question 2 views

Daily Challenge 22.7

(**Due: Tuesday 2/26 at 12:00 noon Eastern**)

You found via Monte Carlo that the volume of an n-ball with radius $R = 1$ is maximized in $n = 5$ dimensions. However, this answer will be different if we change the radius. In this problem, you'll find the dimension which maximizes the volume as a function of radius R.

Recall that the volume of an n-ball of radius R is

$$V(n, R) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^n.$$

We know that the gamma function is defined for any positive real argument, not just integers and half-integers. So promote this volume function to depend on an arbitrary *real* number x,

$$V(x, R) = \frac{\pi^{x/2}}{\Gamma(\frac{x}{2}+1)} R^x.$$

We would like to take the derivative $\frac{dV}{dx}$ and set it equal to zero in order to maximize the volume for a given R. However, it will actually be more convenient to maximize $\log(V)$. Since the logarithm is strictly increasing, the maximum of $\log(V)$ occurs at the same place as the maximum of V.

(a) Compute the derivative $\frac{d}{dx}(\log V(x, R))$.

(b) Define the new function

$$\psi(y) = \frac{\Gamma'(y)}{\Gamma(y)}.$$

Show that $V(x, R)$ is maximized when the equation

$$\psi\left(\frac{x}{2}+1\right) = \log(\pi) + 2\log(R)$$

is satisfied.

(c) This function ψ is called the digamma function and can be called in Wolfram Alpha using the syntax `digamma`.

Plot the equation you found in part (b), namely

$$\psi\left(\frac{x}{2}+1\right) = \log(\pi) + 2\log(R).$$

when $R = 1$, for $0 < x < 10$ using Wolfram Alpha. Verify that the solution is close to $x = 5$, which is the dimension you found to maximize the unit n-ball volume. (Solution here.)

daily_challenge

Updated 1 month ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Refer to dm; I shall type up after launch

Updated 1 month ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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