question 2 views

Daily Challenge 14.3

(**Due: Saturday 9/15 at 12:00 noon eastern**)

No review/reading today -- let's start hammering out some problems which use what we've learned about finding maxima and minima.

This one will go on CD 4, but I haven't made the Overleaf document yet.

(1) Problem: tuning a for a local min.

Determine the real number a with the property that the function $f(x) = x^4 - x^3 - x^2 + ax + 1$has a local *minimum* at the point $x = a$.

(Source: Harvard-MIT math tournament)

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
We know that a local minimum has a first derivative equal to zero when said minimum is inputted, and a positive second derivative; We shall go down the line of derivatives;

$f(x) = x^4 - x^3 - x^2 + ax + 1$
$f'(x) = 4x^3 - 3x^2 - 2x + a$
$f''(x) = 12x^2 - 6x - 2$

We can then look at the first derivative; substitute $x = a$, then $f'(a) = 4a^3 - 3a^2 - a = a(4a^2 - 3a - 1)$and of course we can note that a is a dummy variable and replace all a with x. One potential root of the first derivative is 0, but inserting this into the second derivative yields the negative number -2, so my next idea is to apply the quadratic formula to the element in parentheses; We factor to see $4x^2 - 3x - 1 = (4x+1)(x-1)$and with this info we can pass these root values onto the second derivative and check; We finally see that $x = \frac{-1}{4}$ results in a zero second derivative and a positive second derivative, so the solution to our problem is $a = \frac{-1}{4}$.

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

We would like to tune a so that $f'(a) = 0$ but $f''(a) > 0$. The first derivative is

$$f'(x) = 4x^3 - 3x^2 - 2x + a,$$

so we impose the condition

$$f'(a) = 4a^3 - 3a^2 - a = 0.$$

To solve for a, we factor as $4a^3 - 3a^2 - a = a(4a+1)(a-1)$ So it seems that there are three choices for a, namely $0, 1$, and $-\frac{1}{4}$.

However, we also need $f''(a) > 0$, and one has

$$f''(x) = 12x^2 - 6x - 2.$$

This is negative at $x = 0$ but positive at $x = 1$ and $x = -\frac{1}{4}$. So there are actually two possible choices for a,

$$\boxed{a \in \left\{1, -\frac{1}{4}\right\}}.$$

I am now comparing my answer to the HMMT answer and realized that I typed the problem wrong. In their problem, they asked you to also impose $f(a) = a$ in addition to the local minimum thing. We see that $f(1) = 1$ but $f(-\frac{1}{4}) \neq -\frac{1}{4}$, so if I had written the problem correctly, we would have said that only $a = 1$ works.

Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*