31.4

Recall that, for any periodic function $f(x)$ with period L,

$$\int_0^L f(x)\,dx = \int_a^{a+L} f(x)\,dx$$

Apply this result to ~~thate~~ the statement that

$$a_n = \int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)dx \cdot \frac{2}{L}$$

$$= \int_0^{2\pi} f(x)\cos(nx) \cdot \frac{2}{L}$$

to see that (since $f(x) = x^2$ on $[-\pi, \pi]$), then

$$\ldots = \int_{-\pi}^{\pi} x^2 \cos(nx)\,dx \cdot \frac{2}{2\pi}$$

Integrating by parts with

$A' = \cos(nx)$

$A = \sin(nx)/n$

$B = x^2$

$B' = 2x$

$$\ldots = \frac{1}{\pi}\left(\left[\sin(nx)/n\ x^2\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin(nx)/n\right)$$

mistakenly assumed even/odd state of n.

$$= \frac{1}{\pi}\left(\left(\frac{1}{n}\cdot\pi^2\right) - \left(\frac{-1}{n}\cdot\pi^2\right) - \int \quad ''\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi^2}{n} - \int \quad ''\right)$$

Now evaluate the remaining integral by IBP with

$A' = \sin(nx)/n$

$A = -\cos(nx)/n^2$

$B = 2x$

$B' = 2$

$\pi \to -\pi$

$$\ldots = \frac{1}{\pi}\left(\frac{2\pi^2}{n} - \left(-\int 2\cdot(-\cos(nx)/n^2) + \left[2x \cdot -\cos(nx)/n^2\right]\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi^2}{n} + 2\int_{-\pi}^{\pi} -\cos(nx)/n^2\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi^2}{n} + 2\left[-\sin(nx)/n^3\right]_{-\pi}^{\pi}\right)$$

We see that for all integer n,

$$\left[\sin(nx)/n\ x^2\right]_{-\pi}^{\pi} = 0;\text{ thus}$$

$$\frac{1}{\pi}\left(\left[\sin(nx)/n\ x^2\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin(nx)/n\ 2x\ dx\right)$$

equals

$$\frac{-2}{\pi}\int \sin(nx)/n\ x\ dx = \frac{-2}{\pi n}\int_{-\pi}^{\pi} x \sin(nx)\,dx$$

Integrate by parts with

$B' = \sin(nx)$

$B = -\cos(nx)/n$

$A = x$

$A' = 1$

to see that

$$\ldots = \frac{-2}{\pi n}\left(\left[x \cdot -\cos(nx)/n\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos(nx)/n\right)$$

$$= \frac{-2}{\pi n}\left(\left[-x\cos(nx)/n\right] + \frac{1}{n^2}\left[\sin(nx)\right]_{-\pi}^{\pi}\right)$$

Once again, for integer n $[\sin(nx)]_{-\pi}^{\pi} = 0$, thus

$$\ldots = \frac{2}{\pi n^2}\left[x\cos(nx)\right]_{-\pi}^{\pi}$$

For odd n,

$$\ldots = \frac{2}{\pi n^2}\cdot(\pi\cdot -1 + \pi\cdot -1) = \frac{-4}{n^2}$$

and for even n,

$$\ldots = \frac{2}{\pi n^2}(\pi\cdot 1 - -\pi\cdot 1) = \frac{4}{n^2}$$

We see that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(0x)\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

and that there is no b_n component since the periodic parabola is even; as such we write

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx). \square$$