question **2** views

Daily Challenge 16.7

(**Due: Saturday 11/3 at 12:00 noon Eastern**)

(1) Problem: integrating a cube.

Prove that $\int_0^b x^3 = \frac{b^4}{4}$ by imitating the calculation used in session 41 to compute $\int_0^b x^2$.

That is, chop up the interval into n equal subintervals and compute the lower and upper sums explicitly, then show that they can be made ϵ-close. Also prove that the value of the integral is indeed $\frac{b^4}{4}$.

As part of your proof, you will need to show that $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$, which can be proven by induction.

daily_challenge

Updated 5 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan pavsxjcferws:
Just as we did in session 41, we begin by breaking this domain of interest into an equally spaced partition with n values, ie $P_n = \{t_0, \cdots, t_n\}$ where $t_i = \frac{i \cdot b}{n}$. We need to prove that the lower and upper sums are equal or ϵ close to eachother, so we refer the the definition of each: The lower sum

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^n (t_{i-1})^3 \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^n (i-1)^3 \cdot \frac{b^3}{n^3} \cdot \left(\frac{i \cdot b}{n} - \frac{(i-1) \cdot b}{n}\right) \\ &= \sum_{i=1}^n (i-1)^3 \cdot \frac{b^3}{n^3} \cdot \frac{b}{n} \\ &= \frac{b^4}{n^4} \sum_{j=0}^{n-1} j^3 \end{aligned}$$

Meanwhile the upper sum, by a nearly identical process:

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^n t_i^3 \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^n i^3 \cdot \frac{b^3}{n^3} \cdot \frac{b}{n} \\ &= \frac{b^4}{n^4} \sum_{j=1}^{n} j^3 \end{aligned}$$

We must find some alternative non-sum way to represent these sums of j^3, and the problem statement gives us that we must show somehow that $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$, and we do so by induction. First when $n = 1$, it is immediately obvious that each side equals 1. We then assume that this equation is true for some $n = k$, and we must prove it for $n = k + 1$. We begin by substituting this into our equation; (induction shall be done later :blobguns:)

We then see by applying our found equation for the sum of cubes that

$$L(f, P_n) = \frac{b^4}{n^4} \cdot \frac{(n-1)^2(n)^2}{4} \text{ and}$$
$$U(f, P_n) = \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4}$$

We then have to somehow prove that

$$\frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{b^4}{n^4} \cdot \frac{n^2(n-1)^2}{4} < \epsilon$$

We then factor out all that we can and see that this breaks down to

$$< br/ > \frac{b^4}{n^4} \cdot \frac{1}{4} \cdot n^2 \cdot \left((n+1)^2 - (n-1)^2\right) < \epsilon$$
$$< br/ > \frac{b^4}{n} < \epsilon$$

Thus by making n sufficiently large this can be made as small as necessary, proving that the upper and lower sums are equal.

Updated 5 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

First I prove a lemma.

Lemma. Let $n \in \mathbb{N}$. Then $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Proof. We proceed by induction. When $n = 1$, the left side is 1 and the right side is $\frac{1^2(1+1)^2}{4} = 1$, so the formula holds in this case.

Now suppose the formula holds when $n = m$. We must do some painful algebra. If $n = m + 1$, the left side is

$$\sum_{k=1}^{m+1} k^3 = \left(\sum_{k=1}^{m} k^3\right) + (m+1)^3$$

because we can split the sum. The first term is equal to $\frac{m^2(m+1)^2}{4}$ because we have assumed the formula holds for $n = m$. Getting a common denominator, this becomes

$$\begin{aligned}\sum_{k=1}^{m+1} k^3 &= \frac{m^2(m+1)^2}{4} + \frac{4(m+1)^3}{4} \\ &= \frac{(m+1)^2\left(m^2 + 4(m+1)\right)}{4} \\ &= \frac{(m+1)^2(m+2)}{4},\end{aligned}$$

but the last line is what our formula predicts when $n = m + 1$. Thus the claim holds for all n by induction. $\square$

Now let's prove the main result.

Proposition. The function $f(x) = x^3$ is integrable on $[0, b]$ and $\int_0^b x^3 = \frac{b^4}{4}$.

Proof. Consider a partition $P_n = \{t_0, \cdots t_n\}$ into equal sub-intervals $t_i = \frac{ib}{n}$. The function $f(x) = x^3$ is increasing on $[0, b]$, so each infimum m_i occurs at the left endpoint of $[t_{i-1}, t_i]$ and each supremum M_i occurs at the right endpoint. Thus the lower sum is

$$\begin{aligned}L(f, P_n) &= \sum_{i=1}^{n} m_i\,(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \left(\frac{(i-1)b}{n}\right)^3 \cdot \frac{b}{n} \\ &= \frac{b^4}{n} \sum_{i=1}^{n} (i-1) \\ &= \frac{b^4}{n^4} \cdot \frac{(n-1)^2(n)^2}{4},\end{aligned}$$

where in the last step we have used our lemma to complete the sum (in particular, re-index to $j = i - 1$ and then apply the formula). Similarly,

$$\begin{aligned}U(f, P_n) &= \sum_{i=1}^{n} M_i\,(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^3 \cdot \frac{b}{n} \\ &= \frac{b^4}{n} \sum_{i=1}^{n} i \\ &= \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4},\end{aligned}$$

where we have used the lemma again.

Now let $\epsilon > 0$ be given; we must choose P_n (i.e. by choosing the number n of sub-partitions) to make $U(f, P_n) - L(f, P_n) < \epsilon$. But by the above calculation, their difference is

$$U(f, P_n) - L(f, P_n) = \frac{b^4}{n^4} \cdot \left(\frac{n^2(n+1)^2}{4} - \frac{(n-1)^2(n)^2}{4}\right) = \frac{b^4}{n}$$

So we simply choose $n > \frac{\epsilon}{b^4}$ to guarantee $U(f, P) - L(f,) < \epsilon$. This completes the first part of the proof, that f is integrable.

To find the value of the integral, we recall that $\int_0^b f = \sup\{L(f, P)\} = \inf\{U(f, P)\}$, where the supremum and infimum run over all partitions. But note that $(n-1)^2n^2 \le n^2 \le (n+1)^2n^2$. Thus, for our partitions P_n,

$$L(f, P_n) \le \frac{b^4}{4} \le U(f, P_n)$$

Since we have proven that the supremum of all lower sums equals the infimum of all upper sums, the above inequality implies that their common value must be $\frac{b^4}{4}$. $\square$

Updated 5 months ago by Christian Ferko

followup discussions *for lingering questions and comments*