A generic continuous time, infinite horizon, deterministic sequential problem (**SP**) can be written:

$V(s_\tau) = \max_{\mathrm{c(t)}} \left\{ \int_{t=\tau}^{t=+\infty} e^{-\rho(t-\tau)} r\big(s(t), c(t)\big) dt \right\}$

$\dot{s}(t) = \mu\big(s(t), c(t)\big)$

$s(\tau) = s_\tau$ given

where $s(t)$ is the state vector, $c(t)$ the control vector, $r(s, c)$ the return function, $\rho \geq 0$ the discount rate, $V(s)$ the optimal value, $\mu(s, c)$ the transition function, and s_τ the initial condition.

The corresponding functional equation (**HJB**) can be written:

$\rho V(s) = \max_{\mathrm{c}} \{J(c, s, V)\} = \max_{\mathrm{c}} \{r(s, c) + V_s \times \mu(s, c)\}$

The FOC: $J_c(c, s, V) = 0 \Rightarrow c(s, V)$ gives the policy function in terms of the unknown value function.

The user should make sure the appropriate SOCs hold.

Combining the HJB with the FOC we get a **DE**: $V(s) = J(c(s, V), s, V) = r\big(s, c(s, V)\big) + V_s \times \mu\big(s, c(s, V)\big)$

In general, DE has multiple solutions $V(s)$, and only one of them is the optimal value function in SP. The goal of this note is to try to understand how to restrict DE to ensure the solution is the optimal value function.

The examples in this note can also be solved by solving the Euler-Lagrange (**EL**) equation (the continuous time analog to the Euler Equation), which in generic differentiable problems with 1 state & 1 control variable can be written:

$r_s = -\frac{r_c}{\mu_c}(\rho - \mu_s) + (\mu_c)^{-2}\big((r_{cc}\dot{c} + r_{cs}\dot{s})\mu_c - r_c(\mu_{cc}\dot{c} + \mu_{cs}\dot{s})\big)$

Note that the corresponding recursive costate variable is $\lambda(s) = V_s(s)$ and TS $\lambda(t) = -r_c\big(c(t), s(t)\big)/\mu_c\big(c(t), s(t)\big)$.

Following Sethi 2022, the standard terminal condition (TC) for this subset of control problems is

$TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0.$

Proposition 1: **DE** has a unique viscosity solution $V(s)$, which is also the solution to **SP**.

This allows for weak solutions (value functions with non-differentiable kinks).

Reference: Crandall & Lions 1983 etc.

Proposition 2: DE can be solved with an FD scheme that converges to the unique viscosity solution under 3 conditions monotonicity/consistency/stability.

Reference: Barles & Souganidis 1991, Tourin 2013 etc.

Q: what theorem tells us that if a candidate solution to DE satisfies an appropriate state boundary inequality, then it must be the unique viscosity solution?

For each solution $V(s)$ of DE, we obtain the policy function from the FOC $c(s) = c\big(s, V(s)\big)$ and then we combine the policy function with the law of motion for the state variable together with the initial condition:

$\dot{s} = \mu\big(s, c(s)\big), s(0) = s_0$ the solution to this initial value ODE is a TS $s(t)$.

Next, we obtain the TS for the control variable $c(t) = c\big(s(t)\big)$ and the multiplier $\lambda(t) = -r_c(c, s)/\mu_c(c, s)$. There are several methods that use the TS corresponding to solution $V(s)$ of DE to verify it is also the solution to SP.

This note will study several problems where DE has multiple solutions. For each example there will be at least 5 ways to rule out the non-viscosity solution to DE.

0	For LQR problems (examples 1 & 2) the solution to SP is the Stabilizing solution of the continuous time Algebraic Riccati Equation (ARE).
1	$TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0$ This only for the TS corresponding to the solution to SP.
2	$\int_0^\infty e^{-\rho t} r(s,c)dt = V(s_0)$ This only for the TS corresponding to the solution to SP.
3	Some authors say the solution to SP should satisfy: $\lim e^{-\rho t}V\big(s(t)\big) \geq 0$ for any admissible plan $\lim e^{-\rho t}V\big(s(t)\big) = 0$ for the optimal plan
4	$V(s)$ is a viscosity solution of DE. Any other solution to DE is either not a viscosity supersolution, or subsolution.
5	Boundary Inequality: $\dot{s} = \dot{s}\big(s_{min}, c(s_{min}, V_s(s_{min}))\big) \geq 0$ Only the solution to SP satisfies BI. **<u>Q</u>**: in general, how do you set s_{min}? It makes perfect sense for problems where we know that $s(t) \to s_{ss}$, then if $s < s_{ss} \Rightarrow \dot{s}(s) > 0$. What about for problems (such as consumption saving) where for some parameters $s(t) \to \pm\infty$?

