question **2** views

Daily Challenge 14.2

(**Due: Friday 9/14 at 12:00 noon eastern**)

Let's do another exercise to drive home the relationship between a function's derivatives and its graph.

(1) The derivatives of f contain valuable information about the graph of f.

In session 32, we will complete our toolkit of techniques for using the derivatives of f to extract information about the graph of f itself.

Although I now come dangerously close to beating a dead horse, I will repeat the key points again because they are so important.

- First derivative.
  - The graph of f is *increasing* on an interval if $f'(x) \geq 0$ on that interval.
  - The graph of f is *decreasing* on an interval if $f'(x) \leq 0$ on that interval.
  - A point where $f'(x) = 0$ is called a *critical point* of f.
- Maxima and minima.
  - We say that a point x is a *local maximum* of f if there exists some number δ such that x is a maximum point of f on the interval $(x - \delta, x + \delta)$. A similar definition applies for *local minimum*.
  - If x is a local maximum or local minimum of f, and if $f'(x)$ is defined, then $f'(x) = 0$.
  - If f is continuous on $[a, b]$ and differentiable on (a, b), then f achieves its maximum and minimum on $[a, b]$, and both occur at points x where either (i) $f'(x) = 0$, (ii) $x = a$ or $x = b$, or (iii) $f'(x)$ is undefined.
  - Second derivative test. Let f be C^2. If $f'(x) = 0$ and $f''(x) > 0$, then x is a local minimum of f. If $f'(x) = 0$ and $f''(x) < 0$, then x is a local maximum of f.
- Second derivatives.
  - A function f is said to be *convex* (or "concave up") on an interval if, for every a, b in the interval, the line segment between the points $(a, f(a))$ and $(b, f(b))$ lies entirely above the graph of f.
  - Similarly, a function f is said to be *concave* (or "concave down") on an interval if, for every a, b in the interval, the line segment between the points $(a, f(a))$ and $(b, f(b))$ lies entirely below the graph of f.
  - If f is C^2 and $f''(x) > 0$ on an interval, then f is convex on that interval; if $f''(x) < 0$ on an interval, then f is concave on that interval.
  - A point x where $f''(x) = 0$ is called an *inflection point*.

We could, in principle, continue onward and come up with interpretations for the third derivative, fourth derivative, and so on; this is rarely done in introductory calculus classes, but ask me if you want to talk about it anyway.

(2) Problem: the function kitchen, redux.

(This problem has several deliverables, marked in bold.)

Cook up a function f with the following properties:

- f is increasing on $(\infty, -2)$ and on $(3, \infty)$,
- f is decreasing on $(-2, 3)$,
- f has an inflection point at $x = 1$.

Once you have constructed your function, **write down** f, f', and f''. **Confirm explicitly** that it has the properties listed above (e.g. by checking the sign of the first derivative), and **identify the intervals** on which it is convex and concave. Finally, **sketch the three graphs** of f, f', and f'' together (you may either sketch all three on the same axes, or else stack the three graphs vertically, whichever you prefer).

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
~~I shall start at the first derivative since what we need to satisfy depends on this and the second derivative; we need~~ $f'(x) = 0$ ~~when~~ $x = -2, 3$~~, so we represent the first derivative as the cubic~~ $x^3 - x^2 - 6x$~~. The derivative of this does not equal zero when~~ $x = 1$~~, but regardless the derivative as is would be~~ $3x^2 - 2x - 6$~~, and we want to find some c we can distribute to all of these such that~~ $3cx^2 - 2cx - 6c = 0; \text{ when } x = 1$ ~~We see that when~~ $x = 1$ ~~then~~ $3c - 2c - 6c$~~.~~

~~We can operate on the knowledge that a polynomial~~ $f'(x)\ (x^3 - x^2 - 6x)$ ~~has the roots we are interested in having for a first derivative; we can take the derivative of this to see~~ $f''(x) = (3x^2 - 2x - 6)$~~, and we can distribute to this some term that results in the second derivative being zero when x = 1 but also doesn't change the sign of the first; first we shall try multiplying by~~ $(x-1)$~~;~~ $(3x^2 - 2x - 6)\ (x - 1) = (3x^3 - 2x^2 - 6x) + (3x^2 - 2x - 6) = 3x^3 + x^2 - 8x - 6$ ~~and we then put this in desmos to see if it has the properties of interest NOPE IT DOESN'T CUS IM FUCKING RETARDED; let's try again with~~ $(1 - x)$~~.~~

