

A generic continuous time, infinite horizon, deterministic sequential problem (**SP**) can be written:

$$V(s_\tau) = \max_{c(t)} \left\{ \int_{t=\tau}^{t=+\infty} e^{-\rho(t-\tau)} r(s(t), c(t)) dt \right\}$$

$$\dot{s}(t) = \mu(s(t), c(t))$$

$$s(\tau) = s_\tau \text{ given}$$

where $s(t)$ is the state vector, $c(t)$ the control vector, $r(s, c)$ the return function, $\rho \geq 0$ the discount rate, $V(s)$ the optimal value, $\mu(s, c)$ the transition function, and s_τ the initial condition.

The corresponding functional equation (**HJB**) can be written:

$$\rho V(s) = \max_c \{J(c, s, V)\} = \max_c \{r(s, c) + V_s \times \mu(s, c)\}$$

The FOC: $J_c(c, s, V) = 0 \Rightarrow c(s, V)$ gives the policy function in terms of the unknown value function.

The user should make sure the appropriate SOCs hold.

Combining the HJB with the FOC we get a **DE**: $V(s) = J(c(s, V), s, V) = r(s, c(s, V)) + V_s \times \mu(s, c(s, V))$

In general, DE has multiple solutions $V(s)$, and only one of them is the optimal value function in SP. The goal of this note is to try to understand how to restrict DE to ensure the solution is the optimal value function.

The examples in this note can also be solved by solving the Euler-Lagrange (**EL**) equation (the continuous time analog to the Euler Equation), which in generic differentiable problems with 1 state & 1 control variable can be written:

$$r_s = -\frac{r_c}{\mu_c}(\rho - \mu_s) + (\mu_c)^{-2}((r_{cc}\dot{c} + r_{cs}\dot{s})\mu_c - r_c(\mu_{cc}\dot{c} + \mu_{cs}\dot{s}))$$

Note that the corresponding recursive costate variable is $\lambda(s) = V_s(s)$ and TS $\lambda(t) = -r_c(c(t), s(t))/\mu_c(c(t), s(t))$.

Following Sethi 2022, the standard terminal condition (TC) for this subset of control problems is

$$TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0.$$

Proposition 1: **DE** has a unique viscosity solution $V(s)$, which is also the solution to **SP**.

This allows for weak solutions (value functions with non-differentiable kinks).

Reference: Crandall & Lions 1983 etc.

Proposition 2: DE can be solved with an FD scheme that converges to the unique viscosity solution under 3 conditions: monotonicity/consistency/stability.

Reference: Barles & Souganidis 1991, [Tourin 2013](#) etc.

Q: what theorem tells us that if a candidate solution to DE satisfies an appropriate state boundary inequality, then it must be the unique viscosity solution?

In general, for each example there will be at least 4 ways to rule out the non-viscosity solution to DE.

0: for LQR problems (examples 1 & 2) the optimal solution is the Stabilizing solution of the continuous time Algebraic Riccati Equation (ARE).

1: for these examples, each solution $V(s)$ to DE, will give a time-series (TS) solution. Only the optimal value function gives $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0$.

{Q: what is the analog to this for stochastic problems? Stochastic maximum principle?}

2: for these examples, each solution $V(s)$ to DE, will give a TS solution. We can plug the TS solution into the return function and compute the implied value function. Then we can check if the implied value function is the same as the solution to DE.

Q: where is it proven that this only holds for the viscosity solution?

3. if you can prove a solution to DE $V(s)$ is either not a viscosity supersolution or subsolution, then it cannot be a solution to DE. In general, this can be hard to verify.

4. the solution $V(s)$ to DE, which also solves SP , must satisfy the state constraint boundary inequality (BI), the other solutions do not. In general, this is much easier to verify.

IMPORTANT: I don't fully understand the BI $\dot{s}(s_{min}, V_s(s_{min})) \geq 0$.

It makes perfect sense for problems where know that $s(t) \rightarrow s_{ss}$, then if $s < s_{ss} \Rightarrow \dot{s}(s) > 0$.

What about for problems (such as consumption saving) where for some parameters $s(t) \rightarrow \pm\infty$?

