question

2 views

Daily Challenge 11.1

(**Due: Monday 7/30 at 12:00 noon Eastern**. Note that there is no challenge due on 7/29; this is a make-up day for 10.6, 10.7, and matrices challenge 3.)

I lied; I will add one more consolidation document 2 problem on using continuity to get more practice on this. This is problem 8 and you will present it at the tutorial on Wednesday.

(1) Problem: discontinuity and the converse of IVT.

(a) Define a function f by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0 \end{cases}.$$

Show that f is not continuous on $[-1, 1]$.

(b) Show that f satisfies the *conclusion* (not the hypotheses) of the intermediate value theorem on $[-1, 1]$. That is, show that if f takes on two values somewhere on $[-1, 1]$, then it takes on every value in between.

(c) Now consider some different function g. Suppose that g also satisfies the *conclusion* of the intermediate value theorem, and that g takes on each value *only once*. Prove that g is continuous.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Simply by looking at this piecewise function, we can see that we must show that f is discontinuous at 0 (just by the way f is enticingly constructed). To do so, we have to somehow show that $\lim_{x\to 0} f(x) \neq f(x)$. As well, by looking at Desmos one can see that the function f begins increasingly rapidly oscillating as $x \to 0$. Since this function has a range of $[1, -1]$ for the domain $[1, -1]$, we can simply set epsilon sufficiently small, after all by the definition of limit it only need be true that $\epsilon > 0$; we shall set $\epsilon = \frac{1}{2}$. Suppose by way of contradiction that there exists δ such that $|f(x) - L| < \epsilon$ is true for the domain $[-\delta, \delta]$ that is implied in this limit. We can exploit the periodicity of the sine function and see that that we can choose integers m and n sufficiently large such that $x_1 = \frac{1}{2\pi n + \frac{\pi}{2}}$ and $x_2 = \frac{1}{2\pi n + \frac{3\pi}{2}}$ are in our domain $[-\delta, \delta]$. We can then see regardless of m and n then $f(x_1) = 1$ and $f(x_2) = -1$, contradicting and showing us that as $x \to 0$ the limit does not exist, and therefore f is not continuous on $[1, -1]$.

b: To show that "if f takes on two values somewhere $[-1, 1]$, then it takes on every value in between," we must consider the potential placements of two values a and b. First let $[a, b]$ be a non-null subset of $[-1, 1]$. First if $0 \notin [a, b]$, then it is automatically true that f is continuous for $[a, b]$ as f is only discontinuous at 0. Therefore by the intermediate value theorem, we can choose any number y between $f(a)$ and $f(b)$ and there exists some $c \in (a, b)$ such that $f(c) = y$, pretty generic for now. In the case where $0 \in [a, b]$, we must show that for y in between $f(a)$ and $f(b)$, there exists a $c \in (a,b)$ such that $f(c) = y$. We must show that this c exists. We have $-1 \leq f(a),f(b) \leq 1$, then in turn $-1 \leq y \leq 1$. We can use the inverse sine function to do our bidding here as we are operating within it's domain; apply $\sin^{-}1(y)$ to get that there exists some number c' such that $\sin(c') = y$. Once again we can exploit the periodicity of the sine function and let $c = \frac{1}{2 \pi n + c'}$ where n is large enough such that $c \in [a,b]$. We then have that $f(c) = \sin(2 \pi n + c') = \sin(c') = y$, and therefore $f(c) = y$ and the conclusion of the intermediate value theorem is true for this function f. $\Box$

