question 2 views

Daily Challenge 16.3

(**Due: by session 40 at the very latest**)

This is question 13 on CD 4.

(1) Problem: Racetracks.

The *racetrack theorem* states that, if f and g are differentiable functions with $f(a) = g(a)$ for some a and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$.

(a) Explain in words what this theorem is saying and why it is called the racetrack theorem.

[Hint: think of f and g as representing the position of two cars on a racetrack as functions of time.]

(b) Prove the racetrack theorem.

(c) Use the racetrack theorem to prove the following result.

Proposition. Suppose f is a twice-differentiable function with $f(0) = 0$, $f(1) = 1$, and $f'(0) = f'(1) = 0$. Prove that $|f''(x)| \geq 4$ for some $x \in [0, 1]$.

[Hint: break it up into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

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Updated 6 months ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

(a) Following the hint, we interpret f and g as the positions of two cars as functions of time. The condition $f'(x) > g'(x)$ says that car f is always moving faster than car g, while the assumption $f(a) = g(a)$ says that the cars are at the same position along the racetrack at some time a. The conclusion, $f(x) > g(x)$ for all $x > a$, says that car f will be further along the racetrack than car g at all times later than a.

This is intuitively obvious: if two cars start at the same position, but the first car is always moving faster than the second, then at any later time the first car will have covered more ground than the second.

(b) Define $h(x) = f(x) - g(x)$, so that $h'(x) = f'(x) - g'(x)$. Then our assumptions can be re-stated as $h(a) = 0$ and $h'(x) > 0$ for all $x > a$, and we wish to prove that $h(x) > 0$ for all $x > a$.

Suppose by way of contradiction that $h(x) < 0$ at some point $x > a$. By the mean value theorem, there exists a point $c \in (0, x)$ such that $h'(c) = \frac{h(x)-h(0)}{x-0}$. But we have assumed $h(x) < 0$, so this means $h'(c) < 0$. This contradicts the assumption that $h'(x) > 0$ for all $x > a$. $\square$

(c) Following the hint, we split the interval into two pieces $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Since $f(0) = 0$ and $f(1) = 1$, we know by the intermediate value theorem that f has to hit $\frac{1}{2}$ somewhere between 0 and 1. This either happens before $x = \frac{1}{2}$ or afterwards.

First assume that $f\left(\frac{1}{2}\right) \geq \frac{1}{2}$. Suppose by way of contradiction that $f''(x) < 4$ for all $x \in \left[0, \frac{1}{2}\right]$. Apply the racetrack theorem to the functions $f'(x)$ and $g(x) = 4x$, which we can do since $f'(0) = g'(0) = 0$ and $f''(x) < g'(x) = 4$. The conclusion of the racetrack theorem guarantees that $f'(x) < g(x) < 4x$. On the other hand, we can also apply the racetrack theorem to $f(x)$ and $h(x) = 2x^2$, since $f(0) = h(0) = 0$ and $f'(x) < h'(x) = 4x$. This gives $f(x) < h(x)$, which means $f\left(\frac{1}{2}\right) < h\left(\frac{1}{2}\right) = \frac{1}{2}$, which contradicts our original assumption that $f\left(\frac{1}{2}\right) \geq \frac{1}{2}$. Thus there must be some point $a \in \left[0, \frac{1}{2}\right]$ where $f(a) \geq 4$.

Now we consider the second case, $f\left(\frac{1}{2}\right) < \frac{1}{2}$. It will be convenient to replace f by a new function $g(x) = 1 - f(1-x)$. We see that $g(0) = 0$, $g(1) = 1$, and $g'(x) = g'(1-x)$, which implies $g'(0) = g'(1) = 0$. Further, one has that $g\left(\frac{1}{2}\right) = 1 - g\left(\frac{1}{2}\right) > \frac{1}{2}$. Then we can simply apply the first half of our proof to g, which guarantees that $g''(a) > 4$ for some $a \in [0, \frac{1}{2}]$. But $f''(x) = -g''(1-x)$, so this actually means that $f''(a) < -4$ at some point a, which completes the proof. $\square$.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*