question 2 views

Daily Challenge 17.4

(**Due: Thursday 11/8 at 12:00 noon Eastern**)

Let's do a slightly harder problem which will generalize the integrals of polynomials we've seen before. I'll scaffold the proof for you and this will go on CD 5 when I get around to writing it.

(1) Problem: integrating any monomial.

In this problem we'll use a clever trick to find $\int_a^b x^p\,dx$ when $0 < a < b$ any for any $p > 0$. Instead of using equally-spaced partitions like before, we'll use partitions $P_n = \{t_0, \cdots, t_n\}$ for which the *ratios* $r = \frac{t_i}{t_{i-1}}$ are all equal.

(a) Prove that, for the partitions P_n described above, we have

$$t_i = a \cdot c^{i/n}$$

where $c = \frac{b}{a}$.

(b) Let $f(x) = x^p$. Show that

$$\begin{aligned} U(f,P) &= a^{p+1}\left(1 - c^{1/n}\right)\sum_{i=1}^{n}\left(c^{\frac{p+1}{n}}\right)^i \\ &= \left(a^{p+1} - b^{p+1}\right)c^{\frac{p+1}{n}}\frac{1-c^{-\frac{1}{n}}}{1-c^{\frac{p+1}{n}}} \\ &= \left(b^{p+1} - a^{p+1}\right)c^{\frac{p}{n}}\frac{1}{1+c^{1/n}+\cdots+c^{p/n}}. \end{aligned}$$

You may use, without proof, that $1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$for any $r \in \mathbb{R}$.

(c) Use the above results to prove that

$$\int_a^b x^p\,dx = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

daily_challenge

Updated 5 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
(a): We see by constructing a few partitions there is a pattern to be found;
$P_n = \left\{a, r\cdot a, r^2\cdot a, \cdots r^{n-1}a, b = r^n\cdot a\right\}$
We then see that

$$\frac{b}{a} = r^n$$
$$\sqrt[n]{\frac{b}{a}} = r$$

Then by inserting this ratio back into our partition, we see that $t_i = a\cdot\left(\sqrt[n]{\frac{b}{a}}\right)^i$, which of course "simplifies" to $t_i = a\cdot c^{i/n}$.

(b): We begin by writing down the definition of upper sum to see that
$U(f,P) = \sum_{i=1}^n f(t_i)\cdot(t_i - t_{i-1})$where we have ignored $i = 0$ due to the infimum on the interval being 0. We then see by substituting in our found value for t_i that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n}(a\cdot c^{i/n})^p\cdot(a\cdot c^{i/n} - a\cdot c^{(i-1)/n}) \\ &= \sum_{i=1}^{n}(ac^{i/n})^p\cdot a(c^{i/n} - \cdot c^{(i-1)/n}) \end{aligned}$$

We break the $c^{(i-1)/n}$ into its *FOOKIN* components and see that it equals
$c^{i/n}\cdot c^{-1/n}$
and therefore we have

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n}(ac^{i/n})^p \cdot a(c^{i/n} - c^{i/n}\cdot c^{-1/n}) \\ &= \sum_{i=1}^{n}(ac^{i/n})^p \cdot a(c^{i/n})(1-c^{-1/n}) \\ &= a^{p+1}(1-c^{-1/n})\sum_{i=1}^{n}(c^{p+1/n})^i \end{aligned}$$

We have finally reached line 1 of (b), and to apply our sum rule, we need to reindex i to $i-1$, fairly easy.

$$= a^{p+1}(1-c^{-1/n})c^{p+1/n}\sum_{i=0}^{n-1}(c^{p+1/n})^i$$

We can now apply the rule of sum of exponents definition/rule we are given ($1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r}$) to see that

$$\begin{aligned} a^{p+1}(1-c^{-1/n})c^{p+1/n}\sum_{i=0}^{n-1}(c^{p+1/n})^i &= a^{p+1}(1-c^{-1/n})c^{p+1/n}\frac{1-(c^{p+1/n})^n}{1-c^{p+1/n}} \\ &= a^{p+1}(1-c^{-1/n})c^{p+1/n}\frac{(1-(c^{p+1}))}{1-c^{p+1/n}} \\ &= a^{p+1}(1-c^{p+1})c^{p+1/n}\frac{(1-c^{-1/n})}{1-c^{p+1/n}} \\ &= (a^{p+1}-b^{p+1})c^{p+1/n}\frac{(1-c^{-1/n})}{1-c^{p+1/n}} \\ &= (a^{p+1}-b^{p+1})c^{p/n}\frac{(c^{1/n}-1)}{1-c^{p+1/n}} \end{aligned}$$