TOC of examples:

1: LQ from Viscosity solutions for Dummies $s_{ss} = c_{ss} = 0$	$r(s,c) = -3s^2 - .5c^2, \rho = 1, \mu(s,c) = c$ DE has two quadratic solutions. The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s,c)dt$, satisfies BI
2: LQ, Non-Hayashi investment $c_{ss} = \frac{z}{\rho+\delta} - 1, s_{ss} = \frac{c_{ss}}{\delta}$	$r(s,c) = zs - c - .5c^2, \mu(s,c) = c - \delta s$ DE has two quadratic solutions. {One is affine.} The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s,c)dt$, satisfies BI
3: Hayashi investment No SS. $s(t) \to \{0, s_0, \infty\}$	$r(s,c) = zs - c - .5c^2/s, \mu(s,c) = c - \delta s.$ DE has two linear solutions. The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s,c)dt$, satisfies BI
4: Consumption saving $c_{ss} = 0, s_{ss} = \frac{c_{ss}}{r} = 0$	$r(s,c) = \frac{c^{1-\gamma}}{1-\gamma}, \mu(s,c) = rs - c$ DE has a concave & an affine solution. {Possibly others too.} The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s,c)dt$, satisfies BI
5: NGM	$r(s,c) = \frac{c^{1-\gamma}}{1-\gamma}, \mu(s,c) = s^\alpha - \delta s - c$ DE has a concave solution. {Possibly others too.} The viscosity sol: Satisfies TC, $V(s_0) = \int_0^\infty e^{-\rho t} r(s,c)dt$, satisfies BI

Example 1 (LQ example from “Viscosity solutions for Dummies”):

SP	$V(s_\tau) = \max_{c(t)} \left\{ \int_\tau^\infty e^{-1(t-\tau)} \left(-3s(t)^2 - \frac{1}{2}c(t)^2 \right) dt \right\}$ $\dot{s}(t) = c(t)$, $s(\tau) = s_\tau$ given. Here $\rho = 1$. Note: the return function is bounded in c: $r(s,c) \le r(s,0) = -3s^2$. Note: the return function is bounded in s: $r(s,c) \le r(0,c) = -.5c^2$.
HJB, FOC DE LQR	$1V(s) = \max_c \{J(s,c,V)\} = \max_c \{-3s^2 - .5c^2 + V_s \times c\} \Rightarrow c(s,V) = V_s(s)$ DE $\Rightarrow V(s) = -3s^2 + 0.5\left(V_s(s)\right)^2$ LQR: Exactly 2 quadratic solutions to DE: $\{V(s) = -s^2, V(s) = 1.5s^2\}$
BI	$s(t) \ge s_{min}, \forall t > 0$ Q: how do you know to set $s_{min} < s_{ss} = 0$? BI: $\dot{s} = \dot{s}\left(s_{min}, V_s(s_{min})\right) = c\left(s_{min}, V_s(s_{min})\right) = V_s(s_{min}) \ge 0$
EL	$\dot{c}(t) = c(t) + 6s(t)$ & $\dot{s}(t) = c(t)$ & $s(0) = s_0$ {EL & LOM & IC} $c_{ss} = s_{ss} = 0$ {unique steady state} $\ddot{s}(t) = \dot{s}(t) + 6s(t), s(0) = s_0$ {Combine EL & LOM} $\Rightarrow s(t) = (s_0 - s_\infty)e^{-2t} + (s_\infty)e^{3t}$ $\Rightarrow c(t) = \dot{s}(t) = -2(s_0 - s_\infty)e^{-2t} + 3(s_\infty)e^{3t}$ Note: $\lambda(t) = V_s\left(s(t)\right) = c(t)$ {TS costate variable} $TC(t) \equiv e^{-1t}\lambda(t)s(t) \to (s_0 - s_\infty)(s_\infty) + 3(s_\infty)^2\infty = 0 \Leftrightarrow s_\infty = 0$

Compare the 2 quadratic solutions to DE:

