Analysis of Eigenmodes of Drums and Direct methods for Time Independent Diffusion Equation

Scientific Computupting Assignment 3

Kaixin Hu

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1 Introduction

1.1 Eigenmodes of drums/membranes of different shapes

There is a famous interesting topic: 'Can you hear the shape of the drum ?', or more specifically, whether the membrane shape can be identified uniquely from its eigenvalue spectrum. The answer recently turned out to be no in general, but this report will still try to hear the difference between some simple membranes. Therefore, in this report, the wave equation on a two-dimensional elastic material will be solved by separating the solution into two functions, with respect to time and space variables independently. This report will focus on the eigenmodes and the eigenfrequencies or resonance frequencies of the system with different shapes.

1.2 Direct methods for solving steady state problems

In the recently previous exercises the iterative methods was used to solve the Time independent Diffusion Equation $\bigtriangledown^2 u = 0$. Now in this report direct methods will be used, where a matrix for the $\bigtriangledown^2$ will be constructed, a matrix equation of the type Mu = b will then be formed, and solved with standard library methods with Python.

2 Theory

2.1 Eigenmodes of drums/membranes of different shapes

The wave equation in two dimension space has the form:

$$\frac{\partial^2 u}{\partial t^2} = D \bigtriangledown^2 u \tag{1}$$

The method of separation of the solution is used. This method look for a solution in the form:

$$u(x, y, t) = v(x, y)T(t) \tag{2}$$

, where the time and space dependencies can be separated in two independent functions. Not every u(x, y, t) can be separated this way, rather this method state that we are specifically looking for a u of this form due to the nice properties. Inserting this in the wave equation and moving all t-dependencies to the left and all x and y dependencies to the right, I get:

$$\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = c^2 \frac{1}{v(x,y)} \bigtriangledown^2 v(x,y) \tag{3}$$

The left hand side depends only on t, the right hand side only on x and y. Thus both sides must equal some constant K independent of x, y, and t: Now the left hand side and the right hand side can be treated independently. We'll start with the left hand side:

$$\frac{\partial^2 T(t)}{\partial t^2} = Kc^2 T(t) \tag{4}$$

If K < 0, it has an oscillating solution:

$$T(t) = Acos(c\lambda t) + Bsin(c\lambda t); \tag{5}$$

where $\lambda^2 = K$. And $\lambda > 0$ can be assumed without loss of generality. If K > 0, the solution either grows or decays exponentially, and this case is not interesting at the moment. If K = 0, the solution is a constant or a linear function of x and y, and has to be v = 0 due to the boundary conditions.

And the right-hand side:

$$\bigtriangledown^2 v(x,y) = Kv(x,y) \tag{6}$$

is an eigenvalue problem. The solutions to this problem give the eigenmodes v and eigenfrequencies λ. A way to find v numerically is to discretize Eq (6), to obtain a set of linear equations, one for each discretization point. If this system of equations is written in the form $Mv = Kv$, where M is a matrix, v a vector of v(x, y) at the grid points and K is a scalar constant, the eigenvalue. This matrix eigenvalue problem will be solved with library 'numpy.linalg' in Python. The numerical method to discretize Eq (6) will be illustrated in the following Method part.

2.2 Direct methods for solving steady state problems [1]

For the Time-independent diffusion equations, the solution at every point in the problem domain depends on all of the boundary data, and consequently an approximate solution must be computed everywhere simultaneously, rather than being generated step by step with time-stepping procedure. The iterative methods have been explored in the first assignment. While in this report, discretiation of the Time-independent diffusion equation results in a single system of algebraic equations to be solved for some finite-dimensional approximation to the solution. Finite difference method can be used to replace the Time independent Diffusion Equation $\bigtriangledown^2 u = 0$ with second-order centered difference approximation at each interior mesh point to obtain the finite difference equations. And then a matrix equation of the type Mu = b is formed. Specific procedure to form matrix M and the vector b will be illustrated in the Methods part.

3 Method

3.1 Eigenmodes of drums/membranes of different shapes

As it explained in the Theory part, the analysis of eigenmode problems finally turns out to be the problem of discretization of Eq (6), and corresponding eigenvalue problem. Take c = 1, and the system is analysed with three types of domain: square, rectangle, and circle.

