question 2 views

Daily Challenge 24.4

(**Due: Saturday 3/16 at 12:00 noon Eastern**)

Last time we introduced the *n-th central moment* of a probability distribution $f(x)$, often written as μ_n:

$$\mu_n = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx.$$

Here the number μ (with no subscript) is the expectation value, defined as

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx.$$

The second central moment is called the *variance* and usually written as σ^2:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx.$$

The square root of the variance, $\sigma = \sqrt{\sigma^2}$, is called the *standard deviation*.

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(1) Problem: Variance of the Gaussian.

Consider again the Gaussian distribution,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

When we first discussed this, μ and σ were just two arbitrary numbers that we chose for writing the distribution $f(x)$. But as you know, we have cleverly adjusted the numerical factors everywhere so that this number σ^2 works out to be the variance:

$$\sigma^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$$

This calculation is so important that I will ask you to prove it in three separate ways.

(**Part a**) Change variables to $u = \frac{(x-\mu)^2}{2\sigma^2}$ in the integral, then express the result in terms of the gamma function

$$\Gamma(z) = \int_0^{\infty} u^{z-1} e^{-u}\,du$$

and simplify the result using the fact $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$.

(Yes, this is identical to the calculation I asked you to do in the diffusion DC. You may copy the steps from your previous calculation verbatim if you like, but I will ask you to actually write it out again rather than re-scanning the same work.)

(**Part b**) Do the same calculation by changing variables to $u = \frac{x-\mu}{\sqrt{2}\sigma}$, then integrating by parts with $a = u$ and $db = ue^{-u^2}\,du$, as we did on slide 30 last time.

(Again, copy the steps if you want but please actually write it out.)

(**Part c**) Now for a cute, different way. Begin with the equation

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1,$$

which is true because any probability distribution integrates to 1.

Now differentiate both sides with respect to σ. That is, differentiate and treat σ as a variable but treat x and μ as constants.

The right side becomes zero because it is the derivative of a constant. On the left side, use the product rule and the ability to move the derivative past the integral sign.

After simplifying (you might want to multiply both sides by some constants to clear part of the denominator), you will find the equation

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left((x-\mu)^2 - \sigma^2\right) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 0.$$

Moving the $-\sigma^2$ term in parentheses to the opposite side, this means

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \sigma^2 \cdot \underbrace{\left(\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx\right)}_{=1}.$$

The thing in parentheses on the right side is just 1, since it's the integral of the Gaussian again (and every PDF integrates to 1). So the right side, including the overall constant, is just σ^2.

But this means

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \sigma^2,$$

which now gives us a third way to compute the variance of the Gaussian.

daily_challenge

Updated 29 days ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

dank

Updated 27 days ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

yellow boi

Updated 28 days ago by Christian Ferko

followup discussions *for lingering questions and comments*