AM 120 Notes

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Contents

1 September 6 Lecture

There are two main problems that we will learn how to handle in this class.

1. Find $x \in R^n$ such that $Ax = b$. A is m by n matrix, $b \in R^n$ vector
2. Find x and λ such that $Ax = \lambda x$

Example $\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

There are 3 ways to solve:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \Rightarrow y = 2, x = -1$
2. $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix}$ $detA = -3$
   $x = A^{-1}b \Rightarrow \frac{1}{-3} \begin{bmatrix} +3 \\ -6 \end{bmatrix}$
3. Kramer's rule

Summary Topics covered in next 3 classes:

1. Geometric interpretation of solving linear systems
2. Matrix notation (LU factorization)
3. Singular cases (no solution, multiple soln's)
4. Efficient way to solve $Ax = b$ using computers

1.1 Geometric interpretation

Example Graphical method:

Row interpretation (plot lines on coordinate system):

$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$ Solution: $x = 2, y = 3$

Column interpretation:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Example 3 by 3 system:

Each row represents a plane:

2u + v + w = 5

4u - 6v + 0 = -2

-2u + 7v + 2w = 9

Remember: inner product of vector with another vector equals 0 $\Rightarrow$ orthogonal.

Column interpretation:

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Example Overdetermined system:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Solution: $c = 1, d = 1$

In 4 dimensions, the rows represent 3-spaces, which are 'flat' relative to 4 dimensional space. If we intersect $(x, y, z, t = 0)$ with $(x, y, z = 0, t)$, two three spaces, we get (x, y) plane.

$$a_1 u + a_2 v + a_3 w + a_4 z = b$$

$$A = (a_1 | a_2 | a_3 | a_4)$$

1.2 Algorithmic approach

Generalizing to n by n. How to solve $Ax = b$ in a way that scales well? Gaussian elimination (row reduction).

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v + 0 &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

$$\Rightarrow \begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14 \end{aligned}$$

$$\Rightarrow \begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}$$

$\Rightarrow v = 1, u = 1$

We need a process that takes:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

...this $Ax = b$ problem and transforms it to a $Ux = \hat{b}$ problem. We can get an upper triangular matrix, and obtain solution by back substitution.

Problems One issue that could arise is if the bottom row is all 0s: infinitely many solutions.

2 September 11 Lecture

Last class:

- Introduced first central problem of linear algebra: solving linear equations
- Studied column and row interpretation of linear systems
- Introduced Gaussian elimination

Example (from previous class) Row/Column interpretation.

2u + v + w = 5

4u - 6v + 0 = -2

-2u + 7v + 2w = 9

Row: Three planes intersecting. Column: linear combination of three vectors

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We were trying to figure out how to transform matrix A into an upper triangular matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

Matrix operations Addition is associative: $A+B+C = (A+B)+C = A+(B+C)$

Multiplication: dimension $m \times n$ multiplied by $n \times p$ results in $m \times p$ matrix. $AB \neq BA$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 8 & 7 \end{bmatrix}$$

Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & \dots & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \dots \\ \sum_{i=1}^n a_{ni}x_i \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & ... & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + ... + a_nx_n$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & ... & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ Ab_1 & Ab_2 \\ | & | \end{bmatrix}$$

Row reduction In matrix form

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

1. Subtract 2 times row 1 to row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{E_{21}} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}_A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

2. Subtract -1 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{E_{31}} E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

3. Subtract -1 times row 2 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{E_{32}} E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Originally we wanted to solve $Ax = b$. Now we have:

$$E_{32}E_{31}E_{21}A = U$$

where U is an upper triangular matrix.

$$E_{32}E_{31}E_{21}Ax = Ux$$

Let's let $E_{32}E_{31}E_{21} = L$. Then, we have

$$L^{-1}A = U$$
$$A = LU$$
$$Ux = C = E_{32}E_{31}E_{21}b$$

Now we can solve by back substitution.

$$L^{-1} = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Matrix inverse properties:

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A_1A_2...A_n)^{-1} = A_n^{-1}...A_2^{-1}A_1^{-1}$$

So we have:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$

Row reduction matrices A matrix that subtracts l times row j from row i is such that it includes $-l$ in row i, column j.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}_L \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}_U$$

L is lower triangular and U is upper triangular.

