question 2 views

Daily Challenge 6.2

(**Due: Wednesday 5/30 at 12:00 noon Eastern**)

Review

Yesterday we saw that the logarithm obeys an identity which is somehow "dual" to the identity $2^{x+y} = 2^x 2^y$, namely

$$\log_a(xy) = \log_a(x) + \log_a(y).$$

In short, exponential functions turn addition into multiplication, and logarithmic functions turn multiplication into addition.

There are two other log identities that we will need; I will prove the first one for you.

Proposition (log of a power). Let a, x, y be real numbers with $a > 0$ and $x > 0$. Then $\log_a(x^y) = y\log_a(x)$.

Proof. First define variables $u = \log_a(x^y)$ and $v = \log_a(x)$. Since $\log$ is the inverse function of the exponential, this means $a^u = x^y$ and $a^v = x$. Plugging the second equation into the first, we see $a^u = (a^v)^x$, which we can simplify using the power-to-a-power rule to find

$$a^u = a^{vy} = a^{y\log_a(x)}.$$

Setting the exponents equal, we conclude that $u = y\log_a(x)$, so $\log_a(x^y) = y\log_a(x)$. $\square$

Note that this rule came from the power-to-a-power rule for exponents, just as the $\log(ab)$ rule came from the exponential-of-a-sum rule.

Problem

(a) Solve the equation $\log_5\left(25^{3x}\right) = 10$ for x. [Hint: use the log-of-a-power rule. It may help that $25 = 5^2$.]

(b) The last identity we need is the *change-of-base formula*, which you will derive here.

Let a, b, x be positive reals. Find an equation relating $\log_b(x)$ to $\log_a(x)$ and $\log_b(a)$.

[Scaffold: Suppose $\log_b(x) = y$. This means that $b^y = x$. Now take the log of both sides of this equation with a *different* base, say a. Use the log-of-a-power rule to simplify. Your result should be an equation that expresses $\log_b(x)$ in terms of $\log_a(x)$ and $\log_a(b)$.]

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: By the log-of-a-power rule, $\log_5\left(25^{3x}\right) = 10$ can be "changed" to $3x \times \log_5(25) = 10$.

$\log_5(25)$ is 2, so $6x = 10$, and finally $x = \frac{5}{3}$.

b: I begin by taking the scaffold/hint in stride, and define $y = \log_b(x)$. By the definition of logarithm, this means that $b^y = x$. Once again, I take a hint and take the logarithm of each side with the base a, $\log_a(b^y) = \log_a(x)$.

I apply the log-of-a-power rule and get that $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) Solve the equation $\log_5\left(25^{3x}\right) = 10$ for x. [Hint: use the log-of-a-power rule. It may help that $25 = 5^2$.]

Solution. Writing $25 = 5^2$, the equation becomes

$$\begin{aligned} \log_5\left(\left(5^2\right)^{3x}\right) &= 10 \\ \log_5\left(5^{6x}\right) &= 10 \end{aligned},$$

where in the second step we have used the power-to-a-power rule. But by definition, $\log_5(5^y) = y$ for any $y \in \mathbb{R}$. Thus

$$\begin{aligned} 6x &= 10, \\ \implies x &= \frac{5}{3}. \end{aligned}$$

(b) The last identity we need is the *change-of-base formula*, which you will derive here.

Let a, b, x be positive reals. Find an equation relating $\log_b(x)$ to $\log_a(x)$ and $\log_b(a)$.

[Scaffold: Suppose $\log_b(x) = y$. This means that $b^y = x$. Now take the log of both sides of this equation with a *different* base, say a. Use the log-of-a-power rule to simplify. Your result should be an equation that expresses $\log_b(x)$ in terms of $\log_a(x)$ and $\log_a(b)$.]

Solution. Following the scaffold, we define $\log_b(x) = y$, so that $b^y = x$. Take the logarithm of each side with base a to find

$$\log_a(b^y) = \log_a(x).$$

By the log-of-a-power rule, we know that $\log_a(b^y) = y \log_a(b)$, so

$$y \log_a(b) = \log_a(x).$$

Now we replace y using its original definition, $y = \log_b(x)$, to conclude

$$\begin{aligned} &\log_b(x) \cdot \log_a(b) = \log_a(x), \\ &\implies \log_b(x) = \frac{\log_a(x)}{\log_a(b)}. \end{aligned}$$

This is the *change-of-base formula*. It relates two logarithms of the same argument, x, but with different bases, a and b.

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*