AN INTRODUCTION TO INFINITE PRODUCTS

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ABSTRACT. In this paper, we assemble what is a cursory look at the theory of the infinite product and associated techniques, as well as how they relate to other mathematical concepts. We also discuss functions that are defined in terms of an infinite product, and infinite products of complex sequences.

CONTENTS

1. INTRODUCTION.

We are going to be considering the nature of the *product* in the following text. An infinite product is defined as the multiplication of all terms in a sequence, and likewise, a partial product is the multiplication of all terms up to an index b:

$$\prod_{n=a}^{b} s_n = s_a \cdot s_{a+1} \cdot \dots \cdot s_b, \tag{1.1}$$

and an infinite product

$$\prod_{n=a}^{\infty} s_n = \lim_{b\to\infty} \prod_{n=a}^{b} s_n. \tag{1.2}$$

A few basic facts are immediately (albeit only intuitively) clear; the terms of an infinite sequence have to go to 1 for any convergent result to be found (where the obvious result is if the sequence goes to 0, or any individual term is 0).

The convergence of these infinite product is discussed, as well as how to determine whether they converge using pre-existing methods for sums.

Also discussed are functions defined in terms of an infinite product, infinite products of complex sequences, the Dedekind eta function and its special features, and evaluating these products in Python.

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2. Basic definitions and results.

Definition 2.1. Let $\{a_n\}$ be a sequence of nonzero real numbers. An infinite product

$$\prod_{n=1}^{\infty} a_n = a_1 \cdot a_2 \cdot a_3 \cdot \cdots \tag{2.1}$$

is said to *converge* if the sequence of partial products

$$p_k = \prod_{n=1}^{k} a_n = a_1 \cdot \cdots \cdot a_k \tag{2.2}$$

converges to a nonzero real number p. That is,

$$\lim_{k\to\infty} \prod_{n=1}^{k} a_n = p. \tag{2.3}$$

In this case, we say

$$p = \prod_{n=1}^{\infty} a_n. \tag{2.4}$$

If an infinite product does not converge, it is said to *diverge*.

Proposition 2.2. We wish to prove the following: If $\prod_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} b_n = 1$.

Proof. Begin by noticing that the notation of the partial product allows us to write

$$\frac{\prod_{n=1}^{m} b_n}{\prod_{n=1}^{m-1} b_n} = b_m = \frac{p_m}{p_{m-1}}. \tag{2.5}$$

Take the limit of each side; then since shifting does not have an effect on the limit,

$$\lim_{m\to\infty} b_m = \lim_{m\to\infty} \frac{p_m}{p_{m-1}} = \frac{l}{l} = 1\,. \tag{2.6}$$

□

Theorem 2.3. An infinite product

$$\prod_{n=1}^{\infty} b_n, \tag{2.7}$$

with all $b_n > 0$, converges if and only if the series

$$\sum_{n=1}^{\infty} \log\left(b_n\right) \tag{2.8}$$

converges.

Proof. We wish to prove that $\prod b_n$ converges if and only if $\sum \log b_n$ converges. For $m > 1$, denote

$$p_m = \prod_{n=1}^{m} b_k, \tag{2.9}$$

$$s_m = \sum_{n=1}^{m} \log(b_k). \tag{2.10}$$

Recall the basic algebraic fact that $\exp(\log(x)) = x$; then by log rules

$$\exp(s_m) = p_m. \tag{2.11}$$

This relation between the partial sums is incredibly powerful; we see that, just by taking the limit of each side,

$$\text{if } \lim_{m\to\infty} p_m = p, \text{ then } p = \lim(\exp(s_m)). \tag{2.12}$$

It is clearly true that $\lim(\exp(s_m))$ converges to p; then $\exp(s_m)$ converges implies that s_m converges. Likewise, if $\lim \exp(s_m) = \exp(s)$, then clearly $\lim(p_m)$ converges and equals $\exp(s)$. □

Proposition 2.4. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\prod_{n=1}^{\infty}(1+a_n)$ converges. Similarly, if $0 \leq a_n < 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\prod_{n=1}^{\infty}(1-a_n)$ converges.

