Chapter 1

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1.1 Probability spaces, measures, and σ-algebras

Problem 1.1.4

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space and A, B, A_i events in $\mathcal{F}$. Prove the following properties of every probability measure.

a

Monotonicity: If $A \subset B$ then $\boldsymbol{P}(A) \leq \boldsymbol{P}(B)$.

If A is not a proper subset of B, then $A = B$ and equality follows because $\boldsymbol{P}$ is a function. Otherwise, A partitions B into A and $B \backslash A$. Then by countable subadditivity,

$$\boldsymbol{P}(B) = \boldsymbol{P}(A) + \boldsymbol{P}(B \backslash A) \geq \boldsymbol{P}(A)$$

by the nonnegativity of measure.

b

Sub-additivity:

$$A \subset \cup_i A_i \longrightarrow \boldsymbol{P}(A) \leq \sum_i \boldsymbol{P}(A_i)$$

Let $B_i = \cup_{k=1}^i A_k$, and $C_i = B_i - B_{i-1}$, letting $B_0 = \emptyset$. Then $\cup_i A_i = \cup_i C_i$, and $A \subset \cup_i C_i$. By monotonicity, $\boldsymbol{P}(A) \leq \boldsymbol{P}(\cup_i C_i)$. Since the C_i are disjoint, $\boldsymbol{P}(\cup_i C_i) = \sum_{i=1} \boldsymbol{P}(C_i)$. Since $C_i = \cup_{j=1}^i A_j - \cup_{k=1}^{i-1} A_k$, $C_i \subset A_i$, so by monotonicity,

$$\boldsymbol{P}(A) \leq \sum_{i=1} \boldsymbol{P}(C_i) \leq \sum_{i=1} \boldsymbol{P}(A_i)$$

c

Continuity from Below:

$A_i \uparrow A$ implies $\boldsymbol{P}(A_i) \uparrow \boldsymbol{P}(A)$.

Note that $\boldsymbol{P}(A_i)$ is weakly increasing.

Define $B_i = A_i - A_{i-1}$, and let $A_0 = \emptyset$. Then $A_i = \cup_{k=1}^{i} B_i$. Then by disjointedness $\boldsymbol{P}(A_i) = \sum_{k=1}^{i} \boldsymbol{P}(B_k)$ implies

$$A = \cup_{i=1}^{\infty} = \cup_{i=1}^{\infty} \cup_{k=1}^{i} B_k = \cup_{i=1}^{\infty}$$

and by countable additivity,

$$\boldsymbol{P}(A) = \sum_{k=1}^{\infty} \boldsymbol{P}(B_k)$$

Since the range of $\boldsymbol{P}$ is $[0,1]$, $|\boldsymbol{P}(A)| < \infty$ and the series $\sum_{k=1}^{\infty} \boldsymbol{P}(B_k)$ converges to $\boldsymbol{P}(A)$. Since $A_i = \cup_{k=1}^{i} B_k$, $\boldsymbol{P}(A_i)$ is the ith partial sum of the $\boldsymbol{P}(B_k)$ series, which converges. Thus the $\boldsymbol{P}(A_i)$ sequence converges to $\boldsymbol{P}(A)$.

d

Continuity from Above:

$A_i \downarrow A, \cap_i^{\infty} A_i = A$ implies $\boldsymbol{P}(A_i) \downarrow A$.

WORK IN PROGRESS. I'm pretty sure I'm being sloppy when I switch from the unions/intersections in sets and the corresponding limits in probability. I need to fix that up.

We first begin with an obvious lemma.

