note 2 views

Daily Challenge 4.7

Let's go back to including problems tomorrow as we begin week 5 (so the first problem will be posted tomorrow morning and due Tuesday at noon). For today, I'll post another worked example.

Example (source: AoPS Calculus, problem 1.6.3). Find a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution. We already have angle-addition formulas for sine and cosine, namely

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta),\\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).\end{aligned}$$

Since the tangent is defined by $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$, we can divide the two equations above to find

$$\tan(\alpha+\beta)=\frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}.$$

Now we need only massage the right side of this equation to express it in terms of $\tan(\alpha)$ and $\tan(\beta)$. Let's divide the numerator and denominator by $\cos(\alpha)\cos(\beta)$. Whenever you divide while doing algebra, you should check that the quantity you're dividing by is nonzero. In this case, $\cos(\alpha)\cos(\beta)$ will be zero whenever α or β is an odd multiple of $\frac{\pi}{2}$. At such points, tangent is undefined, so our formula will only be valid wherever $\tan(\alpha)$ and $\tan(\beta)$ are defined.

Keeping this in mind, the division gives

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.\end{aligned}$$

Et voilà, we've found a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$, valid for $\alpha,\beta\in\mathbb{R}\setminus\left\{\frac{k\pi}{2}\mid k \text{ is an odd integer}\right\}$.

daily_challenge

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*