29.3

(a): We see by the alternating series test that

$$\sum_{K=1}^{\infty} \frac{(-1)^K}{\sqrt{K}} = \sum_{K=1}^{\infty} (-1)^K a_K$$

has that a_K decreases monotonically, and is strictly positive, and $\lim a_K = 0$; thus

$$\sum_{K=1}^{\infty} \frac{(-1)^K}{\sqrt{K}}$$ converges.

This sequence is absolutely convergent if

$$\sum_{K=1}^{\infty} \left| \frac{(-1)^k}{\sqrt{K}} \right|$$ converges.

$$\dots = \sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$$ by applying abs. value to top and bottom.

Then, by the "All power law sums at once" proof in Session 63, we know that

$$\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$$ diverges due to $\frac{1}{2} = p < 1$; thus, it is conditionally convergent

(b): Once again by Leibniz, see that (since $(-1)^{K^2} = (-1)^K$ ~~due to~~)

$$\sum_{K=1}^{\infty} \frac{(-1)^{K^2} K^2}{K^3+1} = \sum_{K=1} (-1)^K \frac{K^2}{K^3+1} = \sum_{K=1} (-1)^K a_K$$

converges since a_K decreases monotonically; it is absolutely convergent if

$$\sum_{K=1}^{\infty} \left| (-1)^K a_K \right| = \sum_{K=1}^{\infty} a_K$$ converges;

See that

$$\frac{K^2}{K^3+1} < \frac{K^2}{K^3} = \frac{1}{K}$$; hmm, not very useful, let's apply the limit comparison test: Evaluate

$$\lim_{K\to\infty} \left(\frac{1/K}{K^2/K^3+1} \right) = \lim \left(\frac{K^3+1}{K^3} \right) = \lim \left(1 + \frac{1}{K^3} \right) = 1$$; thus since

$\frac{1}{K}$ diverges, $\frac{K^2}{K^3+1}$ diverges.

(c): Let's break this into 3 cases:

- If $r<1$, then a_n converges; ~~begin by then~~ see that
  $\lim(\sqrt[n]{a_n})<1$

  To get some more useful out of this limit, define some variable $s\in\mathbb{R}$ such that $r<s<1$; thus,
  $\lim(\sqrt[n]{a_n})\leq r<s$

  ~~or~~ it is then implied that since the limit is less than s, then
  $\sqrt[n]{a_n}<s \Rightarrow a_n<s^n$, albeit only true for $n>$ some N.

  But by the ~~convergence test~~ comparison test, we notice that $\sum_{k=1}^{\infty} s^k$ converges since $s<1$, thus a_n converges.

- Now we must consider $r>1$; let $r>s>1$ and go through similar steps to notice that
  $a_n > s^{n}$ for $n>N'$

  Then, since $s>1$ and as such $\sum s^n$ diverges, then for $r>1$, a_n diverges by the contrapositive of the comparison test.

- Now for $r=1$; the statement that
  $\lim(\sqrt[n]{a_n})=1$

  We have shown in the past that
  $\lim(\sqrt[n]{n})=1$

  ~~I believe this implies $a_n=n$, but we know by the that~~
  ~~$\sum\frac{1}{n^p}$ diverges for $p\geq1$ and converges for $p<1$, so we apply the~~

~~ratio test to see that (with $b_n = \frac{1}{n^p}$)~~

~~lim (~~

Applying the root test to the series $\sum \frac{1}{n^p}$, we have that

$$\lim\left(\sqrt[n]{\frac{1}{n^p}}\right) = \lim\left(\frac{1}{\sqrt[n]{n}}\right)^p = \left(\frac{1}{\lim \sqrt[n]{n}}\right)^p = \frac{1}{1}^p = 1$$

Thus for p-series (at least) the root test yields inconclusive results, since we know that for $p > 1$ this ~~div~~ converges and for $p \leq 1$ it converges.

Nice.