question 2 views

Daily Challenge 9.7

(~~Due: Monday 7/16 at 12:00 noon Eastern~~)
(**Due: Thursday 7/19 at 12:00 noon Eastern**)

Today we'll continue exploring some of the nice properties that limits have.

(1) Limits can be divided, for nonzero denominator.

So far, we have proven that limits respect most of the ordinary arithmetic operations: for instance, we have

$$\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right),$$

and similarly

$$\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right).$$

You might wonder whether *division* behaves in a similarly intuitive way, assuming that the denominator is non-zero. We now show that it does.

Proposition. Suppose that $\lim_{x\to a} f(x) = F$ and $F \neq 0$. Then $\lim_{x\to a}\left(\frac{1}{f(x)}\right) = \frac{1}{F}$.

Exploration. We will need to bound $\left|\frac{1}{f(x)} - \frac{1}{F}\right|$. Let's first do some algebra parkour, similar to the trick that Spivak used in the proof that $\lim_{x\to a}\frac{1}{x} = \frac{1}{a}$:

$$\left|\frac{1}{f(x)} - \frac{1}{F}\right| = \frac{|f(x)-F|}{|f(x)||F|}.$$

We will first need to handle the troublesome $\frac{1}{|f(x)|}$ piece.

Since we know that $\lim_{x\to a} f(x) = F$, we can find a δ_1 so that $0 < |x-a| < \delta_1$ implies $|f(x)-F| < \frac{|F|}{2}$. In particular, this guarantees that $f(x) \neq 0$ for x in this range.

Given the assumption $|f(x)-F| < \frac{|F|}{2}$, combined with the "reverse triangle inequality" $|b|-|a| \leq |a-b|$ which holds for all $a, b \in \mathbb{R}$ (and which I've proved from the ordinary triangle inequality in a follow-up below), we have

$$|F| - |f(x)| \leq |f(x)-F| < \frac{|F|}{2}.$$

This implies $|f(x)| > \frac{|F|}{2}$. We can invert both sides and reverse the inequality to find

$$\frac{1}{|f(x)|} < \frac{2}{|F|}.$$

Now we're in business -- this handles the troublesome $\frac{1}{|f(x)|}$ factor. When this inequality is satisfied, our original expression becomes

$$\begin{aligned}\left|\frac{1}{f(x)} - \frac{1}{F}\right| &= \frac{|f(x)-F|}{|f(x)||F|} \\ &< \frac{2|f(x)-F|}{|F|^2}.\end{aligned}$$

Now we make a second restriction: if $\epsilon > 0$ is given, choose $\delta_2 > 0$ so that $0 < |x-a| < \delta_2$ implies $|f(x)-F| < \frac{|F|^2\epsilon}{2}$. Taking the smaller of δ_1 and δ_2 will prove the claim.

Proof. Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = F$, we can choose δ_1 and δ_2 so that

$$\begin{aligned}0 < |x-a| < \delta_1 &\implies |f(x)-F| < \frac{|F|}{2}, \\ 0 < |x-a| < \delta_2 &\implies |f(x)-F| < \frac{|F|^2\epsilon}{2}\end{aligned}.$$

Now take $\delta = \min(\delta_1, \delta_2)$. We claim that, if $0 < |x-a| < \delta$, then $\left|\frac{1}{f(x)} - \frac{1}{F}\right| < \epsilon$. Indeed, we have

$$\begin{aligned}\left|\frac{1}{f(x)}-\frac{1}{F}\right| &= \frac{|f(x)-F|}{|f(x)||F|} \\ &< \frac{|F|\epsilon}{2|f(x)|} \\ &< \epsilon,\end{aligned}$$

where in the first step we have used the assumption associated with δ_1 above, and in the second step we have used that $\frac{1}{|f(x)|} < \frac{2}{|F|}$ assuming that $|f(x)-F| < \frac{|F|}{2}$.

Thus $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{F}$, as desired. $\square$

An easy consequence is that we can now evaluate the limit of a *quotient*, assuming the denominator is nonzero, since division is the same as multiplication by a reciprocal and we have already proved that the limit of a product is the product of the limits.

Just as we use the word "proposition" for a small result, and "theorem" for big results, we use the word "corollary" for a claim which follows easily from a result we've already proven.

Corollary. Let f and g be functions with $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, and $G \neq 0$. Then $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{F}{G}$.

Proof. We write the function $\frac{f(x)}{g(x)}$ as $f(x)\cdot\frac{1}{g(x)}$. Then by the rule for the limit of a product,

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \left(\lim_{x\to a}(f(x))\right)\left(\lim_{x\to a}\left(\frac{1}{g(x)}\right)\right).$$

But we have proven above that $\lim_{x\to a}\left(\frac{1}{g(x)}\right) = \frac{1}{G}$, so we conclude that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{F}{G}. \square$$

Note that the assumption that the denominator is non-zero was critical. Later on, we will encounter limits of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

where both $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Of course, our above result will not apply in these cases, and more sophisticated technology will be needed.