Solution 1 (quadratic concave): $V(s) = -s^2$	Solution 2 (quadratic convex): $V(s) = 1.5s^2$
Recursive Solution: $V(s) = -s^2$ {value} $c(s) = -2s$ {policy} $\dot{s}(s) = -2s$ {transition} $\dot{s}'(s) = -2 < 0 \Rightarrow s \to s_{ss}$ {always} $\lambda(s) = V_s(s) = -2s$ $V_{ss}(s) = -2 < 0$ {concave}	Recursive Solution: $V(s) = 1.5s^2$ {value} $c(s) = 3s$ {policy} $\dot{s}(s) = 3s$ {transition} $\dot{s}'(s) = 3 < 0 \Rightarrow s \to s_{ss}$ {never} $\lambda(s) = V_s(s) = 3s$ $V_{ss}(s) = 3 > 0$ {convex}
TS Solution implied by the recursive solution: $\dot{s} = \mu\left(s, c(s)\right) = -2s, s(0) = s_0$ {IVP-ODE} $s(t) = s_0 e^{-2t} \to s_{ss} = 0$ $c(t) = -2s_0 e^{-2t}$ $\lambda(t) = -2s_0 e^{-2t}$ $r(s,c) = -5(s_0)^2 e^{-4t}$ Same as the solution to EL when $s_\infty = 0$.	TS Solution implied by the recursive solution: $\dot{s} = \mu\left(s, c(s)\right) = 3s, s(0) = s_0$ {IVP-ODE} $s(t) = s_0 e^{3t} \to s_{ss} \Leftrightarrow s_0 = s_{ss} = 0$ $c(t) = 3s_0 e^{3t}$ $\lambda(t) = 3s_0 e^{3t}$ $r(s,c) = -7.5(s_0)^2 e^{6t}$ Same as the solution to EL when $s_\infty = s_0$.
0: Stabilizing solution of the continuous time ARE.	0: Anti-Stabilizing solution of the continuous time ARE.
1: $TC(t) \equiv e^{-1t}\lambda(t)s(t) = -2(s_0)^2 e^{-(4+1)t} \to 0$	1:$TC(t) \equiv e^{-1t}\lambda(t)s(t) = 3(s_0)^2 e^{(6-1)t} \to \infty, \forall s_0 \ne 0$
2:$\int_0^\infty e^{-1t} r(s,c)dt = -(s_0)^2 = V(s_0)$ TS implied $V(s)$ is the same as the solution to DE!	2: $\int_0^\infty e^{-1t} r(s,c)dt = -(s_0)^2\infty \ne V(s_0)$ TS implied $V(s)$ is not the same as the solution to DE!
3: $\lim e^{-\rho t} V\left(s(t)\right) = \lim -e^{-\rho t}(s_0)^2 e^{-4t} = 0$	3: $\lim e^{-\rho t} V\left(s(t)\right) = \lim 1.5e^{-\rho t}(s_0)^2 e^{6t} = \infty \ne 0$
4: $V(s) = -s^2$ is a viscosity solution of DE.	4: $V(s) = 1.5s^2$ is not a viscosity supersolution. Let $\phi(s) = as^2$ for $a \in (-\infty, -1.5)$.
5: BI: $V_s(s_{min}) = -2s_{min} \ge 0 \Leftrightarrow s_{min} \le 0$ How do you know to set $s_{min} < s_{ss} = 0$???	5: BI: $V_s(s_{min}) = 3s_{min} \ge 0 \Leftrightarrow s_{min} \ge 0$ Not satisfy BI for $s_{min} < 0$.

Example 2 (LQ, Non-Hayashi Firm Investment) Summary:

SP	$V(s_\tau) = \max_{c(t)}\left\{\int_\tau^\infty e^{-\rho(t-\tau)}\left(zs(t) - c(t) - \frac{1}{2}c(t)^2\right)dt\right\}$ $\dot{s}(t) = c - \delta s$, $s(\tau) = s_\tau$ given. Parameters: $\rho, \delta > 0, z > \rho + \delta$. Let $\rho = \delta = 1, z = 3$. Note: the return function is bounded in c: $r(s,c) \le r(s,-1) = zs + 0.5$. Note: the return function is unbounded in s: $r(s,c) \le r(\infty, c) = \infty$
HJB, FOC DE LQR	$\rho V(s) = \max_c\{zs(t) - c(t) - .5c(t)^2 + V_s \times (c - \delta s)\} \Rightarrow c(s,V) = V_s(s) - 1$ DE $\Rightarrow \rho V(s) = zs + .5(V_s(s) - 1)^2 - \delta s V_s(s)$ LQR: 2 quadratic solutions to DE: $\{V(s) = .125 + 1.5s, V(s) = .5 + 1.5s^2\}$
BI	$s(t) \ge s_{min}, \forall t > 0$ Q: how do you know to set $s_{min} < s_{ss}$? BI: $\dot{s} = \dot{s}(s_{min}, V_s(s_{min})) = V_s(s_{min}) - 1 - \delta s_{min} \ge 0$
EL	$\dot{c} = (\rho + \delta - z) + (\rho + \delta)c$ & $\dot{s} = c - \delta s$ & $s(0) = s_0$ {EL & LOM & IC} $c_{ss} = \frac{z}{\rho+\delta} - 1, s_{ss} = \frac{c_{ss}}{\delta}$ {unique steady state} $\ddot{s}(t) = (\rho + \delta - z) + (\rho + \delta)\delta s(t) + \rho\dot{s}(t), s(0) = s_0$ {Combine EL & LOM} $\Rightarrow s(t) = s_{ss} + e^{-\delta t}(s_0 - s_\infty - s_{ss}) + e^{(\rho+\delta)t}s_\infty$ $\Rightarrow c(t) = c_{ss} + e^{(\rho+\delta)t}(\rho + 2\delta)s_\infty$ Note: $\lambda(t) = V_s(s(t)) = c(t) + 1$ {TS costate variable} $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0 \Leftrightarrow s_\infty = 0$

Compare the 2 quadratic solutions to DE:

