question 2 views

Daily Challenge 10.6

(~~**Due: Friday 7/27 at 12:00 noon Eastern**~~)
(**Due: Saturday 7/28 at 12:00 noon Eastern**)

We're almost done with the first pass of solutions on consolidation document 2. This is the last hard problem, and then we just need to do revisions and finish one or two small lingering sub-parts.

(1) Problem: proving the squeeze theorem.

In this problem, we will prove the squeeze theorem, which is copied below for reference.

Theorem. Let $a \in \mathbb{R}$ and let f, g, h be real-valued functions such that $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ in an open interval containing a (note that the inequality need not hold at a itself). If

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} g(x) = L$ as well.

I will scaffold the proof for you.

(a) First we will prove a useful inequality as an intermediate step.

Suppose that the assumptions of the squeeze theorem hold and let $x \neq a$ be in the interval. We handle two cases separately.

Case one: suppose $g(x) \geq L$. In this case, prove that

$$g(x) - L \leq h(x) - L.$$

Case two: suppose $g(x) \leq L$. In this case, prove that

$$L - g(x) \leq L - f(x)$$

Now explain why these two cases can be combined and re-written more concisely as

$$|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right).$$

(b) Now we can begin the proof itself. Let $\epsilon > 0$ be given. Since we have assumed $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, we can choose two numbers $\delta_f, \delta_h > 0$ such that

$$0 < |x - a| < \delta_f \implies |f(x) - L| < \epsilon,$$
$$0 < |x - a| < \delta_h \implies |h(x) - L| < \epsilon.$$

Complete the proof: what must we do next to show that $\lim_{x \to a} g(x) = L$?

(c) Use the squeeze theorem to prove that

$$\lim_{x \to 0} \left(\frac{1 - \cos(x)}{x} \right) = 0.$$

You may use the inequality we derived in session 14 to prove that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Case one is simple enough, we already know that $g(x) \leq h(x)$ by the beginning of the theorem, and it follows algebraically that subtracting L from both sides is still true, therefore it is true that $g(x) - L \leq h(x) - L$.

Case two took a wee bit more creative thought; we can take the clearly true statement $L = L$ and subtract an inequality we know, in this case $f(x) \leq g(x)$. Via this we receive the inequality we are interested in, $L - g(x) \leq L - f(x)$.

Awesomely enough, we then know that we can compact these phrases into the following inequality: $|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right)$ This is true since in the former case ($|g(x) - L| \leq |h(x) - L|$) as we know that when $g(x) \geq L$, also relatedly and importantly $g(x) - L \geq 0$ it is then true by the definition of absolute value that $|g(x) - L| \leq |h(x) - L|$.The latter case is also true by the definition of absolute value since, when $g(x) \leq L$ and $L - g(x) \leq L - f(x)$, then $|g(x) - L| \leq |f(x) - L|$. We know that $|g(x) - L| \leq |h(x), f(x) - L|$, and it is then true that $|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right)$

b: We begin where the scaffold left off, with the lovely statements $\begin{array}{l} 0 < |x-a| < \delta_f \implies |f(x)-L| < \epsilon, \\ 0 < |x-a| < \delta_h \implies |h(x)-L| < \epsilon. \end{array}$

We can then define a delta for the $g(x)$ function, ie $\delta_g = \min(\delta_f, \delta_h)$. I do not totally understand how to prove it, but it would seem that this function $g(x)$ could be successfully restricted to the smaller of the two domains since it is known that $f(x) \leq g(x) \leq h(x)$ when $x \neq a$.

c: (WIP)

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) We have assumed that $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ in an open interval containing a. Let x be in this interval; then those two inequalities imply that $g(x) - L \leq h(x) - L$ (by subtracting L) and that $L - g(x) \leq L - f(x)$ (by multiplying through by -1 and then adding L).

