question 2 views

Daily Challenge 11.5

(**Due: Thursday 8/16 at 12:00 noon Eastern**)

Today I will skip the review and just post one of the consolidation document 3 problems, which will give you more practice on checking differentiability directly using the definition.

(1) Problem: some absolute value bounds.

This is problem 3 on CD 3; please copy over your solutions when you're done.

(a) Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

(b) Let $\alpha > 1$. If f satisfies $|f(x)| \leq |x|^\alpha$, prove that f is differentiable at zero.

daily_challenge

Updated 8 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:

a: We must show that the function f where $|f(x)| \leq x^2$ for all x is differentiable at zero. First we recall the definition of a derivative, ie $f'(x) = \lim_{h\to 0} \frac{f(0+h)+f(0)}{h}$. We can begin by seeing that for $x = 0$ we have that $|f(0)| \leq 0^2$, therefore $f(0) = 0$. We can apply this to our limit and see that we must show that the limit $\lim_{h\to 0} \frac{f(h)}{h}$ exists at 0. Let $\epsilon > 0$ be given and let $\delta = \epsilon$. We then have that since $|f(x)| \leq x^2$, it is true that when $0 < |h| < \delta$, it is true that $|f(h)| \leq \delta^2$ where $\delta^2 = \epsilon^2$, and we can then see that $\left|\frac{f(h)}{h}\right| \leq \left|\frac{\epsilon^2}{\epsilon}\right| = \epsilon$ and the limit exists, therefore f is differentiable at 0.

b: We see that the way this problem is worded suggests a nearly identical argument to the previous; simply instead of x^2 we have "parabolas" of all powers greater than 1. Once again, we have that $|f(x)| \leq |x|^a$ where $a > 1$. We then have that for $x = 0$ that $|f(0)| \leq |0|^a$ and therefore $f(0) = 0$. We can now refer to the derivative and see that $\lim_{h\to 0} \frac{f(h)}{h}$ for $x = 0$. We must show that this limit exists at zero. Let $\epsilon > 0$ be given. We have by assumption that $\left|\frac{f(h)}{h}\right| \leq \left|h^{a-1}\right|$. Let $\delta = \epsilon^{\frac{1}{a-1}}$. We then have that $\left|\frac{f(h)}{h}\right| \leq \left|\delta^{a-1}\right| = \left(\epsilon^{\frac{1}{a-1}}\right)^{a-1} = \epsilon$. Therefore the limit exists and f is differentiable at 0.

Updated 8 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) Suppose f is a real-valued function satisfying $|f(x)| \leq x^2$ for all x. We consider the limit which defines the derivative $f'(0)$, if the limit exists:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

Applying the assumption $|f(x)| \leq x^2$ at $x = 0$ yields $|f(0)| \leq 0$ which implies $f(0) = 0$. Thus we need to prove that the limit

$$\lim_{h\to 0} \frac{f(h)}{h}$$

exists. Let $\epsilon > 0$ be given and choose $\delta = \epsilon$. If $0 < |h| < \delta$, then by assumption, $|f(h)| \leq \delta^2 = \epsilon^2$, so

$$\left|\frac{f(h)}{h}\right| \leq \left|\frac{\epsilon^2}{\epsilon}\right| = \epsilon,$$

which is what we wanted to show. Thus $\lim_{h\to 0} \frac{f(h)}{h}$ exists and f is differentiable at 0. $\square$

(b) Suppose f is a real-valued function satisfying $|f(x)| \leq x^\alpha$, where $\alpha > 1$, for all x. Again we consider the limit which defines the derivative $f'(0)$, if the limit exists:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

As in part (a), the assumption $|f(x)| \leq x^\alpha$ requires that $|f(0)| \leq (0)^\alpha = 0$ which implies $f(0) = 0$. So we need to prove that the limit

$$\lim_{h\to 0} \frac{f(h)}{h}$$

exists. Let $\epsilon > 0$ be given. By assumption $\left|\frac{f(h)}{h}\right| \leq |h^{\alpha-1}|$, so choose $\delta = \epsilon^{\frac{1}{\alpha-1}}$. Note that this choice is well-defined since we have assumed $\alpha > 1$. Now if $0 < |h| < \delta$,

$$\left|\frac{f(h)}{h}\right| \leq |\delta^{\alpha-1}| = \left(\epsilon^{\frac{1}{\alpha-1}}\right)^{\alpha-1} = \epsilon,$$

hence $\lim_{h\to 0} \frac{f(h)}{h}$ exists and f is differentiable at 0. $\square$

Updated 8 months ago by Christian Ferko

followup discussions *for lingering questions and comments*