c: g is a function satisfying the conclusion of the intermediate value theorem, and takes on each value only once. We shall show that g is continuous by way of contradiction. Suppose by way of contradiction that there exists a point a where $\lim_{x \to a} g(x) \neq g(x)$. We can take the negation of the definition of a limit at a continuous point to get what is meant by a limit not being continuous at a point: "There exists some $\epsilon > 0$ for which it is true that, no matter what $\delta > 0$ you pick, there will always be some values of x where $|x - a| < \delta$ but still $|g(x) - g(a)| > \epsilon$." We can then choose a value of epsilon such that the previous statement is true, and in turn it is true that regardless of how "close" we get to a, then our input x will either have that $g(a) + \epsilon < g(x)$ or $g(x) < g(a) - \epsilon$. Without loss of generality assume the former, and for a newly defined input x_1. Therefore $g(x_1) > g(a) + \epsilon$. From this we then have that there exists a $c \in (a, x_1)$ so that $g(c) = g(a) + \frac{\epsilon}{2}$. We can once again refer to the fact that we have assumed g is discontinuous at some point a, and therefore there exists more values x such that $g(x) > g(a) + \epsilon. We can then say x_2 is another value on this interval (a,c). We have now found that $a < x_2 < c < x_1$ and that $g(x_1), g(x_2) > \epsilon$, and that $g(c) = g(a) + \frac{\epsilon}{2}$. By the IVT, we can see that this information contradicts our claim that g outputs each number only once, as we see that there exists $y_1 \in (x_2, c)$ where $g(y_1) = g(a) + \epsilon$ and another value $y_2 \in (c,x_1)$ where $g(y_2) = g(a) + \epsilon$. This contradiction then verifies our claim that g must be continuous. $\Box$

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

This is a very interesting problem, perhaps my favorite in this chapter! We will see that, even though the *assumptions* of the IVT fail for this function (it is not continuous), it turns out that the *conclusion* still holds.

(a) The function is not continuous on $[-1, 1]$ because it is discontinuous at $x = 0$.

We actually proved this in DC 9.6(b). Let $\epsilon = \frac{1}{2}$; we will show that there is no δ such that, whenever $0 < |x - a| < \delta$, we are guaranteed that $|f(x) - L| < \epsilon$ for any L.

Indeed, if there were such a δ, then the inequality $|f(x) - L| < \epsilon$ would need to hold in the interval $(-\delta, \delta)$. Pick two integers m and n large enough so that $x_1 = \frac{1}{2\pi n + \frac{\pi}{2}}$ and $x_2 = \frac{1}{2\pi n + \frac{3\pi}{2}}$ lie in the interval $(-\delta, \delta)$. Then we see $f(x_1) = 1$ and $f(x_2) = -1$. But this contradicts that $|f(x) - L| < \frac{1}{2}$ for all x in the interval.

This contradiction establishes that f is discontinuous at 0, as desired.

(b) Following the hint, let a, b be two points in $[-1, 1]$ with $a < b$. We consider two cases: case 1 is where $[a, b]$ does not include 0, and case 2 is where $[a, b]$ does include zero.

Case 1. Suppose $[a, b]$ does not include 0. The only point at which f is discontinuous is $x = 0$. In particular, f is continuous on the interval $[a, b]$. Thus, by the intermediate value theorem, if we choose some number y between $f(a)$ and $f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = y$.

Case 2. Suppose $[a, b]$ includes 0 and let y be between $f(a)$ and $f(b)$. We need to show that there exists $c \in (a, b)$ such that $f(c) = y$.

Now we know $-1 \leq f(a) \leq 1$ and similarly for $f(b)$, so $-1 \leq y \leq 1$. This means that we can apply the inverse sine function to y and get some number $\tilde{c}$ with the property that $\sin(\tilde{c}) = y$.

As before, choose an integer n large enough so that the number $c = \frac{1}{2\pi n + \tilde{c}}$ lies in the interval $[a, b]$. By construction, we have

$$f(c) = \sin(2\pi n + \tilde{c}) = \sin(\tilde{c}) = y,$$

which shows that the conclusion of the intermediate value theorem holds.

(c) Let g be as described in the problem statement, and suppose by way of contradiction that g is **not** continuous at some point a.

By taking the negation of the statement "for every $\epsilon > 0$ there exists $\delta > 0$ so that $|x - a| < \delta$ implies $|g(x) - g(a)| < \epsilon$", we see that the statement that g is **not** continuous at a means:

> "there is at least one $\epsilon > 0$ for which it is true that, no matter what $\delta > 0$ you pick, there will always be some values of x with $|x - a| < \delta$ but still $|g(x) - g(a)| > \epsilon$" .