Proof. We see by plugging into the Taylor series for e^x that

$$e^{a_n} = 1 + a_n + (a_n)^2/2! + \cdots . \tag{2.13}$$

Clearly by subtracting $(a_n)^2/2! + \cdots$ on the right side (which is non-negative since $a_n \geq 0$), we get the inequality

$$e^{a_n} \geq 1 + a_n. \tag{2.14}$$

Now let us define the partial product

$$p_k = \prod_{n=1}^{k} (a_n + 1), \tag{2.15}$$

and the partial sum

$$s_k = \sum_{n=1}^{k} a_n. \tag{2.16}$$

We see that

$$e^{s_k} = e^{a_1} \cdot e^{a_2} \cdots e^{a_k}. \tag{2.17}$$

By (2.14),

$$(1 + a_1) \cdot (1 + a_2) \cdots (1 + a_k) \leq e^{a_1} \cdot e^{a_2} \cdots e^{a_k}, \tag{2.18}$$

and thus $p_k \leq e^{s_k}$.

We then see that since for any k, $\sum_{k+1}^{\infty} a_n \geq 0$, then by "extending" (2.18) that

$$p_k \leq \exp\left(\sum_{n=1}^{\infty} a_n\right). \tag{2.19}$$

Now let us write the pattern that

$$\begin{aligned} p_1 &= 1 + a_1, \\ p_2 &= 1 + (a_1 + a_2) + a_1 a_2, \\ p_3 &= 1 + (a_1 + a_2 + a_3) + (a_1 a_2 + a_1 a_3 + a_2 a_3), \end{aligned} \tag{2.20}$$

and see that

$$p_k = 1 + \sum_{i=1}^{k} a_i + \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j + \cdots . \tag{2.21}$$

Then, since $s_k = \sum_{i=1}^{k} a_i$, then $s_k \leq p_k$; thus the partial products p_k are bounded above and below. We also see that p_k is non-decreasing, since all terms $1 + a_n \geq 1$ since $a_n \geq 0$. Thus we can conclude that $\prod_{n=1}^{\infty} a_n + 1$ converges.

Now let us consider the second claim. Begin by noticing that $a_n \to 0$ by the vanishing criterion for sums; then we can pick N large enough that $a_n < 1/2$ for $n > N$. Notice that we are given that $a_m \geq 0$; then follow the series of steps

$$\begin{aligned} a_n &\geq 0 \\ a_n^2 &\geq 0 \\ 1 - a_n^2 &\leq 1 \\ (1+a_n)(1-a_n) &\leq 1 \\ (1-a_n) &\leq (1+a_n)^-1 \end{aligned} \tag{2.22}$$

to get one inequality that shall be useful later, and also recall we can work with

$$\begin{aligned} a_n &\leq 1/2 \\ 1 - 2a_n &\geq 0 \\ a_n - 2a_n^2 &\geq 0 \\ 1 + a_n - 2a_n^2 &\geq 1 \\ (1-a_n) &\geq (1+2a_n)^{-1}. \end{aligned} \tag{2.23}$$

We then see that

$$\prod_{n=1}^{\infty} (1+a_n)^{-1} \leq \prod_{n=1}^{\infty} (1-a_n) \leq \prod_{n=1}^{\infty} (1+2a_n)^{-1}, \tag{2.24}$$

and thus the partial products are bounded; we see that the partial products are non increasing since all terms ≤ 1, and thus we conclude that $\prod_{n=1}^{k} (1-a_n)$ converges. □

3. Examples.

Example 3.1. Begin by recalling that sine has Taylor series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots . \tag{3.1}$$

We divide each side by x to see that

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots . \tag{3.2}$$

Now we propose the following non-rigorous argument (since we haven't proven anything about factoring infinite polynomials): since the polynomial clearly has roots when $\sin(x) = 0$, and thus $x = z\pi$ where z is an integer. We can then write that the infinite sum as equal to

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{4\pi^2}\right) \cdots . \tag{3.3}$$

We see by an "infinite foil" of the right side that x^2 has coefficient

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right). \tag{3.4}$$

By setting the coefficient of x^2 in the sum equal to that seen in the product, we find

$$-\frac{1}{6} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right), \tag{3.5}$$

which implies (by multiplying each side)

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{3.6}$$

We have acquired an interesting result using a non-rigorous argument about infinite products; we'll see how we can rigorize this later.