Square Shape: with side length L

Centered definite difference method is used to approximates $\bigtriangledown^2 v(x, y)$ for non-boundary points as follows:

$$\bigtriangledown^2 v(x, y) = v_{xx} + v_{yy} = \frac{(v_{i+1,j} + v_{i-1,j} + v_{i,j-1} + v_{i,j+1} - 4v_{i,j})}{(\Delta x)^2} \tag{7}$$

And since the boundary is fixed, boundary conditions is v(x, y) = 0 for boundary mesh points. This yields the matrix M in Eq (6) as $\frac{1}{(dx)^2}M'$, M' is shown in Fig 1 for a 4*4 size problem. And the corresponding eigenvector is [v(0,0), v(1,0), v(2,0), v(3,0), v(0,1), v(1,1), v(2,1), v(3,1), v(0,2), v(1,2), v(3,2), (0,3), v(1,3), v(2,3), v(3,3)]. It can be observed from Fig 1 that the entries in the row with respect to the boundary points are all zero, for other rows there are five entries nonzero. In order to build a more accurate model, more points will be introduced in this report, and the M matrix and eigenvector can be formed similarly.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	-4	1	0	0	1	0	0	0	0	0	0
0	0	1	0	0	1	-4	1	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	1	-4	1	0	0	1	0	0
0	0	0	0	0	0	1	0	0	1	-4	1	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 1: Matrix M in a 4*4 size problem for a square shape system

Rectangle Shape: with side length L and 2L

For the shape of rectangle, the scheme to derive matrix M is similar as the shape of square.The rows of M with respect to the boundary points have zero values. Other rows entries are in the similar pattern as it showed in Fig 1. Eigenvectors are in the same order as in the square shape.

Circle Shape: with diameter L The circle shape is placed in a square shape, so the vector v contains points in a square grid. Some of them do not belong to the domain, and for those, $v_k = 0$. This can be written in the matrix form by filling the k-th row of the matrix with 0. Just as what is done to the boundary points in the square and rectangle shape.

Finally, this discretization method turns the solveing K in Eq (6) problem to be solving the eigenvalue problem of $MV = KV$, where M is above mentioned matrix to be solved, and K and V are eigenvalue and eigenvector respectively. There are several functions in numpy, including scipy.linalg.eig(), eigh() and eigs(). Since eigh() is for a Hermitian or symmetric matrix and eigs() returns k eigenvalues and eigenvectors of the square matrix A. So for this eigenvalue problem, eig() is suitable, as the spectrum of eigenfrequencies, depending on the spectrum of eigenvalues, is interesing, all the eigenvalues and eigenvectors are needed.

After K, the eigenvalue, is solved, the Eq (5) is also determined ,as $\lambda^2 = K$. And the eigenfrequency is also determined, as f= $\frac{\sqrt{-K}}{2\pi}$. The solution $u(x, y, t) = v(x, y)T(t)$ then is solved, the eigenmode is determined by the chosen eigenvalues.

In the results part, first, the eigenvectors v for some of the five largest nonzero eigenvalues will be plot , for L = 1. Second, How does the spectrum of eigenfrequencies depend on the size L and number of discretization steps will be explored. Third, animated plots of some eigenmodes for the square system will be plot.

3.2 Direct methods for solving steady state problems

Finite difference method is used to replace the Time independent Diffusion Equation $\bigtriangledown^2 u = 0$ with second-order centered difference approximation at each interior mesh point:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0 \tag{8}$$

Assuming $\Delta x = \Delta y$, we get:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \tag{9}$$

A circular domain with radius 2 and centered on the origin in a two-dimension coordinate system, will be explored here. The matrix is formed similarly as in previous section " Eigenmodes of drum" in Fig (1). The circle shape is placed in a square shape, so the vector v contains points in a square grid in a sequence from left to right and from bottom to the top. And then the values for points not belonging to the domain or on the boundary are set as 0. This can be written in the matrix form by filling the k-th row of the matrix with 0, with the k-th entry in the vector corresponding to the points on or outside the circle domain. But one main difference from previous matrix should be noticed here: at the point (0.6, 1.2) a source is present, the concentration there is 1, so the corresponding row in the matrix should be all zero except for the k'-th entry set as 1, with k'-th entry in the matrix representing the source point. For the b vector, all the entries are 0, except for the k'-th entry, which corresponds to the source, is set as 1. So far a matrix equation of the type Mu = b is formed. Build-in Function scipy.linalg.solve() in Python is then used to solve the equation.

