AM 120 Notes

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September 27, 2012

Contents

1 September 6 Lecture

There are two main problems that we will learn how to handle in this class.

1. Find $x \in R^n$ such that $Ax = b$. A is m by n matrix, $b \in R^n$ vector

2. Find x and λ such that $Ax = \lambda x$

Example $\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

There are 3 ways to solve:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \Rightarrow y = 2, x = -1$

2. $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix}$ $detA = -3$

   $x = A^{-1}b \Rightarrow \frac{1}{-3} \begin{bmatrix} +3 \\ -6 \end{bmatrix}$

3. Kramer's rule

Summary Topics covered in next 3 classes:

1. Geometric interpretation of solving linear systems
2. Matrix notation (LU factorization)
3. Singular cases (no solution, multiple soln's)
4. Efficient way to solve $Ax = b$ using computers

1.1 Geometric interpretation

Example Graphical method:

Row interpretation (plot lines on coordinate system):

$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$ Solution: $x = 2, y = 3$

Column interpretation:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Example 3 by 3 system:

Each row represents a plane:

2u + v + w = 5

4u - 6v + 0 = -2

-2u + 7v + 2w = 9

Remember: inner product of vector with another vector equals 0 $\Rightarrow$ orthogonal.

Column interpretation:

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Example Overdetermined system:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Solution: $c = 1, d = 1$

In 4 dimensions, the rows represent 3-spaces, which are 'flat' relative to 4 dimensional space. If we intersect $(x, y, z, t = 0)$ with $(x, y, z = 0, t)$, two three spaces, we get (x, y) plane.

$$a_1u + a_2v + a_3w + a_4z = b$$

$$A = (a_1|a_2|a_3|a_4)$$

1.2 Algorithmic approach

Generalizing to n by n. How to solve $Ax = b$ in a way that scales well? Gaussian elimination (row reduction).

$$\begin{array}{c} 2u + v + w = 5 \\ 4u - 6v + 0 = -2 \\ -2u + 7v + 2w = 9 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{c} 2u + v + w = 5 \\ -8v - 2w = -12 \\ 8v + 3w = 14 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{c} 2u + v + w = 5 \\ -8v - 2w = -12 \\ w = 2 \\ \hline \end{array}$$

$\Rightarrow v = 1, u = 1$

We need a process that takes:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

...this $Ax = b$ problem and transforms it to a $Ux = \hat{b}$ problem. We can get an upper triangular matrix, and obtain solution by back substitution.

Problems One issue that could arise is if the bottom row is all 0s: infinitely many solutions.

2 September 11 Lecture

Last class:

- Introduced first central problem of linear algebra: solving linear equations
- Studied column and row interpretation of linear systems
- Introduced Gaussian elimination

Example (from previous class) Row/Column interpretation.
2u + v + w = 5
4u - 6v + 0 = -2
-2u + 7v + 2w = 9
Row: Three planes intersecting. Column: linear combination of three vectors

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We were trying to figure out how to transform matrix A into an upper triangular matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

Matrix operations Addition is associative: $A+B+C = (A+B)+C = A+(B+C)$

Multiplication: dimension $m \times n$ multiplied by $n \times p$ results in $m \times p$ matrix. $AB \neq BA$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 8 & 7 \end{bmatrix}$$

Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & \dots & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \dots \\ \sum_{i=1}^n a_{ni}x_i \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ Ab_1 & Ab_2 \\ | & | \end{bmatrix}$$

Row reduction In matrix form

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

1. Subtract 2 times row 1 to row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{E_{21}} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}_A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

2. Subtract -1 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{E_{31}} \quad E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

3. Subtract -1 times row 2 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{E_{32}} \quad E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Originally we wanted to solve $Ax = b$. Now we have:

$$E_{32}E_{31}E_{21}A = U$$

where U is an upper triangular matrix.

$$E_{32}E_{31}E_{21}Ax = Ux$$

Let's let $E_{32}E_{31}E_{21} = L$. Then, we have

$$L^{-1}A = U$$
$$A = LU$$
$$Ux = C = E_{32}E_{31}E_{21}b$$

Now we can solve by back substitution.

$$L^{-1} = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Matrix inverse properties:

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A_1A_2...A_n)^{-1} = A_n^{-1}...A_2^{-1}A_1^{-1}$$

So we have:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$

Row reduction matrices A matrix that subtracts l times row j from row i is such that it includes $-l$ in row i, column j.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}_L \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}_U$$

L is lower triangular and U is upper triangular.