Example 3.2. Let's try to evaluate the following product:

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right). \tag{3.7}$$

Let's start with a quick algebraic manipulation; multiply by 1 to see that

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{\infty} \left(\frac{n^2-1}{n^2}\right). \tag{3.8}$$

Then,

$$\prod_{n=2}^{\infty} (\frac{n^2-1}{n^2}) = \underbrace{\underbrace{\underbrace{\frac{3}{4} \cdot \frac{8}{9}}_{2/3} \cdot \frac{15}{16}}_{5/8} \cdot \frac{24}{25}}_{3/5} \cdot \cdots. \tag{3.9}$$

Each subsequent denominator is "half-cancelled" by the preceding numerator for all consecutive terms, so intuitively we expect the product to go to $\frac{1}{2}$.

Example 3.3. Let's also try to evaluate

$$\prod_{n=1}^{\infty} (1 + x^{2^n}). \tag{3.10}$$

Begin by writing out the first few partial products;

$$\begin{aligned} \prod_{n=1}^{\infty} (1 + x^{2^n}) &= (1 + x^2)(1 + x^4)(1 + x^8) \cdots \\ &= (1 + x^2 + x^4 + x^6)(1 + x^8) \cdots \\ &= (1 + x^4 + x^2 + x^6 + x^8 + x^{10} + x^{12} + x^{14}. \end{aligned} \tag{3.11}$$

We see that our partial products to some N are equal to the geometric series

$$\sum_{n=0}^{N} x^{2^n}. \tag{3.12}$$

Thus the partial products converge to $\frac{1}{1-x^2}$ for $|x| < 1$ and likewise the infinite product converges to that value.

4. Infinite product representations of functions.

Theorem 4.1. The function $\frac{\sin(x)}{x}$ can be represented as the infinite product

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{4\pi^2}\right) \cdots = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{4.1}$$

Proof. Define some number α such that $0 < \alpha < 1$. Then, consider the function $f(x)$ which is given by $\cos(\alpha x)$ on $[-\pi, \pi]$, but every 2π starting at π and $-\pi$, the function on $[-\pi, \pi]$ is extended periodically. Then we want to find the Fourier series of f; there exist no sine terms in the Fourier series since the $f(x)$ in question is even, thus we must only find the cosine coefficients. First, see that

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\alpha x)\, dx \\ &= \frac{1}{2\pi} \frac{1}{\alpha} \left(\sin(\alpha\pi) - \sin(-\alpha\pi)\right). \end{aligned} \tag{4.2}$$

Then since $\sin(-x) = -\sin(x)$,

$$a_0 = \frac{\sin(\pi\alpha)}{\pi\alpha}. \tag{4.3}$$

Now we must solve for all other coefficients a_n:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(\alpha x) \cos(nx)\, dx. \tag{4.4}$$

Unfortunately we cannot apply orthogonality, since $\alpha \neq 1$ or any integer for that matter. We can, however, apply the trig identity

$$2\cos(x)\cos(y) = \cos(x - y) + \cos(x + y) \tag{4.5}$$

to see that

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos(\alpha x - nx) + \cos(\alpha x + nx)\right)\, dx \tag{4.6}$$

$$= \frac{1}{2\pi} \left(\frac{\sin(\alpha x - nx)}{(\alpha - n)} + \frac{\sin(\alpha x + nx)}{(\alpha + n)} \right)_{-\pi}^{\pi} \tag{4.7}$$

$$= \frac{1}{2\pi} \left(\frac{(\alpha + n)\sin(\alpha x - nx) + (\alpha - n)\sin(\alpha x + nx)}{(\alpha - n)(\alpha + n)} \right)_{-\pi}^{\pi}. \tag{4.8}$$

Then, by the trig identity that $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(b)$, we see that

$$a_n = \frac{1}{2\pi} \left(\frac{(\alpha + n)\left(\sin(\alpha x)\cos(-nx) + \sin(-nx)\cos(\alpha x)\right) + (\alpha - n)\left(\sin(\alpha x)\cos(nx) + \sin(nx)\cos(\alpha x)\right)}{\alpha^2 - n^2} \right)_{-\pi}^{\pi}. \tag{4.9}$$

At these endpoints, we see that all occurrences of $\sin(nx) = 0$, and that $\cos(nx) = \cos(-nx) = (-1)^n$; thus

$$\begin{aligned} a_n = \frac{1}{2\pi} \Big(&\left((\alpha + n)\sin(\alpha\pi)(-1)^n + (\alpha - n)\sin(\alpha\pi)(-1)^n\right) \\ &- \left((\alpha + n)\sin(\alpha(-\pi))(-1)^n + (\alpha - n)\sin(\alpha(-\pi))(-1)^n\right) \Big) \Big/ (\alpha^2 - n^2). \end{aligned} \tag{4.10}$$