(2) Problem: composition of functions.

The only property of limits that we have *not* considered is behavior under composition.

Let f and g be real-valued functions and $a, b, c \in \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = c$. Notice that x approaches different points in the two limits!

(a) Show that it is *not* necessarily true that $\lim_{x\to a}(g(f(x))) = c$, by writing down an example for f and g for which the claim is false.

(b) Show that it *is* true that $\lim_{x\to a}(g(f(x))) = c$ if we impose the additional assumption that $g(b) = c$.

Note that these two parts are problem 2(c) and 2(d) on CD 2.

[Hint for (a): try $f(x) = 0$ and give g a point discontinuity at 0.]

[Hint for (b): use two delta-epsilon arguments, one involving g and one involving f. The "epsilon" in your f argument will be the "delta" in your g argument.]

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: It is not necessarily true that $\lim_{x\to a}(g(f(x))) = c$ since we can suppose $f(x) = 0$ and $g(x)$ is the piecewise function $g(x) = \begin{cases} x^2+1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\end{cases}$, and in this case $\lim_{x\to a}(g(f(x))) \neq g(f(a))$.

b: Proof: Suppose $g(b) = c$ and $\lim_{x\to b} g(x) = c$ and therefore the function g is continuous at the point b. Given $\epsilon > 0$ we can then find a δ_g such that $0 \le |x-b| < \delta_f \implies |g(x)-c| < \epsilon$ Next we can suppose $\lim_{x\to a} f(x) = b$ and in turn we can find a δ_f such that $0 < |x-a| < \delta_f \implies |f(x)-b| < \delta_g$. This proves that $\lim_{x\to b} g(f(x)) = c$ as we have shown $f(x)$ can be "fed into" $g(x)$ while remaining true. $\square$

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) Following the hint, let $f(x) = 0$ and $g(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. Then clearly $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, since both functions are constants away from 0. However, we see that $g(f(x)) = 1$ for all x, so $\lim_{x\to 0} g(f(x)) = 1$.

Thus the limit of a composite function does *not* need to behave in the expected way, if the outer function is discontinuous.

(b) Now suppose $g(b) = c$ and $\lim_{x\to b} g(x) = c$, or in other words, that $g(x)$ is continuous at b. Let $\epsilon > 0$ be given. Then we can find a δ_1 with the property that

$$0 \leq |y - b| < \delta_1 \implies |g(y) - c| < \epsilon.$$

Note the all-important $\leq$ rather than $<$ in the inequality for $|y - b|$, which is justified because we have assumed $g(y) = c$.

Now, since $\lim_{x\to a} f(x) = b$, we can find a δ_2 such that

$$0 < |x - a| < \delta_2 \implies |f(x) - b| < \delta_1.$$

In particular, note that we have used the "delta" of the previous argument as the "epsilon" of this argument; this is fine, since the definition of the limit assures us that we can find a delta for any positive number epsilon, and the δ_1 of our first step is just some positive number.

We claim that this proves the result. Indeed, if $0 < |x - a| < \delta_2$, then $|f(x) - b| < \delta_1$ and hence

$$|g(f(x)) - c| < \epsilon,$$

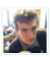
which proves that $\lim_{x\to a} g(f(x)) = c$. $\Box$

Note that it immediately follows from this proof that the composite of two continuous functions is continuous.

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 9 months ago
I should probably prove the "reverse triangle inequality."

Proposition. Let a, b \in \mathbb{R}. Then |b| - |a| \leq | a - b |.

Proof. We begin with the usual triangle inequality: for any x, y \in \mathbb{R}, we have shown that

\begin{align} |x+y| \leq |x| + |y| . \end{align}

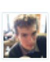
Now let x = b - a and y = a. This gives

\begin{align} | b - a + a | \leq | b - a | + | a | . \end{align}

Moving |a| to the other side, and using the rule |-z| = |z| for z \in \mathbb{R} to rewrite |b-a| = |a-b|, we conclude

\begin{align} |b| - | a | \leq | a - b | . \; \Box \end{align}

Resolved Unresolved



Christian Ferko 8 months ago
Feedback so far:

> a: It is not necessarily true that \lim_{x \to a} \left(g (f (x)) \right) = c since we can suppose f(x) = 0 and g(x) is the piecewise function g(x) = \begin{cases} x^2+1 &\text{ if } x \neq 0 \\ 0 &\text{ if } x = 0 \end{cases}, and in this case \lim_{x \to a} \left(g (f (x)) \right) \neq g(f(a)).

I think you've missed the point; you claim \lim_{x \to a} \left(g (f (x)) \right) \neq g(f(a)) but it *is* true that \lim_{x \to a} \left(g (f (x)) \right) = g(f(a)).

The point is that \lim_{x \to a} \left(g (f (x)) \right) \neq \lim_{x \to b} g(x).

This is a 3/6 for now?