question 2 views

Daily Challenge 9.2

(**Due: Wednesday 7/11 at 12:00 noon Eastern**)

Today we will make another limit-related definition so that our vocabulary is less restrictive, then work towards proving some "arithmetic" properties of limits.

(1) A limit, if it exists, determines a number.

Yesterday, we defined the symbols

$$\lim_{x\to a} f(x) = L$$

to mean

$$\text{for all } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that } 0 < |x-a| < \delta \text{ implies } |f(x) - L| < \epsilon.$$

We only defined the entire combination of symbols "$\lim_{x\to a} f(x) = L$" together.

Suppose we wanted to also define the symbols "$\lim_{x\to a} f(x)$" appearing alone, without the "$= L$" at the end. One might attempt the following.

Definition. Suppose that $\lim_{x\to a} f(x) = L$. Then we define $\lim_{x\to a} f(x)$ as the number L.

Although this sounds reasonable, we could *not* have written down this definition before yesterday's daily challenge!

The problem occurs when we speak about "the" number L, which implies that there is only one such number. If it were possible that both $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$ for two distinct numbers L and L', then the above definition would not be consistent.

However, you proved in yesterday's problem that, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$, then we *must* have $L = L'$. Now that we know this, the above definition is sensible!

The alert reader will notice that, since we now have two definitions, the expression "$\lim_{x\to a} f(x) = L$" can be interpreted in two different ways:

1. If we use the original definition from daily challenge 9.1, this entire collection of symbols is short for the sentence "for any $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$."

2. If we use the new definition from today, then we interpret the symbols "$\lim_{x\to a} f(x)$" as "the unique number K, if it exists, with the property that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - K| < \epsilon$", we interpret the equals sign $=$ as the usual equality of real numbers, and we interpret L in the usual way as the number L.

Thankfully, the two interpretations are logically equivalent to one another, so our new definition is consistent in the theory we are building.

(2) Mathematics is built up carefully from simple definitions.

If you are not accustomed to doing serious mathematics, the above discussion may seem nit-picky.

You might ask "Why should we be so careful to define every word and symbol precisely, and prove obvious results like the statement that a function cannot approach two different points at once?"

Answer: proper mathematics is done using the **axiomatic method**.

This means that, when developing a new subject like calculus, we begin with the smallest and simplest set of assumptions and definitions, and then build up all of the conclusions step-by-step.

When you are asked to prove something, like the claim that a function cannot approach two different limits at the same point, the goal is not just to know that this claim is true **but to know that it can be proven from our definitions**.

Indeed, if we had chosen a different definition of limit, this statement could have been *false* -- or it could have even been *independent* of our definition, which means we can neither prove it true nor prove it false! One purpose of writing out the proof is to understand which of these possibilities holds.

There are several advantages to this method.

- **Elegance**. Mathematics trains your mind to build a complicated structure like calculus through logical deduction from a small number of starting assumptions. I find this deeply beautiful.

- **Modularity**. Suppose we wanted to generalize calculus by changing some of our definitions or assumptions. This could require a lot of work: we may have to build a new theory from the ground up!

  With the axiomatic method, though, we save a lot of effort. Since we have clearly listed our starting assumptions and definitions, we can simply keep the results which relied only on assumptions that we have *not* changed, and throw away all results which depend on assumptions that we *have* changed.

For example, if you go on to study topology, you will find that it *is* possible for a function to have two different limits at the same point, if the range of such a function sits in a so-called *non-Hausdorff space*.

- **Discovery**. Sometimes you find that you cannot prove something true, nor prove it false, using the definitions and assumptions you've written down.

  Often this means that your assumptions apply equally well to two different "parallel universes," one where the statement is true and one where it is false.

  The most famous example of this is in geometry. Euclid started with a small number of basic assumptions, called *postulates*, from which one can derive all of the plane geometry you learn in school. One of these, Euclid's fifth postulate, is

  "Given a line L and a point p not on L, at most one line parallel to L can be drawn through p."

