question 2 views

Daily Challenge 15.3

(**Due: Monday 9/24 at 12:00 noon eastern**)

No challenge due today (Sunday) -- let's skip a day to give you time to catch up. Hopefully we can get back on track by Monday.

Today we have some nice proofs using MVT. This is question 10 on CD 4.

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(1) Problem: the mean value theorem and bounded derivatives.

(a) Suppose f is a differentiable function and m is a real number such that $f'(x) \geq m$ for all $x \in [a, b]$. Prove that $f(b) \geq f(a) + m(b-a)$.

(b) Suppose that f is a differentiable function such that $|f'(x)| \leq 1$ for all $x \in [-1, 1]$. Write an inequality relating $f(1)$ and $f(-1)$.

daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a): Was done on the consolidation document, shall do b here just to have an idea of what I still need to do.

(b): First a little basic thing, by the definition of absolute value, $|f'(x)| \leq 1 \implies -1 \geq f'(x) \leq 1.$ We can then apply the fact we found in the previous problem, and see that for $a = -1$ and $b = 1$, then $f(1) \geq f(-1) + -(1 - (-1))$, therefore $f(1) \geq f(-1) - 2$. Similarly, an alternative version of the above proves gives us the info that it is also true that $f(1) \leq f(-1) + (1 - (-1)) = f(-1) + 2$ Thus $f(-1) - 2 \leq f(1) \leq f(-1) + 2$

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) Suppose by way of contradiction that $f(b) < f(a) + m(b-a)$. By algebra, this means $\frac{f(b)-f(a)}{b-a} < m$.

By the mean value theorem, there exists a point $c \in (a, b)$ such that the tangent line slope at $(c, f(c))$ equals the secant line slope through $(a, f(a))$ and $(b, f(b))$. But the secant line slope is

$$\text{secant line slope } = \frac{f(b) - f(a)}{b - a} < m,$$

where in the last step we used the inequality derived in the first paragraph above. But there can be no point where the tangent line slope is less than m, since $f'(x) \geq m$ on the interval by assumption. □

(b) By directly applying our result from part (a) with $a = -1$, $b = 1$, and $m = -1$, we get the inequality $f(1) \geq f(-1) - 2$.

However, we can do better than this. It is clear that reversing the signs of the inequalities in the argument of part (a) gives an entirely analogous result if there is an *upper* bound on the derivative $f'(x)$ in an interval. This gives the inequality $f(1) \leq f(-1) + 2$.

Combining these two results, we arrive at the inequality

$$f(-1) - 2 \leq f(1) \leq f(-1) + 2.$$

This is the most we can say, so it is our final answer. □

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*