question 2 views

Daily Challenge 22.3

(**Due: Friday 2/22 at 12:00 noon Eastern**)

Let's review the definitions of supremum and infimum.

(1) Bounds, suprema, and infima.

Definition. We say that a set of real numbers A is *bounded above* if there exists a number x such that

$$x \geq a \text{ for every } a \in A.$$

Such a number x is called an *upper bound* for A.

Any set with an upper bound will have many upper bounds. For instance, if $A = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, then clearly 200 is an upper bound for A since 200 is greater than every element in A. But 2 is also an upper bound for A, as is 1, as is $\frac{55}{2}$.

It is useful to define a unique notion of upper bound that avoids this ambiguity.

Definition. A number x is a *least upper bound* or *supremum* of A if

1. x is an upper bound of A, and
2. if y is an upper bound of A, then $x \leq y$.

In other words: the supremum of a set A is the smallest number x with the property that every number $a \in A$ is less than or equal to x.

These two properties are the *only* definition of supremum.

- The supremum of a set A is *not* the largest element in a set: for instance, if $A = (0, 1)$, then $\sup(A) = 1$ but clearly 1 is not in the set.
- The supremum of a set is *not* the "next number that comes after the set." There is no notion of "next number" in a continuum; for example, what is the "next number" after π?

Said differently, any proof that uses the notion of the supremum $\sup(A)$ of a set must use the two properties that (1) the supremum is an upper bound, and (2) that the supremum is the smallest upper bound. If you have not used both properties, something has gone wrong.

Example 1. Let A be a nonempty set of real numbers and let $\alpha = \sup(A)$. Let $\epsilon > 0$ be given. Prove that the exists some element $a \in A$ such that $a > \alpha - \epsilon$.

Proof 1. Suppose by way of contradiction that there were *no* element $a \in A$ such that $a > \alpha - \epsilon$. This is another way of saying that every element $a \in A$ satisfies $a \leq \alpha - \epsilon$, which means that the number $\alpha - \epsilon$ is an upper bound of A. But $\alpha - \epsilon$ is smaller than α, which contradicts that α is the smallest upper bound of A. $\square$

Example 2. Let $A = (0, 1)$. Prove, directly from the definition, that $\sup(A) = 1$.

Proof 2. To show that $\sup(A) = 1$, we must prove two things: (1) that 1 is an upper bound for A, and (2) that 1 is the smallest upper bound for A.

First, by the definition of open interval we have $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$. This means that, for any $a \in A$, we have $a < 1$. In particular, we have $a \leq 1$, which means that 1 is an upper bound for A.

Now we show that 1 is the *smallest* upper bound. No other number y with $0 < y < 1$ could be an upper bound for A, since the number $\frac{y+1}{2}$ belongs to A and it is not true that $y \geq \frac{y+1}{2}$. Therefore 1 is the least upper bound, proving that it is the supremum. $\square$

We have an entirely analogous definition for lower bounds.

Definition. A set of real numbers A is *bounded below* if there exists a number x such that $x \leq a$ for every $a \in A$.

Definition. A number x is the *greatest lower bound* or *infimum* of a set A if

1. x is a lower bound of A, and
2. if y is any other lower bound of A, then $x \geq y$.

(2) Problem: a Spivak exercise on suprema/infima.

This daily challenge has two parts.

Part I: carefully read section (1) above. Seriously. Read it slowly and don't gloss over definitions or examples. Make sure you understand everything and could explain it to someone else.

Part II: complete this Spivak problem.

- (a) Suppose $A \neq \emptyset$ is bounded below. Let A_{-} denote the set of all $-x$ for x in A:

$$A_- = \{-x \mid x \in A\}.$$

Prove that $A_- \neq \emptyset$, that A_- is bounded above, and that $-\sup(A_-)$ is the greatest lower bound of A.

- (b) If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Show that $B \neq \emptyset$, that B is bounded above, and that $\sup(B)$ is the greatest lower bound of A.

Answer on Overleaf: https://www.overleaf.com/1231232126rckrscxfchyf

daily_challenge

Updated 1 month ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

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Updated 1 month ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) First we prove that $A_- \neq \emptyset$. Since $A \neq \emptyset$, there exists at least one $a \in A$, which means that the corresponding number $-a$ is an element of A_-, and hence A_- is nonempty.

Next we show that A_- is bounded above. Since A is bounded below, there exists some x such that $x \leq a$ for all $a \in A$. Multiplying by -1, this means that $-x \geq -a$ for all $a \in A$. But this means that $-x \geq b$ for all $b \in A_-$, so $-x$ is an upper bound for A_-.

Finally, let $\alpha_- = \sup(A_-)$ be the least upper bound of A_-. Then α_- is an upper bound for A_-, which means $\alpha_- \geq b$ for all $b \in A_-$, and hence $-\alpha_- \leq a$ for all $a \in A$. Thus the number $-\alpha_-$ is a lower bound of A. It is also the greatest lower bound of A, since if there were any other $\beta > -\alpha_-$ such that $\beta \leq a$ for all $a \in A$, then we would have $-\beta \geq b$ for all $b \in A_-$ but $-\beta < \alpha_-$, contradicting that α_- is the least upper bound of A_-. We conclude that $-\alpha_-$ is the infimum of A.

(b) First, $B \neq \emptyset$ because A is assumed to be bounded below, so there exists at least one x such that $x \leq a$ for all $a \in A$; this x is an element of B.

Let a be any element of a. Then B is bounded above by a (that is, every $b \in B$ satisfies $b \leq a$), since we cannot have any lower bound of A which is greater than an element of A.

Now let $\alpha = \inf(A)$; we will argue that $\alpha = \sup(B)$ (clearly B has a supremum, since every set of real numbers with an upper bound also has a least upper bound, by the completeness property). As usual, we must prove two things: that α is an upper bound of B, and that it is the least upper bound.

1. (Proof that α is an upper bound of B.) This is just from the definition of infimum: since α is the greatest lower bound of A, if $b \in B$ is any other lower bound of A, we have $b \leq \alpha$.

2. (Proof that α is the smallest upper bound of B). Suppose α' were another upper bound of B with $\alpha' > \alpha$. This would mean that the greatest lower bound of A is at least $\alpha' > \alpha$. But this contradicts that α was the greatest lower bound of A.

Thus we conclude that $\sup(B) = \inf(A)$. $\square$

Updated 1 month ago by Christian Ferko

followup discussions *for lingering questions and comments*