282

(a): ~~Be~~gin with

$$\lim_{n\to\infty}\left(\frac{n}{n+1}-\frac{n+1}{\cancel{A}}\right)$$

then multiply by 1,

$$\ldots=\lim\left(\frac{n}{n+1}-\frac{(n+1)(\frac{1}{n}+1)}{n(\frac{1}{n}+1)}\right)=\lim\left(\frac{n-(1+n+\frac{1}{n}+1)}{n+1}\right)$$

$$=\lim\left(\frac{-2-\frac{1}{n}}{n+1}\right)$$

Both $\lim(-2-\frac{1}{n})$ and $\lim(n+1)$ don't equal zero in the range, so

$$\ldots=\frac{\lim(-2-\frac{1}{n})}{\lim(n+1)}\overset{?}{=}\frac{-2}{\cancel{n+1}\infty}=0$$

By addition and noting that ~~x = n~~ n dominates 1.

(b): Let's do some casework; ~~we~~ we wish to evaluate

$$\lim\left(\frac{a^n-b^n}{a^n+b^n}\right)$$

• Start with $|a|>|b|$; multiply top and bottom by $\frac{1}{a^n}$

$$\ldots=\lim\left(\frac{1-\frac{b^n}{a^n}}{1+\frac{b^n}{a^n}}\right)$$

then split the limit:

$$\ldots=\lim\frac{1}{1+\frac{b^n}{a^n}}-\lim\frac{b^n/a^n}{1+\frac{b^n}{a^n}}$$

The right limit clearly goes to zero, since $|a|>|b|$ tells us that $b^n/a^n\to 0$. Similarly, we can split

$$\lim\frac{1}{1+\frac{b^n}{a^n}}=\ldots=\frac{1}{1+\lim\frac{b^n}{a^n}}=1,$$ so we have

$$\lim\left(\frac{a^n-b^n}{a^n+b^n}\right)=1-0=1 \text{ if } |a|>|b|$$

• Now for the second case: $|a|<|b|$; divide ~~at~~ top and bottom by b^n

$$\lim\left(\frac{a^n-b^n}{a^n+b^n}\right)=\lim\left(\frac{\frac{a^n}{b^n}-1}{\frac{a^n}{b^n}+1}\right)=\frac{\lim(\frac{a^n}{b^n})-1}{\lim(\frac{a^n}{b^n})+1}$$, since $|a|<|b|$, then

$$\ldots=\frac{0-1}{0+1}=-1$$

(c): ~~Name~~ some Cauchy sequence a_n; ~~then by the definition of cauchy sequence $\lim(a_n) = L$~~ thus

$a_{n_1}, a_{n_2}, a_{n_3}, \dots \to L$;

then we wish to show that

$\lim(a_n) = L.$

Let $\epsilon > 0$ be given; then choose N_j such that

$|a_{n_j} - L| < \frac{\epsilon}{2}$ for $n_j > N_j$.

Then since we are told a_n is Cauchy, choose N_n such that

$|a_n - a_m| < \frac{\epsilon}{2}$ where $n, m > N_n$

Then let $N' = \max(N_j, N_n)$; then for $n > N'$,

$|a_n - L| \leq |a_n - a_{N_{j+1}}| + |a_{N_{j+1}} - L| < \epsilon$; thus

$|a_n - L| < \epsilon \Rightarrow \lim(a_n) = L$

(d): We are told ~~told~~ that $a_n \to L$; let a_{n_j} be a subsequence of $(a_n)_{n=1}^{\infty}$.
Let $\epsilon > 0$ be given; then by the definition of convergence there exists N such that

$|a_n - L| < \epsilon$ for $n > N$;

then since $a_n < a_{n+1}$, there exists J such that $n_j > N$ for $j > J$.
Thus $|a_{n_j} - L| < \epsilon$ for $j < J$ and we conclude that
$\lim_{j \to \infty}(a_{n_j}) = L. \square$