

AM 120 Notes

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1 September 6 Lecture

There are two main problems that we will learn how to handle in this class.

1. Find $x \in R^n$ such that $Ax = b$. A is m by n matrix, $b \in R^n$ vector
2. Find x and λ such that $Ax = \lambda x$

Example
$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

There are 3 ways to solve:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \Rightarrow y = 2, x = -1$

$$2. A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} \det A = -3$$

$$x = A^{-1}b \Rightarrow \frac{1}{-3} \begin{bmatrix} +3 \\ -6 \end{bmatrix}$$

3. Kramer's rule

Summary Topics covered in next 3 classes:

1. Geometric interpretation of solving linear systems
2. Matrix notation (LU factorization)
3. Singular cases (no solution, multiple soln's)
4. Efficient way to solve $Ax = b$ using computers

1.1 Geometric interpretation

Example Graphical method:

Row interpretation (plot lines on coordinate system):

$$\begin{array}{l} 2x - y = 1 \\ x + y = 5 \end{array} \quad \text{Solution: } x = 2, y = 3$$

Column interpretation:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Example 3 by 3 system:

Each row represents a plane:

$$2u + v + w = 5$$

$$4u - 6v + 0 = -2$$

$$-2u + 7v + 2w = 9$$

Remember: inner product of vector with another vector equals 0 \Rightarrow orthogonal.

Column interpretation:

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Example Overdetermined system:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Solution: $c = 1, d = 1$

In 4 dimensions, the rows represent 3-spaces, which are ‘flat’ relative to 4 dimensional space. If we intersect $(x, y, z, t = 0)$ with $(x, y, z = 0, t)$, two three spaces, we get (x, y) plane.

$$a_1u + a_2v + a_3w + a_4z = b$$

$$A = (a_1|a_2|a_3|a_4)$$

1.2 Algorithmic approach

Generalizing to n by n . How to solve $Ax = b$ in a way that scales well? Gaussian elimination (row reduction).

$$\begin{array}{rcl} 2u + v + w & = & 5 \\ 4u - 6v + 0 & = & -2 \\ -2u + 7v + 2w & = & 9 \\ \hline 2u + v + w & = & 5 \\ \Rightarrow -8v - 2w & = & -12 \\ 8v + 3w & = & 14 \\ \hline 2u + v + w & = & 5 \\ \Rightarrow -8v - 2w & = & -12 \\ w & = & 2 \\ \hline \Rightarrow v = 1, u = 1 \end{array}$$

We need a process that takes:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

...this $Ax = b$ problem and transforms it to a $Ux = \hat{b}$ problem. We can get an upper triangular matrix, and obtain solution by back substitution.

Problems One issue that could arise is if the bottom row is all 0s: infinitely many solutions.

2 September 11 Lecture

Last class:

- Introduced first central problem of linear algebra: solving linear equations
- Studied column and row interpretation of linear systems
- Introduced Gaussian elimination

Example (from previous class) Row/Column interpretation.

$$2u + v + w = 5$$

$$4u - 6v + 0 = -2$$

$$-2u + 7v + 2w = 9$$

Row: Three planes intersecting. Column: linear combination of three vectors

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We were trying to figure out how to transform matrix A into an upper triangular matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

Matrix operations Addition is associative: $A+B+C = (A+B)+C = A+(B+C)$

Multiplication: dimension $m \times n$ multiplied by $n \times p$ results in $m \times p$ matrix. $AB \neq BA$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 8 & 7 \end{bmatrix}$$

Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \dots \\ \sum_{i=1}^n a_{ni}x_i \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ Ab_1 & Ab_2 \\ | & | \end{bmatrix}$$

Row reduction In matrix form

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

1. Subtract 2 times row 1 to row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{E_{21}} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}_A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

2. Subtract -1 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{E_{31}} E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

3. Subtract -1 times row 2 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{E_{32}} E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Originally we wanted to solve $Ax = b$. Now we have:

$$E_{32}E_{31}E_{21}A = U$$

where U is an upper triangular matrix.

