question 2 views

Daily Challenge 19.3

(**Due: Thursday November 22 at 12:00 noon Eastern**)

(1) Problem: Stress-testing the FTC.

Let f be integrable on $[a, b]$, let $c \in (a, b)$, and let

$$F(x) = \int_a^x f \quad , \quad a \leq x \leq b.$$

For each of the following statements, give either a proof (if the statement is true) or a counterexample (if the statement is false).

(a) F must be differentiable at c.

(b) If f is differentiable at c, then F is differentiable at c.

(c) If F is differentiable at c, we have $F'(c) = f(c)$.

(d) If f' is continuous at c, then F' is continuous at c.

daily_challenge

Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a): Nein. Let f be a piecewise function where for $x < 0$ it returns -1, and for $x \geq 0$ it returns 1. We see then see that if $a = 0$, then $F(x) = |x|$, which we then see has a kink at $x = 0$ and is of the form of an absolute value. Thus F is not differentiable.

(b) This is true. If f is differentiable at c, then it is also continuous at c. Dangerously similar to christians solution, we see by the FTC that $F'(c) = f(c)$.

(c): Nein. Let $f(x) = x$ at all points except 1, where it is equal to zero; We then see that $F(x) = x^2 + C$, but going in the opposite direction we get a simple line x rather than a piecewise function.

(d) We see that since f' is continuous at c and in turn by the definition of continuity an ϵ small range around it is also continuous, then f is continuous at c and we can conclude that F', being equivalent to f, is also continuous.

Updated 4 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

This entire problem is supposed to drive home the assumptions in the first fundamental theorem of calculus, which I repeat here for convenience.

Theorem (first FTC). Let f be integrable on $[a, b]$ and define F on $[a, b]$ by

$$F(x) = \int_a^x f.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

Note in particular the continuity assumption. Now we return to the problem.

(a) False. Consider $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$. Then f is integrable on $[-1, 1]$ and $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$. Thus F has a kink at $x = 0$ and is not differentiable there.

(b) True. If f is differentiable at some point $c \in \mathbb{R}$, it is certainly continuous at c. Therefore by the fundamental theorem of calculus we have $F'(c) = f(c)$.

(c) False. Let $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. Then f is integrable (we proved that changing the value at a single point does not affect integrability) and the integral is $F(x) = 0$ everywhere. But clearly $F'(0) = 0$ which is not equal to $f(0) = 1$. Thus the FTC can fail if f has discontinuities.

(d) True. If f' is continuous at c, then $f'(x)$ is defined for all x in an interval around c (proof: pick any small ϵ and then note that the derivative exists and is ϵ-close to $f'(c)$ on an interval $[c - \delta, c + \delta]$ for some δ). But this means that f is continuous on an interval around c, so indeed $F'(x) = f(x)$ on this interval and thus F' is continuous at c.

Updated 4 months ago by Christian Ferko

followup discussions *for lingering questions and comments*