4 Results

4.1 Eigenmodes of drums/membranes of the square shape

4.1.1 Plots of eigenvectors

Fig 2 shows the eigenvectors v with respect to the five largest nonzero eigenvalues in the square systems. Parameters are set as follows:

Diffusion coefficient: c=1;

Domain size: L=1;

discretization length: dx=dy=0.02;

Parameters in Eq 5: A=B=1;

It can be observed that the higher the frequency, the larger number of peaks for the eigenvector spectrum will be. This indicates the higher the frequency, the shorter the wave length, which can be verified in the animation made for this kind of system. "animation-square f=0.7070.mp4" shows only one wave pattern, "animation-square f=1.1174.mp4" shows two wave patterns in the same size domain, and "animation-square f=1.4133.mp4","animation-square f=1.5790.mp4" show four wave patterns in the domain.

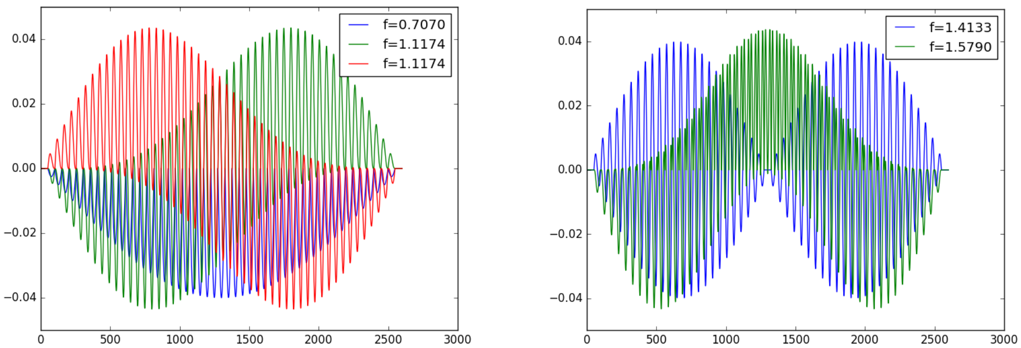


Figure 2: Plots of eigenvectors v for the five largest nonzero eigenvalues, for the shape of square. Parameters: L=1,dx=dy=0.02.

4.1.2 Plots of spectrum of eigenfrequencies

Rrelationship between spectrum of eigenfrequencies and domain size L, and relationship betweenvnumber of discretization steps in a square system are shown in Fig 3. The first graph shows the spectrum of eigenfrequencies is the same as L increases but the discretisation step length is kept the same, which means that the number of mesh points increases correspondingly. The second graph shows the spectrum of eigenfrequencies expands with smaller size, with the discretisation number fixed. The third graph shows that the spectrum of eigenfrequencies also expands in the system of same domain size, but more discretisation points.

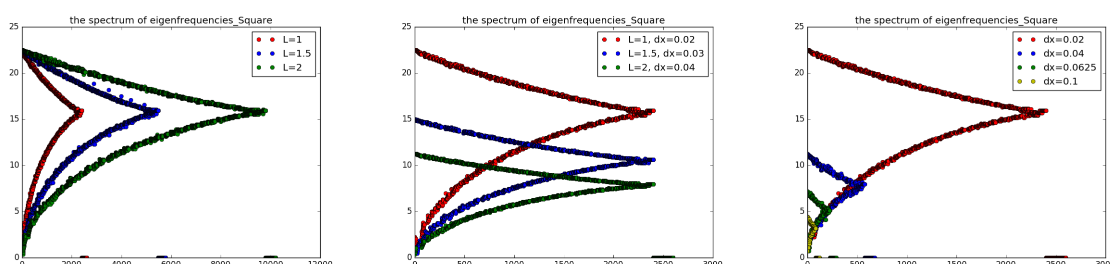


Figure 3: Spectrum of eigenfrequencies with respect to different parameters, for the shape of square.
Left graph: various size L, with discretisation step length is set equally as 0.02.
Middle graph: various size L, with discretisation number is set equally as 51*51.
Right graph: size L set equally as 1, but with various discretisation length and discretisation number.

