24.2

(a): This is intuitively true, since there is a 50/50 distribution of where the particles making up a density come from; left or right. In this case, a particle ~~at T to t~~ coming into a space from ~~tef~~ the left is from $x+\Delta x$, and from the right is $x-\Delta x$, thus ~~the probability dens~~ our density has model

$$P(x, t+\Delta t) = \cancel{\tfrac{1}{2}P(x-\Delta x, t) + \tfrac{1}{2}(P(t+\Delta}$$
$$= \frac{1}{2}P(x+\Delta x, t) + \frac{1}{2}P(x-\Delta x, t)$$

(b): Let us focus on handwriting. We define
$$D = \frac{(\Delta x)^2}{2\Delta t}$$
and consider the equation seen in (a):
$$P(x, t+\Delta t) = \frac{1}{2}P(x+\Delta x, t) + \frac{1}{2}P(x-\Delta x, t)$$
then subtract $P(x,t)$ from each side
$$'' \quad - P(x,t) = \quad '' \quad - P(x,t)$$
then multiply each side by D:
$$\frac{(P(x,t+\Delta t)-(P(x,t)))(\Delta x)^2}{2\Delta t} = D\left(\quad '' \quad - P(x,t)\right)$$
$$\Rightarrow$$
$$\frac{P(x,t+\Delta t)-P(x,t)}{\Delta t} = D\,\frac{P(x+\Delta x,t)+P(x-\Delta x,t)-2P(x,t)}{(\Delta x)^2}$$

(c) Take the limit of each sides as $\Delta t \to 0$ and $\Delta x \to 0$;
$$\frac{d\,P(x,t)}{dt} = \lim_{\Delta x \to 0}\left(\frac{P(x-\Delta x,t)+P(x+\Delta x,t)-2P(x,t)}{\Delta x^2}\right)$$

Fasting Notes

German

~~We must simplify~~

$$\cancel{\lim_{\Delta x \to 0}\left(\frac{P(x-\Delta x, t) + P(x+\Delta x, t) - 2P(x,t)}{\Delta x^2}\right)}$$

~~We can rewrite the numerator~~

$$\cancel{P(x+\Delta x, t) + P(x-\Delta t) - 2P(x,t) =}$$

$$\cancel{P(x+\Delta x, t) - P(x,t) + (}$$

Begin with

$$\lim_{\Delta x \to 0}\left(\frac{P(x-\Delta x, t) + P(x+\Delta x, t) - 2P(x,t)}{\Delta x^2}\right) =$$

$$\lim_{\Delta x \to 0}\left(\frac{1}{\Delta x}\left(\frac{P(x+\Delta x) - P(x,t)}{\Delta x} - \left(\frac{P(x,t) - P(x-\Delta x)}{\Delta x}\right)\right)\right)$$

Algebraically. Then, the interior terms simplify to

$$\lim_{\Delta x \to 0}\left(\frac{1}{\Delta x}\left(P'(x,t) - P'(x-\Delta x, t)\right)\right) =$$

$$\lim_{\Delta x \to 0}\frac{P'(x,t) - P'(x-\Delta x, t)}{\Delta x} = P''(x,t) = \frac{\partial^2 P(x,t)}{\partial x^2}$$

(d): Let's begin by taking the derivative of each side of the given equation with respect to t:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi D}} \cdot t^{-1/2} \cdot \left(e^{\frac{-x^2}{4Dt}}\right)\right)$$

$$= \frac{1}{\sqrt{4\pi D}}\left(\frac{1}{2}\frac{1}{\sqrt{t^3}} \cdot e^{\frac{-x^2}{4Dt}} + \frac{1}{\sqrt{t}} \cdot e^{\frac{-x^2}{4Dt^2}}\right)$$

Now we take the derivative with respect to x twice of

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{\frac{-x^2}{4Dt}}$$

$$\frac{\partial P(x,t)}{\partial x} = \frac{1}{\sqrt{4\pi Dt}} e^{\frac{-x^2}{4Dt}} \cdot \frac{-2x}{4Dt}$$

$$\frac{\partial^2 P(x,t)}{\partial x^2} = \frac{1}{\sqrt{4\pi Dt}} \left(e^{\frac{-x^2}{4Dt}} \cdot \left(\frac{-2x}{4Dt}\right)^2 + e^{\frac{-x^2}{4Dt}} \cdot \frac{-2}{4Dt} \right)$$

242D

Let's do this right. We are told that

$$P(x,t) = \cancel{D\frac{\partial^2 P(x,t)}{\partial x^2}} = \frac{1}{\sqrt{4\pi D t}} \operatorname{Exp}\left(\frac{-x^2}{4Dt}\right)$$

and we must show that

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}$$

We shall prove this explicitly.

