TURING MACHINES

Introduction to Turing Machine

Finite Automata

- Regular Languages

Pushdown Automata

- Context-Free Languages

Finite Automata

- Regular Languages

Pushdown Automata

- Context-Free Languages

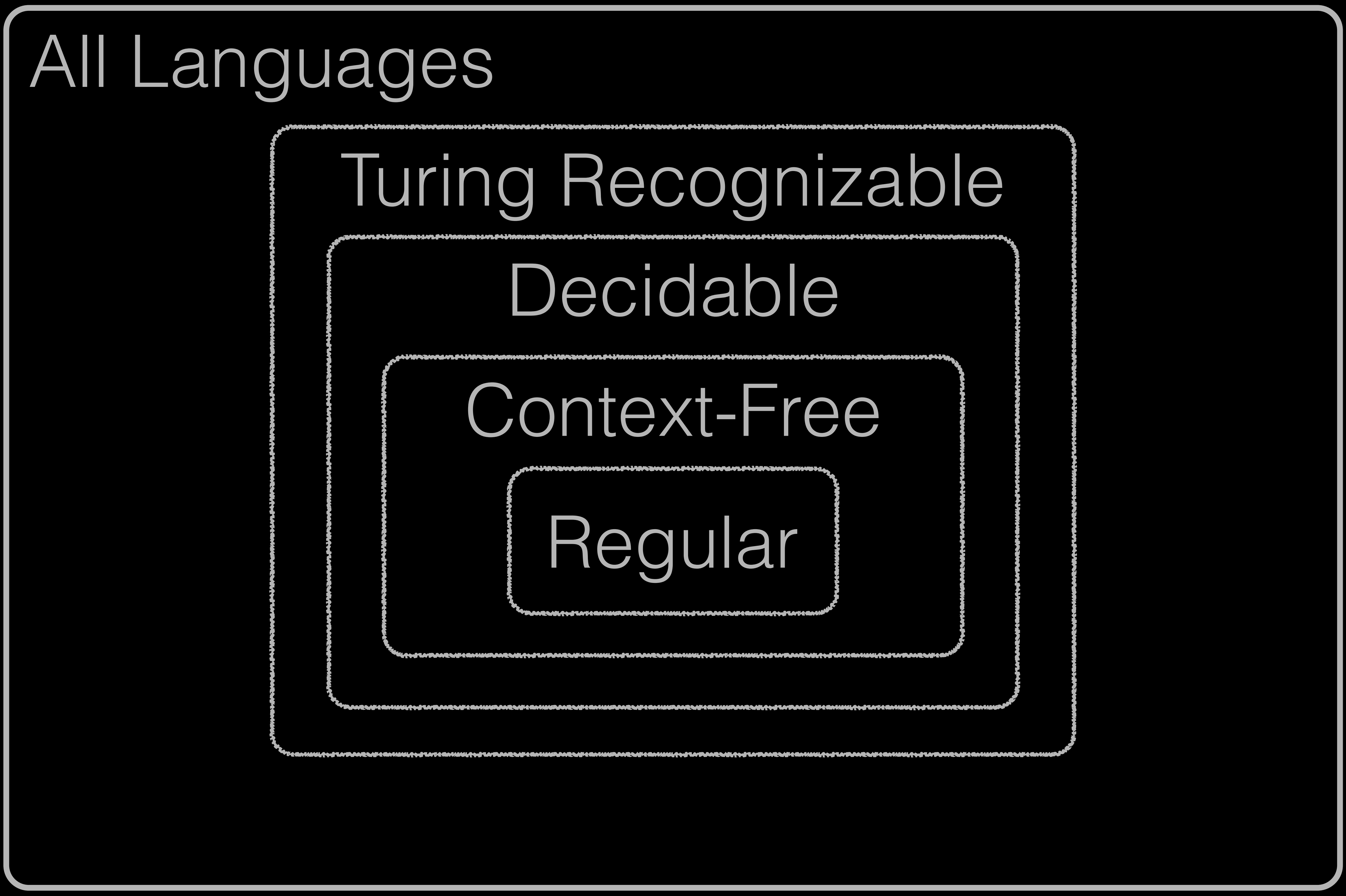
TM: Turing Machines

- A new model of computation
- Not much more elaborate
- A “model” for all computers

TM: Turing Machines

- “decidable” languages
- “Turing recognizable” languages
- languages that are not “Turing recognizable”

CLASSES OF LANGUAGES



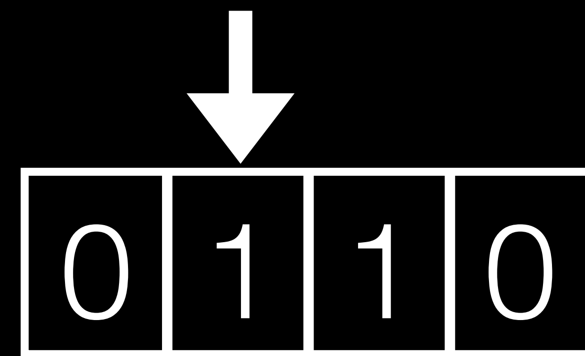
Note: There are variations in the exact definition from textbook to textbook.

All variations are **equivalent.**

DATA STRUCTURES

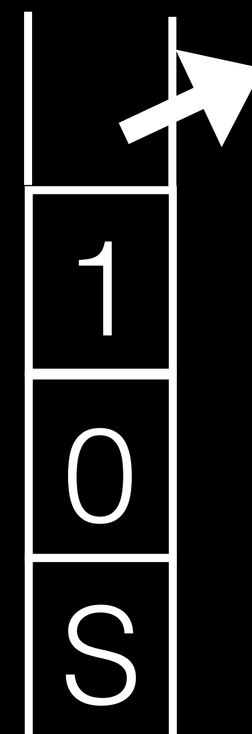
FSM

- the input string



PDA

- the input string
- a stack



TM

- a “tape”

Typical: $\Sigma = \{0, 1\}$

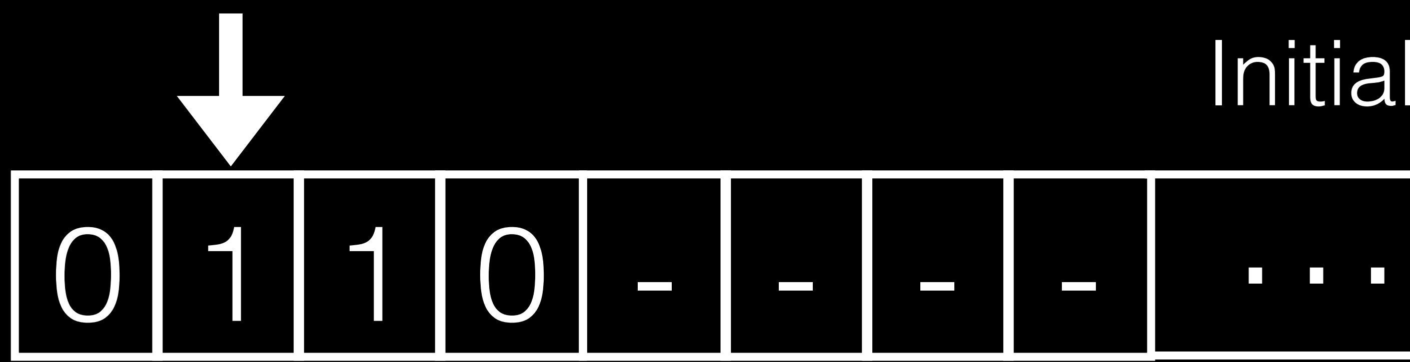
But also common: $\{0, 1, a, b, X, Y, \$\}$

The "blank" symbol is special

$$\text{\_} \notin \Sigma$$

TAPE ALPHABET

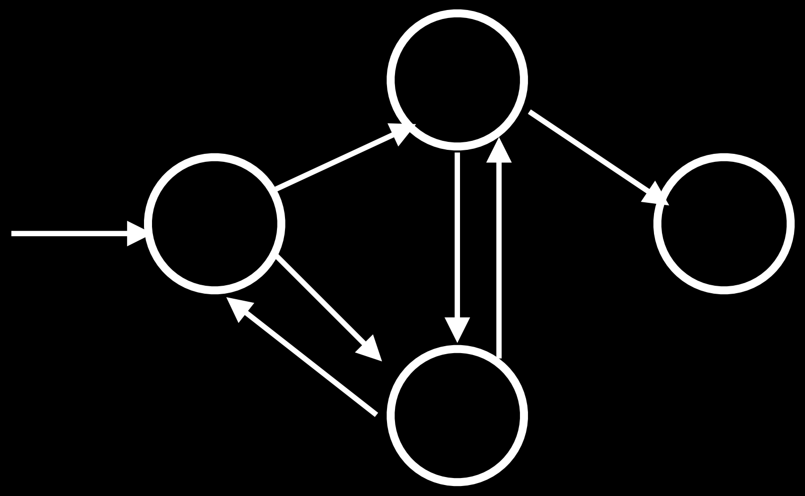
Initial configuration



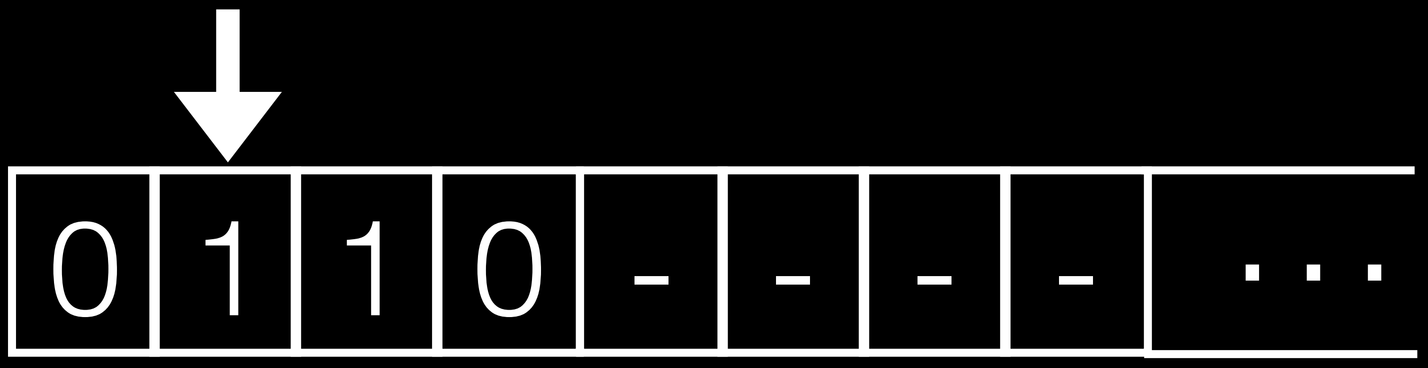
The current position (the tape head)

- initially at the leftmost cell
- can move left or right
- can read (“scan”) the current symbol
- can write the symbol to current position

TAPE ALPHABET



The control portion
Similar to a FSM or PDA
The “program”
Deterministic

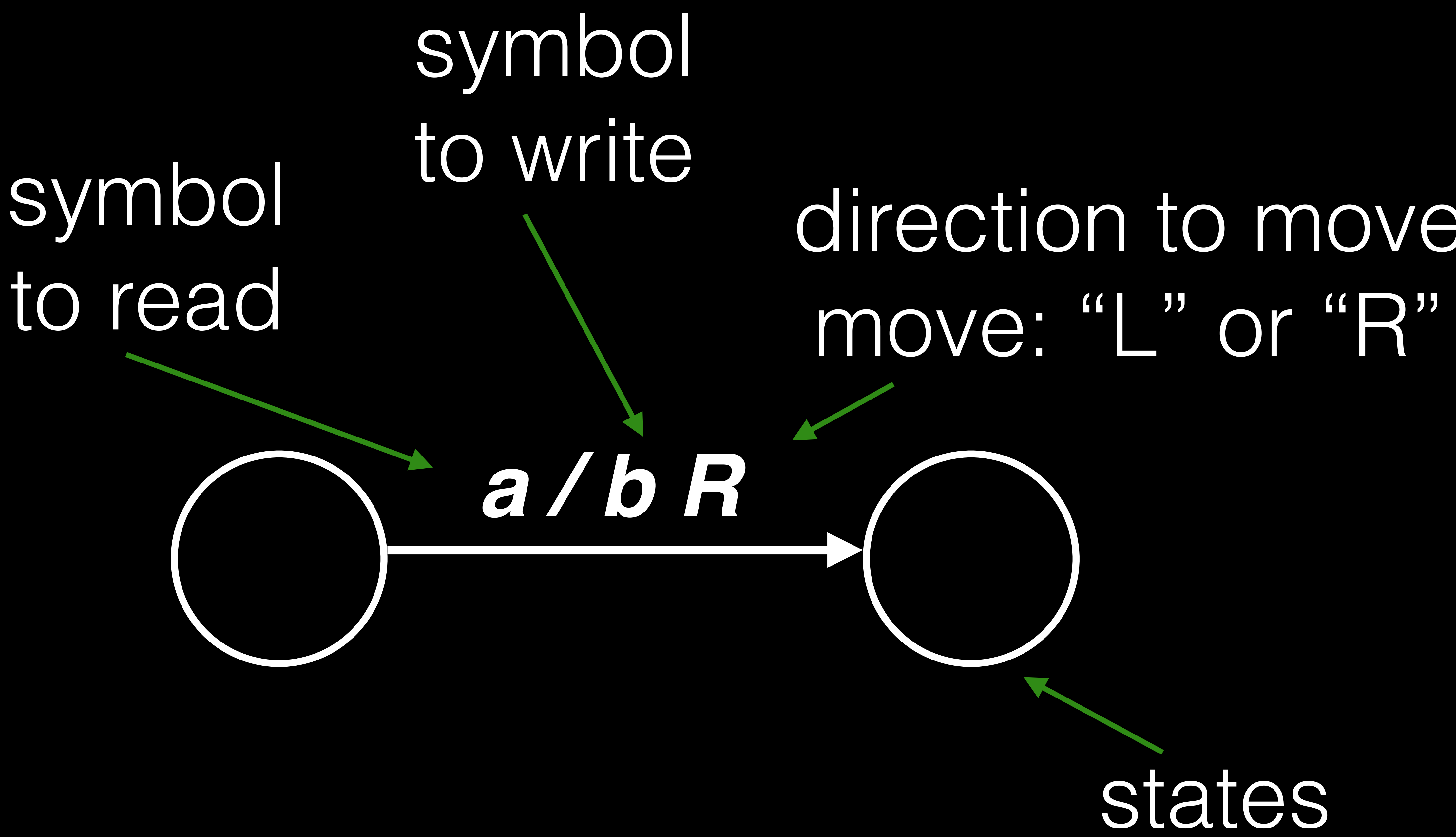


RULES OF OPERATION

At each step of the computation:

- Read the current symbol
- Update (i.e. write) the same cell
- Move exactly one cell either left or right
  - If we are at the left end of the tape and trying to move left, then do not move; stay at the left end.

RULES OF OPERATION



Don’t want to update the cell?

- Just write the same symbol.

RULES OF OPERATION

- Control is with a sort of finite state machine.
- Initial state
- Final states: exactly 2 final states
  - The “accept” state
  - The “reject” state

RULES OF OPERATION

Computation can…

- **halt and "accept"**
  - Whenever the machine enters the "accept" state, computation immediately halts
- **halt and "reject"**
  - Whenever the machine enters the "reject" state, computation immediately halts
- "**loop**"
  - The machine fails to halt

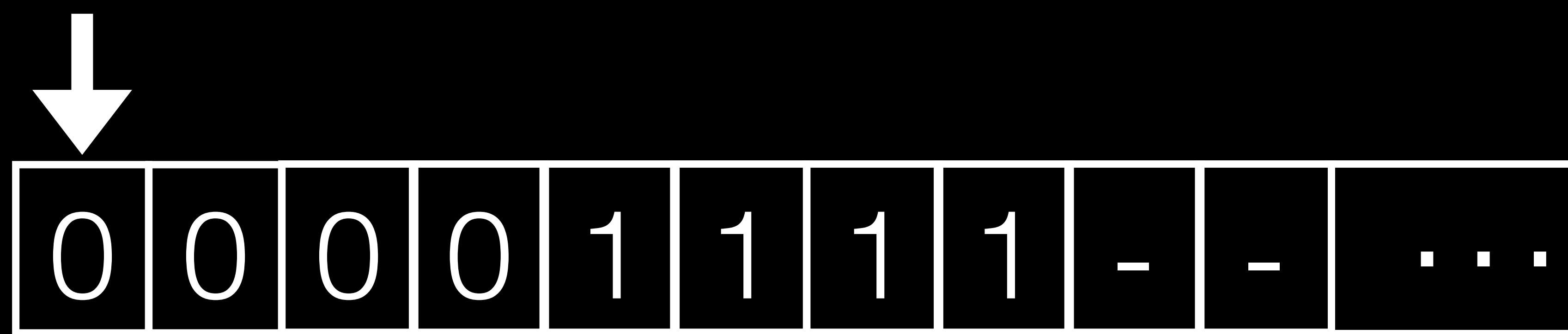
RULES OF OPERATION

The TM is **deterministic**.

Turing Machine Examples

EXAMPLE

$$L = 0^n 1^n$$



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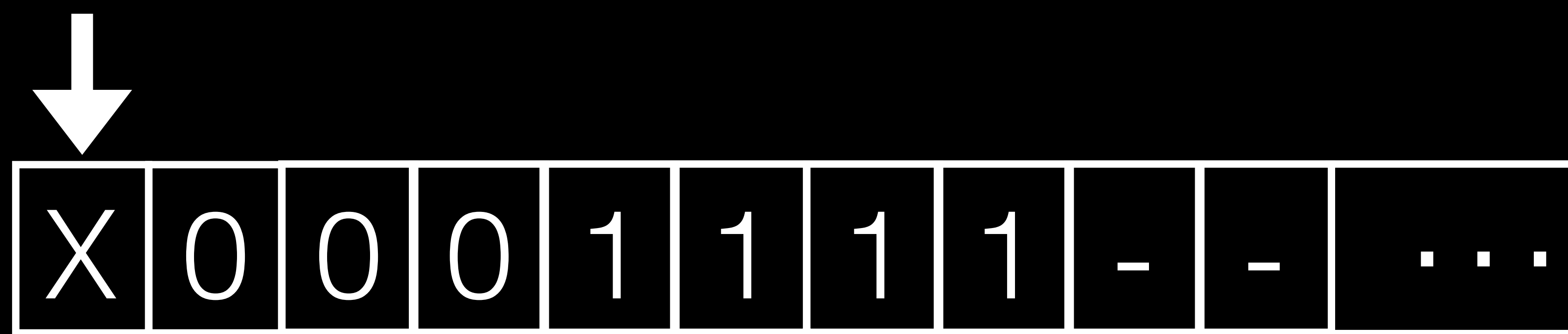

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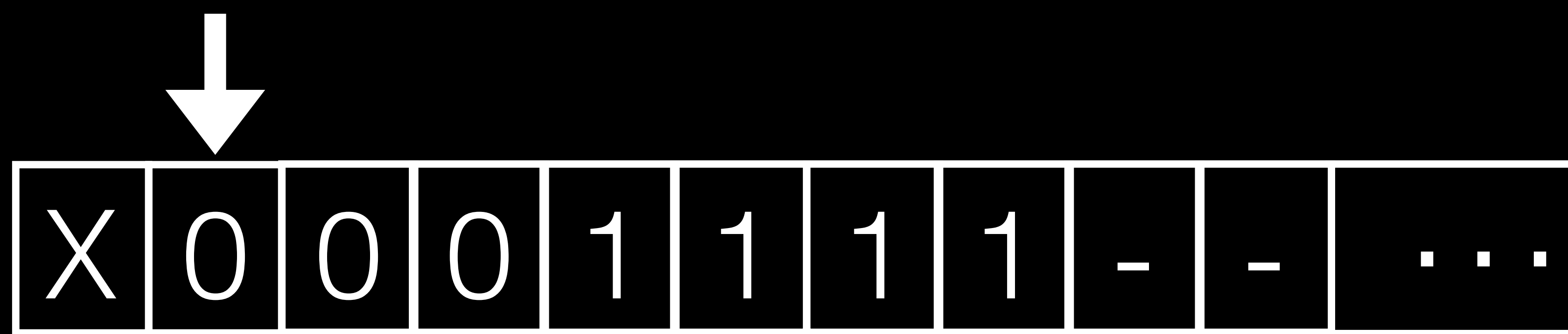

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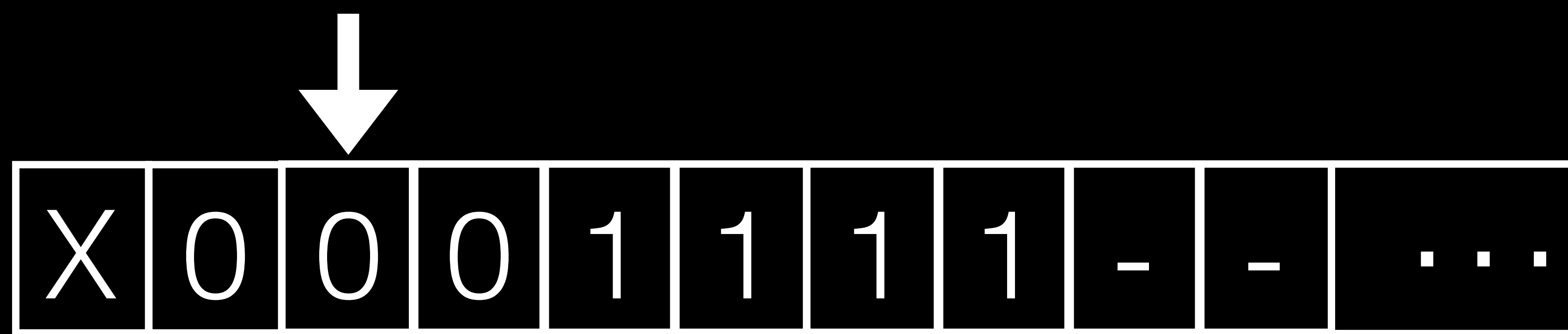

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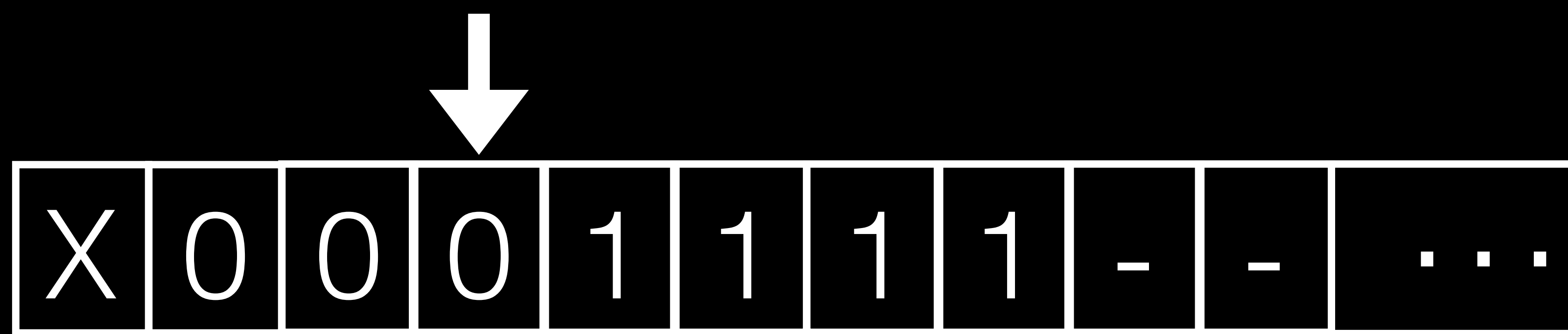

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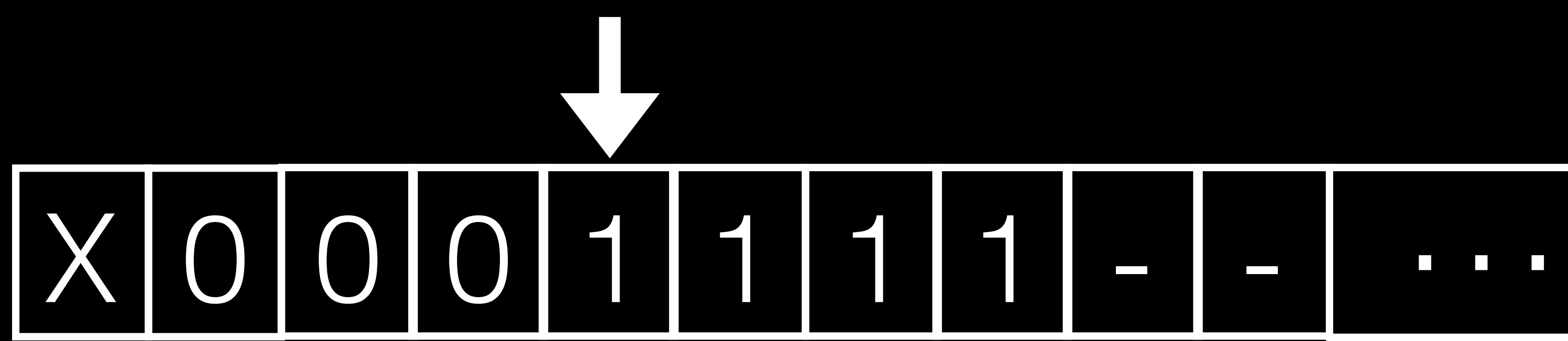

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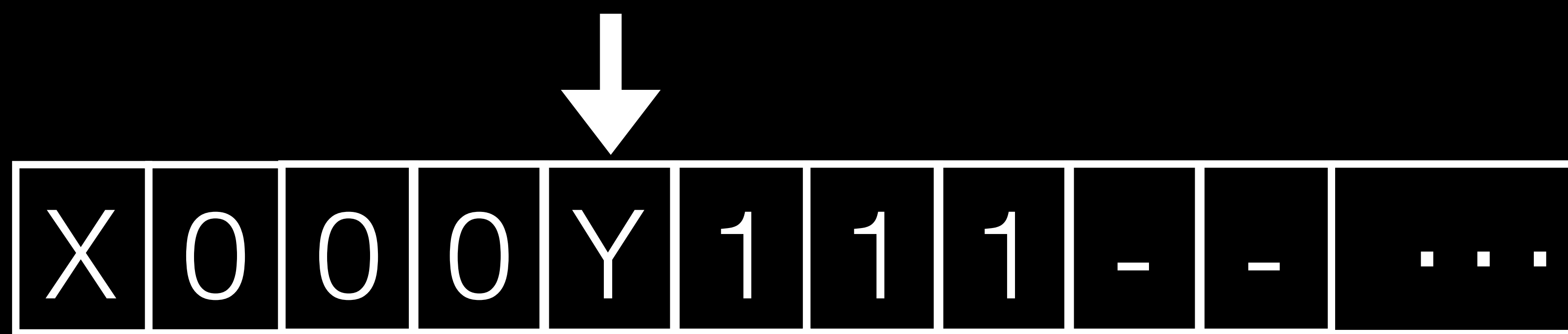

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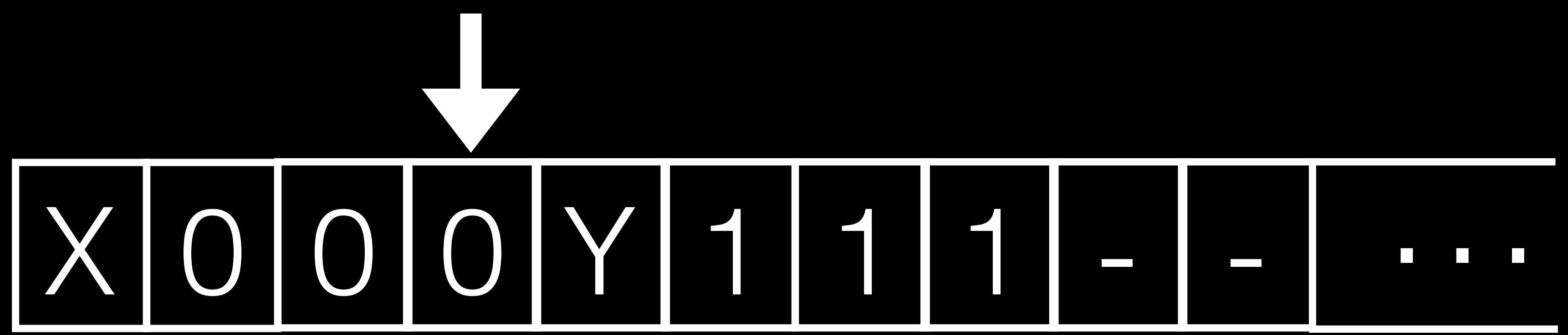

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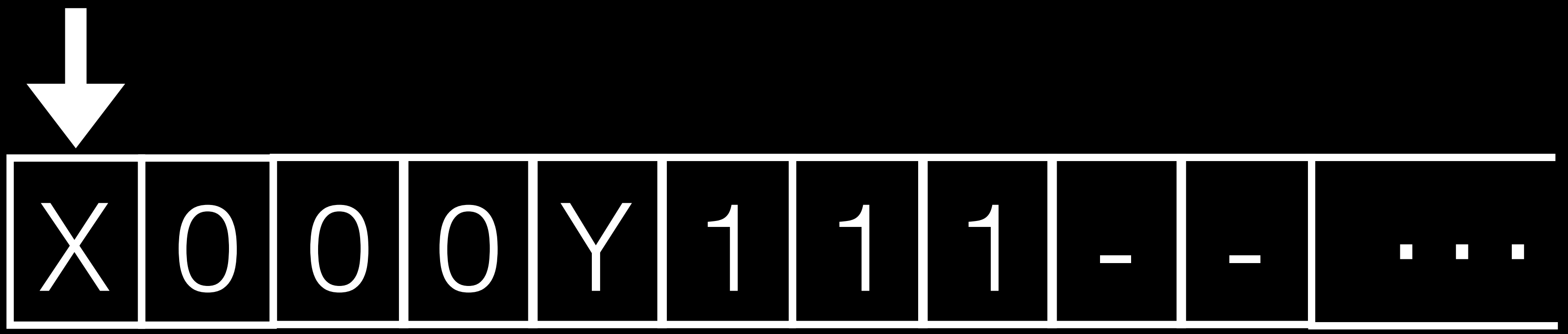

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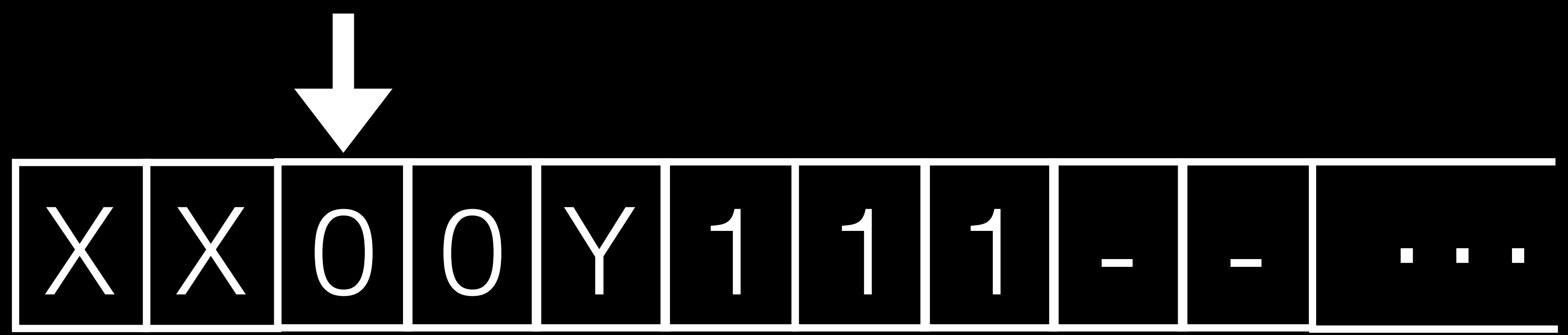