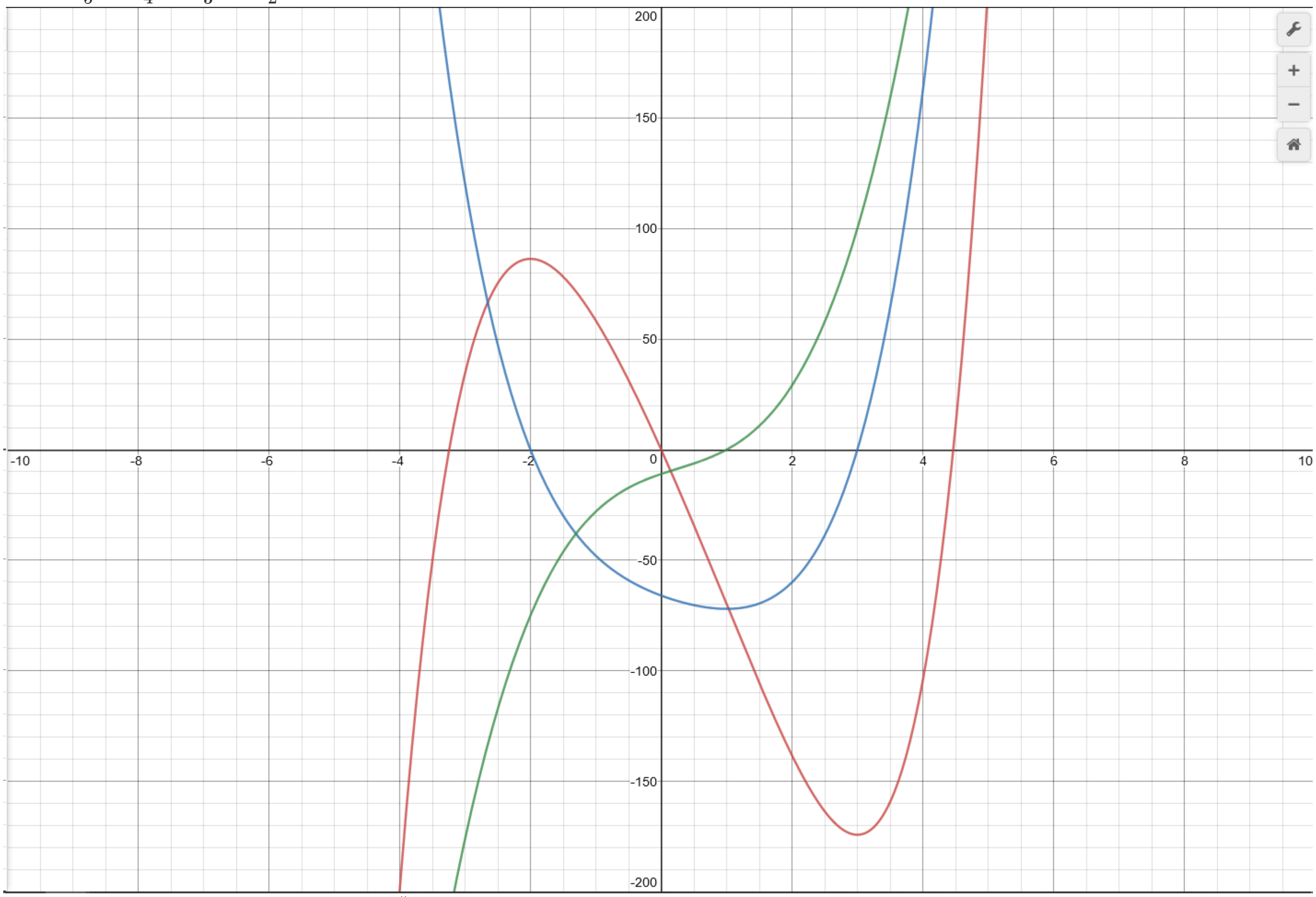
$\left(3x^2-2x-6\right)(1-x) = (3x^3-2x^2-6x)+(3x^2-2x-6) = 3x^3+x^2-8x-6$

Attempt No. 3: We shall attempt this time by taking the method that Christian suggests; finding some way of multiplying the first derivative in such a way that the second derivative is 0 when $x=1$. We see that a polynomial that satisfies that traits looked for in the first two bullet points is $p(x)=(x-2)(x+3)=x^2-x-6$ so we want to find some $q(x)$ such that $f'(x)=(x^2-x-6)q$, and $f''(x)=(2x-1)q+(x^2-x-6)q'$We see that we want some function that is always positive, as to not mess with the sign of our base polynomial; we allow this always positive function to be x^2+c where c is some value greater than 0. We can insert 1 into our equation and begin solving for c.
$f''(x)=(2(1)-1)q(1)+(1-1-6)q'(1)$ or after simplification $q(1)+-6q'$; since this is a function of our choosing we can "demand" that $6g'(1)=g(1)$, in which case we see by inserting $x=1$ that $12(1)=1+c$, or $c=11$. We then work back; and see by substituting this $c=11$ that

$f''(x)=(2x-1)(x^2+11)+(x^2-x-6)(2x)$

$f'(x)=(x^2-x-6)(x^2+11)=(x^4-x^3-6x^2)+(11x^2-11x-66)=x^4-x^3+5x^2-11x-66$

$f(x)=\dfrac{1}{5}x^5-\dfrac{1}{4}x^4+\dfrac{5}{3}x^3-\dfrac{11}{2}x^2-66x$



And finally, by referring to the green line representing $f''(x)$, f is concave when $x<1$ and convex when $x>1$.

