question **2** views

Daily Challenge 26.7

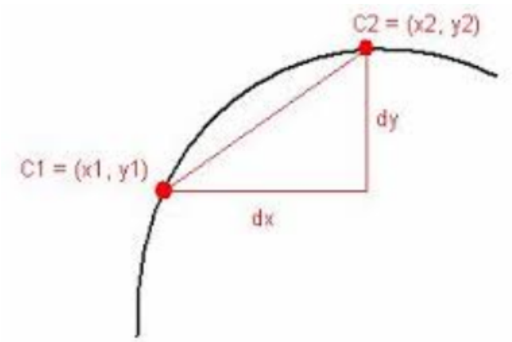
(**Due: Friday 4/12 at 12:00 noon Eastern**)

One of the most important applications of the calculus of variations is finding *geodesics*, or paths which minimize length.

You've already shown in DC 26.3 that geodesics in the ordinary (x, y) plane are straight lines. Let's quickly review how the argument works. Although we proved rigorously in a previous session that the arc length of a curve $y(x)$ between a and b is given by

$$L = \int_a^b \sqrt{1 + (y'(x))^2}\, dx,$$

it is useful to keep in mind the following "quick and dirty" (i.e. non-rigorous but intuitively correct) way of deriving the formula. Imagine chopping up your curve into many small segments. We approximate the length of each segment by a straight line between the endpoints as in the following figure:



By Pythagoras, the length of the line segment is $\sqrt{dx^2 + dy^2}$. Adding up the contributions from the many small line segments, the total length is roughly

$$L = \sum_{i=1}^{N} \sqrt{dx_i^2 + dy_i^2}.$$

At this stage, the symbols dx_i and dy_i represent finite lengths (*not* differentials or derivatives), so we are free to pull a factor of dx_i out of the square root:

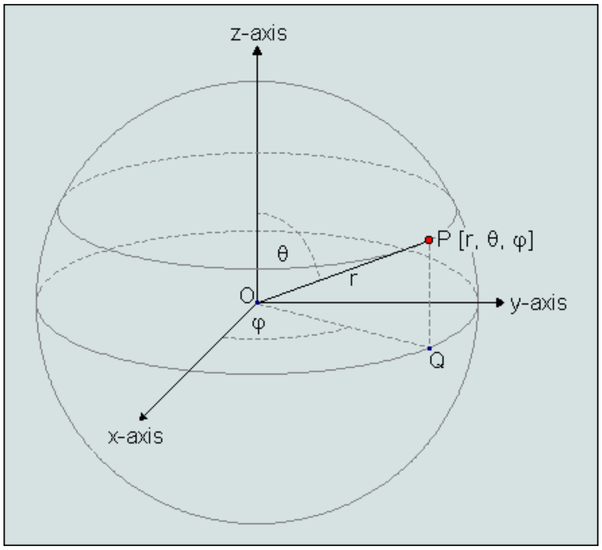
$$L = \sum_{i=1}^{N} \sqrt{1 + \left(\frac{dy_i}{dx_i}\right)^2}\, dx_i.$$

Now for the sketchy part: take the limit as all of the dx_i become very small. The ratios $\frac{dy_i}{dx_i}$ tend to the honest-to-goodness derivative $\frac{dy}{dx}$ evaluated at a point on the interval, while the sum converges to an integral (we are not proving these statements here, but they can be made precise). Thus

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

as claimed.

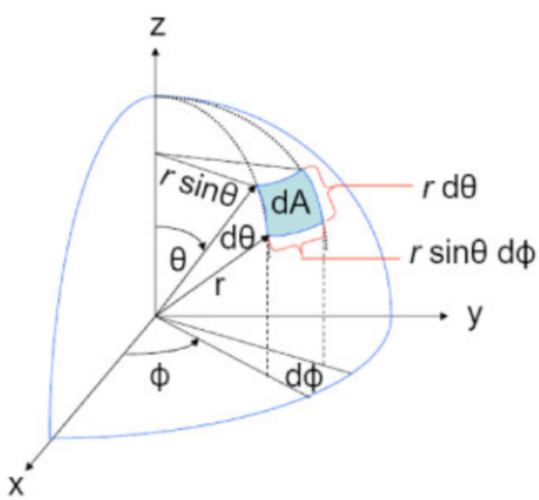
(**Part a**) Now suppose we are on a sphere of radius r. We label points on the sphere with two angles θ and ϕ, where θ measures the angle with the z axis and ϕ measures the angle between the projection of the point onto the (x, y) plane and the x axis, as shown below.



