question 2 views

Daily Challenge 15.4

(**Due: Tuesday 9/25 at 12:00 noon eastern**)

Here's another nice optimization problem, though not as tricky as the ladder-in-a-hallway question. It's question 15 on CD 4.

(1) Problem: minimizing a sum of areas.

A wire of length 10 meters is to be cut into two pieces. The first piece will then be folded to form a square, while the second piece will be bent into a circle. What is the minimum total area that can be enclosed?

daily_challenge

Updated 6 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

We see that the amount remaining of wire after removing a length for the square can be represented as $10-l$, which implies we have l length to work with for the perimeter of the square and $10-l$ to work with for the circle. A square has 4 sides of equal length, therefore the area of our square can be represented $A_s=\left(\frac{l}{4}\right)^2$, and the area of a circle in terms of our circumference is $A_c=\pi(\frac{10-l}{2\pi})^2=\pi(\frac{100+l^2-20l}{4\pi^2})=\frac{100+l^2-20l}{4\pi}$ We add these to see that the cumulative area is $\frac{l^2}{16}+\frac{100+l^2-20l}{4\pi}=A=\frac{1}{16}l^2+\frac{1}{4\pi}l^2-\frac{5}{\pi}l+\frac{25}{\pi}=\frac{1+4\pi^{-1}}{16}l^2-\frac{5}{\pi}l+\frac{25}{\pi}$ This is an upwards opening parabola, so we know that the vertex is where this function is at its minimum at $\frac{-b}{2a}$, therefore this function has its minimum at $\frac{40}{\pi+4}$, about 5.6 meters. :wow:

Updated 6 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Suppose that we use a length ℓ for the square and the remaining length $10-\ell$ for a circle. This means that the perimeter of the square is $P=\ell$ and the circumference of the circle is $C=10-\ell=2\pi r$.

In other words, the side length of the square is $\frac{\ell}{4}$ and the radius of the circle is $\frac{10-\ell}{2\pi}$. Thus the total area is

$$A(\ell)=\left(\frac{\ell}{4}\right)^2+\pi\left(\frac{10-\ell}{2\pi}\right)^2=\frac{\ell^2}{16}+\frac{100-20\ell+\ell^2}{4\pi}.$$

Expanding and collecting like terms,

$$A(\ell)=\left(\frac{1}{16}+\frac{1}{4\pi}\right)\ell^2-\frac{5}{\pi}\ell+\frac{25}{\pi}.$$

This is an upward-opening parabola, so the minimum occurs at the vertex $\ell=-\frac{b}{2a}$ where $b=-\frac{5}{\pi}$ and $a=\left(\frac{1}{16}+\frac{1}{4\pi}\right)$. Thus the minimum occurs at

$$\ell_{\min}=\frac{5}{2\pi\left(\frac{1}{16}+\frac{1}{4\pi}\right)}.$$

Numerically, this is about 5.6 meters. So we should use a bit more than half of the wire on the square and the rest on the circle.

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*