Updated 7 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

We want the function's first derivative to be positive on (-\infty, -2) and on (3, \infty), and negative on (-2, 3). So our first guess is to make the first derivative a parabola:

\begin{align} f'(x) \overset{?}{=} (x-2) (x+3) = x^2 - x - 6 . \end{align}

However, we also want f to have an inflection point at x=1. This means we need f''(1) = 0, but the above parabola doesn't have that property! We see that it has f''(x) = 2 x - 1, so it would give f''(1) = 1.

In fact, you can convince yourself that *no* parabola of the above form would work, since the vertex will always lie halfway between the roots (at x = \frac{1}{2}), while we want the second derivative to vanish at x=1.

Let's try *multiplying* the above function by something which will give the desired inflection point at x=1. Call the thing we multiply by g(x). So our new guess is something of the form

\begin{align} f' (x) \overset{?}{=} g(x) (x-2) (x + 3) = g(x) \left(x^2 - x - 6 \right) . \end{align}

Now, we don't want to multiply by anything which changes the sign of f'(x), since we carefully built our guess so that it is positive and negative in the right places. So whatever function g(x) we pick, it had better be positive everywhere.

However, we also want an inflection point at x=1, so the second derivative should be zero there. By the product rule, the second derivative of our above guess is

\begin{align} f''(x) \overset{?}{=} g'(x) \left(x^2 - x - 6 \right) + g(x) \left(2x - 1 \right) . \end{align}

At x=1, this is

$$\begin{align} f''(1) \overset{?}{=} g'(1) \left(- 6 \right) + g(x) \left(2 - 1 \right) = - 6 g'(1) + g(1) \overset{\text{want}}{=} 0 . \end{align}$$

Now we need some creativity: can we find a function g(x) which is positive everywhere, but which also has 6 g'(1) = g(1)? There are many options -- for instance, we could take g(x) = e^{\frac{1}{6} x}, which is clearly always positive and has g'(1) = \frac{1}{6} e^{\frac{1}{6}} but g(1) = e^{\frac{1}{6}}.

A perhaps simpler (i.e. polynomial) choice is to make g(x) = x^2 + c for some positive number c. Then g will always be positive, as desired. To find c, we demand that 6 g'(1) = g(1):

$$\begin{align} g'(1) = 2 \quad , \quad g(1) = 1 + c . \end{align}$$

To get 6 g'(1) = 12 to be equal to 1+c, we see that we need c=11.

So if we go with the polynomial choice, our final guess for the derivative is

$$\begin{align} f'(x) \overset{?}{=} \left(x^2 + 11 \right) (x-2) (x+3) = x^4 - x^3 + 5 x^2 - 11 x - 66 . \end{align}$$

Of course, this means the original function should look like

$$\begin{align} f(x) = \frac{1}{5} x^5 - \frac{1}{4} x^4 + \frac{5}{3} x^3 - \frac{11}{2} x^2 - 66 x . \end{align}$$

We should explicitly verify that this has the desired properties.

$$\begin{align} f'(x) &= (x^2 + 1) (x + 2) (x - 3) , \\ f''(x) &= 4 x^3 - 3 x^2 + 10 x - 11 . \end{align}$$

Indeed, we see that f''(1) = 4 - 3 + 10 - 11 = 0, so there is an inflection point at x=1. Likewise, by inspection we see that f' is positive on (-\infty, -2) and on (3, \infty), and negative on (-2, 3).

We know that x=1 is a root of the second derivative, so factor it out to find f''(x) = (x-1) (4 x^2 + x + 11). The second factor is always positive, so f'' > 0 if x>1 and f'' < 0 if x<1. This means that the original function is concave on (- \infty, 1) and convex on (1, \infty).

Finally, I show the plots below.

In[27]:= $f[x\_] := \frac{1}{5} x^5 - \frac{1}{4} x^4 + \frac{5}{3} x^3 - \frac{11}{2} x^2 - 66 x;$

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In[28]:= Plot[{f[x], f'[x], f''[x]}, {x, -5, 5}, PlotLegends → "Expressions"]
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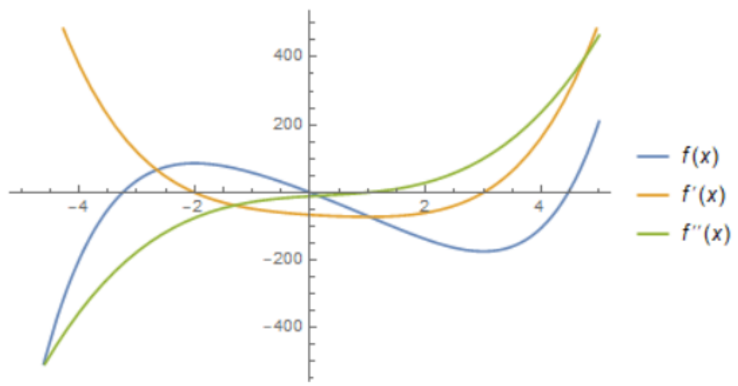
Out[28]=



Updated 7 months ago by Christian Ferko

followup discussions *for lingering questions and comments*