Then since $\sin(-x) = -\sin(x)$ and cancellations of terms with n coefficients,

$$a_n = \frac{1}{2\pi}\left(\frac{4\alpha\sin(\alpha\pi)(-1)^n}{\alpha^2 - n^2}\right) \tag{4.11}$$

$$= \frac{2\alpha\sin(\alpha\pi)(-1)^n}{\pi(\alpha^2 - n^2)}, \tag{4.12}$$

and thus we can assemble the known a_0 and factor out the constant part of the a_n and conclude that

$$f(x) = \frac{2\alpha\sin(\pi\alpha)}{\pi}\left(\frac{1}{2\alpha^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2 - n^2}\cos(nx)\right). \tag{4.13}$$

Now, recalling that $f(x) = \cos(\alpha x)$ on $[-\pi, \pi]$, then by letting $x = \pi$ see that

$$\cos(\alpha\pi) = \frac{2\alpha\sin(\pi\alpha)}{\pi}\left(\frac{1}{2\alpha^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2 - n^2}\cos(n\pi)\right), \tag{4.14}$$

and dividing each side by $\sin(\alpha\pi)$ and noting $\cos(nx)^2 = (-1)^{2n} = 1$ then

$$\cot(\alpha\pi) = \frac{1}{\alpha\pi} + \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha^2 - n^2}, \tag{4.15}$$

which implies

$$\cot(\alpha\pi) - \frac{1}{\alpha\pi} = \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha^2 - n^2}. \tag{4.16}$$

Now we want to integrate over α from 0 to t; we can only integrate the sum if it is uniformly convergent over the open-ended range in question though. Begin by noting that for $\alpha \in (0, t)$, $\alpha < t$ and since $\alpha < n$ we have

$$|\alpha^2 - n^2| = n^2 - \alpha^2 > n^2 - t^2, \tag{4.17}$$

which implies

$$\left|\frac{1}{\alpha^2 - n^2}\right| \leq \frac{1}{n^2 - t^2}. \tag{4.18}$$

Thus we can choose the majorants

$$M_n = \frac{1}{n^2 - t^2}, \tag{4.19}$$

which has a convergent sum since all $M_n < \frac{1}{n^2}$, a series known to have convergent sum. Thus by Weierstrass M the original sum converges uniformly, and we can integrate each term individually:

$$\int_0^t \left(\cot(\alpha\pi) - \frac{1}{\alpha\pi}\right) d\alpha = \int_0^t \left(\frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha^2 - n^2}\right) d\alpha \tag{4.20}$$

$$= \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\int_0^t \left(\frac{1}{\alpha^2 - n^2}\right) d\alpha. \tag{4.21}$$

For the left side, recall $\int \cot(x)\,dx = \log(\sin(x))$. For the right side, notice the resemblance to a derivative of log:

$$\int_0^t \left(\cot(\alpha\pi) - \frac{1}{\alpha\pi}\right) d\alpha = \log\left(\frac{\sin(\pi t)}{\pi t}\right), \tag{4.22}$$

and

$$\int_0^t \left(\frac{1}{\alpha^2 - n^2}\right) d \ = \log(1 - \frac{t^2}{n^2}) \tag{4.23}$$

allows us to assemble

$$\log\left(\frac{\sin(\pi t)}{\pi t}\right) = \sum_{n=1}^{\infty} \log\left(1 - \frac{t^2}{n^2}\right). \tag{4.24}$$

Finally, let $x = \pi t$; then

$$\log\left(\frac{\sin(x)}{x}\right) = \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{\pi^2 n^2}\right). \tag{4.25}$$

Take the exponential of each side and notice that sums are turned into products. We finally conclude that

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right). \tag{4.26}$$

□

Remark 4.2. As written, we do not need to worry about checking whether the (4.26) converges (as a results of starting with a convergent Fourier series). Suppose however, that you started your proof somewhere in the middle of this mess. How would you prove that the product converges? Let's look back to proposition (2.4), which says that, if $0 \le a_n < 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\prod_{n=1}^{\infty}(1-a_n)$ converges; now consider $a_n = \frac{x^2}{\pi^2 n^2}$; we assume $x < \pi$, so $a_n < \frac{1}{n^2}$. By the comparison test since $\frac{1}{n^2}$ converges, $\sum a_n$ converges and thus the product converges.

