Daily Challenge 4.1

(**Due: Tuesday 5/15 at 12:00 noon Eastern**)

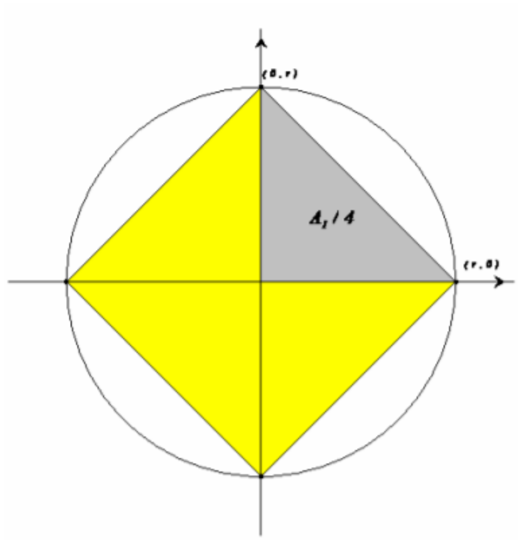
We begin week four!

Review

In our last meeting, I used the half-angle formula for sine to obtain a recursive formula for the areas of inscribed polygons in the method of exhaustion; by "recursive", I mean that this equation gives the area A_{n+1} at the $(n+1)$-th step in terms of the previous area A_n at the n-th step.

In this exercise, you'll derive that recursive formula and understand the strange "nested square roots" we saw in the first meeting.

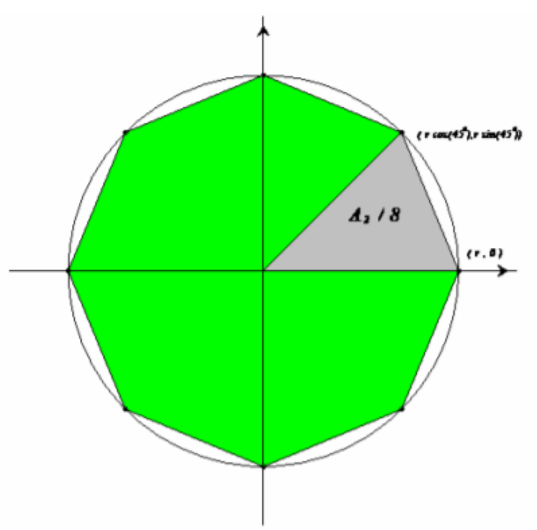
To begin, let's look at the first few inscribed polygons to help us guess the pattern. The first case, $n = 1$, is the square, which we break into 4 triangles.



Every triangle has a base $b = 1$ and a central angle of $\frac{\pi}{2}$, so its height is $h = \sin\left(\frac{\pi}{2}\right)$. There are four of them, so the total area is

$$A_1 = \underbrace{4}_{\#\text{ triangles}} \times \left(\frac{1}{2} \times 1 \times \underbrace{\sin\left(\frac{\pi}{2}\right)}_{\text{height of triangle}}\right) = 2.$$

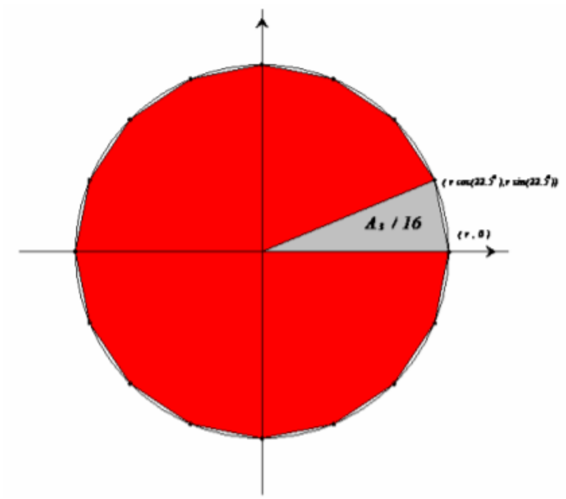
Next at $n = 2$, we double the number of sides, which gives an octagon:



Now there are 8 triangles, each with a base $b = 1$ and height $h = \sin\left(\frac{\pi}{4}\right)$, so

$$A_2 = \underbrace{8}_{\#\text{ triangles}} \times \left(\frac{1}{2} \times 1 \times \underbrace{\sin\left(\frac{\pi}{4}\right)}_{\text{height of triangle}} \right) = 2\sqrt{2}.$$

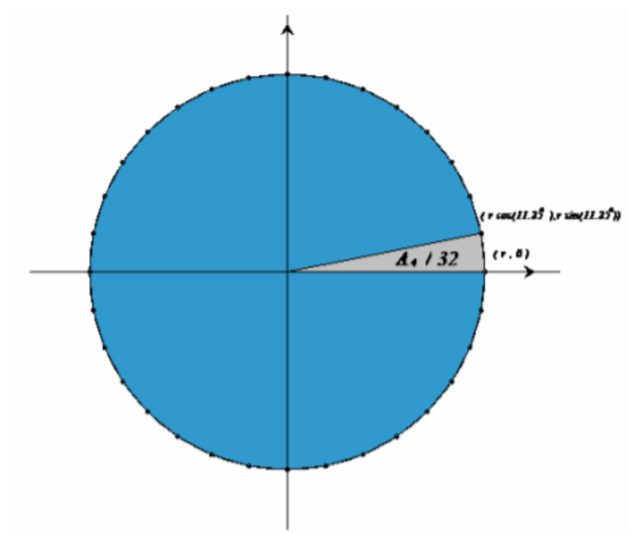
Next we go to $n = 3$, or the 16-gon.



