question 3 views

Daily Challenge 3.6

(**Due: Sunday 5/13 at 12:00 noon Eastern**)

Review

Yesterday, we used the ideas of images and pre-images to understand why trigonometric functions fail to be invertible unless we restrict their domains. Let's recall the definitions of these beasts.

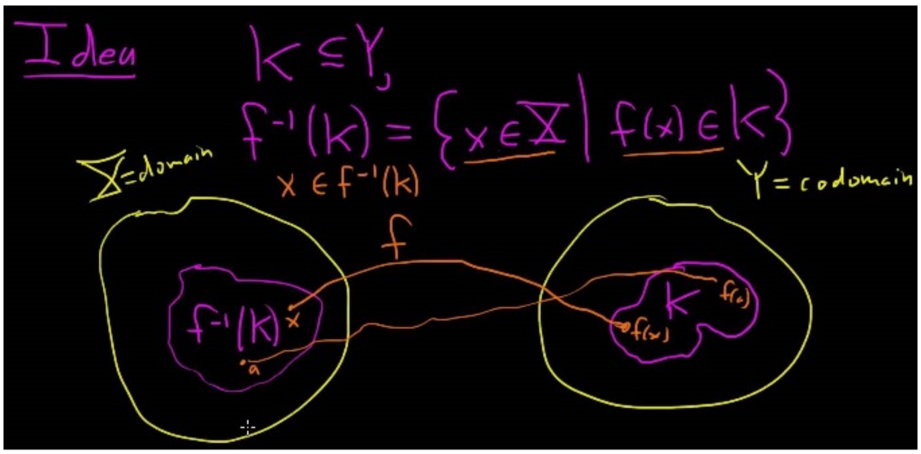
Definition. Let $f : X \to Y$ be a function from a set X to a set Y. Let K be a subset of Y. Then the *preimage* or *inverse image* of K under f, written $f^{-1}(K)$ is the set of all elements in X that are mapped into K:

$$f^{-1}(K) = \{x \in X \mid f(x) \in K\}\,.$$

Now let L be a subset of X. The *image* of L, written $f(L)$, is the set of all elements of Y obtained by applying the function f to an element of L:

$$f(L) = \{y \in Y \mid y = f(\ell) \text{ for some } \ell \in L\}\,.$$

Here is a screenshot which I have shamelessly stolen from Khan Academy:



This illustrates the *pre-image* of a subset K in the codomain; the pre-image $f^{-1}(K)$ is drawn on the left, and contains everything that gets mapped into K.

[An aside: when I first learned this subject, I naturally thought that images were more important than inverse images, since doing something forward seems more fundamental than doing it backward. But I later changed my mind and now believe the inverse image is more fundamental. For instance, one uses the idea of inverse image to state the general definition of continuity in topology.]

Understanding images and inverse images precisely will give us more practice writing set-theoretic proofs.

Example. Let f be a function and $A \subset \mathrm{Dom}(f)$. What is the relationship between A and $f^{-1}(f(A))$? Prove your assertions.

Exploration. Although the notation might tempt us to conjecture that $f^{-1}(f(A)) = A$, let's be a bit more careful.

If A sits in the domain, it pushes forward to a subset $f(A)$ in the codomain. However, there could be other inputs in the domain (outside of A) which also map into $f(A)$.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, and let A be the interval $[0, 1]$. Then $f(A) = [0, 1]$. But the preimage of $[0, 1]$ is larger: $f^{-1}([0, 1]) = [-1, 1]$, since the negative numbers in $[-1, 0]$ also get mapped into the interval $[0, 1]$. In other words, our example is

$$f : \mathbb{R} \to \mathbb{R} \quad , \quad f(x) = x^2 \quad , \quad A = [0, 1],$$
$$f^{-1}(f([0, 1])) = [-1, 1].$$

This shows that $f^{-1}(f(A)) \neq A$, in general, but we are asked to find the relationship between these sets. If the two sets are not equal, perhaps the relationship is that one is a subset of the other: $A \subseteq f^{-1}(f(A))$.

Let's see if we can prove this idea carefully.

Argument. Let f be a function and $A \subset \mathrm{Dom}(f)$. We claim that $A \subset f^{-1}(f(A))$. To show this, we will suppose that $a \in A$ and then prove that a also belongs to $f^{-1}(f(A))$.

