question 3 views

Daily Challenge 9.1

(Due: Tuesday 7/10 at 12:00 noon Eastern)

The information below is the most important content that I've written on Piazza yet, as far as the development of the theory of calculus is concerned. To emphasize this, I will stop labeling things as "Review" and start titling each section by its content.

This reading is longer and more challenging than others, but I am confident in your ability to understand all of it. Questions are especially welcome.

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(1) The word "limit" has just one specific meaning.

Before the break, we introduced the central definition of calculus: whenever we speak the words

> "the limit of $f(x)$ as x approaches a equals L"

or whenever we write the symbols

> "$\lim_{x \to a} f(x) = L$"

what we really mean is the longer sentence

> "for any $\epsilon > 0$ I can find a $\delta > 0$ such that, if $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$".

It is important to realize that *this is the only use of the word "limit" which has meaning so far.*

Although we have some intuitive idea that a limit describes what happens when a function's input "gets close to" some point, **this intuition is banned** from any formal argument.

Words like "gets close to" absolutely cannot appear in a proof.

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(2) Math definitions are like Python variables.

Think of the collection of symbols "$\lim_{x \to a} f(x) = L$" as a variable in Python. A variable is just a *placeholder* for something.

If you wrote a program and declared `x=5` on some line, then later wrote an expression like `x>10`, Python would automatically unpack the symbol `x` into the value `5` and convert the phrase `x>10` into the value `False`.

Likewise, in any theorem or proof, you should view the symbols "$\lim_{x \to a} f(x) = L$" as a variable like `x`, which your brain should unpack into the sentence "for any $\epsilon > 0$ I can find a $\delta > 0$ such that, whenever $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$".

I cannot stress this enough, so I will say it again.

> **Important**: The symbols $\lim_{x \to a} f(x) = L$ are simply an abbreviation for the statement "for any $\epsilon > 0$ I can find a $\delta > 0$ such that, whenever $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$".
>
> They have *no other meaning* than this.

Notice that we have only defined the symbols "$\lim_{x \to a} f(x) = L$" collectively, as one unit. We have *not* yet offered any definition for "$\lim_{x \to a} f(x)$", without the "$= L$" at the end.

That means that, until we write a new definition, the symbols "$\lim_{x \to a} f(x)$" are officially meaningless to us. We only know how to talk about the entire combination "$\lim_{x \to a} f(x) = L$".

This seems strange, but is totally sensible within the Python analogy: if I define a variable `number_one` and then later try to use a variable called `number`, Python will throw an error. Likewise, we have only defined "$\lim_{x \to a} f(x) = L$", so if we try to talk about "$\lim_{x \to a} f(x)$", the logical theory that we are building will also throw.

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(3) Proofs may use only definitions and mathematical reasoning.

Here is an example problem about limits, along with a correct solution.

Proposition. Suppose that $\lim_{x\to a} f(x) = L$. Then $\lim_{h\to 0} f(a+h) = L$.

Proof. Let $\epsilon > 0$ be given. By the definition of limit, we must prove that we may find some number δ_1 with the property that, if $|h| < \delta_1$, then $|f(a+h) - L| < \epsilon$.

However, since we have assumed $\lim_{x\to a} f(x) = L$, we may find some δ_2 such that, whenever $0 < |x-a| < \delta_2$, it holds that $|f(x) - L| < \epsilon$, where ϵ is the same as in the preceding paragraph.

We choose $\delta_1 = \delta_2$. To see that this satisfies the definition of the limit, suppose that $0 < |h| < \delta_1$. Then certainly $0 < |(a+h) - a| < \delta_1$, since we have simply added zero in the form $a - a$ inside the absolute value. Applying the condition that $0 < |x-a| < \delta_2$ implies $|f(x) - L| < \epsilon$ when $x = a+h$, this means that $|f(a+h) - L| < \epsilon$, as desired. $\square$

The above problem is mostly an exercise in understanding what the words mean. Let's review the solution.