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial t}\frac{1}{\sqrt{4\pi D t}} \exp\left(\frac{-x^2}{4Dt}\right)$$

$$= \left(\frac{1}{\sqrt{4\pi D t}}\right)' \cdot \exp\left(\frac{-x^2}{4Dt}\right) + \frac{1}{\sqrt{4\pi D t}}\left(\exp\frac{-x^2}{4Dt}\right)'$$

We break this into pieces for clarity:

$$\left(\frac{1}{\sqrt{4\pi D t}}\right)' = \frac{1}{\sqrt{4\pi D}} \cdot \left(t^{-1/2}\right)'$$

$$= \frac{1}{\sqrt{4\pi D}} \cdot \frac{1}{\sqrt{t^3}}$$

$$\left(\exp\left(\frac{-x^2}{4Dt}\right)\right)' = \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-x^2}{4Dt^2}$$

Thus,

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{\sqrt{4\pi D t^3}} \cdot \exp\left(\frac{-x^2}{4Dt}\right) + \frac{1}{\sqrt{4\pi D t}} \cdot \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-x^2}{4Dt^2}$$

Now we have to take two derivatives with respect to x of the given equation.

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24.2 e-i

e: We are given to compute the formula

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right) dx$$; pull out the constant

$$= \int_{-\infty}^{\infty} x^2 \exp\left(\frac{-x^2}{4Dt}\right) dx \cdot \frac{1}{\sqrt{4\pi Dt}}$$; apply integration by parts

We integrate by parts where

$A' = \exp\left(\frac{-x^2}{4Dt}\right)$

$A = \cancel{\exp\left(\frac{-4Dt}{x^2}\right)} = \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-4Dt}{x^2}$

$B = x^2$

$B' = 2x$

then,

$$E[x^2] = \left[\exp\left(\frac{-x^2}{4Dt}\right) - 4Dt\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-8Dt}{x} dx$$

$$= 8Dt \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{1}{x} dx$$; simplify

We integrate by parts again with

$A' = \exp\left(\frac{-x^2}{4Dt}\right)$

$A = \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-4Dt}{x^2}$

$B = \frac{1}{x}$

$B' = \frac{-1}{x^2}$

then,

$$\ldots = 8Dt \left(\left[\exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-4Dt}{x^2} \cdot \frac{1}{x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{4Dt}\right) \cdot \frac{-4Dt}{x^2} \cdot \frac{-1}{x^2} dx \right.$$

This just made this integral unnessarily complicated, so let's try a different method. Begin ~~at~~ again with

~~$E[x^2]$~~ $\int$

Let's replicate the hint since this is a long and hard fai.

Mk.2 e: We are given to compute the integral

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right) dx$$

$$= \frac{1}{\sqrt{4\pi Dt}} 2\int_{0}^{\infty} x^2 \exp\left(\frac{-x^2}{4Dt}\right) dx$$

The hint then suggests we split this for clarity;

$$= \frac{1}{\sqrt{4\pi Dt}} 2\int_{0}^{\infty} x \exp\left(\frac{-x^2}{4Dt}\right) x\, dx$$

then u-sub where

$$u = \frac{x^2}{4Dt} \Rightarrow x = \sqrt{4Dtu}$$

$$du = \frac{x}{2Dt} dx \Rightarrow dx\,(x) = 2Dt\, du$$

then,

$$\ldots = \frac{1}{\sqrt{4\pi Dt}} \cdot 2\int_{0}^{\infty} \sqrt{4Dt}\,\sqrt{u}\, \exp(-u)\, 2Dt\, du$$

Pull out scalars

$$= \frac{1}{\sqrt{\pi}} \cdot 2Dt \cdot 2\int_{0}^{\infty} u^{1/2} \exp(-u)\, du$$

then by the definition of the gamma function where $z = \frac{3}{2}$,

$$= \frac{1}{\sqrt{\pi}} \cdot 4Dt\, \Gamma\left(\frac{3}{2}\right) ; \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$= 2Dt$$

i: 1: By decreasing delta X in my function or increasing delta Y.
2: Make one direction slightly more pronounced or give it a high probability of occuring.
3: Randomly choose an axis and to follow it by ± 1