Proving the formula (4.26) for sine was an absolute mess, so let's simply go over some other functions that can also be defined by infinite products. First, cosine:

$$\cos(x) = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2 \pi^2}\right). \tag{4.27}$$

We can also write $\Gamma(x)$ (a continuous form of the factorial, if you have not seen it before) as an infinite product:

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{(1 + 1/n)^x}{(1 + \frac{z}{n})}. \tag{4.28}$$

We can also consider the Riemann zeta function, normally defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{4.29}$$

as an infinite product that iterates over n prime:

$$\zeta(x) = \prod_{n \text{ prime}} \frac{1}{1 - n^{-x}}. \tag{4.30}$$

5. Infinite products of complex numbers.

In the unfortunate scenario that you have not been introduced to complex numbers, we present the following summary. A complex number z is defined as some point in the complex plane $\mathbb{C}$ which has two axes, real and imaginary. The number $z = (x, y) = x + iy$, in a manner akin to vector notation. We define complex number addition as adding/subtracting the components on each axis, as well as a new function *modulus*:

$$|z| = |(x, y)| = \sqrt{x^2 + y^2}. \tag{5.1}$$

Modulus measures the distance between the origin and some point in the complex plane, and is always positive. We define complex number multiplication geometrically: the result is found at the sum of the angles of z_1 and z_2 from the $+x$ axis, and the distance from the origin is the product of the moduli of z_1 and z_2. All of this is nearly identical to polar coordinates. Now, how would we talk about the convergence of an infinite sequence in the complex plane?

Definition 5.1. An infinite sequence of complex numbers converges to l if, for every $\epsilon > 0$, there is a natural number N such that for $n > N$,

$$|a_n - l| < \epsilon. \tag{5.2}$$

Just like the real case, we can define convergence of a complex infinite product.

Definition 5.2. An infinite product

$$\prod_{n=1}^{\infty} z_n \tag{5.3}$$

converges if

$$\lim_{k\to\infty} p_k = \lim_{k\to\infty} \prod_{n=1}^{k} z_n = l. \tag{5.4}$$

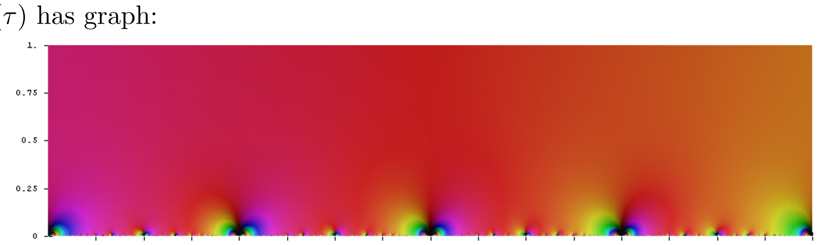
6. The Dedekind eta function.

First, some motivation: The Dedekind Eta function is a *modular form*, or a function that exists on a torus. Each point on the torus is mapped to a value, and the function is periodic on that torus; you can loop through the inside or outside and get the same result. We define the *Dedekind eta function* by

$$\eta(\tau) = \exp\left(\frac{\pi i \tau}{12}\right) \prod_{n=1}^{\infty} \left(1 - e^{2ni\pi\tau}\right), \tag{6.1}$$

where τ is a complex number with positive imaginary component.

$\eta(\tau)$ has graph:



Where this is a *color map*, where the darkness/lightness of a color represents the modulus of $\eta(\tau)$ at τ, and the color itself represents the argument. The black values are where modulus is (or nearly is) zero, and so on.

Theorem 6.1. For τ complex in the half plane with $\tau = x + iy$ and $y > 0$,

$$\eta\left(\frac{-1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau). \tag{6.2}$$

(Also assume we are considering $z^{1/2} > 0$ if $z > 0$.)

