question 2 views

Daily Challenge 20.4

Try these problems from the 2007 Integration Bee.

(a) $\int \left(2\log(x) + (\log x)^2\right)\, dx$. Hint: try a "rearranged" version of integration by parts. Does the integrand look like the derivative of a product?

(b) $\int \frac{2x^3-1}{x^4+x}\, dx$. Hint: factor the denominator, but not all the way.

(c) $\int \sin(\sqrt[3]{x})\, dx$. Hint: try $u = \sqrt[3]{x}$.

(d) $\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}\, dx$. Try $x = u^6$.

daily_challenge

Updated 4 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

(a): The easiest way to an answer is to faff about and slowly refine it, so we shall start with something that one could gloss over and not question; let's see what goes wrong when we take the derivative of $\log^2(x)$; We see by the chain rule that
$\frac{d}{dx}\log^2(x) = \frac{2}{x}\log(x)$
Alright, how about we try adding an x to the front to cancel out the one that appear here? We see by the product & chain rules that
$\frac{d}{dx}x\log^2(x) = 2\log(x) + log^2(x)$
Oh. That got it perfectly. :lovely:

(b): We factor the denominator:

Ignore the following.

$$\int \frac{2x^3-1}{x^4+x}\, dx = \int \frac{2x^3-1}{x(x^3+1)}\, dx$$

Maybe we can add zero in some peculiar way? Let's add $\cdots + 2 - 2$and see where that gets us. I've tried the following:

$$\int \frac{2x^3-1}{x(x^3+1)} = \int \frac{2x^3+2}{x(x^3+1)}\, dx - \int \frac{-3}{x(x^3+1)}$$
$$= 2\log(x) + 3\int \frac{1}{x^4+x^3}$$

Here we go again with these fooookin polynomials in exponents.
Ignore everything up to here.

We have that

$$\int \frac{2x^3-1}{x^4+x}\, dx = \int \frac{2x^3-1}{x(x^3+1)}\, dx$$

We then add zero in the form of $\pm(x^3+1)$, thus

$$\int \frac{2x^3-1}{x(x^3+1)}\, dx = \int \frac{2x^3-1}{x(x^3+1)}\, dx$$
$$= \int \frac{3x^3}{x(x^3+1)} + \int \frac{-1-x^3}{x(1+x^3)}$$
$$= \int \frac{3x^2}{(x^3+1)} - \int \frac{1}{x}$$

We then shall apply u-sub to the left integral, where $u = x^3$ and $du = 3x^2 dx$, meanwhile the right term is simply equal to $\log(x)$.
$\int \frac{du}{(u+1)} - \log(x) = \log(x^3+1) - \log(x) + C$

(c): We do as the hint suggests, and let $u = \sqrt[3]{x}$ and $du = \frac{1}{3}\cdot(x)^{\frac{-2}{3}}\, dx = \frac{1}{3}\cdot(u)^{-2} dx$, which implies $dx = 3u^2\, du$. Thorugh consecutive applications of the IBP, where we are moving the derivative of the trig function onto the monomial, we see that

$$\int 3u^2\sin(u) = -3u^2\cos(u) + 6u\sin(u) + 6\cos(u)$$

and we substitute back in the $u = \sqrt[3]{x}$ to see

$$\int \sin(\sqrt[3]{x})\, dx = -3(\sqrt[3]{x})^2\cos(\sqrt[3]{x}) + 6(\sqrt[3]{x})\sin(\sqrt[3]{x}) + 6\cos(\sqrt[3]{x})$$

(d): We let $x = u^6$ as suggested, and thus $dx = 6u^5 du$. We insert these values:

$$\begin{aligned}\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}} &= \int \frac{u^{-3}}{1+u^2} 6u^5\, du \\ &= 6 \int \frac{u^2+1-1}{1+u^2} \\ &= 6\left(\int \frac{u^2+1}{1+u^2} - \int \frac{1}{1+u^2}\right) \\ &= 6\,(u - \arctan(u))\end{aligned}$$

Updated 2 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) If you see a sum of two terms in the integrand of an integration bee question, it's probably the derivative of a product.