1. Compute LU factorization
2. Solve for c in $Lc = b$ (forward substitution)
3. Solve for x in $Ux = c$ (back substitution)

We want to solve $Ax = b$. We factor to get $LUx = b$. First we find c such that $Lc = b$

2.1 General Example

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = b_1/l_{11} \\ c_2 = b_2 - b_1l_{21}/l_{11} \\ c_3 = b_3 - l_{31}b_1 - l_{32}(b_2 - b_1l_{21}) \end{matrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = c_3/u_{33} \\ x_2 = \frac{1}{u_{22}}(c_2 - u_{23}c_3/u_{33}) \\ x_1 = ... \end{matrix}$$

3 September 13 Lecture

Announcements

- Matlab tutorials (sections)
- Final projects
  - Adjustment based on class size
  - Pairs
- Assignment 1 due Fri @ 7pm in Pierce 303
- Collaboration policy

From last time:

- Linear equations $\rightarrow$ Matrix notation
- Column j of $AB = Ab_j$

$$A \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_n \end{bmatrix}$$

- Introduced the LU factorization of square matrix A (see general example at end of last lecture)

$$Ax = b \Rightarrow LUx = b$$

1. Find LU
2. Solve for c in $Lc = b$
3. Solve for x in $Ux = c$

Example LU factorization

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

1. Subtract 2 times row 1 to row 2

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Subtract 3 times row 1 to row 2

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{E_{31}} \quad E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

3. Subtract 2 times row 2 to row 3

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E} \quad 32E_{31}E_{21}A = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

$$L^{-1} = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0-2 & 1 & 0-3 & -2 & 1 \end{bmatrix}$$

$$L^{-1}A = U$$
$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Generalizing LU factorization To $n \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & a_{n3} & ... & a_{nn} \end{bmatrix}$$

1. Introduce zeros below a_{11} by subtracting multiples of row 1

2. Use multipliers $l = \frac{a_{i1}}{a_{11}}$

3. Repeat 1 and 2 for a_{22}^*, a_{33}^*, ...

Step 1:

$$\begin{array}{c|cccc}
a_{11} & a_{12} & a_{13} & ... & a_{1n} \\
\hline
0 & a^*_{22} & a^*_{23} & ... & a^*_{2n} \\
... & ... & ... & ... & ... \\
0 & a^*_{n2} & a^*_{n3} & ... & a^*_{nn}
\end{array}$$

Step 2:

$$\begin{array}{c|c|ccc}
a_{11} & a_{12} & a_{13} & ... & a_{1n} \\
\hline
0 & a^*_{22} & a^*_{23} & ... & a^*_{2n} \\
\hline
... & 0 & a^*_{33} & ... & a^*_{3n} \\
... & ... & ... & ... & ... \\
0 & 0 & a^*_{n3} & ... & a^*_{nn}
\end{array}$$

How many operations does this algorithm use?

$$\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

4 September 18 Lecture

To review: solving $Ax = b$:

1. Find $LU = A$
2. Solve for c in $Lc = b$ (forward substitution)
3. Solve for x in $Ux = c$ (back-subst)

Multipliers to find U are entries of L.
What is the # of operations needed to get LU factorization?

$$\approx \frac{n^3 - n}{3}$$

Forward substitution Number of operations:
$(n-1) + (n-2) + ...(1) \approx O(n^2)$
Back substitution is similar process (also $O(n^2)$). Most time consuming place is step 1.

Algorithm Failure This $Ax = b$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, our algorithm won't find the answer because it can't switch rows. If the algorithm fails we have two options:

1. We need to rearrange rows
2. No solution
3. Infinitely many solutions

Example of (2):

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example of (3):

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Fact $\det(A) = \det(LU) = \det(L)\det(U)$

$$\det(U) = \prod_{i=1}^{n} u_{ii}$$

Example Consider this:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9998 \end{bmatrix}$$

$$\Rightarrow x_2 = \frac{9998}{9999}$$

$$0.0001x_1 + \frac{9998}{9999} = 1$$

$$\Rightarrow x_1 = \frac{10000}{9999}$$

If we do all of this with limited precision (say 3 digits), we do the following:

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -10^4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10^4 \end{bmatrix}$$

$$\Rightarrow x_2 = 1$$

Then if we use the first equation, we get

$$\Rightarrow x_1 = 0$$

This is called **catastrophic cancellation**.

