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243de

(d): We are told to find the value N such that

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \begin{cases} N\sqrt{R^2-x^2} & \text{if } -R \le x \le R \\ 0 & \text{if else} \end{cases}$$

$$= \int_{-R}^{R} N\sqrt{R^2-x^2}\,dx$$

Then we substitute $x = R\sin(\theta) \Rightarrow dx = R\cos(\theta)$, and the endpoints become $[\arcsin(-1), \arcsin(1)] = [\frac{-\pi}{2}, \frac{\pi}{2}]$

$$= N\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{R^2(1-\sin^2(\theta)}\; R\cos(\theta)\,d\theta$$

$$= NR^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\,d\theta$$

then recall $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$ implies

$$= \frac{NR^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta)$$

$$= \frac{NR^2}{2}\left(+\pi + \left[\frac{1}{2}\sin(2\theta)\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$1 = \frac{NR^2(+\pi)}{2}$$

$$N = \frac{+2}{R^2\pi}$$

(e): We are to evaluate the integral

$$\frac{2}{\pi R^2}\int_{-R}^{R} x^2 \left(\sqrt{R^2-x^2}\right) dx$$

Trig-sub; $x = R\sin(\theta)$ $\quad dx = R\cos(\theta)$

$$= \frac{2}{\pi R^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2 \sin^2(\theta)\sqrt{R^2(1-\sin^2(\theta))}\; R\cos(\theta)\, d\theta$$

$$= \frac{R^2 2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\theta)\cos^2(\theta)$$

Recall $\sin(2\theta) = 2\sin(\theta)\cos(\theta) \Rightarrow \dfrac{\sin^2(2\theta)}{4} = \sin^2(\theta)\cos^2(\theta)$

$$= \frac{R^2 2}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta$$

Then by the cosine double angle identity; $\cos(2\theta) = 1 - 2\sin^2(\theta)$

$$\sin^2(2\theta) = \frac{1-\cos(4\theta)}{2}$$

$$= \frac{R^2 1}{4\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(4\theta)\right)$$

$$= \frac{R^2 1}{4\pi}\left(\pi - \frac{1}{4}\Big[\sin(4\theta)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$= \frac{R^2 \cdot 1}{4}$$