31.7

(a): Begin by plugging into

$x^3 - x + \epsilon = 0$

$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3)$

Then,

$0 = (x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3))^3 - (x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3))$

$= \cancel{(x_0^2 + \epsilon^2 x_1^2 + \epsilon x_0 x_1 + \epsilon^2 x_0 x_2}$

Ignore all terms with ϵ-power ≥ 2

$= (x_0 + \epsilon x_1 + \epsilon^2 x_2)(x_0 + \epsilon x_1 + \epsilon^2 x_2)\cdots$

$= (x_0^2 + 2\epsilon x_0 x_1 + 2\epsilon^2 x_0 x_2)(x_0 + \epsilon x_1 + \epsilon^2 x_2)$

$= (\underbrace{x_0^3} + \underbrace{\epsilon x_0^2 x_1} + \underbrace{\epsilon^2 x_0^2 x_2} + \underbrace{2\epsilon x_0^2 x_1} + 2\epsilon^2 x_0 x_1^2 +$

$\underbrace{2\epsilon^2 x_0^2 x_2}) - (\underbrace{x_0} + \underbrace{\epsilon x_1} + \underbrace{\epsilon^2 x_2}) + \underbrace{\epsilon}$

Implies

$0 = x_0^3 - x_0 \quad \Rightarrow x_0 = 0, -1, 1$

$0 = 3x_0^2 x_1 - x_1 + 1 \Rightarrow$ if $x_0 = 0$: $x_1 = 1$; if $x_0 = -1$: $x_1 = \frac{-1}{2}$; if $x_0 = 1$: $x_1 = \frac{-1}{2}$

$0 = 3x_0^2 x_2 - x_2 + 3x_0 x_1^2$; if $x_0 = 0$: $x_2 = 0$; if $x_0 = -1$: $x_2 = +3/8$; if $x_0 = 1$: $x_2 = -3/8$

$x_0 = -1$

$= +3x_2 - x_2 - 3x_1^2$

Hold ff up.

$= \cancel{4x_2}\, 2x_2 - 3\left(\frac{-1}{2}\right)^2$

$\frac{3}{4} = 2x_2$

$\frac{3}{8} = x_2 = \frac{3}{8}$

$x_0 = 1$

$0 = 3x_2 - x_2 + 3 \cdot \frac{1}{4}$

$\frac{-3}{4} = 2x_2$

$\frac{-3}{8} = x_2$

~~Let's try again with~~

$$\cancel{0 = 3 X_0 X_2 - X_2 + 2 X_0 X_1^2}$$

~~If $X_0 = 0$, $X_2 = 0$.~~

~~If $X_0 = 1$,~~

$$\cancel{0 = 3 X_2 - X_2 - 2 X_1^2}$$

We then have the roots

$$x = \epsilon + O(\epsilon^3)$$

$$x = -1 + \frac{-1}{2}\epsilon + 3/8\,\epsilon + O(\epsilon^3)$$

$$x = 1 + \frac{-1}{2}\epsilon - 3/8\,\epsilon + O(\epsilon^3)$$

(b): The hint suggests completing the square to see that

$$\frac{-1}{2} m^2 x^2 + J x = \cancel{\frac{-m}{2}\left(x - \frac{J}{m}\right)^2 +}$$

$$= \frac{-m^2}{2}\left(X - \frac{-J}{-m}\right)^2 - \frac{J^2}{-2m}$$

$$= \frac{-m^2}{2}\left(X - \frac{J}{m^2}\right)^2 + \frac{J^2}{2m^2}$$

Now let's look back to the original integral to see that

$$\int_{-\infty}^{\infty} \exp\left(\frac{-1}{2} m^2 x^2 + J x\right) = \int_{-\infty}^{\infty} \exp\left(\frac{-m^2}{2}\left(X - \frac{J}{m^2}\right)^2\right)\exp\left(\frac{J^2}{2m^2}\right)$$

$$= \exp\left(\frac{J^2}{2m^2}\right)\int_{-\infty}^{\infty} \exp\left(-\frac{m^2}{2}\left(X - \frac{J}{m^2}\right)^2\right)$$

The hint suggests u-sub; u-sub

$$u = \sqrt{\frac{m^2}{2}}\left(X - \frac{J}{m^2}\right)$$

$$du = \sqrt{\frac{m^2}{2}}\, dx \text{ ; thus}$$

$$\ldots = \sqrt{\frac{2}{m^2}}\exp\left(\frac{J^2}{2m^2}\right)\int_{-\infty}^{\infty} \exp(-u^2)$$

$$= \sqrt{\frac{2\pi}{m^2}}\exp\left(\frac{J^2}{2m^2}\right)$$

(c) Begin with the integral

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}m^2x^2 + Jx - \frac{\lambda}{4!}x^4\right)dx$$

$$=\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}m^2x^2 + Jx\right)\exp\left(\frac{-\lambda}{4!}x^4\right)$$

and using the Taylor series for exp(x),

$$=\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}m^2x^2 + Jx\right)\cancel{\exp}\left(1-\frac{\lambda}{4!}x^4+\frac{1}{2}\frac{\lambda}{4!}x^8+\cdots\right)$$

The hint then suggests noticing that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}m^2x^2 + Jx\right)x^{4n}\,dx = \left(\frac{d}{dJ}\right)^{4n}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}m^2x^2+Jx\right)$$

By replacing the x^{4n} with our series for exp,

$$\cdots = \left(1-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8-\cdots\right)\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}x^2m^2+Jx\right)dx$$

$$= \qquad = \qquad \sqrt{\frac{2\pi}{m^2}}\exp\left(\frac{J^2}{2m^2}\right)$$

by (b); then working in reverse,

$$\cdots = \exp\left(-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4\right)\exp\left(\frac{J^2}{2m^2}\right)\sqrt{\frac{2\pi}{m^2}} = Z(J).$$