27.5

with limit

(1): We let $\{a_n\}$ be ~~the limit~~ some function(?) L, when n is large. This is defined in the lesson as

$$\text{given } \epsilon > 0,\ a_n = L + \epsilon \text{ for } n \gg 1$$

• The limit converges if the limit L exists; otherwise it is divergent. A limit that exists can be written as

$$\lim_{n \to \infty} a_n = L \quad \text{where } a_n \Rightarrow L$$

(2): In more traditional notation,

$$a_n \underset{\epsilon}{\approx} L$$

is the same as

$$\text{given } \epsilon > 0, \text{ the limit exists if } |a_n - L| < \epsilon$$

by seeing that we want them to have very little difference. The same idea could be satisfied with

$$(a_n - L)^2 < \epsilon^2$$

(3): $\lim a_n = L$ if given $\epsilon > 0$, $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$.

Building this up in three successive stages:

~~Given~~

$a_n \underset{\epsilon}{\approx} L$ (a_n approximates to L within ϵ)

for $n \gg 1$, (the approximation holds for all a_n far enough into the sequence

given $\epsilon > 0$, (the approximation can be made as close as desired, provided we go far enough into the sequence - in general a smaller ϵ necessitates going further into the sequence)

(4): According to the definition seen in (3), we must show

(2) given $\epsilon > 0$, ~~$\frac{n-1}{n+1}$~~ $\frac{n-1}{n+1} \underset{\epsilon}{\approx} 1$ for $n \gg 1$

We begin by examining the size of the difference and simplifying it:

$$\left| \frac{n-1}{n+1} - 1 \right| = \left| \frac{-2}{n+1} \right| = \frac{2}{n+1}$$

We want to show this difference is small if $n \gg 1$. Use the inequality laws:

$\frac{2}{n+1} < \epsilon$ if $n+1 > \frac{2}{\epsilon}$, ie if $n > N$, where $N = \frac{2}{\epsilon} - 1$

this proves (2), in view of the definition of (for $n \gg 1$).

This argument can be written on one line (it's ungrammatical, but easier to write, print, and read this way)!

Solution: Given $\epsilon > 0$, $\left| \frac{n-1}{n+1} - 1 \right| = \frac{2}{n+1} < \epsilon$, if $n > \frac{2}{\epsilon} - 1$

(5): • The heart of a limit proof is in the approximation statement; ie in getting a small upperestimate for $|a_n - L|$.

• In giving the proofs you must exhibit ~~the phrase~~ a value for the N that lurks in the phrase "for $n \gg 1$."

• Think of a limit demon ~~who~~ whose only purpose in life is to make it hard for you to show that limits exist; it always picks unpleasantly small values for ϵ. Your task is, given any ϵ the limit demon hands you, to find a corresponding N (depending on ϵ) such that $a_n \underset{\epsilon}{\approx} L$ for $n > N$