question 4 views

Daily Challenge 2.4

(**Due: Friday 5/4 at 12:00 noon Eastern**)

Yesterday, we saw how to prove that one set is a subset of another, $A \subseteq B$, by considering any element $a \in A$ and proving that $a \in B$. Today we'll consider the longer issue of proving set equality.

Review

Two sets A and B are equal if and only if they contain exactly the same elements. More precisely, we say that $A = B$ if every element of A is an element of B and every element of B is an element of A.

Sometimes it is possible to check equality of small, finite sets by looking at the elements explicitly. For instance, if $A = \{x \in \mathbb{Z} \mid 3 \leq x < 9 \text{ and } x \text{ is odd }\}$, while $B = \{y \in \mathbb{Z} \mid 3 \leq x \leq 10 \text{ and } x \text{ is prime }\}$, we can show that $A = B$ by simply noting that $A = \{3, 5, 7\}$ and $B = \{3, 5, 7\}$.

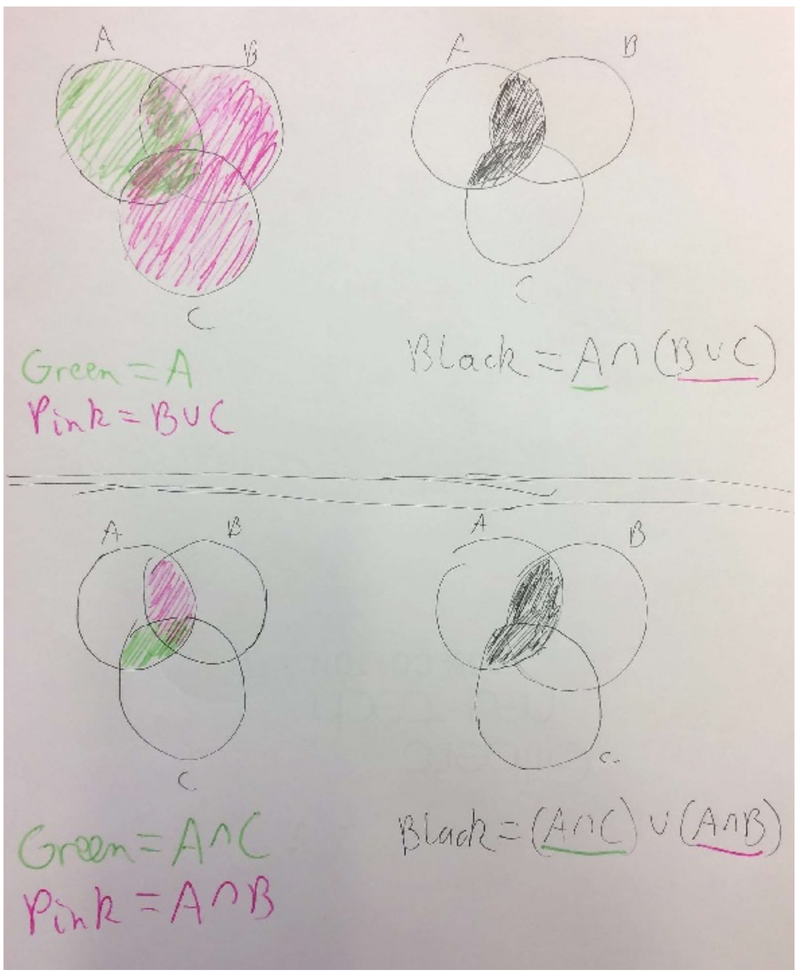
However, in almost all interesting cases, we cannot list the elements explicitly, so we need a different strategy for proving set equality. We can restate the above definition as follows:

$$(A = B) \iff (A \subseteq B \text{ and } B \subseteq A)\,.$$

That is, to prove that two sets $A = B$ are equal, we must prove two separate claims: first assume $a \in A$ and show that $a \in B$ (which proves $A \subseteq B$), and second assume $b \in B$ and prove $b \in A$ (so $B \subseteq A$).

Theorem. If A, B, C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Exploration. Do I believe that this is true? To convince myself, I look at the appropriate Venn diagrams:



The intersection of the two sets at the top right appears to be equivalent to the union of the two sets on the bottom right. Furthermore, the proposed equality works in some simple cases: for instance, if $A = B = C$, the left side is A and the right side is A, which checks out. I'm starting to believe that this could hold in general.

How do I prove it? I know the proof will have two parts: first showing that the set on the left side is a subset of the set on the right side, and then vice-versa. Each of these parts will boil down to using the properties of logical operations like "and" and "or".

It may help me to write down the definitions of union and intersection in terms of logical "and" and "or":

$$\begin{aligned} B \cup C &= \{x \mid x \in B \text{ or } x \in C\}\,, \\ A \cap B &= \{x \mid x \in A \text{ and } x \in B\}\,. \end{aligned}$$

Okay, now I'm ready to start the proof.

Argument. For convenience, define $X = A \cap (B \cup C)$ and $Y = (A \cap B) \cup (A \cap C)$. We prove that $X = Y$ in two steps.

1. First we prove that $X \subseteq Y$.

   Let $x \in X$. By the definitions of union and intersection, this means

   $(x \in A) \text{ and } (x \in B \text{ or } x \in C)$.

   The logical operator "and" will be true when both of its statements are true, so we know *both* that x belongs to A, *and* that it must belong to at least one of B and C. In other words, the logical statement

   $(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

   must hold. But this statement is equivalent to saying $x \in (A \cap B) \cup (A \cap C)$, using the definition of union and intersection. This is the definition of the set Y, so every element of X is also an element of Y, which means $X \subseteq Y$.

