question 3 views

Daily Challenge 11.2

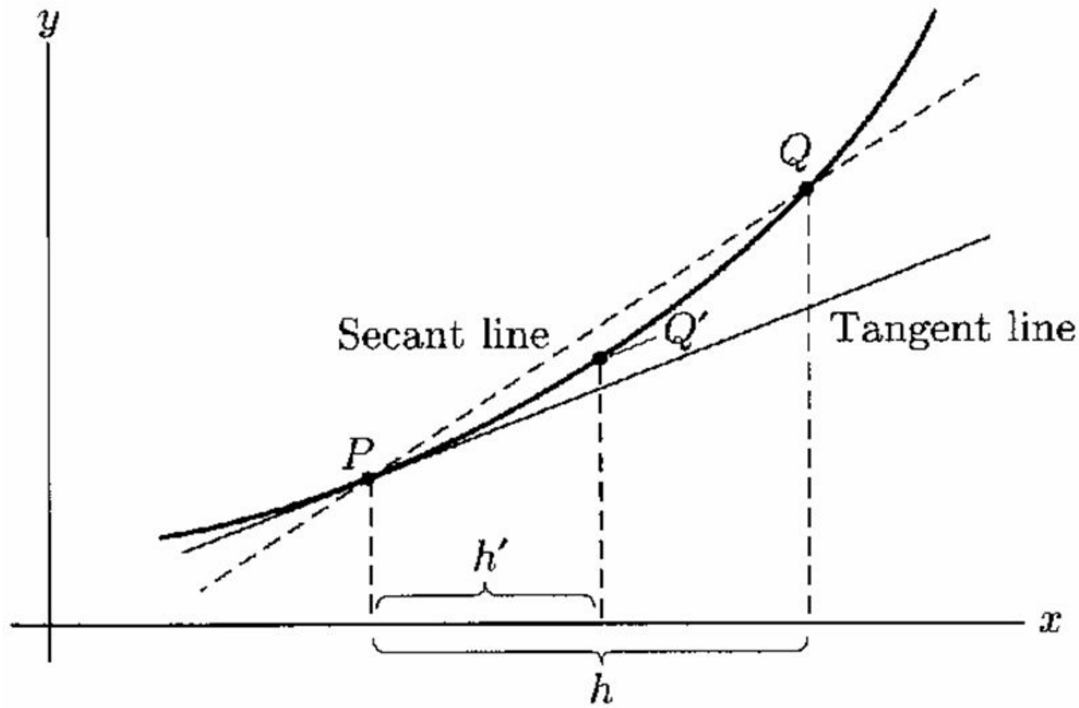
(**Due: Sunday 8/12 at 12:00 noon Eastern**)

We now begin our discussion of derivatives, the first of two major tools (differentiation and integration) which are built on the foundation of limits. The power of these tools is what gives calculus its distinctive flavor.

(1) A derivative is a particular limit of a quotient of differences.

We saw in the last couple of sessions that the idea of a derivative is to find the slope of a *tangent line* to a graph at a particular point by first approximating it with a *secant line* and then taking a limit as the approximation gets better.

I hasten to warn the reader that the proper definition of "tangent line" is somewhat subtle; we saw that the naive idea to call it "the line ℓ which touches the graph of a function f at a single point $(a, f(a))$" does **not** work. Thus I will postpone the rigorous definition for the moment and revisit it once we have defined the derivative itself, but the intuitive (read: non-rigorous) idea is encoded in the following image.



As we take Q closer to P, the accuracy with which the slope of the secant line approximates the slope of the tangent line increases.

In the above figure, we wish to compute the slope of the "tangent line" to the graph at the point P (I put "tangent line" in scare quotes to emphasize that I have not actually defined it yet).

As a first approximation, we find the slope of the *secant line*, which is the unique line ℓ passing through two points P and Q on the graph. (You will recall from a geometry course that we can always draw a unique line between any two points on the plane.)

Let the coordinates of the point P be $(a, f(a))$ and those of Q be $(a+h, f(a+h))$. Then the slope of the secant line through P and Q is

$$\text{slope of secant line through } P, Q = \frac{f(a+h) - f(a)}{h},$$

which is the usual rise over run formula. An expression of this form, $\frac{f(a+h)-f(a)}{h}$, is called a *difference quotient*.

As the figure suggests, we can get a better approximation to the "tangent line" slope by choosing a closer point $Q' = (a+h', f(a+h'))$, where $h' < h$. The slope of this new secant line is

$$\text{slope of secant line through } P, Q' = \frac{f(a+h') - f(a)}{h'}.$$

You now see the point: *we have written down an approximation to the quantity of interest, and also specified a technique which makes the approximation better*. The key conceptual idea of calculus is that we can take a limit as this approximation-improvement procedure is repeatedly carried out, which will produce the exact answer, if one exists.