1. Compute LU factorization
2. Solve for c in $Lc = b$ (forward substitution)
3. Solve for x in $Ux = c$ (back substitution)

We want to solve $Ax = b$. We factor to get $LUx = b$. First we find c such that $Lc = b$

2.1 General Example

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{array}{c} c_1 = b_1/l_{11} \\ c_2 = b_2 - b_1 l_{21}/l_{11} \\ c_3 = b_3 - l_{31}b_1 - l_{32}(b_2 - b_1 l_{21}) \end{array}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{array}{c} x_3 = c_3/u_{33} \\ x_2 = \frac{1}{u_{22}}(c_2 - u_{23}c_3/u_{33}) \\ x_1 = ... \end{array}$$

3 September 13 Lecture

Announcements

- Matlab tutorials (sections)
- Final projects
  - Adjustment based on class size
  - Pairs
- Assignment 1 due Fri @ 7pm in Pierce 303
- Collaboration policy

From last time:

- Linear equations $\rightarrow$ Matrix notation
- Column j of $AB = Ab_j$

$$A \begin{bmatrix} | & | & & | \\ b_1 & b_2 & ... & b_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & ... & Ab_n \end{bmatrix}$$

- Introduced the LU factorization of square matrix A (see general example at end of last lecture)

$$Ax = b \Rightarrow LUx = b$$

1. Find LU
2. Solve for c in $Lc = b$
3. Solve for x in $Ux = c$

Example *LU* factorization

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

1. Subtract 2 times row 1 to row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{E_{21}} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Subtract 3 times row 1 to row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}_{E_{31}} E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

3. Subtract 2 times row 2 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}_{E} 32E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{U}$$

$$L^{-1} = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0-2 & 1 & 0-3 & -2 & 1 \end{bmatrix}$$

$$L^{-1}A = U$$
$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Generalizing *LU* factorization To $n \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

1. Introduce zeros below a_{11} by subtracting multiples of row 1

2. Use multipliers $l = \frac{a_{i1}}{a_{11}}$
3. Repeat 1 and 2 for a^*_{22}, a^*_{33}, ...

Step 1:

$$\begin{array}{c|cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \hline 0 & a^*_{22} & a^*_{23} & \dots & a^*_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a^*_{n2} & a^*_{n3} & \dots & a^*_{nn} \end{array}$$

Step 2:

$$\begin{array}{c|c|ccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \hline 0 & a^*_{22} & a^*_{23} & \dots & a^*_{2n} \\ \hline \dots & 0 & a^*_{33} & \dots & a^*_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & a^*_{n3} & \dots & a^*_{nn} \end{array}$$

How many operations does this algorithm use?

$$\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

4 September 18 Lecture

To review: solving $Ax = b$:

1. Find $LU = A$
2. Solve for c in $Lc = b$ (forward substitution)
3. Solve for x in $Ux = c$ (back-subst)

Multipliers to find U are entries of L.
What is the # of operations needed to get LU factorization?

$$\approx \frac{n^3 - n}{3}$$

Forward substitution Number of operations:
$(n-1) + (n-2) + ...(1) \approx O(n^2)$

Back substitution is similar process (also $O(n^2)$). Most time consuming place is step 1.

Algorithm Failure This $Ax = b$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, our algorithm won't find the answer because it can't switch rows. If the algorithm fails we have two options:

1. We need to rearrange rows
2. No solution
3. Infinitely many solutions

Example of (2):

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example of (3):

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Fact $\det(A) = \det(LU) = \det(L)\det(U)$

$$\det(U) = \prod_{i=1}^{n} u_{ii}$$

Example Consider this:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9998 \end{bmatrix}$$

$$\Rightarrow x_2 = \frac{9998}{9999}$$

$$0.0001x_1 + \frac{9998}{9999} = 1$$

$$\Rightarrow x_1 = \frac{10000}{9999}$$

If we do all of this with limited precision (say 3 digits), we do the following:

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -10^4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10^4 \end{bmatrix}$$

$$\Rightarrow x_2 = 1$$

Then if we use the first equation, we get

$$\Rightarrow x_1 = 0$$

This is called **catastrophic cancellation**.

5 September 20 Lecture

First part of AM120 is to solve $Ax = b$ for arbitrary $||A|| = n$.

$$u_{11} = a_{11}, u_{22} = a_{22}$$

Pseudocode did not have 0s in L and U. Second part of code is given L and b, should output c. Third part takes U and c and outputs x.

Assignment 2 Due on Monday morning (9am).