We assume that a given limit exists, namely that $\lim_{x\to a} f(x) = L$, and we translate this assumption into its precise definition in terms of epsilons and deltas. That is,

$$\text{Assume: for every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon.$$

Then we wish to prove that a different limit exists, i.e. that $\lim_{h\to 0} f(a+h) = L$, which means

$$\text{Want to show: for every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |h| < \delta \implies |f(a+h) - L| < \epsilon.$$

Then we invoke the existence of the first limit: if you hand me an ϵ, I pick the corresponding δ so that $0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$. This sentence is true for *any* value of x. In particular, it holds when $x = a+h$. That means

$$0 < |(a+h) - a| < \delta \implies |f(a+h) - L| < \epsilon\ .$$

But this is exactly what we want to prove.

The actual algebraic manipulation needed to obtain the second condition was quite simple: we merely re-wrote the condition $0 < |h| < \delta_1$ as $0 < |(a+h) - a| < \delta_1$. The main hurdle in solving this problem is conceptual.

Next let's consider the following proposition and a proof which is *incorrect*.

Proposition 1. The limit of $f(x) = x$ as x approaches 2 equals 2.

Incorrect proof. The function $f(x) = x$ always returns an output value equal to the input value. If we plug in 2, then $f(2) = 2$. Likewise, if we plug in any value close to 2, like 1.999, the function will also output a value close to 2. Therefore the output of $f(x)$ gets close to 2 as the input gets close to 2, which means that $\lim_{x\to 2} f(x) = 2$. $\square$

This is not a proof.

To clarify, I don't mean that the statements are *wrong*; every sentence in this fake proof is actually true. Rather, it is not even wrong: this attempt is not even an incorrect proof because it does not have the necessary precision to qualify as a proof.

Imagine that I was asked to write a Python program to find prime numbers, and instead I wrote an explanation describing how to check primality by looking for divisors. The explanation is not *wrong* -- it may even be quite good, as far as explanations are concerned -- but it is simply not a Python program.

Returning to the example above, the incorrect proof seems not to understand that the words

> "The limit of $f(x) = x$ as x approaches 2 equals 2"

are **not ordinary English words that we can write sentences about**; the entire sentence, taken together as one entity, is simply a placeholder for the mathematical statement

> "for any $\epsilon > 0$ I can find a $\delta > 0$ such that, whenever $0 < |x-2| < \delta$, it is true that $|f(x) - 2| < \epsilon$".

Here is a correct proof.

Proposition 1. The limit of $f(x) = x$ as x approaches 2 is equal to 2.

Proof. Let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. Whenever $0 < |x-2| < \delta$, we also have $0 < |f(x) - 2| < \delta$, since $f(x) = x$. But because $\delta = \epsilon$, this means that $|f(x) - 2| < \epsilon$. By the definition of the limit, then, we conclude that $\lim_{x\to 2} f(x) = 2$. $\square$

This proof understands the definition of the limit. One can imagine the author's thoughts during the exploration phase, before writing the above argument. Perhaps he reads the sentence

> "The limit of $f(x) = x$ as x approaches 2 equals 2"

and thinks to himself

> "Aha, so this collection of characters is really shorthand for the statement that, for any $\epsilon > 0$, one can find a $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|f(x) - 2| < \epsilon$.
>
> To prove that this is true, I had better describe how to pick such a number δ, if ϵ is given, which satisfies the given property. Only when I have described such a procedure have I actually proven the statement."

This is how you should think before writing a proof: ask yourself "what mathematical statement are these words a placeholder for, and how do I show that this mathematical statement is true?"

(4) Misunderstanding definitions leads to incorrect proofs.

Here is another claim and two incorrect proofs.

Speaking imprecisely, the claim roughly says "a function cannot have two different limits at the same point." Of course, we have not defined the words "two different limits" or "the same point", so the preceding sentence is officially meaningless. We must state this more carefully.

Proposition 2. Let f be a real-valued function and let $a, L, L' \in \mathbb{R}$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$. Then $L' = L$.

