28.6

(a): Apply the limit comparison test; evaluate by comparing to $\frac{1}{n}$

$$\lim n \sin\left(\frac{1}{n}\right) = \lim \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

Since this goes to $0/0$, we can apply L'Hopital's rule;

$$= \lim \frac{\cos\left(\frac{1}{n}\right) \cdot -1 \cdot \frac{1}{n^2}}{-1 \cdot \frac{1}{n^2}} = \lim\left(\cos\left(\frac{1}{n}\right)\right) = 1$$

Since the ~~harr~~ harmonic series is divergent, then

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges.

(b): We see that

$$\sum_{n=1}^{k} a_n = b_k - b_0$$

Then by taking the limit of each side and applying $b_k \to 0$,

$$\sum_{n=1}^{\infty} a_n = 0 - b_0 = -b_0$$

Now, solve for A and B in

$$\cancel{\frac{A}{n} + \frac{B}{n^2} = \frac{1}{n^2+n}},$$

$$\Downarrow$$

$$\cancel{An + B = 1}$$

$$\frac{A}{n} + \frac{B}{n+1} = \frac{1}{n^2+1}$$

$$\Downarrow$$

$$An + A + Bn = 1$$

$$A + B = 0$$

$$A = 1$$

$$\Downarrow$$

$B = -1$; thus,

$$\frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$$

Then, ~~we must compute~~ for $a_n = \frac{1}{n} - \frac{1}{n+1}$, we must compute

$= b_{n-1} - b_n$

$$\sum_{n=1}^{\infty} a_n = b_0 = 1 \longleftrightarrow b_n = \frac{1}{n+1}$$

(c) We see that since $\left(\frac{2}{5}\right)^n$ and $\left(\frac{3}{5}\right)^n$ converge individually,

(1) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n} = \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$, then apply the sum of the geometric series for $r<1$, to see that

$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$, thus

$$(1)\ldots = \frac{5}{3}\cdot\frac{2}{5} + \frac{5}{2}\cdot\frac{3}{5} = \cancel{\frac{10}{6}} \cancel{+} \cancel{\frac{15}{6}} =$$

$$= \frac{2}{3} + \frac{3}{2}$$

$$= \frac{4}{6} + \frac{9}{6}$$

$$= \frac{13}{6} \quad \square$$