The Squeeze Theorem & Application of IVT

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August 1, 2018

Beginning the proof of the Squeeze Theorem.

We have the lovely theorem below to prove today:
Theorem: Let $A \in \mathbb{R}$ and let f, g, h be real valued function such that $f(x \leq g(x) \leq h(x)$. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} h(x) = L$

It's algebra for now.

Our proof begins by finding a useful inequality. Assuming the functions we are operating on abide by our hypothesis, we then have two cases to work with:

- $g(x) \geq L$: We already know that $g(x) \leq h(x)$, so we can simply subtract our limit L from each side to receive that $g(x) - L \leq h(x) - L$.
- $g(x) \leq L$: We know from the assumptions that $g(x) \leq f(x)$, so we can take that it is true reflexively that $L = L$, and subtract from this $g(x) \leq f(x)$ to receive $L - g(x) \leq L - f(x)$

It's slightly less algebra now.

Once again, in our former case since $g(x) \geq L$ and in turn $g(x) - L \geq 0$, it is true that $g(x) - L = |g(x) - L| \leq |h(x) - L| = h(x) - L$.

In the second case we in turn have by the definition of absolute value that since $g(x) - L \leq 0$, then $L - g(x) = |g(x) - L| \leq |f(x) - L| = L - f(x)$.

We have then shown that $|g(x) - L|$ is less than two unique statements, and it is then true that $|g(x) - L| \leq \max\left(|h(x) - L|, |f(x) - L|\right)$

It's calculus now.

We have assumed in our hypothesis that the functions f and g have limits, therefore for all $\epsilon > 0$ we have that there exists δ_f and δ_g such that
$0 < |x - a| < \delta_f \implies |f(x) - L| < \epsilon$ and as well
$0 < |x - a| < \delta_h \implies |h(x) - L| < \epsilon$. We can then set δ_g (the delta we are using for the g function's limit) to be equal to the minimum of these two deltas, ie $\delta_g = \min(\delta_f, \delta_h)$. We can now show that $|g(x) - L| < \epsilon$. Suppose that $0 < |x - a| < \delta_g$. We refer back to our *useful* inequality and know that $|g(x) - L| \leq \max(|h(x) - L|, |f(x) - L|)$. However, since $\delta_g \leq \delta_f, \delta_h$, it is in then true that $|h(x) - L| < \epsilon$ and $|g(x) - L| < \epsilon$, allowing us to conclude that since $|g(x) - L| \leq \max(\epsilon, \epsilon)$, in turn $|g(x) - L| < \epsilon$ and therefore $\lim_{x \to a} f(x) = L$. $\square$

Our workhorse function

Theorem: Define a function f by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0 \end{cases} \tag{1}$$

Then f is not continuous for the domain $[1, -1]$.

Exploration:

Simply by looking at this piecewise function, we can see that we must show that f is discontinuous at 0 (just by the way f is enticingly constructed). To do so, we have to somehow show that $\lim_{x \to 0} f(x) \neq f(x)$. As well, by looking at Desmos one can see that the function f begins increasingly rapidly oscillating as $x \to 0$. Since this function has a range of $[1, -1]$ for the domain $[1, -1]$, we can simply set epsilon sufficiently small, after all by the definition of limit it only need be true that $\epsilon > 0$.

Beginning the proof

We shall set $\epsilon = \frac{1}{2}$. Suppose by way of contradiction that there exists δ such that $|f(x) - L| < \epsilon$ is true for the domain $[-\delta, \delta]$ that is implied in this limit. We can exploit the periodicity of the sine function and see that that we can choose m and n sufficiently large such that

$$x_1 = \frac{1}{2\pi n + \frac{\pi}{2}} \text{ and } x_2 = \frac{1}{2\pi n + \frac{3\pi}{2}}$$

are in our domain $[-\delta, \delta]$. We can then see regardless of m and n then $f(x_1) = 1$ and $f(x_2) = -1$, contradicting and showing us that as $x \to 0$ the limit does not exist, and therefore f is not continuous on $[1, -1]$. $\square$

The conclusion of the IVT, part 0

Theorem: Let f be the piecewise function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0 \end{cases} \tag{2}$$

Then f satisfies the conclusion of the Intermediate Value Theorem.

The conclusion of the IVT, part 1

To show that this function f is consistent with the conclusion of the IVT (if f takes on two values somewhere $[-1, 1]$, then it takes on every value in between) we must consider the potential placements of two values a and b. First let $[a, b]$ be a non-null subset of $[-1, 1]$. First if $0 \notin [a, b]$, then it is automatically true that f is continuous for $[a, b]$ as f is only discontinuous at 0. Therefore by the intermediate value theorem, we can choose any number y between $f(a)$ and $f(b)$ and there exists some $c \in (a, b)$ such that $f(c) = y$, pretty generic for now.

The conclusion of the IVT, part 2

In the case where $0 \in [a, b]$, we must show that for y in between $f(a)$ and $f(b)$, there exists a $c \in (a, b)$ such that $f(c) = y$. We must show that this c exists. We have $-1 \leq f(a), f(b) \leq 1$, then in turn $-1 \leq y \leq 1$. We can use the inverse sine function to do our bidding here as we are operating within it's domain; apply $\sin^{-}1(y)$ to get that there exists some number c' such that $\sin(c') = y$. Once again we can exploit the periodicity of the sine function and let $c = \frac{1}{2\pi n + c'}$ where n is large enough such that $c \in [a, b]$. We then have that $f(c) = \sin(2\pi n + c') = \sin(c') = y$, and therefore $f(c) = y$ and the conclusion of the intermediate value theorem is true for this function f. $\square$

g is a function satisfying the conclusion of the intermediate value theorem, and takes on each value only once. We shall show that g is continuous by way of contradiction. Suppose by way of contradiction that there exists a point a where $\lim_{x\to a} g(x) \neq g(x)$. We can take the negation of the definition of a limit at a continuous point to get what is meant by a limit not being continuous at a point: "There exists some $\epsilon > 0$ for which it is true that, no matter what $\delta > 0$ you pick, there will always be some values of x where $|x - a| < \delta$ but still $|g(x) - g(a)| > \epsilon$." We can then choose a value of epsilon such that the previous statement is true, and in turn it is true that regardless of how "close" we get to a, then our input x will either have that $g(a) + \epsilon < g(x)$ or $g(x) < g(a) - \epsilon$.

Without loss of generality assume the former, and for a newly defined input x_1. Therefore $g(x_1) > g(a) + \epsilon$. From this we then have that there exists a $c \in (a, x_1)$ so that $g(c) = g(a) + \frac{\epsilon}{2}$. We can once again refer to the fact that we have assumed g is discontinuous at some point a, and therefore there exists more values x such that $g(x) > g(a) + \epsilon$. We can then say x_2 is another value on this interval (a, c). We have now found that $a < x_2 < c < x_1$ and that $g(x_1), g(x_2) > \epsilon$, and that $g(c) = g(a) + \frac{\epsilon}{2}$. By the IVT, we can see that this information contradicts our claim that g outputs each number only once, as we see that there exists $y_1 \in (x_2, c)$ where $g(y_1) = g(a) + \epsilon$ and another value $y_2 \in (c, x_1)$ where $g(y_2) = g(a) + \epsilon$. This contradiction then verifies our claim that g must be continuous. $\square$