Incorrect proof A. We are given that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$. By the transitive property, if the left sides are equal, the right sides are equal, so $L = L'$. $\square$

No. If a student in one of my UChicago courses wrote this, I would give 0 points and ask him to come talk to me.

I agree that, given two equations $p = q$ and $q = s$, we can conclude that $p = s$. But that is *totally irrelevant* in this proof.

In the context of this proof, the symbol "$\lim_{x\to a} f(x) = L$" **is not an equation** between one object $\lim_{x\to a} f(x)$ and another object L, like the equation $p = q$. We have not even *defined* the symbols "$\lim_{x\to a} f(x)$" appearing alone, without the "$= L$ " at the end.

Do not let the equals sign fool you; until we state another definition, the symbols $\lim_{x\to a} f(x) = L$ only mean one thing:

> "for any $\epsilon > 0$ I can find a $\delta > 0$ such that, whenever $0 < |x - a| < \delta$, it is true that $|f(x) - L| < \epsilon$".

That's all they mean, and *only* the precise combination of symbols $\lim_{x\to a} f(x) = L$ has this meaning. Until we argue otherwise, it is not an equation; it is a placeholder for this mathematical sentence.

(Aside: *after* we've completed this proof, we are free to define the symbols $\lim_{x\to a} f(x)$ as the unique number L such that the statement $\lim_{x\to a} f(x) = L$ is true. But this is a different definition; so far, we have defined the symbols $\lim_{x\to a} f(x) = L$, but not the symbols $\lim_{x\to a} f(x)$.)

Here is a slightly less bad, but still very incorrect, proof.

Proposition 2. Let f be a real-valued function and let $a, L, L' \in \mathbb{R}$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$. Then $L' = L$.

Incorrect proof B. Suppose, by way of contradiction, that $L' \neq L$. Since $\lim_{x\to a} f(x) = L$, we know that the function $f(x)$ gets close to L as x approaches a. Likewise, since $\lim_{x\to a} f(x) = L'$, we know that the function $f(x)$ gets close to L' as x approaches a. But if $L \neq L'$, the function cannot get close to both numbers at once.

Indeed, if we find a δ so that $|f(x) - L| < \epsilon$, then with that same δ we cannot have $|f(x) - L'| < \epsilon$. Since the second limit exists, we could also find a δ' so that $|f(x) - L'| < \delta'$, but we have no guarantee that $\delta = \delta'$. Therefore the limit does not exist. $\square$

No. If a math or physics major wrote this, I would award something like 3 points out of 10.

The first paragraph of this incorrect proof might be appropriate for the exploration phase, since it correctly captures the intuition that a function cannot get arbitrarily close to two different numbers at once. But any sentence involving words like "gets close" are absolutely inappropriate in a proof.

The second paragraph of this incorrect proof is total nonsense; it completely misunderstands the roles of δ and ϵ in the definition of the word "limit".

In fact, this second paragraph sounds a lot like one math professor's description of an underperforming student:

> He was so lost that his homework assignments were neither right nor wrong— they were simply nonsense. He merely recycled math terms that he wrote down during class discussions without even knowing their meaning. It was as if he were writing a poem in a language that he himself did not understand.

Punchline: never write proofs that look like the second paragraph above. Don't "bullshit" and write words without understanding their meaning. It is better to write a partial proof, where you clearly understand each step and can explain where you got stuck, than to fake it and write a poem in a language that you don't understand.

(5) Problem: correctly proving proposition 2.

Read the first four sections above *slowly* and *carefully*. I recommend reading the whole thing twice. If you don't understand this, you will be lost as we move forward.

Once the above is crystal clear, **write a logically correct proof of proposition 2**. I have reproduced it below for convenience.

Proposition 2. Let f be a real-valued function and let $a, L, L' \in \mathbb{R}$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$. Then $L' = L$.