$$E_{32}E_{31}E_{21}Ax = Ux$$

Let's let $E_{32}E_{31}E_{21} = L$. Then, we have

$$\begin{aligned} L^{-1}A &= U \\ A &= LU \\ Ux &= C = E_{32}E_{31}E_{21}b \end{aligned}$$

Now we can solve by back substitution.

$$L^{-1} = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Matrix inverse properties:

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} \\ (A_1A_2\dots A_n)^{-1} &= A_n^{-1}\dots A_2^{-1}A_1^{-1} \end{aligned}$$

So we have:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$

Row reduction matrices A matrix that subtracts l times row j from row i is such that it includes $-l$ in row i , column j .

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}_L \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}_U$$

L is lower triangular and U is upper triangular.

1. Compute LU factorization
2. Solve for c in $Lc = b$ (forward substitution)
3. Solve for x in $Ux = c$ (back substitution)

We want to solve $Ax = b$. We factor to get $LUx = b$. First we find c such that $Lc = b$

2.1 General Example

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = b_1/l_{11} \\ c_2 = b_2 - b_1 l_{21}/l_{11} \\ c_3 = b_3 - l_{31} b_1 - l_{32} (b_2 - b_1 l_{21}) \end{array}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = c_3/u_{33} \\ x_2 = \frac{1}{u_{22}}(c_2 - u_{23}c_3/u_{33}) \\ x_1 = \dots \end{array}$$

3 September 13 Lecture

Announcements

- Matlab tutorials (sections)
- Final projects
 - Adjustment based on class size
 - Pairs
- Assignment 1 due Fri @ 7pm in Pierce 303
- Collaboration policy

From last time:

- Linear equations \rightarrow Matrix notation
- Column j of $AB = Ab_j$

$$A \begin{bmatrix} | & | & \dots & | \\ b_1 & b_2 & \dots & b_n \\ | & | & & | \end{bmatrix} = [Ab_1 \quad Ab_2 \quad \dots \quad Ab_n]$$

- Introduced the LU factorization of square matrix A (see general example at end of last lecture)

$$Ax = b \Rightarrow LUx = b$$

1. Find LU
2. Solve for c in $Lc = b$
3. Solve for x in $Ux = c$

Example LU factorization

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

1. Subtract 2 times row 1 to row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{E_{21}} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Subtract 3 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}_{E_{31}} E_{21} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

3. Subtract 2 times row 2 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}_E 32 E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_U$$

$$L^{-1} = E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 & -3 & -2 & 1 \end{bmatrix}$$

$$\begin{aligned} L^{-1} A &= U \\ L &= E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} \\ L &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \end{aligned}$$

Generalizing LU factorization To $n \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

1. Introduce zeros below a_{11} by subtracting multiples of row 1

2. Use multipliers $l = \frac{a_{i1}}{a_{11}}$
3. Repeat 1 and 2 for $a_{22}^*, a_{33}^*, \dots$

Step 1:

$$\begin{array}{c|cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \hline 0 & a_{22}^* & a_{23}^* & \dots & a_{2n}^* \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_{n2}^* & a_{n3}^* & \dots & a_{nn}^* \end{array}$$

Step 2:

$$\begin{array}{c|c|ccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \hline 0 & a_{22}^* & a_{23}^* & \dots & a_{2n}^* \\ \hline \dots & 0 & a_{33}^* & \dots & a_{3n}^* \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & a_{n3}^* & \dots & a_{nn}^* \end{array}$$

How many operations does this algorithm use?

$$\sum_{k=1}^n k^2 - \sum_{k=1}^n k = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

4 September 18 Lecture

To review: solving $Ax = b$:

1. Find $LU = A$
2. Solve for c in $Lc = b$ (forward substitution)
3. Solve for x in $Ux = c$ (back-subst)

Multipliers to find U are entries of L .

What is the # of operations needed to get LU factorization?

$$\approx \frac{n^3 - n}{3}$$

Forward substitution Number of operations:

$$(n-1) + (n-2) + \dots + 1 \approx O(n^2)$$

Back substitution is similar process (also $O(n^2)$). Most time consuming place is step 1.

