question 2 views

Daily Challenge 13.7

(**Due: Wednesday 9/12 at 12:00 noon eastern**)

(1) The MVT guarantees a point where the derivative equals the secant line slope on an interval.

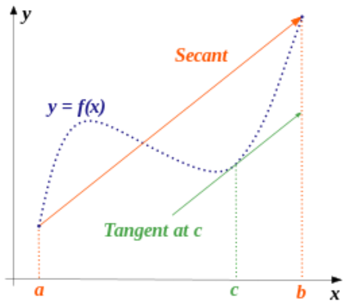
In our proof that the sign of the derivative controls whether a function is increasing or decreasing, we had to develop a very powerful result known as the *mean value theorem*.

Along with the intermediate value theorem and extreme value theorem, the MVT is among the most overpowered results in calculus. I repeat it here for convenience.

Theorem (MVT). Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Graphically, this is telling us that there must be *some* point in the interval at which the slope of the tangent line to f matches the slope of the secant line through the endpoints $(a, f(a))$ and $(b, f(b))$.



One can use this to obtain some surprisingly strong results. I provide two examples.

Example. Suppose a driver passes a point on the road at $t = 0$, then passes a second point on the road 70 miles away at time $t = 1$ hour. The speed limit along the road is 65 miles per hour. Prove that the driver exceeded the speed limit at some time.

Proof. Let $f(t)$ be the position of the car at time t, so that $f(0) = 0$ and $f(1 \text{ hour}) = 70 \text{ miles}$. Then the slope of the secant line on the interval is

$$\frac{f(b) - f(a)}{b - a} = 70 \frac{\text{miles}}{\text{hour}}.$$

If the function $f(t)$ is continuous and differentiable (which it should be, unless cars can teleport), then we may apply the mean value theorem to find that there was a time $t_0 \in (0, 1)$ at which $f'(t) = 70 \text{ mph}$. At the time t_0, the car exceeded the speed limit. $\square$

Here is a second example which is less playful.

Example. Let f be differentiable on (a, b) and continuous on $[a, b]$, and suppose that $f'(x) = 0$ for all $x \in (a, b)$. Prove that f must be a constant function.

Proof. Consider any two points $c, d \in (a, b)$ with $c < d$. By restricting to the interval $[c, d]$, we know that f is still continuous and differentiable on $[c, d]$, and thus we may apply the mean value theorem to conclude that there must exist a point $z \in (c, d)$ such that

$$f'(z) = \frac{f(d) - f(c)}{d - c}.$$

However, we have by assumption that $f'(z) = 0$, so we immediately conclude that $f(d) = f(c)$. But the two points d and c were completely arbitrary, so the function f must take on the same value at every point in $[a, b]$ (that is, f is constant). $\square$

(2) Problem: polynomial roots.

Recall that we say a polynomial $p(x)$ has n *distinct roots* if there are n different numbers x_i such that $p(x_i) = 0$. For instance, the polynomial $p(x) = x^2$ has only one distinct root (namely $x = 0$), while $p(x) = (x - 1)(x + 2)$ has two distinct roots (here they are $x = 1$ and $x = -2$).

Clearly, by the fundamental theorem of algebra, a polynomial of degree n can have at most n distinct roots (so long as we demand that the polynomial is not simply the constant function $p(x) = 0$, which has infinitely many roots). But it can certainly have fewer than n distinct roots, if a root occurs with multiplicity, as we see with examples like $p(x) = x^2$.

Challenge: suppose that f is a degree n polynomial with n distinct roots. Show that f' has exactly $n - 1$ distinct roots.

(Hint: apply the mean value theorem where the endpoints are roots of the original polynomial f.)

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
We are given that $p'(x)$ is an n degree polynomial with n distinct roots. Let a be the smallest of the distinct roots, and b be the largest of the distinct roots. We know that polynomials are continuous and differentiable at all points; We can then apply the mean value theorem to see that there exists some point c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. We see that since $f(b) = f(a) = 0$, then it must be true $f'(c) = 0$ and therefore c is a root of the derivative. We see that we can choose various pairs of the original distinct roots and find a point in that range where the derivative is zero. Intuitive idea follows: like a fork, there is one point in each "gap" between roots where the derivative equals zero, and as a result of that derivative being zero it is a distinct root of the derivative's graphed equation.

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Proof. Let $p(x)$ be a polynomial of degree n with n distinct roots. Enumerate the roots $x_1, x_2, \cdots, x_n$ from smallest to largest.

Because $p(x)$ is a polynomial, it satisfies the hypotheses of the mean value theorem everywhere, so we may apply the MVT to the interval $[x_1, x_2]$ to find that there exists some $y_1 \in (x_1, x_2)$ where

$$f'(y_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0,$$

where in the last step we have used the assumption that $f(x_2) = 0 = f(x_1)$.

But clearly we may apply the argument again to the interval $[x_2, x_3]$, to $[x_3, x_4]$, and so on, up to $[x_{n-1}, x_n]$. In this way we construct a sequence of $n-1$ points $y_1, \cdots, y_{n-1}$ with the property that $f(y_i) = 0$.

Thus the derivative $f'(x)$ has at least $n-1$ distinct roots, the y_i constructed above. On the other hand, by the power rule we know that $f'(x)$ is a polynomial of degree $n-1$, so by the fundamental theorem of algebra it has a total of $n-1$ complex roots counting multiplicity. In particular, this means that it can have *at most* $n-1$ distinct roots.

We conclude that $f'(x)$ has precisely $n-1$ distinct roots, as claimed. $\square$

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*