29.2

(a): We see that, for some N we choose,

$$\left|\sum_{n=1}^{N} a_n\right| \leq \sum_{n=1}^{N} |a_n|$$ by the triangle inequality; likewise

$$\sum_{n=1}^{N} |a_n| \leq \sum_{n=1}^{\infty} |a_n|$$

since we are simply adding the $|a_k|$ where $k \in [N+1, \infty]$ and $k \in \mathbb{N}$; $|a_k| \geq 0$ by the absolute

We see by the definition of absolute convergence that, given $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, $\sum_{n=1}^{\infty} |a_n|$ converges.

~~By taking the limit of as $N \to \infty$ of~~

$$\cancel{\left|\sum_{n=1}^{N} a_n\right| \leq \sum |a_n|}$$

We then see by the statement that "taking limits preserves inequalities" that for $N \to \infty$ of

$$\left|\sum_{n=1}^{N} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|, \text{ then}$$

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$$

(b): ~~Let the subsequence $b_j = a_{n_j}$; we can then bound the sum of the sum $|b_j| + \dots + |b_k|$ by noting~~

$$\cancel{|b_j| + \dots + |b_k| \leq \sum_{n=j}^{k} a_n}$$

~~We see that, for $\epsilon = \sum_{n=j}^{k} a_{n_j}$ then cauchy~~

We let the sequence $(b_j) = (a_{n_j})$ where n_j is a sequence representing the indices in the subsequence; Then, for $|b_j| + \dots + |b_k|$ to satisfy cauchy, we need to make it sufficiently small when j or k might be big; see that $\dots = |a_{n_j}| + \dots + |a_{n_k}|$, but this does not have to operate over $[j, k]$,

so $|a_{n_j}| + \dots + |a_{n_k}| \leq |a_{n_j}| + |a_{n_{j+1}}| + \dots |a_{n_{k-1}}| + |a_{n_k}|$

By including all non-negative values in the range. We can then make this ~~sequence~~ sum small by noting that a_n obeys Cauchy's criterion; thus we can make ~~small (a~~

$|a_{n_j}| + |a_{n_{j+1}}| + \dots + |a_{n_k}|$ small, and thus

$|b_j| + \dots + |b_K|$ small; thus,

$\sum_{n=1}^{\infty} b_n$ converges and converges absolutely.

If a_n is a sequence with a conditionally convergent sum, then let $p_n = a_n$ if $a_n \geq 0$ and 0 otherwise; likewise, let $q_n = a_n$ if $a_n < 0$ and 0 otherwise.

~~Suppose by way of contradiction that one of these subsequences~~

We see that both p_n and q_n must both be unbounded partial sums, since

p_n and q_n bounded $\implies$ absolutely convergent (we know this is false)

p_n only 1 is bounded $\implies$ unbounded partial sums (contradicts known convergence)

Thus both must be unbounded. ~~Th~~ In turn, simply pick

$\sum_{n=1}^{\infty} p_n$,

This must be unbounded as we saw before, so the result of (b) is false for conditionally convergent.

(c): We know that given $\sum a_n$ is convergent, then ~~$\lim_{n\to\infty}$~~ given $\epsilon > 0$

$\lim_{n\to\infty} a_n = 0$; thus there exist some $N \in \mathbb{N}$ such that

$n > N \Rightarrow |a_n| < \epsilon$. Simply demand $\epsilon > 0$ and we know there exists N such that $|a_n| < 1$ is satisfied. We know by intuition that, for $n > N$ such that $|a_n| < 1$, it must be true that

~~$a_n^2 = |a_n|^2 < 1$ and $a_n^2 < |a_n| < 1$~~ $|a_n| |a_n| < 1 |a_n|$; thus

$$0 \leq |a_n|^2 = a_n^2 < |a_n| < 1$$

by the comparison test,

$$a_n^2 = |a_n|^2 < |a_n| \Rightarrow \sum_{n=N}^{\infty} a_n^2 \text{ converges}$$

We can safely ignore the first $N-1$ terms since we proved in past (dc 28.4) that "Changing finitely many terms in a series does not affect convergence." Thus we conclude

$$\sum_{n=\infty}^{\infty} a_n^2 \text{ Converges.}$$