question 2 views

Daily Challenge 10.5

(**Due: Thursday 7/26 at 12:00 noon Eastern**)

I think we're almost ready to move on from limits; perhaps we can plan the British-style tutorial for chapter 2 on Wednesday 8/1.

(1) Continuous functions obey powerful theorems.

As we saw in the "Three Hard Theorems" session, continuous functions are highly constrained: they satisfy certain reasonable-sounding properties, like being forced to "hit all outputs" between two endpoints on a closed interval.

More precisely, these nice properties are encoded in the follow three results.

Theorem (intermediate value theorem). Let f be a continuous, real-valued function defined on an interval $[a, b]$ with $f(a) \neq f(b)$, and let y be a real number between $f(a)$ and $f(b)$. Then there exists some $c \in (a, b)$ such that $f(c) = y$.

Theorem (boundedness theorem). Let f be a continuous, real-valued function defined on an interval $[a, b]$. Then there exists some $M \in \mathbb{R}$ such that $f(c) \leq M$ for all $c \in [a, b]$.

Theorem (extreme value theorem). Let f be a continuous function defined on a closed interval $[a, b]$. Then f attains a maximum on $[a, b]$; that is, there exists a number $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.

The proofs of these theorems were given in session 17; they used both continuity and the completeness property of the real numbers in a key way.

One common application of these theorems is to prove that a solution to some equation exists. To do this, one tries to write down a continuous function $g(x)$ which is zero when the desired equation is satisfied, argue that $g(x)$ is continuous on some interval, and then show that g changes sign on the interval.

Example. Prove that there exists a real number x such that $x^{179} + \frac{163}{1+x^2+\sin^2(x)} = 119$.

Solution. This function looks hopelessly complicated, but we only need to prove the *existence* of a solution. Define the function

$$g(x) = x^{179} + \frac{163}{1 + x^2 + \sin^2(x)} - 119.$$

So, if we can show that $g(c) = 0$ for some c, we're done.

Now, we've proven that polynomials, constants, and the sine function are continuous, and that sums and reciprocals of continuous functions are continuous (so long as the denominator is nonzero, which it must be here, since it is the sum of 1 and two non-negative quantities). So g is continuous.

Next note that $g(0) = 163 - 119 > 0$ but $g(1) = 1 + \frac{163}{2+\sin^2(1)} - 119$. I don't know what $\sin^2(1)$ is, but it's certainly non-negative, so

$$\frac{163}{2 + \sin^2(1)} < \frac{163}{2} < 82,$$

so $g(1) < 83 - 119 < 0$. So $g(0)$ is positive but $g(1)$ is negative. Since g is continuous on $[0, 1]$, by the intermediate value theorem there exists some number $c \in (0, 1)$ so that $g(c) = 0$. $\square$

(2) Problem: using continuity.

Answer the following three questions; you may use the intermediate value theorem, boundedness theorem, and extreme value theorem.

Note that this is question 5 on consolidation document 2.

(a) Show that if f is continuous on $[a, b]$, then f attains a minimum on $[a, b]$.

(b) Suppose that f is continuous on $[a, b]$ and $f([a, b]) \subseteq \mathbb{Q}$ (that is, the image of $[a, b]$ under f lies in the rationals). What can we conclude about f?

[Hint: you may use, without proof, the fact that between any two rationals numbers p and q there exists an irrational number.]

(c) Let f be a continuous function with domain $[0, 1]$ and range $[0, 1]$. Prove that f must have a *fixed point*: that is, there exists some $a \in [0, 1]$ such that $f(a) = a$.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
a: We must show that if f is continuous on $[a, b]$, then f attains a minimum on the closed interval $[a, b]$. We begin by defining that $g(x) = -f(x)$. It is then true by the bounded theorem that within the interval $[a, b]$ the function g attains a maximum, and therefore f attains a minimum in this interval.

b: We begin with the information that f is continuous on the interval $[a, b]$. We then know, assuming $a \neq b$, there exists a value $y \in [a, b]$ such that $y \in (\mathbb{R} \setminus \mathbb{Q})$. There is only one continuous function that can take in an irrational and output a rational, and that is a constant function, ie $f(x) = c$ where c is rational.

c: f is a continuous function with the domain and range $[0, 1]$. It is then true by the intermediate value theorem that there exists some value y where $0 \leq y \leq 1$ such that $f(y) = y$.

Still working lol

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

My responses follow.

(a) Suppose f is continuous on $[a, b]$ and let $g(x) = -f(x)$. Then g is also continuous on $[a, b]$, since it is a constant function -1 times the continuous function f. By the extreme value theorem, g achieves its maximum on $[a, b]$. But the maximum of g will be the minimum of $f = -g$, so f achieves its minimum on $[a, b]$, as desired.

(b) We claim that f must be constant.

To prove this, suppose by way of contradiction that $f([a, b]) \subseteq \mathbb{Q}$ but that f is not constant. This means that there are two points x_1 and x_2 such that $f(x_1) = q_1$ and $f(x_2) = q_2$ but that $q_1 \neq q_2$.

Now, since f is continuous on $[a, b]$, it is also continuous on $[x_1, x_2]$, and $f(x_1) = q_1 \neq q_2 = f(x_2)$, so the hypotheses of the intermediate value theorem apply here. By the hint, we may use without proof that there exists some irrational number y between the two rational numbers q_1 and q_2. Using the intermediate value theorem, then, there is a point $c \in (x_1, x_2)$ such that $f(c) = y$.

But this means that $f(c)$ is irrational, which contradicts that $f([a, b]) \subseteq \mathbb{Q}$.

(c) We consider two cases.

Case one: either $f(0) = 0$ or $f(1) = 1$. In this case, either 0 is a fixed point or 1 is a fixed point, so we're done.

Case two: both $f(0) > 0$ and $f(1) < 1$. Define the new function $g(x) = f(x) - x$. Then g is continuous on $[0, 1]$, but $g(0) > 0$ and $g(1) < 0$, so by the intermediate value theorem there exists $c \in (0, 1)$ so that $g(c) = 0$. But this means $f(c) = c$, as desired.

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*