And finally by working in the opposite direction as last time from the given rule,

$$U(f,P) = \left(b^{p+1}-a^{p+1}\right)c^{p/n}\frac{1}{1+c^{1/n}+\cdots+c^{p/n}}.$$

(c): We have found the upper sum in a useful form, so let us look at the lower sum

$$L(f,P) = \sum_{i=1}^{n}(a\cdot c^{i-1/n})^p\cdot(a\cdot c^{i/n} - a\cdot c^{i-1/n})$$

This is familiar; let us pull a $c^{-p/n}$ out of the first term and the sum; We then see that

$$\begin{aligned} L(f,P) &= c^{-p/n}\sum_{i=1}^{n}(a\cdot c^{i/n})^p\cdot(a\cdot c^{i/n} - a\cdot c^{i-1/n}) \\ &= c^{-p/n}U(f,P) \end{aligned}$$

We then see that if this is true, then what we see in the final step of (b) means that

$$L(f,P) = \left(b^{p+1}-a^{p+1}\right)\cdot\frac{1}{1+c^{1/n}+\cdots+c^{p/n}},$$

due to powers cancelling.

Our next valid step to prove that this integral is equal to the value of interest is to show that the upper and lower sums squeeze some value, but to do so we must show that $U(f,P) - L(f,P) < \epsilon$where ϵ is small. We see that

$$U(f,P)-L(f,P) = \left(b^{p+1}-a^{p+1}\right)\cdot\left(c^{p/n}-1\right)\left(\frac{1}{1+c^{1/n}+\cdots+c^{p/n}}\right),$$

This can be made less than ϵ since, as n grows large, our "middle" term $\left(c^{p/n}-1\right)/to0$, and due to the multiplication the entirety shall go to zero. Another term is also in terms of n; the third term shall go to $1 \text{ as } n \to \inf$, and we conclude that as $U-L$ can be made sufficiently small.

We can finally see (and I spent about 5 minutes straight understanding why this is true) that

$$L(f,P) = \left(b^{p+1}-a^{p+1}\right)\cdot\frac{1}{1+c^{1/n}+\cdots+c^{p/n}} \le \frac{\left(b^{p+1}-a^{p+1}\right)}{p+1} \le \left(a^{p+1}-b^{p+1}\right)c^{p/n}\,\frac{1}{1+c^{1/n}+\cdots+c^{p/n}} = U(f,P)$$

and once again as if squeezed, the infimum of the upper sum, and sup of the lower meet at the value we would like to prove. (one could expect explanation :swet:)

Updated 5 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

(a) Suppose the ratios $$\frac{t_i}{t_{i-1}} = r$$ are all equal. The product of all of them is therefore

$$\begin{align} \frac{b}{a} = \frac{t_n}{t_0} = \frac{t_n}{t_{n+1}} \cdot \frac{t_{n-1}}{t_{n-2}} \cdot \cdots \cdot \frac{t_1}{t_0} = r^n . \end{align}$$

Thus the common ratio is $$r = \left(\frac{b}{a} \right)^{1/n}$$. We have that every $$\frac{t_i}{t_{i-1}} = r = \left(\frac{b}{a} \right)^{1/n}$$, which means $$\frac{t_i}{a} = \left(\frac{b}{a} \right)^{i/n}$$, and hence

$$\begin{align} t_i = a c^{i/n} \end{align}$$

where $$c= \frac{b}{a} $$.

(b) The supremum occurs at the right endpoint on each interval, since $$[a,b]$$ is an interval of positive numbers and any $$x^p$$ with $$p>0$$ is increasing on such an interval. But the right endpoints are $$t_i = a c^{i/n}$$ by part (a). Thus

$$\begin{align} U(f, P) &= \sum_{i=1}^{n} M_i \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \left(a c^{i/n} \right)^p \cdot \left(a c^{i/n} - a c^{(i-1)/n} \right) \\ &= a^{p+1} (1 - c^{-1/n}) \sum_{i=1}^{n} \left(c ^{(p+1)/n} \right)^i \\ &= a^{p+1} (1 - c^{-1/n}) c^{(p+1)/n} \sum_{i=0}^{n-1} \left(c^{(p+1)/n} \right)^i . \end{align}$$

