Nitsche-type Mortaring for Maxwell's Equations

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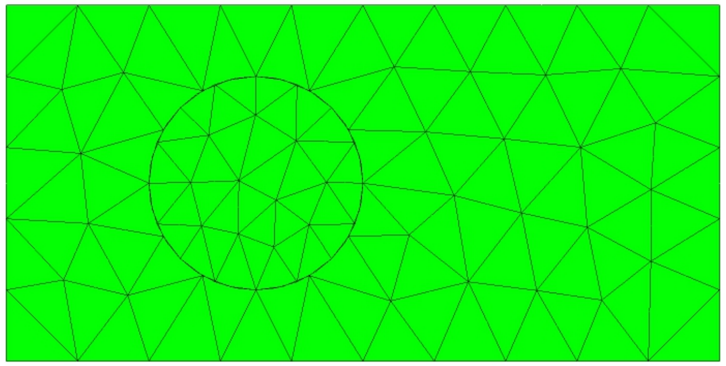
Problem setup

Domain decomposition on non-matching meshes:



Contents:

- Method for scalar model problem
- Nitsche-method for Maxwell's equation
- Numerical results

A model problem

Poisson equation:

$$\begin{aligned} -\Delta u &= f \qquad \text{on } \Omega_1 \cup \Omega_2 \\ u &= 0 \qquad \text{at } \partial(\overline{\Omega_1 \cup \Omega_2}) \end{aligned}$$

Interface conditions on Γ:

$$\begin{aligned} [u] := u_1 - u_2 &= 0 \\ \partial_{n_1} u_1 + \partial_{n_2} u_2 &= 0 \end{aligned}$$

Mortar method

Pose the constraint as additional equation.

Find $u \in H^1_{0,D}(\Omega_1) \times H^1_{0,D}(\Omega_1)$ and $\lambda \in H^{-1/2}(\Gamma)$ such that

$$
\begin{array}{lcccll}
\int_{\Omega_1 \cup \Omega_2} \nabla u \cdot \nabla v & + & \int_\Gamma [v] \lambda & = & \int f v & \forall\, v \\
\int_\Gamma [u] \mu & & & = & 0 & \forall\, \mu
\end{array}
$$

The Lagrange parameter λ is the normal flux $\partial_n u$.

Requires stability condition for finite element spaces (LBB).

Leads to an indefinite system matrix.

Nitsche / Discontinuous Galerkin method

Allows discontinuous approximation by keeping extra boundary terms:

Find $u \in H^1_{0,D}(\Omega_1) \times H^1_{0,D}(\Omega_1)$ such that

$$\int_{\Omega_1 \cup \Omega_2} \nabla u \cdot \nabla v - \int_\Gamma \partial_n u \, [v] - \int_\Gamma \partial_n v \, [u] + \alpha \int_\Gamma [u] \, [v] = \int f v \qquad \forall \, v$$

with soft penalty term $\alpha \sim p^2/h$ sufficiently large.

No extra stability condition is required.

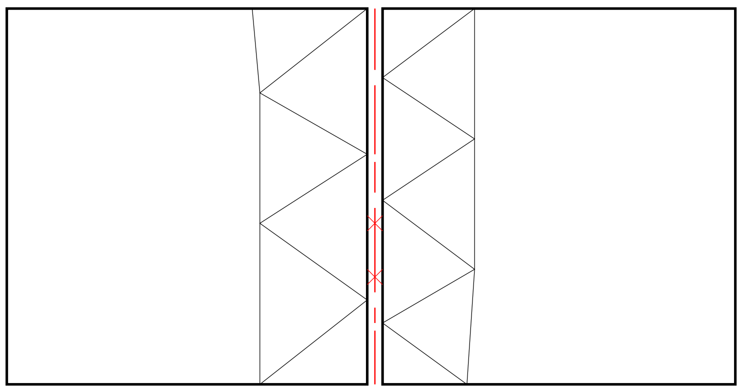
Leads to a symmetric positive definite stiffness matrix.