TOC of examples:

1: LQ from Viscosity solutions for Dummies $s_{ss} = c_{ss} = 0$	$r(s, c) = -3s^2 - .5c^2, \rho = 1, \mu(s, c) = c$ DE has two quadratic solutions. The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt$, satisfies BI
2: LQ, Non-Hayashi investment $c_{ss} = \frac{z}{\rho + \delta} - 1, s_{ss} = \frac{c_{ss}}{\delta}$	$r(s, c) = zs - c - .5c^2, \mu(s, c) = c - \delta s$ DE has two quadratic solutions. {One is affine.} The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt$, satisfies BI
3: Hayashi investment No SS. $s(t) \rightarrow \{0, s_0, \infty\}$	$r(s, c) = zs - c - .5c^2/s, \mu(s, c) = c - \delta s$. DE has two linear solutions. The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt$, satisfies BI
4: Consumption saving $c_{ss} = 0, s_{ss} = \frac{c_{ss}}{r} = 0$	$r(s, c) = \frac{c^{1-\gamma}}{1-\gamma}, \mu(s, c) = rs - c$ DE has a concave & an affine solution. {Possibly others too.} The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt$, satisfies BI
5: NGM to do	

In the first two examples:

The TS implies by the viscosity solution will converge to the SS. The TS implies by the non-viscosity solutions will diverge. This is not true in general.

In the consumption savings model (and Hayashi firm investment problem), for certain parameters the TS implied by the viscosity solution also diverges!

Example 1 (LQ example from “Viscosity solutions for Dummies”):

SP	$V(s_\tau) = \max_{c(t)} \left\{ \int_\tau^\infty e^{-1(t-\tau)} \left(-3s(t)^2 - \frac{1}{2}c(t)^2 \right) dt \right\}$ $\dot{s}(t) = c(t), s(\tau) = s_\tau \text{ given. Here } \rho = 1.$ <p>Note: the return function is bounded in c: $r(s, c) \leq r(s, 0) = -3s^2$.</p> <p>Note: the return function is bounded in s: $r(s, c) \leq r(0, c) = -.5c^2$.</p>
HJB, FOC DE LQR	$1V(s) = \max_c \{J(s, c, V)\} = \max_c \{-3s^2 - .5c^2 + V_s \times c\} \Rightarrow c(s, V) = V_s(s)$ $\text{DE} \Rightarrow V(s) = -3s^2 + 0.5(V_s(s))^2$ $\text{LQR: Exactly 2 quadratic solutions to DE: } \{V(s) = -s^2, V(s) = 1.5s^2\}$
BI	$s(t) \geq s_{min}, \forall t > 0$ Q: how do you know to set $s_{min} < s_{ss} = 0$? $\text{BI: } \dot{s} = \dot{s}(s_{min}, V_s(s_{min})) = c(s_{min}, V_s(s_{min})) = V_s(s_{min}) \geq 0$
EL	$\dot{c}(t) = c(t) + 6s(t) \text{ \& } \dot{s}(t) = c(t) \text{ \& } s(0) = s_0 \text{ \{EL \& LOM \& IC\}}$ $c_{ss} = s_{ss} = 0 \quad \quad \quad \{\text{unique steady state}\}$ $\ddot{s}(t) = \dot{s}(t) + 6s(t), s(0) = s_0 \quad \quad \quad \{\text{Combine EL \& LOM}\}$ $\Rightarrow s(t) = (s_0 - s_\infty)e^{-2t} + (s_\infty)e^{3t}$ $\Rightarrow c(t) = \dot{s}(t) = -2(s_0 - s_\infty)e^{-2t} + 3(s_\infty)e^{3t}$ <p>Note: $\lambda(t) = V_s(s(t)) = c(t)$ \{TS costate variable\}</p> $TC(t) \equiv e^{-1t} \lambda(t) s(t) \rightarrow (s_0 - s_\infty)(s_\infty) + 3(s_\infty)^2 \infty = 0 \Leftrightarrow s_\infty = 0$

Compare the 2 quadratic solutions to DE:

Solution 1 (quadratic concave): $V(s) = -s^2$	Solution 2 (quadratic convex): $V(s) = 1.5s^2$
<p>Recursive Solution:</p> $V(s) = -s^2 \quad \quad \quad \{\text{value}\}$ $c(s) = -2s \quad \quad \quad \{\text{policy}\}$ $\dot{s}(s) = -2s \quad \quad \quad \{\text{transition}\}$ $s'(s) = -2 < 0 \Rightarrow s \rightarrow s_{ss} \text{ \{always\}}$ $\lambda(s) = V_s(s) = -2s$ $V_{ss}(s) = -2 < 0 \quad \quad \quad \{\text{concave}\}$	<p>Recursive Solution:</p> $V(s) = 1.5s^2 \quad \quad \quad \{\text{value}\}$ $c(s) = 3s \quad \quad \quad \{\text{policy}\}$ $\dot{s}(s) = 3s \quad \quad \quad \{\text{transition}\}$ $s'(s) = 3 < 0 \Rightarrow s \rightarrow s_{ss} \text{ \{never\}}$ $\lambda(s) = V_s(s) = 3s$ $V_{ss}(s) = 3 > 0 \quad \quad \quad \{\text{convex}\}$
<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = -2s, s(0) = s_0 \text{ \{IVP-ODE\}}$ $s(t) = s_0 e^{-2t} \rightarrow s_{ss} = 0$ $c(t) = -2s_0 e^{-2t}$ $\lambda(t) = -2s_0 e^{-2t}$ <p>Same as the solution to EL when $s_\infty = 0$.</p>	<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = 3s, s(0) = s_0 \text{ \{IVP-ODE\}}$ $s(t) = s_0 e^{3t} \rightarrow s_{ss} \Leftrightarrow s_0 = s_{ss} = 0$ $c(t) = 3s_0 e^{3t}$ $\lambda(t) = 3s_0 e^{3t}$ <p>Same as the solution to EL when $s_\infty = s_0$.</p>
0: Stabilizing solution of the continuous time ARE.	0: Anti-Stabilizing solution of the continuous time ARE.
1: $TC(t) \equiv e^{-1t} \lambda(t) s(t) = -2(s_0)^2 e^{-(4+1)t} \rightarrow 0$	1: $TC(t) \equiv e^{-1t} \lambda(t) s(t) = 3(s_0)^2 e^{(6-1)t} \rightarrow \infty, \forall s_0 \neq 0$
<p>Plug TS into return fcn: $r(s, c) = -5(s_0)^2 e^{-4t}$</p> <p>Plugin SP: $V(s_0) = \int_0^\infty e^{-1t} r(s, c) dt = -(s_0)^2$</p> <p>2: TS implied $V(s)$ is the same as the solution to DE!</p>	<p>Plug TS into return fcn: $r(s, c) = -7.5(s_0)^2 e^{6t}$</p> <p>Plugin SP: $V(s_0) = \int_0^\infty e^{-1t} r(s, c) dt = -(s_0)^2 \infty$</p> <p>2: If $s_0 \neq 0 \Rightarrow V(s_0) = -\infty \neq 1.5(s_0)^2$</p>
3: $V(s) = -s^2$ is a viscosity solution of DE.	3: $V(s) = 1.5s^2$ is not a viscosity supersolution. Let $\phi(s) = as^2$ for $a \in (-\infty, -1.5)$.
4: BI: $V_s(s_{min}) = -2s_{min} \geq 0 \Leftrightarrow s_{min} \leq 0$ How do you know to set $s_{min} < s_{ss} = 0$???	4: BI: $V_s(s_{min}) = 3s_{min} \geq 0 \Leftrightarrow s_{min} \geq 0$ Not satisfy BI for $s_{min} < 0$.

Example 2 (LQ, Non-Hayashi Firm Investment) Summary:

SP	$V(s_\tau) = \max_{c(t)} \left\{ \int_\tau^\infty e^{-\rho(t-\tau)} \left(zs(t) - c(t) - \frac{1}{2}c(t)^2 \right) dt \right\}$ $\dot{s}(t) = c - \delta s, s(\tau) = s_\tau \text{ given.}$ <p>Parameters: $\rho, \delta > 0, z > \rho + \delta$. Let $\rho = \delta = 1, z = 3$.</p> <p>Note: the return function is bounded in c: $r(s, c) \leq r(s, -1) = zs + 0.5$.</p> <p>Note: the return function is unbounded in s: $r(s, c) \leq r(\infty, c) = \infty$</p>
HJB, FOC DE LQR	$\rho V(s) = \max_c \{ zs(t) - c(t) - .5c(t)^2 + V_s \times (c - \delta s) \} \Rightarrow c(s, V) = V_s(s) - 1$ $\text{DE} \Rightarrow \rho V(s) = zs + .5(V_s(s) - 1)^2 - \delta s V_s(s)$ $\text{LQR: 2 quadratic solutions to DE: } \{V(s) = .125 + 1.5s, V(s) = .5 + 1.5s^2\}$
BI	$s(t) \geq s_{min}, \forall t > 0 \text{ Q: how do you know to set } s_{min} < s_{ss}?$ $\text{BI: } \dot{s} = \dot{s}(s_{min}, V_s(s_{min})) = V_s(s_{min}) - 1 - \delta s_{min} \geq 0$
EL	$\dot{c} = (\rho + \delta - z) + (\rho + \delta)c \text{ \& } \dot{s} = c - \delta s \text{ \& } s(0) = s_0 \text{ \{EL \& LOM \& IC\}}$ $c_{ss} = \frac{z}{\rho + \delta} - 1, s_{ss} = \frac{c_{ss}}{\delta} \quad \{\text{unique steady state}\}$ $\ddot{s}(t) = (\rho + \delta - z) + (\rho + \delta)\delta s(t) + \rho \dot{s}(t), s(0) = s_0 \text{ \{Combine EL \& LOM\}}$ $\Rightarrow s(t) = s_{ss} + e^{-\delta t}(s_0 - s_{ss}) + e^{(\rho + \delta)t}s_{\infty}$ $\Rightarrow c(t) = c_{ss} + e^{(\rho + \delta)t}(\rho + 2\delta)s_{\infty}$ <p>Note: $\lambda(t) = V_s(s(t)) = c(t) + 1$ \{TS costate variable\}</p> $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0 \Leftrightarrow s_{\infty} = 0$

Compare the 2 quadratic solutions to DE:

Solution 1 (quadratic concave): $V(s) = .125 + 1.5s$	Solution 2 (quadratic convex): $V(s) = .5 + 1.5s^2$
<p>Recursive Solution:</p> $V(s) = .125 + 1.5s \quad \{\text{value}\}$ $c(s) = 0.5 \quad \{\text{policy}\}$ $\dot{s}(s) = .5 - s \quad \{\text{transition}\}$ $\dot{s}'(s) = -1 < 0 \Rightarrow s \rightarrow s_{ss} \quad \{\text{always}\}$ $\lambda(s) = V_s(s) = 1.5$ $V_{ss}(s) = 0 \leq 0 \quad \{\text{weakly concave}\}$	<p>Recursive Solution:</p> $V(s) = .5 + 1.5s^2 \quad \{\text{value}\}$ $c(s) = 3s - 1 \quad \{\text{policy}\}$ $\dot{s}(s) = 2s - 1 \quad \{\text{transition}\}$ $\dot{s}'(s) = 2 < 0 \Rightarrow s \rightarrow s_{ss} \quad \{\text{never}\}$ $\lambda(s) = V_s(s) = 3s$ $V_{ss}(s) = 3 > 0 \quad \{\text{convex}\}$
<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = .5 - s, s(0) = s_0 \text{ \{IVP-ODE\}}$ $s(t) = s_{ss} + e^{-\delta t}(s_0 - s_{ss}) \rightarrow s_{ss}$ $c(t) = c_{ss}$ $\lambda(t) = c_{ss} + 1$ <p>Same as the solution to EL when $s_{\infty} = 0$.</p>	<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = 3s, s(0) = s_0 \text{ \{IVP-ODE\}}$ $s(t) = s_{ss} + e^{(\delta + \rho)t}(s_0 - s_{ss}) \rightarrow s_{ss} \Leftrightarrow s_0 = s_{ss}$ $c(t) = c_{ss} + e^{(\rho + \delta)t}(\rho + 2\delta)(s_0 - s_{ss})$ $\lambda(t) = c_{ss} + 1 + e^{(\rho + \delta)t}(\rho + 2\delta)(s_0 - s_{ss})$ <p>Same as the solution to EL when $s_{\infty} = s_0 - s_{ss}$.</p>
0: Stabilizing solution of the continuous time ARE.	0: Not Anti-Stabilizing solution of the continuous time ARE. Does not satisfy LQ sufficient conditions...
1: $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0$	1: $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow \text{sign}(s_0 - s_{ss})\infty + \infty$
<p>Plug TS into return fcn: $r(s, c) = .875 + e^{-\delta t}(3s_0 - 1.5)$</p> <p>Plugin SP: $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt = .125 + 1.5s_0$</p> <p>2: TS implied $V(s)$ is the same as the solution to DE!</p>	<p>Plug TS into return fcn: $r(s, c) = 1 - 0.5c^2$</p> <p>Plugin SP: $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt = -\infty$</p> <p>2: TS implied $V(s)$ is not the same as the solution to DE!</p>
3: $V(s) = .125 + 1.5s$ is a viscosity solution of DE.	3: $V(s) = .5 + 1.5s^2$ is not a viscosity solution.
<p>4: BI: $V_s(s_{min}) = 1.5 \geq 1 + s_{min} \Leftrightarrow s_{min} \leq s_{ss}$</p> <p>How do you know to set $s_{min} < s_{ss} = 0$???</p>	<p>4: BI: $V_s(s_{min}) = 3s_{min} \geq 1 + s_{min} \Leftrightarrow s_{min} \geq s_{ss}$</p> <p>BI not satisfied for $s_{min} < s_{ss}$.</p>