Be sure not to fall into the traps of the first and second incorrect proofs (namely, failing to understand what the symbols $\lim_{x\to a} f(x) = L$ mean).

[Hint: suppose $L \neq L'$, choose two δ's appropriately, and obtain a contradiction. In the final step of your proof, you may find it helpful to use the fact that, for any two real numbers a and b, it is true that $|a+b| \leq |a| + |b|$. This is called the *triangle inequality*, and you will prove it tomorrow.]

daily_challenge

Updated 9 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski:
Proof: Suppose that that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = L'$, but by way of contradiction suppose $L' \neq L$. We can treat the limits of these functions with the same ϵ, in which case I shall subsequently define $\epsilon = \frac{|L'-L|}{2}$.
By the definition of limit, there exists a δ_1 such that $0 < |x-a| < \delta_1 \implies |f(x) - L| < \epsilon$
Similarly, there exists a δ_2 such that $0 < |x-a| < \delta_2 \implies |f(x) - L| < \epsilon$
We can now unite these implications by defining that $\delta = \min(\delta_1, \delta_2)$ and supposing there exists a value x_0 such that $0 < |x_0 - a| < \delta$ is true. We aim to show that the value x_0, while satisfying the previously described condition, does not satisfy the one it implies. Recall $\epsilon = \frac{|L'-L|}{2}$, and therefore $2\epsilon = |L' - L|$. We can now apply the first of the most important rules in algebra, adding zero. We can both add and subtract $f(x_0)$ within this absolute value to receive $2\epsilon = |(L' + f(x_0)) - (L + f(x_0))|$. By the triangle inequality, $|(L' + f(x_0)) - (L + f(x_0))| \leq |f(x_0) - L'| + |f(x_0) - L|$We can now use the implications made by the original limits to see that $|L' + f(x_0)| < \epsilon$ and likewise $|L + f(x_0)| < \epsilon$, therefore these are collectively less that 2ϵ, ie $|L' + f(x_0)| + |L + f(x_0)| < 2\epsilon$ Comparing the start and end of this "chain" shows that $2\epsilon < 2\epsilon$. This is false for obvious reasons, therefore L must equal L'. $\square$

Updated 9 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

Proposition 2. Let f be a real-valued function and let $a, L, L' \in \mathbb{R}$. Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = L'$. Then $L' = L$.

Proof (Christian). Suppose by way of contradiction that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = L'$ but $L' \neq L$.

We define $\epsilon = \frac{|L'-L|}{2}$. By the definition of limit, we may find a number δ_1 such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \epsilon$$

and a number δ_2 such that

$$0 < |x-a| < \delta_2 \implies |f(x) - L'| < \epsilon.$$

Now let $\delta = \min(\delta_1, \delta_2)$ and consider some x_0 with the property $0 < |x_0 - a| < \delta$. We will show that the value $f(x_0)$ satisfies a contradictory condition.

We have the chain of inequalities

$$\begin{aligned} 2\epsilon &= |L - L'| \\ &= |(f(x_0) - L') - (f(x_0) - L)| \\ &\leq |f(x_0) - L'| + |f(x_0) - L| \\ &< 2\epsilon. \end{aligned}.$$

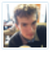
In the first step, we have used that $\epsilon = \frac{|L'-L|}{2}$; in the second step, we have added and subtracted $f(x_0)$ inside the absolute value; in the third step, we have used the triangle inequality $|a+b| \leq |a| + |b|$; in the final step, we have used that $\delta \leq \delta_1$ and $\delta \leq \delta_2$, so $|f(x_0) - L| < \epsilon$ and $|f(x_0) - L'| < \epsilon$ by the two limit assumptions above.

Taken together, this calculation shows $2\epsilon < 2\epsilon$, which yields the desired contradiction. $\square$

Updated 9 months ago by Christian Ferko

followup discussions *for lingering questions and comments*

Resolved Unresolved



Christian Ferko 9 months ago
Good, the student response is about a 5/6 on our rubric -- essentially correct up to small (probably typographical) errors.