4 / 8 /2019

26.3

(a): Given some

$$f(x, y(x), y'(x)) = \sqrt{1+y'(x)^2},$$

and recalling the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

we see that

$$\frac{\partial f}{\partial y} = 0 \text{ and}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+y'(x)^2}} \cdot 2y'(x) = \frac{y'(x)}{\sqrt{1+y'(x)^2}}$$

and in turn

$$\cancel{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{y''(x)\sqrt{1+y'(x)^2} + y'(x)2(1+y'(x)^2)^{-1/2} \cdot y''(x)}{1+y'(x)^2}}$$

Then since

$$+\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

then

$$\frac{\partial f}{\partial y'} = C, \text{ a constant; } \frac{y'(x)}{\sqrt{1+y'(x)^2}} = C$$

thus,

$$y'(x) = C\sqrt{1+y'(x)^2} \Rightarrow y'(x)^2 = C^2(1+y'(x)^2)$$
$$= C^2 + C^2y'(x)^2$$
$$0 = C^2 + (C^2-1)y'(x)^2$$

Refer to page 2

Recall

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}, \quad \cancel{\frac{\pm\sqrt{-4C^2(C^2-1)}}{2C^2}} \quad \frac{\pm\sqrt{-4(C^2-1)C^2}}{2C^2-2} = y'(x)$$

(b): Once again, let

$$f(t, x(t), x'(t)) = \left(\frac{1}{2} m \dot{x}^2 - U(x)\right)$$

and by the Euler-Lagrange equation

$$\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial \dot{x}} = 0,$$

See that

$$\frac{\partial f}{\partial x} = -U'(x) \cdot \dot{x}$$

$$\frac{\partial f}{\partial \dot{x}} = m\dot{x},$$

then algebraically

$$\frac{U(x)}{\partial x} = m\ddot{x}$$

$$F = ma$$

a cont. We have that

$$y'(x) = c\sqrt{1+y'(x)^2}$$

Square each side,

$$y'(x)^2 = c^2\left(1+y'(x)^2\right) \Rightarrow 0 = c^2 + c^2 y'(x)^2 - y'(x)^2$$

$$= (c^2-1)\,y'(x)^2 + 0\,y'(x) + c^2$$

Apply the quadratic formula:

$$y'(x) = \pm \frac{0 + \sqrt{-4(c^2-1)c^2}}{2(c^2-1)} = \pm \frac{i\sqrt{(c^2-1)c^2}}{c^2-1}$$

then

$$y'(x)^2 = \frac{-(c^2-1)c^2}{(c^2-1)^2} = \frac{-c^2}{c^2-1}$$

$$y'(x) = \pm \frac{ic}{\sqrt{c^2-1}}$$