question 4 views

Daily Challenge 2.6

(**Due: Sunday 5/6 at 12:00 noon Eastern**)

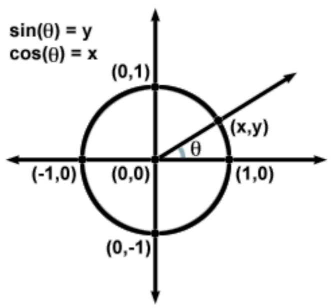
I don't think we've quite mastered proofs yet, but it would be useful to mix up our daily challenges by starting some trigonometry (chapter 3 of that *Make it Stick* book explains that we learn best when our practice is *interleaved* -- that is, rather than drilling the same ideas over and over, it's better to jump around between different subjects).

I'll try to switch between some proofs and some trigonometry over the next week or so.

"Review"

For today, I just want to define the trigonometric functions $\sin(\theta)$ and $\cos(\theta)$. We will often use the Greek letter θ (pronounced THAY-tuh and given in $\LaTeX$ by \theta) to represent angles.

Definition. Consider a circle of radius 1 centered at the origin, $(0,0)$. Draw a line segment which begins at the origin and makes an angle θ with the x-axis (measured counter-clockwise). This line segment intersects the circle at some point (x,y). We define $\cos(\theta)$ to be the x-coordinate of this intersection point, and we define $\sin(\theta)$ to be the y-coordinate.

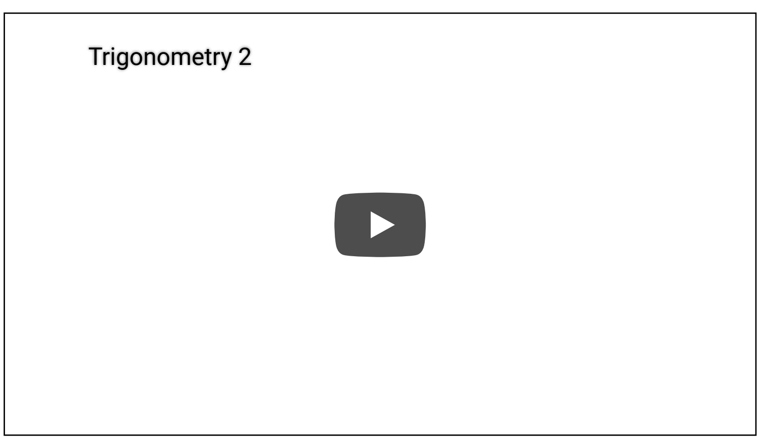


In short: $(\cos(\theta), \sin(\theta))$ is the point on the unit circle that you get by beginning at $(1,0)$ and moving counter-clockwise by an angle θ.

When you have time, check out the following mini-lectures for more details:

1. Part 1 builds some intuition for the definitions given above.

2. Part 2 computes the values of $\cos(\theta)$ and $\sin(\theta)$ for a few easy angles, using the rules for $(45°, 45°, 90°)$ triangles and $(30°, 60°, 90°)$ triangles.



3. Part 3 proves an important identity: for any angle θ, we have $(\sin(\theta))^2 + (\cos(\theta))^2 = 1$.

We will usually use the shorthand $\sin^2(\theta)$ for $(\sin(\theta))^2$ and $\cos^2(\theta)$ for $(\cos(\theta))^2$, but I have written out the long-hand versions for now to avoid introducing too much notation at once.

Problem

Watch the videos above, then try to answer following questions.

(a) Compute the values of $\cos(135^\circ)$ and $\sin(135^\circ)$ by drawing a circle of radius 1 centered at $(0, 0)$, and then finding the x and y coordinates of the point 135° counter-clockwise from the x axis.

(b) We define the sine and cosine of *negative* angles by going clockwise rather than counter-clockwise. Find an equation which expresses $\sin(-\theta)$ in terms of $\sin(\theta)$, and also find one which expresses $\cos(-\theta)$ in terms of $\cos(\theta)$.

[Hint: draw the circle and compare the (x, y) coordinates of two points obtained by going clockwise and counter-clockwise by the same angle. Guess the pattern.]

(c) [**Optional but good practice]** Find a value of the angle θ which makes the following equation true:

$$(\sin(\theta))^2 - \frac{5}{2}\sin(\theta) + 1 = 0.$$

[Hint: Do this problem in two steps. First, define $x = \sin(\theta)$ so the equation becomes a quadratic, and find the roots using your favorite method. Then use our table to find an angle θ so that $\sin(\theta)$ is one of the roots.]

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Solutions (Corbin). Your results go here.

(a)

(b)

(c)

Solutions (Logan). Your results go here.