5 September 20 Lecture

First part of AM120 is to solve $Ax = b$ for arbitrary $||A|| = n$.

$$u_{11} = a_{11}, u_{22} = a_{22}$$

Pseudocode did not have 0s in L and U. Second part of code is given L and b, should output c. Third part takes U and c and outputs x.

Assignment 2 Due on Monday morning (9am).

This Doolittle algorithm can fail:

1. If there is a pivot $= 0$

   (a) System is singular $\Rightarrow \det(A) = 0$. This means there is no solution or infinitely many solutions.

   (b) We can exchange rows and 'cure' system.

$$\det(A) = \det(L)\det(U) = 1\prod_{k=1}^{n} u_{kk}$$

Example From last class:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This had true solution:

$$x_1 = \frac{10000}{9999}, x_2 = \frac{9998}{9999}$$

But with limited precision (three digit arithmetic), we got:

$$x_1 = 0, x_2 = 1$$

What if we switch the rows?

$$\begin{bmatrix} 1 & 1 \\ 0.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 10^{-4} & 1 \end{bmatrix}_L \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - 2 \cdot 10^{-4}t \end{bmatrix}$$

5.1 Finite precision

The computer represents a floating point number with a sign, exponent, and digits for the value itself. When we are talking about n-digit arithmetic, we are referring to the number of digits storing the value.

$$\begin{gathered} U = \text{max exponent} \\ L = \text{lowest exponent} \\ P = \text{mantissa number of digits} \\ \beta = \text{base} \end{gathered}$$

For example, for $L = -1, U = 1, p = 2$ and $\beta = 10$:

$$(\text{sign})d_0.d_1d_2...d_p \times 10^{\text{exponent}}$$

$$\begin{array}{cc} \text{largest} & 9.9 \times 10^1 = 99 \\ \text{smallest (non-zero)} & 1.0 \times 10^{-1} = 0.1 \end{array}$$

Examples of real values in floating point systems:

	β	P	L	U	
IEEE	2	24	-126	123	single
	2	53	-1022	1023	double
HP	10	12	-499	499	

What is the total number of floating point numbers?

$$2(\beta - 1)\beta^{p-1}(u - l + 1) + 1$$

Largest representable number:

$$(\beta - 1).(\beta - 1)...(\beta - 1) \cdot \beta^U$$

Smallest number (absolute value):

$$\beta^L (\text{underflow})$$

Machine precision Note that the difference between the real number and the floating point number chosen depends on exponent.

$$\forall x \in R, \exists \text{fl}(x) = \hat{x} \text{ such that } |x - \hat{x}| \leq \sum |x|$$

Note that this is not really true for all $x \in R$ – only within a certain range. ϵ_{mach} is the largest number s.t. $\text{fl}(1 + \epsilon_{\text{mach}}) > 1$.

Floating point numbers are not associative:

$$A + (B + C)) \neq (A + B) + C$$

6 September 25 Lecture

Not converted from paper

7 September 27 Lecture

Last Class Theory: For a non-singular and square matrix A, $\exists P$ (permutation matrix) that reorders rows of A to avoid zeroes in the pivot positions

$Ax = b$ has a unique solution, and with the rows ordered "in advance:"

$$PA = LU \text{ where } L \text{ and } U \text{ are unique.}$$

Note: If A is singular, no Pcan produce a full set of pivots and elimination fails.

Gaussian elimination with partial pivoting: if a pivot is zero, then A is singular

$$Ax = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ -1 \end{bmatrix} = b$$

$$PA = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\tilde{A} = PA = LU = \begin{bmatrix} 1 & 0 & 00 & 1 & 01/2 & 1/2 & 1 \end{bmatrix}_{\tilde{L}} \begin{bmatrix} 2 & -2 & 10 & 4 & 10 & 0 & 6 \end{bmatrix}_{\tilde{U}}$$

7.1 Ill-conditioned Matrices

The presence of round-off error makes it difficult to identify singular matrices.

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ .999 & 1 \end{bmatrix}_L \begin{bmatrix} 1000 & 999 \\ 0 & -.001 \end{bmatrix}_U$$

$$Ax = b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

With limited precision (5-digit arithmetic), A appears singular, because 0.999(999)=998.00.

$$Ax = \hat{b} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix} = b + \delta b = b + 10^{-2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$$

Small change in b and same A, x changes a lot. We call A 'ill-conditioned,' meaning that its 'condition number,' $k(A)$ is big. This is the definition of condition number:

$$\frac{||\delta x||}{||x||} \leq k(A) \frac{||\delta b||}{||b||}$$

Hilbert matrices are of this kind: changing b a little bit, x changes a lot (they are ill conditioned). Condition number of a singular matrix A is ∞.

