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# Daily Challenge 26.3

~~(**Due: Thursday 4/4 at 12:00 noon Eastern**)~~
(**Due: Monday 4/8 at 12:00 noon Eastern**)

No meetings today or this week, but I'll try to keep up with challenges.

In session 59, we discussed the calculus of variations. Consider a *functional*

$$J[y] = \int_{x_1}^{x_2} L(x, y(x), y'(x))\, dx$$

which takes as input a function $y(x)$ and returns as output a number $J[y]$ obtained by integrating the *Lagrangian* $L$.

We argued that the functional $J$ has a critical point (so it is maximized or minimized) when the *Euler-Lagrange equation*,

$$\frac{\delta L}{\delta y} - \frac{d}{dx}\left(\frac{\delta L}{\delta y'}\right) = 0,$$

is satisfied.

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(**Part a**) Fix two points $(x_1, y_1)$ and $(x_2, y_2)$. The *length functional* for a curve $y(x)$ connecting these two points is

$$L[y] = \int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx,$$

as we argued earlier in this chapter.

Use the Euler-Lagrange equation to show that the length is extremized when

$$\frac{y'(x)}{\sqrt{1+(y'(x))^2}} = C$$

for some constant $C$. (In fact, this is a minimum rather than just an extremum, though you need not justify this.)

Now solve that equation for $y'(x)$ to show that, to minimize the length, we need

$$y'(x) = \pm\frac{C}{\sqrt{1-C^2}}$$

The right side of this equation is just another constant, which we may as well rename $m$. So you have shown we need $y'(x) = m$, or $y = mx + b$ for some $m$ and $b$.

Punchline: you have proven that the curve which minimizes the distance between two points in the plane is a straight line with constant slope $m$. Or, in the hopelessly imprecise but more common wording, "the shortest distance between two points is a straight line."

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(**Part b**) Let $x(t)$ be the position of a particle at time $t$, and suppose the particle is subject to a potential $U(x)$.

The total energy, kinetic plus potential, is of course $E = \frac{1}{2}m\dot{x}^2 + U(x)$. Is there a meaning to the *difference*, i.e. kinetic energy minus potential energy?

Define the *action functional*,

$$S[x] = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - U(x)\right)\, dx.$$

Don't get confused in comparing this equation to the definition above: now $x(t)$ (not $y(x)$) is the function, and $t$ (not $x$) is the independent variable.

Apply the Euler-Lagrange equation to find the conditions on $x(t)$ which extremize the action. Using the definition of force, $F(x) = -\frac{dU}{dx}$, show that your equation can be written as

$$F(x) = m\ddot{x},$$

or $F = ma$.

Punchline: you have derived Newton's second law from a variational principle.

daily_challenge

Updated 5 days ago by Christian Ferko

**the students' answer,** *where students collectively construct a single answer*

kek

Updated 5 days ago by Logan Pachulski

**the instructors' answer,** *where instructors collectively construct a single answer*

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