Proof. We just need to prove this claim for $\tau = iy$, since by analyticity it holds for the entire half-plane with y greater than 0. Let $\tau = iy$, then on each side of (6.2)

$$\eta(i/y) = y^{1/2}\eta(iy), \tag{6.3}$$

and through log rules,

$$\log(\eta(i/y)) - \log(\eta(iy)) = \frac{1}{2}\log(y). \tag{6.4}$$

Now consider $\log(\eta(iy))$ and $\log(\eta(i/y))$; for brevity we shall only consider $\log(\eta(iy))$. First notice that, by log rules,

$$\log(\eta(iy)) = \frac{-\pi y}{12} + \log\prod_{n=1}^{\infty}\left(1 - e^{-2\pi ny}\right), \tag{6.5}$$

and then the log turns the product into a sum:

$$\log(\eta(iy)) = \frac{-\pi y}{12} + \log\sum_{n=1}^{\infty}\left(1 - e^{-2\pi ny}\right). \tag{6.6}$$

Now, recall the Taylor series for $\log(1 + x)$:

$$\log(1 + x) = \sum_{m=1}^{\infty}\frac{x^m}{m}. \tag{6.7}$$

Apply this result to see that

$$\log(\eta(iy)) = \frac{-\pi y}{12} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{e^{-2\pi mny}}{m}. \tag{6.8}$$

Change the order of the sums, and notice that

$$\log(\eta(iy)) = -\frac{\pi y}{12} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-2\pi my}}{1 - e^{-2\pi my}}, \tag{6.9}$$

since we can factor out $\frac{1}{m}$ and $\sum_{n=1}^{\infty} e^{-2\pi my}$ is a geometric series. Then, multiply by 1 (within the sum) in the form $\frac{-e^{2\pi my}}{-e^{2\pi my}}$ to find

$$\log(\eta(iy)) = -\frac{\pi y}{12} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1 - e^{2\pi my}}. \tag{6.10}$$

We can find a similar expression for $\log\eta(i/y)$ by replacing all occurrences of y with $\frac{1}{y}$; thus we want to prove that

$$-\frac{\pi y}{12} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1 - e^{2\pi my}} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1 - e^{2\pi m/y}} = -1/2\log(y). \tag{6.11}$$

To really begin the proof, we can apply the theorems of residue calculus to the left side. The statement of the residue theorem is that for some closed curve C with interior area denoted by a_k,

$$\oint_C f(z)dz = 2\pi i \sum \operatorname{Res}(f, a_k). \tag{6.12}$$

The "residue" is that which in in the coefficient when f is blowing up to infinity at some point in the range Again, consider $y > 0$ and now consider $n \in \mathbb{N}$ and associated $N = n + \frac{1}{2}$; define the helper function

$$F_n(z) = -\frac{1}{8z} \cot(\pi i N z) \cot(\frac{\pi N Z}{y}). \tag{6.13}$$

Then, consider the curve C in the complex plane that is lines traveled counterclockwise (starting on the positive real axis) connecting $y, i, -y,$ and $-i$. Then, F_n has one (set of) pole(s) at $z = ik/N$ where k is any non-zero integer with associated residue

$$\frac{1}{8\pi k} \cot\left(\frac{\pi i k}{y}\right). \tag{6.14}$$

We get this result by rewriting $f_n(z)$ as a multiplication of Taylor series, extracting all terms proportional to $1/z$, and recognizing the now clear residue that is found in the numerators.

For the next poles, we recall the idea that

$$\text{residue} = \lim_{z \to a} \left(f(z) \cdot (z - a)\right) \tag{6.15}$$

For $z = ky/N$ there is a pole with residue

$$-\frac{1}{8\pi k} \cot\left(i\pi k y\right), \tag{6.16}$$

which we acquire by and at $z = 0$ there is a pole with residue

$$i\left(y - y^{-1}\right)/24. \tag{6.17}$$

By summing the contributions from all these residues, we see that $2\pi i$ times aforementioned sum gives us an expression that when $\lim_{n\to\infty}$ is taken, is equal to

$$-\frac{\pi y}{12} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m y}} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}}. \tag{6.18}$$

Thus, we want to show that

$$\lim_{n\to\infty} \int_C F_n(z) dz = \frac{-1}{2} \log(y). \tag{6.19}$$

Begin by considering the function $zF_n(z)$; this function has limit $\frac{1}{8}$ on the edges in the 1st and 3rd quadrants, and limit $\frac{-1}{8}$ on the edges in the 2nd and 4th quadrants.

