Chapter 1

Arthur Chen

September 1, 2022

1.1 Probability spaces, measures, and σ-algebras

Problem 1.1.4

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space and A, B, A_i events in $\mathcal{F}$. Prove the following properties of every probability measure.

Part a

Monotonicity: If $A \subset B$ then $\boldsymbol{P}(A) \leq \boldsymbol{P}(B)$.

If A is not a proper subset of B, then $A = B$ and equality follows because $\boldsymbol{P}$ is a function. Otherwise, A partitions B into A and $B \backslash A$. Then by countable subadditivity,

$$\boldsymbol{P}(B) = \boldsymbol{P}(A) + \boldsymbol{P}(B \backslash A) \geq \boldsymbol{P}(A)$$

by the nonnegativity of measure.

Part b

Sub-additivity:

$$A \subset \cup_i A_i \longrightarrow \boldsymbol{P}(A) \leq \sum_i \boldsymbol{P}(A_i)$$

Let $B_i = \cup_{k=1}^{i} A_k$, and $C_i = B_i - B_{i-1}$, letting $B_0 = \emptyset$. Then $\cup_i A_i = \cup_i C_i$, and $A \subset \cup_i C_i$. By monotonicity, $\boldsymbol{P}(A) \leq \boldsymbol{P}(\cup_i C_i)$. Since the C_i are disjoint, $\boldsymbol{P}(\cup_i C_i) = \sum_{i=1} \boldsymbol{P}(C_i)$. Since $C_i = \cup_{j=1}^{i} A_j - \cup_{k=1}^{i-1} A_k$, $C_i \subset A_i$, so by monotonicity,

$$\boldsymbol{P}(A) \leq \sum_{i=1} \boldsymbol{P}(C_i) \leq \sum_{i=1} \boldsymbol{P}(A_i)$$

Part c

Continuity from Below:

$A_i \uparrow A$ implies $\boldsymbol{P}(A_i) \uparrow \boldsymbol{P}(A)$.

Note that $\boldsymbol{P}(A_i)$ is weakly increasing.

Define $B_i = A_i - A_{i-1}$, and let $A_0 = \emptyset$. Then $A_i = \cup_{k=1}^{i} B_i$. Then by disjointedness $\boldsymbol{P}(A_i) = \sum_{k=1}^{i} \boldsymbol{P}(B_k)$ implies

$$A = \cup_{i=1}^{\infty} = \cup_{i=1}^{\infty} \cup_{k=1}^{i} B_k = \cup_{i=1}^{\infty}$$

and by countable additivity,

$$\boldsymbol{P}(A) = \sum_{k=1}^{\infty} \boldsymbol{P}(B_k)$$

Since the range of $\boldsymbol{P}$ is $[0,1]$, $|\boldsymbol{P}(A)| < \infty$ and the series $\sum_{k=1}^{\infty} \boldsymbol{P}(B_k)$ converges to $\boldsymbol{P}(A)$. Since $A_i = \cup_{k=1}^{i} B_k$, $\boldsymbol{P}(A_i)$ is the ith partial sum of the $\boldsymbol{P}(B_k)$ series, which converges. Thus the $\boldsymbol{P}(A_i)$ sequence converges to $\boldsymbol{P}(A)$.

Part d

Continuity from Above:

$A_i \downarrow A, \cap_{i}^{\infty} A_i = A$ implies $\boldsymbol{P}(A_i) \downarrow A$.

Consider A_i^C. The A_i^C are increasing sets because the A_i are decreasing. $\cap_{i=1}^{\infty} A_i = A$ implies $A^C = (\cap_{i=1}^{\infty} A_i)^C = \cup_{i=1}^{\infty} A_i^C$. By continuity from below, $\boldsymbol{P}(A_i^C) \uparrow \boldsymbol{P}(A^C)$. Because $\boldsymbol{P}(\Omega) = 1$, this implies $\boldsymbol{P}(A_i) \downarrow \boldsymbol{P}(A)$ as desired.

Problem 1.1.5

Prove that a finitely additive non-negative set function μ on a measurable space $(\Omega, \mathcal{F})$ with the 'continuity' property

$$B_n \subset \mathcal{F}, B_n \downarrow \emptyset, \mu(B_n) < \infty \longrightarrow \mu(B_n) \to 0$$

must be countably additive if $\mu(\Omega) < \infty$. Give an example that it is not necessarily so when $\mu(\Omega) = \infty$.

Let $A_n \in \mathcal{F}$ be a countable collection of disjoint, measurable sets. Let $B_k = \cup_{i=k}^{\infty} A_i$.

Since sigma-algebras are closed under countable union, $B_k \in \mathcal{F}$ for all $k \in \mathbb{N}$. Since $\mu(\Omega) < \infty$, monotonicity implies that $\mu(B_k) < \infty$.

Lemma 1 *$B_k \downarrow \emptyset$.*

Proof: *The limit suupremum of B_k is the set of points x such that $x \in B_k$ for infinitely many k. I will show this set is empty.*

B_k is the union of disjoint sets A_i, so $x \in B_k$ for some k implies that $x \in A_j$ for some $j \in \mathbb{N}$. This then implies that $x \in B_k$ for $k \leq k$, and $x \notin B_k$ for $k > j$. Thus x is only in a finite number of B_k. Thus $\limsup B_k = \emptyset$. Since $\liminf B_k \subset \limsup B_k$ and the only subset of the empty set is the empty set, $\limsup B_k = \emptyset$. Since the upper and lower limit equal each other, $\lim B_k = \emptyset$. Since the B_k are weakly decreasing, the lemma is proved. □

We will now prove the main result. For all $n \in \mathbb{N}$, finite additivity means

$$\boldsymbol{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \boldsymbol{P}(A_i) + \boldsymbol{P}(\cup_{i=n+1}^{\infty} A_i) = \sum_{i=1}^{n} \boldsymbol{P}(A_i) + \boldsymbol{P}(B_{n+1})$$

Taking limits, the last term goes to zero, leaving

$$\boldsymbol{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \boldsymbol{P}(A_i)$$

as desired.

