question 2 views

Daily Challenge 23.7

(**Due: Thursday 3/7 at 12:00 noon Eastern**)

In the last session, we saw how to find the total mass of an object with varying density:

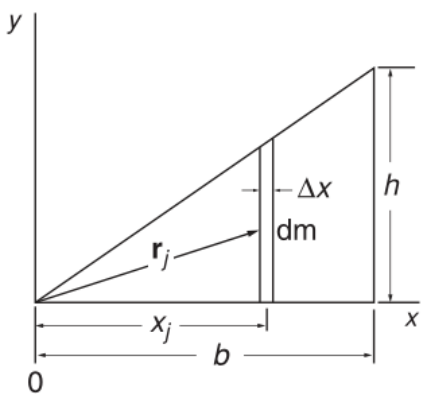
$$M = \begin{cases} \int \lambda(x)\,dx & \text{if one-dimensional} \\ \int \sigma(x,y)\,dx\,dy & \text{if two-dimensional} \\ \int \rho(x,y,z)\,dx\,dy\,dz & \text{if three-dimensional} \end{cases}$$

along with the center of mass coordinates $(x_{\text{com}}, y_{\text{com}}, z_{\text{com}})$, which I will write only in the three-dimensional case for simplicity:

$$\begin{aligned} y_{\text{com}} &= \frac{\int y\rho(x,y,z)\,dx\,dy\,dz}{M} \\ x_{\text{com}} &= \frac{\int x\rho(x,y,z)\,dx\,dy\,dz}{M}, \\ z_{\text{com}} &= \frac{\int z\rho(x,y,z)\,dx\,dy\,dz}{M} \end{aligned}$$

In this problem, we'll find the center of mass for a triangular plate with total mass M, base b, height h, and a small thickness t.

For simplicity, assume the density is uniform and equal to ρ (in units of mass per cubic meter). That means $\rho = \frac{M}{\frac{1}{2}bht}$, since the total volume is $\frac{1}{2}bht$.



(a) Divide the triangle into strips of width Δx parallel to the y axis, as shown in the figure.

The j-th strip at position x_j has its center of mass halfway up, because the plate is uniform, and the total height of the j-th strip is $\frac{x_j h}{b}$ by similar triangles.

Use this to write the position vector $\vec{r}_j$ to the position of the **strip**'s center of mass for the j-th strip. (Answer: $\vec{r}_j = x_j\hat{i} + \frac{x_j h}{2b}\hat{j}$).

(b) Now the center of mass vector is defined by

$$\vec{R}_{\text{com}} = \frac{1}{M}\int \vec{r}\,dm,$$

where

$$\begin{aligned} M &= \rho A t = \frac{1}{2}\rho bht, \\ dm &= \rho t y\,dx = \rho t\frac{xh}{b}\,dx. \end{aligned}$$

Combine the equations above to show that the center of mass integral can be written as

$$\vec{R}_{\text{com}} = \left(\frac{2}{\rho tbh}\right)\int_0^b \vec{r}\rho t\frac{xh}{b}\,dx.$$

(c) Replace $\vec{r} = x\hat{i} + \frac{xh}{2b}$ in the integral of part (b), which is simply the expression for the center of mass for each vertical strip that you found in part (a).

Do the integral. Note that the integral of a vector splits into components:

$$\int \left(f\,\hat{i} + g\,\hat{j}\right)\,dx = \left(\int f\,dx\right)\hat{i} + \left(\int g\,dx\right)\hat{j}\,.$$

Show that the result is $\vec{R}_{\text{com}} = \frac{2}{3}b\hat{i} + \frac{1}{3}h\hat{j}$.

daily_challenge

Updated 1 month ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

~ An instructor (Christian Ferko) endorsed this answer ~

Updated 1 month ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

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