question 2 views

Daily Challenge 14.1

(**Due: Thursday 9/13 at 12:00 noon eastern**)

We begin the fourteenth week of challenges! :wow:

(1) Extrema occur at points satisfying one of three conditions.

We've now proven several results linking the first derivatives to extrema of a function. By an *extremum*, I mean either a maximum or a minimum. For convenience, I restate these definitions.

Definition. Let f be a function and $A \subseteq \mathrm{Dom}(f)$. We say x is a *maximum point* for f on A if

$$f(x) \geq f(y) \text{ for all } y \in A.$$

The number $f(x)$ itself is called the *maximum value* of f on A. (We sometimes say "f has its maximum value on A at x.")

Similarly, we say x is a *minimum point* for f on A if

$$f(x) \leq f(y) \text{ for all } y \in A.$$

The number $f(x)$ is then called the *minimum value* of f on A. (We say "f has its maximum value on A at x.")

Remark. If $A = \mathrm{Dom}(f)$, so that the maximum and minimum points discussed above are truly the largest and smallest values that the function ever achieves, we sometimes call these the *global maximum* and *global minimum* to emphasize that $A = \mathrm{Dom}(f)$. Not every function has global extrema; for instance, $f(x) = x$ gets both arbitrarily large as $x \to \infty$ and arbitrarily negative as $x \to -\infty$.

The first theorem we proved was somewhat weak, but I repeat it here.

Theorem. Let f be any function defined on (a, b). If x is a maximum (or a minimum) point for f on (a, b) and f is differentiable at x, then $f'(x) = 0$.

We proved this in session 30. It has the disadvantage of *assuming* both that x is a maximum or minimum point and that f is differentiable at x. The theorem fails, for instance, when applied to a function like $f(x) = |x|$ on $(-1, 1)$, or even to a function like $f(x) = x^2$ on a closed interval $[-1, 2]$.

Accordingly, we strengthened the theorem by enumerating the cases where it can fail, and by combining it with the extreme value theorem.

Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then f achieves its maximum and minimum values on $[a, b]$, and both occur at points x where one of the following three conditions is satisfied:

1. $f'(x) = 0$,
2. $x = a$ or $x = b$, or
3. the derivative does not exist at x.

(2) Problem: optimization.

Find the maximum and minimum value of the function $f(x) = \frac{1}{x^5+x+1}$ on the closed interval $\left[-\frac{1}{2}, 1\right]$ by explicitly enumerating all points in the three classes above and choosing those that yield that largest and smallest output values.

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski

We shall begin by defining the function in a more friendly form; $f(x) = \left(x^5 + x + 1\right)^{-1}$. We can then apply the chain rule to see that $f'(x) = -\left(x^5 + x + 1\right)^{-2} \cdot (x^4 + 1)$. We see that $x^4 + 1 > 0$ since $x^4 \geq 0$, therefore we can only try to solve for a point where $0 = -\left(x^5 + x + 1\right)^{-2}$. We shall do some algebra parkour.

$$\begin{aligned} 0 &= -\left(x^5 + x + 1\right)^{-2} \\ &= \frac{-1}{\left(x^5 + x + 1\right)^2} \end{aligned}$$

It is impossible for us to insert an x such that this equals zero, so we shall move on to the second option.

We insert $\frac{-1}{2}$ and 1 into the equation; the former results in $\frac{1}{\frac{-1}{32}+\frac{1}{2}+1} = \frac{1}{\frac{47}{32}} = \frac{32}{47}$, while the latter results in $\frac{1}{3}$

Once again we see that there is no way for us to insert an x such that the derivative does not exist, therefore our only options of maximum and minimum are those attained in the second method. The minimum of this function is $\frac{1}{3}$ and the maximum is $\frac{32}{47}$, corresponding in the domain to $\frac{-1}{2}$ and 1.

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

The extrema occur at critical points, boundary points, or points where the derivative is undefined. The derivative is

$$f'(x) = -\frac{5x^4+1}{\left(x^5+x+1\right)^2}.$$

This is defined on $\left[-\frac{1}{2}, 1\right]$ (to see this, note that the denominator is strictly increasing and is nonzero when $x = -\frac{1}{2}$), so we need only consider critical points and endpoints.

However, the numerator $5x^4+1$ is also non-vanishing (it is the sum of two positive quantities), so in fact we only need to look at the endpoints.

We see that $f\left(-\frac{1}{2}\right) = \frac{1}{\left(-\frac{1}{2}\right)^5+\left(-\frac{1}{2}\right)+1} = \frac{32}{15}$ while $f(1) = \frac{1}{3}$. So the maximum is $\frac{32}{15}$ occurring at $x = -\frac{1}{2}$ and the minimum is $\frac{1}{3}$ occurring at $x = 1$.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*