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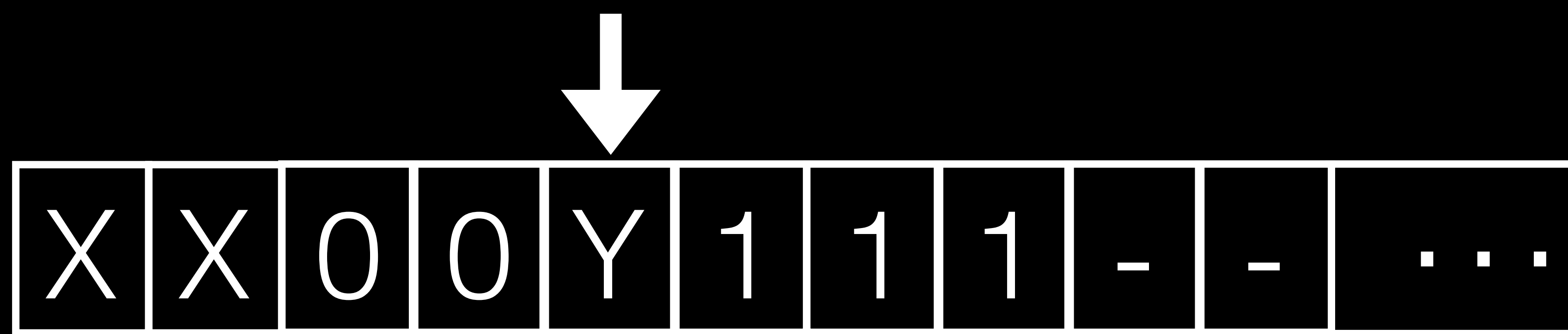

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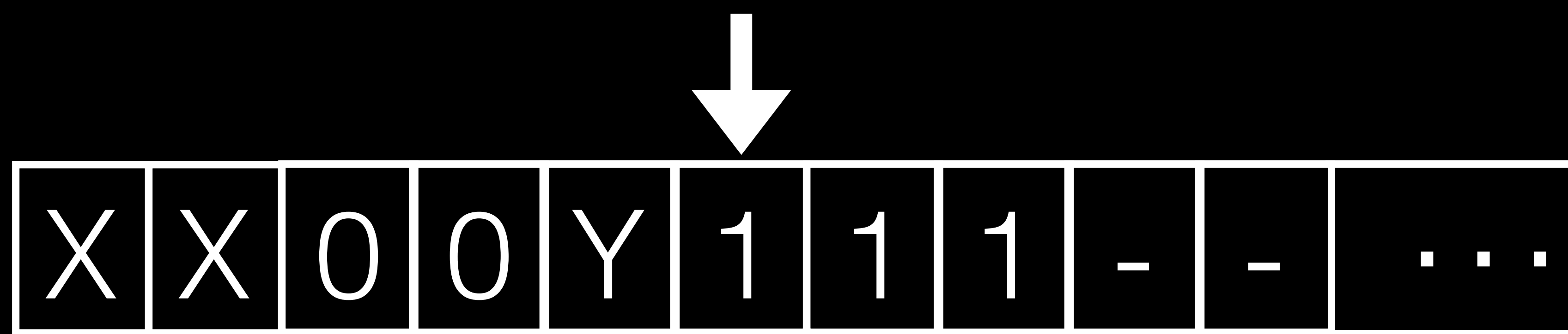

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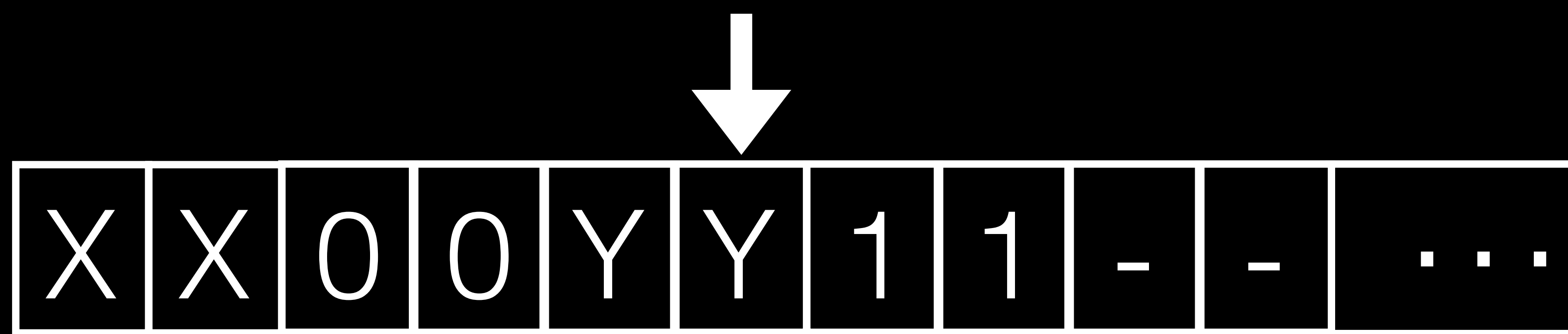

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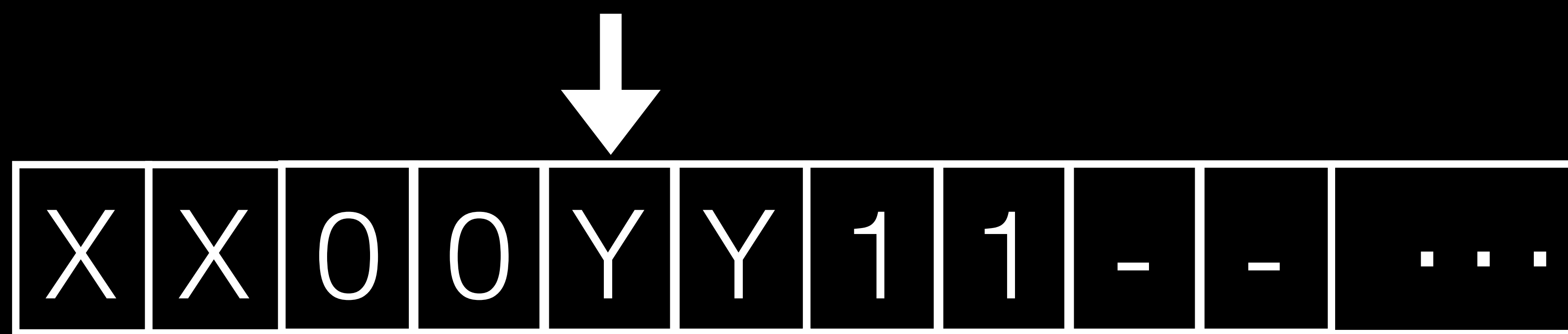

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X	X	0	0	Y	Y	1	1	-	-	...

EXAMPLE

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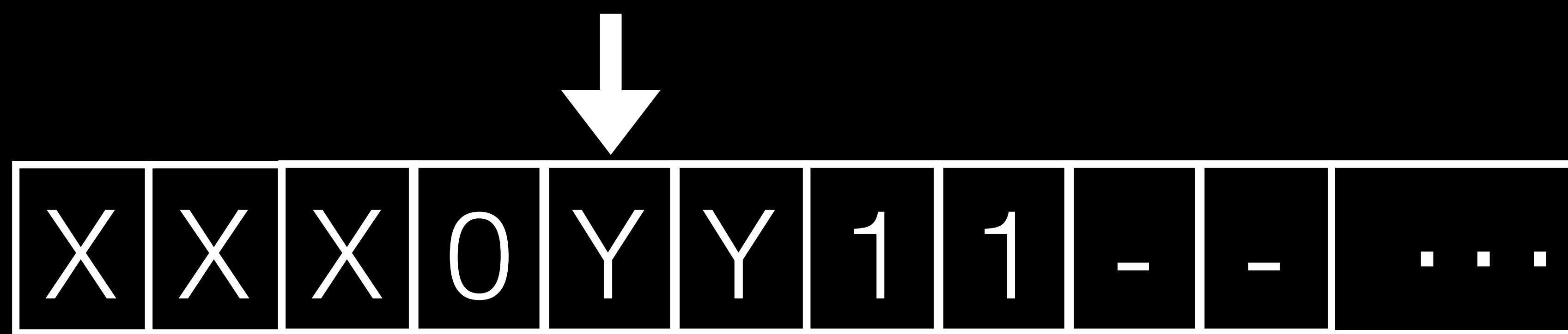

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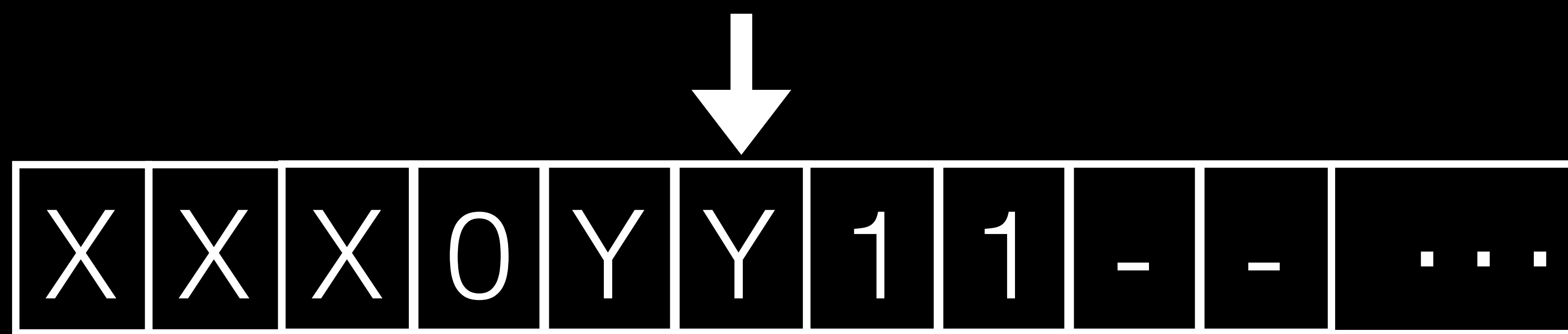

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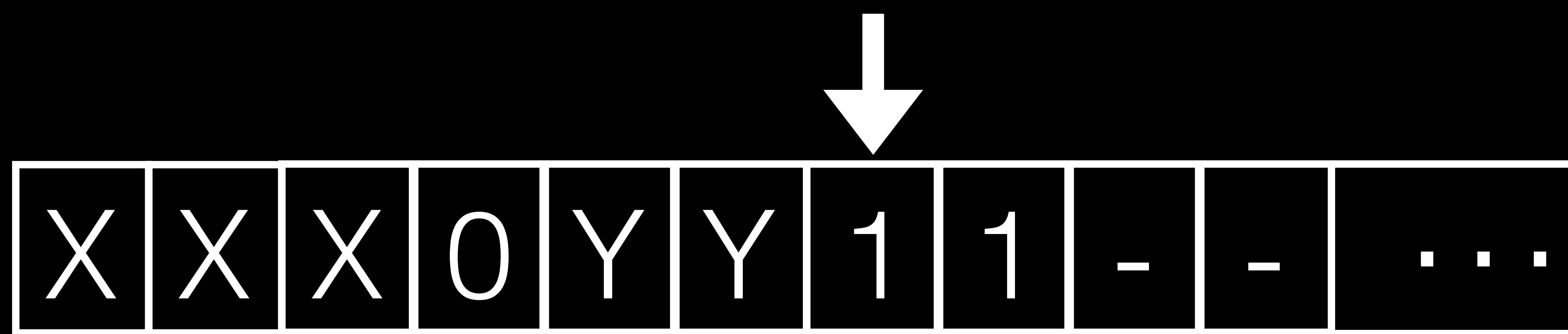

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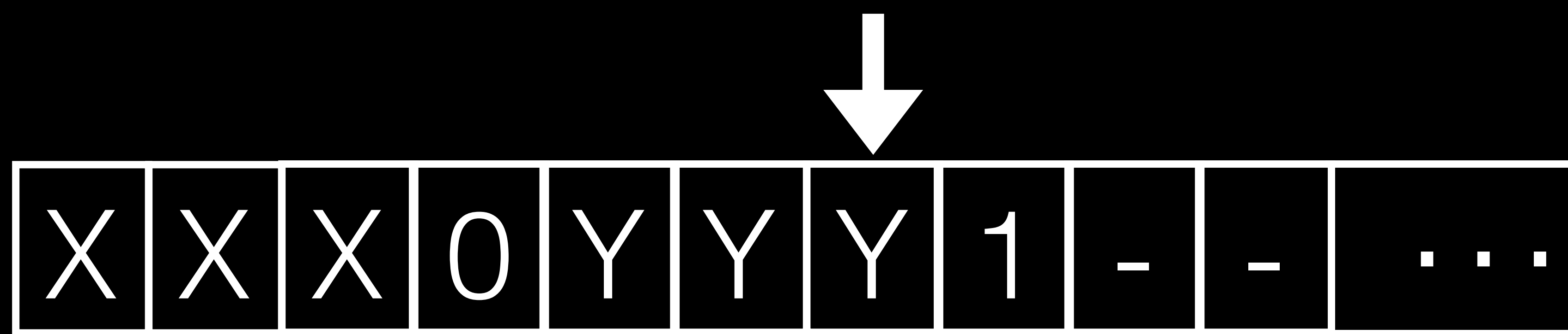

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X	X	X	0	Y	Y	Y	1	-	-	...

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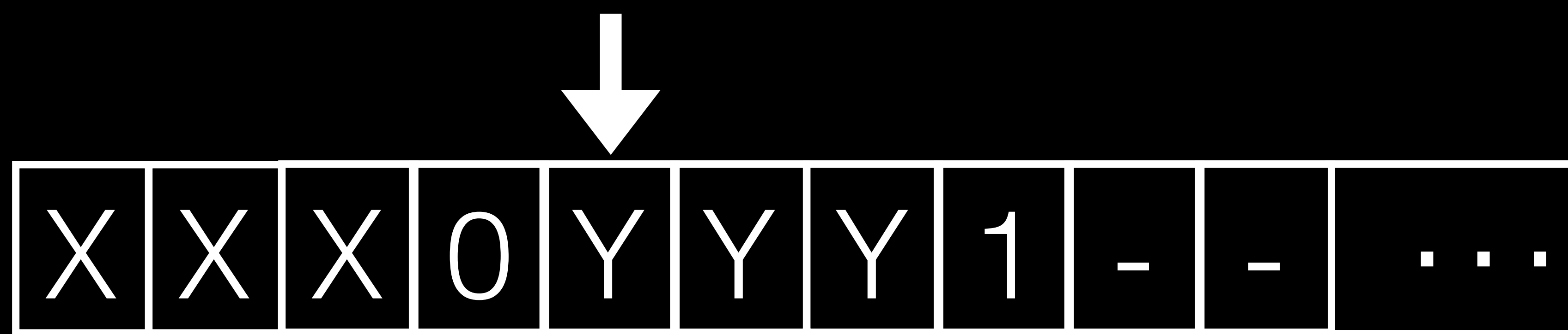

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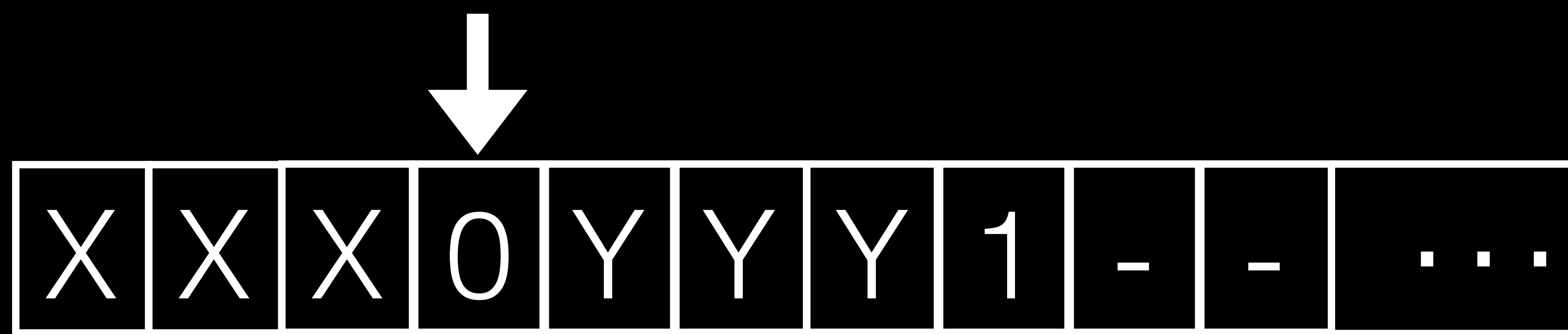

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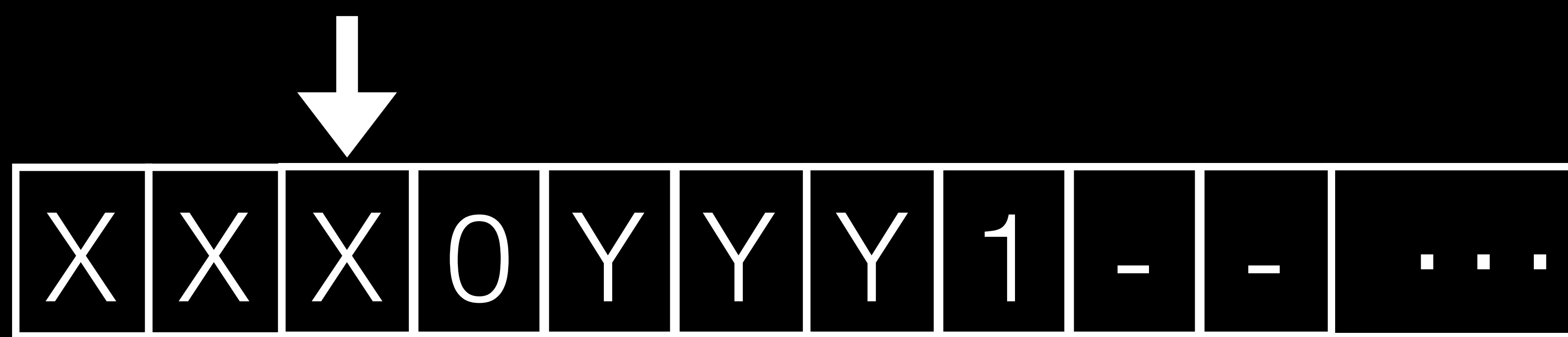

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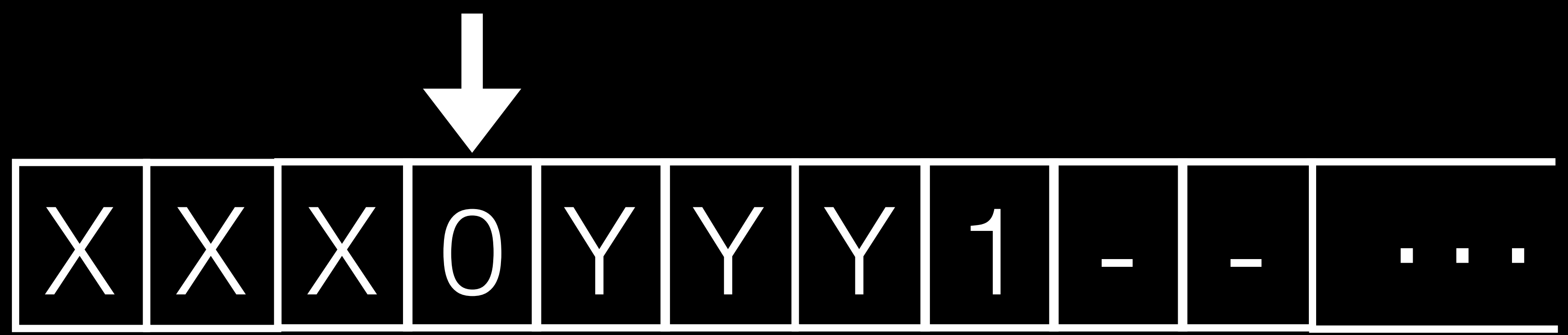

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X	X	X	X	Y	Y	Y	1	-	-	...

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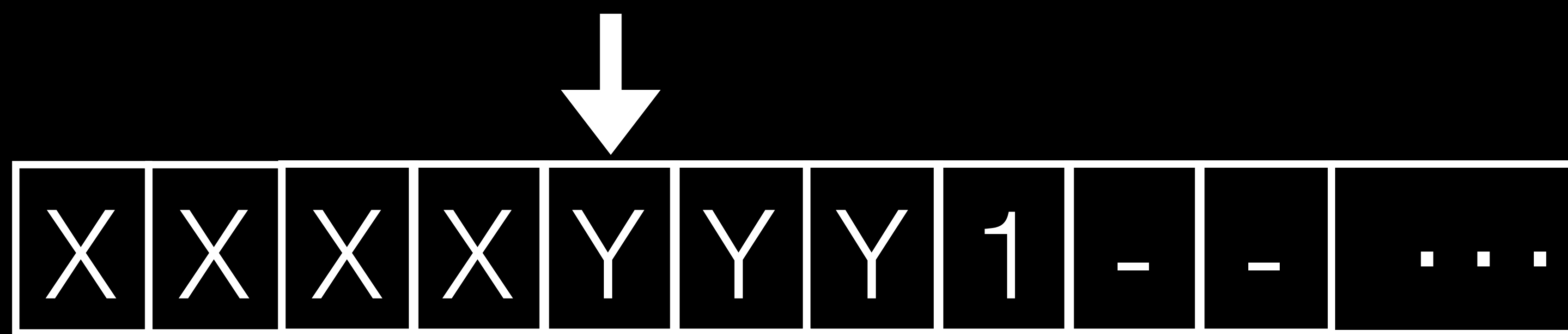

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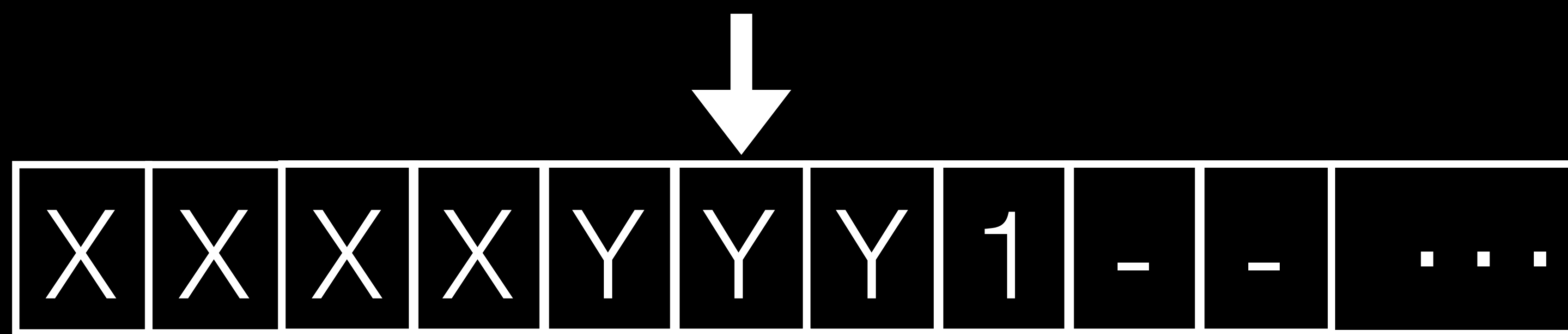

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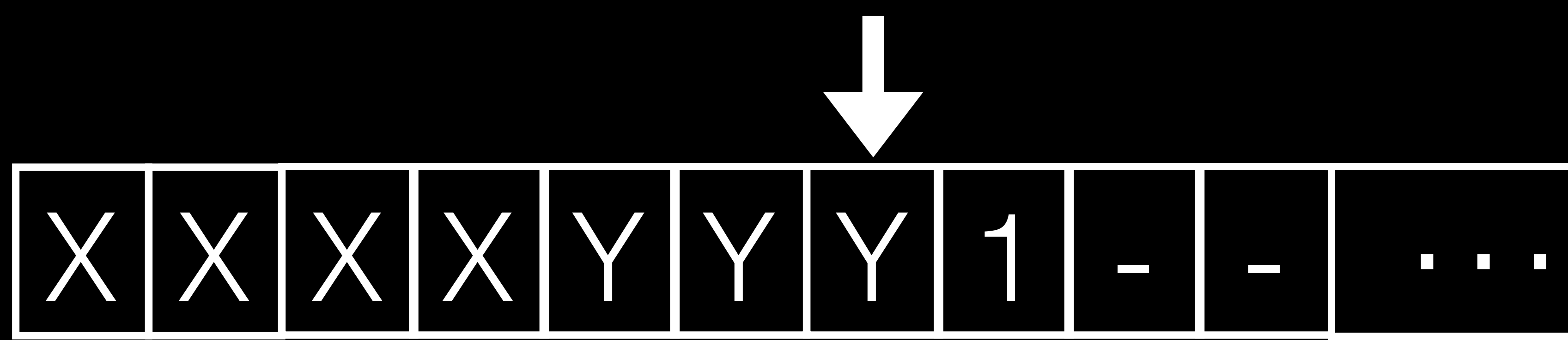

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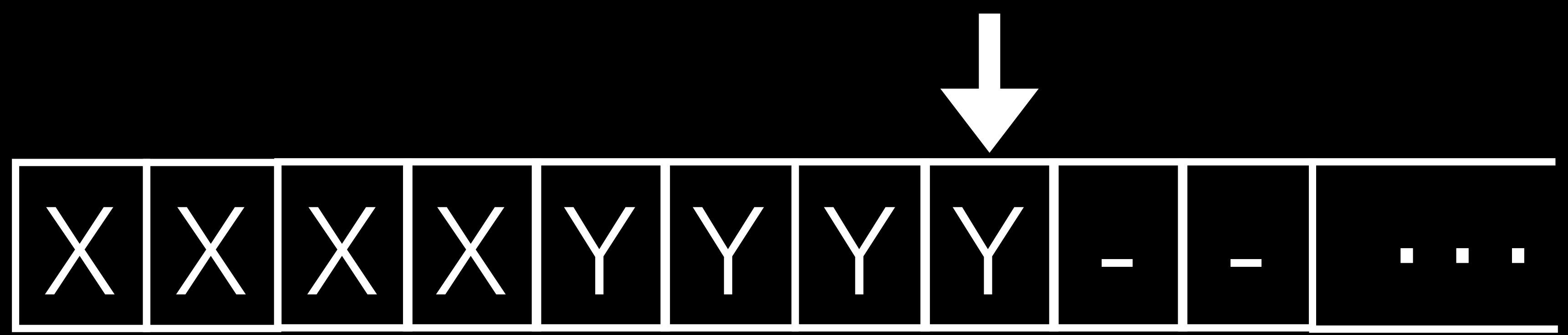

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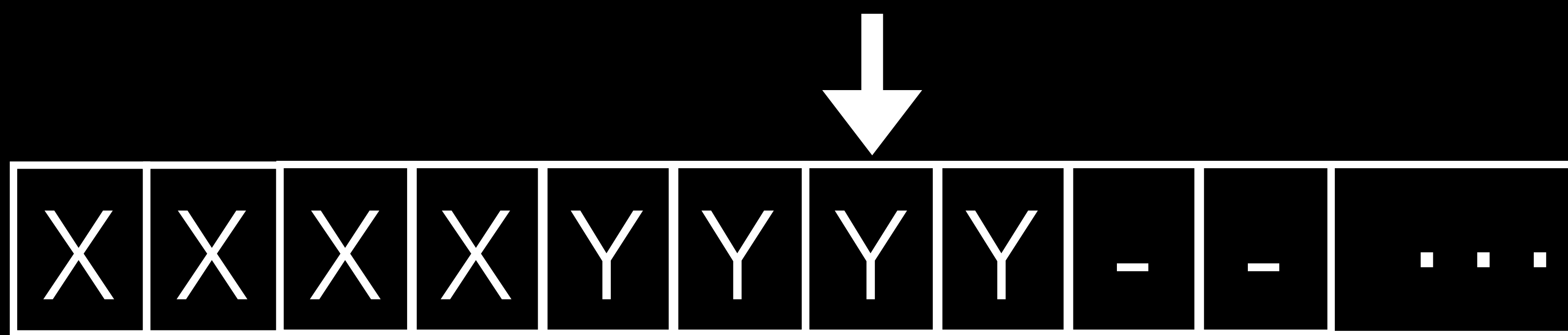

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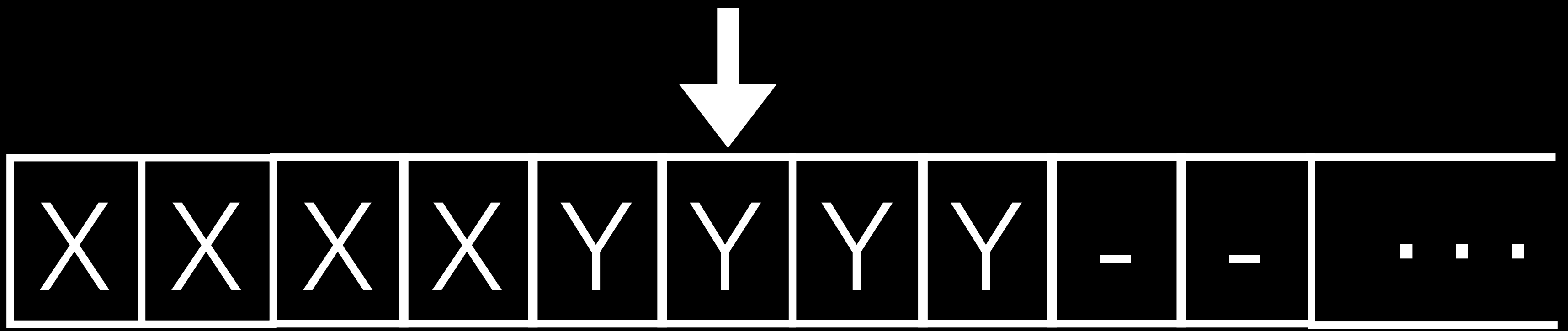

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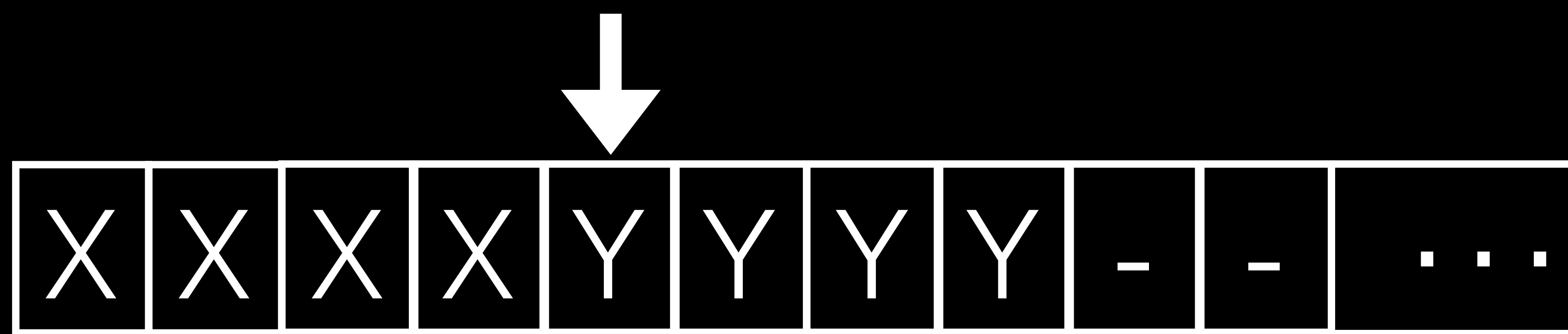

