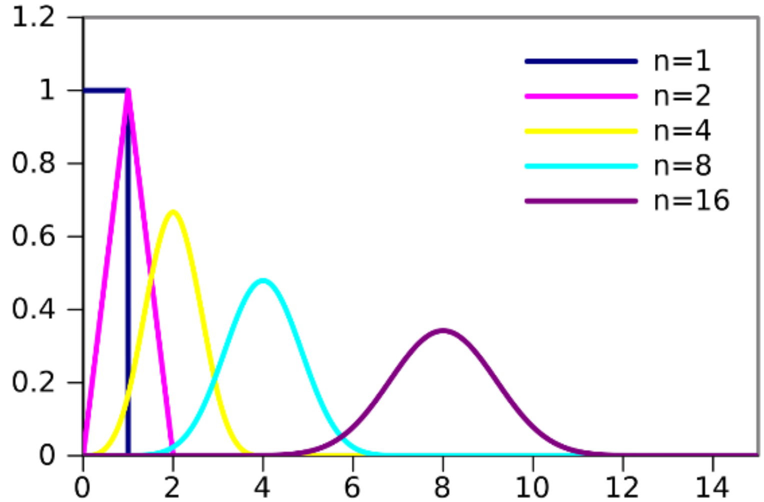
question 2 views

Daily Challenge 24.6

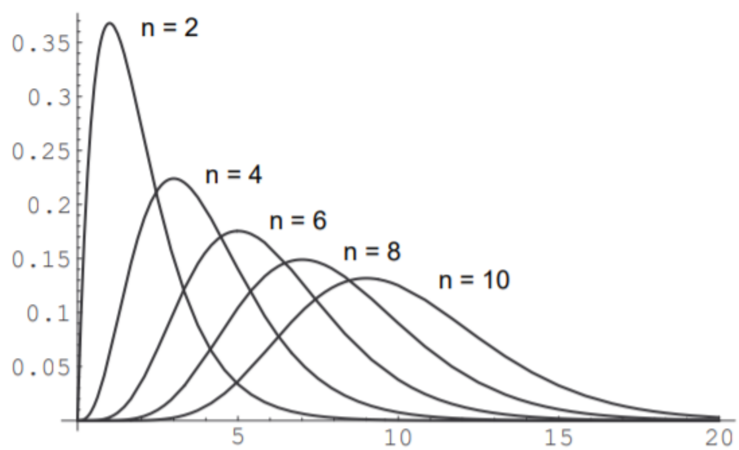
(**Due: Monday 3/18 at 12:00 noon Eastern**)

Last time, we saw that convolving a PDF with itself repeatedly makes it look "more Gaussian."

For instance, here's what happens when you convolve n uniform distributions on $[0, 1]$:



Here's what happens if you convolve n exponentials:



Visually, it indeed seems that repeated convolution "Gaussianizes" something.

So what happens if you convolve a Gaussian with itself? It seems that there is no way to make a Gaussian "more Gaussian", so you might suspect that convolving two Gaussians gives another Gaussian.

(1) Problem: convolving Gaussians.

For simplicity, take a Gaussian with $\mu = 0$ and $\sigma = 1$, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Find the convolution $(f * f)(z)$, which gives the PDF for the sum of two Gaussian-distributed random variables x and y, and show that the convolution is another Gaussian. Use the definition

$$(f * g)(z) = \int_{-\infty}^{\infty} f(x) g(z - x)\, dx.$$

Hint/Solution. Begin with

$$
\begin{aligned}
(f*f)(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right)\exp\left(-\frac{(z-x)^2}{2}\right)dx \\
&= \frac{1}{2\pi}\exp\left(-\frac{z^2}{4}\right)\cdot\int_{-\infty}^{\infty}\exp\left(-\left(x-\frac{z}{2}\right)^2\right)dx
\end{aligned}
$$

The second step above was just foiling out the square in the exponential and doing some algebraic re-arrangement using exponent rules. (But you should be sure to write out the intermediate steps more carefully than I just did.)

Inside the integral, z is considered a constant, so you can massage the integrand to look like the usual $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ story. At the end of the day, you should get

$$
(f*f)(z) = \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right).
$$

This is another Gaussian, as claimed, but note that the spread changed when we took the convolution -- now there is a 4 in the denominator of the exponential, rather than the 2 that appeared in the original function $f(x)$.

daily_challenge

Updated 27 days ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

yeeteth

Updated 26 days ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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