Example 3 (Firm Investment Hayashi) Summary:

SP	$V(s_\tau) = \max_{c(t)} \left\{ \int_\tau^\infty e^{-\rho(t-\tau)} \left(zs(t) - c(t) - \frac{1}{2} \frac{c(t)^2}{s(t)} \right) dt \right\}$ $\dot{s}(t) = c - \delta s, s(\tau) = s_\tau \text{ given.}$ <p>Parameters: $\rho, \delta, z > 0, z \in \left[0, \rho + \delta + \rho\delta + \frac{1}{2}(\delta^2 + r^2)\right]$</p> <p>Note: the return is bounded in c: $r(s, c) \leq r(s, -s) = (z + .5)s$</p> <p>Note: the return is unbounded in s: $r(s, c) \leq r(\infty, c) = \infty$</p>
HJB, FOC DE	$\rho V(s) = \max_c \{zs - c - .5c^2s^{-1} + V_s \times (c - \delta s)\} \Rightarrow c(s, V) = (V_s(s) - 1)s$ $DE \Rightarrow \rho V(s) = (z + .5(V_s - 1)^2 - \delta V_s)s$ $V(s) = \left((1 + \delta + \rho) - \sqrt{(\delta + \rho)^2 + 2(\delta + \rho - z)} \right) s = Q_-s$ $V(s) = \left((1 + \delta + \rho) + \sqrt{(\delta + \rho)^2 + 2(\delta + \rho - z)} \right) s = Q_+s$
BI	$s(t) \geq s_{min}, \forall t > 0$ $BI: \dot{s} = (V_s(s_{min}) - 1 - \delta)s_{min} \geq 0$
EL	<p>EL: Nasty quadratic ODE, no SS.</p> <p>Note: $\lambda(t) = V_s = c(t)/s(t) + 1$ {TS costate variable}</p> $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow ?$

Compare the 2 linear solutions to DE:

Solution 1: $V(s) = Q_-s$	Solution 2: $V(s) = Q_+s$
<p>Recursive Solution:</p> $V(s) = Q_-s \quad \{\text{value}\}$ $c(s) = (Q_- - 1)s \quad \{\text{policy}\}$ $\dot{s}(s) = (Q_- - 1 - \delta)s \quad \{\text{transition}\}$ $\dot{s}'(s) = (Q_- - 1 - \delta) < 0 \quad \{\}$ $\lambda(s) = V_s(s) = Q_-$ $V_{ss}(s) = 0 \leq 0 \quad \{\text{weakly concave}\}$	<p>Recursive Solution:</p> $V(s) = Q_+s \quad \{\text{value}\}$ $c(s) = (Q_+ - 1)s \quad \{\text{policy}\}$ $\dot{s}(s) = (Q_+ - 1 - \delta)s \quad \{\text{transition}\}$ $\dot{s}'(s) = (Q_+ - 1 - \delta) < 0 \quad \{\text{Never}\}$ $\lambda(s) = V_s(s) = Q_+$ $V_{ss}(s) = 0 \leq 0 \quad \{\text{weakly concave}\}$
<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = (Q_- - 1 - \delta)s, s(0) = s_0 \text{ {IVP-ODE}}$ $s(t) = s_0 e^{(Q_- - 1 - \delta)t} \rightarrow \{0, s_0, \infty\}$ $c(t) = (Q_- - 1)s_0 e^{(Q_- - 1 - \delta)t}$ $\lambda(t) = Q_-$ $z \in \left(0, \rho + \delta + \rho\delta + \frac{1}{2}\delta^2\right) \Rightarrow s(t) \rightarrow 0$ $z = \rho + \delta + \rho\delta + \frac{1}{2}\delta^2 \Rightarrow s(t) \rightarrow s_0$ $z \in \left(\rho + \delta + \rho\delta + \frac{1}{2}\delta^2, \rho + \delta + \rho\delta + \frac{1}{2}(\delta^2 + r^2)\right) \Rightarrow s \rightarrow \infty$	<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = (Q_+ - 1 - \delta)s, s(0) = s_0$ $s(t) = s_0 e^{(Q_+ - 1 - \delta)t} \rightarrow \{0, s_0, \infty\}$ $c(t) = (Q_+ - 1)s_0 e^{(Q_+ - 1 - \delta)t}$ $\lambda(t) = Q_+$
1: $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0 \Leftrightarrow Q_- - 1 - \delta - \rho < 0$ {Always}	1: $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow \infty$ {always}
<p>Plug TS into return fcn: $r(s, c)$</p> <p>Plugin SP: $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt = Q_-s_0$</p> <p>2: TS implied $V(s)$ is the same as the solution to DE!</p>	<p>Plug TS into return fcn: $r(s, c)$</p> <p>Plugin SP: $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt$</p> <p>2: TS implied $V(s)$ is not the same as the solution to DE!</p>
3: $V(s)$ is a viscosity solution of DE.	3: $V(s) = .5 + 1.5s^2$ is not a viscosity solution.
4: BI:	4: BI

Example 4 (Consumption Savings) Summary:

SP	$V(s_\tau) = \max_{c(t)} \left\{ \int_\tau^\infty e^{-\rho(t-\tau)} \left(\frac{c(t)^{1-\gamma}-1}{1-\gamma} \right) dt \right\}$ $\dot{s}(t) = rs(t) - c(t), s(\tau) = s_\tau \text{ given, } s(t) \geq 0 \text{ or } \lim e^{-rt}s(t) = 0$ <p>Let: $r, \rho, \gamma > 0$ let $\omega \equiv \frac{r-\rho}{\gamma}$ and note $r - \omega = \rho - (1-\gamma)\omega = \frac{r(\gamma-1)+\rho}{\gamma}$</p> <p>Note: the return function is unbounded $r(s, c) \leq r(s, \infty) = \infty$. Need $s \geq s_{min}$.</p>
HJB, FOC DE LQR	$\rho V(s) = \max_c \left\{ \frac{c^{1-\gamma}-1}{1-\gamma} + V_s \times (rs - c) \right\} \Rightarrow c(s, V) = (V_s(s))^{-1/\gamma}$ $DE \Rightarrow \rho V(s) = \frac{\gamma}{1-\gamma} (V_s(s))^{1-\frac{1}{\gamma}} - \frac{1}{1-\gamma} + V_s(s)rs$ $\text{Sol 1: } V(s) = \frac{1}{\rho(\gamma-1)} + (r-\omega)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$ $\text{Sol 2: } V(s) = B_0 + B_1 s \text{ if } r = \rho \text{ and } \rho B_0 = \frac{\gamma}{1-\gamma} (B_1)^{1-\frac{1}{\gamma}} - \frac{1}{1-\gamma} \text{ and } B_1 > 0$
BI	$s(t) \geq s_{min}, \forall t > 0 \text{ Q: how do you know to set } s_{min} < s_{ss}?$ $\text{BI: } \dot{s} = \dot{s}(s_{min}, V_s(s_{min})) = rs_{min} - (V_s(s_{min}))^{-1/\gamma} \geq 0$ $\text{BI: } \dot{s}(0) = r0 - c(0) \geq 0 \Rightarrow c(0) \leq 0 \Rightarrow c(0) = V_s(0) = 0$
EL	$\dot{c} = \omega c \text{ \& } \dot{s} = rs - c \text{ \& } s(0) = s_0 \quad \{\text{EL \& LOM \& IC}\}$ $c_{ss} = 0, s_{ss} = \frac{c_{ss}}{r} = 0 \quad \{\text{unique steady state}\}$ $\ddot{s}(t) = (r + \omega)\dot{s}(t) - \omega rs(t), s(0) = s_0 \quad \{\text{Combine EL \& LOM}\}$ $\Rightarrow s(t) = e^{\omega t}(s_0 - s_\infty) + e^{rt}(s_\infty)$ $\Rightarrow c(t) = e^{(\omega)t}(r - \omega)(s_0 - s_\infty)$ <p>Note: $\lambda(t) = V_s(s(t)) = e^{(\rho-r)t}((r - \omega)(s_0 - s_\infty))^{-\gamma}$ {TS costate variable}</p> $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0 \Leftrightarrow s_\infty = 0$

