Projectile Mechanics the detailed way

Logan S. Pachulski

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Part 1: The Problem Statement

We want a reasonably straightforward way of quickly graphing the path of some projectile, say a cannonball, through a vacuum and under earth's gravity. The cannonball is fired from the origin at an angle α with velocity v.

Part 2: Making the Functions

We begin by finding a way to write down the locations as functions of time; We know that the only acceleration occurring throughout the shot is the strictly-downwards acceleration due to gravity, so $y(t)'' = -g$. We also see that velocity imparted on our object at time zero can be represented on two axes separately; $x(0)' = v\cos(\alpha)$, $y(o)' = v\sin(\alpha)$. We can then find the positional graphs of each by undifferentiating each to see that $x(t) = v\cos(\alpha)t$ and $y(t) = -\frac{1}{2}gt^2 + v\sin(\alpha)t$.

To show that this is function is a parabola, we must find some way to express y as a function of x rather than t, simplifying this to two dimensions and allowing us to continue. We see that $x = v\cos(\alpha)t \implies t = \frac{x}{v\cos(\alpha)}$. We then see that

$$y(x) = -\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 + v\sin(\alpha)\frac{x}{v\cos(\alpha)}, \tag{1}$$

and by placing the top and bottom of the former element to the second power as suggested and separating, we see that

$$-\frac{1}{2}g\left(\frac{x}{v\cos(\alpha)}\right)^2 = \left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2 \tag{2}$$

We also see by basic algebra that

$$v\sin(\alpha)\left(\frac{x}{v\cos(\alpha)}\right) = \tan(\alpha)x \tag{3}$$

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We conclude that

$$y(x) = \left(-\frac{g}{2v^2\cos^2(\alpha)}\right)x^2 + (\tan(\alpha))\,x, \tag{4}$$

which is a happy little parabola of the form $y(x) = ax^2 + bx + c$, where a, b are 2 and 3 respectively, sans-x, and $c = 0$.

Part 4: Parabolas are making life easier

We see that a parabola is amazingly symmetric. Then the y position will equal zero at two times the x-value of the vertex, which can be found as the point where the $y'(t) = 0$. We see that $y'(t) = -gt + v\sin(\alpha)$, and by solving when this equals zero, we see that $t = \frac{v\sin(\alpha)}{g}$, therefore the horizontal distance can be found by inserting double this into x(t), then the distance can be represented as $\frac{2v^2\sin(\alpha)\cos(\alpha)}{g}$.

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We refer to the range function in terms of some angle α we previously defined, $R(\alpha) = \frac{2v^2 \sin(\alpha)\cos(\alpha)}{g}$. $\frac{2v^2}{g}$ is given to us as a positive, so we simply must maximize $\sin(\alpha)\cos(\alpha)$; We find the derivative of this via the product rule to be $\cos^2(\alpha) - \sin^2(\alpha)$. We then solve for zero, and find thinking of some point where $\cos^2(\alpha) = \sin^2(\alpha)$ that $\alpha = \frac{\pi}{4}$ is a potential answer; our range of interest is $\alpha \in [0, \frac{\pi}{2}]$, where at the endpoints $R = 0$, and there are no undefined points. Then launching an object at 45 degrees is the most efficient, who knew!