question 2 views

Daily Challenge 6.5

(**Due Thursday 6/7 at 11:59 pm Eastern**)

Instead of a review, I will list the learning goals for chapter 1. I think you've already achieved a nonempty subset of these goals, and the consolidation document will help practice the remainder.

Remark. Many learning goals have the form "recall the definition..." or "state the definition...", but this does not mean that one needs to *memorize* these definitions. It is better to be able to reconstruct a definition on the fly by calling up your intuition about a property and then quickly formalizing it in your head.

For instance, I have not memorized the definition of "periodic function", but if you asked me to define it, my internal monologue would sound like:

> Hmm, I know that a periodic function is one that repeats after a given interval, like how $\sin(\theta)$ repeats when we take θ to $\theta + 2\pi$. So I want a definition which says that, whenever I shift the argument of the function by a special amount, the output of the function is unchanged. Aha, I know how to make that rigorous. A function $f(x)$ is *periodic* if there exists some real number k so that f has the property that $f(x+k) = f(x)$ for all $x \in \mathbb{R}$.

Except for a few very common results, like $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, mathematicians rarely memorize; most reconstruct on the spot, as I described above, by rapidly rebuilding a definition from intuition and reasoning.

Chapter 1 Learning Goals.

1. Mathematical proof. The student will be able to:
   - Identify the assumptions and conclusions of a given theorem.
   - Read a proof with proper technique (i.e. actively asking oneself why each statement follows from the previous ones, referring back to the appropriate definitions, etc.)
   - Write direct proofs about elementary structures, such as rational numbers or even and odd integers (examples: prove that the sum of rational numbers is rational; prove that the square of an even integer is even).
   - Write proofs by contradiction about elementary structures (example: prove that the sum of a rational number and an irrational number is irrational).
   - Identify the converse, inverse, and contrapositive of a given statement, and understand that, of these, only the contrapositive is logically equivalent to the original statement.

2. Set theory. The student will be able to:
   - Recall the definitions of set membership ($\in$), null set ($\emptyset$), subset ($\subseteq$), and proper subset ($\subset$), and use these to prove basic results about sets and containment (e.g. show that it is never true that both $B \subset A$ and $A \subset B$ for any sets A and B, or show that the empty set $\emptyset$ is a subset of any set A).
   - Recall the definitions of union ($\cup$), intersection ($\cap$), and set difference ($\setminus$).
   - Prove that two sets are equal by showing that each set is a subset of the other (e.g. prove the distributive laws for intersection and union).
   - Describe a set of interest using the notation $\{x \in \text{ some set } \mid x \text{ satisfies some property }\}$, and understand descriptions of sets presented in this notation.

3. Numbers and intervals. The student will be able to:
   - State the definitions of upper bound, least upper bound (supremum), lower bound, and greatest lower bound (infimum).
   - Recall the definitions of common symbols used for sets of numbers, like $\mathbb{Z}$, $\mathbb{N}$ (I will use the definition of $\mathbb{N}$ *without* zero), $\mathbb{Q}$, and $\mathbb{R}$.
   - Prove basic results about bounds, suprema, and infima (e.g. if A and B are bounded non-disjoint intervals, then $\sup(A \cap B) = \min(\sup(A), \sup(B))$.
   - Recall the definitions of interval, open interval, and closed interval.
   - Prove basic results about intervals (e.g. that an open interval is an interval, or that the intersection of two intervals is an interval).
   - Explain what the *completeness property* of the real numbers means and why it is important.