Lemma 1 $\boldsymbol{P}(B \backslash A) = \boldsymbol{P}(B) - \boldsymbol{P}(A)$

Proof: *A and $B \backslash A$ partition B.* □

Let $B_i = A_1 \backslash A_i$. From the construction, we see that

$$B_i \cup A \uparrow A_1$$

Because B_i and A are disjoint, by continuity from below

$$\boldsymbol{P}(B_i) + \boldsymbol{P}(A) \uparrow \boldsymbol{P}(A_1)$$

We now exploit this fact. Taking the union of all B_i's to take advantage of continuity from below and using DeMorgan's laws,

$$\cup_i^{\infty} B_i = \cup_i^{\infty} A_1 \backslash A_i = A_1 \backslash \cap_i^{\infty} A_i$$

Taking probabilities on both sides and looking at the right side,

$$\boldsymbol{P}(A_1 \backslash \cup_i^{\infty} A_i) = \boldsymbol{P}(A_1) - \lim_{n \to \infty} \boldsymbol{P}(\cap_{n=1}^{i} A_i)$$

Using the left side and continuity from below, we get

$$\boldsymbol{P}(A_1) - \boldsymbol{P}(A) = \boldsymbol{P}(A_1) - \lim_{n\to\infty} \boldsymbol{P}(\cap_{n=1}^{i} A_i)$$

which simplifies to the desired result. Downwards convergence then follows because $\boldsymbol{P}$ being a nonnegative function and $\cap_{n=1}^{i} A_i$ being weakly decreasing imply that $\boldsymbol{P}(\cap_{n=1}^{i} A_i)$ is weakly decreasing.

Problem 1.1.5

Prove that a finitely additive non-negative set function μ on a measurable space $(\Omega, \mathcal{F})$ with the 'continuity' property

$$B_n \subset \mathcal{F}, B_n \downarrow \emptyset, \mu(B_n) < \infty \longrightarrow \mu(B_n) \to 0$$

must be countably additive if $\mu(\Omega) < \infty$. Give an example that it is not necessarily so when $\mu(\Omega) = \infty$.

Let $A_n \in \mathcal{F}$ be a countable collection of disjoint sets. Since the A_n are measurable, we can take $\mu(A_i)$.

Lemma 2 *Let $A_n \in \mathcal{F}$ be a countable collection of disjoint measurable sets, and let μ be a non-negative finitely additive function. Then for all $n \in \mathbb{N}$,*

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \mu(A_i) + \mu(\cup_{i=n+1}^{\infty} A_i)$$

Proof: *The collection of sets $\{A_1, A_2 \dots A_n, \cup_{i=n+1}^{\infty} A_i\}$ is finite, and because sigma-algebras are closed under countable union, $\cup_{i=n+1}^{\infty} A_i \in \mathcal{F}$. The statement then follows because μ is finitely additive.* □

Theorem 3 *Let M be the codomain of μ restricted on the collection $A = \{A_n\}$ - that is, M is the set of values that μ takes when given A_i as input. Then M has a maximal value.*

Proof: *If $\mu(A_i)$ is nonzero only for a finite number of A_i's, then the proof is trivial.*

Suppose that $\mu(A_i) > 0$ for a countable number of A_i's. Suppose that μ does not have a maximal value, and let A_i be an arbitrary set such that $\mu(A_i) = \epsilon > 0$. Then there exists A_j such that $\mu(A_i) < \mu(A_j)$, and A_k such that $\mu(A_j) < \mu(A_k)$, and so forth. Thus by induction, there are a countable number of sets $A_j, A_k \dots$ such that $\mu(A_j) > \epsilon$. Denote this collection B.

$\mu(\Omega) < \infty$, so denote $\mu(\Omega) = M$. Since for all elements of B, $\mu(A_j) > \epsilon$, by the Archimedian property, we can take a finite number of the elements in B such that the sum of their measures is greater than M.

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \mu(A_i) + \mu(\cup_{i=n+1}^{\infty} A_i) > M$$

by the above discussion and the non-negativity of μ. However, $\cup_{i=1}^{\infty} A_i \subset \Omega$, so by the monotonicity of μ, $\mu(\cup_{i=1}^{\infty} A_i) \leq M$. Thus by contradiction, M has a largest element. $\square$

TODO.

Relabel the A_i so that they are ordered in decreasing measure. That is, $\mu(A_1) \geq \mu(A_2) \geq \dots$. We can do this by the above theorem because M has a maximal element and M is totally ordered.

(Formally, I might have to invoke Zorn's lemma here, but I am not strong enough with it to say.)