One approach is to note that $\frac{d}{dx}\left((\log x)^2\right) = \frac{2}{x}\log(x)$, which is almost the first term, and that we can fix the errant $\frac{1}{x}$ through multiplying by x. Then our refined guess is $x(\log x)^2$, which indeed works by the product rule:

$$\frac{d}{dx}\left(x(\log x)^2\right) = (\log x)^2 + 2\log x.$$

A more general way to handle these "derivative of a product" type questions is to use a rearranged chain rule, namely

$$\int (u\, dv + v\, du) = uv,$$

so we're looking for two functions u and v such that $u\, dv = 2\log x\, dx$ and $v\, du = (\log x)^2$. By the above reasoning, we see that $u = x$ and $v = (\log x)^2$ works.

Either way, we conclude that

$$\int \left(2\log x + (\log x)^2\right)\, dx = x(\log x)^2 + C.$$

(b) Following the hint, we factor the denominator as

$$\frac{2x^3-1}{x^4+x} = \frac{2x^3-1}{x(x^3+1)},$$

I want to add zero in the numerator in a way which splits the fraction. Let's try adding and subtracting x^3+1, the thing in the denominator. (You could also use partial fractions if you don't see this trick.)

$$\begin{aligned}\frac{2x^3-1}{x(x^3+1)} &= \frac{2x^3+x^3-1-x^3}{x(x^3+1)} \\ &= \frac{3x^3}{x(x^3+1)} + \frac{-1-x^3}{x(1+x^3)} \\ &= \frac{3x^2}{x^3+1} - \frac{1}{x}.\end{aligned}$$

The first term looks ripe for a u-substitution, since we see that the expression x^3 and its derivative $3x^2$ both appear. Letting $u = x^3$ and $du = 3x^2\, dx$, we have

$$\begin{aligned}\int \frac{2x^3-1}{x(x^3+1)}\, dx &= \int \left(\frac{3x^2}{x^3+1} - \frac{1}{x}\right)\, dx \\ &= \left(\int \frac{du}{u+1}\right) - \log(x) + C \\ &= \log(u+1) - \log(x) + C \\ &= \log\left(\frac{x^3+1}{x}\right) + C.\end{aligned}$$

In the last step, we replaced $u = x^3$ and used the difference-of-logs rule to simplify.

(c) As suggested, we let $u = x^{1/3}$ so $x = u^3$ and $dx = 3u^2\, du$. Then the integral yields to two applications of integration by parts:

$$\begin{aligned}\int \sin\left(x^{1/3}\right)\, dx &= \int 3u^2\ \sin(u)\, du \\ &= -3u^2\cos(u) + \int 6u\cos(u)\, du \\ &= -3u^2\cos(u) + 6u\sin(u) - \int 6\sin(u)\, du \\ &= -3u^2\cos(u) + 6u\sin(u) + 6\cos(u) + C.\end{aligned}$$

Restoring the dependence on $x = u^3$, we've shown

$$\int \sin\left(x^{1/3}\right)\, dx = (-3x^{2/3} + 6)\cos(x^{1/3}) + 6x^{1/3}\sin(x^{1/3}) + C.$$

(d) Using $x = u^6$, $dx = 6u^5\, du$, the integral becomes

$$\int \frac{x^{-1/2}}{1 + x^{1/3}}\, dx = \int \frac{6u^2}{u^2 + 1}\, du.$$

Split the integrand by adding zero in the form $1 - 1$ in the numerator, which gives

$$\begin{aligned} 6\int \frac{u^2}{u^2+1}\, du &= 6\int \left(\frac{u^2+1}{u^2+1} - \frac{1}{u^2+1}\right)\, du \\ &= 6\int \left(1 - \frac{1}{u^2+1}\right)\, du \\ &= 6\,(u - \arctan(u)) + C. \end{aligned}$$

Substituting back in, our result is

$$\int \frac{x^{-1/2}}{1 + x^{1/3}}\, dx = 6\left(x^{1/6} - \arctan(x^{1/6})\right) + C.$$

Updated 2 months ago by Christian Ferko

followup discussions *for lingering questions and comments*