4. Functions. The student will be able to:
   - Recall the definitions of domain, codomain, and range.
   - Identify the domain and range of a given function.
   - Understand composition of functions and recognize the notation $(f \circ g)(x) = f(g(x))$ sometimes used for this.
   - Recall the definition of an inverse function.
   - Prove elementary results about inverse functions from the definition (e.g. if an inverse exists, it is unique).
   - Recall the definitions of image and preimage.
   - Identify the image or inverse image of a given set under a given function.
   - Prove results about image and inverse image (e.g. if $A \subseteq B \subseteq \mathrm{Dom}(f)$, then the image $f(A)$ is necessarily a subset of the image $f(B)$).
   - Understand the set-theoretic representation of the graph of a function $f(x)$, namely as the set of all ordered pairs $(x, f(x))$ for $x \in \mathrm{Dom}(f)$ (see the reading in section 1.4).
   - Explain what the vertical line test and horizontal line test are, and what properties they test for.
   - Prove results involving line tests and the graphs of functions (e.g. prove that, if the graph of a function f intersects a horizontal line $y = b$ in more than one point, f cannot have an inverse).
   - Given the graph of a function $f(x)$, understand how the graph is transformed if we re-scale or add constants (e.g. what does the graph of $2\sin(3x+1)$ look like?).

5. Trigonometry. The student will be able to:
   - State the definitions of $\cos(\theta)$ and $\sin(\theta)$ as the x and y coordinates of a point on the unit circle at angle θ counter-clockwise from the x axis.
   - Recall the values of $\sin(\theta)$ and $\cos(\theta)$ for $\theta \in \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi\right\}$ from memory, and figure out the sine or cosine of any multiple of these angles in another quadrant (e.g. $\sin\left(\frac{5\pi}{4}\right)$) after a moment's thought or drawing the unit circle.
   - Explain why the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, either using the defining equation of a circle of radius 1 or the Pythagorean theorem, and apply this identity when appropriate in problems.
   - Recall Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and be able to use it to simplify expressions involving complex numbers raised to powers (examples: problem 15 on the AoPS pretest, compute i^i or $\sqrt{i}$, etc.).

- Understand the definition of the radian as a dimensionless ratio of a subtended arc length to a radius.
- Convert angles between angles and radians, and vice-versa.
- Know that there *exist* formulas for the sine and cosine of a sum of angles (namely $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$and $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, respectively), and be able to look up and apply these formulas when necessary in problems. (But one does not need to state them from memory.)
- Either produce from memory, or be able to look up, the double-angle identities $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ (which are special cases of the angle addition formulas in the previous learning goal) and be able to apply them in problems.
- Recall the definitions of the trigonometric functions $\tan(\theta), \sec(\theta), \csc(\theta), \cot(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.
- Understand the definitions of the inverse sine and inverse cosine functions, $\sin^{-1}(x)$ and $\cos^{-1}(x)$, including their domains and ranges (especially that their ranges differ).
- Recall the definition of a periodic function.
- Identify whether a given function is periodic; given a periodic function, determine whether it has a period, and if so, what the period is.

6. Exponentials and logarithms. The student will be able to:
   - Recall the definition of the number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and be able to explain its significance by giving an example of a growth process, such as a continuously dividing bacterial colony or continuously compound interest, which is intrinsically tied to e.
   - State the definitions of strictly increasing and strictly decreasing functions, and be able to recognize when a given function has either of these properties.
   - Recall the formal definition of an exponential function (a strictly decreasing, strictly increasing, or constant function defined on the reals satisfying $f(x+y) = f(x)f(y)$ and which only outputs positive values).
   - Prove results about exponential functions from the definition (e.g. that a non-constant exponential function passes the horizontal line test).
   - Understand that the logarithm is the inverse function of an exponential and compute simple expressions involving a logarithms (e.g. $\log_2(64)$).
   - Apply the logarithm-of-a-product, logarithm-of-a-power, and change-of-base formulas to simplify expressions involving logarithms.
   - Recognize that the natural logarithm, with base e, is the standard choice of base for logarithmic functions; recall that we will denote this function as $\log(x)$ but other sources call it $\ln(x)$.

7. Mathematical maturity. The student will be able to:

   - Recognize the difference between heuristic or intuitive (i.e. non-rigorous) reasoning and genuine mathematical proof, and to hold himself to the standard of demanding the latter before being satisfied.

