5 / 16 / 2019

30.2

(a): Let us begin by answering the first part: Does $f_n(x) = \sin^n(x)$ converge uniformly on $[0,a]$ for $a \in (0,\frac{\pi}{2})$? Let's begin by ~~evaluo~~ noticing that pointwise on $[0,\pi/2)$, $\lim_{n\to\infty} f_n(x) = 0$, since at $x=\pi/2$, $f_n(x) = 1$.

Now ~~we evaluate~~ evaluate

$$\frac{d}{dx}|f - f_n| = \frac{d}{dx} f_n \text{ since } f_n \geq 0 \text{ on } [0,\pi/2)$$

$$\frac{d}{dx} f_n = n\cos^{n-1}(x) \stackrel{!}{=} 0$$

$$\cos^{n-1}(x) = 0$$

$$x = \pi/2$$

We notice that this maximum is only achieved in the second part, plugging in see that

$f_n\left(\frac{\pi}{2}\right) = 1$; thus $f_n(x)$ does not converge uniformly on $[0,\pi/2]$; however for part 1 we must only consider $x \in [0,\pi/2)$. Since the ~~supremum is on this range is 1,~~ ~~maxima~~ supremum on this range is 1, then

$$\sin^n(x) < 1 \text{ for } x \in [0,\pi/2)$$

implies we can make n large enough that

$$\sin^n(x) \to 0.$$

(b): As instructor response suggests, we ~~sta~~ want to multiply by 1 to get something that looks more like the hint:

$$f_n(x) = 2^n \sin\left(\frac{1}{3^n x}\right) \cdot \frac{3^n x}{3^n x}$$

~~Then take the limit of each side as $3^n x \to \infty$~~

$$\cancel{\lim f_n(x) = 2^n \lim \left(3^n x \sin\left(\frac{1}{3^n x}\right)\right) \cdot \lim\left(\frac{1}{3^n x}\right) = ? \cdot 1 \cdot 0 = 0}$$

Let's copy the $f_n(x)$ we are working with for clarity:

$$f_n(x) = 2^n \frac{3^n x}{3^n x} \sin\left(\frac{1}{3^n x}\right)$$

$$= \frac{2^n}{3^n x} \frac{\sin(1/(3^n x))}{(3^n x)^{-1}}$$

We see that, since $(3^n x)^{-1} \to 0$ and $x \geq a$, then for large enough n we can desire the ratio

$\cancel{\dots \leq \frac{2^n}{3^n a}}$ ~~We f~~

$$\frac{\sin(1/(3^n x))}{1/(3^n x)}$$ to be < 2, or as close as desired but not equal to 1; thus a valid upper bound would be

$$\dots \leq \frac{2}{a} \cdot \left(\frac{2}{3}\right)^n$$

We see that since $(2/3)^n$ is a geometric series with $r<1$ and $2/a$ is constant, then ~~this~~ by Weierstrass

$$f_n(x) = 2^n \sin\left(\frac{1}{3^n x}\right) < \frac{2}{a} \cdot \left(\frac{2}{3}\right)^n = M_n$$

since M_n is convergent, then $f_n(x)$ converges uniformly.

(c): ~~The hint suggests we consider the~~ define a large ~~real~~ number N and let

$x = \frac{2}{3^N \pi}$; then ~~the sum we~~ consider the sum

$$\sum_{n=N}^{\infty} 2^n \sin\left(\frac{1}{3^n \frac{2}{3^N \pi}}\right)$$

~~We see that~~ from $n = N \to n = \infty$, $\sin(1/\text{junk}) \leq 1$; ~~then this implies~~ that

$$\sum_{n=N}^{\infty} 2^n \sin\left(\frac{1}{3^n \frac{2}{3^N}}\right) \leq \sum_{n=N}^{\infty} 2^n$$

~~The~~

(c): Let's hit this young child one more time. ~~Consider the series~~
Suppose by way of contradiction that the series in question converges uniformly on $(0, \infty)$; then, for every $\epsilon > 0$, ~~we show~~ there exists N such that

$$n > N \Rightarrow \left| \sum_{n=0}^{\infty} f_n - \sum_{n=0}^{N} f_n \right| < \epsilon$$
$$\Downarrow$$
$$\left| \sum_{n=N}^{\infty} f_n \right| < \epsilon$$

But consider the series

$$\sum_{n=N}^{\infty} f_n = \sum_{n=N}^{\infty} 2^n \sin\left(\frac{1}{3^n x}\right)$$

at

$$x = \frac{2}{3^N \pi};$$

$$\ldots = \sum_{n=N}^{\infty} 2^n \sin\left(\frac{\pi}{2} \cdot \frac{1}{3^{n-N}}\right) > \sum_{n=N}^{N} 2^N = 2^N$$

Since for $n > N$ in the left series, we have ~~strict~~ $2^N \cdot \sin(\frac{\pi}{2}) = 2^N$ added to many strictly positive terms. But this inequality means we can make

$\sum_{n=N}^{\infty} f_n$ infinitely large just by making N larger; if it can be made infinitely large, it cannot possibly be made epsilon small. Thus, the series does not converge uniformly on $(0, \infty)$.