24.4

For all of these we begin with the integral

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}(x-u)^2\exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)dx$$

(a): Begin by substituting $u = \frac{(x-u)^2}{2\sigma^2} \Rightarrow du = \frac{x-u}{2\sigma^2}\,dx$

Split ~~one of the~~ the $(x-u)^2$ for clarity:

$$\dots = \int_{-\infty}^{\infty}(x-u)\exp\left(\frac{-(x-u)^2}{2\sigma^2}\right)(x-u)\,dx \cdot \frac{1}{\sqrt{2\pi\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\sqrt{2\sigma^2 u}\exp(-u)\,2\sigma^2\,du$$

$$= \frac{1}{\sqrt{\pi}}\cdot 2\sigma^2\int_{-\infty}^{\infty}u^{(3/2-1)}e^{-u}\,du$$

$$= \frac{1}{\sqrt{\pi}}\cdot 2\sigma^2\,\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}}\cdot 2\sigma^2\cdot\frac{1}{2}\cdot\sqrt{\pi} = \sigma^2$$

(b): Substitute

$u = \frac{(x-u)}{\sqrt{2}\,\sigma} \Rightarrow du = \frac{1}{\sqrt{2}\sigma}dx$ to get

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}2\sigma^2u^2\exp(-u^2)\,\sqrt{2}\sigma\,du$$

$$\dots = \cancel{\frac{1}{\sqrt{\pi}\sigma}2\sigma^2} = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sigma}\cdot\frac{2\sigma^2}{1}\cdot\frac{\sigma}{1}\int_{-\infty}^{\infty}u^2\exp(-u^2)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}u\cdot ue^{-u^2}$$

We now have to solve

$$\frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u \cdot u e^{-u^2} \, du$$

Integrate by parts where

$A' = u e^{-u^2}$

$\cancel{A = u e^{-u^2} \cdot \frac{1}{-u^2}}$, check $\left(\frac{-1}{2}(e^{-u^2})\right)' = e^{-u^2} \cdot -2u \cdot \frac{-1}{2} = u e^{-u^2}$

$A = \frac{-1}{2} e^{-u^2}$

$B = u$

$B' = 1$

$$\ldots = \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[u \frac{-1}{2} e^{-u^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-1}{2} e^{-u^2} \, du \right)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \int_{-\infty}^{\infty} e^{-u^2} \, du$$

$$= \sigma^2$$

(c): We are given the equation

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-u)^2}{2\sigma^2} \right) dx = 1$$

We are told to take the derivative of each side with respect to σ!

$$\frac{d}{d\sigma} \left(\int_{-\infty}^{\infty} \exp\left(-\frac{(x-u)^2}{2\sigma^2} \right) dx \right) = \frac{d}{d\sigma}(1)$$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x-u)^2}{2\sigma^2} \right) \cdot \frac{-(x-u)^2}{2} \cdot (\sigma^{-2})' = 0$$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x-u)^2}{2\sigma^2} \right) \cdot (x-u)^2 \cdot \frac{1}{\sigma^3} \, dx \neq 0$$

We also need to find

$$\left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)' = \left(\frac{1}{\sqrt{2\pi}} \cdot \sigma^{-1} \right)' = \frac{-1}{\sqrt{2\pi}} \cdot \sigma^{-2} = \frac{-1}{\sqrt{2\pi}\,\sigma^2}$$

$$\frac{d}{d\sigma}\left(\cdots\right) = \frac{-1}{\sqrt{2\pi\sigma^2}}\int \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right) + \frac{1}{\sqrt{2\pi\sigma^2}}\int \exp\left(\frac{-(x-u)^2}{2\sigma^2}\right)\frac{(x-u)^2}{\sigma^3}\,dx$$

$$0 = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int -\exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)dx + \int \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)\frac{(x-u)^2}{\sigma^2}\right.$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)(x-u)^2\,dx - \int \sigma^2 \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)\right.$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int \left((x-u)^2 - \sigma^2\right)\exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)$$

Then we see that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}(x-u)^2\exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)dx = \sigma^2\int\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)$$

$$= \sigma^2$$

Proving what we wanted to.