question 2 views

Daily Challenge 13.1

(~~Due: Monday 8/27 at 12:00 noon eastern~~)
(**Due: Friday 8/31 at 12:00 noon eastern**)

Time to start banging out CD 3 problems in earnest. This exericse is question one on the CD; please either work directly in Overleaf or else copy over your answer (and correct the formatting) when you're done.

(1) Quick calculations.

You should be able to compute each of the following derivatives quickly (in under one minute each). Practice until differentiation is as automatic as addition or multiplication.

(a) Differentiate $\sin^3\left(x^2+\log(x)\right)$.

(b) Differentiate $\exp\left((x+1)^2(x+2)\right)$.

Note that we will sometimes use the notation $\exp(x)$ instead of e^x; they mean the same thing. That is, this function can also be written as

$$\exp\left((x+1)^2(x+2)\right) \equiv e^{(x+1)^2(x+2)}.$$

(c) Differentiate $\tan^2\left(\frac{x}{x+1}\right)$.

(d) Differentiate $\left(e^{x^2}\right)^2$.

(e) Find the derivative of $\cos^{-1}(x)$ and simplify until there are no trigonometric functions remaining.

(f) Calculate $\frac{dy}{dx}$ for $x^{1/3}+y^{1/3}=1$ by implicit differentiation (that is, differentiate both sides with respect to x, then solve for $y'(x)$). Then calculate $\frac{dy}{dx}$ in a second way: solve for y explicitly and calculate y' using the chain rule. Confirm that your answers are the same.

(g) Suppose $PV^c = nRT$, and assume $P=P(V)$ with all other letters representing constants. Compute $\frac{dP}{dV}$.

(h) Suppose $F=\frac{mg}{(1+r^2)^{3/2}}$, and assume $F=F(r)$ with all other letters representing constants. Compute $\frac{dF}{dr}$.

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: We see first by the chain rule that $f'(x) = 3\sin^2(x^2+\log(x))\cdot\left(\cos(x^2+\log(x))\right)' \qquad = 3\sin^2(x^2+\log(x))\cdot -\sin(x^2+\log(x))\cdot(2x+\dfrac{1}{x})$

b: We see by the chain rule that

$$\begin{aligned} f'(x) &= \exp((x+1)^2(x+2)\cdot\left(((x+1)^2)'(x+2)+x(x+1)^2\right) \\ &= \exp((x+1)^2(x+2)\cdot\left(2x(x+1)(x+2)+x(x^2+2x+1)\right) \\ &= \exp((x+1)^2(x+2)\cdot\left(2x(x^2+3x+2)+(x^3+2x^2+x)\right) \\ &= \exp((x+1)^2(x+2)\cdot\left((3x^3+8x^2+5x)\right) \end{aligned}$$

c: First by the chain rule where the inner is $\tan(\frac{x}{x+1})$ and the outer is x^2, then

$$\begin{aligned} f'(x) &= 2\tan(\frac{x}{x+1})\cdot(\tan(\frac{x}{x+1}))' \\ &= 2\tan(\frac{x}{x+1})\cdot\sec^2(\frac{x}{x+1})\cdot(\frac{x}{x+1})' \\ &= 2\tan(\frac{x}{x+1})\cdot\sec^2(\frac{x}{x+1})\cdot(\frac{1}{(x+1)^2}) \end{aligned}$$

d: :thinking:

$$\begin{aligned} f'(x) &= ((e^x)^4)' \\ &= (e^{x^2}\cdot e^{x^2})' \\ &= (e^{2x^2})' \\ &= (e^{2x^2})\cdot 4x \\ &= 4xe^{2x^2} \end{aligned}$$

e: We see by the inverse function rule that $f'(x)=\frac{1}{\sin(\cos^{-1}(x))}$. By inserting this denominator into Wolfram Alpha (because I have no better ideas and don't recognize it) we see that $\sin(\cos^{-1}(\theta))=\sqrt{1-x^2}$, therefore $f'(x)=\frac{1}{\sqrt{1-x^2}}$.

f:

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) Differentiate $\sin^3\left(x^2+\log(x)\right)$.

Solution. By the chain rule, the derivative is $3\sin^2\left(x^2+\log(x)\right)\cdot\cos\left(x^2+\log(x)\right)\cdot\left(2x+\frac{1}{x}\right)$

(b) Differentiate $\exp\left((x+1)^2(x+2)\right)$.

Solution. Chain to find the derivative: $\exp\left((x+1)^2(x+2)\right)\cdot\left(2(x+1)(x+2)+(x+1)^2\right)$ which can be factored to $\exp\left((x+1)^2(x+2)\right)\cdot(x+1)\,(3x+5)$ If you prefer, we can write this as

$$e^{(x+1)^2(x+2)}\cdot\left(3x^2+8x+5\right).$$

(c) Differentiate $\tan^2\left(\frac{x}{x+1}\right)$.