Algorithm Failure This $Ax = b$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, our algorithm won't find the answer because it can't switch rows. If the algorithm fails we have two options:

1. We need to rearrange rows
2. No solution
3. Infinitely many solutions

Example of (2):

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example of (3):

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Fact $\det(A) = \det(LU) = \det(L)\det(U)$

$$\det(U) = \prod_{i=1}^n u_{ii}$$

Example Consider this:

$$\begin{aligned} \begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0.0001 & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ -9998 \end{bmatrix} \\ \Rightarrow x_2 &= \frac{9998}{9999} \\ 0.0001x_1 + \frac{9998}{9999} &= 1 \\ \Rightarrow x_1 &= \frac{10000}{9999} \end{aligned}$$

If we do all of this with limited precision (say 3 digits), we do the following:

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -10^4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10^4 \end{bmatrix}$$

$$\Rightarrow x_2 = 1$$

Then if we use the first equation, we get

$$\Rightarrow x_1 = 0$$

This is called **catastrophic cancellation**.

5 September 20 Lecture

First part of AM120 is to solve $Ax = b$ for arbitrary $\|A\| = n$.

$$u_{11} = a_{11}, u_{22} = a_{22}$$

Pseudocode did not have 0s in L and U . Second part of code is given L and b , should output c . Third part takes U and c and outputs x .

Assignment 2 Due on Monday morning (9am).

This Doolittle algorithm can fail:

1. If there is a pivot = 0
 - (a) System is singular $\Rightarrow \det(A) = 0$. This means there is no solution or infinitely many solutions.
 - (b) We can exchange rows and ‘cure’ system.

$$\det(A) = \det(L) \det(U) = 1 \prod_{k=1}^n u_{kk}$$

Example From last class:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This had true solution:

$$x_1 = \frac{10000}{9999}, x_2 = \frac{9998}{9999}$$

But with limited precision (three digit arithmetic), we got:

$$x_1 = 0, x_2 = 1$$

What if we switch the rows?

$$\begin{bmatrix} 1 & 1 \\ 0.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 10^{-4} & 1 \end{bmatrix}_L \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - 2 \cdot 10^{-4}t \end{bmatrix}$$

5.1 Finite precision

The computer represents a floating point number with a sign, exponent, and digits for the value itself. When we are talking about n -digit arithmetic, we are referring to the number of digits storing the value.

$$\begin{aligned} U &= \text{max exponent} \\ L &= \text{lowest exponent} \\ P &= \text{mantissa number of digits} \\ \beta &= \text{base} \end{aligned}$$

For example, for $L = -1, U = 1, p = 2$ and $\beta = 10$:

$$(\text{sign})d_0.d_1d_2\dots d_p \times 10^{\text{exponent}}$$

$$\begin{array}{ll} \text{largest} & 9.9 \times 10^1 = 99 \\ \text{smallest (non-zero)} & 1.0 \times 10^{-1} = 0.1 \end{array}$$

Examples of real values in floating point systems:

	β	P	L	U	
IEEE	2	24	-126	123	single
	2	53	-1022	1023	double
HP	10	12	-499	499	

What is the total number of floating point numbers?

$$2(\beta - 1)\beta^{p-1}(u - l + 1) + 1$$

Largest representable number:

$$(\beta - 1).(\beta - 1)\dots(\beta - 1) \cdot \beta^U$$

Smallest number (absolute value):

$$\beta^L(\text{underflow})$$

Machine precision Note that the difference between the real number and the floating point number chosen depends on exponent.

$$\forall x \in R, \exists \hat{x} \text{ such that } |x - \hat{x}| \leq \sum |x|$$

Note that this is not really true for all $x \in R$ – only within a certain range. ϵ_{mach} is the largest number s.t. $\text{fl}(1 + \epsilon_{\text{mach}}) > 1$.

Floating point numbers are not associative:

$$A + (B + C) \neq (A + B) + C$$