  For many years, people thought that this assumption should be unnecessary; that is, they thought that you should be able to build up all of geometry from the other postulates, discarding this one.

  Remarkably, if you try to develop geometry *without* the fifth postulate, you cannot even prove that rectangles exist. If you try to do so, you end up with so-called Saccheri quadrilaterals.

  The reason for this is amazing: there is a *different* geometry, besides the ordinary kind we learn in school, which satisfies all of Euclid's postulates except the fifth postulate. In this new geometry, there are no quadrilaterals which have all four angles equal to 90°, so one could never have proven the existence of a rectangle.

  This alternative geometry is a *hyperbolic space*, and plays a prominent role in string theory.

  The punchline is that, if we had done geometry in a sloppy way without carefully working up from basic assumptions, we almost certainly would not have discovered the "parallel universe" of hyperbolic space.

(3) Limit proofs involve absolute values.

Now that we have defined the symbols $\lim_{x\to a} f(x)$ as a number, one might wonder what properties such numbers satisfy.

For instance, is it true that

$$\lim_{x\to a}\left(f(x)+g(x)\right)=\left(\lim_{x\to a} f(x)\right)+\left(\lim_{x\to a} g(x)\right),$$

so that we can "add" limits?

This will turn out to be true, but to prove it from our definitions, we will need to manipulate expressions like

$$|f(x)+g(x)-F-G|$$

which involve an absolute value of a sum of several terms.

Therefore, we will take a brief detour to define the absolute value function carefully so that we can prove things about it.

Definition. Let $x\in\mathbb{R}$. The *absolute value* of x, denoted by $|x|$, is the number

$$|x|=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0\end{cases}.$$

Since this function is defined differently in two different cases, most proofs of its properties will involve casework.

Proposition. Let $x\in\mathbb{R}$. Then $|x|\geq x$.

Proof. We consider two cases.

If $x\geq 0$, then $|x|=x$, and it is true that $x\geq x$, so the claim holds in this case.

On the other hand, if $x<0$, then $|x|=-x$. Multiplying both sides of the inequality $x<0$ by -1, we see that $-x>0$. So $x<0<-x=|x|$, and hence $|x|\geq x$ in this case.

Since the claim holds in both cases, and these two cases are the only possibilities, the claim holds in general. $\square$

Note that the proof worked by handling positive and negative numbers separately. The next claim is similar.

Proposition. Let $x\in\mathbb{R}$. Then $|-x|=|x|$.

Proof. There are two cases.

If $x\geq 0$, then $-x\leq 0$. From the definition of absolute value, this means that $|x|=x$ and $|-x|=-(-x)=x$. Thus in this case we have that $|-x|=|x|$.

Next suppose $x < 0$, which means $|x| = -x$. But since $x < 0$ we have $-x > 0$, which means $|-x| = -x$, so the equation also holds in this case.

Since there are only two possibilities (either $x \geq 0$ or $x < 0$) and the equation holds in both possible cases, it is true for any x. $\square$

Here is one more proposition which involves four cases.

Proposition. Let $x, y \in \mathbb{R}$. Then $|xy| = |x||y|$.

Proof. There are four cases.

- Suppose $x \geq 0$ and $y \geq 0$. Then $xy \geq 0$, so $|xy| = xy$. Likewise, $|x| = x$ and $|y| = y$, so the two sides are equal.
- Suppose $x < 0$ and $y \geq 0$. It follows that $xy \leq 0$, so $|xy| = -xy$. We also have $|x| = -x$ and $|y| = y$, so $|x||y| = -xy$, and the two sides are equal.
- Suppose $x \geq 0$ and $y < 0$. Then $xy \leq 0$, so $|xy| = -xy$, while $|x| = x$ and $|y| = -y$, so $|x||y| = -xy$, and the two sides are equal.
- Finally, suppose $x < 0$ and $y < 0$. In this case, $xy > 0$ so $|xy| = xy$, while $|x| = -x$ and $|y| = -y$. But $(-x)(-y) = xy$ since the minus signs cancel, so again the two sides are equal.

