27.7

(a): (1): We use the identity $A-B=\frac{A^2-B^2}{A+B}$ which tells us that,

$$\left|\sqrt{n+1}-\sqrt{n}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}};$$

given $\epsilon > 0$,

$$\frac{1}{2\sqrt{n}} < \epsilon \text{ if } \frac{1}{4n} < \epsilon^2, \text{ ie if } n > \frac{1}{4\epsilon^2}$$

(2): We must show that

$$\lim_{n\to\infty}\left(\frac{n^2+1}{n^2+1}\right) = 1; \; 1$$

Begin exploration by recalling (and letting $\epsilon > 0$ be given)

$\left|\frac{n^2+1}{n^2-1} - 1\right| < \epsilon$ must be satisfied.

$$\left|\frac{n^2+1}{n^2-1} - \frac{n^2-1}{n^2-1}\right| = \left|\frac{2}{n^2-1}\right| < \epsilon$$

$n \gg 1$, so we see that

$$\frac{2}{n^2-1} < \epsilon \Rightarrow \frac{n^2-1}{2} > \frac{1}{\epsilon} \Rightarrow n^2 > \frac{2}{\epsilon}+1 = \frac{2+\epsilon}{\epsilon}$$

or finally;

$$n > \frac{\sqrt{2+\epsilon}}{\sqrt{\epsilon}} = N$$

Thus we conclude that for $n > N = \frac{\sqrt{2+\epsilon}}{\sqrt{\epsilon}}$ the function is approx-imated, and in turn the limit exists.

(b): (1):

- Let $\epsilon > 0$ be arbitrary
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- Assume $n \geq N$
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$

(2): Problem: Show $\lim\left(\frac{n+1}{n}\right) = 1$

Proof ~~Exploration~~: Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ with $N > 1/\epsilon$. To veri
that this choice of N is appropriate, let $n \in \mathbb{N}$ satisfy $n \geq N$. Then, $n \geq N$
implies $n > 1/\epsilon$; or equivalently $1/n < \epsilon$. Finally, this means

$$\left|\frac{n+1}{n} - 1\right| < \epsilon$$

as desired. $\square$

(c): Claim: $\lim\left(3 - \frac{4}{n}\right) = 3$;

Proof:

- Let $\epsilon > 0$ be given.
- Then $n > N$ implies $|a_n - 3| = \left|\left(3 - \frac{4}{n}\right) - 3\right| = \left|\frac{4}{n}\right| = \frac{4}{n} < \epsilon$,
- Hence we have shown that
  for all $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that for all $n > N$, $\left|3 - \frac{4}{n} - 3\right| = \frac{4}{n} < \epsilon$
  thus $\lim\left(3 - \frac{4}{n}\right) = 3$ as desired.