Chapter 1

Arthur Chen

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1.1 Probability spaces, measures, and σ-algebras

Problem 1.1.5

Prove that a finitely additive non-negative set function μ on a measurable space $(\Omega, \mathcal{F})$ with the 'continuity' property

$$B_n \subset \mathcal{F}, B_n \downarrow \emptyset, \mu(B_n) < \infty \longrightarrow \mu(B_n) \to 0$$

must be countably additive if $\mu(\Omega) < \infty$. Give an example that it is not necessarily so when $\mu(\Omega) = \infty$.

Let $A_n \in \mathcal{F}$ be a countable collection of disjoint sets. Since the A_n are measurable, we can take $\mu(A_i)$.

Lemma 1 *Let $A_n \in \mathcal{F}$ be a countable collection of disjoint measurable sets, and let μ be a non-negative finitely additive function. Then for all $n \in \mathbb{N}$,*

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \mu(A_i) + \mu(\cup_{i=n+1}^{\infty} A_i)$$

Proof: *The collection of sets $\{A_1, A_2 \dots A_n, \cup_{i=n+1}^{\infty} A_i\}$ is finite, and because sigma-algebras are closed under countable union, $\cup_{i=n+1}^{\infty} A_i \in \mathcal{F}$. The statement then follows because μ is finitely additive.* $\square$

Theorem 2 *Let M be the codomain of μ restricted on the collection $A = \{A_n\}$ - that is, M is the set of values that μ takes when given A_i as input. Then M has a maximal value.*

Proof: *If $\mu(A_i)$ is nonzero only for a finite number of A_i's, then the proof is trivial.*

Suppose that $\mu(A_i) > 0$ for a countable number of A_i's. Suppose that μ does not have a maximal value, and let A_i be an arbitrary set such that $\mu(A_i) = \epsilon > 0$. Then there exists A_j such that $\mu(A_i) < \mu(A_j)$, and A_k such that $\mu(A_j) < \mu(A_k)$, and so forth. Thus by induction, there are a countable number of sets $A_j, A_k \dots$ such that $\mu(A_j) > \epsilon$. Denote this collection B.

$\mu(\Omega) < \infty$, so denote $\mu(\Omega) = M$. Since for all elements of B, $\mu(A_j) > \epsilon$, by the Archimedian property, we can take a finite number of the elements in B such that the sum of their measures is greater than M.

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \mu(A_i) + \mu(\cup_{i=n+1}^{\infty} A_i) > M$$

by the above discussion and the non-negativity of μ. However, $\cup_{i=1}^{\infty} A_i \subset \Omega$, so by the monotonicity of μ, $\mu(\cup_{i=1}^{\infty} A_i) \leq M$. Thus by contradiction, M has a largest element. $\square$

TODO.

Relabel the A_i so that they are ordered in decreasing measure. That is, $\mu(A_1) \geq \mu(A_2) \geq \dots$. We can do this by the above theorem because M has a maximal element and M is totally ordered.

(Formally, I might have to invoke Zorn's lemma here, but I am not strong enough with it to say.)