Integration of boundary terms

Both methods require to compute integrals of finite element functions from different meshes:

Mortar method: $\int_\Gamma u_2 \mu$

Nitsche method: $\int_\Gamma v_1 \partial_n u_2$



Requires the calculation of an intersection mesh

Complicated implementation, in particular on curved interfaces in 3D

Hybrid Nitsche method - derivation

Introduce a new variable for the primal unknown on the interface:

$$\lambda := u|_\Gamma$$

Multiply by test-functions, and integrate by parts:

$$\sum_i \int_{\Omega_i} \nabla u \cdot \nabla v - \int_\Gamma \partial_n u \, v = \int f v \qquad \forall \, v \in H^1(\Omega_1) \times H^1(\Omega_2)$$

Use continuity of $\partial_n u$ and introduce single-valued test function μ on interface:

$$\sum_i \int \nabla u \cdot \nabla v - \int_\Gamma \partial_n u \, (v - \mu) = \int f v \qquad \forall \, v \, \forall \, \mu$$

Use $u = \lambda$ on Γ to symmetrize and stabilize with $\alpha \sim p^2/h$.

$$\sum_i \int \nabla u \cdot \nabla v - \int_\Gamma \partial_n u \, (v - \mu) - \int_\Gamma \partial_n v \, (u - \lambda) + \alpha \int_\Gamma (u - \lambda)(v - \mu) = \int f v \qquad \forall \, v \, \forall \, \mu$$

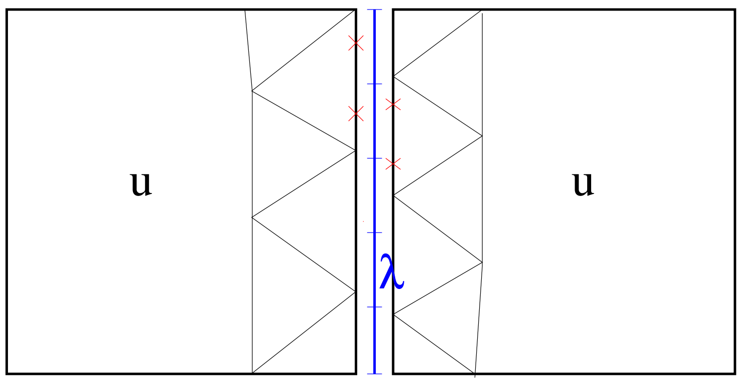
For α chosen right, the discrete formulation is stable independent of the choice of fe spaces for u and λ.

Discretizing and numerical integration

In general, numerical integration is still difficult.

We propose to use smooth B-spline functions for discretizing the hybrid variable λ. This allows

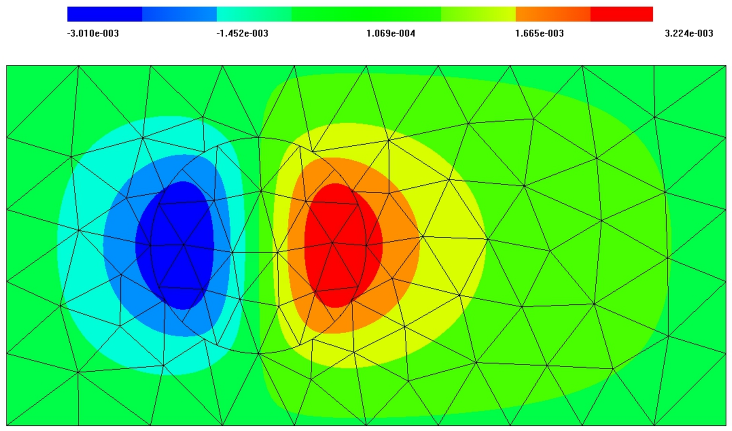
- evaluation in global coordinates
- efficient numerical integration by Gauss-rules on the surface elements



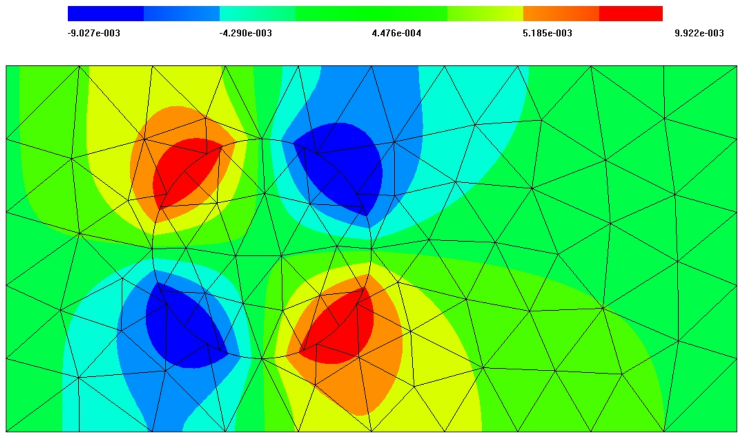
Numerical experiments in 2D

$f = x$ in circle, else $f = 0$.

