question

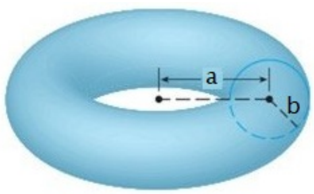
Daily Challenge 21.4

(Due: Friday 2/15 at 12:00 noon eastern)

(1) Problem: volume of a torus.

After the sphere, the *torus* (the shape of a bagel or donut) is perhaps the most important shape in string theory. Like the sphere, it can be generalized to higher-dimensional versions; these are useful in so-called *string compactifications*.

In this problem, you will compute the volume of a torus with radius a and cross-sectional radius b:

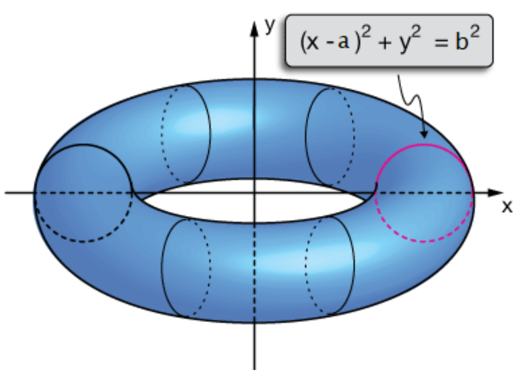


This shape is obtained by rotating a circle of radius b around a line in the same plane as the circle, where a is the distance between the line and center of the circle.

(a) Note that a circle of radius b centered at $(a, 0)$ has the equation

$$(x-a)^2 + y^2 = b^2.$$

Solve this for y, including both the positive and negative signs on the square root; the positive sign gives the upper half of a cross-section of the torus (shown red below), and the negative sign gives the lower half.



(b) First consider the upper half of a cross-section (positive root in part (a)). Slice the torus into a cylindrical shell at a fixed value of x. You may want to draw a picture to help visualize this.

Write down an integral which adds up the volume contributions from these cylindrical shells. You may use that the area of a cylinder of radius r and height h is $2\pi rh$. (Hint: your integral should run from $x = a - b$ to $x = a + b$).

(c) Double the integral you wrote down in (b) to account for the lower half of the cross-section. Make the substitution $u = x - a$ and evaluate the resulting integral (one of the terms vanishes by symmetry; if you can explain why, you need not compute it!).

You should find that the volume of the torus is $2\pi^2 ab^2$.

daily_challenge

Updated 1 month ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachuk:
(a): This is a simple bit of algebra;
$$(x-a)^2 + y^2 = b^2 \implies y = \pm\sqrt{b^2 - (x - a^2)^2}$$

(b): We would like to write down a formula to find the area of a cylindrical cross section of the top half of a torus a distance from the center; recall that the a cylinder has wall surface area $2\pi rh$, where we see by the equation we found in (a) that $h = y = \sqrt{b^2 - (x - a^2)^2}$ and simply we simply substitute $r = x$:

$A(x) = 2\pi x\sqrt{b^2 - (x - a^2)^2}$

We can now integrate over x from $a - b \to a + b$

$\frac{1}{2}V_t(a, b) = 2\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x - a)^2}\, dx$

Where of course the leading $\frac{1}{2}$ is because this is the top half of the torus.

(c): We begin evaluating this integral by doubling each side of the equation we left off on in (b), since that only considered one half, the top half, of our torus.

$V_t(a, b) = 4\pi \int_{a-b}^{a+b} x\sqrt{b^2 - (x - a)^2}\, dx$

We now $u - sub$, where we let $u = x - a$ and we don't have to consider the du since it is 1.

$V_t(a, b) = 4\pi \int_{-b}^{b} (u + a)\sqrt{b^2 - u^2}\, du$

We now trig-sub where we let $u = b\sin(\theta)$ and thus $du = b\cos(\theta)\, d\theta$. Thus,

$V_t(a, b) = 4\pi \int_{-\pi/2}^{\pi/2} (b\sin(\theta) + a)b\cos(\theta)\sqrt{b^2(1 - \sin^2)}\, d\theta$

Recall that $1 - \sin^2(z) = \cos^2(z)$, and pass everything through the square root.

$V_t(a, b) = 4\pi b^2 \int_{-\pi/2}^{\pi/2} (b\sin(\theta) + a)\cos^2(\theta)\, d\theta$

We then distribute,

$4\pi b^2 \int_{-\pi/2}^{\pi/2} b\sin(\theta)\cos^2(\theta)\, d\theta + 4\pi b^2 \int_{-\pi/2}^{\pi/2} a\cos^2(\theta)\, d\theta$

We see by graphing the first term that it is odd for the range of interest, and thus is equal to zero. We just need to find

$\cdots = 4\pi b^2 a \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta$

We also recall the cosine double angle formula $\cos(2x) = 2\cos^2(x) - 1 \implies \cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$

$$\begin{aligned}
\cdots &= 2\pi b^2 a \int_{-\pi/2}^{\pi/2} \cos(2\theta) + 1\, d\theta \\
&= 2\pi b^2 a \Big[\frac{1}{2}\sin(2\theta) + \theta\Big]_{-\pi/2}^{\pi/2} \\
&= 2\pi b^2 a(\pi/2 - (-\pi/2)) \\
&= 2\pi^2 b^2 a
\end{aligned}$$

:clap:

Updated 1 month ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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followup discussions *for lingering questions and comments*