For an example of how this breaks down when $\mu(\Omega) = \infty$, let $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$, and μ be finitely-additive with $\mu(\emptyset) = 0$ and the following properties:

- If $A_n \in \mathcal{F}$ is an event such that A is a singleton that only contains $n \in N$, then $\mu(A_n) = \frac{1}{2^n}$.
- If $A \in \mathcal{F}$ is infinite, $\mu(A) = \infty$.

Thus, $\mu(\cup_{i=1}^{\infty} A_i) = \mu(\Omega) = \infty$, but $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = 2$. Thus μ is not countably additive.

Problem 1.1.8

Suppose $\boldsymbol{P}$ is non-atomic and $A \in F$ with $\boldsymbol{P}(A) > 0$.

Part a

Show that for every $\epsilon > 0$, we have $B \subset A$ such that $0 < \boldsymbol{P}(B) < \epsilon$.

If $\epsilon \geq \boldsymbol{P}(A)$, then the proof is trivial. Otherwise, we will construct a sequence of decreasing subsets A_n such that eventually, $\boldsymbol{P}(A_n) < \epsilon$. Let $A_1 = A$. Because $\boldsymbol{P}$ is nonatomic, there exists $B_1 \subset A, B_1 \in \mathcal{F}$ such that $\boldsymbol{P}(B_1) < \boldsymbol{P}(A_1)$. Since sigma-algebras are closed under complements, $A_1 \backslash B_1$ is also measurable, with $\boldsymbol{P}(B_1) + \boldsymbol{P}(A_1 \backslash B_1) = \boldsymbol{P}(A_1)$. Because of this, we know that $\min(\boldsymbol{P}(B_1), \boldsymbol{P}(A_1 \backslash B_1)) \leq \frac{1}{2}\boldsymbol{P}(A_1)$. Denote the subset with the smaller measure as A_2, and repeat this process to create A_i. Note that the A_is are measurable by induction.

Since at each step, $\boldsymbol{P}(B_i) < \boldsymbol{P}(A_i)$, $\boldsymbol{P}(B_i), \boldsymbol{P}(A_i \backslash B_i) > 0$. Thus, we construct a decreasing sequence $A_1 \supset A_2 \supset \dots$ such that at each step, $0 < \boldsymbol{P}(A_i) \leq \frac{\boldsymbol{P}(A)}{2^i}$. Eventually, for large enough i, we will get a A_i with $\boldsymbol{P}(A_i) < \epsilon$.

Part b

Prove that if $0 < a < \boldsymbol{P}(A)$, then there exists $B \subsetneq A$ with $\boldsymbol{P}(B) = a$.

I will follow the hint used in the notes. Let $\epsilon_n \downarrow 0$ be arbitrary but fixed. Define inductively the numbers x_n and sets $G_n, H_n \in \mathcal{F}$ as follows.

Let $H_0 = \emptyset$ and $H_n = \cup_{k<n} G_k$. What we're going to do is repeatedly take slices out of A. Denote these slices G_i. H_n is the union of all the previous slices, and we keep taking slices G_i to slowly increase $\boldsymbol{P}(H_n)$ up to a, with ϵ_n controlling the rate of convergence. We construct these sets inductively. We start with the base case.

We first figure out what the 'maximum' possible slice is. Let $x_n = \sup\{\boldsymbol{P}(G) : G \subset A\backslash H_n, \boldsymbol{P}(H_n \cup G) \leq a\}$. In other words, x_n is limiting value of the possible Gs at this nth step, such that $\boldsymbol{P}(H_{n+1})$ will be $\leq a$.

This is where the non-atomic assumption comes in. $A \in \mathcal{F}$ by assumption, and $H_0 = \emptyset \in \mathcal{F}$ trivially. At the $n = 1$ step, $A\backslash H_1 = A$ is measurable. By the assumption that $\boldsymbol{P}$ is nonatomic and from Part a, there exists a $G \subset A$ such that $\boldsymbol{P}(G) < a$. Thus this supremum is over a nonempty set, and x_1 is a real number.

We now determine the actual G_1 we take. Because x_1 is the supremum of possible G_1 subsets we can take, we can take a G_1 such that $\boldsymbol{P}(G_1)$ is arbitrarily close to x_1. This is where the ϵ_1 comes in; it controls how arbitrarily close to x_1 we want $\boldsymbol{P}(G_1)$ to be.

Thus, let G_1 be a set $G_1 \subset A\backslash H_1 = A$ such that $\boldsymbol{P}(H_1 \cup G_1) \leq a$ and $\boldsymbol{P}(G_1) \geq (1 - \epsilon_1)x_1$. As discussed above, this construction is well defined. We thus have a G_1 with the following properties:

- $\boldsymbol{P}(H_1 \cup G_1) \leq a$
- **SOMETHING ABOUT HOW THE G1 ARE GETTING CLOSER TO a**

TODO

inductively, $H_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, implying $A\backslash H_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. Now using the non-atomic assumption, there exists at $G \subset A\backslash H_n$ such that $\boldsymbol{P}(G) < \boldsymbol{P}(A\backslash H_n) = \boldsymbol{P}(A) - \boldsymbol{P}(H_n)$.