question 2 views

Daily Challenge 26.5

(**Due: Wednesday 4/10 at 12:00 noon Eastern**)

In an earlier session, I claimed that the shape assumed by a free-hanging rope or chain suspended between two endpoints and subject to gravity is the catenary,

$$y = a\cosh\left(\frac{x-b}{a}\right),$$

where a and b are constants determined by the endpoints. I never proved this, but let's do so today.

We know that a physical system seeks to minimize its potential energy, since the definition $F = -\frac{dU}{dx}$ means that a system experiences a force which pushes it towards lower potential energy. The gravitational potential energy of a *particle* is

$$U = mgh.$$

However, a chain or rope is a continuous mass distribution with some linear density λ. The units of λ are mass per unit length. Thus the potential energy density u -- that is, the gravitational potential energy per unit length of rope -- is

$$u(x) = \lambda g h(x),$$

where $h(x)$ is the height of the small segment of rope at position x.

Note that $u(x)$ has units of energy *per unit length*. The potential energy of a small length of rope with length ds is approximately $u(x)\,ds$.

(**Part a**) Write an integral for the total potential energy of a rope with constant density λ, suspended between $x = -\ell$ and $x = \ell$, and whose height above the ground at position x is given by some function $y(x)$.

You will need to integrate the potential energy density $u(x)$, which gives the potential energy *per unit length*, multiplied by an expression which gives the arc length of a small segment ds of rope. You have seen this latter expression before, when we computed the arc length of a curve.

[Answer: $U = \lambda g \int_{-\ell}^{\ell} y(x)\sqrt{1+(y'(x))^2}\,dx$.]

(**Part b**) To minimize the potential energy, it suffices to find a critical point of the functional

$$J[y] = \int_{-\ell}^{\ell} y(x)\sqrt{1+(y'(x))^2}\,dx$$

Write down the Euler-Lagrange equation for this functional. Evaluate the derivatives and simplify.

(Part c) Show that the Euler-Lagrange equation is satisfied if we demand that

$$y = a\sqrt{1+(y')^2},$$

where a is some constant.

[Side note: this technique of "guess and check" is common in solving equations of this type, for reasons we will discuss in the final chapter. When writing formal mathematical prose, however, one uses the more dignified-sounding term "ansatz" rather than "guess and check", as in "We make the ansatz $y = a\sqrt{1+(y')^2}$ and show that this satisfies the given equation.]

(**Part d**) Square both sides of the equation from part (c) and re-arrange terms to show that $y' = \pm\frac{1}{a}\sqrt{y^2-a^2}$. We will take the positive root.

(**Part e**) Divide both sides of your equation from (d) by $\sqrt{y^2-a^2}$. Then integrate both sides with respect to x. Use the substitution $u = y$ on the left side to change integration variables from dx to dy. You should find the equation

$$\int \frac{dy}{\sqrt{y^2-a^2}} = \frac{1}{a}\int dx,$$

where I have suppressed the bounds of integration to minimize clutter (to wit, the integral on the left runs from $y(-\ell)$ to $y(\ell)$, and the one on the right runs from $-\ell$ to ℓ, but this will not be important).

(**Part f**) The integral on the left side can be evaluated using a hyperbolic trig substitution. Let

$$y = a\cosh(u)$$

and use the identity

$$\cosh^2(u) - 1 = \sinh^2(u).$$

to evaluate the integral on the left. Don't forget that $\frac{d}{du}\cosh(u) = \sinh(u)$, which you will need to convert dy to du. The sinches in the top and bottom cancel so the integral turns into $\int du = u$ after the change of variables. The right-side integral is trivial. So you should find

$$u = \frac{1}{a}(x-b),$$

where b is some constant of integration. Plug the definition $y = a\cosh(u)$ back in and solve for y to conclude

$$y(x) = a\cosh\left(\frac{x-b}{a}\right).$$

Punchline: you have now proven that a hanging rope acted upon by the gravitational force assumes the shape of a *catenary*, whose functional form is described by the hyperbolic cosine. This result just comes from energy minimization.

daily_challenge

Updated 5 days ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

beet

Updated 2 days ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

yellow boi

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followup discussions *for lingering questions and comments*