As well, $F_n(z)$ is uniformly bounded on C, and thus we see that

$$
\begin{aligned}
\lim_{n\to\infty}\int_C F_n(z)dz &= \lim_{n\to\infty}\int_C F_n(z)\frac{z}{z}dz \\
&= \frac{1}{8}\left(\int_i^y + \int_{-i}^{-y} - \left(\int_{-i}^{y} + \int_i^{-y}\right)\right)\frac{dz}{z} \\
&= \frac{1}{4}\left(\int_y^i - \int_{-i}^{y}\right)\frac{dz}{z} \\
&= \frac{1}{4}\left(\left(\frac{\pi}{2} - \log(y)\right) - \left(\log(y) + \frac{\pi i}{2}\right)\right) \\
&= \frac{-1}{2}\log(y).
\end{aligned}
\tag{6.20}
$$

□

7. Numerically computing infinite products.

One can numerically evaluate infinite products in Python using the mpmath library, which can evaluate infinite and finite sums and products alike; it does so using *Richardson extrapolation* and *Shanks transformation*. Both strive to increase the rate of convergence of a sequence. First consider the Shanks transformation:

The Shanks transformation takes in some sequence A and returns some sequence S through the function

$$S(A_n) = A_{n+1} - \frac{(A_{n+1} - A_n)^2}{(A_{n+1} - A_n) - (A_n - A_{n-1})}. \tag{7.1}$$

where S converges considerably faster, and such a transformation can be applied infinitely many times.

Richardson extrapolation is considerably more complicated, so I'll summarize it:

(1) Consider some function dependent on the function you want to approximate and some small h.
(2) Define some other function that depends on the function seen in (1) and some small variable t.
(3) The function written in (2) then has a quicker converging sequence.

Now what results can we get from mpmath? Consider the Wallis product:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right). \tag{7.2}$$

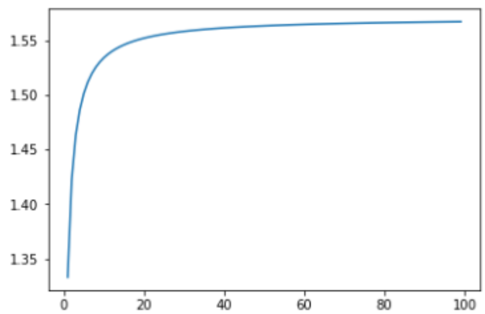
And let's make a graph showing an application of mpmath to see the graph of the partial products as n increases:

```
y_vals = []

for val in x_vals:
    y_vals.append(float(mpmath.nprod(lambda n: ((2*n)/(2*n-1) ) * (2*n/(2*n+1)), [1,val])))

plt.plot(x_vals, y_vals)
```

```
[<matplotlib.lines.Line2D at 0x27f31fc0dd8>]
```



Similarly, we can consider the infinite product associated with log(2):

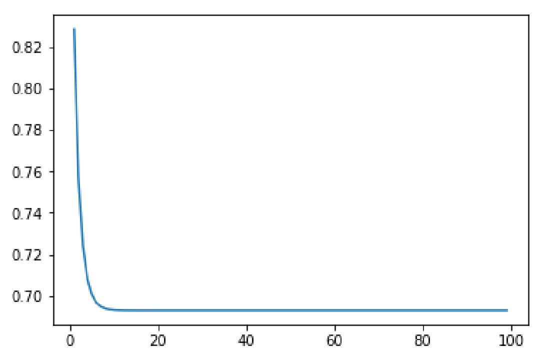
```
x_vals = range(1,100)

y_vals = []

for val in x_vals:
    y_vals.append(float(mpmath.nprod(lambda n: 2 / (1 + 2**(1/(2**n))), [1,val])))
```

```
plt.plot(x_vals,y_vals)
```

```
[<matplotlib.lines.Line2D at 0x27f31e1f160>]
```



References

1. Dikran Karagueuzian. Math 478 (Real Analysis) course notes. http://people.math.binghamton.edu/dikran/478/Ch6.pdf.
2. Tom M. Apostol. “Modular Functions and Dirichlet Series in Number Theory.”
3. Michael Spivak. “Calculus.”
4. George B. Arfken and Hans J. Weber. “Mathematical Methods for Physicists.”
5. Carl M. Bender and Steven A. Orszag. “Advanced Mathematical Methods for Scientists and Engineers I.”