We are now ready to state the definition.

Definition. Let f be a real-valued function and $a \in \mathbb{R}$. We say that f is *differentiable at a* if the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If it exists, we call its value the *derivative* of f at a, which we write as $f'(a)$ or $\frac{df}{dx}\Big|_a$.

(2) The graph of a line has a constant derivative, namely its slope.

Let's revisit a sanity-check calculation we did two sessions ago.

Let $f(x) = mx + b$ be a function whose graph is a line. We will attempt to compute its derivative by writing down the definition and checking whether the limit exists. This is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

First I replace both occurrences of f by the definition -- that is, I replace $f(a+h)$ by $m(a+h)+b$ where I have simply inserted the argument $a+h$ in place of the dummy variable x as usual, and so on. This gives

$$\begin{aligned} f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \frac{m(a+h) + b - (ma+b)}{h} \\ &= \lim_{h\to 0} \frac{mh}{h}. \end{aligned}$$

We recall that the value of a limit is insensitive to the behavior of the function precisely at the limit point, so we are free to assume $h \neq 0$ and cancel $\frac{mh}{h} = h$ inside the limit. This gives

$$f'(a) = m.$$

Thus the derivative of a line is a constant equal to the slope of that line; in other words, the "tangent line" to a line is just that line itself.

Now is probably a good time to pay the piper and actually define "tangent line" properly.

Definition. The *tangent line* to the graph of a function f at a point a is the line ℓ passing through the point $(a, f(a))$ whose slope is equal to $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, if that limit exists. If the limit does not exist, we say that there is no tangent line to f at a.

(3) Problem: using the definition of the derivative.

(a) Let $f(x) = x^2$. Working directly from the definition, compute $f'(a)$. Justify any intermediate steps by citing the appropriate theorems.

(b) Let $c \in \mathbb{R}$ and let f be a function which is differentiable everywhere. Prove, directly from the definition of derivative, that $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$. In other words, prove that $(cf)'(a) = cf'(a)$ for any $a \in \mathbb{R}$.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: First, recall by the definition of derivative that $f'(x) = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h}$. We can then foil and see that $\lim_{h\to 0} \frac{(a+h)^2-a^2}{h} = \lim_{h\to 0} \frac{2ah+h^2}{h}$ (the a^2's cancel out). We can then divide by h as we do not have to account for $h = 0$, and get $\lim_{h\to 0} \frac{2ah+h^2}{h} = \lim_{h\to 0} 2a + h$, which in turn as the limit of a sum is the sum of the limits, we have that $\lim_{h\to 0} 2a + h = \lim_{h\to 0} 2a + \lim_{h\to 0} h = 2a$, as 2a does not contain the variable h and is not affected by the limit, meanwhile $\lim_{h\to 0} h = 0$, because this function is a polynomial and is therefore continuous and the limit equals the solution normally. We then conclude that $f'(x) = 2a$. $\square$

b: We are given the information that f is a function that is differentiable (and in turn continuous) everywhere. Given a constant $c \in \mathbb{R}$, we must somehow show that $(cf(x))' = c(f(x))'$, or otherwise $c \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{c(f(a+h))-c(f(a))}{h}$. We can factor out the c in the numerator of the right side to get that

$\lim_{h\to 0} \frac{c(f(a+h))-c(f(a))}{h} = \lim_{h\to 0} \frac{c(f(a+h)-f(a))}{h}$. We can take the c out of the numerator and apply the product rule of limits and get $\lim_{h\to 0}(c) \times \lim_{h\to 0} \frac{(f(a+h)-f(a))}{h}$, and since the former limit is a constant not containing h then we have what we want to prove, that $c \times \lim_{h\to 0} \frac{(f(a+h)-f(a))}{h} = (cf(x))' = c(f(x))'$. $\square$

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) By definition,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

In this case, $f(x) = x^2$, so we have

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2}{h} \\ &= \lim_{h\to 0} (2a+h) \\ &= 2a. \end{aligned}$$

In the second step, we have canceled h from the numerator and denominator, which is permitted because we can assume $h \neq 0$ in the limit. In the final step, we have used that $2a + h$ is a polynomial in h, and polynomials are continuous. Thus $f'(a) = 2a$.

(b) Using the definition of the derivative,

$$\begin{aligned} (cf)'(a) &= \lim_{h\to 0} \frac{cf(a+h) - cf(a)}{h} \\ &= \left(\lim_{h\to 0} c\right)\left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right) \\ &= cf'(a). \end{aligned}$$

In the second step, we have used that both limits exist (the first is the limit of a constant; the second exists because we have assumed f is differentiable) to split the limit of the product into the product of two limits. The last step proves the claim. $\square$

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*