Solution 1 (quadratic concave): $V(s) = .125 + 1.5s$	Solution 2 (quadratic convex): $V(s) = .5 + 1.5s^2$
Recursive Solution: $V(s) = .125 + 1.5s$ {value} $c(s) = 0.5$ {policy} $\dot{s}(s) = .5 - s$ {transition} $\dot{s}'(s) = -1 < 0 \Rightarrow s \to s_{ss}$ {always} $\lambda(s) = V_s(s) = 1.5$ $V_{ss}(s) = 0 \le 0$ {weakly concave}	Recursive Solution: $V(s) = .5 + 1.5s^2$ {value} $c(s) = 3s - 1$ {policy} $\dot{s}(s) = 2s - 1$ {transition} $\dot{s}'(s) = 2 < 0 \Rightarrow s \to s_{ss}$ {never} $\lambda(s) = V_s(s) = 3s$ $V_{ss}(s) = 3 > 0$ {convex}
TS Solution implied by the recursive solution: $\dot{s} = \mu(s, c(s)) = .5 - s, s(0) = s_0$ {IVP-ODE} $s(t) = s_{ss} + e^{-\delta t}(s_0 - s_{ss}) \to s_{ss}$ $c(t) = c_{ss}$ $\lambda(t) = c_{ss} + 1$ $r(s,c) = .875 + e^{-\delta t}(3s_0 - 1.5)$ Same as the solution to EL when $s_\infty = 0$.	TS Solution implied by the recursive solution: $\dot{s} = \mu(s, c(s)) = 3s, s(0) = s_0$ {IVP-ODE} $s(t) = s_{ss} + e^{(\delta+\rho)t}(s_0 - s_{ss}) \to \infty \Leftrightarrow s_0 \ne s_{ss}$ $c(t) = c_{ss} + e^{(\rho+\delta)t}(\rho + 2\delta)(s_0 - s_{ss})$ $\lambda(t) = c_{ss} + 1 + e^{(\rho+\delta)t}(\rho + 2\delta)(s_0 - s_{ss})$ $r(s,c) = 1 - 0.5c^2$ Same as the solution to EL when $s_\infty = s_0 - s_{ss}$.
0: Stabilizing solution of the continuous time ARE.	0: Not Anti-Stabilizing solution of the continuous time ARE. Does not satisfy LQ sufficient conditions…
1: $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0$	1:$TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to \text{sign}(s_0 - s_{ss})\infty + \infty$
2: $\int_0^\infty e^{-\rho t} r(s,c)dt = .125 + 1.5s_0 = V(s_0)$ TS implied $V(s)$ is the same as the solution to DE!	2: $\int_0^\infty e^{-\rho t} r(s,c)dt = -\infty \ne V(s_0)$ TS implied $V(s)$ is **not** the same as the solution to DE!
3: $\lim e^{-\rho t}V(s(t)) = 0$	3: $\lim e^{-\rho t}V(s(t)) = \infty \ne 0$
4: $V(s) = .125 + 1.5s$ is a viscosity solution of DE.	4: $V(s) = .5 + 1.5s^2$ is not a viscosity solution.
5: BI: $V_s(s_{min}) = 1.5 \ge 1 + s_{min} \Leftrightarrow s_{min} \le s_{ss}$ How do you know to set $s_{min} < s_{ss} = 0$???	5: BI: $V_s(s_{min}) = 3s_{min} \ge 1 + s_{min} \Leftrightarrow s_{min} \ge s_{ss}$ BI not satisfied for $s_{min} < s_{ss}$.

Example 3 (Firm Investment Hayashi) Summary:

SP	$V(s_\tau)=\max_{c(t)}\left\{\int_\tau^\infty e^{-\rho(t-\tau)}\left(zs(t)-c(t)-\frac{1}{2}\frac{c(t)^2}{s(t)}\right)dt\right\}$ $\dot{s}(t)=c-\delta s, s(\tau)=s_\tau$ given. Parameters: $\rho,\delta,z>0, z\in\left[0,\rho+\delta+\rho\delta+\frac{1}{2}(\delta^2+r^2)\right]$ Let $\bar{z}\equiv\rho+\delta+\rho\delta+\frac{1}{2}\delta^2$ Note: the return is bounded in c: $r(s,c)\le r(s,-s)=(z+.5)s$ Note: the return is unbounded in s: $r(s,c)\le r(\infty,c)=\infty$
HJB, FOC DE	$\rho V(s)=\max_c\{zs-c-.5c^2s^{-1}+V_s\times(c-\delta s)\}\Rightarrow c(s,V)=(V_s(s)-1)s$ DE $\Rightarrow \rho V(s)=(z+.5(V_s-1)^2-\delta V_s)s$ $V(s)=\left((1+\delta+\rho)-\sqrt{(\delta+\rho)^2+2(\delta+\rho-z)}\right)s=Q_-s$ $V(s)=\left((1+\delta+\rho)+\sqrt{(\delta+\rho)^2+2(\delta+\rho-z)}\right)s=Q_+s$
BI	$s(t)\ge s_{min}, \forall t>0$ BI: $\dot{s}=(V_s(s_{min})-1-\delta)s_{min}\ge 0$
EL	EL: Nasty quadratic ODE, no SS. Note: $\lambda(t)=V_s=c(t)/s(t)+1$ {TS costate variable} $TC(t)\equiv e^{-\rho t}\lambda(t)s(t)\to ?$

Compare the 2 linear solutions to DE:

Solution 1: $V(s)=Q_-s$	Solution 2: $V(s)=Q_+s$
Recursive Solution: $V(s)=Q_-s$ {value} $c(s)=(Q_--1)s$ {policy} $\dot{s}(s)=(Q_--1-\delta)s$ {transition} $\dot{s}'(s)=(Q_--1-\delta)<0$ {} $\lambda(s)=V_s(s)=Q_-$ $V_{ss}(s)=0\le 0$ {weakly concave}	Recursive Solution: $V(s)=Q_+s$ {value} $c(s)=(Q_+-1)s$ {policy} $\dot{s}(s)=(Q_+-1-\delta)s$ {transition} $\dot{s}'(s)=(Q_+-1-\delta)>0$ {Never} $\lambda(s)=V_s(s)=Q_+$ $V_{ss}(s)=0\le 0$ {weakly concave}
TS Solution implied by the recursive solution: $\dot{s}=\mu(s,c(s))=(Q_--1-\delta)s, s(0)=s_0$ {IVP-ODE} $s(t)=s_0e^{(Q_--1-\delta)t}\to 01_{z<\bar{z}}+s_01_{z=\bar{z}}+\infty 1_{z>\bar{z}}$ $c(t)=(Q_--1)s_0e^{(Q_--1-\delta)t}$ $\lambda(t)=Q_-$ $r(s,c)=(z-Q_-+1-.5(Q_--1)^2)s=\omega s(t)$	TS Solution implied by the recursive solution: $\dot{s}=\mu(s,c(s))=(Q_+-1-\delta)s, s(0)=s_0$ $s(t)=s_0e^{(Q_+-1-\delta)t}\to\{\infty\}$ $c(t)=(Q_+-1)s_0e^{(Q_--1-\delta)t}$ $\lambda(t)=Q_+$ $r(s,c)=(z-Q_++1-.5(Q_+-1)^2)s=\omega s(t)$
1: $TC(t)\equiv e^{-\rho t}\lambda(t)s(t)\to 0$ $\Leftrightarrow Q_--1-\delta-\rho<0$ {always}	1: $TC(t)\equiv e^{-\rho t}\lambda(t)s(t)\to\infty$ {always}
2: $\int_0^\infty e^{-\rho t}r(s,c)dt=Q_-s_0=V(s_0)$	2: $\int_0^\infty e^{-\rho t}r(s,c)dt=\omega s_0\infty\ne V(s_0)$
3: $\lim e^{-\rho t}V(s(t))=0$	3: $\lim e^{-\rho t}V(s(t))\ne 0$
4: $V(s)$ is a viscosity solution of DE.	4: $V(s)=.5+1.5s^2$ is not a viscosity solution.
5: BI:	5: BI

Example 4 (Consumption Savings) Summary:

SP	$V(s_\tau) = \max_{c(t)} \left\{ \int_\tau^\infty e^{-\rho(t-\tau)} \left(\frac{c(t)^{1-\gamma}}{1-\gamma} \right) dt \right\}$ $\dot{s}(t) = rs(t) - c(t)$, $s(\tau) = s_\tau$ given, $s(t) \geq 0$ or $\lim e^{-rt} s(t) = 0$ Let: $r, \rho, \gamma > 0$ let $\omega \equiv \frac{r-\rho}{\gamma}$ and note $r - \omega = \rho - (1-\gamma)\omega = \frac{r(\gamma-1)+\rho}{\gamma}$ Note: the return function is unbounded $r(s,c) \leq r(s,\infty) = \infty$. Need $s \geq s_{min}$.
HJB, FOC DE	$\rho V(s) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + V_s \times (rs - c) \right\} \Rightarrow c(s,V) = \left(V_s(s)\right)^{-1/\gamma}$ DE $\Rightarrow \rho V(s) = \frac{\gamma}{1-\gamma} \left(V_s(s)\right)^{\frac{\gamma-1}{\gamma}} + V_s(s) rs$ Sol 0: $V(s) = 0$ solves DE but implies $c = \infty$ {rule this out} Sol 1: $V(s) = (r-\omega)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$ Sol 2: $V(s) = B_0 + B_1 s$ if $r = \rho$ and $\rho B_0 = \frac{\gamma}{1-\gamma} (B_1)^{1-\frac{1}{\gamma}}$ and $B_1 > 0$
BI	$s(t) \geq s_{min}, \forall t > 0$ Q: how do you know to set $s_{min} < s_{ss}$? BI: $\dot{s} = \dot{s}\left(s_{min}, V_s(s_{min})\right) = rs_{min} - \left(V_s(s_{min})\right)^{-1/\gamma} \geq 0$ BI: $\dot{s}(0) = r0 - c(0) \geq 0 \Rightarrow c(0) \leq 0 \Rightarrow c(0) = V_s(0) = 0$
EL	$\dot{c} = \omega c$ & $\dot{s} = rs - c$ & $s(0) = s_0$ {EL & LOM & IC} $c_{ss} = 0, s_{ss} = \frac{c_{ss}}{r} = 0$ {unique steady state} $\ddot{s}(t) = (r+\omega)\dot{s}(t) - \omega r s(t), s(0) = s_0$ {Combine EL & LOM} $\Rightarrow s(t) = e^{\omega t}(s_0 - s_\infty) + e^{rt}(s_\infty)$ $\Rightarrow c(t) = e^{(\omega)t}(r-\omega)(s_0 - s_\infty)$ Note: $\lambda(t) = V_s\left(s(t)\right) = e^{(\rho-r)t}\left((r-\omega)(s_0 - s_\infty)\right)^{-\gamma}$ {TS costate variable} $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0 \Leftrightarrow s_\infty = 0$