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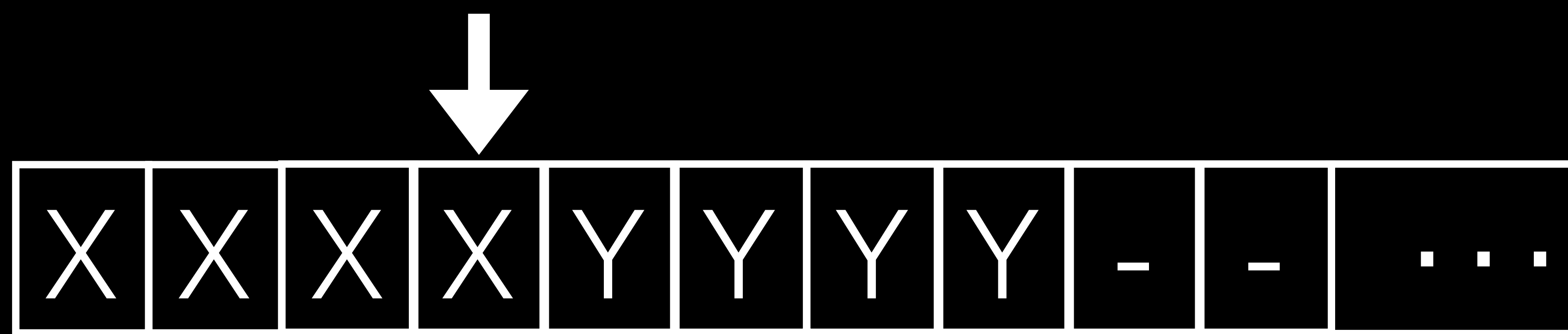

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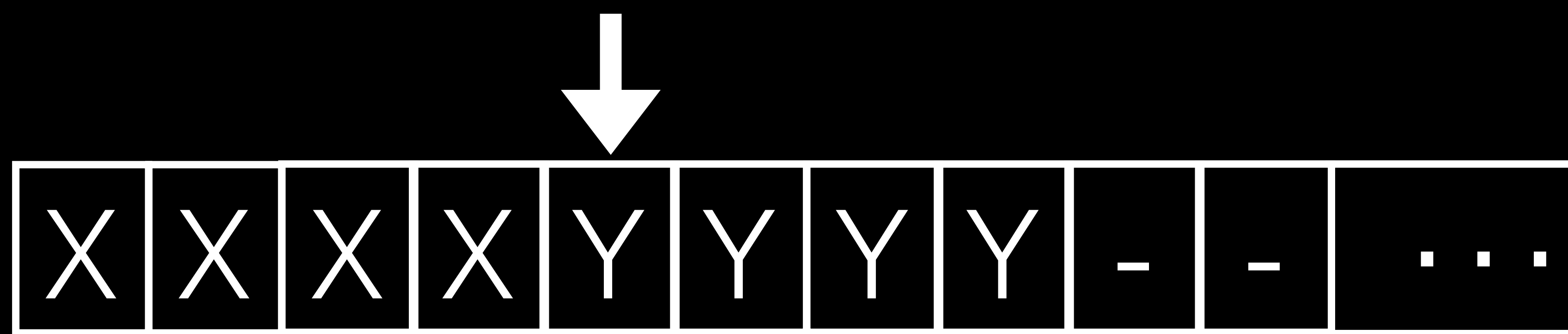

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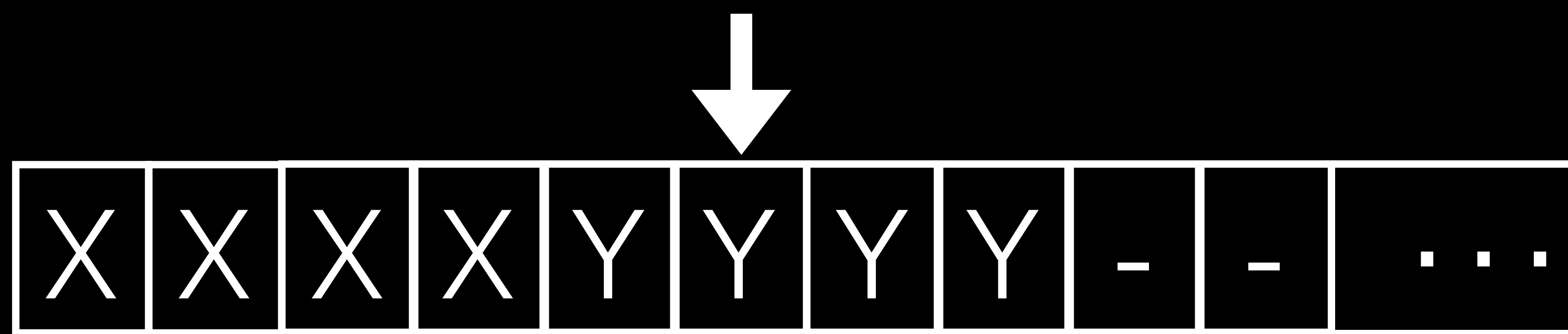

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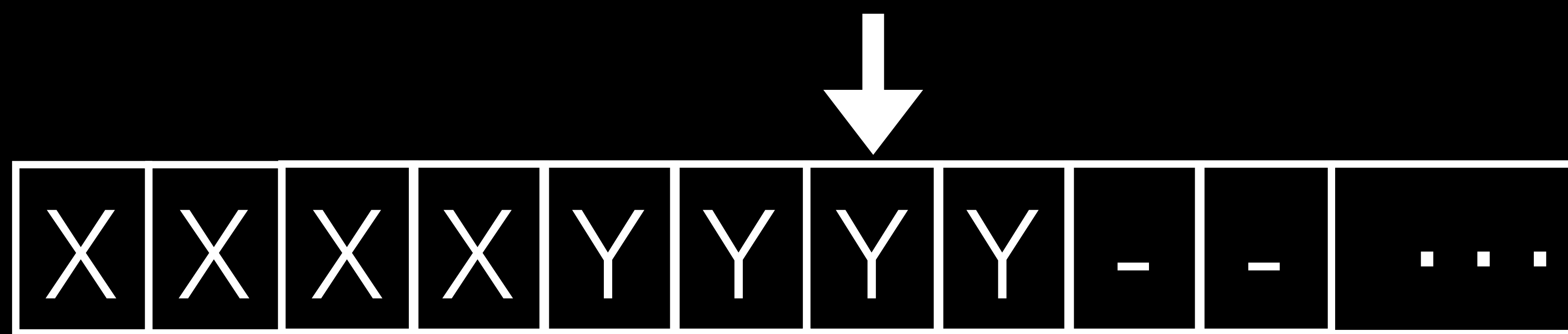

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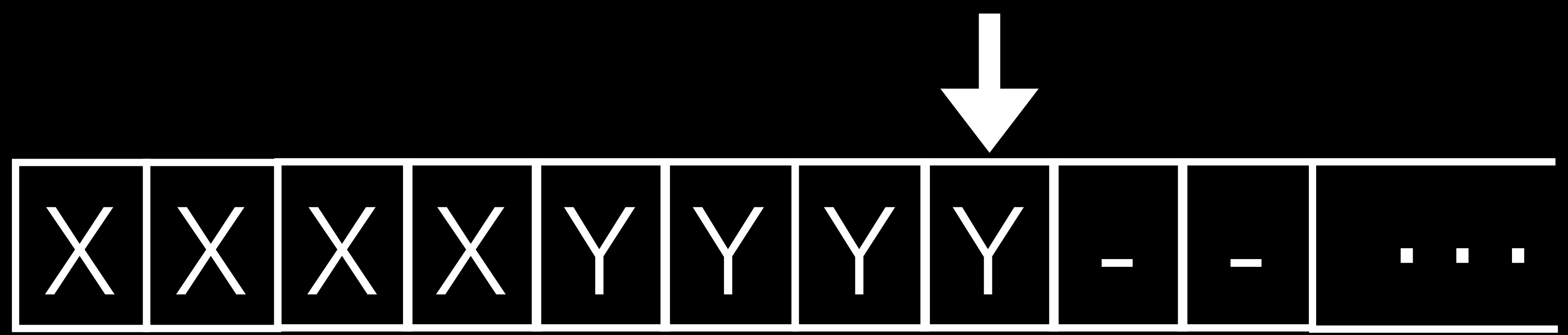

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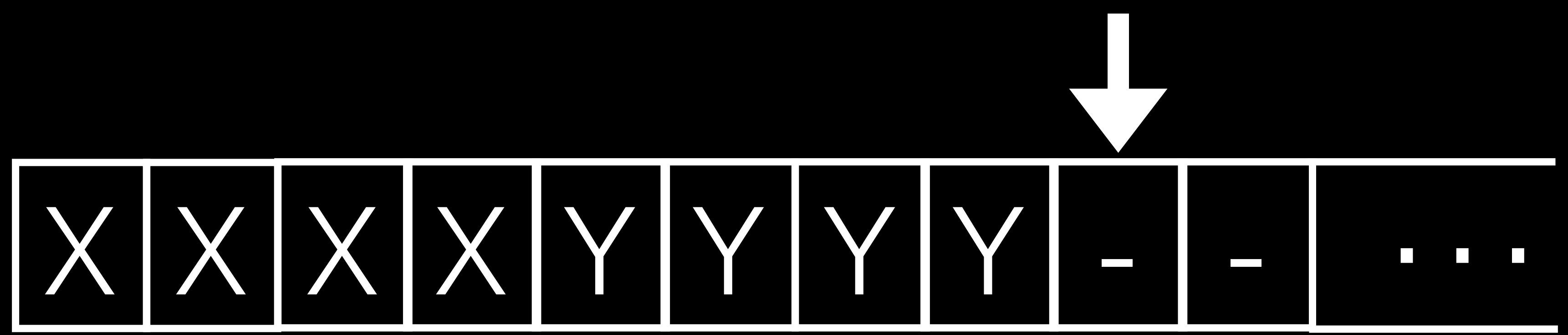

Accept!

EXAMPLE

$L = 0^n 1^n$

Algorithm:

1. Change 0 to X
2. Move right to first 1. If none, reject.
3. Change 1 into Y.
4. Move left to leftmost 0
5. Repeat (1) until no more 0s.
6. Make sure no more 1s remain.

EXAMPLE

$L = 0^n 1^n$

Computation History:

0 0 0 0 1 1 1 1

X 0 0 0 1 1 1 1

X 0 0 0 Y 1 1 1

X X 0 0 Y 1 1 1

. . .

X X X X Y Y Y Y

Tape Alphabet:

$\Gamma = \{0, 1, X, Y, —\}$

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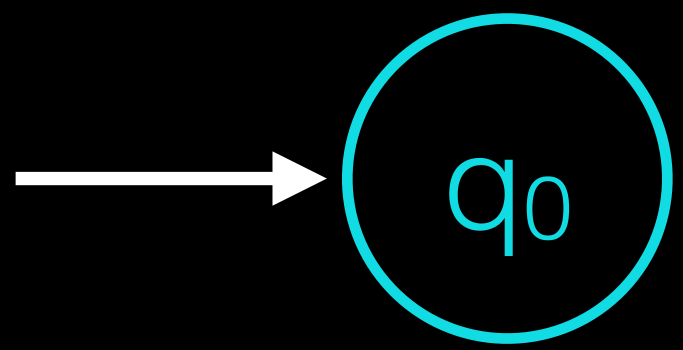

0 0 0 1 1 1

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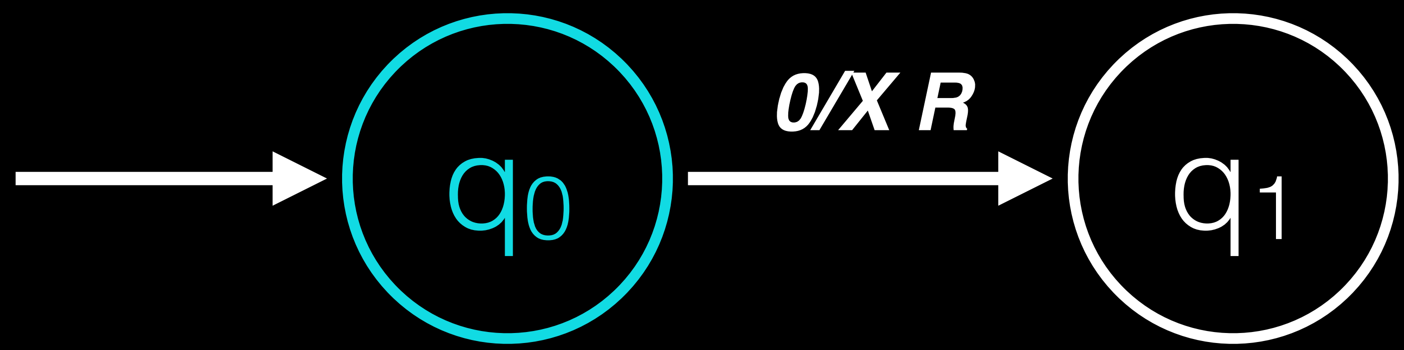

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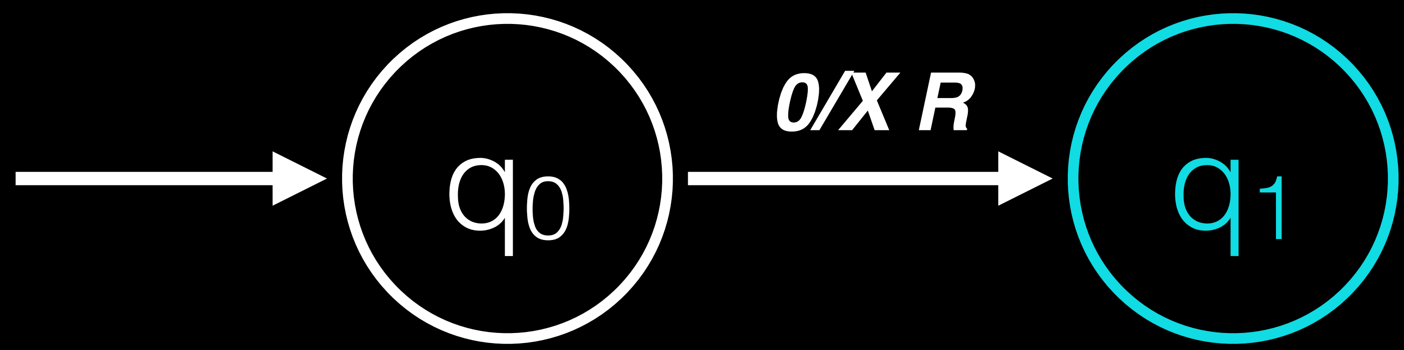

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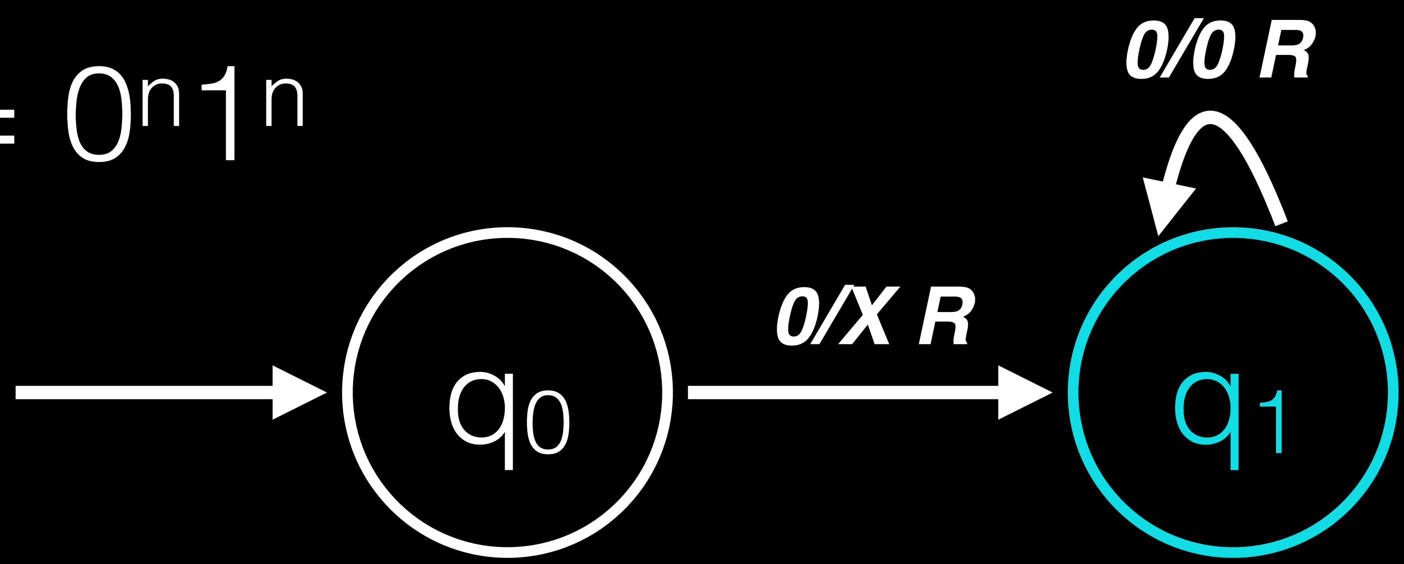

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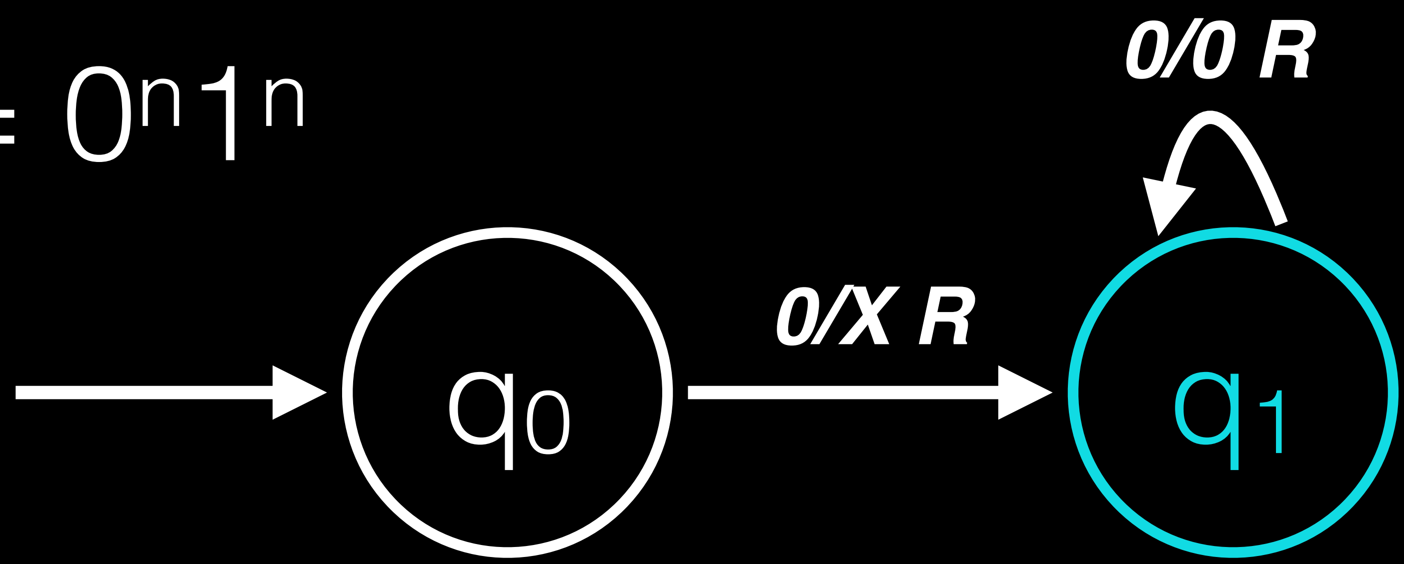

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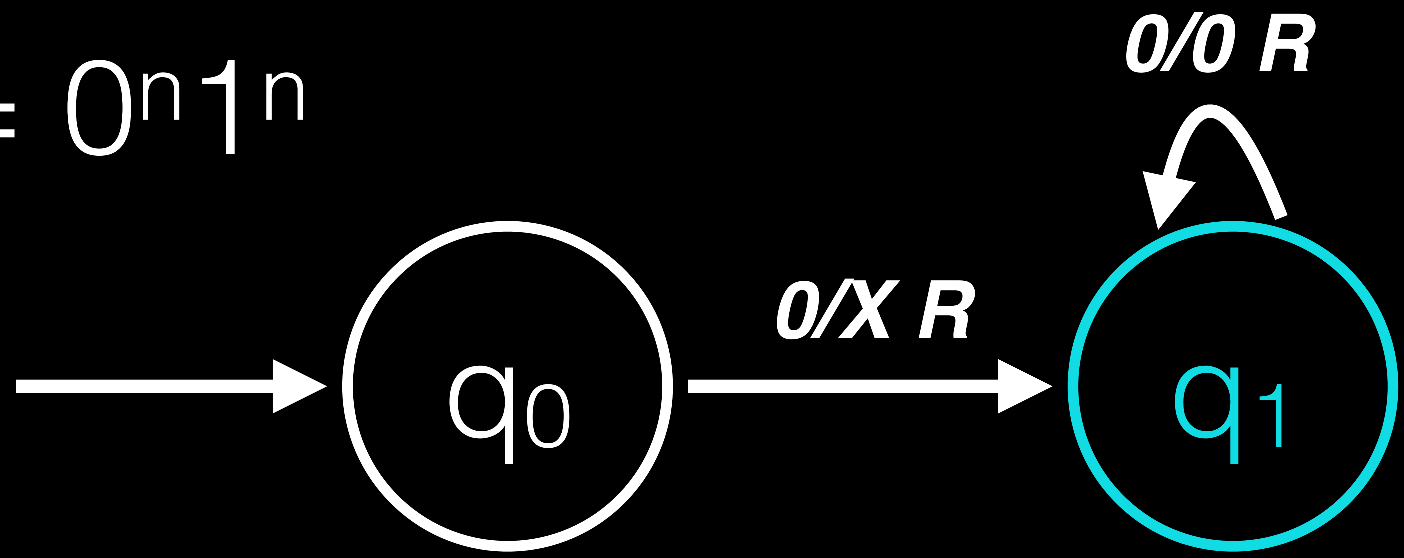

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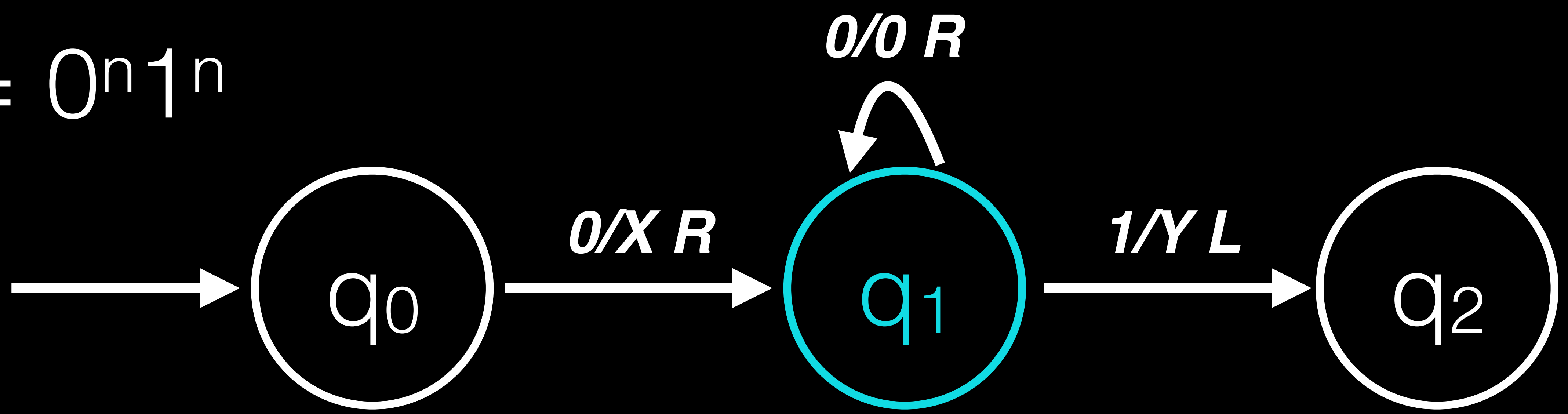

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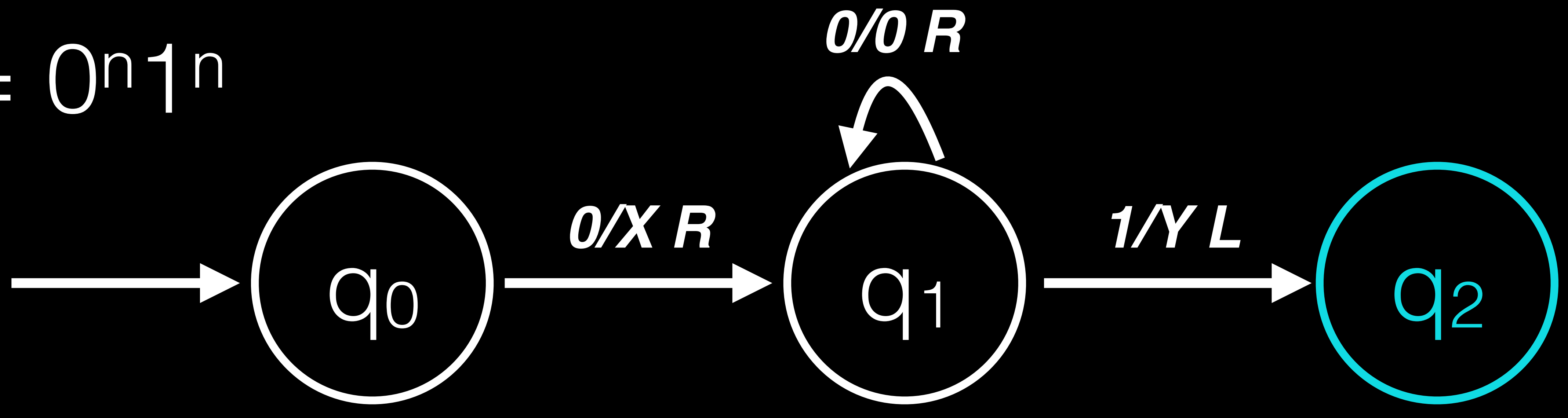

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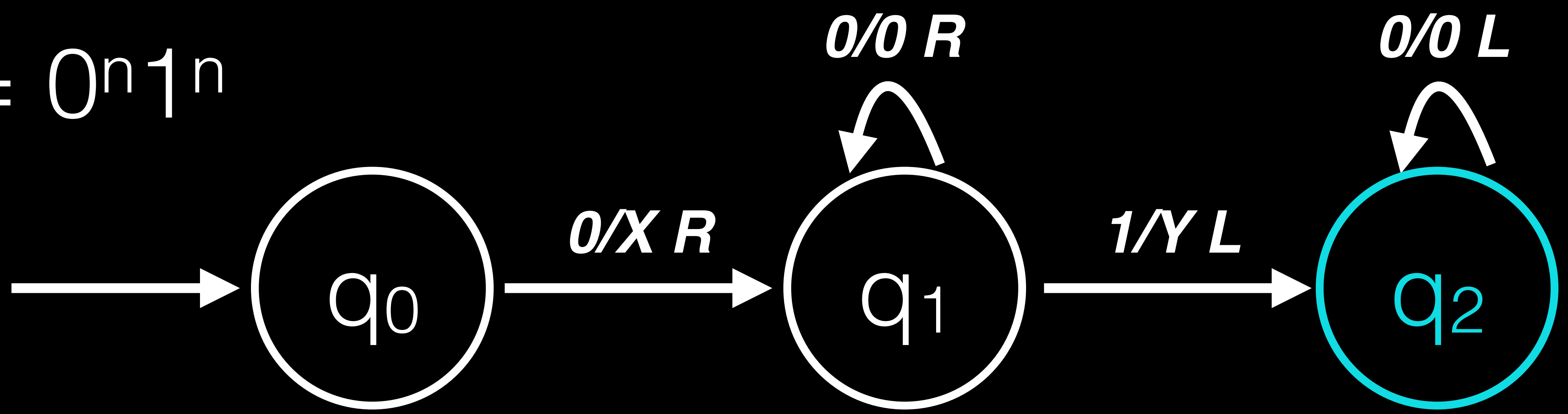

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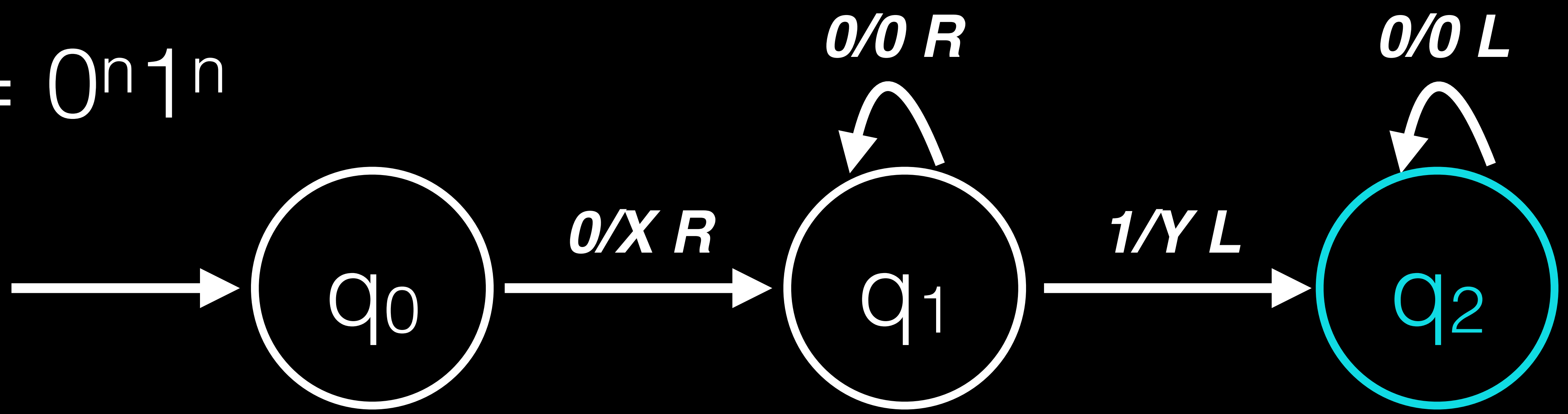

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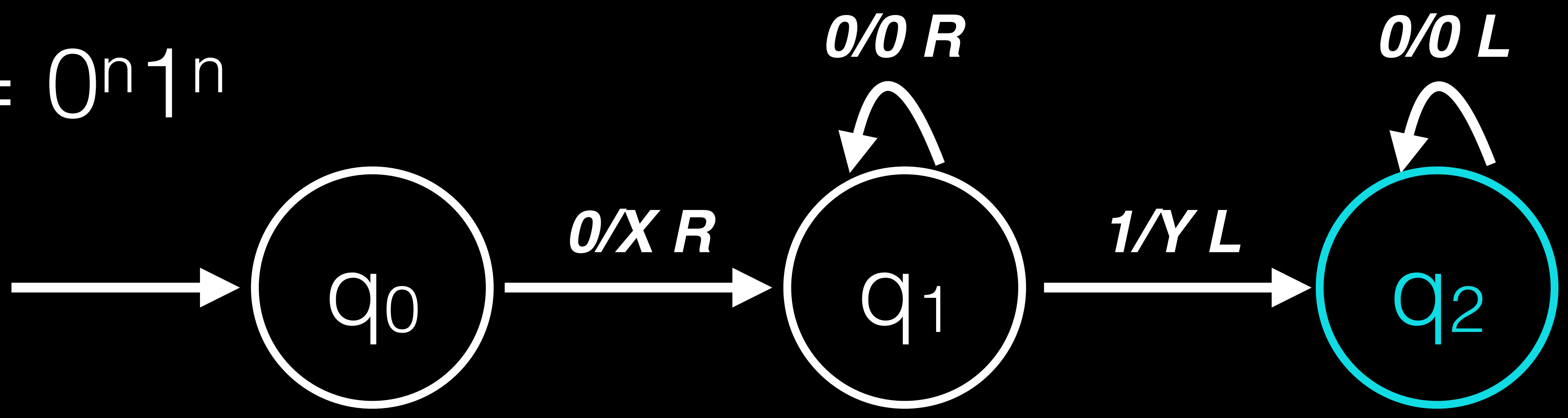

