26.5

(a): We are told ~~to multiply~~ to integrate the multiplication of the potential energy by the arc length of a small segment.
Potential energy $= u(x) = \lambda g\, y(x)$ by the problem statement.
Arc length $= \sqrt{1+y'(x)}$ by recall; thus,

$$U = \int_{-L}^{L} \lambda g\, y(x) \sqrt{1+y'(x)}\, dx$$

$$= \lambda g \int_{-L}^{L} y(x)\sqrt{1+y'(x)}$$

(b): We apply the Euler-Lagrange equation to the result of (a):
Let

$$f(x, y(x), y'(x)) = y(x)\sqrt{1+y'(x)^2}$$

and recall

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) = 0;$$

then

$$\sqrt{1+y'(x)^2} - \frac{\partial}{\partial x}\left(\frac{1}{2} y(x) \cdot (1+y'(x)^2)^{-1/2} \cdot 2y'(x)\right) = 0$$

$$\sqrt{1+y'(x)^2} = \cancel{\tfrac{1}{4} y'(x) \cdot (1+y'(x)^2)^{-3/2}}.$$

$$= \frac{1}{2} y'(x)\,(1+y'(x)^2)^{-1/2} \cdot 2y'(x)$$

$$+ \frac{-1}{4} y(x) \cdot (1+y'(x)^2)^{-3/2} \cdot 4y'(x)^2$$

$$+ \frac{1}{2} y(x) \cdot (1+y'(x)^2)^{-1/2} \cdot 2y''(x)$$

Jeebus.

~~(c):~~ $\cancel{y(x) = y'(x)^2 (y(x)^{-1} - y(x) \cdot y'(x)^2 \cdot y^{-3} + ay''(x)}$

$\cancel{= \frac{y'(x)^2}{ay(x)} - \frac{y'(x)^2}{ay(x)^2} + \frac{ay''(x)}{a}}$

(c): By substituting, we get

$$\frac{y}{a} = \frac{ay'(x)^2}{y(x)} - y(x)\cdot\left(\frac{a}{y}\right)^3 \cdot y'^2 + y(x)\,\frac{a}{y}\cdot y''(x)$$

$$= \frac{ay'(x)^2}{y(x)} - \frac{a^3}{y^2}\cdot y'^2 + a y''(x)$$

Maybe ~~wont~~ we should just substitute this into the Euler-Lagrange equation seen in (b) instead:

$$\frac{y}{a} - \frac{\partial}{\partial x}\left(y\cdot y'\cdot\left(\frac{y}{a}\right)^{-1}\right) = 0 = \frac{y}{a} - \frac{\partial}{\partial x}\left(a\,y'(x)\right)$$

$$\Longrightarrow$$

$$\frac{y}{a} = a\,y''(x) \Rightarrow y = a^2 y''(x)$$

Now ~~let~~ we take two derivatives of y:

$$y = a\sqrt{1+y'(x)^2}$$

$$y' = \tfrac{1}{2}a\left(1+y'(x)^2\right)^{-1/2}\cdot 2y'\cdot y'' = a^2 y^{-1} y' y'' \Rightarrow \boxed{y = a^2 y''}$$

$$\cancel{y'' = \frac{-1}{4}a\left(1+y'(x)^2\right)^{-3/2}\cdot 2y'(x) + \frac{1}{2}a\left(1+y'(x)^2\right)^{-1/2} 2y''(x)}$$

$$\cancel{y'' = \frac{-1}{2}y^{-3}}$$

$$\cancel{y'' = y^{-2}y'y'' + y^{-1}y''^2 + y^{-1}y'y'''}$$

By the boxed equation, we see that the ansatz is valid.

(d): We see by (c) that

$$y^2 = a^4 y''^2$$

but $y = a\sqrt{1+y'(x)^2}$, so

$$a^2(1+y'(x)^2) = a^4 y''^2 \Rightarrow 1+y'(x)^2 = a^2 y''^2$$

By substituting $\frac{y^2}{a^2} = a^2 y''^2$,

$$y' = \pm\sqrt{\frac{y^2}{a^2} - 1\cdot\frac{a^2}{a^2}}$$

$$= \pm\frac{1}{a}\sqrt{y^2 - a^2}$$

(e): We see by (d) that

$$\frac{y'}{\sqrt{y^2-a^2}} = \frac{1}{a}$$

then integrate with respect to x of each sides on $[-L, L]$;

(2): $$\int_{-L}^{L} \frac{y'}{\sqrt{y^2-a^2}}\,dx = \int_{-L}^{L} \frac{1}{a}\,dx = \frac{1}{a}(x-b)$$ where b is a constant of integration;

u-sub on the left side $u = y(x)$
$du = y'(x)\,dx$,

then

(3): $$\int_{-y(L)}^{y(L)} \frac{1}{\sqrt{u^2-a^2}}\,du = \frac{1}{a}\Big[x\Big]_{+L}^{L} = \int_{-y(L)}^{y(L)} \frac{1}{\sqrt{y^2-a^2}}\,dy \quad (1)$$

(f): We then u-sub the right side of (1) where $y = a\cosh(u)$
$dy = a\sinh(u)\,du$

$$\int_{y(-L)}^{y(L)} \frac{1}{\sqrt{a^2(\cosh^2(u)-1)}}\, a\sinh(u)\,du = \int_{y(-L)}^{y(L)} \frac{1}{a\sinh(u)} \cdot a\sinh(u)\,du$$
$$= \int_{y(-L)}^{y(L)} du = u$$

We assemble equations 2&3 and see that

$$u = \frac{1}{a}(x-b)$$

then since $y = a\cosh(u)$,

$$y(x) = a\cosh\left(\frac{x-b}{a}\right).$$