question **2** views

Daily Challenge 16.1

(**Due: by session 40 at the very latest)**

In this challenge you'll practice some applications of L'Hopital's rule. This is question 5 on CD 4.

(1) Problem: Some L'Hopital calculations.

Find the following limits.

- $\lim_{x\to 1} \frac{x^2-1}{\log(x)}$
- $\lim_{x\to 0} \frac{\cos^2(x)-1}{x^2}$
- $\lim_{t\to 0} \frac{\sin(at)}{\sin(bt)}$, where a and b are nonzero real numbers.
- Explain what is wrong with the following application of L'Hopital's rule:

  $$\lim_{x\to 1} \frac{x^3+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{3x^2+1}{2x-3} = \lim_{x\to 1} \frac{6x}{2} = 3.$$

  Then find the true value of the limit.

daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

A few basic calculations to get back in the swing of things (finally).

A: We can apply L'Hopital's rule here since the top and bottom go to zero as $x \to 1$; We see that $\lim_{x\to 1} \frac{x^2-1}{\log(x)} = \lim_{x\to 1} \frac{2x}{\frac{1}{x}} = \lim_{x\to 1} 2x^2$. This is a polynomial and is therefore continuous, and the value of this limit equals 2.

B: We can apply L'Hopital's rule here since the top and bottom go to zero as $x \to 0$; We take the derivative of the numerator and denominator of this quotient of interest and see by the chain rule that $\lim_{x\to 0} \frac{\cos^2(x)-1}{x^2} = \lim_{x\to 0} \frac{2\cos(x)\cdot - \sin(x)}{2x}$; this also abides by the assumption of our lovely Frenchman, so we see $\lim_{x\to 0} \frac{2\cos(x)\cdot - \sin(x)}{2x} = \lim_{x\to 0} \frac{-2\sin(x)\cdot - \sin(x) - 2\cos^2(x)}{2}$, and this final limit isn't discontinuous as $x \to 0$, so $L = -1$

C: L'Hopital's rule is satisfied; we continue and can see that $\lim_{t\to 0} \frac{\sin(at)}{\sin(bt)} = \lim_{t\to 0} \frac{\cos(at)\times a}{\cos(bt)\times b}$. We then see that this is equal to $\frac{a}{b}$.

D: This application of L'Hopital's rule is incorrect, we see in the second step that L'Hopital's rule is applied even though the top and bottom do not approach zero at the limit. I shall shortcut and note that neither approach zero, and neither are close to zero, so I simply insert the value and we see that the result matches the instrcutor value, -4.

Updated 6 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

- Since $1^2-1=0$ and $\log(1)=0$, both the numerator and denominator vanish at the limit point. Further, the functions in the top and bottom are both differentiable on intervals containing $x=1$. Thus we may apply L'Hopital's rule. This gives

  $$\lim_{x\to 1} \frac{x^2-1}{\log(x)} = \lim_{x\to 1} \frac{2x}{\frac{1}{x}} = 2.$$

- Again the numerator and denominator go to zero at the point in question but are differentiable everywhere, so by L'Hopital,

  $$\lim_{x\to 0} \frac{\cos^2(x)-1}{x^2} = \lim_{x\to 0} \frac{-2\cos(x)\sin(x)}{2x}.$$

  This result can be split into a product of two limits,

  $$\lim_{x\to 0} \frac{-2\cos(x)\sin(x)}{2x} = -\left(\lim_{x\to 0} \cos(x)\right) \cdot \left(\lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)\right),$$

  since we have proven that cosine is continuous and that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, and that one can split a limit of a product into the product of the limits assuming that both limits exist.

Doing this, we see immediately that the result is -1.

Another way to see this: go back to the limit $\lim_{x\to 0} \frac{-2\cos(x)\sin(x)}{2x}$. This result is another quotient where the top and bottom go to zero, yet both functions are again differentiable. So it seems that the hypothesis of L'Hopital's rule are still satisfied, and we can differentiate again:

$$= \lim_{x\to 0} \frac{-2\left(-\sin^2(x)+\cos^2(x)\right)}{2} = -1.$$

- Again the hypotheses of L'Hopital's rule are satisfied, so one finds

$$\lim_{t\to 0} \frac{\sin(at)}{\sin(bt)} = \lim_{t\to 0} \frac{a\cos(at)}{b\cos(bt)} = \frac{a}{b}.$$

- The first step is correct because both numerator and denominator go to zero at the point in question. Indeed, $1^3+1-2=0$ and $1^2-3\cdot 1+2=0$

  However, the second step is incorrect because now neither the numerator nor denominator goes to zero; the top goes to 4 and the bottom goes to -1. We cannot apply L'Hopital's rule in this case.

  The correct limit is obtained by applying our theorems about limits of quotients in the second step. This gives the true limiting value of -4.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*