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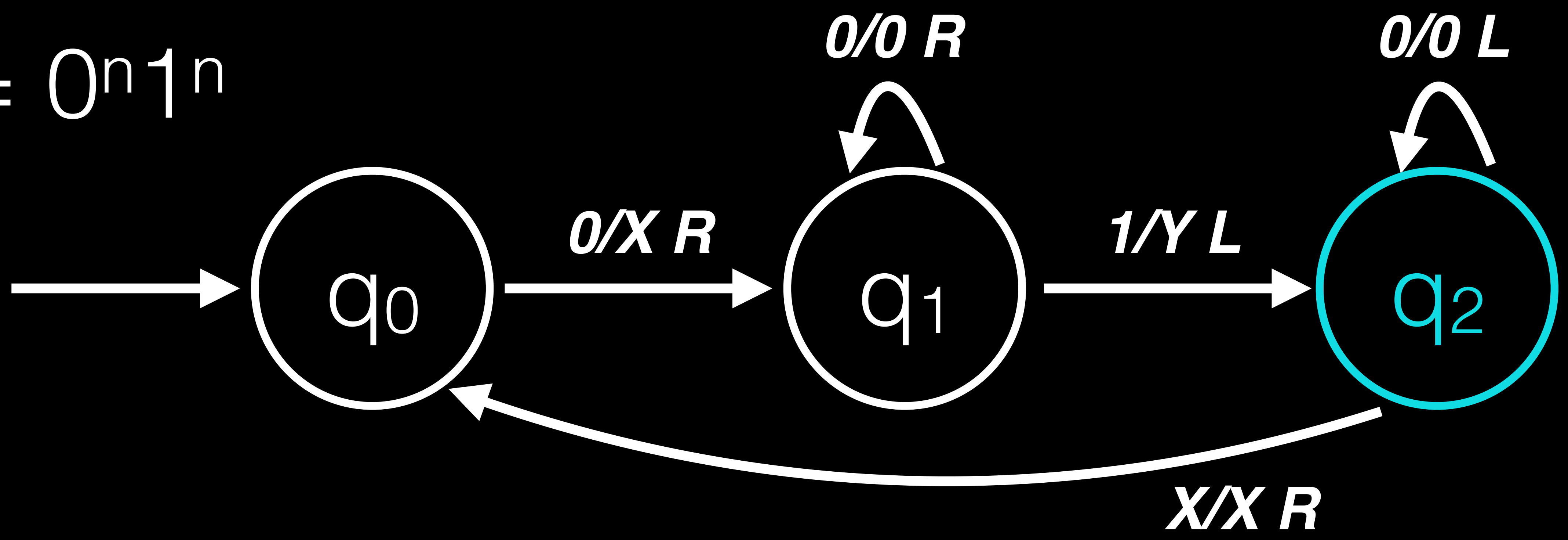

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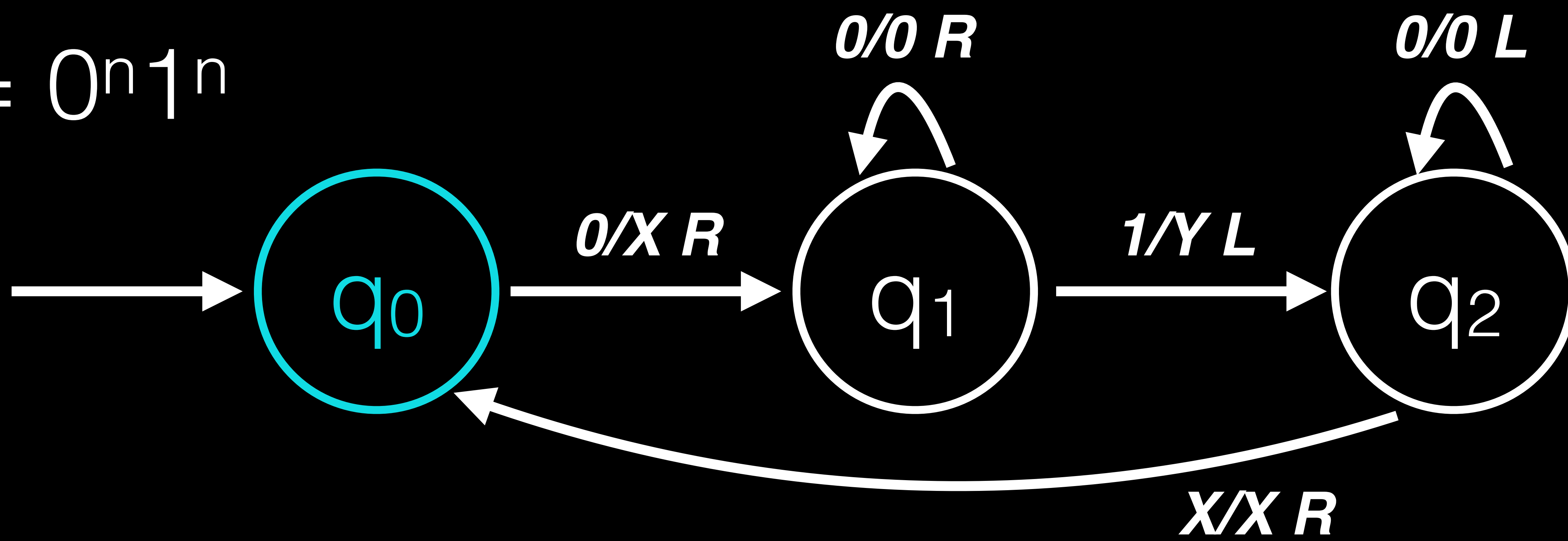

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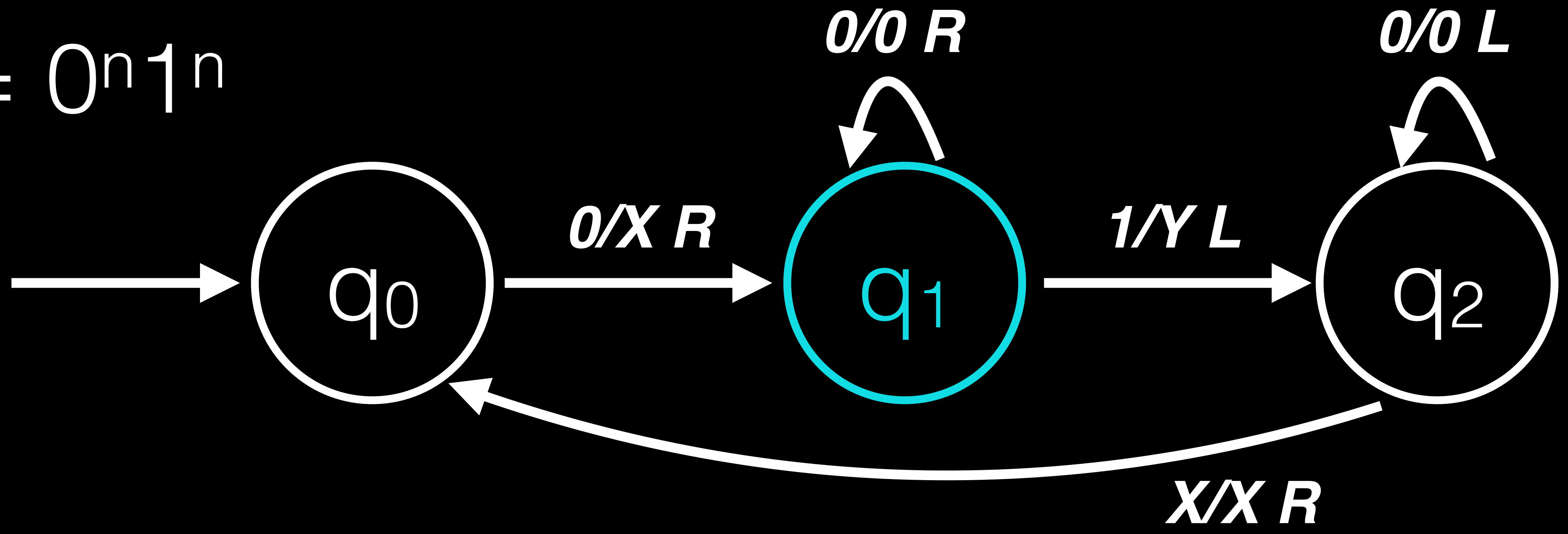

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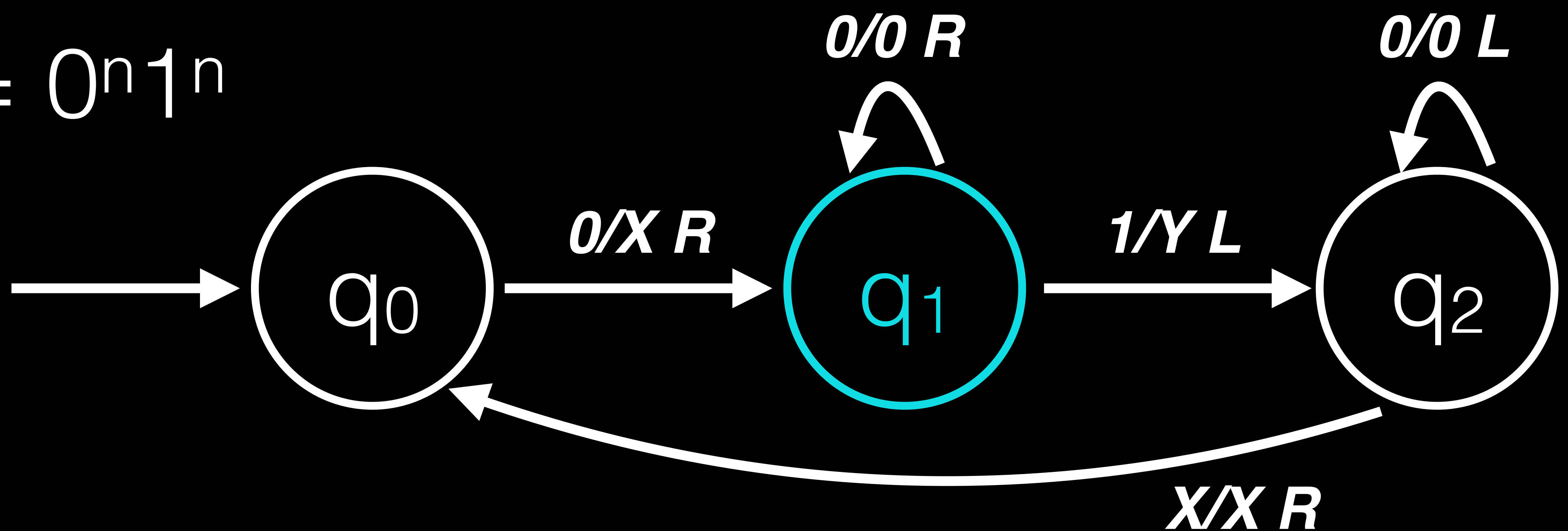

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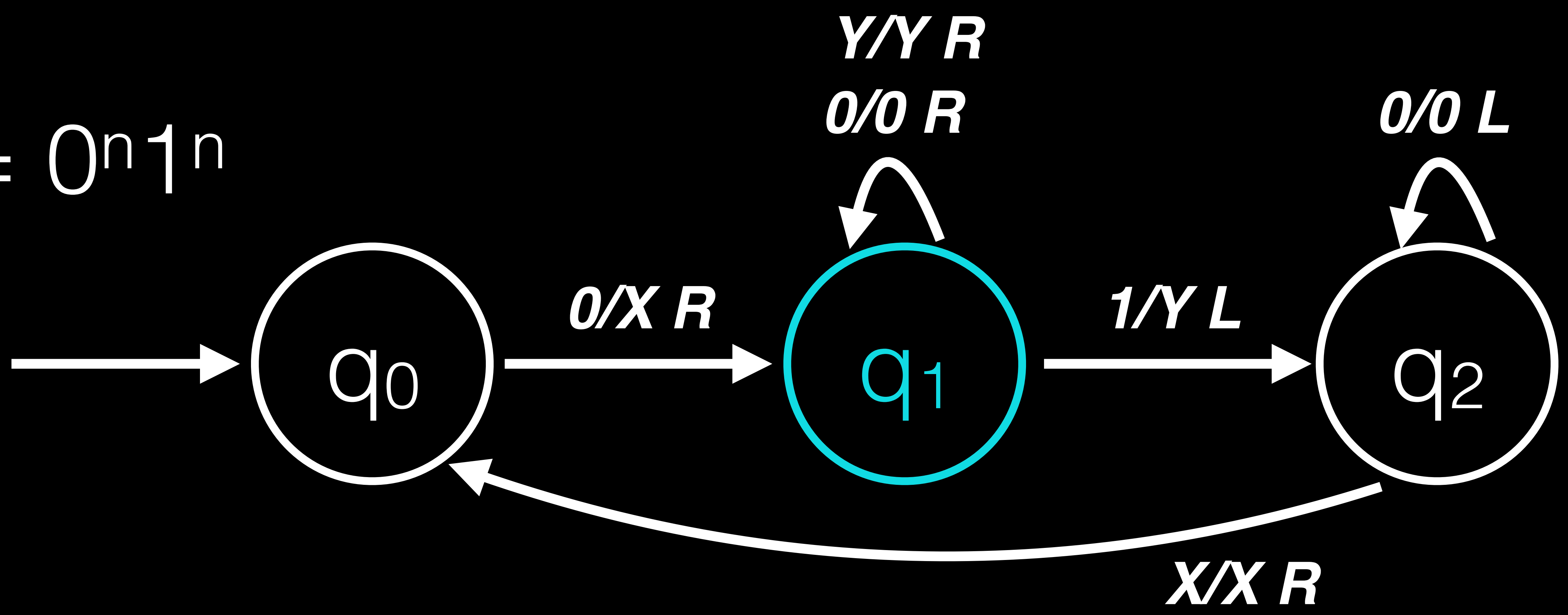
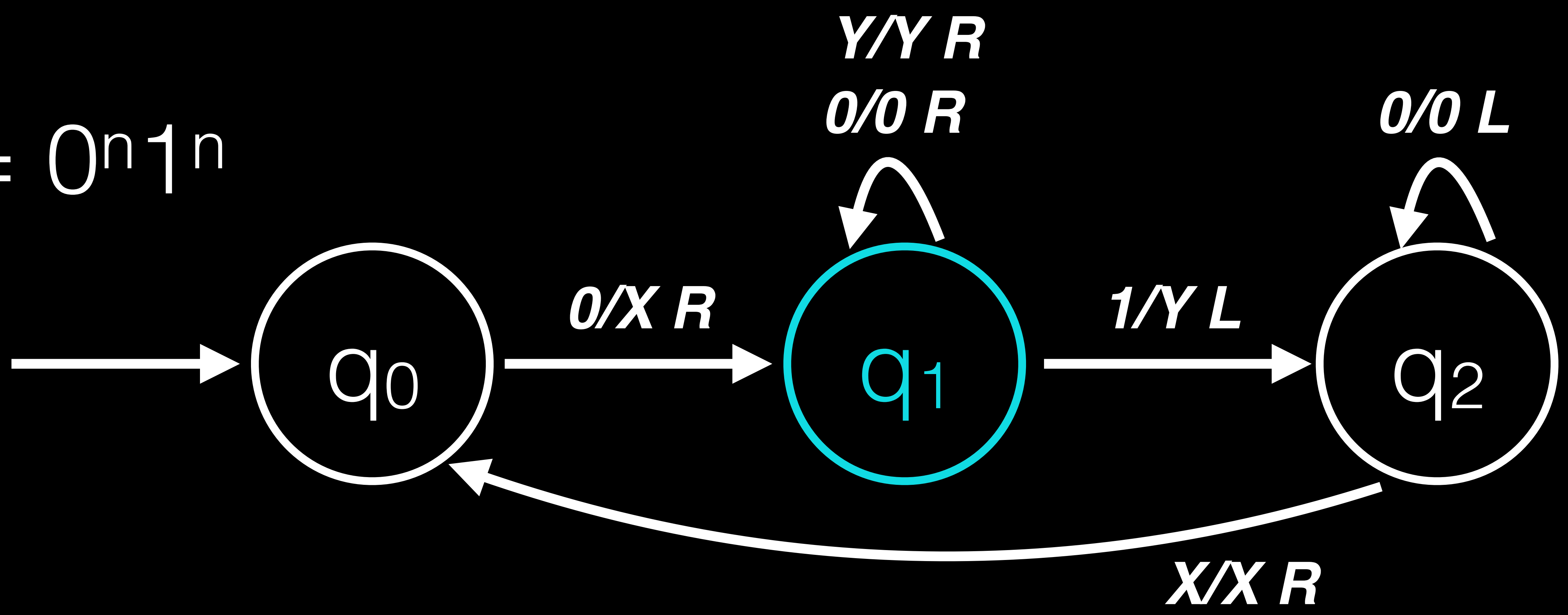


X X 0 Y 1 1

EXAMPLE

$L = 0^n 1^n$



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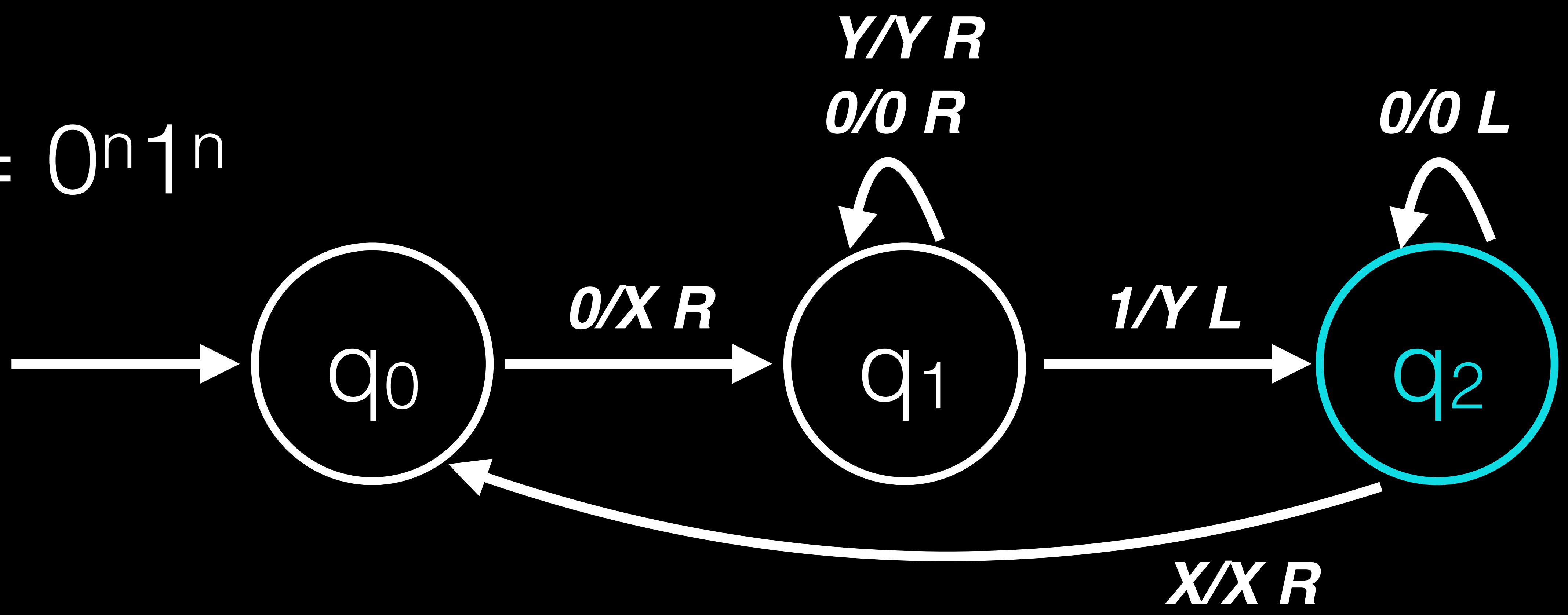

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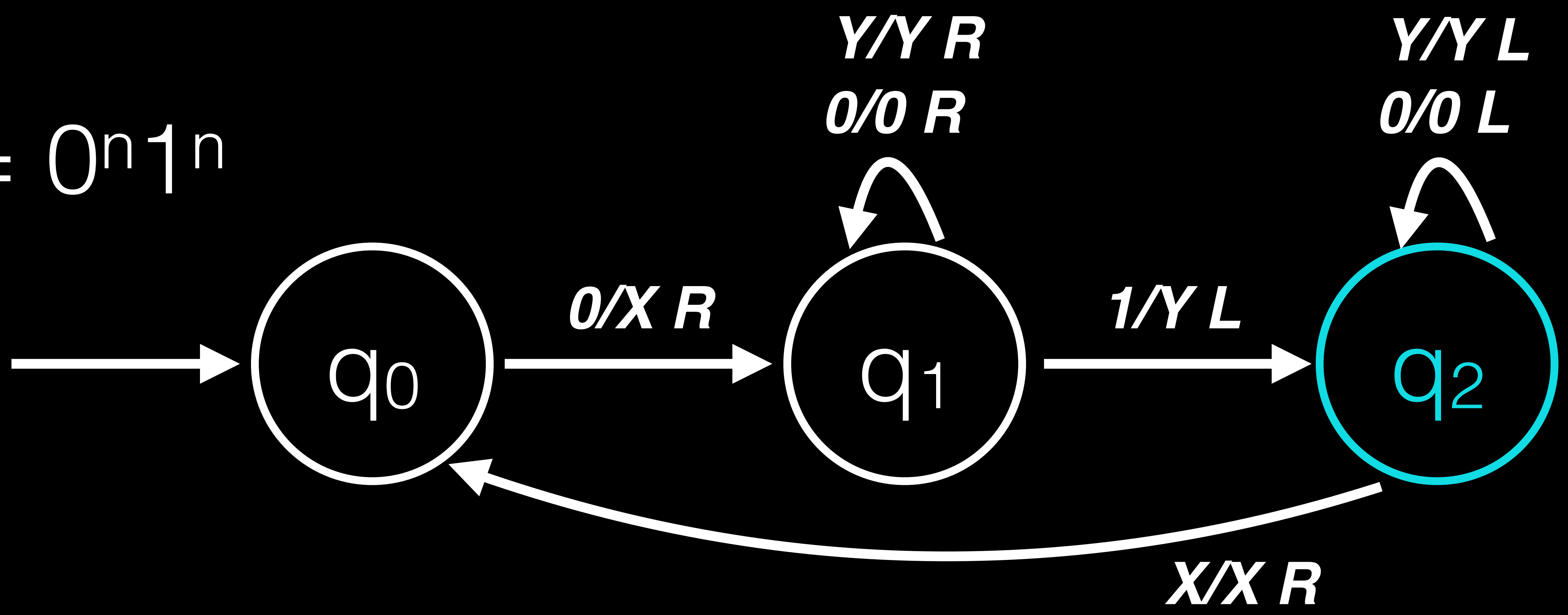

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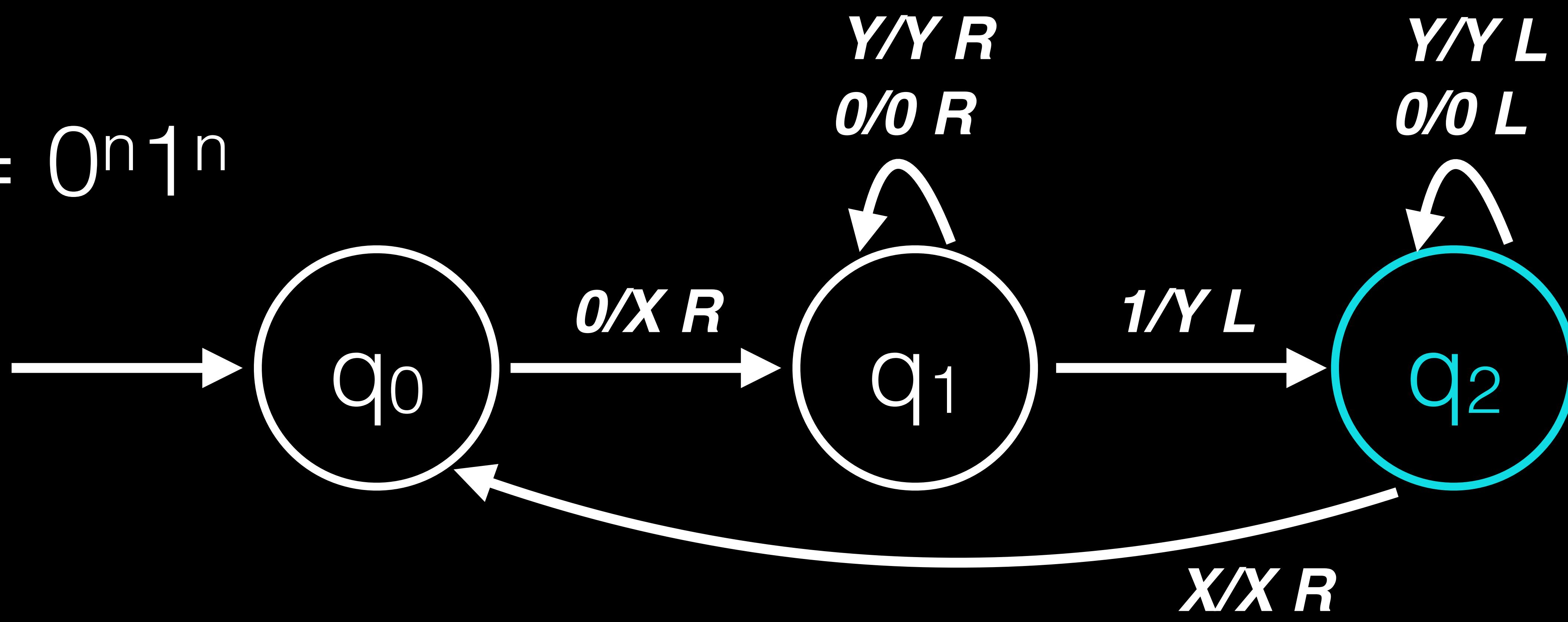

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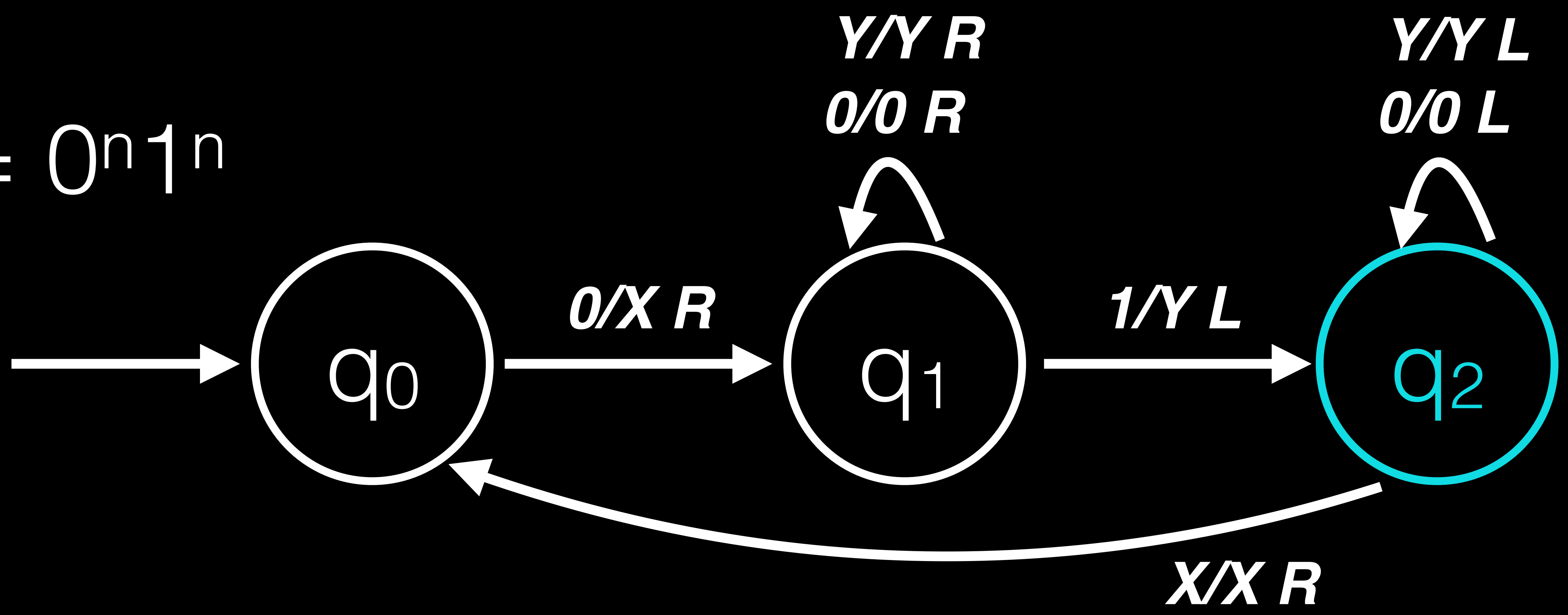

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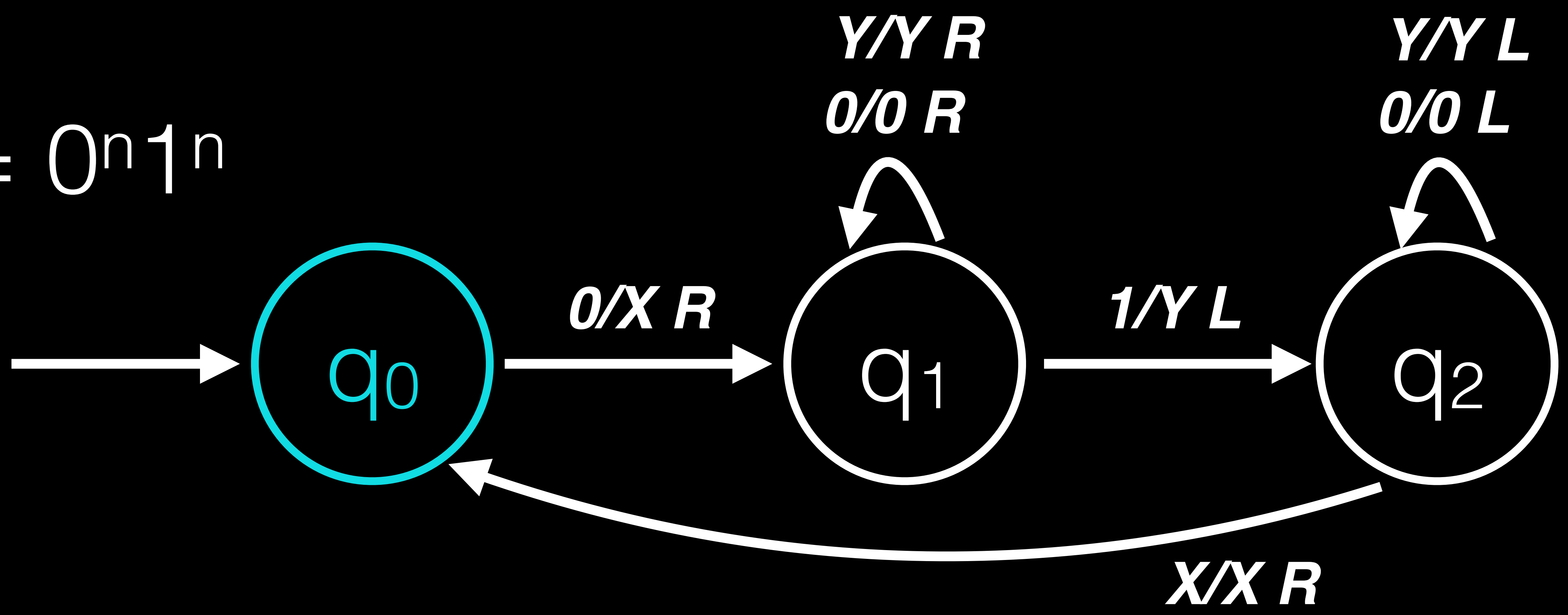

