question 2 views

# Daily Challenge 16.5

(**Due: Thursday November 1 at 12:00 noon Eastern**)

Since the definition of integral involves suprema and infima, it will be nice to prove some simple properties of these objects. I meant to ask these questions in chapters 1 and 2 but foolishly removed them.

## (1) Sup and inf review.

First, let me remind you of the definitions.

**Definition**. Let $A \subset \mathbb{R}$. If a number $x$ has the property that $x > a$ for every $a \in A$, then $x$ is called an *upper bound* for $A$.

If a set has one upper bound, it has infinitely many. For instance, if $A = [1, 2]$ then clearly $x_1 = 2$ is an upper bound but so is $x_2 = 3$, $x_3 = 4, \cdots$.

**Definition**. Let $A \subset \mathbb{R}$. A number $x$ is the *least upper bound* or *supremum* of $A$ if it satisfies the following two properties.

1. $x$ is an upper bound of $A$; and
2. if $y$ is an upper bound of $A$, then $x \leq y$.

Recall we proved in chapter 1 that there can only be one least upper bound of $A$, justifying my use of the phrase "the least upper bound" (rather than "a least upper bound").

Analogous definitions hold for *lower bound* and *greatest lower bound* (i.e. *infimum*).

## (2) Problem: connecting sup/inf with $\epsilon$'s.

Let $A \subset \mathbb{R}$ and suppose $\alpha = \sup(A)$, $\beta = \inf(A)$ (in particular, we assume that both the supremum and infimum exist). Prove the following:

1. For any $\epsilon > 0$, there exists $x \in A$ such that $\alpha - \epsilon < x$.
2. For any $\epsilon > 0$, there exists $y \in A$ such that $\beta + \epsilon > y$.

[Hint: don't work hard; each of these should be a two-line proof by contradiction. Suppose the contrary and then show that it contradicts the definition of supremum/infimum.]

daily_challenge

Updated 5 months ago by Christian Ferko

**the students' answer,** *where students collectively construct a single answer*

Logan pachasudf:
1: Suppose by way of contradiction that "there exists at least one $\epsilon > 0$ for which no $x \in A$ satisfies $\alpha - \epsilon < x$". However, this statement is most certainly false; if such an $\epsilon$ existed, then we see that $\alpha$ was not a true least upper bound, since $\alpha - \epsilon < x$, of course this contradicts us being told that $\alpha = \sup(A)$, thus it must be true that for any $\epsilon > 0$, there exists $x \in A$ such that $\alpha - \epsilon < x$.

2: Once again, suppose by way of contradiction that there exists at least one $\epsilon > 0$ for which no $y \in A$ satisfies $\beta + \epsilon > y$. This is also false for a similar reason as prior; if such an $\epsilon$ existed, then we would see that $\beta$ was not a valid least upper bound as given. Then it must be true that for any $\epsilon > 0$, there exists $x \in A$ such that $\beta + \epsilon > x$.

Updated 5 months ago by Logan Pachulski

**the instructors' answer,** *where instructors collectively construct a single answer*

(1) Suppose the contrary: say there were some $\epsilon > 0$ so that $\alpha - \epsilon \geq x$ for all $x \in A$. By definition, this means that $\alpha - \epsilon$ is an upper bound for $A$. But we also have $\alpha - \epsilon < \alpha$, which means that $\alpha - \epsilon$ is smaller than the least upper bound $\alpha = \sup(A)$, a contradiction.

(2) Likewise, suppose by way of contradiction that there were an $\epsilon > 0$ so that $\beta + \epsilon \leq y$ for all $y \in A$. Then $\beta + \epsilon$ is a lower bound for $A$, but it is larger than the greatest lower bound $\beta = \inf(A)$, a contradiction. $\square$

Updated 5 months ago by Christian Ferko

**followup discussions** *for lingering questions and comments*

Resolved Unresolved

**Christian Ferko** 5 months ago
Your first step is wrong; you negated the statement incorrectly.

We're trying to prove that, for every $\epsilon > 0$, there exists some $x \in A$ such that $\alpha - \epsilon < x$. The negation of this statement is that there exists at least one $\epsilon > 0$ for which no $x \in A$ satisfies $\alpha - \epsilon < x$.

Also the comment about "open or closed" doesn't make sense. $A$ is an arbitrary set of real numbers.

**Logan Pachulski** 5 months ago Let $\epsilon = 2 * |(\sup(A) - \inf(A))|$?

**Christian Ferko** 5 months ago What? You need to prove this **for all epsilon**. You can't just pick your favorite epsilon.

**Logan Pachulski** 5 months ago Are we trying to prove the negation true or false?

**Logan Pachulski** 5 months ago :thinking:

**Christian Ferko** 5 months ago How does proof by contradiction work? You want to prove statement $P$ is true. To do so, you assume that the negation $\text{not}(P)$ is true, then you show that something false results.

In this case, we want to prove that for all $\epsilon > 0$, there exists $x \in A$ so $\alpha - \epsilon < x$.

We assume the contrary: suppose there exists at least one $\epsilon > 0$ so that it is *not true* that there exists $x \in A$ so $\alpha - \epsilon < x$. We don't get to pick what that $\epsilon$ is; we're just assuming it exists, by way of contradiction.

Show that this assumption contradicts the definition of least upper bound.

**Logan Pachulski** 5 months ago Okay.

**Logan Pachulski** 5 months ago It seems I could have been slightly more precise, but I'm happy with my work :thinking:

**Christian Ferko** 5 months ago Yeah, I'd say your solution is like a 4.5/6 on our old scale. Looks good to me.

Your original argument had the right idea but wasn't really phrased correctly (you said something like $\alpha - \epsilon \geq x$ but didn't specify that it was true for all $x$, and then said something like $x = \sup(A)$ which I didn't quite understand but I think you meant something analogous to the response above).

Resolved Unresolved

**Christian Ferko** 5 months ago
pachasudf :thinking: