note 2 views

Daily Challenge 4.5

We continue with no-problem days. Instead, I'll write up two worked examples for you -- as you read, try to judge whether you would have been able to answer the question yourself, had the solution not been provided.

Example 1. Suppose the quadratic $ax^2 + bx + c = 0$ has only real roots, and that a, b, c are all positive. Prove that both of the roots are negative.

Comment: This problem can be solved using either a direct proof or a proof by contradiction. I'll show you both.

Solution 1a (direct proof). Let a, b, c be positive reals and suppose $ax^2 + bx + c$ has only real roots. By the quadratic formula, the two roots are

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}.$$

Since we have assumed both roots are real, it must be true that $b^2 \geq 4ac$, for otherwise the quantity under the square root would be negative (giving an imaginary part). But this means that $\sqrt{b^2 - 4ac} \leq b$, so

$$-b \pm \underbrace{\sqrt{b^2 - 4ac}}_{\leq b} \leq 0.$$

Since the numerator and denominator of our expression for $x_\pm$ are both positive, we conclude that the roots $x_\pm$ themselves must also be positive. $\square$

Solution 1b (contradiction). Let a, b, c be positive reals and suppose $ax^2 + bx + c$ has only real roots. We proceed by contradiction: suppose that a root x_+ of the quadratic were positive. Since x_+ is a root, this means

$$ax_+^2 + bx_+ + c = 0.$$

But the left side is a sum of three positive numbers, and hence positive, so it cannot be equal to zero. This is a contradiction, so we conclude that both roots are negative. $\square$

Example 2. Compute $\sin\left(\frac{5\pi}{12}\right)$.

Solution 2. This is not one of the standard angles for which we have memorized the values of sine and cosine, so we will need to be a bit clever. Note that $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$, so

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right).$$

On the other hand, we have the angle-addition formula

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha),$$

so

$$\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right).$$

These *are* familiar angles that we've memorized: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$, and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. Therefore,

$$\begin{aligned}\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}.\end{aligned}$$

In the last line, we have gotten a common denominator and combined fractions.

daily_challenge

Updated 11 months ago by Christian Ferko

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