(a) $\cos(135^\circ) = \frac{-1}{\sqrt{2}}$; $\sin(135^\circ) = \frac{1}{\sqrt{2}}$

(b) After thinking, one can notice that the result of $\sin(-\theta)$ is the opposite/negative of the result of $\sin(\theta)$ therefore the sin of a negative degree can be expressed in terms of $-\sin(\theta) = \sin(-\theta)$ Identically, $-\cos(\theta) = \cos(-\theta)$

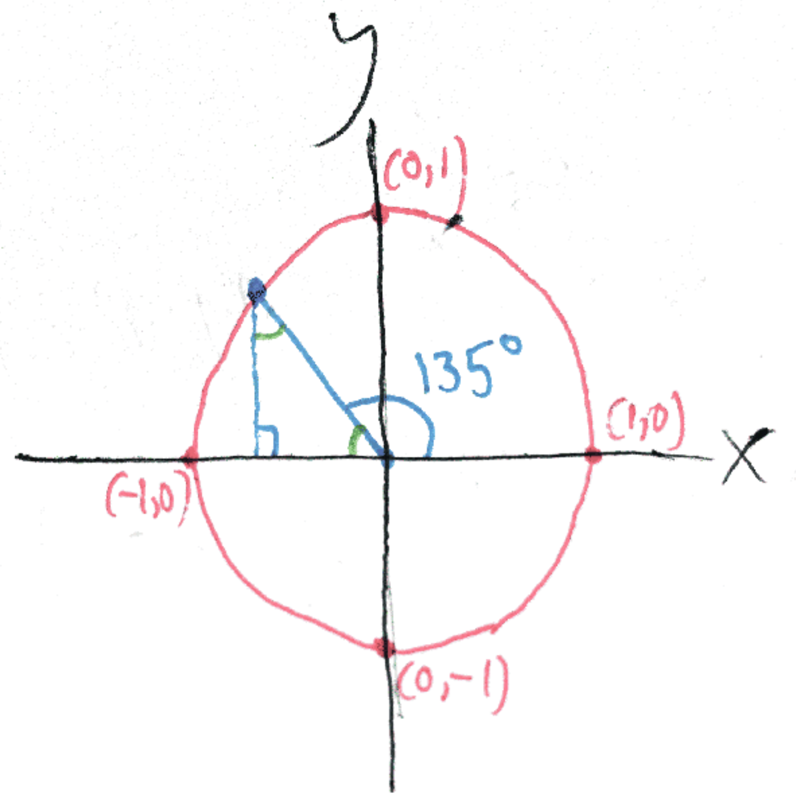
(c)

Updated 11 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Solutions (Christian).

(a) By definition, the sine and cosine of 135° are the y and x coordinates, respectively, of the point on the unit circle obtained by going 135° counter-clockwise. The picture looks like this:

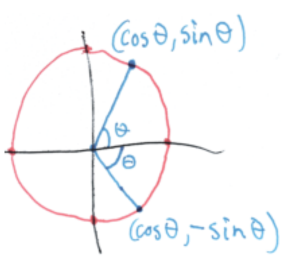


The two green angles each measure 45°, so this is a $(45^\circ, 45^\circ, 90)$ triangle. We recall that the lengths of such a triangle come in the ratio $1 : 1 : \sqrt{2}$, where the longer side is the hypotenuse. Here the hypotenuse has length 1, since it is the radius of our circle, so the other two legs each measure $\frac{1}{\sqrt{2}}$.

The y coordinate, then, is $\frac{1}{\sqrt{2}}$, while the x coordinate is $-\frac{1}{\sqrt{2}}$ since it lies to the left of the origin:

$$\sin(135^\circ) = \frac{1}{\sqrt{2}} \quad , \quad \cos(135^\circ) = -\frac{1}{\sqrt{2}}.$$

(b) Let's draw a diagram with two points, one obtained by going counter-clockwise by some angle θ and the other obtained by going clockwise by the same angle θ:



We see that going *clockwise* by an angle θ gives us a point which is the mirror image of the point we get from going *counter-clockwise* by the same angle θ. By "mirror image", I mean that the two points lie on opposite sides of the x axis. This means that the two points will have the same x coordinate, but their y coordinates will differ by a minus sign. Thus we conclude that

$$\sin(-\theta) = -\sin(\theta) \quad , \quad \cos(-\theta) = \cos(\theta).$$

(c) Following the hint, we first let $x = \sin(\theta)$. The equation becomes

$$x^2 - \frac{5}{2}x + 1 = 0.$$

This can be factored as

$$\left(x - \frac{1}{2}\right)(x - 2) = 0,$$

which has the roots $x = 2$ and $x = \frac{1}{2}$.

Now we need to find a value of $x = \sin(\theta)$ which gives one of these roots. Since $\sin(\theta)$ is the y coordinate of a point on a circle of radius 1, it can never equal 2 (it is, at most, equal to 1). But we can obtain $\sin(\theta) = \frac{1}{2}$ when $\theta = 30^\circ$. Thus we've found a solution to the given equation:

$\theta = 30^\circ$.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*