

27.4

(a) Let's start by evaluating $f'(x), f''(x), f'''(x)$ given $f(x) = \arctan(x)$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$f''(x) = -2x(1+x^2)^{-2}$$

$$f'''(x) = -2(1+x^2)^{-2} + -2x(1+x^2)^{-3} \cdot 2x$$

Thus, if

$$f(0) = p(0)$$

$$0 = a_0$$

$$f'(0) = p'(0)$$

$$1 = a_1$$

$$f''(0) = p''(0)$$

$$0 = 2a_2$$

$$f'''(0) = p'''(0)$$

$$-2 = 6a_3$$

and thus

$$\arctan(x) \approx x - \frac{1}{3}x^3$$

(b) Refer to solh_27.4-Arctan_Polynomial.png.

(c) We wish to find a Taylor series by looking at the derivative of the function in question. We see that

$$\frac{d}{dx} (\arctan(x)) = \frac{1}{1+x^2}$$

Then consider the sum

$$S = 1 + y + y^2 + \dots$$

$$= 1 + y(1 + y + y^2 + \dots)$$

$$S = 1 + yS \Rightarrow (1-y)S = 1$$

$$S = \frac{1}{1-y}$$

If we let $y = -x^2$, then

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

and by integrating each side:

$$\cancel{111} \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

(d) If we let $x=1$, then since $\arctan(1) = \frac{\pi}{4}$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

~~Aside: The square~~