question 2 views

Daily Challenge 10.4

(**Due: Wednesday 7/25 at 12:00 noon Eastern**)

I will skip the reading portion of the challenge today so that you can focus on one of the consolidation document problems.

(1) Problem: Stars over Babylon.

This is question 6 on consolidation document 2.

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ , with } p, q \in \mathbb{Z} \text{ in lowest terms} \end{cases} .$$

This function is called *stars over Babylon* or *Thomae's function*.

(a) Prove that $\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$.

[Hint: re-read the argument in DC 10.3 which showed that the function $g(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N}_+ \\ 0 & \text{otherwise} \end{cases}$ is continuous at zero. Use a similar strategy here; in particular, if $\epsilon > 0$ is given, pick some number N so that $\frac{1}{N} < \epsilon$. Then note that there are only finitely many rational numbers with denominators at most N; pick δ small enough so that you don't hit any of them.]

(b) Using your result from (a), show that $f(x)$ is continuous at every irrational point and discontinuous at every rational point in $\mathbb{R}$.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: Proof: We begin as the hint suggests; let $\epsilon > 0$ be given. We can then choose a number N such that $N \in \mathbb{Z}$ so that $\frac{1}{N} < \epsilon$. It is then true that the list of rational numbers with denominators up to N is finite. In turn we can choose a value $\frac{p}{q}$ in this set that is the closest to our target value a. We can then say $\delta = |\frac{p}{q} - a|$, and it is then true that if $0 < |x - a| < \delta$, then either x is irrational or $\frac{1}{M} < \frac{1}{N} < \epsilon$.

b: We have shown in our previous proof that if
 x is irrational then $\lim_{x \to a} f(x) = 0 = f(x)$, therefore $f(x)$ is continuous as irrational x. Similarly, it is true that for rational x that $\lim_{x \to a} f(x) = 0 \neq f(x)$ since f doesn't output zero at any rationals.

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) Let $a \in \mathbb{R}$; we claim that $\lim_{x \to a} f(x) = 0$. Indeed, let $\epsilon > 0$ be given. Find some integer N such that $\frac{1}{N} < \epsilon$. List all of the rationals with denominators up to N:

$$\frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \cdots ; \frac{1}{N}, \cdots , \frac{N-1}{N}; \cdots .$$

There are only finitely many of these, so pick whichever one $\frac{p}{q}$ is closest to a (that is, the one for which $|\frac{p}{q} - a|$ is smallest). Then define $\delta = |\frac{p}{q} - a|$. By construction, if $0 < |x - a| < \delta$, then either x is irrational or $f(x) < \frac{1}{N} < \epsilon$.

(b) We have shown that $\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$. At irrational points, f equals zero and is thus equal to its limit; by definition, it is continuous at these points. But at rational points, f is not equal to zero, so it is discontinuous at these points. This proves the claim.

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*