Pick a value of ϵ for which the preceding paragraph is true. Then no matter how close we get to a, there will either be some inputs x with either $g(x) > g(a) + \epsilon$ or $g(a) < g(a) - \epsilon$. Without loss of generality, assume it is the former, and pick x_1 so $g(x_1) > g(a) + \epsilon$.

Now we have also assumed g satisfies the conclusion of the intermediate value theorem. Since $g(x_1) > g(a) + \epsilon$, this means there exists a $c \in (a, x_1)$ so that $g(c) = g(a) + \frac{\epsilon}{2}$.

Next we apply the discontinuity assumption again: no matter how close we get to a, there will always be more x values with $g(x) > g(a) + \epsilon$. Find another such value on the interval (a, c) and call this value x_2 (labeled as y in the hint).

To summarize so far: we have found three values x_1, x_2, c with $a < x_2 < c < x_1$ and such that $g(x_2) > g(a) + \epsilon$, $g(x_1) > g(a) + \epsilon$, and $g(c) = g(a) + \frac{\epsilon}{2}$. (See the picture in the hint.)

Finally we have a contradiction: the function g must take on some values twice, but we have assumed that it outputs each number at most once.

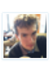
To see why g takes on some values twice, apply the IVT to find one number $y_1 \in (x_2, c)$ with $g(y_1) = g(a) + \epsilon$ and a second number $y_2 \in (c, x_1)$ with $g(y_2) = g(a) + \epsilon$. (Roughly speaking, from the picture we see that $g(x)$ must cross the horizontal line at $g(a) + \epsilon$ twice, but we assumed it hits each value at most once.)

This contradiction establishes the claim. $\square$

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 8 months ago

- Hint 1 for (a): Continuity fails at $x = 0$. Why?
- Hint 2 for (a): To prove continuity fails at $x = 0$, let $\epsilon = \frac{1}{2}$. Prove that you can never find an appropriate δ. It might be helpful to review DC 9.6 (b).
- Hint 1 for (b): Let a, b be two points in $[-1, 1]$. There are two cases. Case 1 is where $[a, b]$ does not include 0; that is, either a and b are both positive or they are both negative. In this case you can actually use the IVT (even though f is discontinuous) if you read the assumptions carefully.
- Hint 2 for (b): Case 2 is where $[a, b]$ includes 0. You need to show that f achieves all values between $f(a)$ and $f(b)$ on $[a, b]$. But certainly $-1 \leq f(a) \leq 1$ and $-1 \leq f(b) \leq 1$. How does that help you? DC 9.6 (b) might be helpful again.
- Hint 1 for (c): Suppose by way of contradiction that g is discontinuous at some point a. The first step is translating the statement that g is **not** continuous into epsilon-delta language.

  If g **is** continuous at a, then for every $\epsilon > 0$ there exists $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$

  Thus if g is **not** continuous at a, there exists at least one $\epsilon > 0$ for which there is no $\delta > 0$ such that every x with $|x - a| < \delta$ has $|f(x) - f(a)| < \epsilon$

  Untangling the logic in the previous paragraph, we see that this means there exists some $\epsilon > 0$ such that, for any $\delta > 0$, we can always find some value of x which is δ-close to a but such that $|f(x) - f(a)| > \epsilon$

- Hint 2 for (c): Suppose by way of contradiction that g is discontinuous at some point a. By the reasoning in hint (1), for some $\epsilon > 0$, there are values of x arbitrarily close to a such that $g(x) > g(a) + \epsilon$ (or $g(x) < g(a) - \epsilon$, but assume it's this way WLOG).

  If g satisfies the IVT, you can find another point x between x and a with $g(c) < g(a) + \epsilon$. Use this to get a contradiction. In particular, contradict the assumption that g takes on each value only once.

- Hint 3 for (c): Use this picture to get the idea, then make the argument rigorous