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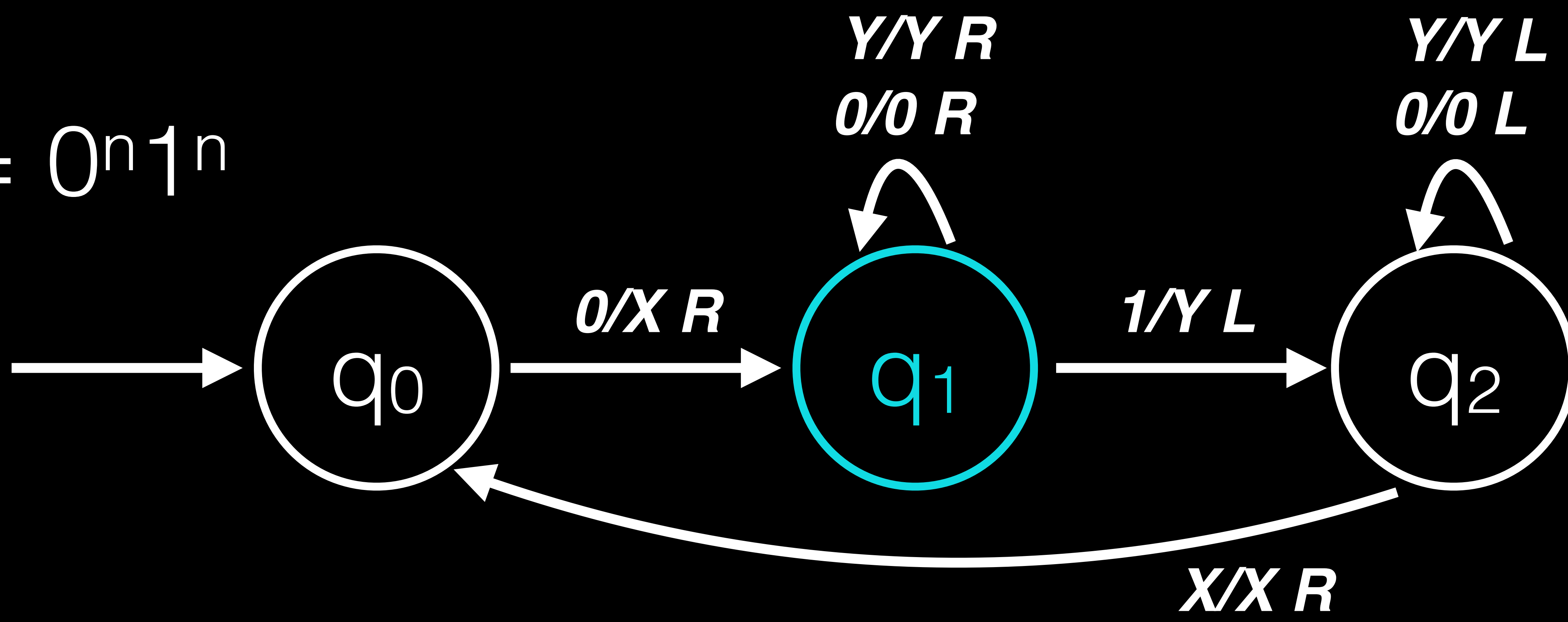

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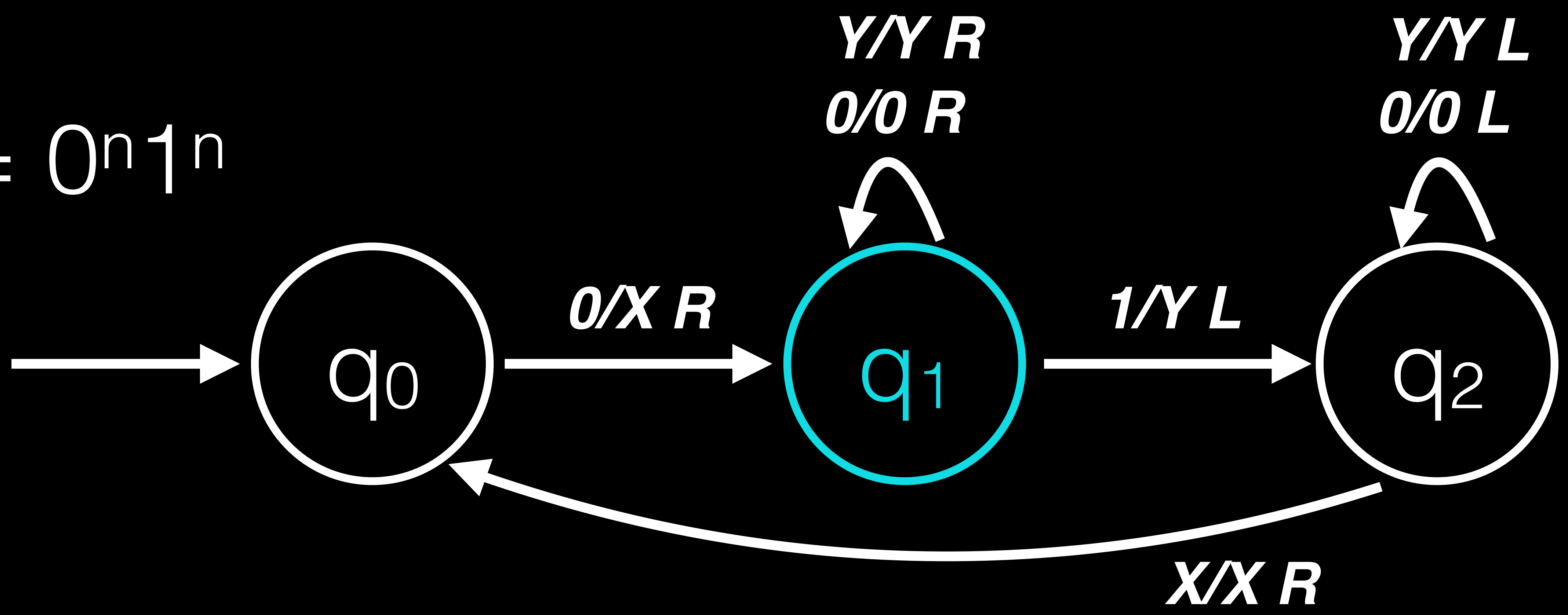

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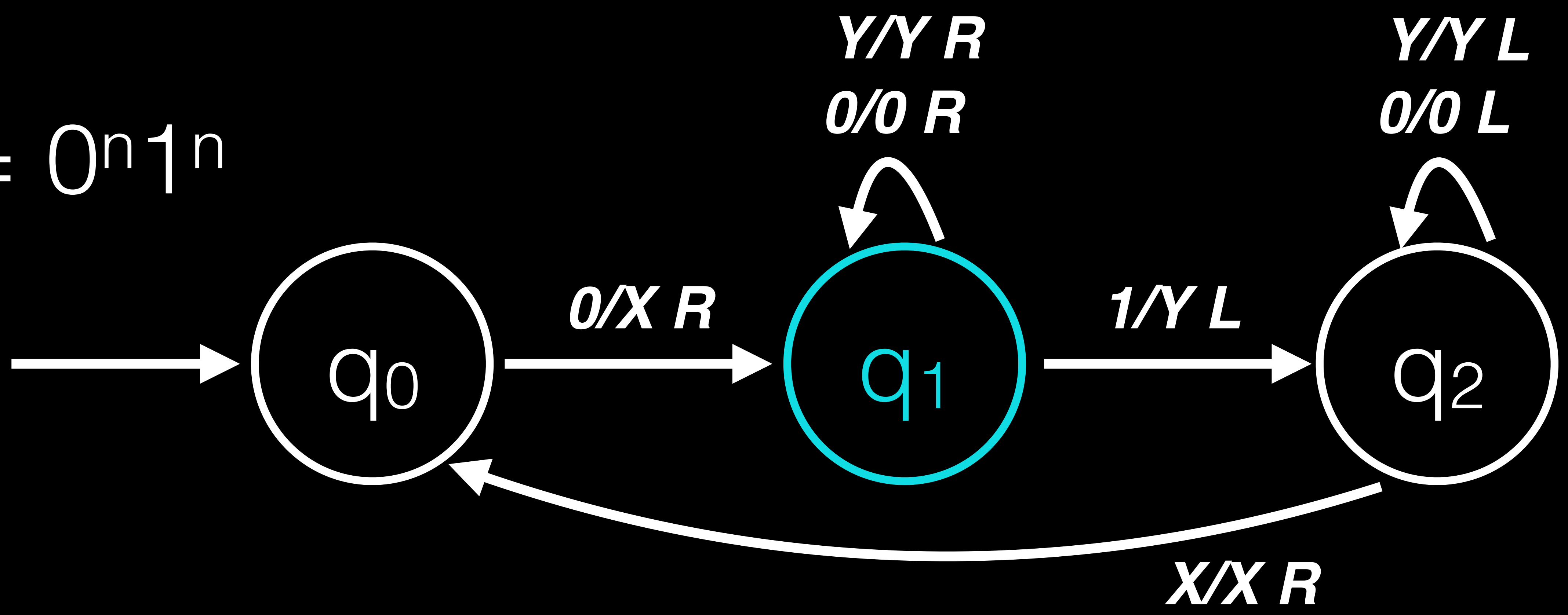

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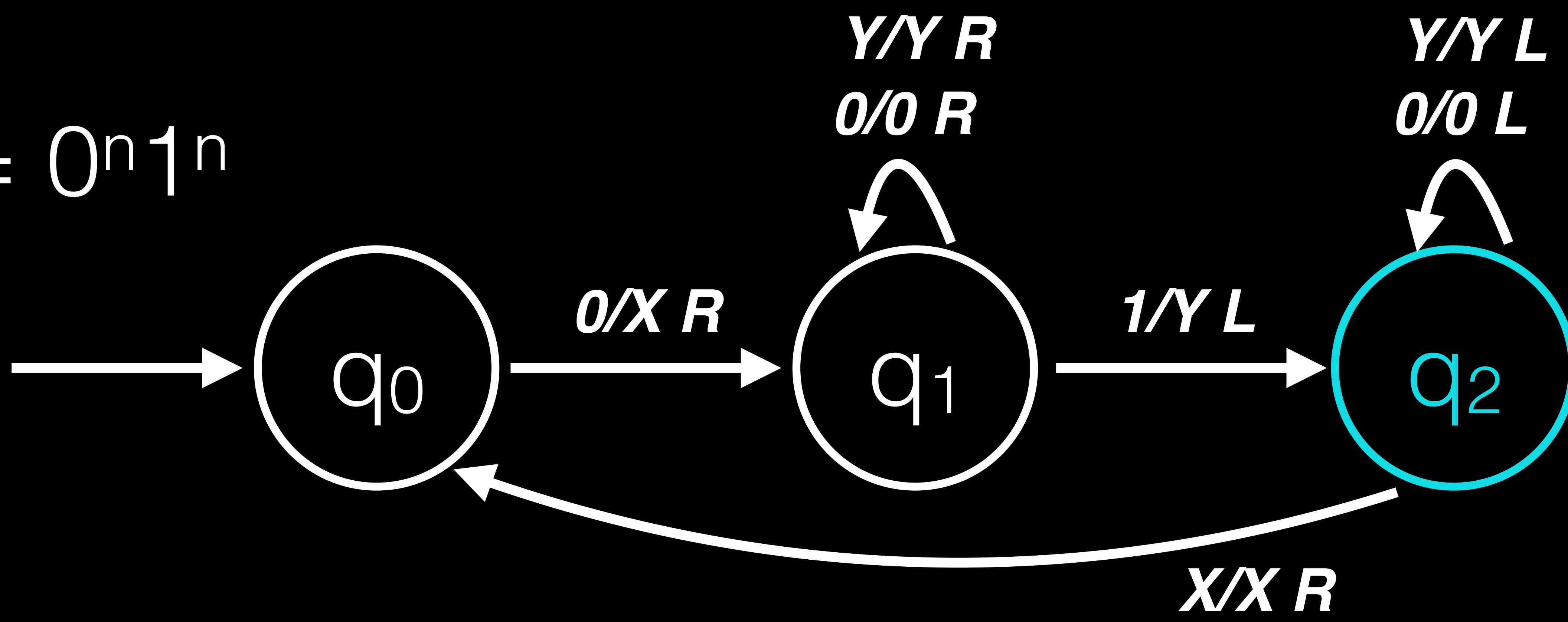

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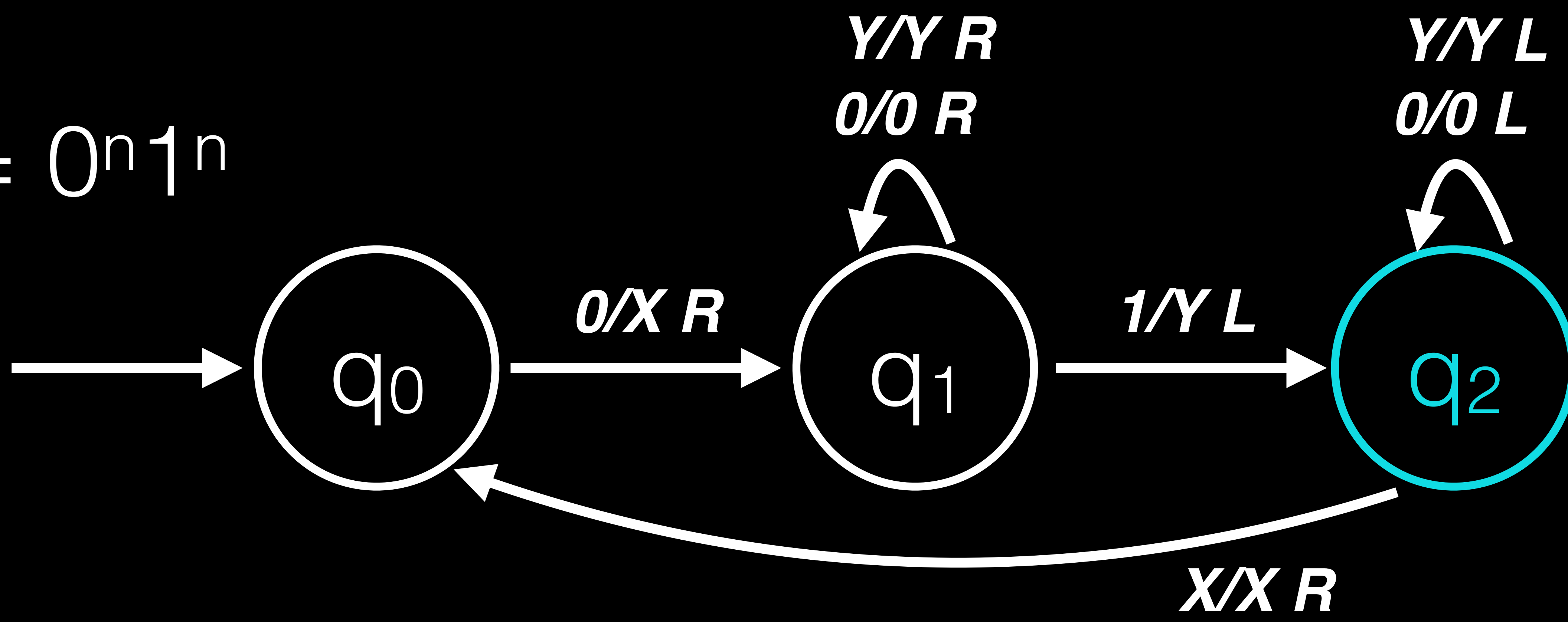

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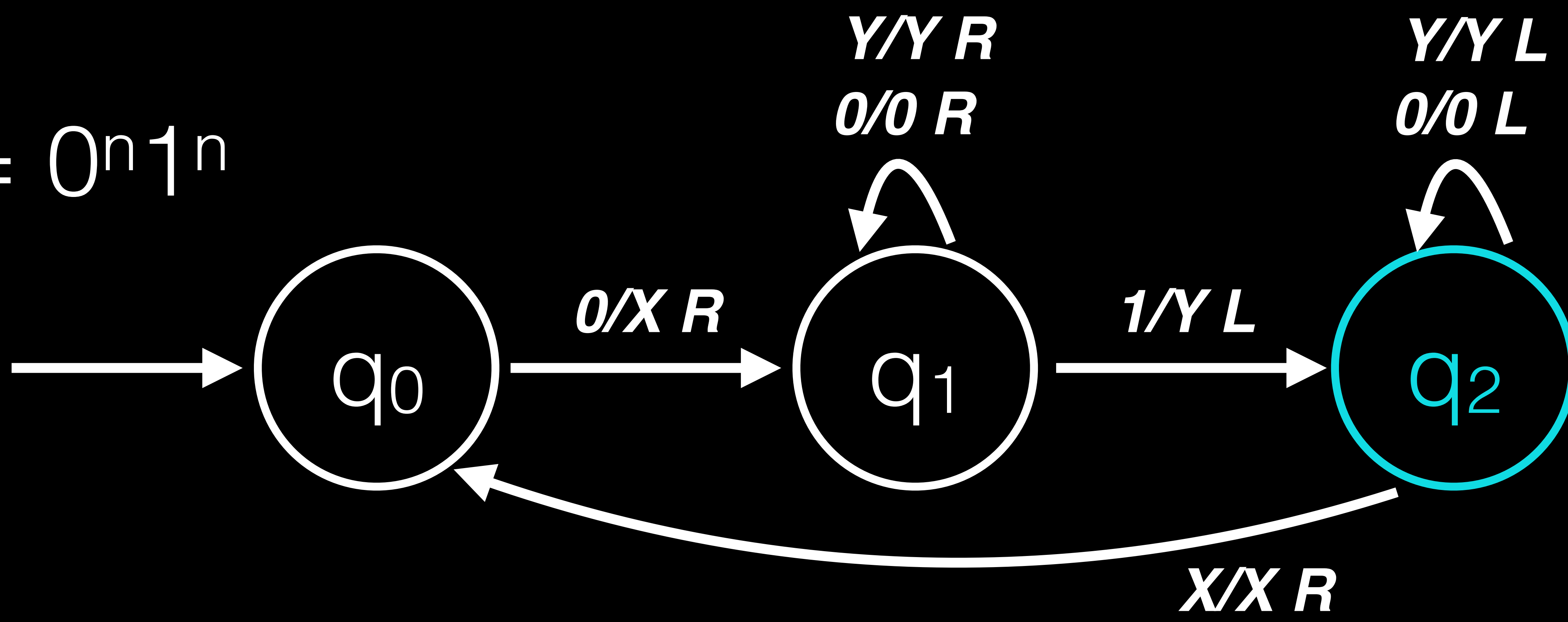

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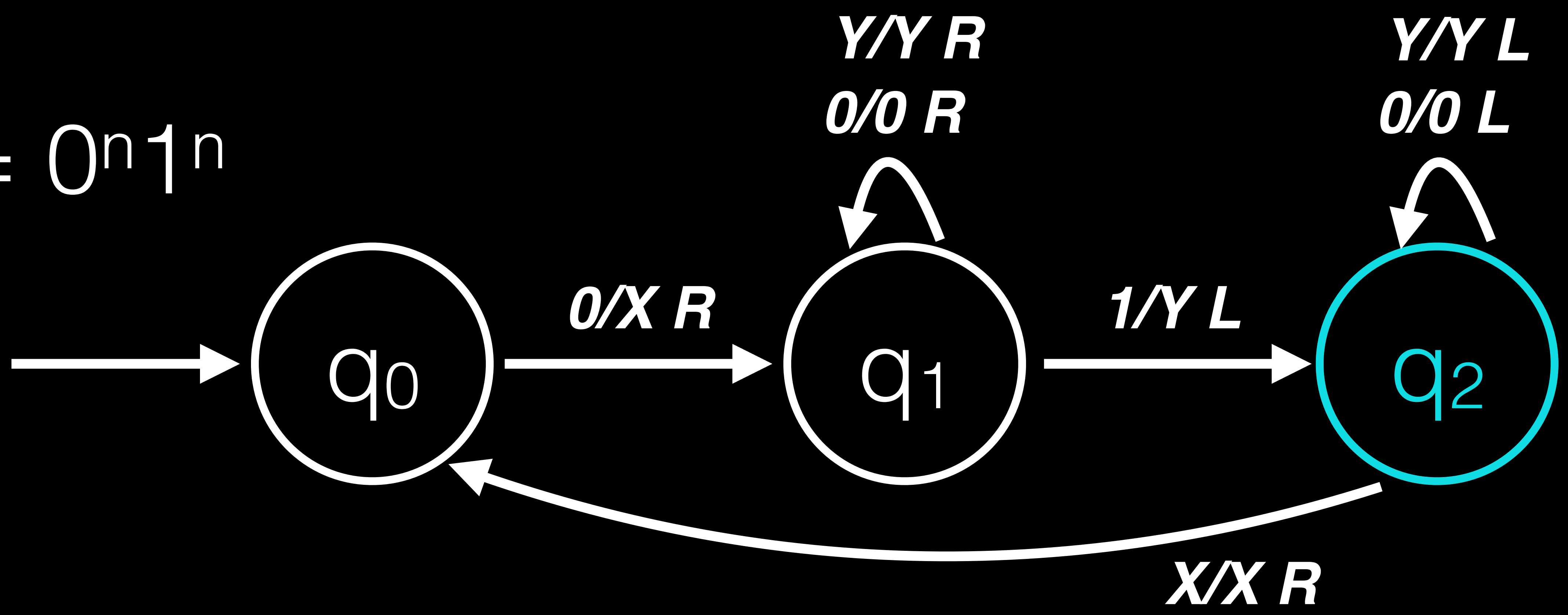

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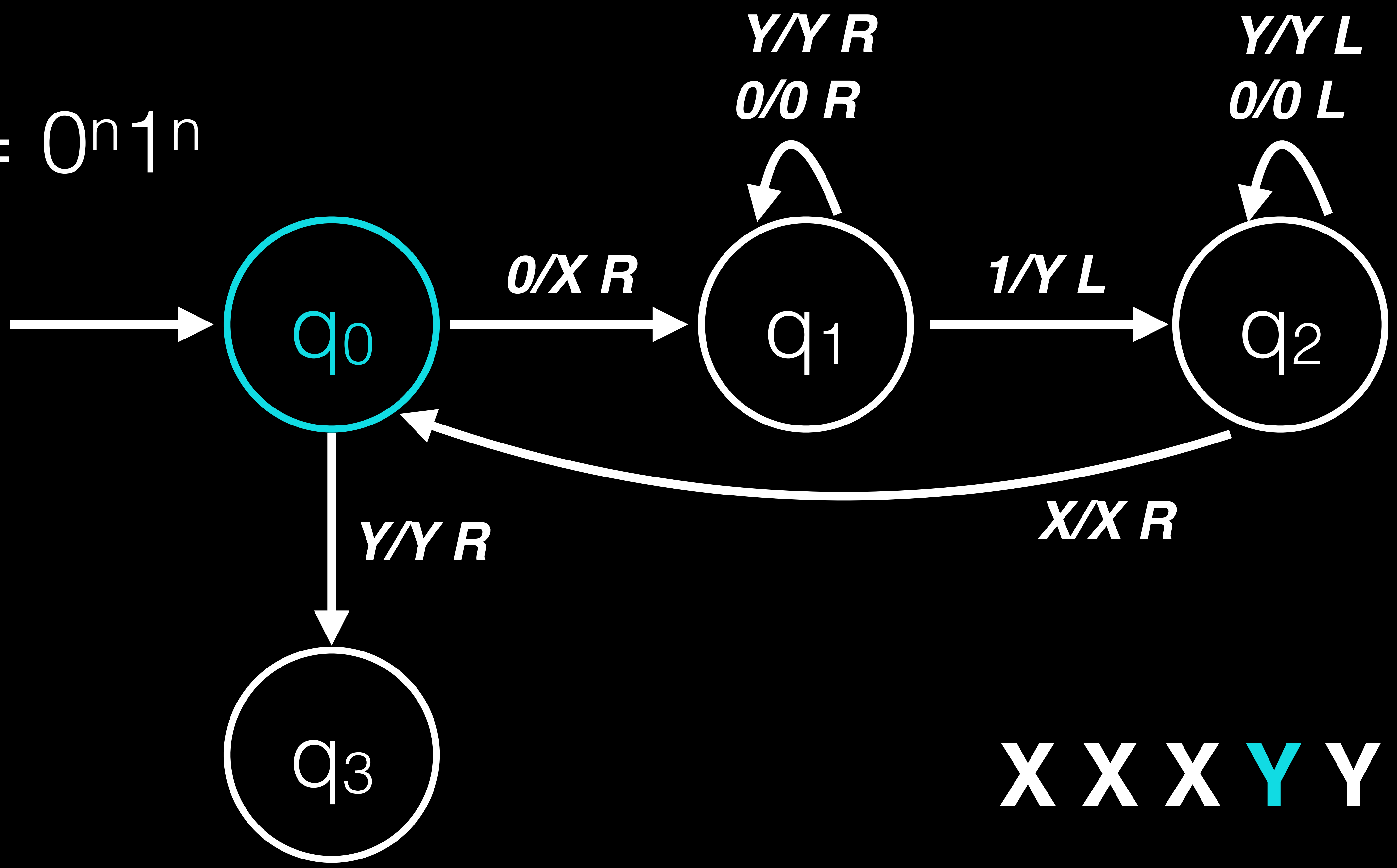

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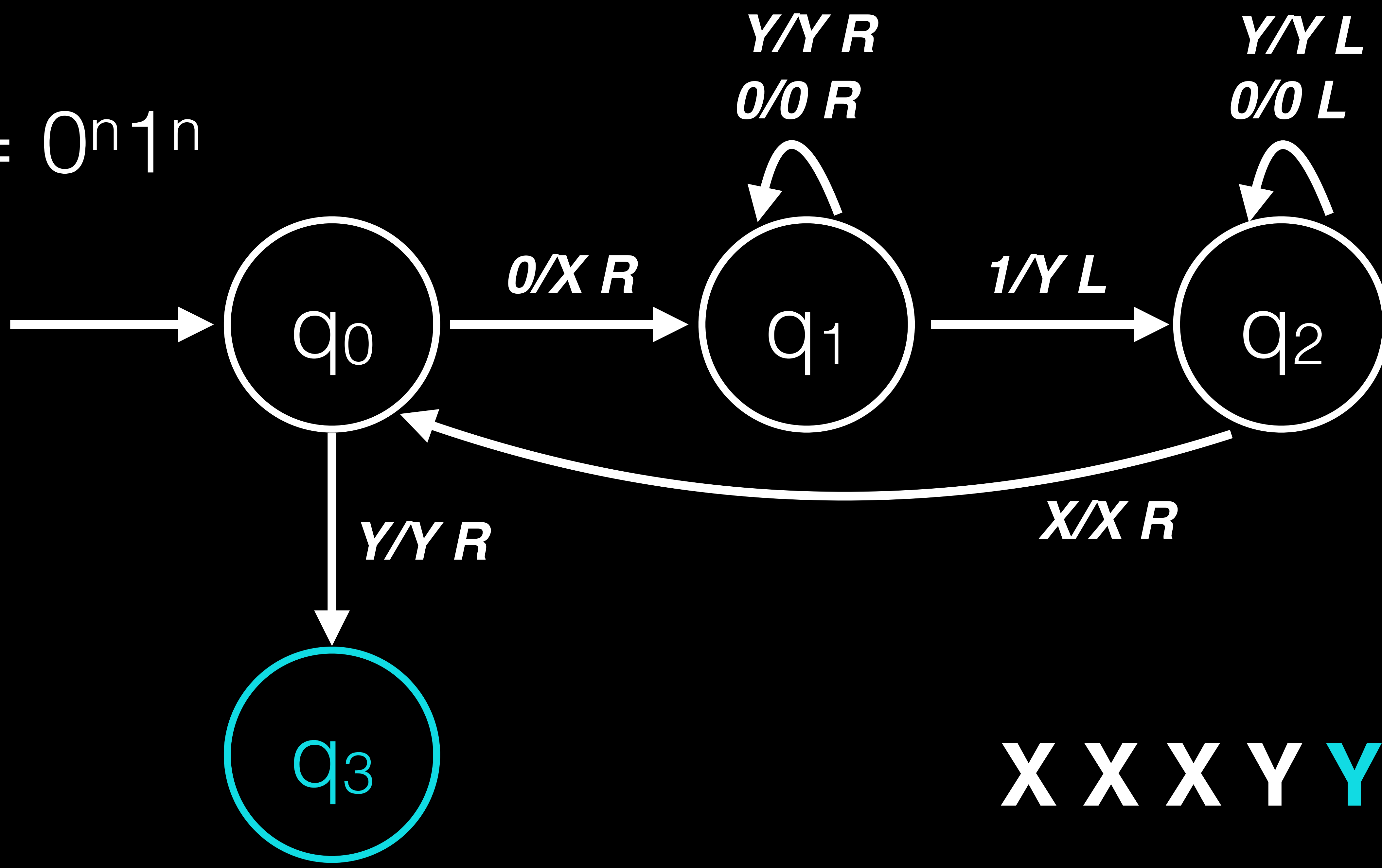

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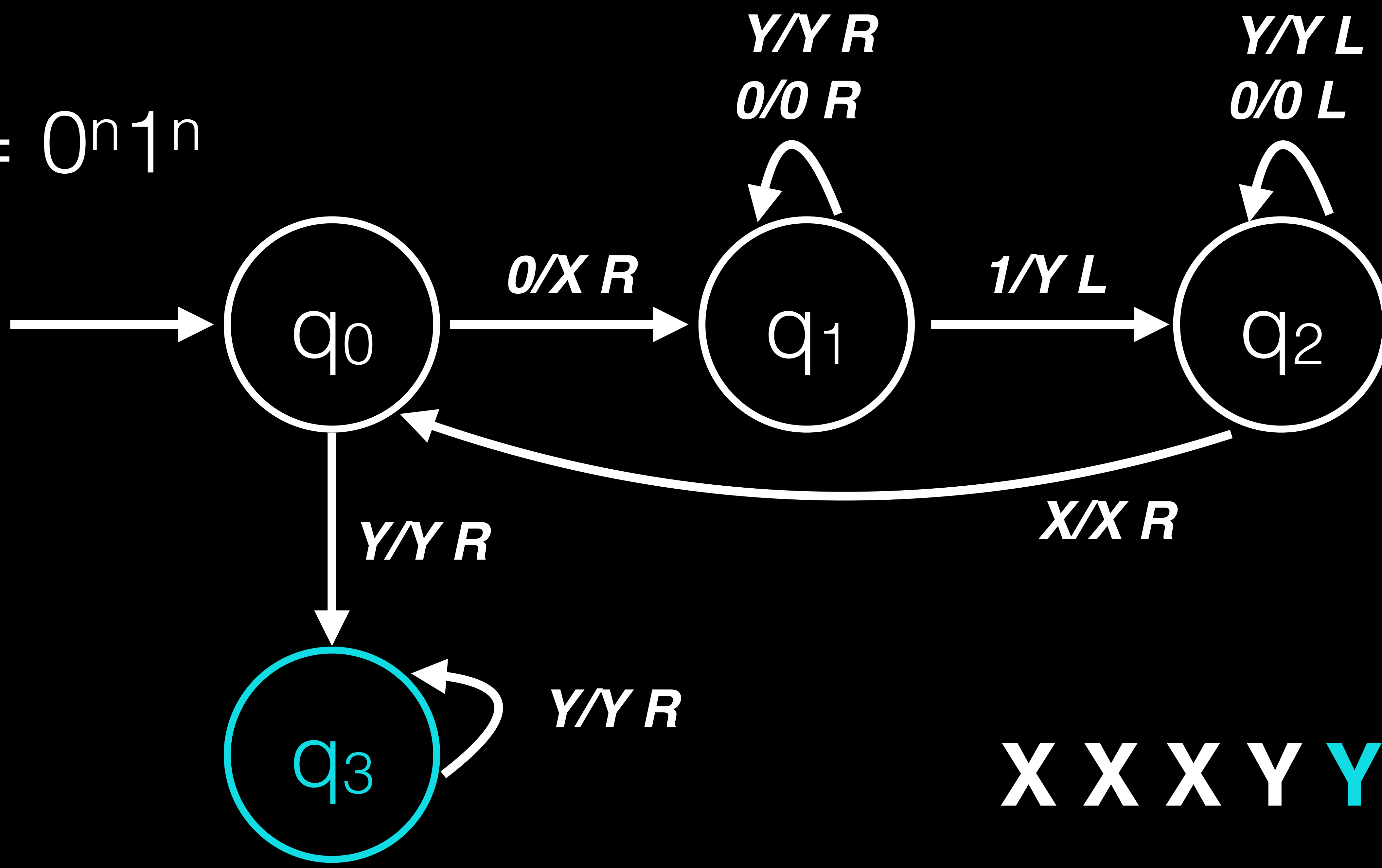