Since the proposed equation holds in all four possible cases, it is true for all x and y. $\square$

Notice that this proves my answer to your question in meeting 13, when you asked whether $\left|\frac{f(x)-L}{c}\right| = \frac{|f(x)-L|}{|c|}$, assuming $c \neq 0$. (Replace the variable x in the proposition by $f(x) - L$, and replace y by $\frac{1}{c}$, and then the two statements are the same.)

(4) Problem: the triangle inequality.

The problem has a "proof-reading part" and a "proof-writing part." In the second part, you will prove the *triangle inequality*, which (in its higher-dimensional form) is among the most famous inequalities in mathematics.

- (a) Read the three absolute value proofs in section (3) carefully. Pay particular attention to (i) the way each proof uses the definition of absolute value, and (ii) how these proofs handle each of multiple possible cases.

  By now, you know that writing a proof is a bit like playing a game of chess: there are certain moves that you are allowed to make at each step, and other moves which are "illegal" either because they are not logically justified or because they are not mathematically rigorous.

  If one uses only "legal steps," it is guaranteed that the conclusion of your theorem will be true whenever the hypotheses are true. This allows us to build up a rock-solid structure of logically sound statements, starting from very humble beginnings, with the certainty that our entire theory is true whenever the foundational assumptions are true.

  (b) Prove the following proposition, making sure that each sentence in your proof is a valid step of mathematical reasoning.

  Proposition. Let a, b be real numbers. Then $|a + b| \leq |a| + |b|$.

  If your proof does not use the piecewise definition of absolute value above in a central way, or if it does not explicitly handle multiple cases, it is certainly incorrect.

  [Hint: you need only consider two cases, namely $a + b \geq 0$ and $a + b < 0$. It will be helpful to use the results, proven above, that $|x| \geq x$ and $|-x| = |x|$ for any x.]

daily_challenge

Updated 9 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
Proof: We aim to prove that $|a+b| \leq |a| + |b|$ when a and b are real numbers.We begin by defining $c = a + b$

- $c \geq 0$, in this case the absolute value of c will be c by the definition of absolute value. Now we can start a chain: $|c| = c = a + b$. We can apply a recent proof $|x| \geq x$,and make $a + b \leq |a| + |b|$, which now allows us to attach the ends and create $|a + b| \leq |a| + |b|$.
- $c < 0$, in this case we can apply that since $c < 0$, then $|c| = -c = (-a) + (-b)$. Since $|x| \geq x$, then $|-b| = |b| \geq -b$ and $|-a| = |a| \geq -a$, therefore $(-a) + (-b) \leq |a| + |b|$, attaching the parts of this chain yields $|c| = -c = (-a) + (-b) \leq |a| + |b|$, therefore $|a + b| \leq |a| + |b|$.

We have now proven both possible cases true, therefore for all real numbers a and b, $|a + b| \leq |a| + |b|$. $\square$

Updated 9 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Proposition. Let a, b be real numbers. Then $|a + b| \leq |a| + |b|$.

Proof (Christian). We consider two cases.

- First suppose $a + b \geq 0$. Then by the definition of absolute value, $|a + b| = a + b$. But we have proven above that $|x| \geq x$ for all x, so in particular, $|a| \geq a$ and $|b| \geq b$. Thus $a + b \leq |a| + |b|$ and hence $|a + b| \leq |a| + |b|$, so the triangle inequality holds in this case.

- Now say $a+b<0$. The definition of absolute value prescribes that $|a+b|=-a-b$. We have proven that $|-x|=x$ for all x, so $|-a|=|a|$ and $|-b|=|b$. Furthermore, we know that $|y|\geq y$ for all y, so applying this to the two cases in the preceding sentence gives $-a\leq|-a|=|a|$ and $-b\leq|-b|=|b|$. Thus

  $$|a+b|=-a-b\leq|-a|+|-b|=|a|+|b|,$$

  so the triangle inequality holds in this case as well.

These are the only two possibilities, so it follows that $|a+b|\leq|a|+|b|$ for all $|a|$ and $|b|$. $\square$

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*