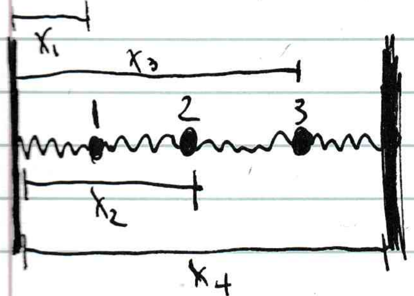
36.2

(a): Recalling that for a spring of length x, $F_{spring} = -kx$, let's sketch our system:



The forces on 1, 2, and 3 are the sum of the forces on each side:

$$F_1 = m\ddot{x}_1 = -kx_1 - k(\underset{x_1 - x_2}{\cancel{x_2 - x_1}}) \qquad = -kx_2$$

$$F_2 = m\ddot{x}_2 = -k(\underset{x_1 - x_2}{\cancel{x_2 - x_1}}) - k(\underset{x_2 - x_3}{\cancel{x_3 - x_2}}) \quad \text{and} = kx_1 - kx_3$$

$$F_3 = m\ddot{x}_3 = -k(\underset{x_2 - x_3}{\cancel{x_3 - x_2}}) - k(x_3). \qquad = kx_2 - 2kx_3$$

(b): This system of equations can be encoded by

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{\begin{bmatrix} -2k & -k & 0 \\ k & 0 & -k \end{bmatrix}}$$

$$= \frac{1}{m}\begin{bmatrix} 0 & -k & 0 \\ k & 0 & -k \\ 0 & k & -2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Making the ansatz

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{i\omega t}$$ then the prior equation becomes

$$-\omega^2 \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \cancel{e^{i\omega t}} = \frac{k}{m}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \cancel{e^{i\omega t}}$$

Now that I corrected that sign error, we see that by factoring and moving constants to be with the matrix,

$$\frac{K}{m}\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{\partial^2}{\partial t^2}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

We then apply the suggestion in the problem statement to let

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix}e^{iwt} \quad \text{and thus}$$

$$\frac{K}{m}\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix}= w^2\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix}$$

which implies w^2 is an eigenvector of that scalar-matrix multiple. For speed's sake, let $a = K/m$; thus we see that we can find the eigenvectors by solving for w^2 in

$$\det\left(\begin{bmatrix}2a-w^2 & -1a & 0\\ -1a & 2a-w^2 & -1a\\ 0 & -1a & 2a-w^2\end{bmatrix}\right)=0$$

Which is going to be atrocious;

$$(2a-w^2)\det\begin{pmatrix}2a-w^2 & -a\\ -a & 2a-w^2\end{pmatrix} + a\det\begin{pmatrix}-a & -a\\ 0 & 2a-w^2\end{pmatrix} = 0$$

equals

$$(2a-w^2)\left((2a-w^2)^2 - a^2\right) + (-a^2)(2a-w^2) = 0$$

which then equals

$$4a^3 - 10a^2w^2 + 6aw^4 - w^6 = 0$$

which can be factored as

$$(w^4 - 4aw^2 + 2a^2)(-w^2 + 2a) = 0$$

For the latter part we get

$$w = \pm\sqrt{2a},$$

while applying the quadratic formula for

$$(w^4 - 4aw^2 + 2a^2) = 0$$

gets us

$$w^2 = 2a \pm \sqrt{2a} \Rightarrow w = \pm\left(2\sqrt{a} \pm \sqrt{2a}\right)$$

(c) Then for the values of w that don't just differ by a scalar, we want to find the eigenvectors V_1, V_2, V_3 s.t.

$$M V_1 = 2a V_1,$$

$$M V_2 = \cancel{\sqrt{2+\sqrt{2}}\sqrt{a}\, V_2} = (2+\sqrt{2})\, a\, V_2$$

$$M V_3 = \cancel{\sqrt{2-\sqrt{2}}\sqrt{a}\, V_3} = (2-\sqrt{2})\, a\, V_3$$

or

$$\begin{bmatrix} 2a & -a & 0 \\ -a & 2a & -a \\ 0 & -a & 2a \end{bmatrix}\begin{bmatrix} 1 \\ A_2 \\ A_3 \end{bmatrix} = 2a\begin{bmatrix} 1 \\ A_2 \\ A_3 \end{bmatrix} \Rightarrow A_2 = 0 \Rightarrow A_3 = -1.$$

Likewise,

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ A_2 \\ A_3 \end{bmatrix} = (2+\sqrt{2})\begin{bmatrix} 1 \\ A_2 \\ A_3 \end{bmatrix} \Rightarrow A_2 = -\sqrt{2} \Rightarrow A_3 = 1$$

And finally

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ A_2 \\ A_3 \end{bmatrix} = (2-\sqrt{2}) \begin{bmatrix} 1 \\ A_2 \\ A_3 \end{bmatrix} \Rightarrow A_2 = \sqrt{2} \Rightarrow A_3 = 1 .$$

(d): We can rewrite the general solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} c_1 e^{i\sqrt{2}\sqrt{a}t} + \cdots$$

using that for some A and ϕ,

$$c_1 e^{iwt} + \bar{c}_1 e^{iwt} = A\cos(wt+\phi).$$