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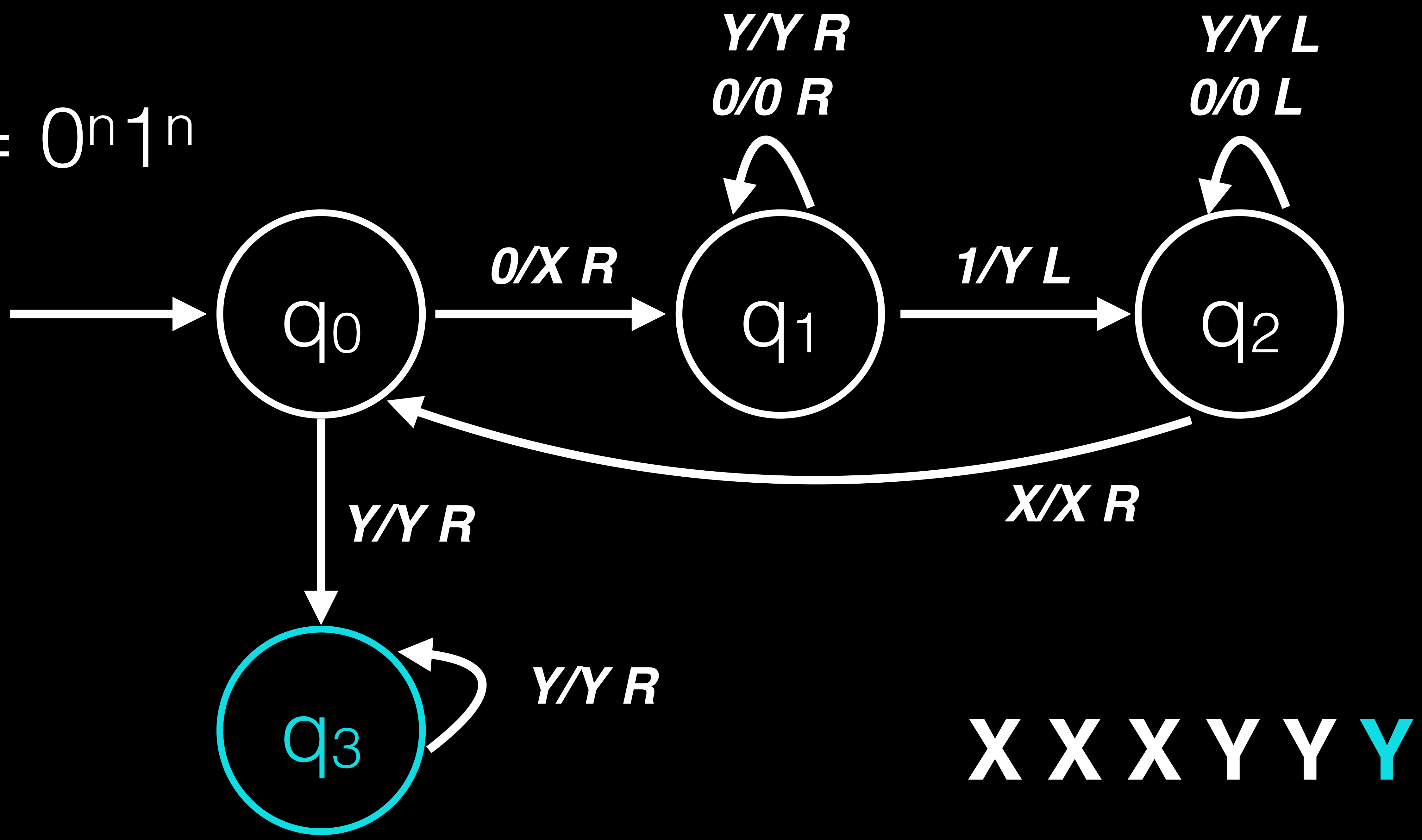

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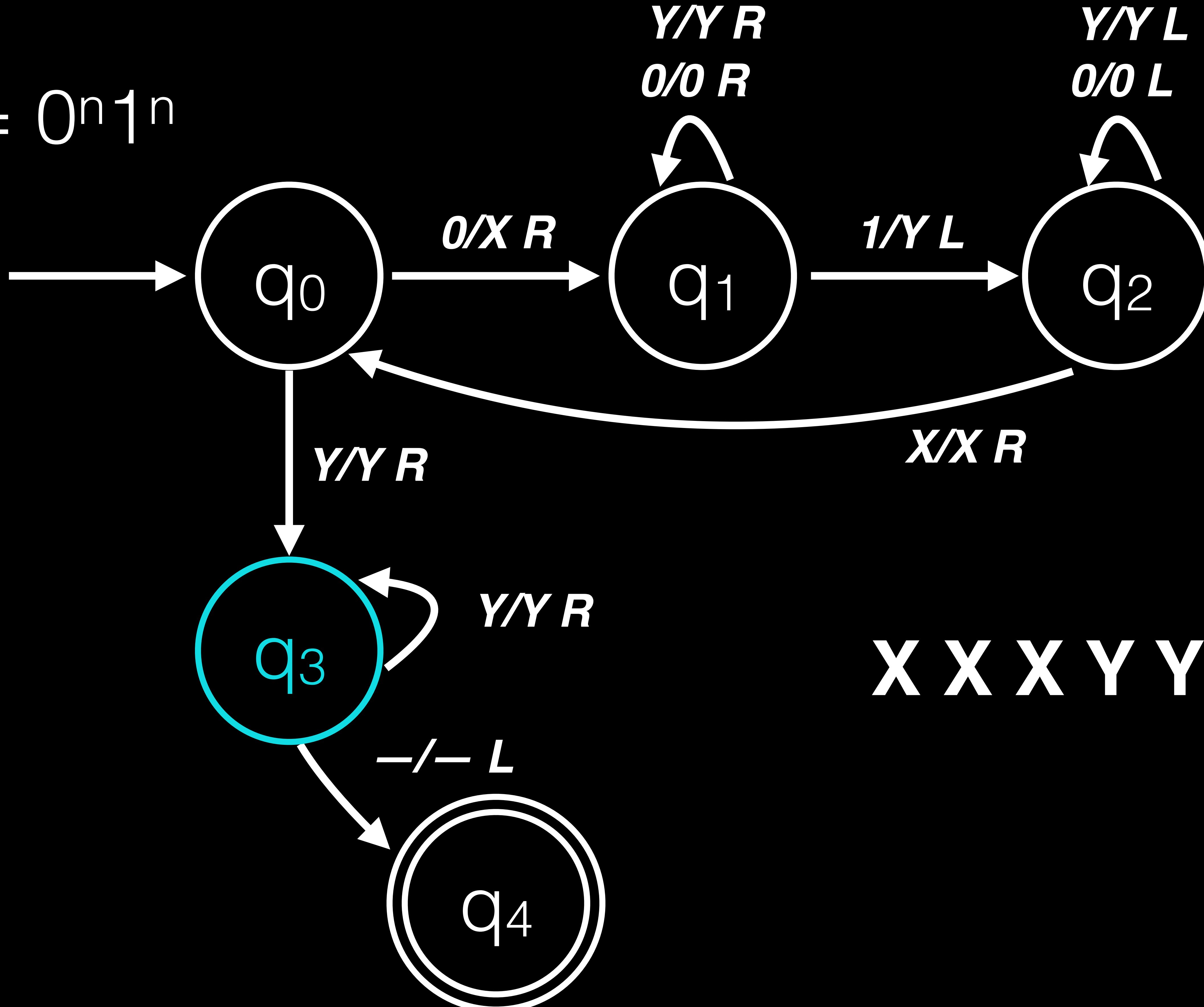

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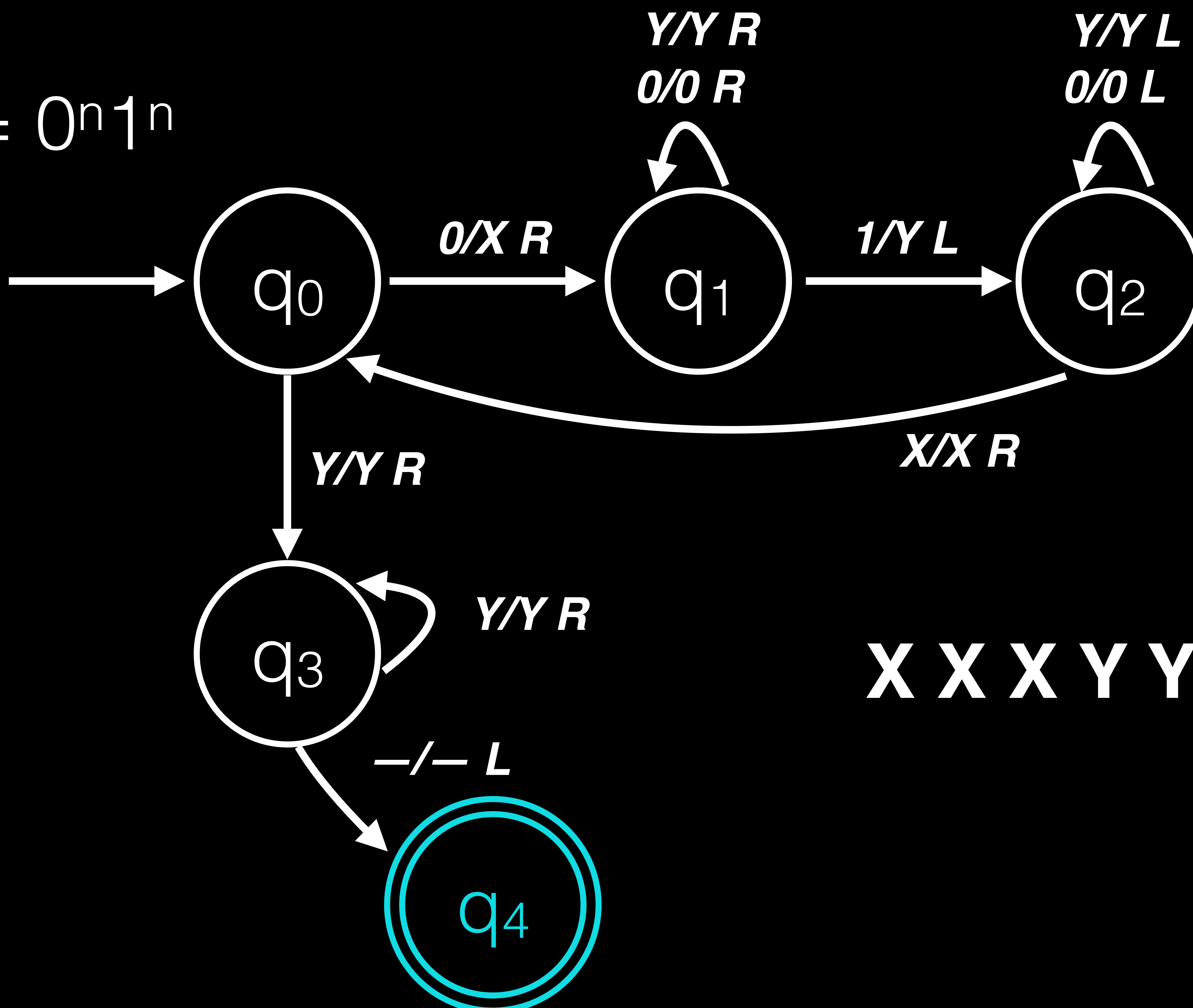

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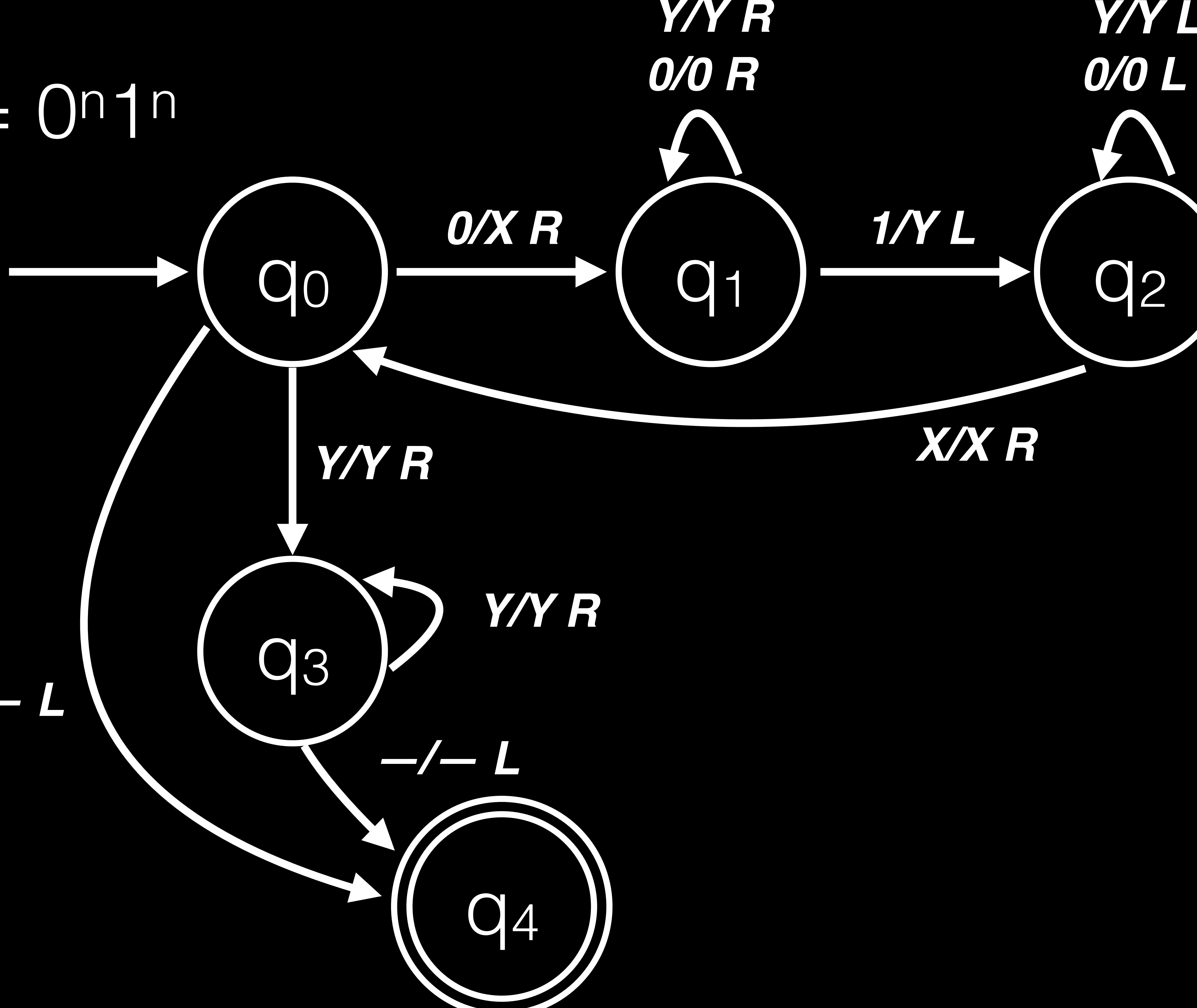


EXAMPLE

Questions:

- ⋆ Is this machine correct?
- ⋆ Does it work?
- ⋆ Does it contain bugs?
  - TMs model computers
  - In this way they are similar!

Definition of TMs and Related Language Classes

FORMAL DEFINITION

$(Q, \Sigma, \Gamma, \delta, q_0, q_{ACCEPT}, q_{REJECT})$

- Q = Set of states
- Σ = Input alphabet
- Γ = Tape Alphabet
  - Often we need a few extra symbols to make computation easier. $\Sigma \subseteq \Gamma$
  - The input cannot contain a blank. $- \notin \Sigma$ and $- \in \Gamma$
- q_0 = Initial state $q_0 \in Q$
- q_{ACCEPT} $q_{ACCEPT} \in Q$
- q_{REJECT} $q_{REJECT} \in Q$
- $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ Transition Function

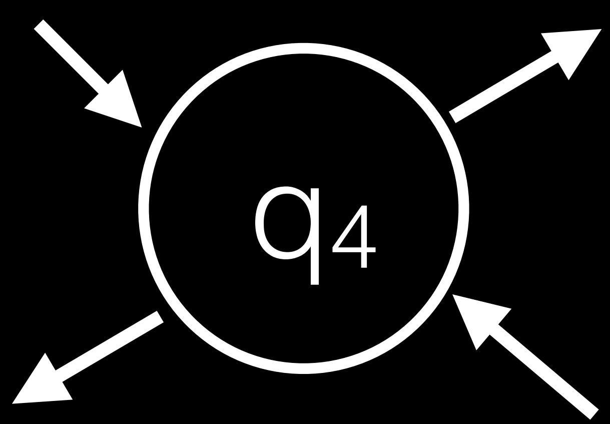
CONFIGURATION

- Gives the entire state of the machine.
- Snapshot of execution at some step.

CONFIGURATION

Need:

- Contents of the tape
- Location of the “tape head”
- Current state



A configuration is a string like this:

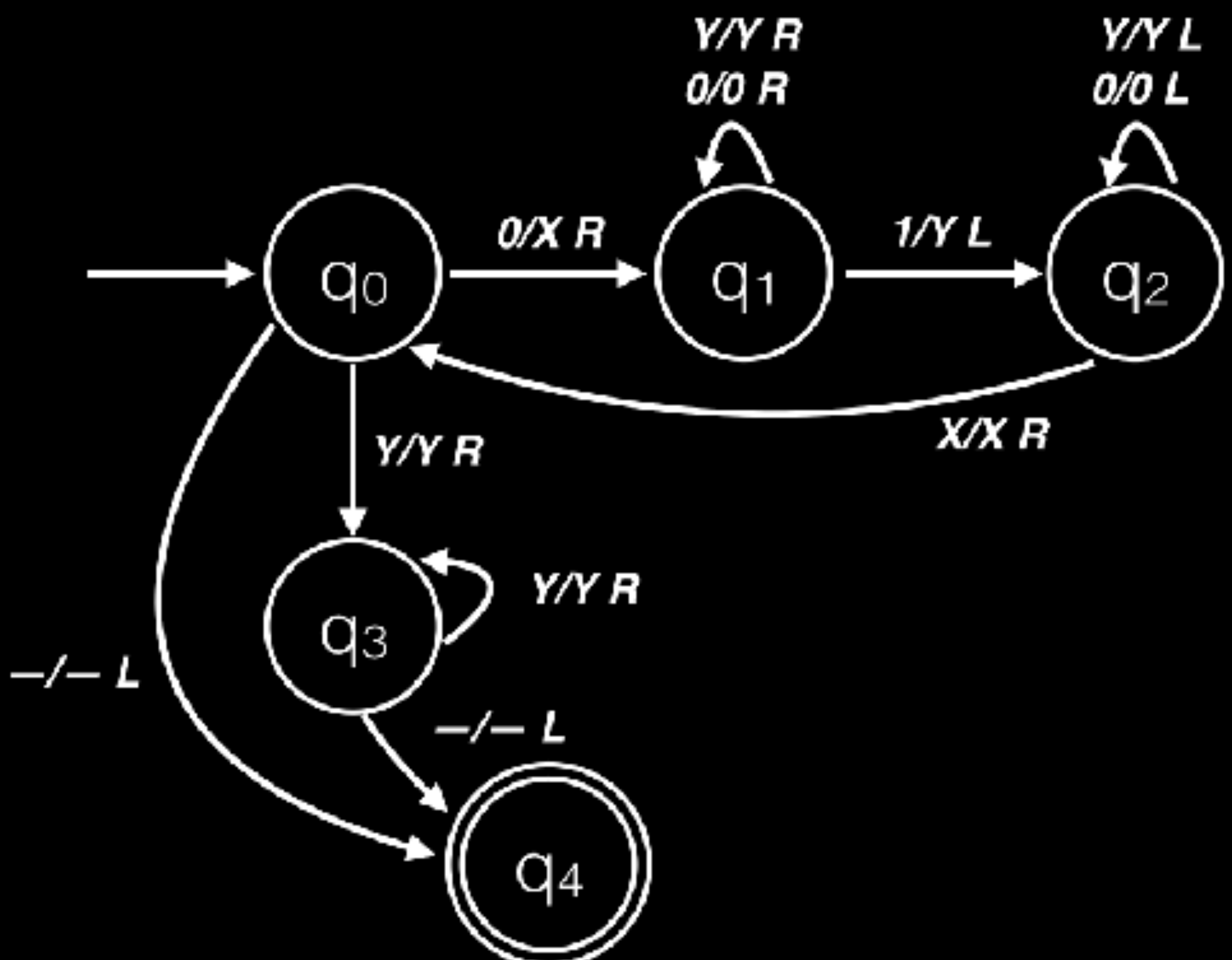
$$0\ 0\ 1\ 1\ q_4\ 0\ 1\ 1$$

A sequence of configurations, starting with the **start configuration**, and ending with an **[Accepting]* configuration** and containing only legal transitions provide a c**omputation history**.

** Rejecting*

EXAMPLE

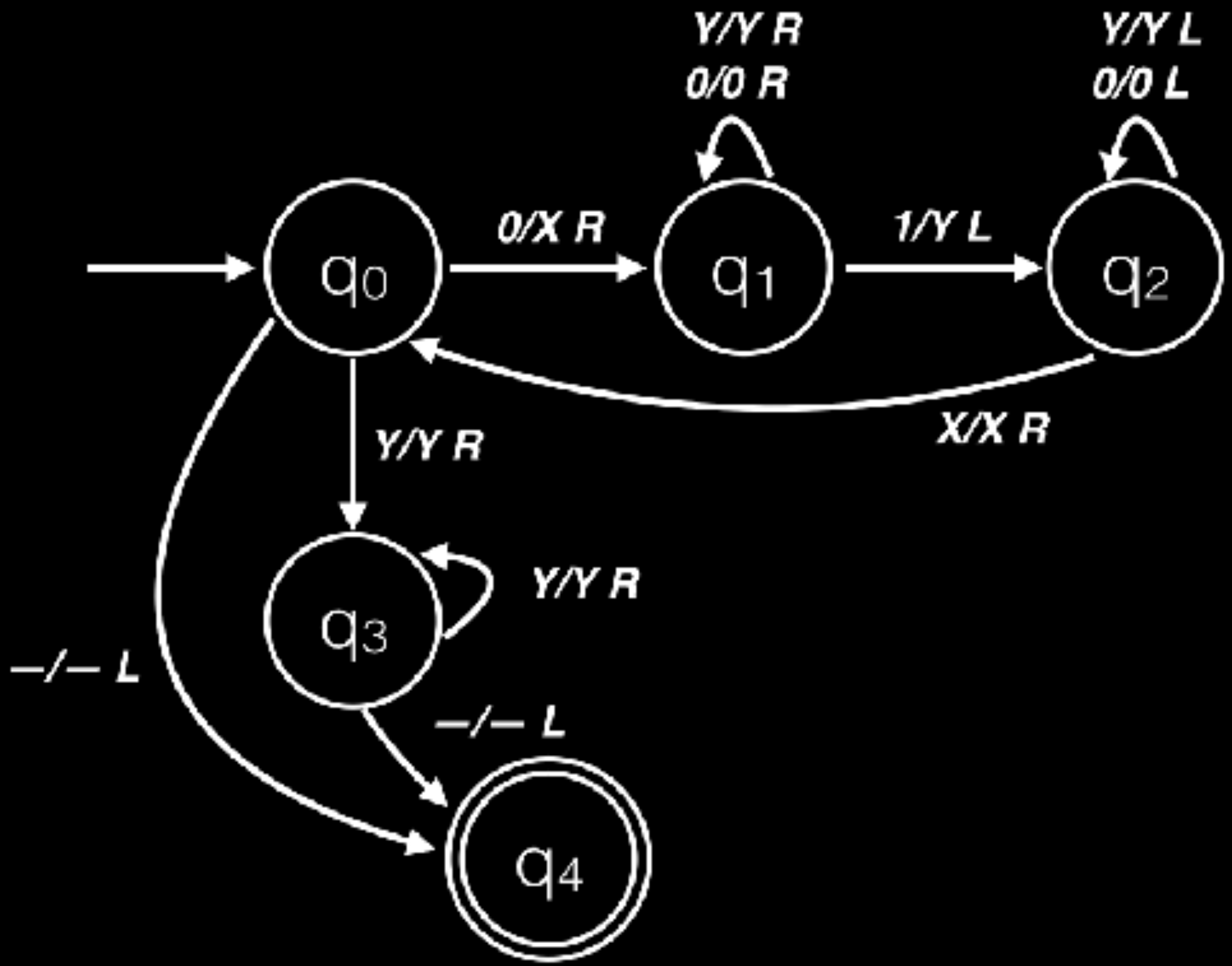
$L = 0^n1^n$, given the string 0011.



EXAMPLE

L = 0^n1^n, given the string 0011.

$q_0$0011



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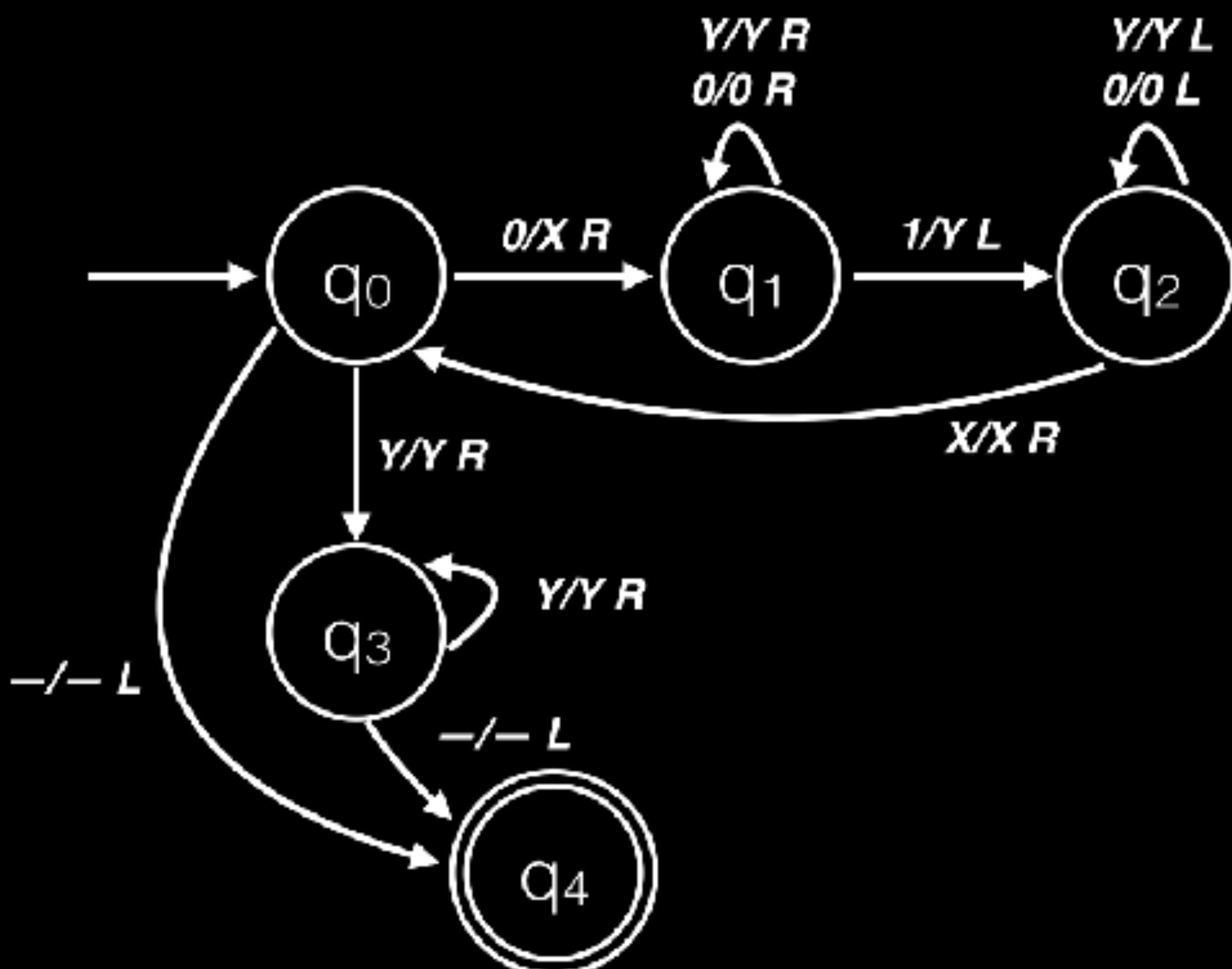
$q_0$0011
X$q_1$011



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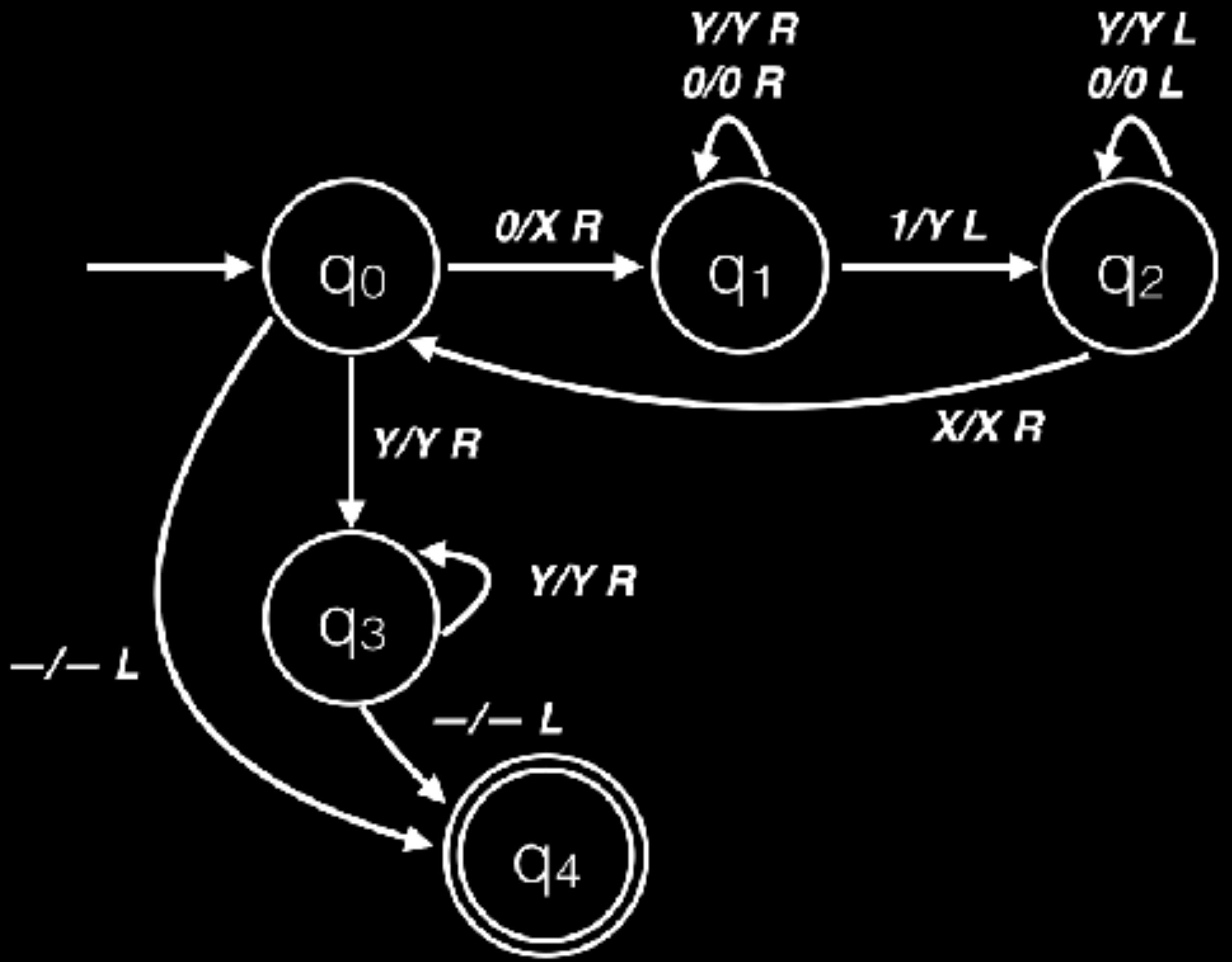

