question 2 views

Daily Challenge 7.6

(**Due: Sunday 6/10 at 12:00 noon Eastern**)

Here is another problem which you'll first write up on Piazza and then transfer to the consolidation document. The differences with ordinary daily challenges include:

1. I will not write a solution (you might choose to present the solution at the British-style tutorial session), but you may ask for as many hints as you like;
2. your solution must be written in $\LaTeX$ rather than pen-and-paper; and
3. consolidation problems are ineligible for weekly skip, 30-minute rule, or submitting partial progress. We must keep working until your solution is totally correct.

Problem

"The shortest path between two truths in the real domain passes through the complex domain." - Jacques Hadamard

(a) Prove the angle-addition formulas $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$and $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$.

[Scaffold: To do this, you should begin by writing the expression $e^{i(\alpha+\beta)}$ in two ways, then set the real and imaginary parts of the two resulting expressions equal. You may use Euler's formula.]

[Scaffold 2: Start by writing $e^{i(\alpha+\beta)}=e^{i\alpha}e^{i\beta}$, which is true due to the exponential-of-a-sum rule. Then apply Euler's formula $e^{i\theta}=\cos(\theta)+i\sin(\theta)$ three times, once with $\theta=\alpha+\beta$ to replace the $e^{i(\alpha+\beta)}$ on the left side, once with $\theta=\alpha$ to replace $e^{i\alpha}$, and once with $\theta=\beta$ to replace $e^{i\beta}$. Proceed.]

(b) Using your result from (a), derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

[I've actually given you the solution for this here.]

(c) Using your result from (b), find the value of $\tan\left(\frac{5\pi}{12}\right)$.

[Hint: cook up two angles α and β which add up to $\frac{5\pi}{12}$ and whose tangents you know, then apply the formula in (b).]

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
A:
Exploration: I begin my exploration by referring to Euler's formula as the scaffold suggests. For later reference I shall place it here:
$e^{i\theta}=\cos(\theta)+i\sin(\theta)$.
I also have a faint memory of applying the Pythagorean theorem in proving this, and as such I'll put it here for later reference as well:
$sin^2(\theta)+cos^2(\theta)=1$.
Since I have seen this before, I need not convince myself of the truth of this equation. I can now begin my proof.

Proof:
We begin by defining $\theta=(\alpha+\beta)$. I believe that I can now substitute this value for θ in Euler's formula, and receive the equation
$e^{i(\alpha+\beta)}=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$
However, we can improve this by applying the product of exponents rule, in turn providing me with $e^{i(\alpha)}\times e^{i(\beta)}$ which can now evolve into a much more useful equation by now applying Euler's formula to each of the expressions, and get
$e^{i(\alpha)}\times e^{i(\beta)}=(\cos(\alpha)+i\sin(\alpha))\times(\cos(\beta)+i\sin(\beta))$
I can now apply foil and get
$e^{i(\alpha)}\times e^{i(\beta)}=(\cos(\alpha)\cos(\beta))+(cos(\alpha)i\sin(\beta))+(\cos(\beta)i\sin(\alpha))-(\sin(\alpha)\sin(\beta))$
Our next step is to group what we received from foiling.
$\Big(\cos(\alpha)\cos(\beta))-(\sin(\alpha)\sin(\beta)\Big)+i\Big((\cos(\alpha)\sin(\beta))+(\cos(\beta)\sin(\alpha))\Big)$
I now take Christian's advice and set the equation I originally received and set it equal to the one I have just found.
$\cos(\alpha+\beta)+i\sin(\alpha+\beta)=\Big(\cos(\alpha)\cos(\beta))-(\sin(\alpha)\sin(\beta)\Big)+i\Big((\cos(\alpha)\sin(\beta))+(\cos(\beta)\sin(\alpha))\Big)$
From this, we can now conclude the following:
$\cos(\alpha+\beta)=\Big(\cos(\alpha)\cos(\beta))-(\sin(\alpha)\sin(\beta)\Big)$
$i\sin(\alpha+\beta)=i\Big((\cos(\alpha)\sin(\beta))+(\cos(\beta)\sin(\alpha))\Big)$
which then simplifies to
$\sin(\alpha+\beta)=\Big((\cos(\alpha)\sin(\beta))+(\cos(\beta)\sin(\alpha))\Big)$

B: (Ah damnit, it has taken me 2.5 hours to do A, LOL)
Since $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$, I believe that I can simply divide the angle addition formula of sine by the angle addition formula of cosine, therefore
$\tan(\alpha+\beta)=\frac{\Big(\cos(\alpha)\sin(\beta)+\cos(\beta)\sin(\alpha)\Big)}{\Big(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\Big)}$
I rearrange this to make my intention more clear:

$$\tan(\alpha+\beta)=\frac{\left(\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\right)}{\left(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\right)}$$

Once again, I take hint from Chris and apply divide every one of these value by $\cos(\alpha)\cos(\beta)$ to get the stupendously large equation:

$$\tan(\alpha+\beta)=\frac{\left(\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\right)}{\left(\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}\right)}$$

This "huge af" equation can now simplify to $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-(\tan(\alpha)\tan(\beta))}$.

C: $tan(\frac{5\pi}{12})=\tan(\frac{\pi}{4}+\frac{\pi}{6})$, which thanks to the formula I just derived equals:

I can now define that $\tan(\alpha)=1$ and $\tan(\beta)=\frac{1}{\sqrt{(3)}}$

Therefore $\tan(\alpha+\beta)=\frac{1+\frac{1}{\sqrt{(3)}}}{1-\frac{1}{\sqrt{(3)}}}$

I don't know if this can be simplified further but it'd probably be an abomination.

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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followup discussions *for lingering questions and comments*