28.4

(a): Claim: Suppose that $|a_k| \leq b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converges. Then $\sum_{k=1}^{\infty} a_k$ converges as well

Proof: We are told that $\sum_{k=1}^{\infty} b_k$ converges and that $|a_k| \leq b_k$ for $k \in \mathbb{N}$; let $\epsilon > 0$ be given.

Since $\sum_{k=1}^{\infty} b_k$ converges, then the series satisfies ~~the~~ Cauchy's Criterion; thus there exists ~~$n, m \in \mathbb{N}$~~ $N \in \mathbb{N}$ such that for $n > m \geq N$,

$$(1) \qquad \left|\sum_{k=1}^{\infty} b_k\right| < \epsilon$$

We let $n > m \geq N$ without loss of generality; then

$$\left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| \quad \text{by the triangle inequality with}$$

$$|x_1 + \cdots + x_k| \leq |x_1| + \cdots + |x_k|$$

$$\leq \sum_{k=m+1}^{n} b_k \quad \text{since } |a_k| \leq b_k$$

$$< \epsilon \quad \text{by (1).}$$

Claim: If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge to A and B respectively, then $\sum_{k=1}^{\infty} a_k + b_k$ converges to $A+B$.

Proof: We are told that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge to A and B respectively; By the definition of series convergence, ~~the~~ let

$$s_n = \sum_{k=1}^{n} a_k, \quad t_n = \sum_{k=1}^{n} b_k$$

and see that

$$\lim(s_n) = A, \quad \lim(t_n) = B$$

Then by the sum property of limits of sequences,

$$\lim(s_n + t_n) = \lim s_n + \lim t_n = A + B$$

Then since
$$s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k + b_k$$

we conclude that
$$\sum_{k=1}^{\infty} (a_k + b_k) = A + B.$$

(b): Statement: Prove that the harmonic ~~sequence~~ series
$$\sum_{k=1}^{\infty} \frac{1}{k}$$
diverges.

Exploration/Proof: Begin by finding the negation of Cauchy's Criterion:
A series
$$\sum_{k=1}^{\infty} a_k$$
~~is di~~ is divergent if and only if ~~for~~ there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$,
$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| > \epsilon$$
for all $n > N$ and $p \geq 1$. Now to begin our proof: Let $\epsilon = \frac{1}{2}$ and $N \in \mathbb{N}$ be given. Then see that for $N+1 < 2N$,
$$\sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2n} = \frac{1}{2} = \epsilon,$$
since $\frac{1}{2n}$ is the maximum value on the list and N such additions of it ~~sums~~ may be present.

(c): We are told to let N_0 be the integer at which $b_k = a_k$ for all $k \geq N$; Then recall the ~~two~~ sequence of ~~have associated~~ Cauchy Criteria $-n$: $\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$ for $n \geq N_a$, ~~$\left|\sum_{k=m+1}^{n} b_k\right| < \epsilon$ for $n > N_2$~~ applied to b_k.
We simply let an overall $N = \max(N_a, N_b)$; then N has these sequences converge ~~and to the~~ and to the same value.