If $\frac{||\delta x||}{||x||} > 1$ we don't expect to find a solution close to the one we were looking for.

In numberical calculations, singular matrices are indistinguishable from ill-conditioned matrix.

$$-\tfrac{d^2u}{dx^2} = f(x), 0 \leq x \leq 1$$
$$u(0) = c_1 u(1) = c_2$$

$$u(x+h) = u(x) + hu'(x) + h^2\frac{u''(x)}{2} + ... + h^k\frac{d^k u}{dx^k}$$

Discretize interval into points x_i, solve for value of u at each point. At each point we can solve the problem as a linear algebra problem. We ignore higher terms, and say:

$$u'(x_i) \approx \frac{u(x_i+h) - u(x_i)}{h}$$

$$-u''(x_i) \approx -\frac{u(x_{i+1} - 2u(x_i) + u(x_{i-1}))}{h^2} = f(x_i)$$

This results in a large matrix:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & -1 & \\ & & & & \ddots \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \\ \vdots \\ f(x_m) \end{bmatrix}$$

8 October 2 Lecture

Last class Condition numbers. Blah blah blah blah blah.

We have implemented solving $Ax = b$ the same way that MATLAB's "\" function works. Now we move on to other things.

8.1 Over/underconstrained Systems

$$-\frac{d^2u}{dx^2} = f(x), u(0) = \alpha, u(1) = \beta$$

Approximation of second derivative at discrete point x_i:

$$-\frac{d^2u(x_i)}{dx^2} \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))}{h^2}$$

$$h^2 f(x_i) \approx u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))$$

At the boundary:

$$h^2 f(x_1) \approx u(x_2) - 2u(x_1) + \alpha$$

We got this from this, ignoring smaller terms:

$$u(x+h) = u(x) + \frac{du}{dx}(x)h + \frac{d^2u}{dx^2}\frac{h^2}{2} + ...$$

This yields:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & -1 & \\ & & & & \ddots \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_m) \end{bmatrix} = \begin{bmatrix} -\alpha + f(x_1)h^2 \\ f(x_2)h^2 \\ \vdots \\ -\beta + f(x_m)h^2 \end{bmatrix}$$

Need to solve $Ax = b$. Now we are studying methods that have to do with matrices that are not square! First, we'll say $m < n$:

$$\begin{bmatrix} & & A & & \end{bmatrix}_{m\times n} \begin{bmatrix} \\ \\ x \\ \\ \\ \end{bmatrix}_{n\times 1} = \begin{bmatrix} b \end{bmatrix}_{m\times 1}$$

The left side of A has an $m \times m$ square section. This is underdetermined, so there are many solutions. What is $m > n$:

$$\begin{bmatrix} \\ \\ & A & \\ \\ \\ \end{bmatrix}_{m\times n} \begin{bmatrix} \\ x \\ \\ \end{bmatrix}_{n\times 1} = \begin{bmatrix} \\ \\ \\ b \\ \\ \\ \\ \end{bmatrix}_{m\times 1}$$

This is an overconstrained system, which may not have a solution. A real example of this is fitting a line to points in a least-squares sense. If each point is at (t_i, y_i), and we want to find a line $y = mt + b$, then we solve:

$$\begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \\ t_m & 1 \end{bmatrix}_A \begin{bmatrix} m \\ b \end{bmatrix}_x = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}_b$$

8.2 Vector Spaces

Vector spaces have two operations:

[Example: $\mathbb{R}^n$]

1. if $x, y \in \mathbb{R}^n$, $x + y = z \in \mathbb{R}^n$
2. if $x \in \mathbb{R}^n, \alpha \in \mathbb{R}$, $\alpha x \in \mathbb{R}^n$

These properties must hold:

Addition: for $x, y, z \in \mathbb{R}^n$

1. Commutativity: $x + y = y + x$
2. Associativity: $x + (y + z) = (x + y) + z$
3. $\exists!$[1] zero vector s.t. $x + 0 = x \forall x \in \mathbb{R}^n$
4. $\exists! -x \in \mathbb{R}^n$ s.t. $x + (-x) = 0 \forall x \in \mathbb{R}^n$

   Scalar multiplication

5. $1x = x$: 1 is scalar
6. $(c_1 c_2)x = c_1(c_2 x)$

[1] '!' indicates there exists some unique zero vector

7. $c(x + y) = cx + cy$

8. $(c_1 + c_2)x = c_1x + c_2x$

A **subspace** is a non-empty subset of a vector space that satisfies all these properties and all linear combinations stay in the subspace.