4.2 Eigenmodes of drums/membranes of the rectangle shape

4.2.1 Plots of eigenvectors

Fig 4 shows the eigenvectors v with respect to the five largest nonzero eigenvalues in rectangle systems. Parameters are set as follows:

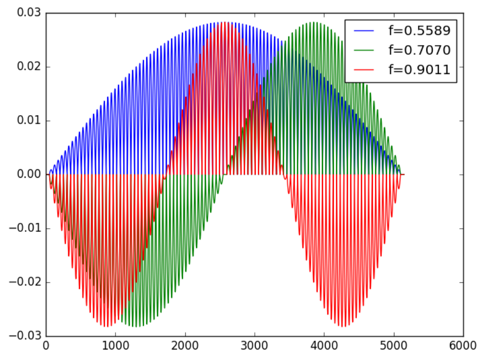
Diffusion coefficient: c=1;

Domain size: Lx=2,Ly=1;

discretization length: dx=dy=0.02;

Parameters in Eq 5: A=B=1;

It can be observed that the pattern of eigenvector plots varies as frequency varies. And they are also quite different from those in square system.



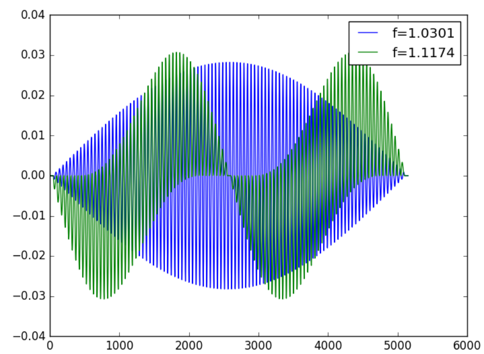


Figure 4: Plots of eigenvectors v for the five largest nonzero eigenvalues, for the shape of rectangle.Parameters: Ly=1,Lx=2,dx=dy=0.02.

4.2.2 Plots of spectrum of eigenfrequencies

Rrelationship between spectrum of eigenfrequencies and domain size L, and relationship betweenvnumber of discretization steps in a rectangle system are shown in Fig 5. The first graph shows the spectrum of eigenfrequencies is the same as L increases in a certain range, with the discretisation step length is kept the same. But as L is set to 2 or larger, the eigenvalues collapse to zeros. This may be due to the memory limitation during computing for my laptop. The second graph shows the spectrum of eigenfrequencies expands with smaller size, with the discretisation number fixed. The third graph shows that the spectrum of eigenfrequencies also expands in the system of same domain size, but more discretisation points. The spectrum of eigenfrequencies, with respect to various numerical-method-related parameter-settings shows similar phenomena as that of square system.

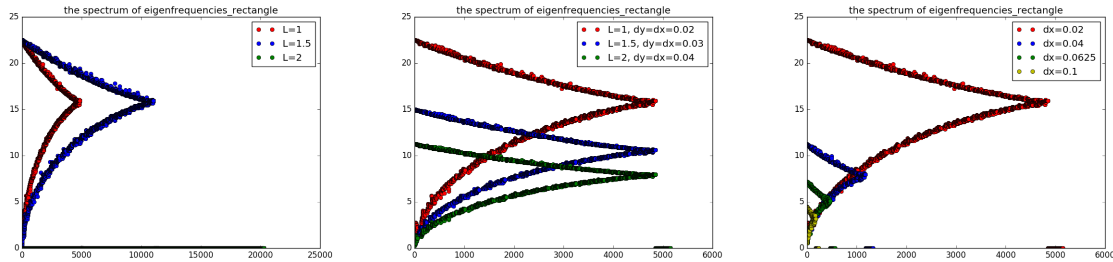


Figure 5: Spectrum of eigenfrequencies with respect to different parameters, for the shape of rectangle. Left graph: various size L, with discretisation step length is set equally as 0.02.
Middle graph: various size L, with discretisation number is set equally as 51*51.
Right graph: size L set equally as 1, but with various discretisation length and discretisation number.

4.3 Eigenmodes of drums/membranes of the circle shape

4.3.1 Plots of eigenvectors

Fig 6 shows the eigenvectors v with respect to the five largest nonzero eigenvalues in circle systems. Parameters are set as follows:

Diffusion coefficient: c=1;

Domain size: Diameter=L=1;

discretization length: dx=dy=0.02;

Parameters in Eq 5: A=B=1;

It can be observed that the pattern of eigenvector plots varies as frequency varies. And they are also quite different from those in square and rectangle system.

