31.6

(a): Plug ~~in~~ $x=\pi$ into

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

to see that

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot(-1)^n \quad \text{since } n \text{ is integer}$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{since } (-1)^{2n}=1$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

(b): By replacing sine with the first term in the Taylor series, and putting all other terms into $O(x^2)$ see that

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\cdot\left(\frac{n\pi x}{L} + O(x^2)\right)$$

$$= \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\cdot\frac{n\pi x}{L} + O(x^2)$$

The n's cancel and we have a constant inside a sum; if $x=0$ then the constant is 0 and the series might converge. For $x\neq 0$ though we have a constant in the sum and that most certainly doesn't go to zero ever, thus the sum is divergent for $x\neq 0$.

(c): The hint suggests writing

$$f(x)^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(a_n\cos\left(\frac{2\pi nx}{L}\right) + b_n\sin\left(\frac{2\pi nx}{L}\right)\right)\left(a_m\cos\left(\frac{2\pi mx}{L}\right) + b_m\sin\left(\frac{2\pi mx}{L}\right)\right)$$

Integrate each side from $0\to L$ and foil

$$\int_0^L f(x)^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(a_n a_m\frac{L}{2}\delta_{nm} + b_n b_m\frac{L}{2}\delta_{nm} + 0 + 0\right)$$

For $n \neq m$, the values in the sum will be zero, but for $n = m$ we can collapse the sum to

$$\sum_{n=0}^{\infty} \frac{L}{2}\left(a_n^2 + b_n^2\right) = \int_0^L f^2(x)$$

$$\sum_{n=0}^{\infty} (a_n^2 + b_n^2) = \frac{2}{L}\int_0^L f^2(x)$$