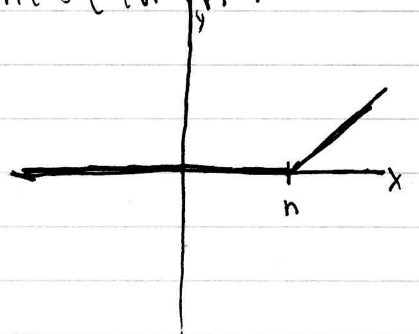
30.1

(a): Let's sketch the scenario



I.e.; we have a line at zero until it connects to a line with x-intercept n. We then see that

$$\lim_{n\to\infty} f_n(x) = f(x) = 0$$

since we can ~~continue to change n~~ make n large enough that $n > b \Rightarrow f([a,b]) = 0$; this of course does not apply for the interval $[-\infty, \infty]$, since we can't make n large enough for that rightend point to be handled; ~~so this~~ ~~f_n~~ Thus $f_n(x)$ has pointwise limit 0 ~~for~~ over any $[a,b]$ s.t. $a, b \in \mathbb{R}$. It converges uniformly as a result as well.

(b): ~~Ti~~ We see that $\lim (x^n - x^{2n}) = 0 - 0$ for $0<x<1$, thus the pointwise limit is 0. ~~Let's find the maximum of~~ ~~$|f - f_n|$~~ on $[0,1]$;

$\dots = |0 - f_n| = f_n$ since $f_n > 0$ for all n, then the max occurs where

$$f_n' = 0 \;;\; f_n' = n x^{n-1} - 2n x^{2n-1} = 0$$

$$n x^{n-1} = 2n x^{2n-1}$$

$$\frac{x^{n-1}}{x^{2n-1}} = 2$$

$$\frac{1}{x^n} = 2 \;;\; x^n = \frac{1}{2}$$

$$x = \sqrt[n]{\frac{1}{2}} = [illegible] = \frac{1}{2} e^{1/n}$$

(a): See in what remarks of the original (b) why the pointwise limit goes to zero; now we want to demonstrate that

$f_n(x) = x^n - x^{2n}$ converges uniformly.

The hint tells us to begin by finding the maximum of

$|f(x) - f_n(x)|$ on $[0,1]$.

Let's do so by first noting that the maximum occurs where the derivative is zero. See that $f(x) = 0$ by the pointwise argument then

$$\frac{\partial}{\partial x}\left|0 - f_n(x)\right| = \frac{\partial}{\partial x}|f_n(x)|$$

Then since $f_n(x)$ is strictly non-negative (small number - smaller number),

$$\ldots = \frac{\partial}{\partial x} f_n(x) = n x^{n-1} - 2n x^{2n-1} \stackrel{!}{=} 0$$

$$n x^{n-1} = 2n x^{2n-1}$$

$$\frac{x^{n-1}}{x^{2n-1}} = 2$$

$(n-1) + (-2n+1) = -n$;

$$\frac{1}{x^n} = 2$$

$$x = \frac{1}{\sqrt[n]{2}}$$

Then plug into

$$f\left(\frac{1}{\sqrt[n]{2}}\right) = \left(\left(\frac{1}{2}\right)^{1/n}\right)^n - \left(\left(\frac{1}{2}\right)^{1/n}\right)^{2n} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

The pointwise limit of zero cannot possibly be made ε-close to this 1/4 jump that must occur, so $f_n(x)$ does not converge uniformly.

(c): Instructor response has informed me that the "bounding seque-ce" we ~~ao~~ want to apply via Weierstrass cannot depend on x, which makes sense because it ~~shou~~ (the bounding sequence) should only be a function of n.

We see that, considering the x-domain $[-\rho, \rho]$, ~~$|x| < \rho$~~

$$-\rho \leq x \leq \rho \Rightarrow |x| \leq \rho;$$

then by algebra,

$$|x|^n \leq \rho^n$$

But the function $f_n(x)$ making our series is defined as

$$f_n(x) = x^n, \text{ so}$$

$$|f_n(x)| \leq \rho^n$$

since absolute values can move through this power, as a result of

$$|n \cdot n \cdots n \cdot n| = |n| \cdot |n| \cdots$$

Thus, a valid majorant for $f_n(x)$ is $M_n = \rho^n$. By the assumpt that $0 < \rho < 1$,

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \rho^n$ is a geometric series equal to $\frac{1}{1-\rho}$.

Then since

$$|f_n(x)| \leq \rho^n = M_n$$

and M_n has a convergent sum, then it is true that

$$\sum_{n=0}^{\infty} x^n \text{ converges uniformly on } [-\rho, \rho]$$

by Weierstrass-M.