question **2** views

Daily Challenge 19.6

(**Due: Monday November 26 at 12:00 noon eastern**)

A while back, you proved that the *Schwarzian derivative*,

$$\mathscr{D}f = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

has the very nice property that it annihilates so-called *fractional linear transformations*, or functions of the form $f(x) = \frac{ax+b}{cx+d}$. That is

$$\mathscr{D}\left(\frac{ax+b}{cx+d}\right) = 0.$$

In this problem, you will prove the converse: any function whose Schwarzian is zero must be a fractional linear transformation. In other words,

$$\mathscr{D}f = 0 \implies f = \frac{ax+b}{cx+d} \text{ for some } a, b, c, d.$$

(1) Problem: Revisiting the Schwarzian.

I will scaffold the proof for you.

(a) Suppose that $\mathscr{D}f = 0$. Prove that $\frac{(f'')^2}{(f')^3}$ is a constant function. [Hint: take the derivative and show that it's zero.]

(b) Let $u = f'$. Show that $u^{-3/2} \cdot u' = C$ for some constant C. [Hint: this is just rewriting your conclusion in (a).]

(c) Take an anti-derivative of your equation in (b) to show that $-2u^{-1/2} = Cx + D$ for some constant D.

[Hint: anti-differentiating the right side gives Cx, trivially. How do you anti-differentiate $u^{-3/2}u'$? In other words, what function $w(x)$ has the property that $w'(x) = u^{-3/2}u'(x)$?]

(d) Solve to find $u(x) = \frac{4}{(Cx+D)^2}$. But we defined $u(x) = f'(x)$, so take another anti-derivative to find $f(x)$. This will introduce one more constant E.

Explain why the result for f that you get is a fractional linear transformation. [Hint: get a common denominator to write it in the form $\frac{ax+b}{cx+d}$. What are a, b, c, d in terms of C, D, E?]

daily_challenge

Updated 4 months ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

(a) Consider the expression $g = \frac{(f'')^2}{(f')^3}$. By the quotient rule and chain rule, its derivative is

$$g' = \frac{2(f'')(f''')(f')^3 - 3(f')^2(f'')(f'')^2}{(f')^4}.$$

Splitting the numerator into two terms and canceling common factors of f' from the top and bottom, this is

$$g' = 2\frac{f''f'''}{f'} - 3\frac{(f'')^3}{(f')^2} = 2 \cdot f''(x) \cdot \mathscr{D}f,$$

where in the last step we have recognized the definition of the Schwarzian. We have assumed that $\mathscr{D}f = 0$, so it follows that $g' = 0$. But the only functions with vanishing first derivative are constant functions, so g is a constant.

(b) We showed in part (a) that $g = \frac{(f'')^2}{(f')^3} = k$ for some constant k. Replacing the denominator by a negative exponent, this is

$$(f'')^2(f')^{-3} = k.$$

Now define $u = f'$ so $u' = f''$. In terms of u, the above equation reads

$$(u')^2 u^{-3} = k,$$

or after taking a square root on each side, $u'u^{-3/2} = \sqrt{k}$. Define a new constant $C = \sqrt{k}$, since the square root of an arbitrary constant is some other arbitrary constant. Then we've shown

$$u'u^{-3/2} = C,$$

as desired.

(c) We wish to take an anti-derivative with respect to x, or in other words, we would like to find

$$\int u'(x)(u(x))^{-3/2}\,dx.$$

This integral admits the painfully obvious u-substitution $u = u$, $du = u'(x)\,dx$, after which

$$\int u'(x)(u(x))^{-3/2}\,dx = \int u^{-3/2}\,du = -2u^{-1/2},$$

where we have applied the power rule. Meanwhile, the integral of the right side of the equation is

$$\int C\,dx = Cx + D,$$

where we have chosen a new arbitrary constant D. Thus we've shown

$$-2u^{-1/2} = Cx + D.$$

(d) In part (c) we showed that $\frac{1}{\sqrt{u}} = -\frac{1}{2}(Cx + D)$, or $\sqrt{u} = -\frac{2}{Cx+D}$, or

$$u(x) = 4(Cx + D)^{-2}.$$

But we began by defining $u(x) = f'(x)$, so our equation is really

$$f'(x) = 4(Cx + D)^{-2},$$

so we can take an indefinite integral on each side to find

$$f(x) = 4\left(\int (Cx + D)^{-2}\,dx\right) + E,$$

where E is a new constant. By the u-substitution $u = Cx + D$, $du = C\,dx$, we see

$$\begin{aligned}\int (Cx + D)^{-2}\,dx &= \int u^{-2}\,\frac{du}{C}\\ &= -\frac{1}{C}u^{-1}\\ &= -\frac{1}{C}(Cx + D)^{-1},\end{aligned}$$

so we have found

$$f(x) = -\frac{4}{C}\cdot\frac{1}{Cx + D} + E.$$

Now it is a simple matter to get a common denominator:

$$\begin{aligned}f(x) &= \frac{-\frac{4}{C}}{Cx + D} + \frac{E(Cx + D)}{Cx + D}\\ &= \frac{ECx + \left(-\frac{4}{C} + ED\right)}{Cx + D}.\end{aligned}$$

This means that $f(x) = \frac{ax+b}{cx+d}$ is a fractional linear combination for some choice of the constants a, b, c, d. If I've done the algebra correctly, it appears that $c = C$, $d = D$, $a = EC$, and $b = -\frac{4}{C} + ED$ is one such choice that works. Note that this choice is not unique.

In any case, we've proven that any function f whose Schwarzian derivative vanishes must have the form $f(x) = \frac{ax+b}{cx+d}$.

Updated 2 months ago by Christian Ferko

followup discussions *for lingering questions and comments*