25.4

We will be computing the Fourier transform of the function

$f(x) = \cancel{} = e^{-a^2 x^2}$

Recall the definition of the Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2}\, e^{-ikx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx - a^2 x^2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2\left(x + \frac{ik}{2a^2}\right)^2 - \frac{k^2}{4a^2}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-k^2}{4a^2}\right) \cancel{dk} \cdot \int_{-\infty}^{\infty} \exp^{-a^2\left(x + \frac{ik}{2a^2}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-k}{4a^2}\right) \cdot \int_{-\infty}^{\infty} \exp\left(-a^2\left(x + ik/2a^2\right)^2\right) dx$$

then u-sub $u = x + ik/2a^2$ and thus $dx = du$

$$\ldots = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{k}{4a^2}\right) \cdot \int_{-\infty}^{\infty} \exp\left(-a^2 u^2\right) du$$

$$= \quad '' \quad \int_{-\infty}^{\infty} \exp\left(-1\cdot(a\,u)^2\right)$$

u-sub $s = a u \Rightarrow ds = a\, du$

$$= \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{k}{4a^2}\right) \int_{-\infty}^{\infty} \exp(-s^2)$$

$$= \frac{1}{a\sqrt{2}} \exp\left(-\frac{k}{4a^2}\right)$$