5 13 2019

29.6

(a): We know ~~th~~ by the integral test that if the series of some sequence exists, then the integral exists; We must show that

$$\sum_{n=1}^{\infty} \frac{e^n}{n^n} \text{ exists.}$$

Notice that

$\lim_{n \to \infty} \frac{e^n}{n^n} = 0$ since n^n grows faster than e^n.

Then one necessary ~~necess ary~~ condition is satisfied for this to converge.

Now to show

$$\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n \text{ converges.}$$

Apply the limit test where we compare to the known convergent geometric series $\left(\frac{1}{2^n}\right)$

Evaluate it ~~lim~~

$$\lim\left(\frac{\frac{1}{2^n}}{e^n/n^n}\right) = \lim\left(\frac{n^n}{2^n e^n}\right) = \lim\left(\left(\frac{n}{2e}\right)^n\right)$$

Clearly the limit blasts off into infinity due to variable numerator and constant denominator; thus

$$\lim\left(\frac{1/2^n}{e^n/n^n}\right) = \infty \Rightarrow \sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n \text{ converges.}$$

(b): We see by the integral test that

$$\sum_{n=2}^{\infty} \frac{1}{\log^{\log(n)}(n)} \text{ converges if } \lim_{c \to \infty} \int_2^{c} \frac{1}{(\log^{\log(x)}(x))} dx \text{ exists.}$$

Let's try u-subbing $u = \log(x)$, $du = \frac{1}{x} dx$, $dx = e^u du$

$$\ldots \lim_{c \to \infty} \int_2^{c} \frac{e^u}{u^u} du$$

We proved that this integral exists in (a); thus the sum

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)^{\log(n)}} \text{ converges.}$$

(c): Apply the integral test to see that ~~if~~

$\sum_{n=2}^{\infty} \frac{1}{\log(n)^{\log(\log(n))}}$ converges if $\lim_{c \to \infty} \int_2^{c} \frac{1}{\log(n)^{\log(\log(n))}}$ exists

u-sub $u = \log(x)$, $du = \frac{1}{x} dx$, $dx = e^u du$,

$\ldots \lim_{c \to \infty} \int_2^{c} \frac{e^u}{u^{\log(u)}}$

Consider that ratio on the inside

$\frac{e^x}{x^{\log(x)}}$; ~~let y = log(x~~ see that $x = e^{\log(x)}$

then

$\frac{e^x}{x^{\log(x)}} = \exp(x(1 - \log^2(x)/x)))$

see that

$\lim(1 - \log^2(x)/(x)) = 0$ by L'Hopital;

then

$\lim \frac{e^x}{\log x^{\log x}} = \lim e^x = \infty$; thus e^x dominates $x^{\log(x)}$ and the integrals ~~diverges.~~ limit does not converge/exist; thus the series in question does not exist.