Chapter 1

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1.1 Probability spaces, measures, and σ-algebras

Problem 1.1.4

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space and A, B, A_i events in $\mathcal{F}$. Prove the following properties of every probability measure.

a

Monotonicity: If $A \subset B$ then $\boldsymbol{P}(A) \leq \boldsymbol{P}(B)$.

If A is not a proper subset of B, then $A = B$ and equality follows because $\boldsymbol{P}$ is a function. Otherwise, A partitions B into A and $B \backslash A$. Then by countable subadditivity,

$$\boldsymbol{P}(B) = \boldsymbol{P}(A) + \boldsymbol{P}(B \backslash A) \geq \boldsymbol{P}(A)$$

by the nonnegativity of measure.

b

Sub-additivity:

$$A \subset \cup_i A_i \longrightarrow \boldsymbol{P}(A) \leq \sum_i \boldsymbol{P}(A_i)$$

Let $B_i = \cup_{k=1}^{i} A_k$, and $C_i = B_i - B_{i-1}$, letting $B_0 = \emptyset$. Then $\cup_i A_i = \cup_i C_i$, and $A \subset \cup_i C_i$. By monotonicity, $\boldsymbol{P}(A) \leq \boldsymbol{P}(\cup_i C_i)$. Since the C_i are disjoint, $\boldsymbol{P}(\cup_i C_i) = \sum_{i=1} \boldsymbol{P}(C_i)$. Since $C_i = \cup_{j=1}^{i} A_j - \cup_{k=1}^{i-1} A_k$, $C_i \subset A_i$, so by monotonicity,

$$\boldsymbol{P}(A) \leq \sum_{i=1} \boldsymbol{P}(C_i) \leq \sum_{i=1} \boldsymbol{P}(A_i)$$

c

Continuity from Below:

$A_i \uparrow A$ implies $\boldsymbol{P}(A_i) \uparrow \boldsymbol{P}(A)$.

Note that $\boldsymbol{P}(A_i)$ is weakly increasing.

Define $B_i = A_i - A_{i-1}$, and let $A_0 = \emptyset$. Then $A_i = \cup_{k=1}^{i} B_i$. Then by disjointedness $\boldsymbol{P}(A_i) = \sum_{k=1}^{i} \boldsymbol{P}(B_k)$ implies

$$A = \cup_{i=1}^{\infty} = \cup_{i=1}^{\infty} \cup_{k=1}^{i} B_k = \cup_{i=1}^{\infty}$$

and by countable additivity,

$$\boldsymbol{P}(A) = \sum_{k=1}^{\infty} \boldsymbol{P}(B_k)$$

Since the range of $\boldsymbol{P}$ is $[0,1]$, $|\boldsymbol{P}(A)| < \infty$ and the series $\sum_{k=1}^{\infty} \boldsymbol{P}(B_k)$ converges to $\boldsymbol{P}(A)$. Since $A_i = \cup_{k=1}^{i} B_k$, $\boldsymbol{P}(A_i)$ is the ith partial sum of the $\boldsymbol{P}(B_k)$ series, which converges. Thus the $\boldsymbol{P}(A_i)$ sequence converges to $\boldsymbol{P}(A)$.

d

Continuity from Above:

$A_i \downarrow A, \cap_i^{\infty} A_i = A$ implies $\boldsymbol{P}(A_i) \downarrow A$.

Consider A_i^C. The A_i^C are increasing sets because the A_i are decreasing. $\cap_{i=1}^{\infty} A_i = A$ implies $A^C = (\cap_{i=1}^{\infty} A_i)^C = \cup_{i=1}^{\infty} A_i^C$. By continuity from below, $\boldsymbol{P}(A_i^C) \uparrow \boldsymbol{P}(A^C)$. Because $\boldsymbol{P}(\Omega) = 1$, this implies $\boldsymbol{P}(A_i) \downarrow \boldsymbol{P}(A)$ as desired.

Problem 1.1.5

Prove that a finitely additive non-negative set function μ on a measurable space $(\Omega, \mathcal{F})$ with the 'continuity' property

$$B_n \subset \mathcal{F}, B_n \downarrow \emptyset, \mu(B_n) < \infty \longrightarrow \mu(B_n) \to 0$$

must be countably additive if $\mu(\Omega) < \infty$. Give an example that it is not necessarily so when $\mu(\Omega) = \infty$.

Let $A_n \in \mathcal{F}$ be a countable collection of disjoint, measurable sets. Let $B_k = \cup_{i=k}^{\infty} A_i$.

Since sigma-algebras are closed under countable union, $B_k \in \mathcal{F}$ for all $k \in \mathbb{N}$. Since $\mu(\Omega) < \infty$, monotonicity implies that $\mu(B_k) < \infty$.

Lemma 1 *$B_k \downarrow \emptyset$.*

Proof: *The limit suupremum of B_k is the set of points x such that $x \in B_k$ for infinitely many k. I will show this set is empty.*

B_k is the union of disjoint sets A_i, so $x \in B_k$ for some k implies that $x \in A_j$ for some $j \in \mathbb{N}$. This then implies that $x \in B_k$ for $k \leq k$, and $x \notin B_k$ for $k > j$. Thus x is only in a finite number of B_k. Thus $\limsup B_k = \emptyset$. Since $\liminf B_k \subset \limsup B_k$ and the only subset of the empty set is the empty set, $\limsup B_k = \emptyset$. Since the upper and lower limit equal each other, $\lim B_k = \emptyset$. Since the B_k are weakly decreasing, the lemma is proved. $\square$

We will now prove the main result. For all $n \in \mathbb{N}$, finite additivity means

$$\boldsymbol{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \boldsymbol{P}(A_i) + \boldsymbol{P}(\cup_{i=n+1}^{\infty} A_i) = \sum_{i=1}^{n} \boldsymbol{P}(A_i) + \boldsymbol{P}(B_{n+1})$$

Taking limits, the last term goes to zero, leaving

$$\boldsymbol{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \boldsymbol{P}(A_i)$$

as desired.

For an example of how this breaks down when $\mu(\Omega) = \infty$, let $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$, and μ be finitely-additive with $\mu(\emptyset) = 0$ and the following properties:

- If $A_n \in \mathcal{F}$ is an event such that A is a singleton that only contains $n \in N$, then $\mu(A_n) = \frac{1}{2^n}$.
- If $A \in \mathcal{F}$ is infinite, $\mu(A) = \infty$.

Thus, $\mu(\cup_{i=1}^{\infty} A_i) = \mu(\Omega) = \infty$, but $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = 2$. Thus μ is not countably additive.