question

2 views

Daily Challenge 11.8

(**Due: Sunday 8/19 at 12:00 noon eastern**)

In session 24, we added the derivative of cosine, $\frac{d}{dx}(\cos(x)) = -\sin(x)$, to our collection of derivatives of trigonometric functions. We have already proven, of course, that $\frac{d}{dx}(\sin(x)) = \cos(x)$.

Today we will complete the collection by differentiating the other four trigonometric functions.

(1) The quotient rule gives $\left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}$.

As we proved a couple of sessions ago, using the "one-over-f" rule and the product rule, the derivative of a quotient satisfies

$$\left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}$$

I will not repeat the proof here; instead, we will see how to use this to differentiate the cotangent function.

Example. Let $f(x) = \cot(x)$. Find $f'(x)$, where it is defined.

Solution. We recall that $\cot(x) = \frac{\cos(x)}{\sin(x)}$, and we know that the derivative of $\cos(x) = -\sin(x)$, while the derivative of $\sin(x)$ is $\cos(x)$. By the quotient rule, then,

$$\begin{aligned} f'(x) &= \frac{(\cos)'(x)\sin(x) - (\sin)'(x)\cos(x)}{\sin^2(x)} \\ &= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \\ &= -\csc^2(x). \end{aligned}$$

In the last step, we have used the Pythagorean identity and recalled the definition $\csc(x) = \frac{1}{\sin(x)}$.

Thus the derivative of $\cot(x)$ is $-\csc^2(x)$, which is defined so long as $\sin(x) \neq 0$. But this is precisely where the original function $\cot(x)$ is defined. □

(2) Other trigonometric functions can be differentiated with the chain rule.

To differentiate a function like $\sec(x) = \frac{1}{\cos(x)}$, we could use the same strategy as in section (1) and apply the quotient rule, where the numerator is here the constant function 1. But it is faster to write $\sec(x) = (\cos(x))^{-1}$ and use the chain rule.

Example. Let $f(x) = \sec(x)$. Find $f'(x)$, where it is defined.

Solution. We write $f(x) = (\cos(x))^{-1}$. By the chain rule,

$$\begin{aligned} f'(x) &= -(\cos(x))^{-2} \cdot (-\sin(x)) \\ &= \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x). \end{aligned}$$

Thus the derivative of $\sec(x)$ is $\sec(x)\tan(x)$; both are defined where $\cos(x) \neq 0$.

(3) Problem: other trigonometric functions.

Using the chain rule or quotient rule, find the derivatives of:

1. $\tan(x)$
2. $\csc(x)$

where they are defined.

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Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: Begin with the definition of tangent, $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Then by the quotient rule we have that $\tan'(a) = \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{(cos(x))^2}$. We also know that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$, so we apply this to our previous result and see $\tan'(a) = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$ Then by Pythagoras we have that $\cos^2(x) + \sin^2(x) = 1$, therefore $\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \frac{1}{\cos(x)} \cdot \frac{1}{\cos(x)} = \sec(x) \cdot \sec(x) = \sec^2(x)$. We can then conclude that $\tan'(x) = \sec^2(x)$.

b: Recall the definition of co-secant; $\csc(x) = \frac{1}{\sin(x)}$. Then by the quotient rule we have that $\csc'(x) = \frac{0 - \cos(x)}{\sin^2(x)}$, and in turn $\frac{0-\cos(x)}{\sin^2(x)} = \frac{-1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} = -\csc(x) \cdot \cot(x)$

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(1) Let $f(x) = \frac{\sin(x)}{\cos(x)}$. By the quotient rule,

$$
\begin{aligned}
f'(a) &= \frac{\cos(a)\cos(a) - (-\sin(a))\sin(a)}{\cos^2(a)} \\
&= \frac{1}{\cos^2(a)} \\
&= \sec^2(a).
\end{aligned}
$$

So the derivative of $\tan(x)$ is $\sec^2(x)$.

(2) By the chain rule,

$$
\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\left((\sin(x))^{-1}\right) \\
&= -(\sin(x))^{-2}\cos(x) \\
&= -\csc(x)\cot(x).
\end{aligned}
$$

Thus the derivative of $\csc(x)$ is $-\csc(x)\cot(x)$.

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followup discussions *for lingering questions and comments*