question 2 views

Daily Challenge 11.3

~~(Due: Monday 8/13 at 12:00 noon eastern)~~
(Due: Tuesday 8/14 at 12:00 noon eastern)

(1) The binomial theorem lets us raise a sum to an integer power.

In section (2) I will appeal to a useful result called the binomial theorem, which one should really learn in algebra II -- although one usually doesn't, unless the course uses the AoPS algebra textbook (section 11.4). The claim is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The symbol $\sum_{k=0}^{n}$ means "add up the terms you would get by plugging in $k=0$, $k=1$, and so on, up to $k=n$, into whatever you see to the right." That is, thinking Pythonically,

$$\sum_{k=0}^{n} \text{thing}(k)$$

$$\updownarrow$$

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sum( [thing(k) for k in range(n+1)] )
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The symbol $\binom{n}{k}$ means $\frac{n!}{k!(n-k)!}$. The expression $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$is called the *factorial* of n: it means to multiply together all integers between 1 and n inclusive (except in the special case of zero, where we define $0! = 1$). For instance, $3! = 3 \cdot 2 \cdot 1 = 6$.

You will use the binomial theorem in a second proof of the power rule, and to prove the Leibniz rule in problem 9 of consolidation document 3.

For now, we will content ourselves with an example: let's use the binomial theorem to write out $(x+y)^4$. Of course, we could do this by brute force by repeatedly FOILing, but let's illustrate how the formula works. We have

$$\begin{aligned}(x+y)^4 &= \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k} \\ &= \binom{4}{0} x^0 y^{4-0} + \binom{4}{1} x^1 y^{4-1} + \binom{4}{2} x^2 y^{4-2} + \binom{4}{3} x^3 y^{4-3} + \binom{4}{4} x^4 y^{4-4} \\ &= y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + x^4.\end{aligned}$$

In the first step, I have expanded out the "sigma notation $\sum_{k=0}^{4}$ using its definition as a sum of terms where we successively replace k by 0, then 1, etc.

In the second step, I have evaluated all of the "binomial coefficients". For instance, $\binom{4}{1} = \frac{4!}{1!3!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(1)(3 \cdot 2 \cdot 1)} = 4$. Similarly,

$$\binom{4}{2} = \frac{4!}{2!}2! = \frac{4 \cdot 3 \cdot 2}{2 \cdot 2} = 6,$$

and so on.

We conclude that the binomial theorem gives us the expansion

$$(x+y)^4 = y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + x^4.$$

You might recognize these coefficients $(1, 4, 6, 4, 1)$ from the fifth row of Pascal's triangle:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

That is, of course, no accident.

It will not strain the reader's imagination to see that the binomial theorem will be useful in obtaining the derivative of x^n, whose difference quotient looks like $\frac{(a+h)^n - a^n}{h}$, since it gives a nice formula for $(a+h)^n$.

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(2) The derivative gives the slope of the tangent line.

We saw yesterday that the *tangent line* to a graph f at a point a is defined as the line ℓ through the point $(a, f(a))$ which has slope given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if that limit exists.

It is sometimes interesting to find the explicit equation for the tangent line, i.e. both the slope m and the intercept b for the equation of the line ℓ given in the form $y = mx + b$. We can do this by first computing the derivative, and then using standard algebra I techniques to find b.

Example. Working directly from the definition, find the equation of the tangent line to the curve $y = x^4 + 3$ at the point $(1, 4)$.