$q_0$0011
X$q_1$011
X0$q_1$11

EXAMPLE

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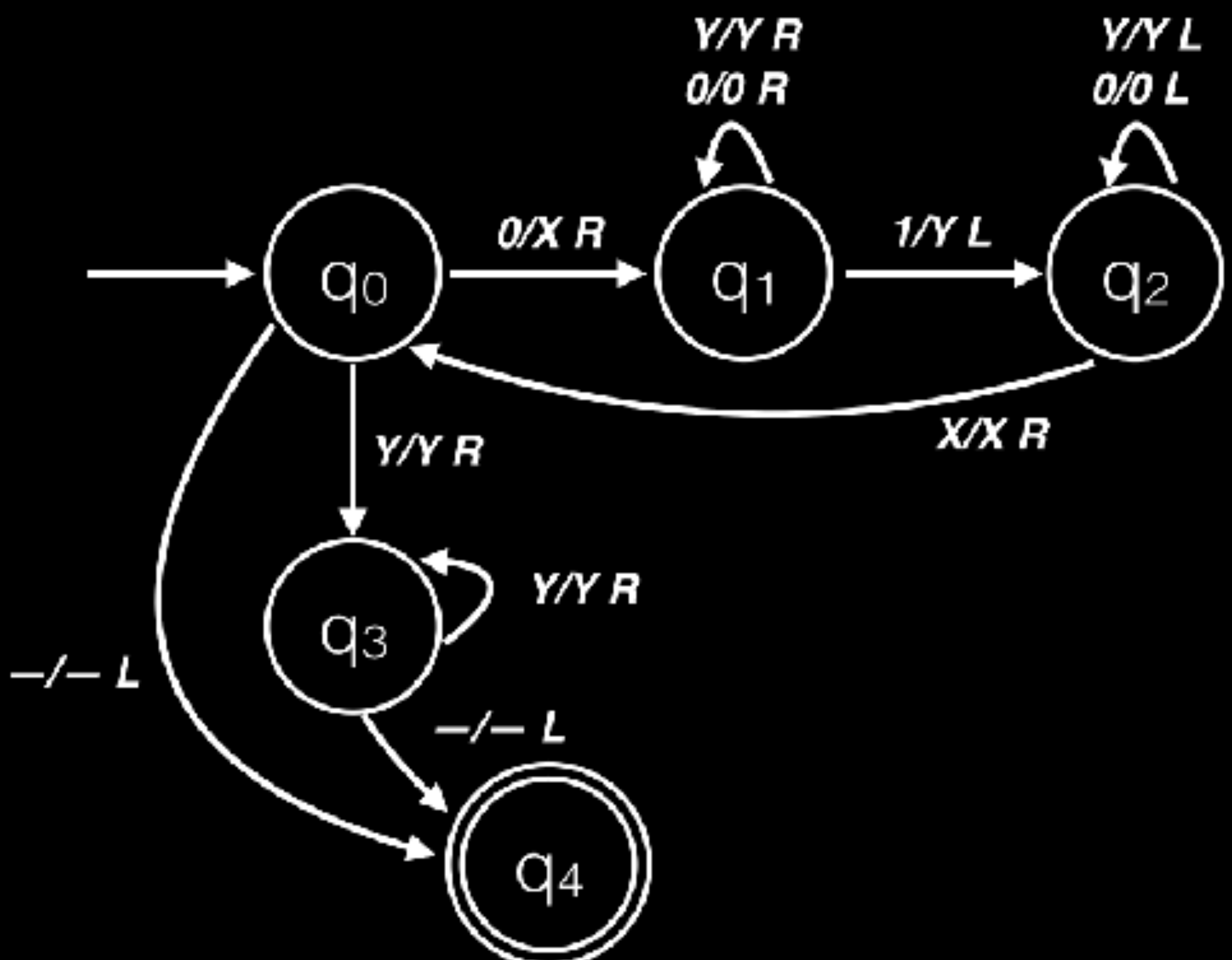


$q_0$0011

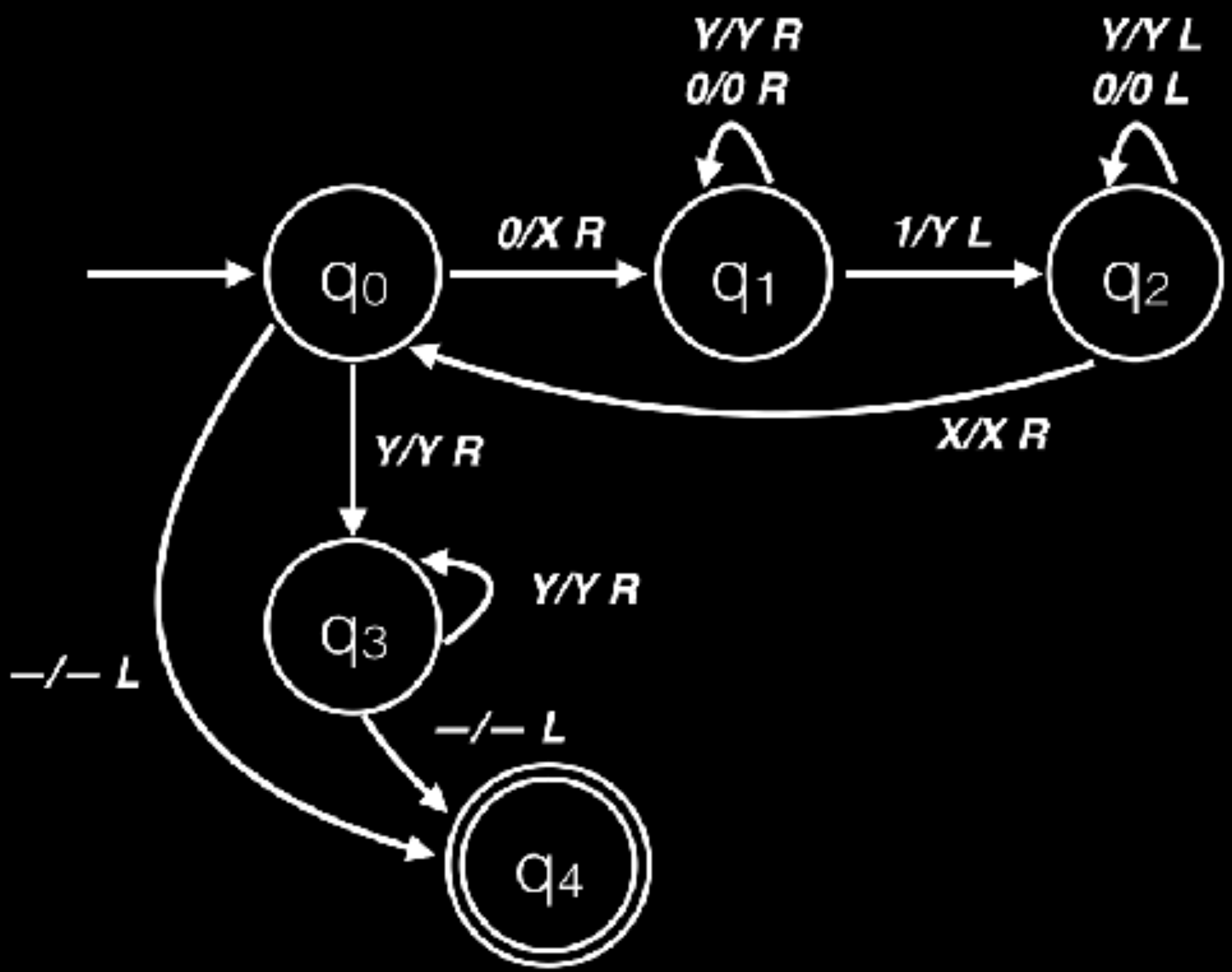
X$q_1$011

X0$q_1$11

X$q_2$0Y1

EXAMPLE

$L = 0^n1^n$, given the string 0011.



$q_0 0011$

$Xq_1 011$

$X0q_1 11$

$Xq_2 0Y1$

$q_2 X0Y1$

EXAMPLE

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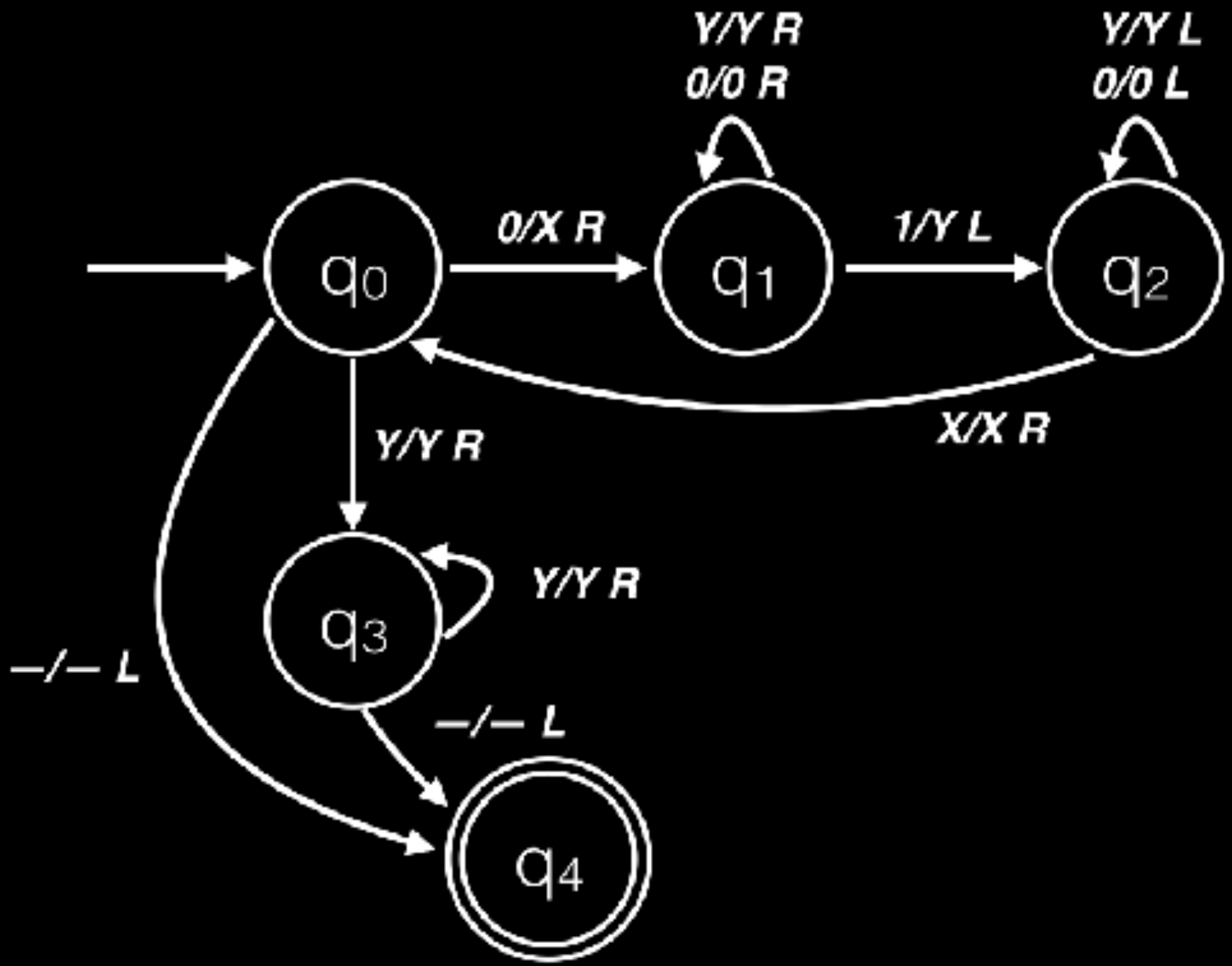

$q_0$0011
X$q_1$011
X0$q_1$11
X$q_2$0Y1
q_2X0Y1
X$q_0$0Y1

EXAMPLE

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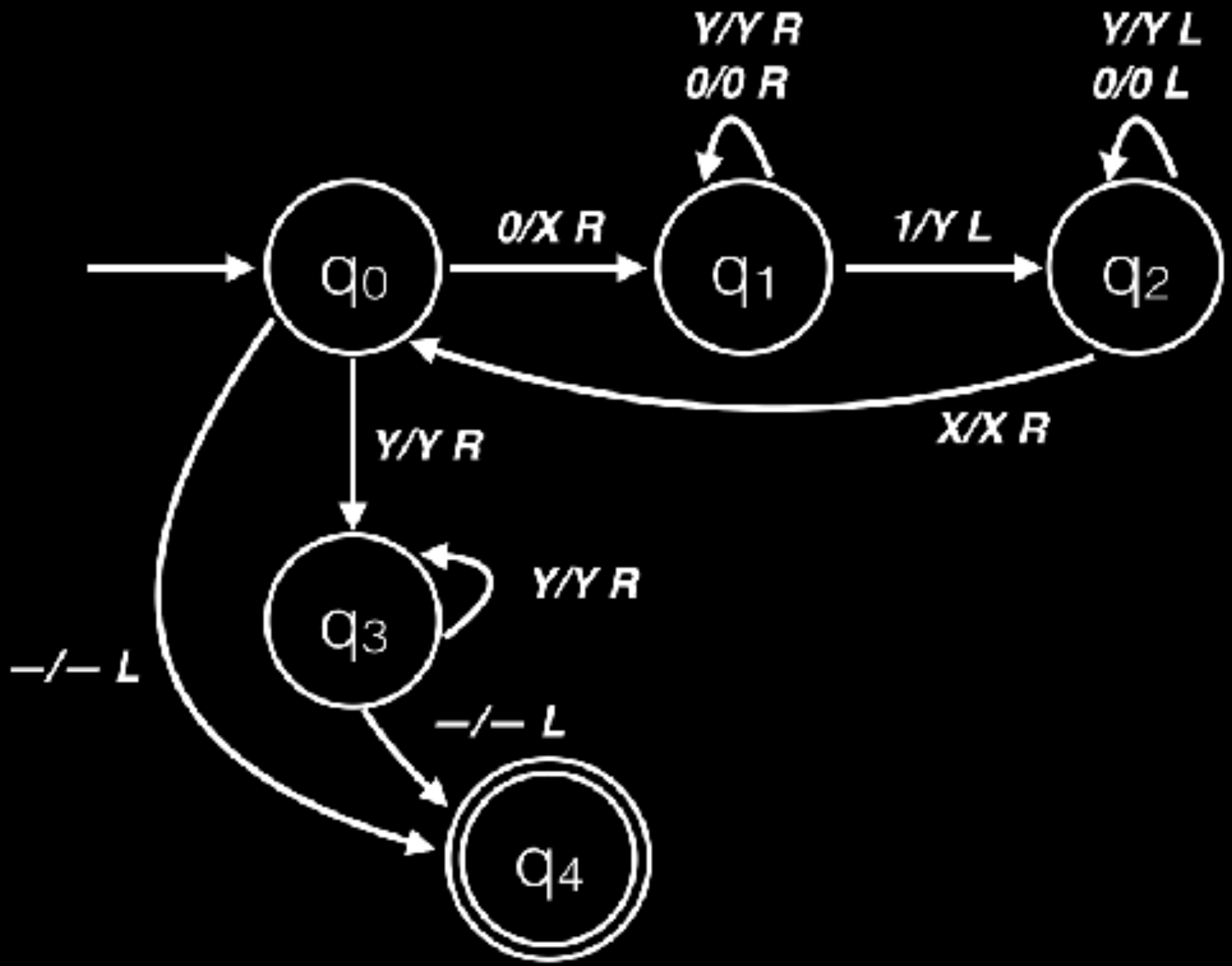

$q_0 0011$
$Xq_1 011$
$X0q_1 11$
$Xq_2 0Y1$
$q_2 X0Y1$
$Xq_0 0Y1$
$XXq_1 Y1$

EXAMPLE

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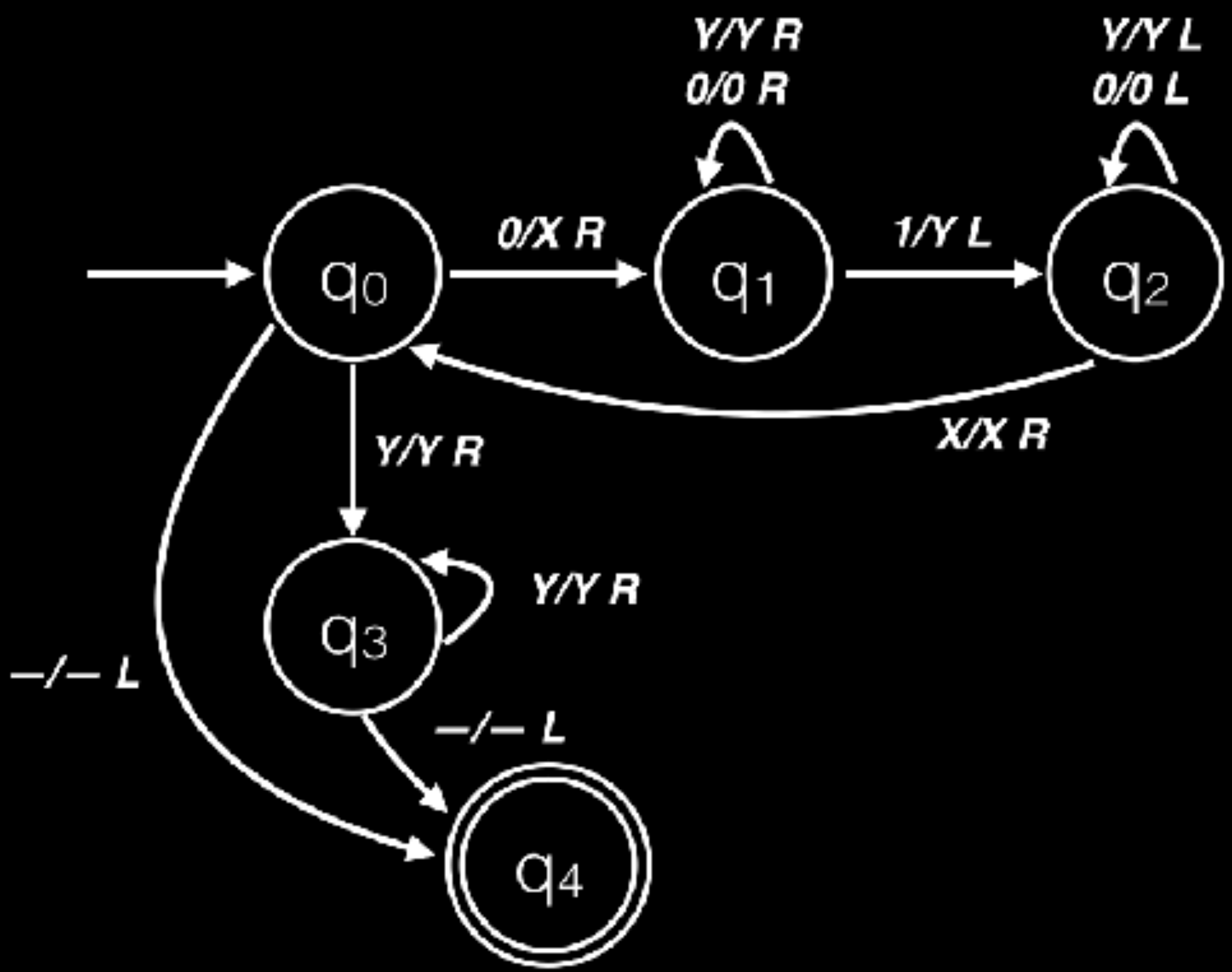

$q_0$0011
X$q_1$011
X0$q_1$11
X$q_2$0Y1
q_2X0Y1
X$q_0$0Y1
XXq_1Y1
XXY$q_1$1

EXAMPLE

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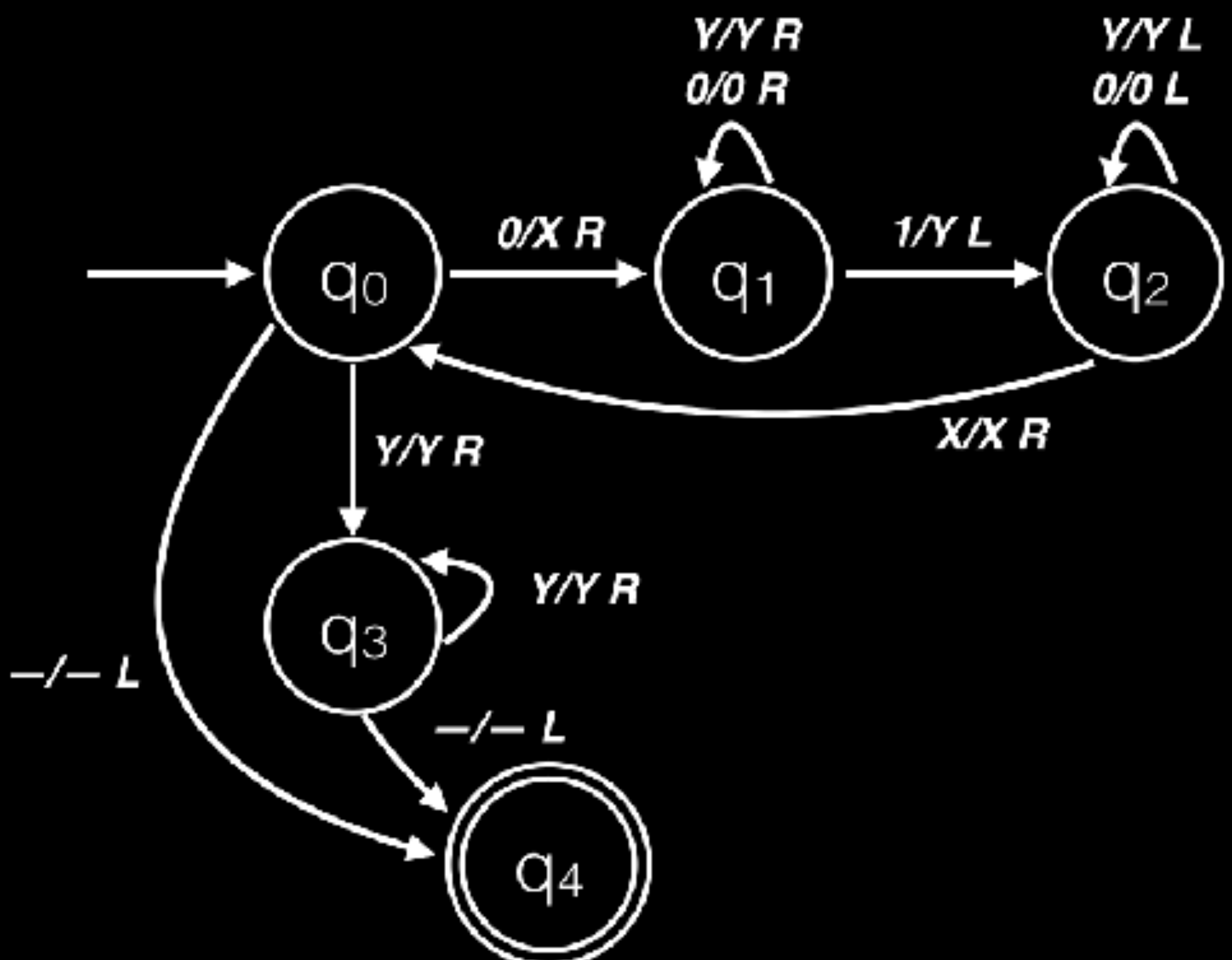

$q_0 0011$
$Xq_1 011$
$X0q_1 11$
$Xq_2 0Y1$
$q_2 X0Y1$
$Xq_0 0Y1$
$XXq_1 Y1$
$XXYq_1 1$
$XXq_2 YY$

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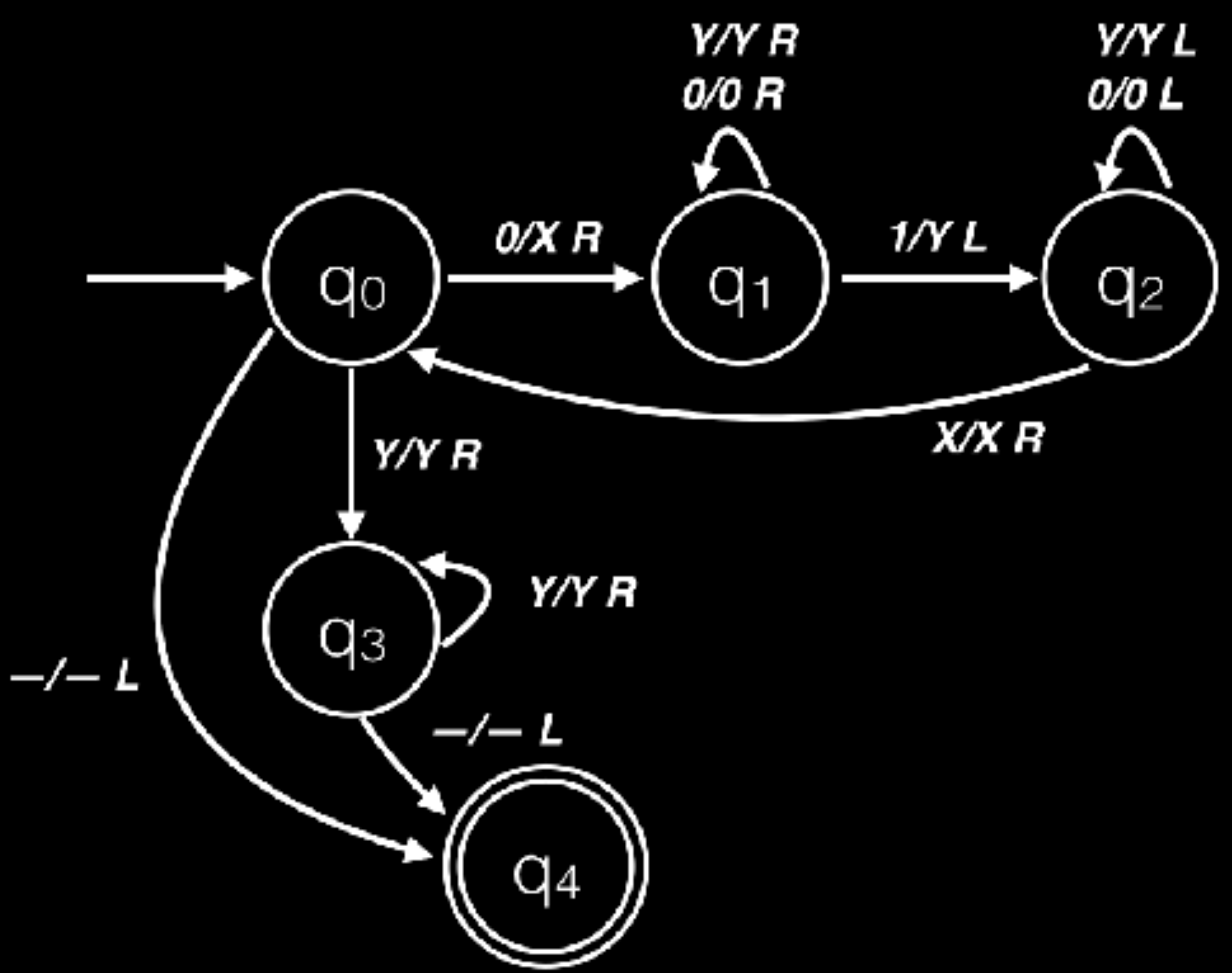

$q_0 0011$
$Xq_1 011$
$X0q_1 11$
$Xq_2 0Y1$
$q_2 X0Y1$
$Xq_0 0Y1$
$XXq_1 Y1$
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$Xq_2 XYY$

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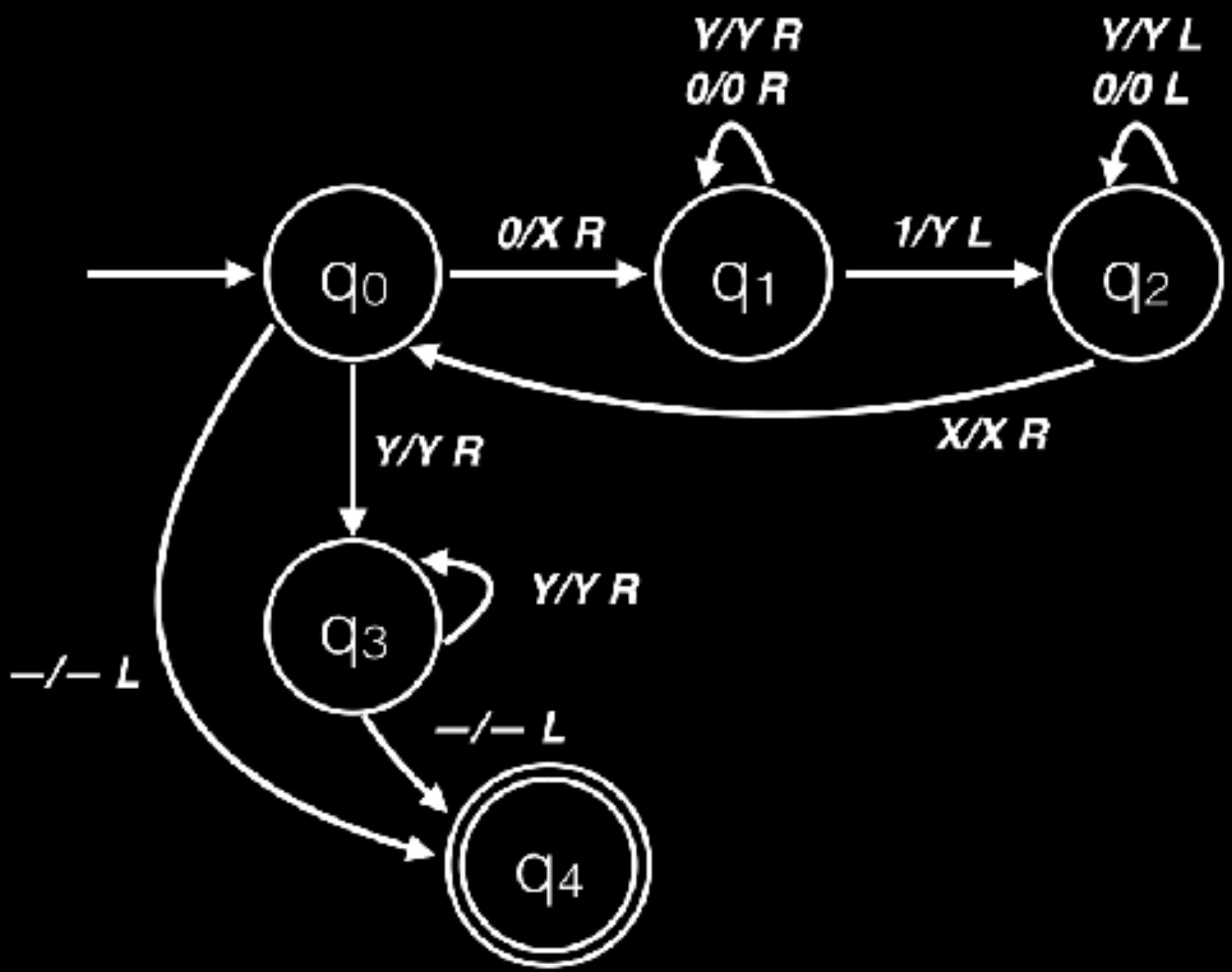

$q_0 0011$
$Xq_1 011$
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$Xq_2 0Y1$
$q_2 X0Y1$
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$XXq_1 Y1$
$XXYq_1 1$
$XXq_2 YY$
$Xq_2 XYY$
$XXq_0 YY$