This Doolittle algorithm can fail:

1. If there is a pivot $= 0$
   (a) System is singular $\Rightarrow \det(A) = 0$. This means there is no solution or infinitely many solutions.
   (b) We can exchange rows and 'cure' system.

$$\det(A) = \det(L)\det(U) = 1\prod_{k=1}^{n} u_{kk}$$

Example From last class:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This had true solution:

$$x_1 = \frac{10000}{9999}, x_2 = \frac{9998}{9999}$$

But with limited precision (three digit arithmetic), we got:

$$x_1 = 0, x_2 = 1$$

What if we switch the rows?

$$\begin{bmatrix} 1 & 1 \\ 0.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 10^{-4} & 1 \end{bmatrix}_L \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - 2 \cdot 10^{-4} t \end{bmatrix}$$

5.1 Finite precision

The computer represents a floating point number with a sign, exponent, and digits for the value itself. When we are talking about n-digit arithmetic, we are referring to the number of digits storing the value.

$$\begin{gathered} U = \text{max exponent} \\ L = \text{lowest exponent} \\ P = \text{mantissa number of digits} \\ \beta = \text{base} \end{gathered}$$

For example, for $L = -1, U = 1, p = 2$ and $\beta = 10$:

$$(\text{sign}) d_0.d_1 d_2 ... d_p \times 10^{\text{exponent}}$$

$$\begin{array}{cc} \text{largest} & 9.9 \times 10^1 = 99 \\ \text{smallest (non-zero)} & 1.0 \times 10^{-1} = 0.1 \end{array}$$

Examples of real values in floating point systems:

	β	P	L	U	
IEEE	2	24	-126	123	single
	2	53	-1022	1023	double
HP	10	12	-499	499	

What is the total number of floating point numbers?

$$2(\beta - 1)\beta^{p-1}(u - l + 1) + 1$$

Largest representable number:

$$(\beta - 1).(\beta - 1)...(\beta - 1) \cdot \beta^U$$

Smallest number (absolute value):

$$\beta^L (\text{underflow})$$

Machine precision Note that the difference between the real number and the floating point number chosen depends on exponent.

$$\forall x \in R, \exists \mathrm{fl}(x) = \hat{x} \text{ such that } |x - \hat{x}| \leq \sum |x|$$

Note that this is not really true for all $x \in R$ – only within a certain range. ϵ_{mach} is the largest number s.t. $\mathrm{fl}(1 + \epsilon_{\text{mach}}) > 1$.

Floating point numbers are not associative:

$$A + (B + C)) \neq (A + B) + C$$

6 September 27 Lecture

Last Class Theory: For a non-singular and square matrix A, $\exists P$ (permutation matrix) that reorders rows of A to avoid zeroes in the pivot positions

$Ax = b$ has a unique solution, and with the rows ordered "in advance:"

$$PA = LU \text{ where } L \text{ and } U \text{ are unique.}$$

Note: If A is singular, no Pcan produce a full set of pivots and elimination fails.

Gaussian elimination with partial pivoting: if a pivot is zero, then A is singular

$$Ax = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ -1 \end{bmatrix} = b$$

$$PA = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\tilde{A} = PA = LU = \begin{bmatrix} 1 & 0 & 00 & 1 & 01/2 & 1/2 & 1 \end{bmatrix}_{\tilde{L}} \begin{bmatrix} 2 & -2 & 10 & 4 & 10 & 0 & 6 \end{bmatrix}_{\tilde{U}}$$

6.1 Ill-conditioned Matrices

The presence of round-off error makes it difficult to identify singular matrices.

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ .999 & 1 \end{bmatrix}_L \begin{bmatrix} 1000 & 999 \\ 0 & -.001 \end{bmatrix}_U$$

$$Ax = b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix} \to x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

With limited precision (5-digit arithmetic), A appears singular, because 0.999(999)=998.00.

$$Ax = \hat{b} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix} = b + \delta b = b + 10^{-2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$$

Small change in b and same A, x changes a lot. We call A 'ill-conditioned,' meaning that its 'condition number,' $k(A)$ is big. This is the definition of condition number:

$$\frac{||\delta x||}{||x||} \leq k(A) \frac{||\delta b||}{||b||}$$

Hilbert matrices are of this kind: changing b a little bit, x changes a lot (they are ill conditioned). Condition number of a singular matrix A is ∞.

If $\frac{||\delta x||}{||x||} > 1$ we don't expect to find a solution close to the one we were looking for.

In numberical calculations, singular matrices are indistinguishable from ill-conditioned matrix.

$$-d^2u \; over dx^2 = f(x), 0 \leq x \leq 1$$
$$u(0) = c_1 u(1) = c_2$$

$$u(x+h) = u(x) + hu'(x) + h^2 \frac{u''(x)}{2} + ... + h^k \frac{d^k u}{dx^k}$$

Discretize interval into points x_i, solve for value of u at each point. At each point we can solve the problem as a linear algebra problem. We ignore higher terms, and say:

$$u'(x_i) \approx \frac{u(x_i + h) - u(x_i)}{h}$$

$$-u''(x_i) \approx -\frac{u(x_{i+1} - 2u(x_i) + u(x_{i-1}))}{h^2} = f(x_i)$$

This results in a large matrix:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & -1 & \\ & & & & \ddots \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \\ \vdots \\ f(x_m) \end{bmatrix}$$