So let $a \in A$. Then $f(a) \in f(A)$, by the definition of image (the image is the set of all $f(x)$ for $x \in A$, so $f(a) \in f(A)$ because $a \in A$). Likewise, $a \in f^{-1}(f(a))$ by the definition of inverse image, since we have shown that a maps to an element $f(a) \in f(A)$.

Thus every element of A is also in $f^{-1}(f(a))$, so $A \subseteq f^{-1}(f(A))$. □

Problem

Try to answer (with proof) the following image and inverse image questions.

(a) Let f be a function and $B \subseteq \mathrm{Cod}(f)$. What is the relationship between B and $f(f^{-1}(B))$? Prove your assertions.

(b) Show that, if $A \subseteq B$ and f is a function whose domain includes B, then $f(A) \subseteq f(B)$. If instead A is a *proper* subset, so that $A \subset B$, is it necessarily true that $f(A) \subset f(B)$? If so, prove it; if not, give a counterexample.

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Exploration (Logan). Your reasoning and thoughts go here.

(a)

(b)

Argument (Logan). Your polished proofs go here.

(a)

(b)

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the instructors' answer, *where instructors collectively construct a single answer*

Exploration (Christian).

(a) Intuitively, I feel like this result should be the *opposite* of the one we considered in the example above. In that case, roughly speaking, we saw that pushing forward and then pulling back by a function can give you a *bigger* set than the one you started with, since the function might not be one-to-one. That is, we proved

$$A \subseteq f^{-1}(f(A)).$$

In this case, we're considering the reverse order: how is $f(f^{-1}(B))$ related to B? This time, it seems like the result should be *smaller*, in general. It is important that B is a subset of the *codomain* and not the *range*: when we first pull B back to $f^{-1}(B)$, we get all elements of the domain that map into B, but then we push this result forward and get a set $f(f^{-1}(B))$ which is a subset of the range. This can be smaller than B.

For example, let's think about $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ again. Let $B = [-1, 1]$. Then the pre-image $f^{-1}(B)$ is $[-1, 1]$, but $f(f^{-1}(B)) = [0, 1] \neq B$.

(b) This sounds very reasonable: if A sits inside B, and $f(A)$ is the set of all outputs of the function f when I feed it an element of A, then $f(A)$ should sit inside $f(B)$ (since every output of f when I feed it an element of A is also an output of f when I feed it an element of B).

I think this should be a straightforward direct proof; we'll just use the definition of images, and the definition of subset.

I don't think the claim should still be true when we replace subsets by proper subsets; think about the constant function $f(x) = 1$, for instance.

Argument (Christian).

(a) Let f be a function and $B \subseteq \mathrm{Cod}(f)$. We claim that $f(f^{-1}(B)) \subseteq B$. To prove this, we will suppose $a \in f(f^{-1}(B))$ and prove that $a \in B$.

So let $a \in f(f^{-1}(B))$. By the definition of image, this means $a = f(x)$ for some $x \in f^{-1}(B)$. But if $x \in f^{-1}(B)$, then by the definition of inverse image, this means $f(x) \in B$. Thus $a \in B$, so we have proved that $f(f^{-1}(B)) \subseteq B$. $\Box$

(To see why $f(f^{-1}(B))$ is only a *subset* of B, but not equal to B, see the $f(x) = x^2$ example in the exploration above.)

(b) Let $A \subseteq B$ and suppose f is a function whose domain includes B. We claim that $f(A) \subseteq f(B)$. To prove this, we will assume $x \in f(A)$ and prove that $x \in f(B)$.

So assume x \in f(A). By the definition of image, this means x = f(a) for some a \in A. But A \subseteq B, so if a \in A, we also have a \in B. Thus x = f(a) for some element a \in B, which means x \in f (B) (again, by the definition of image).

The claim is *not* true if subsets are replaced by proper subsets. To show that it is not true, we need to construct a counterexample where the claim is false. So consider f: \mathbb{R} \to \mathbb{R} defined by f(x) = 1. Let A = \left[0, 1 \right] and B = \left[0, 2 \right]. Then A \subset B, but f(A) = \left\{ 1 \right\} = f(B), so it is not true that f(A) \subset f(B), only that f(A) \subseteq f(B). \Box

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followup discussions *for lingering questions and comments*