question 4 views

Daily Challenge 1.3

(**Due: Thursday 4/26 at 12:00 noon Eastern**.)

The reading for Thursday's meeting will introduce you to several proof strategies, including proof by contradiction. I'll give you a head start with today's daily challenge.

"Review"

Proof by contradiction is the second-most-common proof technique (after direct proof). The strategy is:

1. Assume that the **conclusion** (the statement you wish to prove) is *not true*.
2. Starting from the assumption that your conclusion is not true, apply a sequence of logical steps to arrive at a statement which is false. This false statement is the **contradiction**.
3. You have shown that, if your conclusion is not true, this would imply something false. Therefore, the conclusion must have been true all along.

This is best illustrated with an example. First recall that an integer n is said to be **even** if $n = 2a$ for some $a \in \mathbb{Z}$, and an integer m is **odd** if $m = 2b + 1$ for some $b \in \mathbb{Z}$.

Theorem. An even integer cannot be equal to an odd integer.

Proof. Assume, by way of contradiction, that $n = m$ but that n is even and m is odd.

By the definitions of even and odd, then, we can write $n = 2a$ for some integer a, and $m = 2b + 1$ for some integer b. Since $n = m$, this means that

$$2a = 2b + 1,$$

where a and b are integers. Now subtract $2b$ from both side of the equation and divide by 2 to find that

$$a - b = \frac{1}{2}\,.$$

This cannot be true! We know that a and b are integers (remember that these are the positive and negative "whole numbers"), and the difference of two integers can never be a non-integer like $\frac{1}{2}$. Thus we have reached a contradiction, so the original supposition (that $n = m$) must have been false.

We conclude that an even integer can never be equal to an odd integer. $\square$

To recap, the big picture of the above proof was:

1. Replace the general statement "an even integer cannot equal an odd integer" with the specific statement "$n \neq m$ if n is even and m is odd."
2. To argue by contradiction, assume the opposite: say that $n = m$ where n is even and m is odd.
3. Un-pack the definitions of even and odd to replace n and m with the expressions $2a$ and $2b + 1$.
4. Show that the equation $n = m$ leads to the absurd conclusion that the difference of two integers is a non-integer, which is the desired contradiction.

You may also recall that Problem 1.5 in reading assignment 1 (section 1.2 of the AoPS calc book) used a proof by contradiction to show that there is no $x \in \mathbb{Q}$ such that $x^2 = 2$, or in other words, that the square root of 2 is irrational.

Here is another example.

Theorem. There is no smallest positive real number.

Proof. Assume, by way of contradiction, that there were a smallest positive real number x. This means that $x \in \mathbb{R}$, and that there exists no other $y \in \mathbb{R}$ such that $0 < y < x$ (if there were such a y, then x would not be the smallest).

Now consider the number $x' = \frac{x}{2}$. This number x' is still positive, since we may divide both sides of the inequality $x > 0$ to find $\frac{x}{2} > 0$. And x' is still a real number, since the reals are closed under division (as long as the denominator is nonzero).

But x' is also smaller than x. To see this, note that $\frac{1}{2} < 1$ and multiply both sides of the inequality by x to get $\frac{x}{2} < x$, or $x' < x$.

Therefore, we have shown that $0 < x' < x$. But this contradicts our assumption that x is the smallest real number, which means there exists no y such that $0 < y < x$. This contradiction establishes the claim: there can be no smallest real number. $\square$.

I cannot help but include one more famous example.

Theorem. There are infinitely many primes.

Proof. Assume, to get a contradiction, that there were finitely many primes $p_1, \cdots, p_n$. Now consider the number

$$p' = p_1 \times p_2 \times \cdots \times p_n + 1.$$

This new number p' is not divisible by any of the numbers $p_1, \cdots, p_n$ (the first term is divisible by all of them, but 1 is not divisible by any of them). Therefore, the only prime factors of p' are 1 and itself. But this means that p' is prime, and it is greater than all of the other numbers $p_1, \cdots, p_n$, so our original assumption that these p's were all of the primes must have been false. This is the contradiction.

We conclude that there are infinitely many primes, as desired. $\square$

That, my friends, is a *beautiful* proof.

Problem

Now try to prove the following:

Theorem. If x is rational and y is irrational, then $x+y$ is irrational.

Here is a rough plan that you might follow:

- Assume, on the contrary, that $x+y$ is *rational*.
- Use the definition of rational numbers to show that y must therefore be rational.
- This contradicts the assumption that y is irrtational. Therefore, conclude that $x+y$ is irrational.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Proof (Corbin). Your proof goes here.
To begin we must assume that $x+y$ is rational. This means that $x+y \in \mathbb{Q}$. Next we have to say that $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$. Let's re-write x as a fraction, $x = \frac{p_1}{q_2}$ such that p and q are integers. This also means that we can write $x+y$ as a fraction, $x+y = \frac{p_2}{q_2}$. Alas from here I cannot see how I would work forwards from here, the only way I can think of is showing that x could be something such as $x = \frac{1}{2}$ and y would be $y = \frac{1}{3}$ and we must attempt to add these. In doing so we will always get an irrational result, but I don't think this is a case that can always apply.

Proof (Logan) -
Assume, by way of contradiction, x + y is rational, ie $x+y \in \mathbb{Q}$. First, I define x as rational; $x \in \mathbb{Q}$, and y as irrational; $y \in \mathbb{I}$. However, as in the past observed and proven, to receive a rational answer in addition, the values must both be irrational or both rational. This definition shows that x or y must both be rational or irrational for the statement to be true. However, this is not true by the theorem we are given, and one can conclude that since $x+y \notin \mathbb{Q}$, then $x+y \in \mathbb{I}$. (Note that $\mathbb{Q}$ represents the set of all rational numbers, and $\mathbb{I}$ represents the set of all irrational numbers.)

Updated 11 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Proof (Christian). Say x is rational and y is irrational, and suppose (to get a contradiction) that $x+y$ is rational.

Since x is rational, we can write $x = \frac{p}{q}$ for two integers p and q. And because we have assumed $x+y$ is rational, we can also write $x+y = \frac{m}{n}$ for two integers m and n.

This means that we can write y as the difference between $x+y$ and x, or in other words,

$$y = (x+y) - (x) = \frac{m}{n} - \frac{p}{q}.$$

In the second step, we have used our assumptions that $x+y$ and x can be written as the given fractions. But now we can get a common denominator to find

$$y = \frac{mq - pn}{nq}.$$

The numerator $mq - pn$ is an integer, and the denominator nq is a non-zero integer, so y has been written as a ratio of integers. But this means that y is rational, which contradicts our assumption that y was irrational.

Because the assumption that $x+y$ is rational leads to a contradiction, we conclude that $x+y$ must be irrational. $\square$

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*