Compare the 2 solutions to DE:

Solution 1 (concave): $V(s) = \frac{1}{\rho(\gamma-1)} + (r-\omega)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$	Solution 2 (weakly concave): $V(s) = B_0 + B_1 s, B_1 > 0$
<p>Recursive Solution:</p> $V(s) = \frac{1}{\rho(\gamma-1)} + (r-\omega)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma} \quad \{\text{value}\}$ $c(s) = (r-\omega)s \quad \{\text{policy}\}$ $\dot{s}(s) = \omega s \quad \{\text{transition}\}$ $\dot{s}'(s) = \omega < 0 \Rightarrow s \rightarrow s_{ss} \quad \{\text{if } r < \rho\}$ $\lambda(s) = V_s(s) = (r-\omega)^{-\gamma}(s)^{-\gamma}$ $V_{ss}(s) = -\gamma(r-\omega)^{-\gamma}(s)^{-\gamma-1} \leq 0 \text{ \{weakly concave\}}$	<p>Recursive Solution:</p> $V(s) = B_0 + B_1 s \quad \{\text{value}\}$ $c(s) = (B_1)^{-1/\gamma} \quad \{\text{policy}\}$ $\dot{s}(s) = rs - (B_1)^{-1/\gamma} \quad \{\text{transition}\}$ $\dot{s}'(s) = r < 0 \Rightarrow s \rightarrow s_{ss} \quad \{\text{never}\}$ $\lambda(s) = V_s(s) = B_1$ $V_{ss}(s) = 0 \leq 0 \quad \{\text{weakly concave}\}$
<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = \omega s, s(0) = s_0 \text{ \{IVP-ODE\}}$ $s(t) = s_0 e^{\omega t} \rightarrow \{s_{ss} = 0, +\infty\}$ $c(t) = (r-\omega)s_0 e^{\omega t}$ $\lambda(t) = ((r-\omega)s_0)^{-\gamma} e^{-\gamma\omega t}$ <p>Same as the solution to EL when $s_\infty = 0$.</p>	<p>TS Solution implied by the recursive solution:</p> $\dot{s} = \mu(s, c(s)) = rs - (B_1)^{-1/\gamma}, s(0) = s_0 \text{ \{IVP-ODE\}}$ $s(t) = \frac{(B_1)^{-1/\gamma}}{r} + e^{rt} \left(s_0 - \frac{(B_1)^{-1/\gamma}}{r} \right) \rightarrow \left\{ \frac{(B_1)^{-1/\gamma}}{r}, \infty \right\}$ $c(t) = (B_1)^{-1/\gamma}$ $\lambda(t) = B_1$
1: $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow 0$	1: $TC(t) \equiv e^{-\rho t} \lambda(t) s(t) \rightarrow \infty$
<p>Plug TS into return fcn: $r(s, c)$</p> $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt = \frac{1}{\rho(\gamma-1)} + (r-\omega)^{-\gamma} \frac{(s_0)^{1-\gamma}}{1-\gamma}$ <p>2: TS implied $V(s)$ is the same as the solution to DE!</p>	<p>Plug TS into return fcn: $r(s, c)$</p> $V(s_0) = \int_0^\infty e^{-\rho t} r(s, c) dt = \frac{(B_1)^{1-1/\gamma}-1}{\rho(1-\gamma)} \neq B_0 + B_1 s$ <p>2: TS implied $V(s)$ is not the same as the solution to DE!</p>
3: $V(s)$ is a viscosity solution of DE.	3: $V(s)$ is not a viscosity solution.
4: BI: $c(0) = (r-\omega)0 = 0$	4: BI: $c(0) = (B_1)^{-1/\gamma} = 0$ contradicts $B_1 > 0$