Note that the quantity $|g(x) - L|$ is given by $g(x) - L$ when $g(x) \geq L$ or by $L - g(x)$ when $g(x) < L$. We know that the first of these quantities, $g(x) - L$, is upper-bounded by $h(x) - L$, and that the second of these quantities, $L - g(x)$, is upper-bounded by $L - f(x)$. Further, we have proved that $y \leq |y|$ for all real numbers y, so $h(x) - L \leq |h(x) - L|$ and $L - f(x) \leq |f(x) - L|$.

Thus the absolute value itself must be upper-bounded by the larger of these two bounds, i.e.

$$|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right),$$

which is what we wanted to prove.

(b) Let $\epsilon > 0$ be given and choose δ_f, δ_h as in the problem statement.

Now pick $\delta_g = \min\left(\delta_f, \delta_h\right)$. We claim that, whenever $0 < |x - a| < \delta_g$, we will have $|g(x) - L| < \epsilon$.

To prove this, suppose that $0 < |x - a| < \delta_g$. By part (a), we have the bound

$$|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right),$$

but since $\delta_g \leq \delta_f$ and $\delta_g \leq \delta_h$, we have

$$|h(x) - L| < \epsilon \quad , \quad |f(x) - L| < \epsilon,$$

because of the way we chose δ_f and δ_g above. Therefore

$$|g(x) - L| \leq \max\left(\epsilon, \epsilon\right) = \epsilon,$$

which establishes that $\lim_{x \to a} g(x) = L$.

(c) We wish to prove that $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$. To do this, we will need to find two functions f and h such that

$$f(x) \leq \frac{1 - \cos(x)}{x} \leq h(x)$$

for all x in an interval containing 0 (but not at 0).

First multiply the top and bottom of our fraction by $1 + \cos(x)$ to see that

$$\begin{aligned} \lim_{x \to 0} \frac{1 - \cos(x)}{x} &= \lim_{x \to 0} \left(\frac{(1 - \cos(x))\,(1 + \cos(x))}{x\,(1 + \cos(x))} \right) \\ &= \lim_{x \to 0} \frac{1 - \cos^2(x)}{x\,(1 + \cos(x))}. \end{aligned}$$

Recall that we proved, by comparing the areas of three triangles, that

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

for all x in an interval around 0. Replace $\sin(x)$ by $\sqrt{1 - \cos^2(x)}$, which is true by the Pythagorean identity, to find

$$\cos(x) \leq \frac{\sqrt{1 - \cos^2(x)}}{x} \leq 1$$

All of these quantities are positive, so we may square each piece of the inequality to obtain

$$\cos^2(x) \leq \frac{1 - \cos^2(x)}{x^2} \leq 1$$

Multiply through by x and divide by $1 + \cos(x)$ to find

$$\underbrace{\frac{x\cos^2(x)}{1+\cos(x)}}_{f(x)} \leq \frac{1-\cos^2(x)}{x(1+\cos(x))} \leq \underbrace{\frac{x}{1+\cos(x)}}_{h(x)}.$$

This is the desired squeezing. The function on the left is a quotient of continuous functions (since we have proven that cosine is continuous), so its limit at zero is simply

$$\lim_{x\to 0} \frac{x\cos^2(x)}{1+\cos(x)} = 0,$$

and similarly for the function on the right:

$$\lim_{x\to 0} \frac{x}{1+\cos(x)} = 0.$$

By the squeeze theorem, then, we conclude that

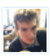
$$\lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = 0.$$

This is exactly the limit we were trying to evaluate. $\square$

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 8 months ago

As a side note, there is actually a faster way to prove the result in (c).

Once we get to the step

\begin{align} \lim_{x \to 0} \frac{1 - \cos^2 (x) }{x (1 + \cos (x)) } \end{align}

we can simply apply the Pythagorean identity in the numerator to write this as

\begin{align} \lim_{x \to 0} \frac{\sin(x) }{x (1 + \cos (x)) } = \left(\lim_{x \to 0} \frac{\sin(x) }{x } \right) \left(\lim_{x \to 0} \frac{1}{1 + \cos (x) } \right) \end{align}

where the step of splitting the limit of a product into a product of two limits is justified because we have proven that both of these limits exist.

Now we can immediately evaluate since we have proven that the first limit equals 0.