question 4 views

Daily Challenge 2.5

(**Due: Saturday 5/5 at 12:00 noon Eastern**)

Let's try something new: I will ask you to analyze a claim and a supposed "proof", then assign one of three grades.

- Assign a grade of A (excellent) if the claim and proof are both correct (even if the proof is not the most elegant or the way you would have done it).
- Award a grade of C (partial credit) for a proof that is largely correct but contains one or two incorrect statements or justifications.
- Give a grade of F (failure) if the claim is false, the main idea of the proof is wrong, or most of the statements are incorrect.

For any grade besides A, explain what parts of the proof are incorrect and why.

I'll grade three examples for you and ask you to grade two more.

Example 1

Claim. Suppose x is a positive real number. The sum of x and its reciprocal is greater than or equal to 2; that is, $x+\frac{1}{x}\geq 2$.

"Proof." Multiplying by x, we get $x^2+1\geq 2x$. By algebra,

$$x^2-2x+1\geq 0,$$

which is the same as

$$(x-1)^2\geq 0\cdot$$

Any real number squared is greater than or equal to zero, so $x+\frac{1}{x}\geq 2$ is true. □

Grade: F. The claim is correct, but the proof is trash -- it's a circular argument! In the first step, the author takes the *conclusion* that he's trying to prove (namely, that $x+\frac{1}{x}\geq 2$) and multiplies both sides by x. We cannot use the result we'd like to show in the proof itself.

A correct proof, for instance, might proceed by contradiction: assume $x+\frac{1}{x}<2$, then apply the same steps above to arrive at $(x-1)^2<0$, which is false. Alternatively, one could begin with the observation that $(x-1)^2\geq 0$ and reverse the steps presented in the incorrect proof.

Example 2

Claim: Suppose A, B, C are sets. If $A\subseteq B$, then $A\setminus C\subseteq B\setminus C$.

"Proof." Assume $A\subseteq B$ and suppose $x\in A$. Then $x\in B$, since $A\subseteq B$. For any set C, $x\in A$ and $x\notin C$. Then $x\in B$ and $x\notin C$. Thus $x\in A\setminus C$ and $x\in B\setminus C$. Therefore, $A\setminus C\subseteq B\setminus C$. □

Grade: C. This proof begins by supposing $x\in A$, but later asserts that $x\notin C$ with no justification. This was a strange choice: beginning with the step $x\in A$ is not helpful in proving that $A\setminus C\subseteq B\setminus C$. Rather, the author should have begun with the sentence:

> "Assume $A\subseteq B$ and suppose $x\in A\setminus C$."

With this change, the remainder of the proof would have been correct.

In general, to prove that $X\subseteq Y$, one must begin by choosing an arbitrary element of X (for instance, by saying "Let $x\in X$.") and then showing that this element also belongs to Y.

Example 3

Claim. If x is any real number, then either $\pi-x$ or $\pi+x$ is irrational.

"Proof." It is known that π is an irrational number; that is, π cannot be written in the form $\frac{a}{b}$ for integers a and b. Let $x\in\mathbb{R}$. Suppose, by way of contradiction, that both $\pi-x$ and $\pi+x$ are rational. Since the sum of two rational numbers is always rational, we have that

$$(\pi-x)+(\pi+x)=2\pi$$

is rational, so $2\pi=\frac{a}{b}$ for integers a and b. This means $\pi=\frac{a}{2b}$ so π is rational. This is a contradiction. Therefore, at least one of $\pi-x$ or $\pi+x$ is irrational. □

Grade: A. This proof is correct as written.

Problem

Your turn: assign grades to each of the following "proofs", and explain which parts are incorrect if you assign a grade other than A.

(a) **Claim**. Suppose m is an integer. If m^2 is odd, then m is odd.

"Proof." Assume m is odd. Then $m=2k+1$ for some integer k. Therefore,

$$m^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1,$$

which is odd because it is of the form $2a+1$ for an integer a (specifically, $a=2k^2+2k$). Therefore, if m^2 is odd, m is odd. □

(b) **Claim**. If A, B, C are sets, and $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

"Proof." Suppose x is any object. If $x \in A$, then $x \in B$, since $A \subseteq B$. If $x \in B$, then $x \in C$, since $B \subseteq C$. Thus, $x \in C$. Therefore, $A \subseteq C$. $\square$

daily_challenge

Updated 11 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Grades (Corbin).

(a) Grade:

Explanation of incorrect parts (if grade isn't an A):

(b) Grade:

Explanation of incorrect parts (if grade isn't an A):

Grades (Logan).

(a) Grade: C
I give this proof a C because while it does make good logical points, it is algebraically flawed in the statement $4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

(b) Grade: A

Updated 11 months ago by Logan Pachulski and Christian Ferko

the instructors' answer, *where instructors collectively construct a single answer*

Grades (Christian).

(a) Grade: **F**.

Explanation: The "proof" is entirely backwards; it shows the converse of the theorem rather than the theorem itself.

The claim was that

$$\text{if} \left(m^2 \text{ is odd} \right) \text{ , then } \left(m \text{ is odd} \right) .$$

However, the "proof" begins by assuming that m is odd and then proves that m^2 is odd. In other words, the argument establishes that

$$\text{if} \left(m \text{ is odd} \right) \text{ , then } \left(m^2 \text{ is odd} \right) .$$

Thus this "proof", while logically correct, doesn't actually argue for the claim which it wanted to show.

(b) Grade: **C**.

Explanation: This "proof" has the right idea, but it is sloppy in speaking about the variable x and thus is technically incorrect.

To show that $A \subseteq C$, we must consider some element $x \in A$ and prove that $x \in C$. But this "proof" begins by saying

> "Let x be any object."

In the remainder of the proof, the author establishes other implications involving x in individual sentences, but these hold only "locally" and not "globally". For instance, the author later says

> "If $x \in A$, then $x \in B$, since $A \subseteq B$."

This is logically correct, but we haven't actually assumed that $x \in A$ in the proof. In particular, the later statement

> "Thus, $x \in C$."

is totally unjustified since we haven't assumed anything about x.

The proof would be completely correct if the first sentence were replaced with "Let $x \in A$."

Updated 11 months ago by Christian Ferko

followup discussions *for lingering questions and comments*