Compare the 2 solutions to DE:

Solution 1 (concave): $V(s) = (r-\omega)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$	Solution 2 (weakly concave): $V(s) = B_0 + B_1 s, B_1 > 0$
Recursive Solution: $V(s) = (r-\omega)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$ {value} $c(s) = (r-\omega)s$ {policy} $\dot{s}(s) = \omega s$ {transition} $\dot{s}'(s) = \omega < 0 \Rightarrow s \to s_{ss}$ {if $r < \rho$} $\lambda(s) = V_s(s) = (r-\omega)^{-\gamma}(s)^{-\gamma}$ $V_{ss}(s) = -\gamma(r-\omega)^{-\gamma}(s)^{-\gamma-1} \leq 0$ {weakly concave}	Recursive Solution: $V(s) = B_0 + B_1 s$ {value} $c(s) = (B_1)^{-1/\gamma}$ {policy} $\dot{s}(s) = rs - (B_1)^{-1/\gamma}$ {transition} $\dot{s}'(s) = r < 0 \Rightarrow s \to s_{ss}$ {never} $\lambda(s) = V_s(s) = B_1$ $V_{ss}(s) = 0 \leq 0$ {weakly concave}
TS Solution implied by the recursive solution: $\dot{s} = \mu\left(s, c(s)\right) = \omega s, s(0) = s_0$ {IVP-ODE} $s(t) = s_0 e^{\omega t} \to \{s_{ss} = 0, +\infty\}$ $c(t) = (r-\omega)s_0 e^{\omega t}$ $\lambda(t) = \left((r-\omega)s_0\right)^{-\gamma} e^{-\gamma\omega t}$ $r(s,c) = \frac{\left((r-\omega)s_0\right)^{1-\gamma}}{1-\gamma} e^{\omega(1-\gamma)t}$ Same as the solution to EL when $s_\infty = 0$.	TS Solution implied by the recursive solution: $\dot{s} = \mu\left(s, c(s)\right) = rs - (B_1)^{-1/\gamma}, s(0) = s_0$ {IVP-ODE} $s(t) = \frac{(B_1)^{-1/\gamma}}{r} + e^{rt}\left(s_0 - \frac{(B_1)^{-1/\gamma}}{r}\right) \to \left\{\frac{(B_1)^{-1/\gamma}}{r}, \infty\right\}$ $c(t) = (B_1)^{-1/\gamma}$ $\lambda(t) = B_1$ $r(s,c) = \frac{(B_1)^{1-1/\gamma}}{(1-\gamma)}$
1: $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0$	1: $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to \infty$
2: $\int_0^\infty e^{-\rho t} r(s,c)dt = (r-\omega)^{-\gamma} \frac{(s_0)^{1-\gamma}}{1-\gamma} = V(s_0)$	2: $\int_0^\infty e^{-\rho t} r(s,c)dt = \frac{(B_1)^{1-1/\gamma}}{\rho(1-\gamma)} \neq V(s_0)$
3: $\lim e^{-\rho t} V\left(s(t)\right) = 0$	3: $\lim e^{-\rho t} V\left(s(t)\right) \neq 0$
4: $V(s)$ is a viscosity solution of DE.	4: $V(s)$ is not a viscosity solution.
5: BI: $c(0) = (r-\omega)0 = 0$	5: BI: $c(0) = (B_1)^{-1/\gamma} = 0$ contradicts $B_1 > 0$

Example 5 (NGM) Summary:

SP	$V(s_\tau) = \max\limits_{c(t)} \left\{ \int_\tau^\infty e^{-\rho(t-\tau)} \left(\frac{c(t)^{1-\gamma}}{1-\gamma} \right) dt \right\}$ $\dot{s}(t) = s(t)^\alpha - \delta s(t) - c(t)$, $s(\tau) = s_\tau$ given, $s(t) \geq 0$ Let: $\alpha, \delta, \rho, \gamma > 0$. Note: the return function is unbounded $r(s,c) \leq r(s,\infty) = \infty$. Need $s \geq s_{min}$.
HJB, FOC DE	$\rho V(s) = \max\limits_{c} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + V_s \times (s^\alpha - \delta s - c) \right\} \Rightarrow c(s,V) = \left(V_s(s)\right)^{-1/\gamma}$ DE $\Rightarrow \rho V(s) = \frac{\gamma}{1-\gamma} \left(V_s(s)\right)^{\frac{\gamma-1}{\gamma}} + V_s(s) s^\alpha - \delta V_s(s) s$ Sol 1: $V(s) = \frac{1}{\rho}(\phi)^{-\gamma} + (\phi)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}, \phi \equiv \frac{\rho+\delta(1-\gamma)}{\gamma}$ for $\gamma = \alpha \in (0,1)$ Sol 2: $V(s) = (\phi)^{-\gamma} \frac{(s)^{1-\alpha\gamma}}{(1-\alpha\gamma)}, \phi \equiv 1 - \frac{1}{\gamma}$ for $\gamma > 1/\alpha > 1$ Are there other non-zero non-viscosity closed form solutions?
BI	$s(t) \geq s_{min}, \forall t > 0$ Q: how do you know to set $s_{min} < s_{ss}$? BI: $\dot{s} = \dot{s}\left(s_{min}, V_s(s_{min})\right) = (s_{min})^\alpha - \delta s_{min} - \left(V_s(s_{min})\right)^{-1/\gamma} \geq 0$
EL	$\dot{c} = \frac{\alpha s^{\alpha-1} - \delta - \rho}{\gamma} c$ & $\dot{s} = s^\alpha - \delta s - c$ & $s(0) = s_0$ {EL & LOM & IC} $s_{ss} = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}, c_{ss} = (s_{ss})^\alpha - \delta s_{ss}$ {unique steady state} $\ddot{s}(t) = \alpha s^{\alpha-1}\dot{s} - \delta \dot{s} - \left(\frac{\alpha s^{\alpha-1} - \delta - \rho}{\gamma}\right)(s^\alpha - \delta s - \dot{s}), s(0) = s_0$ Note: $\lambda(t) = V_s = \left(c(t)\right)^{-\gamma}$ {TS costate variable} $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0 \Leftrightarrow s_\infty = 0$

Compare the 2 solutions to DE:

Solution 1: $V(s) = \frac{1}{\rho}(\phi)^{-\gamma} + (\phi)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$ Let: $\phi \equiv \frac{\rho+\delta(1-\gamma)}{\gamma}, \gamma = \alpha \in (0,1)$	Solution 2: $V(s) = (\phi)^{-\gamma} \frac{(s)^{1-\alpha\gamma}}{(1-\alpha\gamma)}$ Let $\phi \equiv 1 - \frac{1}{\gamma} = 1 - \frac{\alpha\delta}{\delta+\rho}$ & $\gamma = \frac{\delta+\rho}{\alpha\delta} > 1$
Recursive Solution: $V(s) = \frac{1}{\rho}(\phi)^{-\gamma} + (\phi)^{-\gamma} \frac{(s)^{1-\gamma}}{1-\gamma}$ {value} $c(s) = \phi s$ {policy} $\dot{s}(s) = s^\alpha - (\delta + \phi)s$ {transition} $\dot{s}'(s) = \alpha s^{\alpha-1} - (\delta+\phi) < 0$ $\lambda(s) = V_s(s) = (\phi)^{-\gamma}(s)^{-\gamma}$ $V_{ss}(s) = -\gamma(\phi)^{-\gamma}(s)^{-\gamma-1} \leq 0$ {weakly concave}	Recursive Solution: $V(s) = (\phi)^{-\gamma} \frac{(s)^{1-\alpha\gamma}}{(1-\alpha\gamma)}$ {value} $c(s) = \phi s^\alpha$ {policy} $\dot{s}(s) = (1-\phi)s^\alpha - \delta s$ {transition} $\dot{s}'(s) = \alpha(1-\phi)s^{\alpha-1} - \delta < 0$ $\lambda(s) = V_s(s) = (\phi)^{-\gamma}(s)^{-\alpha\gamma}$ $V_{ss}(s) = -\alpha\gamma(\phi)^{-\gamma}(s)^{-\alpha\gamma-1} \leq 0$ {weakly concave}
TS Solution implied by the recursive solution: $\dot{s} = s^\alpha - (\delta+\phi)s, s(0) = s_0$ {IVP-ODE} $s(t) = \left(s_{ss}^{1-\alpha} + (s_0^{1-\alpha} - s_{ss}^{1-\alpha})e^{-t(1-\alpha)(\delta+\phi)}\right)^{\frac{1}{1-\alpha}} \to s_{ss}$ $c(t) = \phi s(t)$ $\lambda(t) = (\phi)^{-\gamma}\left(s(t)\right)^{-\gamma} \to \lambda_{ss}$ $r(s,c) = \frac{\phi^{1-\gamma}}{1-\gamma}\left(s_{ss}^{1-\alpha} + (s_0^{1-\alpha} - s_{ss}^{1-\alpha})e^{-t(1-\alpha)(\delta+\phi)}\right)$	TS Solution implied by the recursive solution: $\dot{s} = (1-\phi)s^\alpha - \delta s, s(0) = s_0$ {IVP-ODE} $s(t) = \left(s_{ss}^{1-\alpha} + (s_0^{1-\alpha} - s_{ss}^{1-\alpha})e^{-t(1-\alpha)\delta}\right)^{\frac{1}{1-\alpha}} \to s_{ss}$ $c(t) = \phi s(t)^\alpha$ $\lambda(t) = (\phi)^{-\gamma}\left(s(t)\right)^{-\alpha\gamma} \to \lambda_{ss}$ $r(s,c) = \frac{\phi^{1-\gamma}}{1-\gamma}\left(s_{ss}^{1-\alpha} + (s_0^{1-\alpha} - s_{ss}^{1-\alpha})e^{-t(1-\alpha)\delta}\right)^{\frac{\alpha-\alpha\gamma}{1-\alpha}}$
1: $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0$	1: $TC(t) \equiv e^{-\rho t}\lambda(t)s(t) \to 0$
2: $\int_0^\infty e^{-\rho t} r(s,c)dt = \frac{1}{\rho}(\phi)^{-\gamma} + (\phi)^{-\gamma}\frac{(s_0)^{1-\gamma}}{1-\gamma} = V(s_0)$	2: $\int_0^\infty e^{-\rho t} r(s,c)dt = (\phi)^{-\gamma}\frac{(s)^{1-\alpha\gamma}}{(1-\alpha\gamma)} = V(s_0)$
3: $\lim e^{-\rho t}V\left(s(t)\right) = 0$	3: $\lim e^{-\rho t}V\left(s(t)\right) = 0$
4: $V(s)$ is a viscosity solution of DE.	4: $V(s)$ is a viscosity solution of DE.
5: BI:	5: BI:

In general: $\dot{s} = s^\alpha - \delta s - c(s)$

If you can write: $\dot{s} = As^\alpha - Bs$
Define the capital-output ratio $z \equiv s/s^\alpha = s^{1-\alpha}$
then $\dot{z} = (1-\alpha)(s^{-\alpha})\dot{s} = (1-\alpha)(s^{-\alpha})As^\alpha - (1-\alpha)(s^{-\alpha})Bs$
then $\dot{z} = (1-\alpha)A - (1-\alpha)(s^{1-\alpha})B$
then $\dot{z} = (1-\alpha)A - (1-\alpha)Bz, z(0) = z_0$
$\Rightarrow z(t) = \frac{A}{B} + e^{-(1-\alpha)Bt}\left(z_0 - \frac{A}{B}\right) \to \frac{A}{B}$
$\Rightarrow s(t) = \left(\frac{A}{B} + e^{-(1-\alpha)Bt}\left((s_0)^{1-\alpha} - \frac{A}{B}\right)\right)^{\frac{1}{1-\alpha}} \to \left(\frac{A}{B}\right)^{\frac{1}{1-\alpha}} = s_{SS} = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$

Two sub-cases for $c(s)$:
(a) linear in output $c(s) = \phi s^\alpha \Rightarrow \dot{s} = (1-\phi)s^\alpha - \delta s$
$\Rightarrow s_\infty = \frac{1-\phi}{\delta} = \frac{\alpha}{\delta+\rho} \Rightarrow \phi = 1 - \frac{\alpha\delta}{\delta+\rho}$
$\Rightarrow \frac{\dot{c}}{c} = \frac{\phi\alpha s^{\alpha-1}}{\phi s^\alpha}\dot{s} = \alpha\frac{\dot{s}}{s} = \frac{\alpha s^{\alpha-1}-\delta-\rho}{\gamma} \Rightarrow \dot{s} = \frac{\alpha s^\alpha - (\delta+\rho)s}{\alpha\gamma} = \frac{1}{\gamma}s^\alpha - \frac{(\delta+\rho)}{\alpha\gamma}s == (1-\phi)s^\alpha - \delta s$
$\Rightarrow \phi = 1 - \frac{1}{\gamma}, \gamma = \frac{(\delta+\rho)}{\alpha\delta}$
Together: $\phi = 1 - \frac{1}{\gamma} = 1 - \frac{\alpha\delta}{\delta+\rho}$ & $\gamma = \frac{\delta+\rho}{\alpha\delta} > 1$

$\frac{\dot{c}}{c} = \frac{\alpha s^{\alpha-1}-\delta-\rho}{\gamma} \Rightarrow \frac{\phi\alpha s^{\alpha-1}}{\phi s^\alpha}\dot{s} = \frac{\alpha}{s}\dot{s} = \frac{\alpha s^{\alpha-1}-\delta-\rho}{\gamma} \Rightarrow \dot{s} = \frac{1}{\gamma}s^\alpha - \frac{(\rho+\delta)}{\alpha\gamma}s$
(b) linear in capital $c(s) = \phi s \Rightarrow \dot{s} = s^\alpha - (\delta+\phi)s$

(a) $\frac{c(s)}{s^\alpha} = \phi, \frac{c(s)}{s} = \phi s^{\alpha-1}$
(b) $\frac{c(s)}{s^\alpha} = \phi s^{1-\alpha}, \frac{c(s)}{s} = \phi$

{Q: what is the analog to this (EL/TC) for stochastic problems? Stochastic maximum principle?}
Q: where is it proven that this only holds for the viscosity solution? $V(s_0) = \int_0^\infty e^{-\rho t}r(s,c)dt$
-Suppose $V(s)$ is a non-viscosity solution to DE, is it still possible that $V(s_0) = \int_0^\infty e^{-\rho t}r(s,c)dt$?