question 2 views

Daily Challenge 15.1

(**Due: Friday 9/21 at 12:00 noon eastern**)

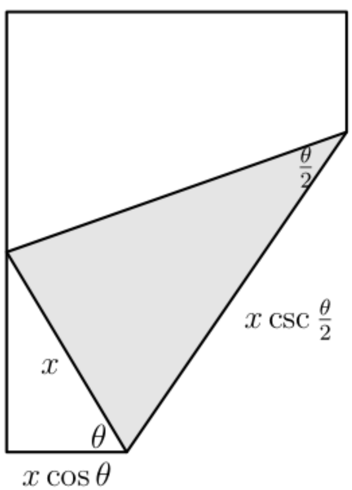
Today let's get back to a nice optimization question which appears as problem 11 on CD 4. Feel free to work directly in Overleaf if you prefer.

(1) Problem: folding a sheet of paper.

A standard 8.5 inches by 11 inches piece of paper is folded so that one corner touches the opposite long side, as shown in the picture below. What is the minimum length of the crease?

[Hint: let θ be the angle between the fold and the bottom of the page, as in the picture. Write down a function $f(\theta)$ that gives the length of the crease as a function of θ. Then find the minimum of $f(\theta)$.

Don't forget that the length and width are given because the paper is $8.5 \text{ in} \times 11 \text{ in}!$]



daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

One can find the values shown in the image to be true through basic geometry. We also see that the length of the crease as determined x and the lower leg of the white triangle add up to 8.5 inches as we are told in the description that this paper is 8.5 inches wide, so we see that $8.5 = x + x\cos(\theta)$, which then implies $x = \frac{8.5}{1+\cos(\theta)}$. The length of the crease is defined as $x\csc(\frac{\theta}{2})$, so we substitute and see that the length of the crease in terms of theta is $L(\theta) = \frac{8.5}{1+\cos(\theta)} \cdot \csc(\frac{\theta}{2})$. We see that as $\theta \to \frac{\pi}{2}$ then $L(\frac{\pi}{2} = 8.5\sqrt{2}$ and on the opposite end of the spectrum, as $\theta \to 0$ then $L \to \inf$ by the definition of cosecant. (Though by basic visualization it'll be 11, but that is not pertinent). We must now consider critical points, where the derivative is zero. We begin by taking the derivative of each side with respect to θ; then we see as an initial step that

$$L'(\theta) = \left(\frac{8.5}{1+\cos(\theta)}\right)' \cdot \csc(\frac{\theta}{2}) + \frac{8.5}{1+\cos(\theta)} \cdot \left(\csc(\frac{\theta}{2})\right)'$$

which in turn through the quotient rule, product rule, chain rule, and all of our lovely tools of the trade equals

$$8.5\frac{(1+\cos(\theta)) \cdot \left(-\frac{1}{2}\csc\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right) - \left(\csc\left(\frac{\theta}{2}\right)(-\sin(\theta))\right)}{(1+\cos(\theta))^2}$$

We need to find some point where this is equal to zero, and we really have it in for us today. We see that for our interval $[0, \frac{\pi}{2}]$ there are no points where the denominator is zero, therefore there are no undefined points. We now know that we must find some θ where the numerator equals zero, ie solve

$$(1+\cos(\theta)) \cdot \left(-\frac{1}{2}\csc\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right) - \left(\csc\left(\frac{\theta}{2}\right)(-\sin(\theta))\right) = 0$$

Once again we can apply our knowledge that we are in operating on $[0, \frac{\theta}{2}]$ and see that for this domain there are no points where $1 + \cos(\theta) = 0$ UGH WHY

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Define the length x of the fold and the angle θ between the fold and bottom of the paper, as shown in the diagram above. Then the length of the crease is $L(x, \theta) = x\csc\left(\frac{\theta}{2}\right)$, as we see from the picture.

We would like to minimize L. However, as it stands, L depends on the two variables x and θ. We should eliminate one of these so that L depends on only one variable.

To do so, note that this is an 8.5 inch by 11 inch sheet of paper, and the sum $x + x\cos(\theta)$ gives the width of the paper; thus $x + x\cos(\theta) = 8.5$, or

$$x = \frac{8.5}{1+\cos(\theta)},$$

where I have suppressed the units ("inches"). Plug this expression for x into the length L to find

$$L(\theta) = \frac{8.5\csc\left(\frac{\theta}{2}\right)}{1+\cos(\theta)}.$$

Now we're in good shape; we try to minimize $L(\theta)$ on the interval $[0, \frac{\pi}{2}]$. The minimum occurs at either (i) an endpoint, (ii) a critical point, or (iii) a point where $L'(\theta)$ is undefined.