2. Now we prove $Y \subseteq X$.

   Suppose $y \in Y$. This means that

   $(y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C)$

   is a true statement. The logical "or" means at least one of the left and right statements must be true, but in either of these, it holds that $y \in A$, so this must be true generally. Then it must also be true that either $y \in B$ or $y \in C$. Converting this to logical operators, we have

   $(y \in A) \text{ and } (y \in B \text{ or } y \in C)$.

   By definition, this means that $y \in A \cap (B \cup C) = X$, so we have shown that every element of y is also an element of X. Thus $Y \subseteq X$.

Since $Y \subseteq X$ and $X \subseteq Y$, we have $X = Y$, as desired. $\square$

These proofs aren't technically challenging, by which I mean that one doesn't need to invent new ideas or methods to solve them. They all boil down to the properties of "and", "or", and (if set differences are involved) "not". Proving these statements rigorously is mostly an exercise in attention to detail and writing out all of the steps carefully.

Problem

Prove the following, including your thoughts and scratch work during exploration, then your polished final argument (if you find one).

Theorem. If A, B, C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Use the strategy above: write two separate sub-proofs, first showing the left side is a subset of the right, and then showing the right side is a subset of the left.

I've actually already given you the solution to this problem in pset 1 (solution 2.1 here). There's another answer in the official AoPS solution manual (solution 1.1.2 here) and a third in the "Instructor's Answer" below. If you get stuck, go ahead and look at one of these solutions, but be sure to rewrite the argument in your own words and understand the steps.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Exploration (Corbin) - I need to prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ I'm going to use the above mentioned approach of splitting this into two sub-proofs. I'm going to define $X = A \cup (B \cap C)$ and $Y = (A \cup B) \cap (A \cup C)$. I'm gonna draw some Venn Diagrams just to show that this statement is true and then I will build my proof around these. IMG_5115.JPG

Argument (Corbin) -

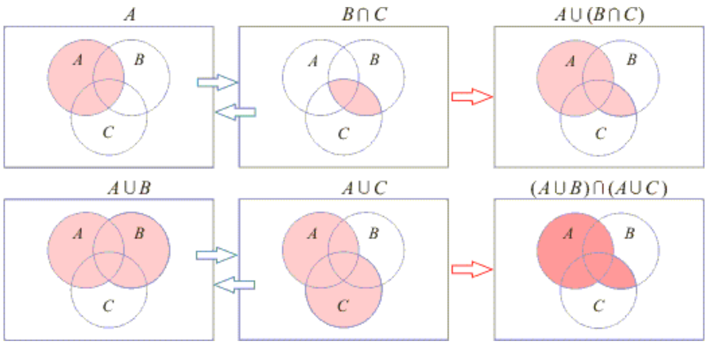
Exploration (Logan). Your thoughts and scratch work go here.

Argument (Logan). Your proof goes here.

Updated 8 months ago by Corbin and 2 others

the instructors' answer, *where instructors collectively construct a single answer*

Exploration (Christian). First I convince myself that this should be true. The Venn diagram looks like



The two sets on the right, $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$, appear to contain the same shaded regions.

We can also check the relation in a few simple cases, to build intuition. For instance, if $B = C$, then $B \cap C = B$ so the left side is $A \cup B$ and the right side is $(A \cup B) \cap (A \cup B) = A \cup B$, so here the claim works. I think the claim may be generally true; it certainly looks plausible, since it's similar to the previous distributive law proved above.

How will I go about showing this? Since we aim to prove equality of sets, my proof will have two parts: first I show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and then I prove that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Each of these will rely on the definitions of unions and intersections in terms of "and" and "or".

Okay, let's start the proof!

Argument (Christian). Let $X = (A \cup B) \cap (A \cup C)$ and $Y = A \cup (B \cap C)$, so that we aim to show that $X = Y$. We prove this in two steps.

1. First we show $X \subseteq Y$.

   Let $x \in X$. Using the definitions of union and intersection, this means

   $$((x \in A) \text{ or } (x \in B)) \text{ and } ((x \in A) \text{ or } (x \in B)) .$$

   The two statements joined by "and" must both be true. Both statements are true if $x \in A$. If $x \notin A$, the only way both statements can be true is if x belongs to both B and C. This means that the above is logically equivalent to

   $$(x \in A) \text{ or } (x \in B \text{ and } x \in C) .$$

   This statement can be re-written as $x \in A \cup (B \cap C) = Y$. Thus every $x \in X$ also belongs to Y, so $X \subseteq Y$.

2. Now we prove $Y \subseteq X$.

   Suppose $y \in Y$, which means

   $$(y \in A) \text{ or } (y \in B \text{ and } y \in C) .$$

   The "or" statement means that at least one of the two statements must be true. We consider each possibility separately.

   (i) In the case $y \in A$, then it is true that $((y \in A) \text{ or } (y \in B)) \text{ and } ((y \in A) \text{ or } (y \in B))$, since each "or" expression is true when either of the statements are true, and here the first statement in each "or" expression (namely, $y \in A$) is known to be true.

   (ii) In the case $(y \in B \text{ and } y \in C)$, it is again true that $((y \in A) \text{ or } (y \in B)) \text{ and } ((y \in A) \text{ or } (y \in B))$, since the second statement in each "or" expression is true.

   Thus, in all possible cases, it is true that $((y \in A) \text{ or } (y \in B)) \text{ and } ((y \in A) \text{ or } (y \in B))$, which means that $y \in (A \cup B) \cap (A \cup C)$. This shows that $y \in X$, so $Y \subseteq X$.

Since we have shown that $X \subseteq Y$ and $Y \subseteq X$, we conclude that $X = Y$, as claimed. $\square$

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*