Note that in the last step we have re-indexed $$i$$ to $$i-1$$. Now we apply the result that $$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$, which we are told we may use without proof. This gives

$$\begin{align} U(f, P) &= a^{p+1} (1 - c^{-1/n}) c^{(p+1)/n} \sum_{i=0}^{n} \left(c^{(p+1)/n} \right)^i \\ &= a^{p+1} (1 - c^{-1/n}) c^{(p+1)/n} \frac{1 - c^{p+1} }{1 - c^{(p+1)/n} } \\ &= a^{p+1} (1 - c^{p+1}) c^{(p+1)/n} \cdot \frac{1 - c^{-1/n} }{1 - c^{(p+1)/n} } \\ &= \left(a^{p+1} - b^{p+1} \right) c^{(p+1)/n} \frac{1 - c^{-1/n}}{1 - c^{(p+1)/n} } . \end{align}$$

In the second step, we just switched the order of two factors; in the second step, we plugged in $$c = \frac{b}{a}$$ and simplified. But now by distributing the factor of $$c^{1/n}$$ into the numerator of the fraction we conclude

$$\begin{align} U(f, P) &= \left(a^{p+1} - b^{p+1} \right) c^{ p /n} \frac{c^{1/n} - 1}{1 - c^{(p+1)/n} } . \end{align}$$

Finally, we again use the result that $$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$ to conclude

$$\begin{align} U(f, P) &= \left(b^{p+1} - a^{p+1} \right) c^{ p /n} \frac{1}{1 + c^{1/n} + \cdots + c^{p/n} } . \end{align}$$

Note that we used the minus sign from the numerator to switch the order of $$b$$ and $$a$$. This is what we wanted to show.

(c) We need to show that $$U(f,P) - L(f,P)$$ can be made smaller than epsilon. The lower sum is

$$\begin{align} L (f, P) &= \sum_{i=1}^{n} \left(a c^{(i - 1)/n} \right)^p \left(a c^{i/n} - a c^{(i-1)/n} \right) \\ &= c^{- p / n } \sum_{i=1}^{n} \left(a c^{i/n} \right)^p \left(a c^{i/n} - a c^{(i-1)/n} \right), \end{align}$$

since each infimum is achieved at the left endpoint and where we just pulled out a constant factor in the last step. But what remains is precisely $$L(f,P) = c^{-p/n} U(f,P)$$. Thus

$$\begin{align} L (f, P) = \left(b^{p+1} - a^{p+1} \right) \cdot \frac{1}{1 + c^{1/n} + \cdots + c^{p/n} } , \end{align}$$

where we have just multiplied by our result for $$U(f,P)$$ by $$c^{-p/n}$$ and killed the factor of $$c^{p/n}$$ which was there.

Therefore,

$$\begin{align} U (f, P) - L(f, P) = \left(b^{p+1} - a^{p+1} \right) \cdot \left(c^{ p /n} - 1 \right) \left(\frac{1}{1 + c^{1/n} + \cdots + c^{p/n} } \right) . \end{align}$$

As $$n$$ grows large, $$\frac{p}{n}$$ goes to $$0$$ so $$c^{p/n}$$ goes to $$1$$. Thus the first factor $$c^{p/n} - 1$$ can be made as small as we like. Similarly, each of the numbers $$c^{i/n}$$ in the denominator of the second factor goes to zero, so this fraction goes to $$1$$. Overall, the second two factors can be made smaller than any $$\epsilon$$. This shows that the function is integrable.

To find the value of the integral, note that

$$\begin{align} L (f, P) = \left(b^{p+1} - a^{p+1} \right) \cdot \frac{1}{1 + c^{1/n} + \cdots + c^{p/n} } \leq \frac{\left(b^{p+1} - a^{p+1} \right)}{p+1} \leq \left(a^{p+1} - b^{p+1} \right) c^{ p /n} \, \frac{1}{1 + c^{1/n} + \cdots + c^{p/n} } = U(f,P) , \end{align}$$

since there are $$p+1$$ factors in each denominator, each of which is at most $$1$$. Thus if the supremum of the $$L$$'s equals the infimum of the $$U$$'s, both must be equal to $$\frac{\left(b^{p+1} - a^{p+1} \right)}{p+1} $$. We conclude

$$\begin{align} \int_a^b x^p = \frac{\left(b^{p+1} - a^{p+1} \right)}{p+1} . \; \Box \end{align}$$

Updated 5 months ago by Christian Ferko

followup discussions *for lingering questions and comments*