question 2 views

Daily Challenge 25.6

(~~**Due: Friday 3/29 at 12:00 noon Eastern**~~)
(**Due: Saturday 3/30 at 12:00 noon Eastern**)

The Fourier transform of a probability distribution is called the *characteristic function* associated with that PDF.

For instance, a uniform probability distribution on $[a, b]$ is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases},$$

where we chose the constant $\frac{1}{b-a}$ so that the integral is 1 (as it must be, for a PDF).

Its characteristic function is

$$\tilde{f}_X(k) = \int_a^b e^{-ikx} \cdot \left(\frac{1}{b-a}\right)\, dx = \frac{e^{-ikb} - e^{-ika}}{(-ik)(b-a)}.$$

(**Warning**: the standard caveat about complex-number calculus applies. I never proved that you can apply the usual integration rules when there are imaginary or complex numbers around, but this formula turns out to be true. It is important to remember which facts you have proved and which are assumed! Many headaches result from sloppily applying formulas derived for real numbers to new objects and finding that the formulas no longer work.)

We saw in session 58 that the Fourier transform of the square pulse is the sinc function. But we can also interpret the uniform probability distribution as a square pulse. And indeed, when $a = -b$ the above formula reduces to

$$\frac{e^{-ikb} - e^{ikb}}{(-ik)(2b)} = \frac{\sin(bk)}{bk},$$

which is sinc again.

Notice that the characteristic function can also be written as the expectation value of e^{-ikx}:

$$\tilde{f}_X(k) = \int_{-\infty}^{\infty} e^{-ikx} f_X(x)\, dx \equiv E[e^{-ikx}]$$

(**Part a**) Find the characteristic function of the exponential distribution, which we introduced to model waiting times:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Show that the result is

$$\tilde{f}(k) = \frac{\lambda}{\lambda + ik}.$$

(**Part b**) Because the Fourier transform turns derivatives into multiplication, we can take derivatives of the characteristic function to get moments of the PDF.

More precisely: if $f(x)$ is any PDF, consider what happens if we take the derivative of its characteristic function and then set $k = 0$.

$$\begin{aligned} \left[\frac{d}{dk}\tilde{f}(k)\right]_{k=0} &= \left[\frac{d}{dk}\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx\right]_{k=0} \\ &= \left[-i\int_{-\infty}^{\infty} x e^{-ikx} f(x)\, dx\right]_{k=0} \\ &= -i\int_{-\infty}^{\infty} x f(x)\, dx, \end{aligned}$$

where in the last step we used $e^{-i\cdot 0\cdot x} = 1$.

Similarly, it is easy to see that

$$\begin{aligned} \left[\frac{d^n}{dk^n}\tilde{f}(k)\right]_{k=0} &= (-i)^n \int_{-\infty}^{\infty} x^n f(x)\, dx \\ &= (-i)^n E[x^n]. \end{aligned}$$

The punchline is that we can get the n-th moment by taking n derivatives of the characteristic function, setting $k = 0$, and dividing by $(-i)^n$:

$$E[x^n] = \frac{1}{(-i)^n}\left[\frac{d^n}{dk^n}\tilde{f}\left(k\right)\right]_{k=0}.$$

Use this to find the first, second, and third moments of the exponential distribution. That is, find $E[x]$, $E[x^2]$, and $E[x^3]$.

Answer to part (b). You should find

$$E[x] = \frac{1}{\lambda}$$
$$E[x^2] = \frac{2}{\lambda^2}$$
$$E[x^3] = \frac{6}{\lambda^3}.$$

It's pretty easy to see that the pattern is

$$E[x^n] = \frac{n!}{\lambda^n}.$$

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the students' answer, *where students collectively construct a single answer*

green?

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