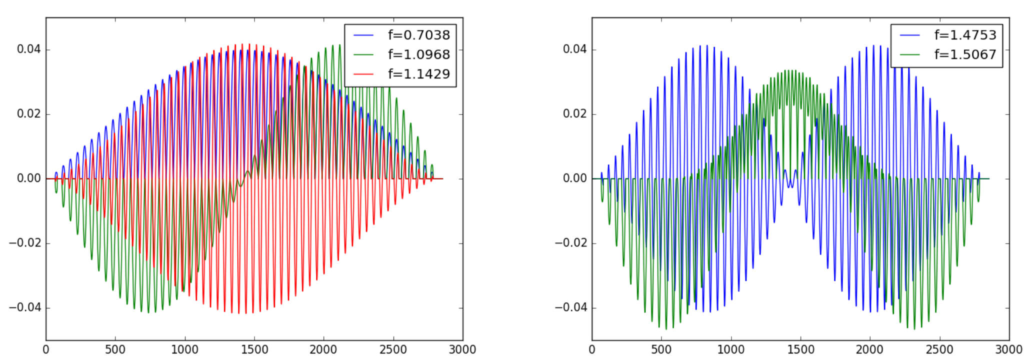


Figure 6: Plots of eigenvectors v for the five largest nonzero eigenvalues, for the shape of circle. Parameters: diameter=L=1,dx=dy=0.02.

4.3.2 Plots of spectrum of eigenfrequencies

Rrelationship between spectrum of eigenfrequencies and domain size L, and relationship between-vnumber of discretization steps in a circle system are shown in Fig 7. The first graph shows the spectrum of eigenfrequencies is the same as L increases but the discretisation step length is kept the same, which means that the number of mesh points increases correspondingly. The second graph shows the spectrum of eigenfrequencies expands with smaller size, with the discretisation number fixed. The third graph shows that the spectrum of eigenfrequencies also expands in the system of same domain size, but more discretisation points. Square, rectangle, and circle system show the same changing behavior of the spectrum of eigenfrequencies as the numerical-method related parameters change.

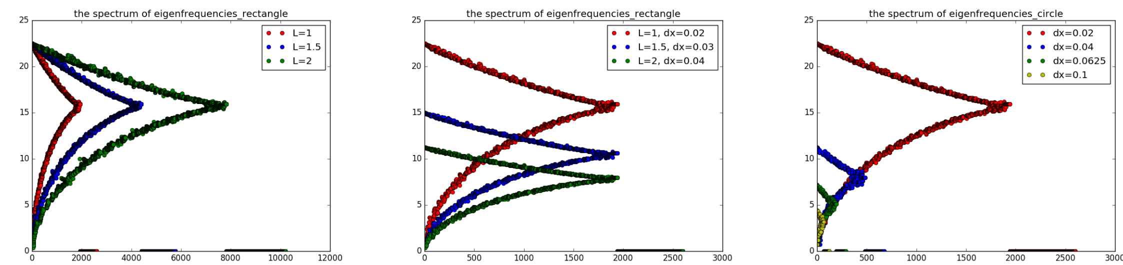


Figure 7: Spectrum of eigenfrequencies with respect to different parameters, for the shape of circle.
Left graph: various size L, with discretisation step length is set equally as 0.02.
Middle graph: various size L, with discretisation number is set equally as 51*51.
Right graph: size L set equally as 1, but with various discretisation length and discretisation number.

4.4 Direct methods for solving steady state problems

Fig 8 shows the steady state concentration u(x, y) in a circle domain with a source point by discretizing the diffusion equation $\bigtriangledown^2 u = 0$. It can be observed that the concentration is decreases as the distance from the source increases.

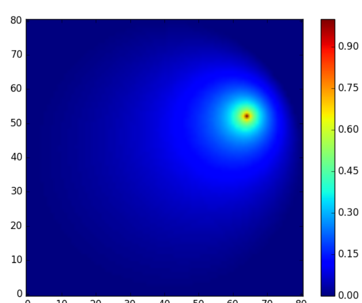


Figure 8: The concentration for the time-independent equation in a circle domain.

References

[1] Michael T Heath. *Scientific computing*. McGraw-Hill New York, 2002.