note 2 views

Daily Challenge 5.4

Let's take another make-up day and try to catch up on daily challenges 4.1, 5.1, 5.2, and 5.3.

The last topic for chapter 1 is logarithms and exponentials, so I'll post a short worked example to remind us of some properties of exponential functions.

Example. Let $f(x) = 2^x$. Answer the following questions, with no proof required: you may use whatever properties of exponents you know.
(a) What are $f(0)$ and $f(1)$?
(b) Write $f(a+b)$ in terms of $f(a)$ and $f(b)$.
(c) Write $f(-a)$ in terms of $f(a)$.
(d) What are the domain and range of f?

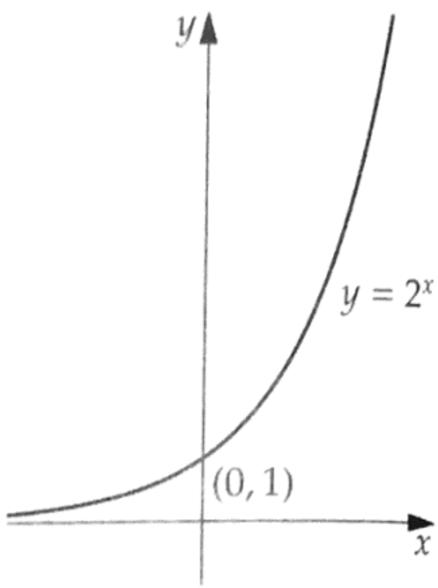
Solutions. (a) We remember that, for any nonzero number x, we have $x^0 = 1$. In particular, $f(0) = 2^0 = 1$.

Likewise, any number raised to the first power is simply that number itself, so $f(1) = 2^1 = 2$.

(b) In general, an exponential of a sum can be decomposed into a product of individual factors as $x^{a+b} = x^a x^b$. Applying this to our function f, we see that $f(a+b) = 2^{a+b} = 2^a 2^b$, which means $f(a+b) = f(a)f(b)$.

(c) A number raised to a negative power is the multiplicative inverse of the number raised to a positive power: in general, $x^{-a} = \frac{1}{x^a}$. This means that $f(-a) = \frac{1}{2^a} = \frac{1}{f(a)}$.

(d) This is easier to visualize if we look at the graph of $f(x)$:



From looking at the graph, we are tempted to say that $\mathrm{Dom}(f) = \mathbb{R}$ and $\mathrm{Rng}(f) = (0, \infty)$. This is indeed true, but we actually haven't said enough to truly define the exponential for all real numbers yet!

The properties we've described above are certainly enough to define $f(x) = 2^x$ for *integer* values of x. For instance, if $x = n$ is a positive integer, then we define 2^n as multiplying 2 by itself n times:

$$f(n) = \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}.$$

If $x = -n$ is a negative integer, we can use the property $f(-x) = \frac{1}{f(x)}$ to define $f(x)$ as

$$f(-n) = \frac{1}{\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}}}.$$

To define the exponential for rational numbers, we could use the "power-to-a-power" rule:

$$f\left(\frac{n}{m}\right) = (2^n)^{\frac{1}{m}}.$$

Thus we can write rigorous definitions of $f(x)$ for all $x \in \mathbb{Q}$. But what do we mean by $f(\pi) = 2^\pi$? What does it mean to "multiply 2 by itself π times"? If we can't rigorously state what we mean by this expression, we cannot claim to have defined $f(x)$ for all reals yet.

Therefore, although we think it should be true that $\mathrm{Dom}(f) = \mathbb{R}$ and $\mathrm{Rng}(f) = (0, \infty)$, we can't actually say for sure based on what we know at this point.

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Updated 10 months ago by Christian Ferko

followup discussions *for lingering questions and comments*