question 4 views

Daily Challenge 1.2

(**Due: Wednesday 4/25 at 12:00 noon Eastern.**)

Also, friendly reminder that the first challenge is due at 12 noon today.

Review

Number systems are like breakups: the good ones have **closure**.

We say that the integers (recall these are $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$) are closed under addition because adding two integers gives you another integer. The integers are *not* closed under division, because dividing two integers -- for instance, dividing 1 by 2 to get $\frac{1}{2}$ -- does *not* always give another integer.

The above paragraph builds intuition, but in math, we need precise definitions for doing proofs. I will define closure in two ways: first with words, and then with symbols.

- A number system is **closed** under some operation if performing that operation on elements of the number system outputs another element of the number system.
- A number system $\mathbb{A}$ is **closed** under some operation T if $T(a_1, a_2, \cdots, a_n) \in \mathbb{A}$ whenever $a_1 \in \mathbb{A}, \cdots, a_n \in \mathbb{A}$.

Last time we proved that the rational numbers are closed under addition. In other words,

Theorem. The sum of two rational numbers is a rational number.

Proof. Let p and q be rational numbers. By the definition of rational numbers, this means we can write $p = \frac{a}{b}$ and $q = \frac{m}{n}$ for some integers a, b, m, n. But then the sum $p + q$ can be written as $\frac{an+mb}{nb}$. The numerator, $an + mb$, is an integer because it is the sum of two products of integers, and the denominator nb is an integer because it is the product of two integers. We also know that the denominator nb cannot be zero since n and b are nonzero, so the result is a well-defined rational. Therefore, $p + q$ can be written as the ratio of two integers, so $p + q \in \mathbb{Q}$. $\square$

You can also check that the rationals are closed under multiplication; the proof is similar (the product of ratios of integers, $\frac{a}{b} \cdot \frac{m}{n} = \frac{am}{bn}$, is another ratio of integers, and we know that $bn \neq 0$ whenever $b \neq 0$ and $n \neq 0$).

Let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the set of *irrational* numbers (this is not standard notation, but we'll use it here anyway). Then $\mathbb{I}$ contains numbers like $\sqrt{2}, \pi, -\sqrt{2}, e$, and so on.

Problem

For each of the following, describe whether you think the statement is true or false. A full proof is not required, but do give one or two sentences explaining your thinking.

1. Is the set $\mathbb{I}$ of *irrational* numbers closed under addition? In other words, is the sum of two irrational numbers always irrational?
2. Is $\mathbb{I}$ closed under multiplication?
3. Is $\mathbb{I}$ closed under the operation of raising numbers to powers? That is: are we guaranteed that, if a and b are irrational numbers, then a^b is always irrational? [Hint: This is one of my favorite arguments. Let $a = \sqrt{2}$ and think about the two numbers a^a and $(a^a)^a$. There are two cases to consider; explain why the question is settled regardless of which case is true.]

The third one is hard; you may need to exercise the 30 minute rule there, but do your best to think about it.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Proof (Corbin) -

1. Yes, because I cannot think of a time that adding two irrational numbers will become rational.
2. No, for example $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$ and $4 \in \mathbb{Z}$
3. Yes? I think $\mathbf{i}$ is closed because a neverending number to the power of a neverending number can't result in a rational number. (I'm gonna use Logan's example here) Because you are multiplying a number by itself to the final decimal place.

Proof (Logan) -

1. Yes, I believe so because there is no scenario where adding two numbers that go on infinitesimally without pattern can add to be rational.
2. No, take for example $\sqrt{2}$ multiplied By $\sqrt{8}$, which yields the result $\sqrt{16} = 4$ and $4 \notin \mathbb{I}$.
3. Yes, $\mathbb{I}$ is closed because taking an irrational number and putting it to a power that is infinitesimally long and without pattern simply can't result in a rational number, because you are multiplying a number by itself to the *final decimal place* (and by this I mean the infinitienth place).

Updated 10 months ago by Corbin and 3 others

the instructors' answer, *where instructors collectively construct a single answer*

(1) No, $\mathbb{I}$ is not closed under addition. For example, add any irrational number to its negative. We see that

$$\underbrace{\sqrt{2}}_{\text{irrational}} + \underbrace{(-\sqrt{2})}_{\text{irrational}} = \underbrace{0}_{\text{rational}},$$

but 0 is rational (since we can write it as, say, $\frac{0}{1}$).

(2) No, $\mathbb{I}$ is also not closed under multiplication. Consider

$$\underbrace{\sqrt{2}}_{\text{irrational}} \times \underbrace{\sqrt{2}}_{\text{irrational}} = \underbrace{2}_{\text{rational}},$$

(3) No, $\mathbb{I}$ is not closed under taking powers.

Let $a = \sqrt{2}$ and $b = \sqrt{2}^{\sqrt{2}}$. There are two cases: either b is rational, or it is irrational. We don't know which is true, so we will handle both cases separately:

1. If b is rational, then we can take an irrational number ($\sqrt{2}$) to an irrational power ($\sqrt{2}$) and get a rational number (b), which is what we wanted to prove, so we're done.
2. If b is irrational, think about $b^a = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$. Using the rules for exponentiation (power to a power is a product of powers), this is

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2.$$

   But this means that $b^a = 2$, and b is irrational and a is irrational but 2 is rational. In this case, therefore, we can also raise an irrational number to an irrational power and get a rational.

Since the claim is true in both cases, and these two cases are the only logical possibilities, it must be true generally.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*