= \frac{a^3 e^{ax}}{ae^{ax}} - \frac{3}{2}\left(\frac{a^2 e^{ax}}{ae^{ax}}\right)^2 \\ &= -\frac{1}{2}a^2. \end{aligned}$$

(b) Show that

$$\mathscr{D}(f \circ g) = ((\mathscr{D}f) \circ g) \cdot (g')^2 + \mathscr{D}g.$$

Solution. Because the Schwarzian involves the third derivative, we will need an expression for $(f \circ g)'''$. We know the first derivative by the chain rule, namely

$$(f \circ g)' = f'(g(x)) \cdot g'(x).$$

Now we take the second derivative using the product rule and chain rule:

$$(f \circ g)'' = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

Finally we take the third derivative, again successively applying the chain and product rules:

$$
\begin{aligned}
(f\circ g)''' &= f'''(g(x))\cdot(g'(x))^3 + f''(g(x))\cdot 2g'(x)g''(x) + f''(g(x))\cdot g'(x)\cdot g''(x) + f'(g(x))\cdot g'''(x)\\
&= f'''(g(x))\cdot(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x).
\end{aligned}
$$

In the last step, we have combined the like terms of $f''(g(x))g'(x)g''(x)$.

Now we have all of the pieces and can compute the Schwarzian. By the definition, we have

$$
\mathscr{D}(f\circ g) = \frac{(f\circ g)'''}{(f\circ g)'} - \frac{3}{2}\left(\frac{(f\circ g)''}{(f\circ g)'}\right)^2.
$$

Now we plug in our above expressions for $(f\circ g)'$, $(f\circ g)''$, and $(f\circ g)'''$. This gives

$$
\begin{aligned}
\mathscr{D}(f\circ g) &= \frac{f'''(g(x))\cdot(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)}{f'(g(x))g'(x)} - \frac{3}{2}\left(\frac{f''(g(x))\cdot(g'(x))^2 + f'(g(x))\cdot g''(x)}{f'(g(x))g'(x)}\right)^2\\
&= \frac{f'''(g(x))\cdot(g'(x))^2}{f'(g(x))} + \frac{3f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\left(\frac{f''(g(x))\cdot g'(x)}{f'(g(x))}\right)^2 + 2\frac{f''(g(x))g''(x)}{f'(g(x))} + \left(\frac{g''(x)}{g'(x)}\right)^2\right).
\end{aligned}
$$

In this step, we have split the fraction $\frac{(f\circ g)'''}{(f\circ g)'}$ into three terms and canceled common factors in each numerator and denominator, then expanded out the square in the second term. Now we distribute the factor of $\frac{3}{2}$ to find

$$
\mathscr{D}(f\circ g) = \frac{f'''(g(x))(g'(x))^2}{f'(g(x))} + \frac{3f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{f''(g(x))g'(x)}{f'(g(x))}\right)^2 - 3\frac{f''(g(x))g''(x)}{f'(g(x))} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.
$$

First, notice that the second and fifth terms cancel.

Next, recall that our goal is to show that this expression is equal to $((\mathscr{D}f)\circ g)\,(g'(x))^2 + \mathscr{D}g$ Thus we want to find a term proportional to $(g'(x))^2$. Examining our above expression, we see that the first and fourth terms are proportional to $(g'(x))^2$. Let's factor the $(g'(x))^2$ out from those terms, obtaining

$$
\mathscr{D}(f\circ g) = \left(\frac{f'''(g(x))}{f'(g(x))} - \frac{3}{2}\left(\frac{f''(g(x))}{f'(g(x))}\right)^2\right)(g'(x))^2 + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.
$$

Aha! Comparing to the definition of the Schwarzian, we see that the term in parenthesis is exactly $(\mathscr{D}f)\circ g$, whereas the second and third term are precisely $\mathscr{D}g$. Thus we conclude

$$
\mathscr{D}(f\circ g) = ((\mathscr{D}f)\circ g)\cdot(g'(x))^2 + \mathscr{D}g.
$$

(c) Show that, if $f(x) = \frac{ax+b}{cx+d}$ where $ad-bc\neq 0$ then $\mathscr{D}f = 0$.

Solution. We compute up to the third derivative:

$$
\begin{aligned}
f'(x) &= \frac{a(cx+d)-c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2},\\
f''(x) &= \frac{-2c(ad-bc)}{(cx+d)^3},\\
f'''(x) &= \frac{6c^2(ad-bc)}{(cx+d)^4}.
\end{aligned}
$$

Note that the first step required the quotient rule because both the numerator and denominator depended on x, but after taking the first derivative, the numerator was a constant; thus we only needed to apply the chain rule in the second and third steps.

With these three expressions in hand, we compute the Schwarzian.

$$
\begin{aligned}
\mathscr{D}f(x) &= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\\
&= \frac{6c^2(ad-bc)}{(cx+d)^4}\,\frac{(cx+d)^2}{ad-bc} - \frac{3}{2}\left(\frac{-2c(ad-bc)}{(cx+d)^3}\,\frac{(cx+d)^2}{ad-bc}\right)^2\\
&= \frac{6c^2}{(cx+d)^2} - \frac{3}{2}\left(\frac{-2c}{cx+d}\right)^2\\
&= 0.
\end{aligned}
$$

So the Schwarzian of $f(x)$ is *identically zero*.

Functions of this form $f(x) = \frac{ax+b}{cx+d}$, where $ad-bc\neq 0$, are important in string theory; they are called **Möbius transformations**. These transformations represent the so-called *conformal Killing group* of the sphere; they represent certain transformations from the Riemann sphere to the Riemann sphere.

You may have noticed that the condition $ad-bc\neq 0$ looks a lot like the statement that some matrix $\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ has nonzero determinant, or in other words, that the matrix is invertible. This is no accident! The group of Mobius transformations can also be represented by 2×2 matrices.

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*