   - Appreciate that the purpose of mathematics is not to simply learn how to calculate, or even *that* something is true, but rather to understand *why* it is true.

   - Automatically engage in productive self-talk when reading mathematics (example here), including asking oneself questions to check understanding, generating and testing examples, producing conjectures, and so on.

Problem

Choose three learning goals from the above list which involve recalling or stating a definition which you cannot yet produce from memory. In the student answer below, **write precise and mathematically rigorous definitions of those three terms**. You may need to refer to chapter 1.

"Precise" means to avoid using vague or colloquial language, name all mathematical objects using appropriate variables, and unambiguously define the term you're speaking about. For instance,

- Bad (imprecise): The set difference of two sets is the set of all elements in one set but not in the other.
- Good (precise): Let A and B be sets. The *set difference* of A and B, written $A \setminus B$, is defined by $A \setminus B = \{a \in A \mid a \notin B\}$.

Another example:

- Bad (imprecise): The inverse image is the set of all elements that get mapped into the set under a function.
- Good (precise): Let f be a function and suppose A is a subset of the codomain of f. Then the *inverse image* or *preimage* of A under f, denoted by $f^{-1}(A)$, is $f^{-1}(A) = \{x \in \mathrm{Dom}(f) \mid f(x) \in A\}$.

A bad definition fails to give names to the objects it speaks about (e.g. "a function" rather than f), uses words when symbols would be more appropriate, and conveys a shaky understanding at best. A good definition is crisp, clear, and is appropriate for use in a formal proof.

daily_challenge

Updated 10 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
1. I have repeatedly demonstrated my inability to remember Euler's Formula, and as such shall provide the formula and a "text spoken" version to give this definition some substance: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, spoken "e to the power i theta equals cosine theta plus i sine theta." For some reason typing that makes me feel better, so I guess it works! :shrug:
2. I understand the inverse trigonometric functions $\sin^{-1}$ and $\cos^{-1}$. However, I do not know each of these functions' domain and range as well as I would like to. In this case as well as I would like to is quick, perhaps with a little thought. I now know to apply the idea of chopping the original function into a single piece to make inverting easier, and conveniently now understand that.
   1. $\sin^{-1}$
      1. Domain: $[-1, 1]$
      2. Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
   2. $\cos^{-1}$
      1. Domain: $[-1, 1]$

2. Range: $[0, \pi]$

3. I have read through this three times and can't find another thing to write on, I guess I have too much self-confidence.

Updated 10 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

For the sake of solidarity, I will choose three definitions that are less familiar to me:

- A subset I of $\mathbb{R}$ is said to be an *interval* if, for any pair of points $a \in I, b \in I$ with $a < b$, we have that $x \in I$ for all $x \in \mathbb{R}$ such that $a < x < b$.
- The *completeness property* of the reals is the result that, if a subset $A \subset \mathbb{R}$ has an upper bound, it necessarily has a least upper bound. Note that the rationals do not have this property; for example, the set $B = \left\{q \in \mathbb{Q} \mid q^2 < 2\right\}$ has an upper bound in $\mathbb{Q}$ but no supremum in $\mathbb{Q}$. This is important because it guarantees that $\mathbb{R}$ has "no holes", so to speak. In other words, $\mathbb{R}$ is closed under the operation of taking suprema. As we will see, this is necessary to have a well-behaved notion of limit.
- Let $f(x)$ be a function and let $A = \{(x, f(x) \mid x \in \mathrm{Dom}(f)\}$ be its graph. The *Horizontal Line Test* asks whether any horizontal line of the form $y = b$ intersects the graph A in at most one point, or equivalently, whether there exist any pair of points $(x_1, y) \in A$ and $(x_2, y) \in A$ with $x_1 \neq x_2$. A function f passes the horizontal line test if and only if there exists an inverse function for f.

Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*