Consider two nearby points $P_1 = (r, \theta, \phi)$ and $P_2 = (r, \theta + d\theta, \phi + d\phi)$on the sphere. Mimic the "quick and dirty" argument above, using the Pythagorean theorem to write down an expression for the distance between P_1 and P_2.

Hint 1: the answer is **not** $\sqrt{d\theta^2 + d\phi^2}$. Indeed, this doesn't even have the right units (angles are dimensionless but we want something with units of length). If you get stuck, go back to a circle of radius r in the plane and figure out the length of an arc which subtends angle $d\theta$.

Hint 2: look at this picture.



The lengths of the two sides of the "circular rectangle" are $r\, d\theta$ and $r \sin(\theta)\, d\phi$. By Pythagoras, what is the length of the diagonal?

Answer: $L = \sqrt{r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2}$.

(**Part b**) Again mimicking the "quick and dirty" approach, let's add up the many contributions from small line segments of the form you found in part (a) and write the total length as

$$L = \sum_i \sqrt{r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2},$$

then imagine that $\phi(\theta)$ is a function that determines the curve, so we can write

$$L[\phi] = r \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2(\theta) \left(\frac{d\phi}{d\theta}\right)^2}\, d\theta.$$

Therefore, to minimize the length of a path on a sphere, we can simply minimize the functional

$$J[\phi] = \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2(\theta) (\phi'(\theta))^2}\, d\theta,$$

where I have removed the factor of r (if you minimize J, you also minimize $L = rJ$ since r is positive).

Write down the Euler-Lagrange equation for this functional. Note that the function $\phi(\theta)$ does not appear (only its derivative $\phi'(\theta)$), so the EL-equation will be of the form $\frac{d}{d\theta}(\text{something}) = 0$. This means $(\text{something}) = \text{constant}$.

More precisely, show that the EL equation implies

$$\frac{\phi' \sin^2(\theta)}{\sqrt{1+\sin^2(\theta)\phi'^2}} = C,$$

where C is some constant.

(Part c) Solve the equation from part (b) for ϕ' to show

$$\phi' = \frac{C}{\sin(\theta)\sqrt{\sin^2(\theta) - C^2}}.$$

Integrate both sides of this equation with respect to θ. On the left side, make the substitution $u = \phi$, $du = \phi'\, d\theta$, and evaluate. The bounds of integration change; assume that $\phi(\theta_1) = \phi_0$ and $\phi(\theta_2) = \phi$.

Show that our equation has now become

$$\phi - \phi_0 = \int_{\theta_1}^{\theta_2} \frac{C}{\sin(\theta)\sqrt{\sin^2(\theta) - C^2}}\, d\theta.$$

(**Part d**) The hard part is doing the integral on the right side. Make the substitution $u = \cot(\theta)$, $du = -\csc^2(\theta)\, d\theta$. Then the integral becomes

$$\int \frac{-C\, du}{\sqrt{1 - C^2 \csc^2(\theta)}}\, d\theta$$
$$= \int \frac{-C\, du}{\sqrt{1 - C^2(1+u^2)}},$$

where I have used a trigonometric identity (which one?), and I am suppressing the bounds of integration. To save writing, define a new variable

$$a = \frac{\sqrt{1-C^2}}{C}.$$

Show that the integral becomes

$$\int \frac{-du}{\sqrt{a^2 - u^2}},$$

which you know how to do, so our overall equation has become

$$\phi - \phi_0 = \arccos\left(\frac{u}{a}\right).$$

(**Part e**) Plug the definition of u in terms of θ back in. Solve. Show that the path which minimizes the length on a sphere is given by the equation

$$\cot(\theta) = a \cos(\phi - \phi_0).$$

This equation describes a curve called a great circle. It is the approximate path that airplanes take when flying between two cities on the earth.

daily_challenge

Updated 2 days ago by Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Click to start off the wiki answer

followup discussions *for lingering questions and comments*