question 2 views

Daily Challenge 19.4

(**Due: Friday November 23 at 12:00 noon Eatern**)

Happy Thanksgiving!

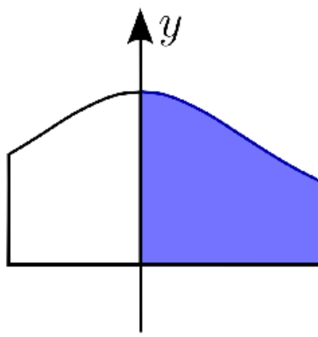
(1) Problem: introducing improper integrals.

The limit

$$\lim_{b\to\infty} \int_a^b f(x)\,dx,$$

if it exists, is written $\int_a^\infty f(x)\,dx$ and called an "improper integral."

Intuitively, it is the total area under f from a "to infinity." For instance, the total area under the curve $f(x) = \frac{1}{1+x^2}$ from 0 to infinity is written as follows:



$$\int_0^\infty \frac{dx}{1+x^2}$$

Calculate the improper integral

$$\int_1^\infty \frac{1}{x^r}\,dx$$

where $r > 0$. This means to compute $\int_1^b x^r\,dr$ for some fixed b and then take the limit as b goes to infinity. For which values of r does the integral converge and diverge? What is the value of the integral where it converges?

daily_challenge

Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

We see that in a previous daily challenge we had a valid method of finding the antiderivative of such a monomial, allow me to find it;

$$\int x^n\,dx = \frac{1x^{n+1}}{n+1} + C$$

Of course, $n = -r$ and we saw in the prior DC that we impose $n \neq -1$ or bad things will happen. Thus,

$$\lim_{b\to\infty}\left(\int_1^b \frac{1}{x^r}\,dx\right) = \lim_{b\to\infty}\left(\frac{1b^{-r+1}}{-r+1} - \frac{1}{-r+1}\right)$$

We then must break this into cases; if $r > 1$, then the former term shall tend to 0 and the integral shall tend to the latter term $-\frac{1}{-r+1} = \frac{1}{r-1}$, therefore it converges.

If $0 < r < 1$, then the former term will tend to infinity, and it diverges.

If $r = 1$ then

$$\int_1^b \frac{1}{x^r}\,dx = \log(b) - \log(1) = \log(b)$$

which, when $b \to \infty$, also goes to ∞, and therefore diverges.

Updated 4 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

First suppose $r \neq 1$. Then for fixed b, we have

$$\int_1^b \frac{1}{x^r}\, dx = \left[\frac{x^{-r+1}}{-r+1}\right]_1^b = \frac{b^{-r+1}}{-r+1} - \frac{1}{-r+1}.$$

If $r > 1$, then the quantity b^{-r+1} tends to zero as $b \to \infty$. In this case, the integral converges to $\frac{1}{r-1}$.

If $0 < r < 1$, then the quantity b^{-r+1} tends to infinity as $b \to \infty$; here the integral diverges.

Finally, we must handle the case $r = 1$ separately. Here we have

$$\int_1^b \frac{1}{x}\, dx = [\log(x)]_1^b = \log(b),$$

which blows up as $b \to \infty$, so this case diverges as well.

Updated 4 months ago by Christian Ferko

followup discussions *for lingering questions and comments*