There are 16 triangles with height $\sin\left(\frac{\pi}{8}\right)$, so

$$A_3 = 16 \times \left(\frac{1}{2} \times 1 \times \sin\left(\frac{\pi}{8}\right) \right) = 4\sqrt{2 - \sqrt{2}}.$$

Once more: at $n = 4$, we go to the 32-gon:



The area is

$$A_4 = 32 \times \left(\frac{1}{2} \times 1 \times \sin\left(\frac{\pi}{16}\right) \right) = 8\sqrt{2 - \sqrt{2 + \sqrt{2}}}.$$

Perhaps you see the pattern: each time, we double the number of triangles and cut the angle appearing in the sine in half.

Problem

(a) Guess the pattern above to write down a formula for the n-th area, A_n. Your result should be a number (which depends on n) times the sine of some angle (which also depends on n), *not* the nested square root form.

Next, we would like to find a formula for the next area, A_{n+1}.

(b) Replace n by $n+1$ in your result from (a) to write down the formula for the area A_{n+1}. Then use the half-angle identity

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}},$$

to rewrite the sine appearing in A_{n+1} in terms of the angle appearing in A_n.

(c) Your result in (b) will involve a cosine of some angle, which we want to eliminate in favor of a sine. We can solve the Pythagorean identity, $\sin^2(\theta) + \cos^2(\theta) = 1$, for the cosine to find

$$\cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}.$$

Substitute this expression for the cosine into your result to part (b).

(d) Rewrite your result from part (c) in terms of the area A_n.

[Hint: get rid of the sine factor by solving your equation from part (a) for the sine in terms of A_n, then plug in.]

Compare your answer to the equation I presented at our last meeting:

$$A_{N+1} = 2^{N+1}\sqrt{\frac{1}{2}\left(1-\sqrt{1-\left(\frac{A_N}{2^N}\right)^2}\right)}.$$

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Solutions (Logan). Your reasoning and results go here.

(a)

(b)

(c)

(d)

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the instructors' answer, *where instructors collectively construct a single answer*

Solution (Christian).

(a) The pattern is straightforward to guess:

$$\begin{aligned} A_n &= 2^{n+1}\left(\frac{1}{2}\times 1\times\sin\left(\frac{\pi}{2^n}\right)\right) \\ &= 2^n\sin\left(\frac{\pi}{2^n}\right). \end{aligned}$$

(b) Replacing n by $n+1$, we find

$$A_{n+1} = 2^{n+1}\sin\left(\frac{\pi}{2^{n+1}}\right).$$

We can rewrite this using the half-angle identity, since

$$\begin{aligned} \sin\left(\frac{\pi}{2^{n+1}}\right) &= \sin\left(\frac{1}{2}\times\frac{\pi}{2^n}\right) \\ &= \sqrt{\frac{1}{2}\left(1-\cos\left(\frac{\pi}{2^n}\right)\right)}. \end{aligned}$$

Therefore,

$$A_{n+1} = 2^{n+1}\sqrt{\frac{1}{2}\left(1-\cos\left(\frac{\pi}{2^n}\right)\right)}.$$

(c) Using the hint, we see that

$$\cos\left(\frac{\pi}{2^n}\right) = \sqrt{1 - \sin^2\left(\frac{\pi}{2^n}\right)}.$$

Then our equation for A_{n+1} becomes

$$A_{n+1} = 2^{n+1}\sqrt{\frac{1}{2}\left(1 - \sqrt{1 - \sin^2\left(\frac{\pi}{2^n}\right)}\right)}.$$

(d) We want to get rid of the ugly $\sin^2\left(\frac{\pi}{2^n}\right)$ appearing in our last equation. Consider our result from part (a), which was

$$A_n = 2^n \sin\left(\frac{\pi}{2^n}\right).$$

Solving for $\sin\left(\frac{\pi}{2^n}\right)$, this is

$$\sin\left(\frac{\pi}{2^n}\right) = \frac{A_n}{2^n}.$$

We plug this into our equation from part (c) to find

$$A_{n+1} = 2^{n+1}\sqrt{\frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{A_n}{2^n}\right)^2}\right)}.$$

This is the desired recursive formula!

Now we understood the nested square roots appearing in our first session:

The square roots come from the half-angle formula for sine, and from solving the Pythagorean identity for cosine.

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*