Example

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} u \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} v$$

9 October 4 Lecture

Use '\' operator in MATLAB from now on unless we specify otherwise. Assignment 4 is now due Tuesday in class.

Last class We discussed vector spaces, with the intention of being able to solve any linear algebra problem, whether it be overdetermined or underdetermined. Vector spaces have two operations, addition and scalar multiplication, with 8 axioms. Note: no notion of proximity or distance (no topology).

Subspace A non-empty subset of a vector space. Closed under addition and scalar multiplication. ($x + y \in$ subspace, $ax \in$ subspace).

$\mathbb{R}^2$	smallest sub-space $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	{zero} element
	largest sub-space	$\mathbb{R}^2$
$\mathbb{R}^3$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	

$$\text{planes and lines that go through } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ are subspaces.}$$

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, Ax = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} x_2$$

Column spaces of A (denoted $C(A)$) is the spaces that contains all linear combinations of the columns of A.

If we have a matrix $\begin{bmatrix} A \end{bmatrix}_{M\times N}$ (with m rows, n columns), then $C(A) \in \mathbb{R}^m$.

b and $\tilde{b} \in C(A), \exists x$ and $\tilde{x}$

$$\begin{aligned} Ax &= b \quad A(x + \tilde{x}) = b + \tilde{b} \\ A\tilde{x} &= \tilde{b} \\ cb & \quad\ \ A(cx) = cAx = cb \end{aligned}$$

Null space:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Null space of A consists of all vectors x such that $Ax = 0$, denoted $N(A) \in \mathbb{R}^n$.

$$Ax = \begin{bmatrix}1\\5\\2\end{bmatrix} x_1 + \begin{bmatrix}0\\4\\4\end{bmatrix} x_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \in N(A), \begin{bmatrix}0\\0\end{bmatrix} \in N(A)$$

Theorem If zero is the only element of $N(A) \Rightarrow$ columns of A are linearly independent.

If $N(A) = \{0\}$ and A is a square matrix $\Rightarrow \exists! x$ such that $Ax = b$ for any b.

Basis for a **vector space** $V\{v_k\}$.

1. v_k's are linearly independent
2. they span V (any $v \in V$ is a linear combination of the basis vectors $\{v_k\}$)

$$\Rightarrow \exists! \text{ way to represent any element of } V$$

$$textdim(V) = \# \text{ of basis vectors}$$

The **complete solution** of a linear system of equations

$$Ax = b \text{ is given by } x = x_p + x_n \text{ (if it exists)}$$

where

$$Ax_p = band Ax_n = 0$$

$$A(x_p + x_n) = b + 0 = b$$

$$A = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} \qquad C(A) = k\begin{bmatrix}1\\2\end{bmatrix} \text{ line}$$

$$N(A) = d\begin{bmatrix}2\\-1\end{bmatrix} \text{ line}$$

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}x_1 & x_2\end{bmatrix} = Ax = b = \begin{bmatrix}3\\6\end{bmatrix}$$

$$x = \begin{bmatrix}3\\0\end{bmatrix}_{x_p} + d\begin{bmatrix}2\\-1\end{bmatrix}_{x_n}$$

Theorem : For any $m \times n$ matrix $A \exists P$ (permutation) and L lower triangular matrix and an $m \times n$ Echelon matrix U such that $PA = LU$

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Let's find LU:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that L is $m \times m$ square, and U is also $m \times n$. Also, we see that U has 2 LI columns.

We already knew that the column space of A lives in $\mathbb{R}^3$, so one column had to be dependent. Now we know that only two vectors in the 4 columns are LI, so the column space is a plane. Null space is also a plane, but it lives in $\mathbb{R}^4$.