Solution. The derivative of $\tan(x)$ is $\sec^2(x)$, so the derivative of $f(x)=\tan^2\left(\frac{x}{x+1}\right)$ is

$$f'(x)=2\tan\left(\frac{x}{x+1}\right)\cdot\sec^2\left(\frac{x}{x+1}\right)\cdot\frac{1}{(x+1)^2}.$$

(d) Differentiate $\left(e^{x^2}\right)^2$.

Solution. Chain rule: one finds $4xe^{x^2}\left(e^{x^2}\right)=4xe^{2x^2}$.

(e) Find the derivative of $\cos^{-1}(x)$ and simplify until there are no trigonometric functions remaining.

Solution. By the inverse function rule, if $f(x)=\cos^{-1}(x)$, then $f'(x)=-\frac{1}{\sin(\cos^{-1}(x))}$. But using the Pythagorean identity, $\sin(\cos^{-1}(x))=\sqrt{1-\cos^2(\cos^{-1}(x))}=\sqrt{1-x^2}$, so

$$f'(x)=-\frac{1}{\sqrt{1-x^2}}.$$

(f) Calculate $\frac{dy}{dx}$ for $x^{1/3}+y^{1/3}=1$ by implicit differentiation (that is, differentiate both sides with respect to x, then solve for $y'(x)$). Then calculate $\frac{dy}{dx}$ in a second way: solve for y explicitly and calculate y' using the chain rule. Confirm that your answers are the same.

Solution. Using implicit differentiation,

$$\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y'=0,$$

which can be solved for y' to find

$$y'=-\frac{y^{2/3}}{x^{2/3}}.$$

On the other hand, we could solve for y explicitly in the original equation, finding $y^{1/3}=1-x^{1/3}$ and hence

$$y=\left(1-x^{1/3}\right)^3.$$

Differentiate to find

$$\begin{aligned} y'(x) &= 3\left(1-x^{1/3}\right)^2\cdot\left(-\frac{1}{3}x^{-2/3}\right)\\ &= -\frac{\left(1-x^{1/3}\right)^2}{x^{2/3}}. \end{aligned}$$

But the numerator is precisely $y^{2/3}$, so the two answers agree.

(g) Suppose $PV^c=nRT$, and assume $P=P(V)$ with all other letters representing constants. Compute $\frac{dP}{dV}$.

Solution. We have $P=nRTV^{-c}$, so

$$\frac{dP}{dV} = -cnRTV^{-c-1}.$$

Note that this can be simplified: substitute into $cnRTV^{-c-1} = cnRTV^{-c}V^{-1} = cPV^{-1}$ using the definition of P, so that

$$\frac{dP}{dV} = -\frac{cP}{v}.$$

(h) Suppose $F = \frac{mg}{(1+r^2)^{3/2}}$, and assume $F = F(r)$ with all other letters representing constants. Compute $\frac{dF}{dr}$.

Solution. Differentiate to find

$$\begin{aligned}\frac{dF}{dr} &= -\frac{3}{2}mg\left(1+r^2\right)^{-5/2} \cdot 2r \\ &= \frac{-3mgr}{(1+r^2)^{5/2}}.\end{aligned}$$

Mathematica verification follows.

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In[19]:= D[(Sin[x])^3, x]
Out[19]= 3 Cos[x] Sin[x]^2

In[21]:= D[ArcCos[x], x]
Out[21]= -1/Sqrt[1 - x^2]

In[22]:= D[(Sin[x^2 + Log[x]])^3, x]
Out[22]= 3 (1/x + 2 x) Cos[x^2 + Log[x]] Sin[x^2 + Log[x]]^2

In[24]:= Simplify[D[Exp[(x + 1)^2 (x + 2)], x]]
Out[24]= E^((1 + x)^2 (2 + x)) (5 + 8 x + 3 x^2)

In[26]:= Simplify[D[(Tan[x/(x + 1)])^2, x]]
Out[26]= 2 Sec[x/(1 + x)]^2 Tan[x/(1 + x)] / (1 + x)^2

In[27]:= D[(E^(x^2))^2, x]
Out[27]= 4 E^(2 x^2) x

In[28]:= D[ArcCos[x], x]
Out[28]= -1/Sqrt[1 - x^2]

In[30]:= Solve[D[x^(1/3) + y[x]^(1/3) == 1, x], y'[x]]
Out[30]= {{y'[x] -> -y[x]^(2/3)/x^(2/3)}}

In[32]:= Solve[D[P[V] V^c == n R T, V], P'[V]]
Out[32]= {{P'[V] -> -c P[V]/V}}

In[33]:= D[F[r] == m g/(1 + r^2)^(3/2), r]
Out[33]= F'[r] == -3 g m r/(1 + r^2)^(5/2)
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Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*