EXAMPLE

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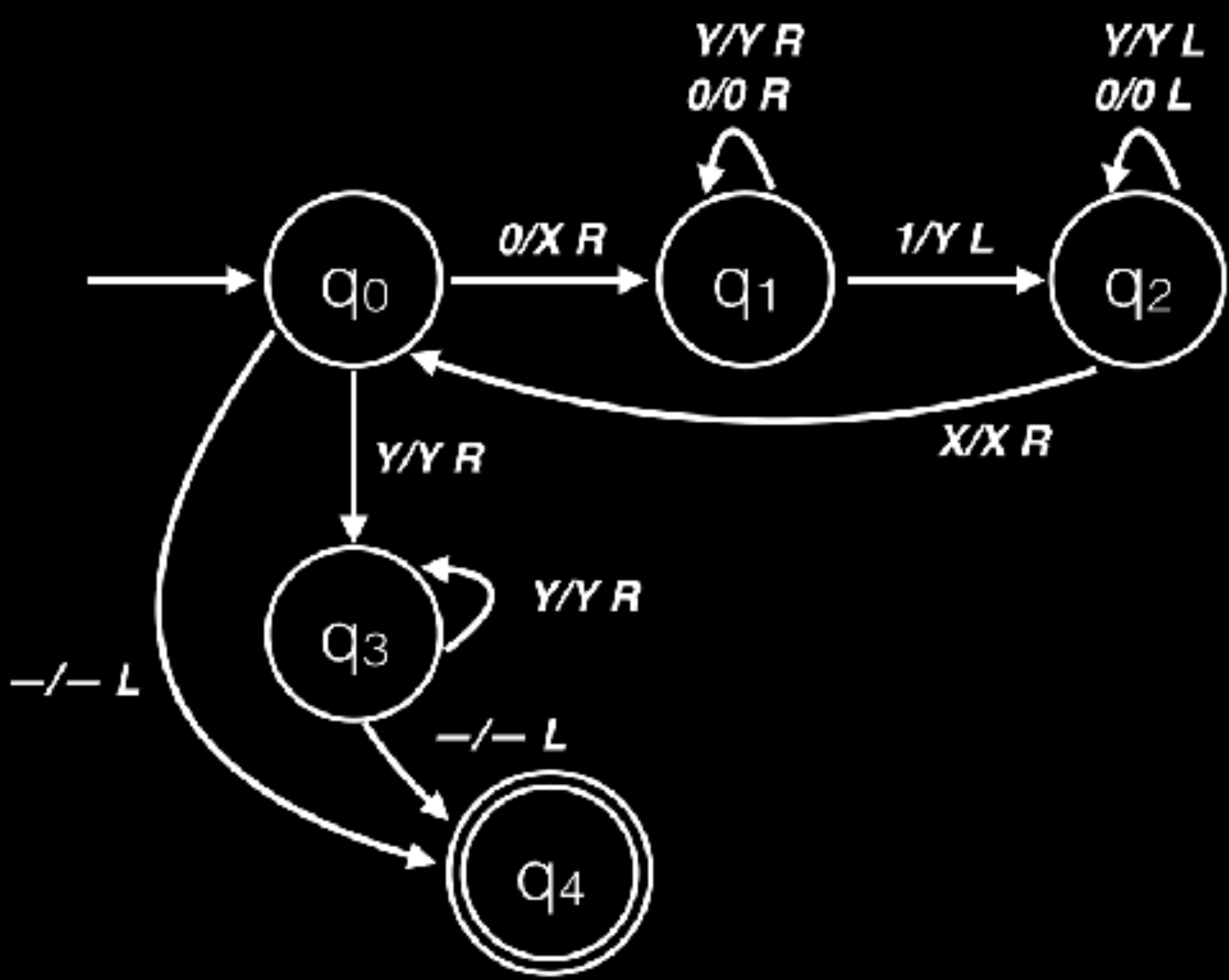

$q_0 0011$
$X q_1 011$
$X0 q_1 11$
$X q_2 0Y1$
$q_2 X0Y1$
$X q_0 0Y1$
$XX q_1 Y1$
$XXY q_1 1$
$XX q_2 YY$
$X q_2 XYY$
$XX q_0 YY$
$XXY q_3 Y$

EXAMPLE

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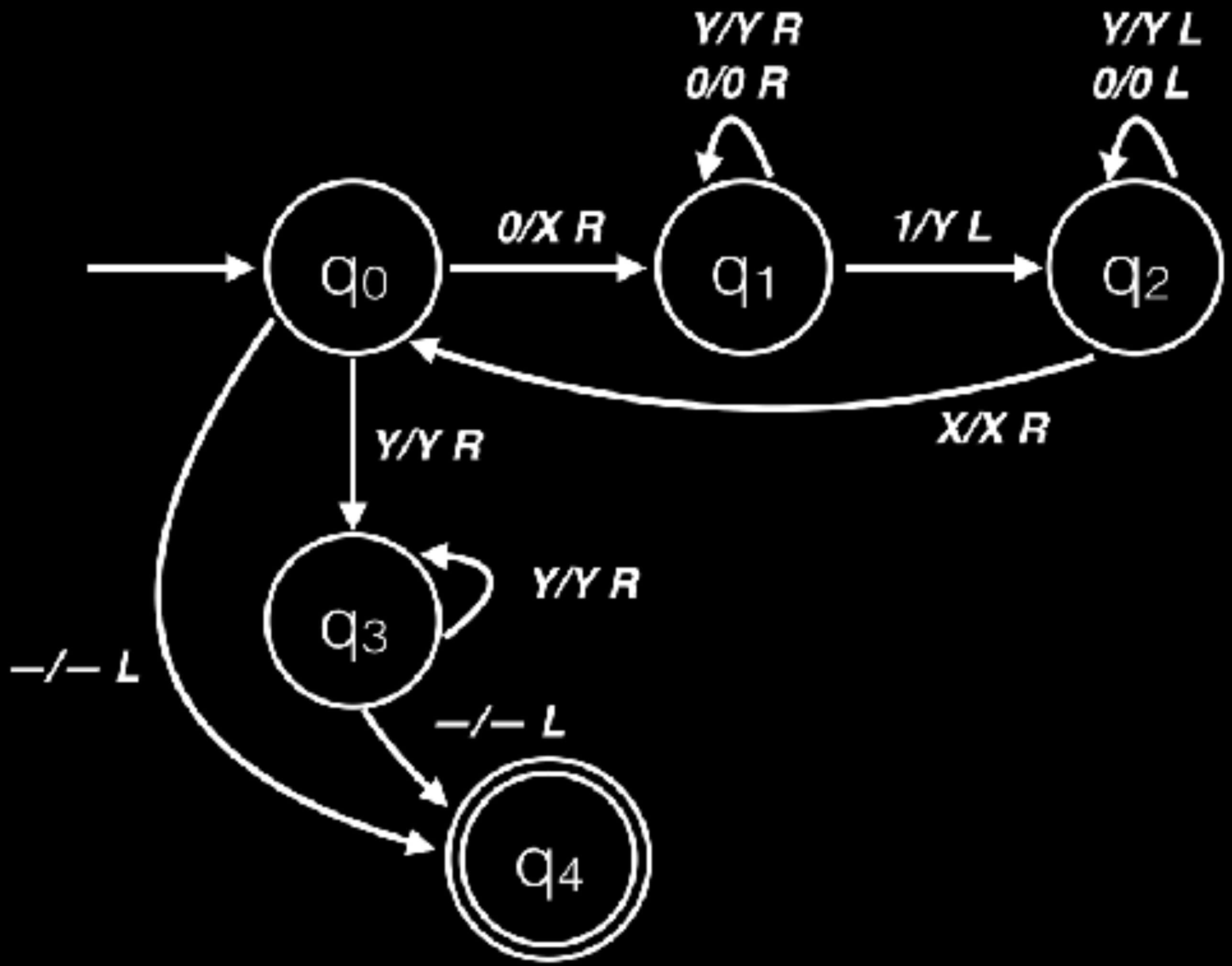

$q_0 0011$
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$X0q_1 11$
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$q_2 X0Y1$
$Xq_0 0Y1$
$XXq_1 Y1$
$XXYq_1 1$
$XXq_2 YY$
$Xq_2 XYY$
$XXq_0 YY$
$XXYq_3 Y$
$XXYYq_3 —$

EXAMPLE

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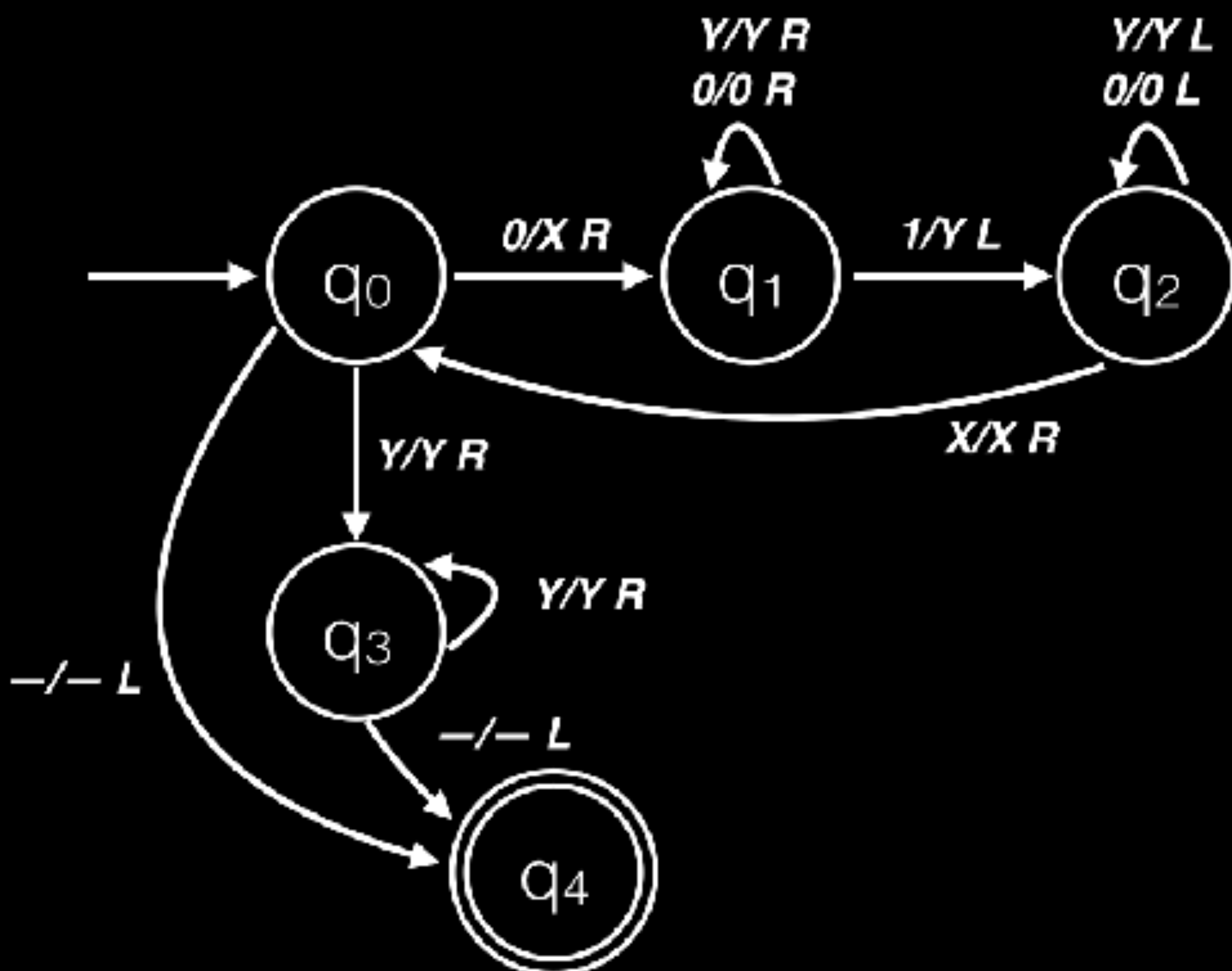

$q_0$0011
X$q_1$011
X0$q_1$11
X$q_2$0Y1
q_2X0Y1
X$q_0$0Y1
XXq_1Y1
XXY$q_1$1
XXq_2YY
Xq_2XYY
XXq_0YY
XXYq_3Y
XXYYq_3—
XXYY—q_4—

1. Decidable Languages

- When given a string as input, the TM will **always halt**.
  - TM will **accept** if it is in L.
  - TM will **reject** if it is not in L.

- also: recursive, computable, solvable

2. Turing Recognizable Languages

- When given a string that is in the language, the TM will always **halt and accept.**
- If not, the TM will either **reject or loop.**

- also: recursively enumerable, RE, partially decidable, semi-decidable

3. Not Turing Recognizable Languages

- Cannot recognize members reliably!

- also: not recursively enumerable, not RE, not partially decidable

USES OF TURING MACHINE

- To **decide** a language
- To **recognize** a language

USES OF TURING MACHINE

- To **decide** a language
- To **recognize** a language
- **To compute a function**

USES OF TURING MACHINE: TO COMPUTE A FUNCTION

1. Computable Function
    - ★ Computable ≅ Decidable
    - ★ Totally computable
    - ★ Defined on all inputs

2. Partially Computable Functions
   - Undefined on some inputs
   - Semi-decidable functions

The Church-Turing Thesis

Several variations on Turing Machines

- One tape or many?
- Infinite on both ends?
- Tiny alphabet {0, 1} or not?
- Can the head stay in the same place?
- Allow nondeterminism

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- One tape or many?
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- Can the head stay in the same place?
- Allow nondeterminism

All variations are equivalent in computing capability!

TMs and Lambda Calculus are also equivalent in power.

CONCLUSION / DEFINITION

Algorithmically Computable

=

Computable by a TM

Turing Machine

$\neq$

Turing Test

Turing Machine Programming Techniques

COMMON PROBLEM

How can we recognize the left end of the tape?

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How can we recognize the left end of the tape?

GOAL: Want to put a special symbol $ on the left end and shift the input over 1 cell to the right.

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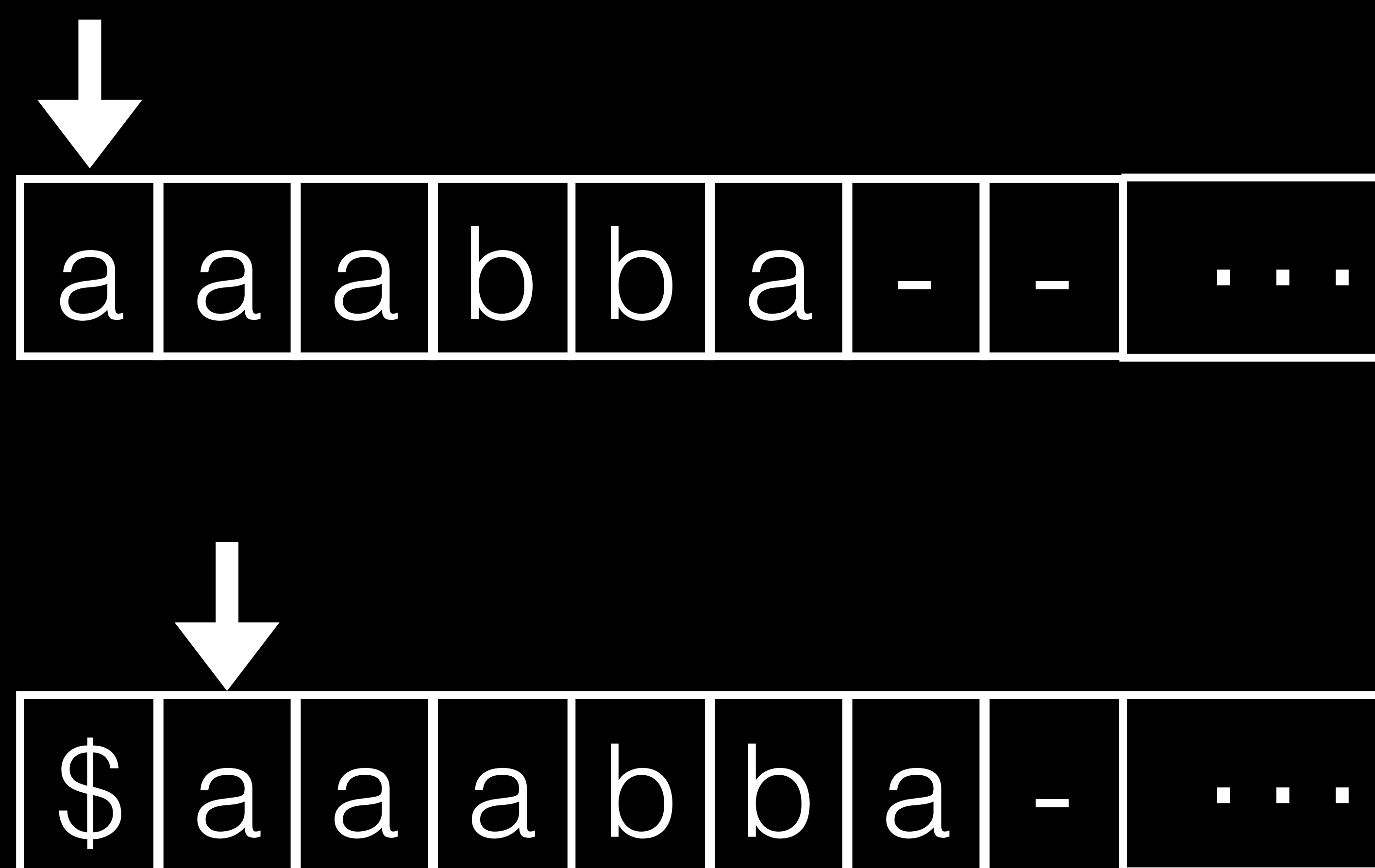
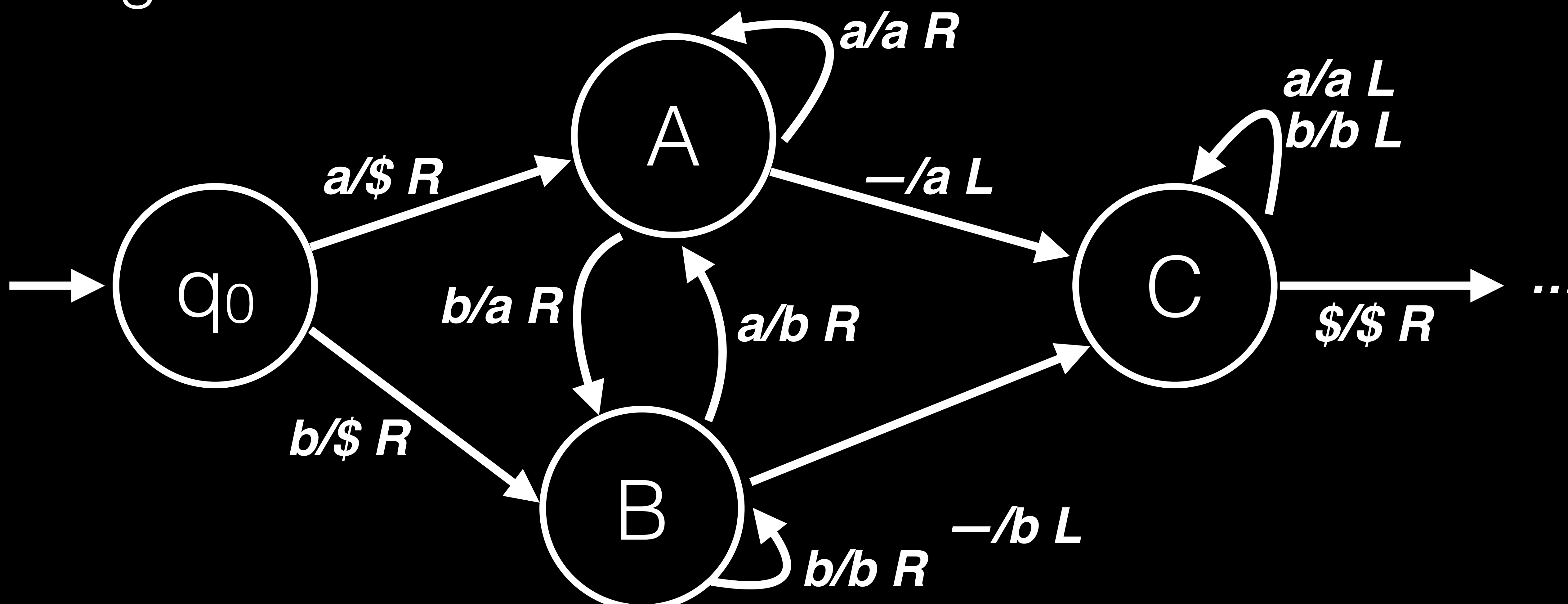


How can we recognize the left end of the tape?

GOAL: Want to put a special symbol $ on the left end and shift the input over 1 cell to the right.



Q: How much TM programming do you need to do?

A: Just enough to get the idea and to convince yourself that all programs or algorithms can be implemented on a Turing Machine.

WRITING PROGRAMS

Machine Code
0110, 1100

↓

Assembly Code
ADD R1, R2, R3

↓

C Code
i = (2 + k) * n

↓

Algorithms
if $S \cap T = \emptyset$…
No implementation details

WRITING PROGRAMS IN TURING MACHINE

Machine Code
0110, 1100

↓

Assembly Code
ADD R1, R2, R3

↓

C Code
i = (2 + k) * n

↓

Algorithms
if $S \cap T = \varnothing$…
No implementation details

Turing Machine
States, Transition Function
(complete TM)

↓

Outline of Algorithm
Still talking about
tape head movement
Data representation

↓

High-level specification of algorithm
No TM specific details

Build a TM to recognize the language $0^n1^n0^n$.

Build a TM to recognize the language $0^n1^n0^n$.

Build a TM to **decide** the language.

Build a TM to recognize the language $0^n1^n0^n$.

This language is **not** context-free. So this will prove

CONTEXT-FREE LANGUAGES $\subset$ DECIDABLE LANGUAGES

proper subset

Build a TM to recognize the language $0^n1^n0^n$.

We already have a TM to turn 0^n1^n into X^nY^n and to decide that language.

PROGRAMMING TECHNIQUE: SUBROUTINE

Build a TM to recognize the language $0^n1^n0^n$.

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Idea: Use that TM as a subroutine!

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Step 1:

0 0 0 0 1 1 1 1 0 0 0 0

X X X X Y Y Y Y 0 0 0 0

Step 2:

Build a similar TM to recognize Y^n0^n

Step 3:

Build the final TM by “gluing” these smaller TMs together into one larger TM.

Idea: Use that TM as a subroutine!

Step 1:

0 0 0 0 1 1 1 1 0 0 0 0

X X X X Y Y Y Y 0 0 0 0

Step 2:

Build a similar TM to recognize $Y^n 0^n$

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PROGRAMMING TECHNIQUE: MARKING SYMBOL

Compare two strings. A TM to decide $\{w\#w \mid w \in \{a, b, c\}\}$.

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Solution:

- Use a new symbol, such as "x"
- Turn each symbol into an "x" after it has been examined.

PROGRAMMING TECHNIQUE: MARKING SYMBOL

Compare two strings. A TM to decide $\{w\#w \mid w \in \{a, b, c\}\}$.

a a b a c # a a b a c

x a b a c # x a b a c

x x b a c # x x b a c

....

x x x x x # x x x x x

Compare two strings. A TM to decide $\{w\#w \mid w \in \{a, b, c\}\}$.

Problem:

Do it nondestructively, without losing the original strings. (Perhaps this task is part of a larger task.)

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Do it nondestructively, without losing the original strings. (Perhaps this task is part of a larger task.)

a = x, b = y, c = z

Compare two strings. A TM to decide $\{w\#w \mid w \in \{a, b, c\}$.

Solution:

Mark each symbol to keep track of what we've already done. Add some new symbols to help.

$a = x$, $b = y$, $c = z$

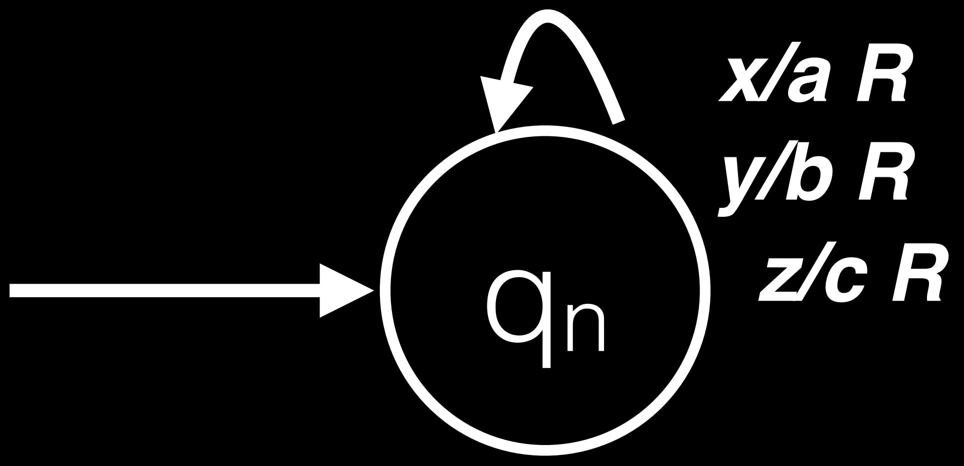
Compare two strings. A TM to decide $\{w\#w \mid w \in \{a, b, c\}$.

a a b a c # a a b a c

...

x x y x z # x x y x z

Later, restore the strings if we need to



PROGRAMMING TECHNIQUE: MARKING SYMBOL

“Mark each symbol with a dot.”

“Remember this location.”

Multitape Turing Machines

THEOREM

Every multitape Turing Machine has an equivalent single-tape Turing Machine.

Equivalent means it decides/recognizes the same languages. It’s not about speed, efficiency or ease of programming.

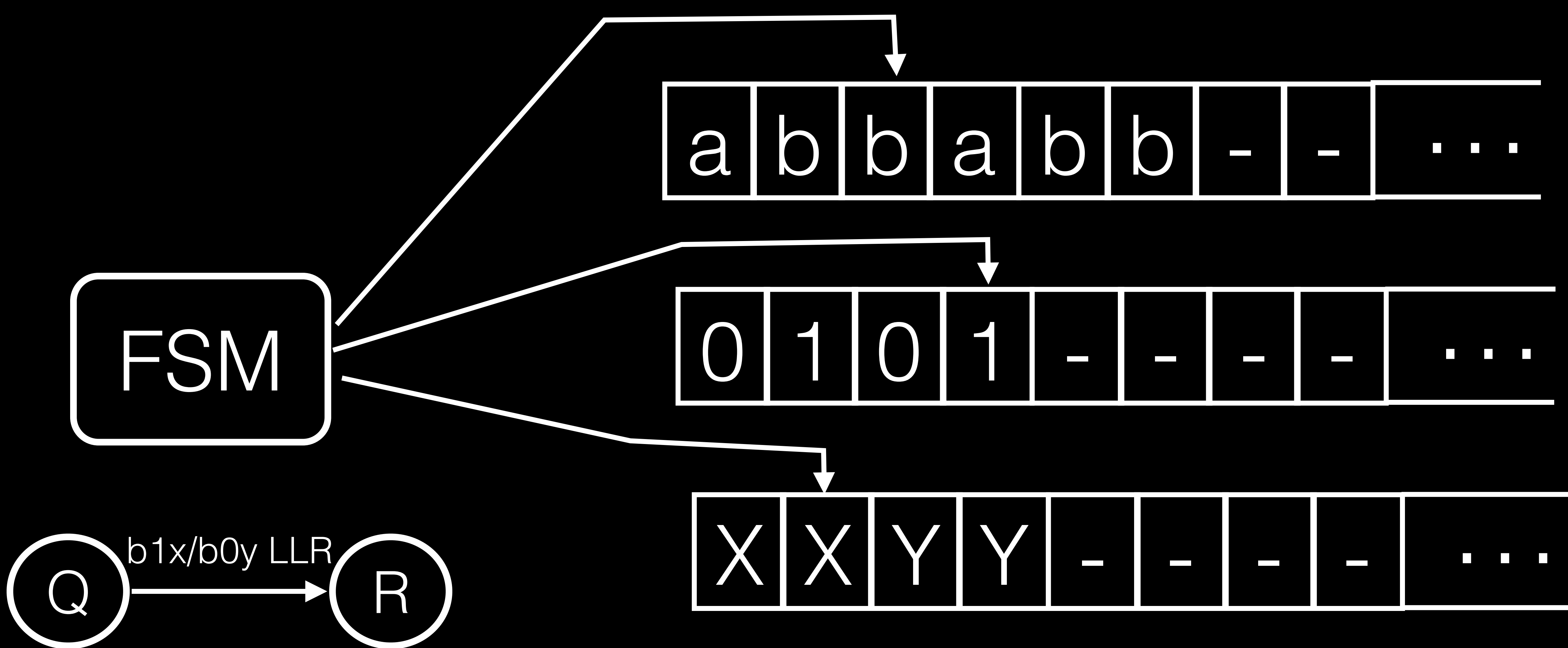
PROOF

Given a multitape TM, show how to build a single-tape TM.

- Need to store all tapes on a single tape. (Show data representation.)
- Each tape has a tape head. (Show how to store that information.)
- Need to transform a move in the multitape TM into a one or more in the single-tape TM.

PROOF

Multitape TM

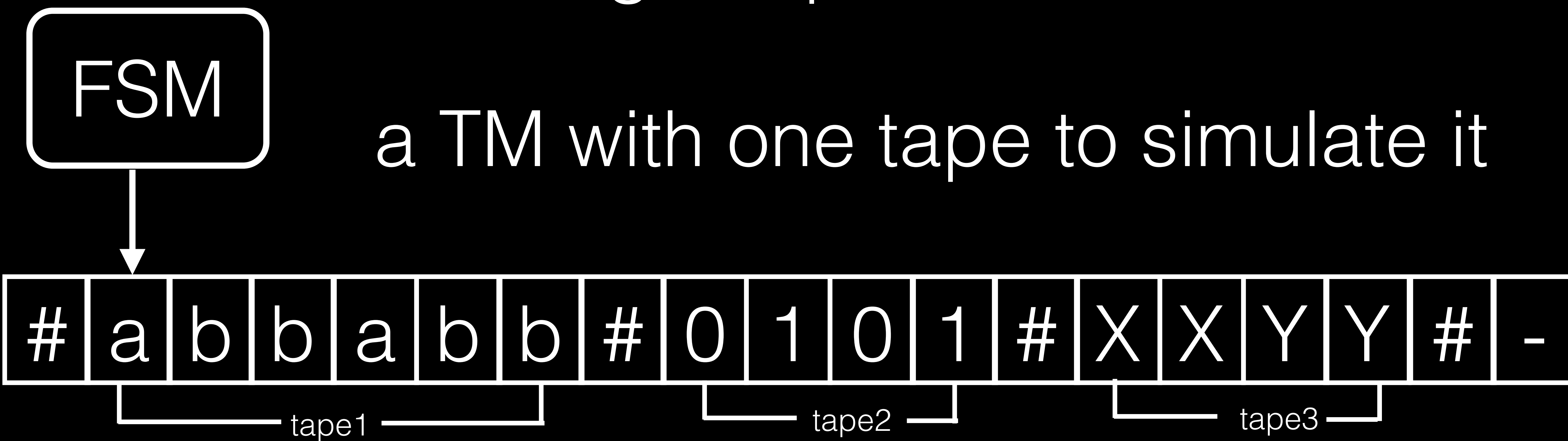


An example machine with k = 3 tapes

PROOF

Single-tape TM

a TM with one tape to simulate it



- Add “dots” to show where head k is.
- To simulate a transition from state Q, we must scan our tape to see which symbols are under the k tape heads.

Nondeterministic Turing Machines

Nondeterminism means that the TM may have more than one choice of action. As usual, a nondeterministic TM (or NTM) accepts a string if some choice of actions lead to that accept state.

Theorem. A nondeterministic TM has the same power as a standard TM.

THEOREM: TM vs NTM

Proof: We show that the NTM can be simulated by a deterministic one.

We need the concept of configuration. We view the calculations of NTM as a tree. The nodes are the configurations of the NTM, and the children of a node are the possible next steps. The NTM accepts the input if there is a branch that leads to an accepting configuration.

The simulator does breadth-first-search of tree.

Turing Machine as Problem Solvers