Another example:

$$\begin{gathered} (t_1, y_1)' = (0,0) \\ (t_2, y_2) = (2,1) \\ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{gathered}$$

$$\begin{gathered} y = P(t) = x_1 + x_2 t \\ y_1 = P(t_1) = x_1 + x_2 t_1 \\ y_2 = P(t_2) = x_1 + x_2 t_2 \end{gathered}$$

We get $P = \frac{1}{2}t$

10 Oct 9 Lecture

The transpose of a matrix A (denoted A^T) is a matrix with columns directly from rows of A (the ith row becomes the ith column of A^T). $(AB)^T = B^T A^T$.

Colummn space of A Denoted $C(A)$. Contains all linear combinations of the columns of A. $C(A)$ is a subspace of $\mathbb{R}^m$.

Complete solution to the problem $Ax = b$ can be expressed as $x = x_p + x_n$. $x_n \in N(A), x_p$ a particular solution.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$
$$Ax_p = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Should know

1. What is a vector space?
2. When a set of vectors are linearly independent?
3. Dimension of a vector space
4. Basis for a vector space

Theorem $N(A)$ contains only the vector iff zero the columns of A are linearly independant. Therefore, if A is square, $\exists! x \text{for any} b$.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}_L \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_U$$

The *row space of* A is the column space of A^T $(C(A^T))$. It is a subspace of $\mathbb{R}^n$.

The left null space of A is the null space of A^T. $y \in N(A^T)$ if $A^T y = 0$ iff $y^T A = 0$.

Let's calculate the null space of A. We've broken A into LU. Since U was obtained by adding and subtracting rows of A, it follows that $N(A) = N(U)$.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_U \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
3w + 3y = 0 &\Rightarrow w = -y \\
u + 3v - 3y + 2y &= 0 \\
u &= y - 3v
\end{aligned}
$$

$$
x \in N(A) = \begin{bmatrix} y - 3v \\ v \\ -y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

The linear combination of these two vectors generates a plane in $\mathbb{R}^4$, the null space of A.

$$
Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_U \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ (b_3 + b_1) - 2(b_2 - 2b_1) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 52b_1 \end{bmatrix}
$$

$$
\iff b_3 - 2b_2 + 5b_1 = 0 \text{ solvability condition}
$$

Now, we choose:

$$
b = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}, \tilde{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_U \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}
$$

$$
\begin{gathered}
3w + 3y = 3 \Rightarrow w = 1 - y \\
u + 3v + 3(1 - y) + 2y = 1 \Rightarrow u = -2 - 3v + y
\end{gathered}
$$

Now we can write

$$\underbrace{\begin{bmatrix}1&3&3&2\\0&0&3&3\\0&0&0&0\end{bmatrix}}_{U}\begin{bmatrix}u\\v\\w\\y\end{bmatrix}=\begin{bmatrix}1\\3\\0\end{bmatrix}\Rightarrow x=\underbrace{\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}}_{x_p}+\underbrace{v\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}+y\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}}_{x_n}$$

Remarks:

- The null space of A, $N(A)$, and the row space of A, $C(A^T)$, are subspaces of $\mathbb{R}^n$.
- The left null space, $N(A^T)$, and column space of A, $C(A)$ are subspace of $\mathbb{R}^m$.

Transform $A \underbrace{\rightarrow}_{\text{using Gauss elim}} U$ we can immediately identify a basis for $C(A^T)$.

$$A=\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, A^T=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}$$

$$C(A)=\begin{bmatrix}1\\0\end{bmatrix} d \text{ line in } \mathbb{R}^2$$

Row space of $A=\begin{bmatrix}1\\0\\0\end{bmatrix}$ c lives in $\mathbb{R}^3$

$N(A)$ lives in $\mathbb{R}^3$: $Ax=0$:

$$A=\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{0}$$

We see that x_1 must be 0, but the other values are free:

$$x_1=0$$
$$x_2=a$$
$$x_3=b$$

$$x=\begin{bmatrix}0\\1\\0\end{bmatrix}a+\begin{bmatrix}0\\0\\1\end{bmatrix}b \text{ a plane in } \mathbb{R}^3$$

Left null space $A^Ty=0$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \vec{0}$$

Dimension of row space corresponds with # of linearly independent rows. $C(A^T)$ dimension is r (r linearly independent rows). If we have $A_{m\times n}$, then $r \leq m$ and $r \leq n$. $C(A)$ dimension is r (even though they live in different spaces).

The dimension of $N(A)$ is $n - r$. Null space lives in $\mathbb{R}^n$

The dimension of $N(A^T)$ is $m - r$.