question 2 views

Daily Challenge 24.3

(**Due: parts (a) - (c) due Thursday 3/14 at 12:00 noon Eastern, parts (d) - (e) due Friday 3/15 at 12:00 noon Eastern**)

You've seen some probability theory for random *scalars*; today I will introduce random *matrices*.

One interesting thing is to generate a random matrix and ask about its eigenvalues. Some matrices can have complex eigenvalues, but symmetric matrices have eigenvalues which are always real numbers. A matrix A is *symmetric* if it satisfies $a_{ij} = a_{ji}$; that is, the entry in row-i-column-j is always equal to the entry in row-j-column-i. For example,

$$S = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

is symmetric, but

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

is not. Another way to say this: a symmetric matrix is equal to its *transpose*, where the transpose A^T is defined by $\left(A^T\right)_{ij} = A_{ji}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

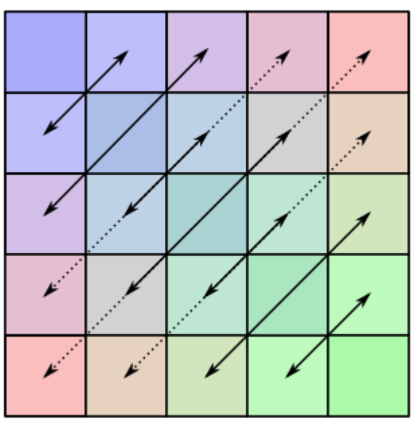
Transpose of a matrix

Clearly the sum of a matrix and its transpose is always symmetric. If we define $S = A + A^T$ for any matrix A, then

$$S^T = \left(A + A^T\right)^T = A^T + A = S,$$

and S is symmetric as claimed.

Schematically, symmetric matrices look like this,



where color is supposed to indicate numerical value.

(**Part a**) Let's start with 2×2 matrices. You can generate a random matrix in numpy by passing the `shape=(2,2)` parameter to any random number generating function. We'll use `numpy.random.normal` in this DC.

```
In [3]: import numpy as np

        np.random.normal(size=(2,2))

Out[3]: array([[ 0.35442529,  0.19919024],
               [-1.88415253,  1.17669365]])
```

However, this matrix is not symmetric. We want to restrict attention to symmetric matrices and study the probability distribution for their eigenvalues, which are real numbers.

Write a function `random_symmetric(n)` which generates a random $n \times n$ matrix A such that $a_{ij} = a_{ji}$. [Hint: you may find the transpose function useful.]

(**Part b**) Write a function which runs the following experiment: it generates a large number, say `n_trials = 1000`, of $n \times n$ matrices. It then finds the eigenvalues of each one (using numpy.linalg.eig, although remember to extract just the piece with eigenvalues and not eigenvectors) and stores them in a big list to return at the end.

Roughly, it should look like

```
def random_matrix_experiment(n_trials = 1000, n=2):

    eig_list = []

    for trial in range(n_trials):
        this_matrix = random_symmetric(n)

        ## Get the eigenvalues and append to eig_list

    return eig_list
```

(**Part c**) A discrete approximation to a probability distribution function is called a *histogram*. Roughly speaking,

$$\lim_{\text{bin size}\to 0} (\text{histogram}) = \text{probability distribution}.$$

Use the matplotlib histogram function to create a plot of the distribution of eigenvalues for your random matrices. Can you guess the shape?

The result actually gets closer to the true shape as you make the size of the matrices larger. Try 10×10 or 20×20 matrices.

(**Part d**) You should find roughly the *semicircle distribution*, whose probability density function is

$$f(x) = \begin{cases} N\sqrt{R^2 - x^2} & \text{if } -R \le x \le R \\ 0 & \text{otherwise} \end{cases},$$

for some constant N which I have intentionally omitted. What must N be, in order for f to be a properly normalized probability distribution?

[Hint: integrate $f(x)$ and set the result equal to 1. Evaluate the integral using the trig sub $x = R\sin(\theta)$. You will get something that looks like a constant times $\int \cos^2(\theta)\, d\theta$ with some endpoints. Substitute using the double-angle identity $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$. Now the integral is easy; when the dust settles, you should find $N = \frac{2}{\pi R^2}$.]

What value of R in the semicircle distribution best describes the histogram you found for the eigenvalues of large $n \times n$ matrices?

(**Part e**) Clearly the average value of the semicircle distribution vanishes by symmetry,

$$\int_{-R}^{R} x\, f(x)\, dx = 0.$$

What is its second moment? That is, what is

$$\int x^2 f(x)\, dx = \frac{2}{\pi R^2}\int_{-R}^{R} x^2\sqrt{R^2 - x^2}\, dx\ ?$$

It is an interesting fact that all even higher moments of the semicircle distribution are related to the so-called "Catalan numbers."

Hint/Solution for (e). Do the trig sub $x = R\sin(\theta)$. Now you have $\frac{2R^2}{\pi}\int \sin^2(\theta)\cos^2(\theta)\, d\theta$, where I'll go back and figure out the endpoints later.

Now recall the double-angle identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, which we can square to find $\sin^2(2\theta) = 4\sin^2(\theta)\cos^2(\theta)$. So now, up to constants, you just need

$$\int \sin^2(2\theta)\, d\theta.$$

Now use the cosine double angle identity, $\cos(2\theta) = 1 - 2\sin^2(\theta)$, with the replacement $\theta \to 2\theta$. THis can be massaged into the form

$$\sin^2(2\theta) = \frac{1-\cos(4\theta)}{2}.$$

Putting the constants back in, you should find

$$\begin{aligned} \frac{2R^2}{\pi}\int \sin^2(\theta)\cos^2(\theta)\,d\theta &= \frac{2R^2}{\pi}\int \frac{1}{8}(1-\cos(4\theta))\ d\theta \\ &= \frac{2R^2}{\pi}\left(\frac{\theta}{8}-\frac{\sin(4\theta)}{32}\right), \end{aligned}$$

and now we need only plug in the endpoints. At the upper endpoint, $x = R$ so $\sin(\theta) = 1$ and $\theta = \frac{\pi}{2}$. At the lower endpoint, $x = -R$ so $\theta = -\frac{\pi}{2}$. So the second term gives us nothing and the first term gives us $\frac{\pi}{8}$, which cancels the π downstairs and kills the 2 upstairs to leave us with $\frac{1}{4}R^2$. Thus the answer is

$$\int x^2 f(x)\,dx = \frac{R^2}{4}.$$

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Updated 1 month ago by Christian Ferko

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that's a fresh meme

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