question 3 views

Daily Challenge 14.5

~~(**Due: Monday 9/17 at 12:00 noon eastern**)~~

(**Due: Tuesday 9/18 at 12:00 noon eastern**)

Let's get our hands dirty with tangent line approximations today. Soon I'll ask you to implement Newton's method in Python.

(1) The tangent line is a good approximation to the function nearby.

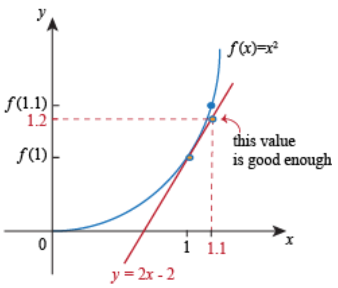
We saw in the parable of Feynman versus the abacus that tangent line approximations are very convenient when one needs a rough estimate of a function.

The idea is this: choose a point a for which you know the value of a function and its derivative. Then write down the equation of the tangent line to f at a. We know the slope of such a line is $f'(a)$ and that the point $(a, f(a))$ lies on the line, so a routine application of kindergarten algebra gives

$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$

for our approximation.

Graphically, we see that the tangent line is only expected to be a "good" approximation to the function near the point a (we will be able to be much more precise about what the word "good" means once we understand Taylor expansions).



Intuitively, we also expect that the approximation will be **too low if** $f''(a) > 0$, as in the picture above, because the graph "curves up and away" from the tangent line, which means the linear approximation is smaller than the true value.

Likewise, we expect the tangent line approximation to be **too high if** $f''(a) < 0$, since in this case the graph "curves down and away" from the tangent line, meaning our result is too large.

(2) Problem: two applications of tangent line approximations.

(a) Estimate $\sqrt{1.1}$ without a calculator. Is your estimate too large or too small? Why?

(b) The *rule of 72* says that, if an amount of money is invested at $r\%$ interest per year, then it will take approximately $\frac{72}{r}$ years for the money to double.

Use a tangent line approximation to explain why this rule is a good estimate for small values of r.

[Hint: first recall, figure out, or look up the formula for the amount of money you would have after t years if you initially invest a principal of P dollars at $r\%$ interest. This is usually discussed in algebra II. Then solve for the value of t which yields a total wealth of $2P$, and approximate that expression for small r.]

daily_challenge

Updated 7 months ago by Christian Ferko

the students' answer, *where students collectively construct a single answer*

Logan Pachulski

a: We see for $a = 1$ that when $f(x) = \sqrt{x}$ then $f(1.1) \approx 1 + \frac{0.5}{1} \cdot (0.1) \approx 1.05$, and this is just slightly too high, since Ione can see that the second derivative at 1 is concave and therefore the function itself is slightly lower than the approximation. A very accurate result nonetheless.

b: We begin by referring the compound interest formula, $B(t) = P \cdot (1 + \frac{r}{100})^{nt}$, and in turn we would like to solve the equation $2P = P \cdot (1 + \frac{r}{100})^t$ or $2 = (1 + \frac{r}{100})^{nt}$. We take the log of each side and see $\log(2) = t \cdot \log(1 + \frac{r}{100})$, and the form of interest follows, $t = \frac{\log(2)}{\log(1 + \frac{r}{100})}$. We can then approximate the expression in the denominator for some small $x = \frac{r}{100}$ and see that $\log(1 + x) \approx \log(1) + x = x$, therefore $t = \frac{\log(2)}{\frac{r}{100}} = \frac{100 \log(2)}{r} \approx \frac{72}{r}$.

Updated 6 months ago by Logan Pachulski

the instructors' answer, *where instructors collectively construct a single answer*

(a) We would like to approximate $f(x)=\sqrt{x}$ near $x=1$. The tangent line approximation is

$$f(1.1)\approx f(1)+f'(1)\cdot 0.1,$$

but $f'(x)=\frac{1}{2\sqrt{x}}$, so $f'(1)=\frac{1}{2}$ and we get

$$f(1.1)\approx 1+(0.1)(0.5)=1.05.$$

To determine whether this approximation is too big or too small, consider the second derivative. Since $f''(1)=-\frac{1}{4}<0$, the graph curves "down and away" from the tangent line, so our estimate is too high.

(b) The formula for one's total wealth after t years having invested principal P with an annually-compounded interest rate of $r\%$ is

$$W(t)=P\left(1+\frac{r}{100}\right)^t.$$

We seek to estimate the *doubling time* t for which the total wealth is twice the principal, i.e. $W=2P$. This means

$$\left(1+\frac{r}{100}\right)^t=2,$$

or taking logs and solving for t,

$$t=\frac{\log(2)}{\log\left(1+\frac{r}{100}\right)}.$$

Now we use a tangent line approximation for $\log(1+x)$ where x is small. Let $f(x)=\log(x)$ so $f'(x)=\frac{1}{x}$. Then

$$\log(1+x)\approx\log(1)+x=x.$$

This gives an approximation for the doubling time, namely

$$t=\frac{\log(2)}{\log\left(1+\frac{r}{100}\right)}\approx\frac{100\log(2)}{r}.$$

But $100\log(2)\approx 69.3$, which is close to 72, which justifies the "rule of 72" that the doubling time is roughly $t\approx\frac{72}{r}$.

You might wonder why we chose 72 rather than 69 (giggity), which is a more accurate estimate. I'm not sure, but my guess is that investment advisers prefer the number 72 because it is divisible by a lot of small numbers, and we typically think about integer interest rates like 2%, 3%, 4%, 6%, etc. The nice thing about 72 is that it is divisible by all of these, so one can get a quick-and-dirty estimate for the doubling time which is an integer number of years (e.g. a 6% interest rate gives an approximate doubling time of $\frac{72}{6}=12$ years).

Updated 6 months ago by Christian Ferko

followup discussions *for lingering questions and comments*