question **4** views

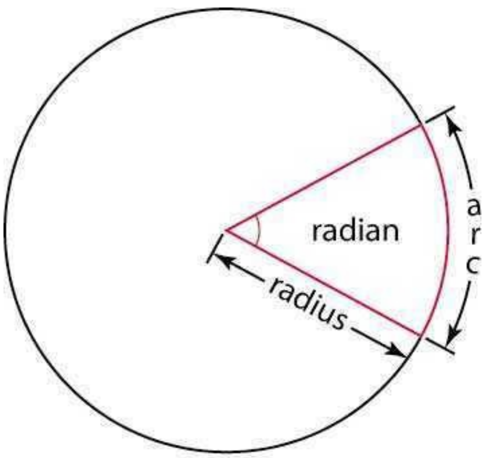
Daily Challenge 3.1

(**Due: Tuesday 5/8 at 12:00 noon Eastern**)

Another week begins!

Review

In our last meeting, we introduced the definition of the radian, a much more natural way of measuring angles. Whereas degrees are somewhat arbitrary (why should a circle have 360° rather than some other number?), radians are built out of a simple ratio of lengths and thus have much nicer mathematical properties.



Definition. Consider a circle of radius r. We draw two radii of the circle, each beginning at the center and going outward to a point on the circumference (these are red in the figure above). Let the angle between the radii be θ (labeled "radian" above).

The shape enclosed by these two radii is called a **circular wedge (**like a slice of pizza). The "crust" part of the pizza is called the **arc length** (labeled "arc" in the figure above") subtended by the angle θ.

We define the **radian measure** of the angle θ as the arc length of the wedge divided by the radius of the circle:

$$\theta = \frac{\text{arc length}}{\text{radius}}.$$

I found this definition much easier to understand after watching the following visualization:

In short: the radian measure of an angle is the fraction of the circumference "swept out" by the angle, divided by the radius.

How can we convert between degrees and radians? We need only figure out the formula for the arc length of a circular wedge in degrees, which is simply the fraction of the circumference swept out by the angle. In other words,

$$L_{\text{arc}} = (2\pi r) \cdot \frac{\theta_{\text{degrees}}}{360^\circ}.$$

For example, an angle of 90° cuts out one-quarter of the circle, so the arc length is one-fourth of the circumference: $L_{\text{arc}} = (2\pi r) \cdot \frac{90^\circ}{360^\circ} = \frac{\pi}{2} r$.

Using this formula for the arc length expressed in degrees, we find

$$\begin{aligned} \theta_{\text{radians}} &= \frac{L_{\text{arc}}}{r} \\ &= 2\pi \cdot \frac{\theta_{\text{degrees}}}{360^\circ}. \end{aligned}.$$

In other words, we have the conversion factor

$$2\pi \text{ radians } = 360^\circ.$$

We can use this formula to convert between degrees and radians: simply multiply your number of degrees by the conversion factor $\frac{2\pi \text{ radians}}{360^\circ}$. For instance,

$$\begin{aligned} 30^\circ &\longrightarrow \frac{\pi}{6} \text{ radians }, \\ 45^\circ &\longrightarrow \frac{\pi}{4} \text{ radians }, \\ 60^\circ &\longrightarrow \frac{\pi}{3} \text{ radians }, \\ 90^\circ &\longrightarrow \frac{\pi}{2} \text{ radians }, \\ &\vdots \end{aligned}$$

Although it's tempting to simply memorize the conversion factor $\frac{2\pi \text{ radians}}{360^\circ}$ and forget about the definition of radians, it is *much more important* to remember that a radian is the ratio of arc length and radius.

Problem

Try the following exercises on radians.

(a) Convert the angles 135°, 210°, and 330° from degrees to radians.

(b) Compute the values $\sin\left(\frac{\pi}{6}\right)$ and $\cos\left(\frac{5\pi}{6}\right)$.

(c) In our meeting, I claimed that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Then I set $\theta = \pi$ and wrote down $e^{i\pi} + 1 = 0$. Explain this step: was I measuring angles in degrees or radians? What angle does $\theta = \pi$ correspond to? What are $\cos(\pi)$ and $\sin(\pi)$?

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Solutions (Corbin). Your results go here.

(a)

(b)

(c)

Solutions (Logan). Your results go here.

(a) $135^\circ = \frac{3\pi}{4}$, $210^\circ = \frac{7\pi}{6}$, $330^\circ = \frac{11\pi}{6}$

(b) $\sin\left(\frac{\pi}{6}\right) = 0.5$, $\sin\left(\frac{5\pi}{6}\right) = 0.5$

(c) In these steps, you were measuring in radians; 180 degrees; $\cos(pi) = -1$, and $\sin(pi) = 0$

Updated 11 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Solutions (Christian).

(a) To convert each of these angles from degrees to radians, we can use the conversion factor $2\pi \text{ rad } = 360^\circ$. Therefore,

$$135^\circ = 135^\circ \times \frac{2\pi \text{ rad}}{360^\circ} = \frac{3\pi}{4},$$
$$210^\circ = 210^\circ \times \frac{2\pi \text{ rad}}{360^\circ} = \frac{7\pi}{6} \quad .$$
$$330^\circ = 330^\circ \times \frac{2\pi \text{ rad}}{360^\circ} = \frac{11\pi}{6}.$$

(b) We recall that the angle $\frac{\pi}{6}$ corresponds to 30°, and we have memorized the value $\sin(30^\circ) = \frac{1}{2}$.

Similarly, the angle $\frac{5\pi}{6}$ puts us in the second quadrant, at an angle of 150° (that is, 30° away from the negative x-axis). We are asked for the cosine of this angle, which is the x coordinate, and $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

(c) We began with Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

where θ is measured in radians. Then we chose $\theta = \pi$, which corresponds to an angle of 180°, putting us at the point $(-1, 0)$ on the unit circle. This means that $\cos(\pi) = -1$ and $\sin(\pi) = 0$. Plugging these into Euler's formula above, we find

$$e^{i\pi} = \underbrace{\cos(\pi)}_{=-1} + i\underbrace{\sin(\pi)}_{=0} = -1,$$

and after adding 1 to both sides, we find the most beautiful equation in mathematics,

$$\boxed{e^{i\pi} + 1 = 0}.$$

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*