Solution. This would be very easy to differentiate with the power rule, but we must resort to the definition. First we compute $f'(a)$. Consider the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^4 + 3 - a^4 - 3}{h}.$$

We will need to expand $(a+h)^4$. Thankfully, we have worked out exactly this expansion in part (1): simply let $x = a$ and $y = h$ to find

$$(a+h)^4 = a^4 + 4ah^3 + 6a^2h^2 + 4a^3h + h^4.$$

After cancelling the terms $a^4 - a^4$ and $3 - 3$ in the numerator, the limit becomes

$$\begin{aligned} \lim_{h \to 0} \frac{4ah^3 + 6a^2h^2 + 4a^3h + h^4}{h} &= \lim_{h \to 0} \left(4ah^2 + 6a^2h + 4a^3 + h^3\right) \\ &= 4a^3, \end{aligned}$$

where in the first step I have canceled a factor of h in the numerator and denominator (since we may assume $h \neq 0$), and in the second step I have used that the expression $\left(4ah^2 + 6a^2h + 4a^3 + h^3\right)$ is a polynomial in h, and we have proven that polynomials are continuous, so the value of the limit is simply obtained by taking $h = 0$.

So, as expected by the power rule, we have shown that $f'(a) = 4a^3$. In particular, the slope of the tangent line to $y = x^4 + 3$ at the point $(1, 4)$ is $4 \cdot (1)^3 = 4$.

Finally, we need to find the intercept of the line. We are given the slope m and a point $(1, 4)$ on the line. Using the point-slope form,

$$y - y_1 = m(x - x_1),$$

one finds

$$y - 4 = +m(x - 1),$$

or substituting $m = 4$, we conclude that the line ℓ is given by

$$y = 4x.$$

(3) Problem: practicing tangent lines and binomials.

(a) Find the equation of the tangent line to the graph $f(x) = \frac{1}{2}x^2$ at the point $(2, 2)$. You may use your result(s) from DC 11.1.

(b) Let n be a positive integer. Find (with proof) the coefficient of the term ha^{n-1} in the expansion of $(a+h)^n$.

[Scaffold: In other words: after completely foiling out, the quantity $(a+h)^n$ will look like $c_0a^n + c_1a^{n-1}h + c_2a^{n-2}h^2 + \cdots + c_nh^n$, where c_i are some numbers. This problem asks you to find c_1, the coefficient of the ha^{n-1} term.

For example, we proved above that $(a+h)^4 = a^4 + 4ah^3 + 6a^2h^2 + 4a^3h + h^4$, so when $n = 4$, the desired coefficient is also 4. In this problem you generalize to find that coefficient for any n, not just $n = 4$.

To prove the result, use the binomial theorem.]

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: We have shown in the past that the derivative of a function $g(x) = x^2$ is $f'(x) = 2x$. Relatedly, we have shown that $(cf(x))' = c(f(x))'$. We can then see that for the function we are given $h(x) = \frac{1}{2}x^2$ then $h'(x) = x$. Therefore the slope of our tangent line at point $(2, 2)$ is 2. We can then solve for b in the equation $y = 2x + b$ given $x, y = 2$ and get the final equation of our tangent line, $y = 2x - 2$.

b: We would like to foil out the given $(a+h)^n$ somehow. We know by the binomial theorem one way we can continue, and see that $(a+h)^n = a^n + \binom{n}{1}ha^{n-1} + \cdots$ and so on, fortunately we needn't go any further. We then have that our $\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$ thanks to the magical knowledge of statistics bestowed upon myself by Mrs. Mauro.

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) In DC 11.1, we proved that the derivative of $g(x) = x^2$ at a is $g'(a) = 2a$. We also proved that the derivative commutes with multiplication by a constant, so since the function of interest is $f(x) = \frac{1}{2}x^2 = \frac{1}{2}g(x)$, its derivative is $f'(a) = a$.

Thus we know that the slope of the tangent line to f at $(2, 2)$ is 2. Using the point-slope form of the line,

$$y - 2 = 2(x - 2)$$

which yields the equation $y = 2x - 2$ of the tangent line.

(b) By the binomial theorem,

$$(a+h)^n = a^n + \binom{n}{1}ha^{n-1} + \cdots,$$

where $\cdots$ represents terms that we do not care about.

Using the definition of the binomial coefficients, we see that

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n.$$

Thus the coefficient of the ha^{n-1} term is n.

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*