We see that L blows up as $\theta \to 0$ since $\csc(0)$ is undefined, so that's not the minimum. The other endpoint is $L(\frac{\pi}{2}) = 8.5\sqrt{2}$.

On to critical points. The derivative is

$$L'(\theta) = 8.5\frac{(1+\cos(\theta)) \cdot \left(-\frac{1}{2}\csc\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right) - \left(\csc\left(\frac{\theta}{2}\right)(-\sin(\theta))\right)}{(1+\cos(\theta))^2}.$$

The denominator never blows up, so this is well-defined everywhere. We look where places where $L'(\theta) = 0$, which occur where the numerator equals zero. That is, we look for values of θ such that

$$0 = (1+\cos(\theta))\left(-\frac{1}{2}\cot\left(\frac{\theta}{2}\right)\right) + \sin(\theta).$$

This is a pain to solve because of the cotangent. Let's substitute using a half-angle identity. We know the half-angle formulas for sine and cosine, namely $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$ and $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$, so divide to find

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} = \frac{\sin(\theta)}{1+\cos(\theta)},$$

where in the last step we multiplied top and bottom by $1+\cos(\theta)$. But cotangent is one-over-tangent, so this means

$$\cot\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{\sin(\theta)}.$$

Substitute this into our equation above for the critical points where $L'(\theta) = 0$ and one finds

$$0 = -\frac{1}{2}\frac{(1+\cos(\theta))^2}{\sin(\theta)} + \sin(\theta).$$

We are free to multiply both sides by $\sin(\theta)$ since this is nonzero on $(0, \frac{\pi}{2})$, which yields

$$\begin{aligned} 0 &= -\frac{1}{2}(1+\cos(\theta))^2 + \sin^2(\theta) \\ &= -\frac{1}{2}(1+\cos(\theta))^2 + 1 - \cos^2(\theta). \end{aligned}$$

In the last step, I replaced $\sin^2(\theta) = 1 - \cos^2(\theta)$ by Pythagoras. This is now a quadratic equation in $\cos(\theta)$! We would like to find θ such that

$$3\cos^2(\theta) + 2\cos(\theta) - 1 = 0,$$

or by the quadratic formula,

$$\cos(\theta) = \frac{-2 \pm \sqrt{2^2 - 4\cdot 3\cdot(-1)}}{2\cdot 3} = \frac{-1\pm 2}{3}.$$

We know the negative root is spurious, since $\cos(\theta) > 0$ on the range we are considering, so the only critical point is $\cos(\theta) = \frac{1}{3}$.

Finally, since the derivative is defined everywhere on $(0, \frac{\pi}{2})$, there are no points of type (iii). So now we need only choose which of the two points that we found (the endpoint $\theta = \frac{\pi}{2}$ and the critical point where $\cos(\theta) = \frac{1}{3}$) that gives the smaller crease.

To do this, we need to evaluate $L(\theta)$ for the value of θ with $\cos(\theta) = \frac{1}{3}$. To do this without resorting to a calculator, one should express $L(\theta)$ in terms of $\cos(\theta)$. Using Pythagoras and the half-angle identities, we have

$$L(\theta) = \frac{8.5 \csc\left(\frac{\theta}{2}\right)}{1+\cos(\theta)}$$
$$= \frac{8.5\sqrt{\frac{2}{1-\cos(\theta)}}}{1+\cos(\theta)},$$

and therefore if $\cos(\theta) = \frac{1}{3}$ the crease length is

$$L(\theta) = \frac{8.5\sqrt{\frac{2}{1-\frac{1}{3}}}}{1+\frac{1}{3}} = 8.5 \cdot \frac{3\sqrt{3}}{4}.$$

This is smaller than the endpoint $\theta = \frac{\pi}{2}$, which gives a crease length of $8.5\sqrt{2}$, since $\sqrt{2} > \frac{3\sqrt{3}}{4}$. Therefore, this value $\theta = \arccos\left(\frac{1}{3}\right)$ is the minimum point and $L(\theta) = 8.5 \cdot \frac{3\sqrt{3}}{4}$ is the minimum length of the crease.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*