Solution u:



Solution $\partial u/\partial x$:

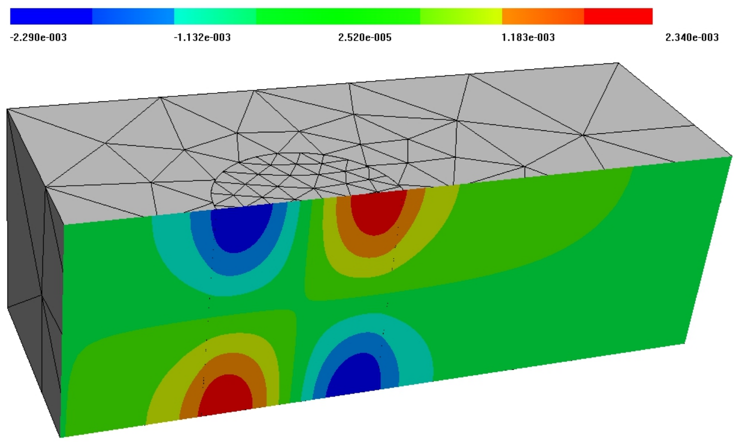


Finite element order $p = 5$.

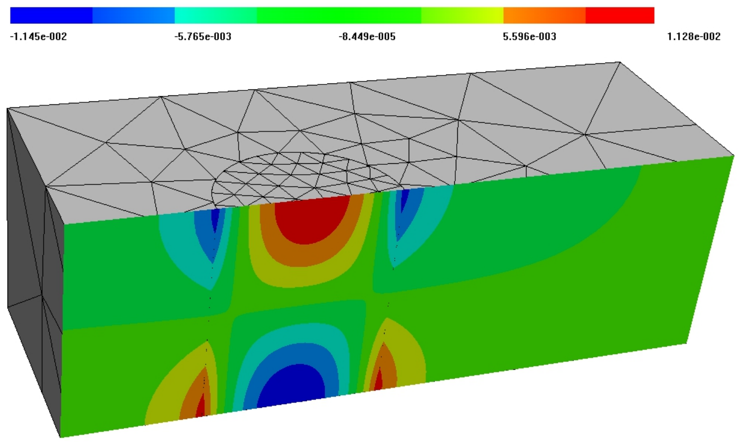
Numerical experiments in 3D

$f = xz$ in cylinder, else $f = 0$.

Solution u:



Solution $\partial u/\partial x$:



Finite element order $p = 4$.

Maxwell's Equations

Time harmonic Maxwell's equations

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u + \kappa u = j \qquad \text{in } \Omega_i$$

with $\kappa = i\omega\sigma - \omega^2\epsilon$, and

$$E = -i\omega u, \quad H = \mu^{-1} \operatorname{curl} u.$$

Transmission conditions

$$\begin{aligned} u_1 \times n_1 + u_2 \times n_2 &= 0, \\ \mu_1^{-1} \operatorname{curl} u_1 \times n_1 + \mu_2^{-1} \operatorname{curl} u_2 \times n_2 &= 0. \end{aligned}$$

Hybrid Nitsche formulation

proceed as in the scalar case:

$$\int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot (v \times n) = \int_{\Omega_i} j \cdot v$$

add symmetry and penalty terms: find (u, λ) such that

$$\sum_{i=1}^{2} \left\{ \int_{\Omega_i} \mu^{-1} \{\operatorname{curl} u \cdot \operatorname{curl} v + \kappa u \cdot v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \cdot [(v - \mu) \times n] \right.$$
$$\left. + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v \cdot [(u - \lambda) \times n] + \frac{\alpha p^2}{\mu h} \int_{\partial\Omega_i} [(u - \lambda) \times n] \cdot [(v - \mu) \times n] \right\} = \int_{\Omega} j \cdot v,$$

where $u, v \in H(\operatorname{curl}, \Omega_1) \times H(\operatorname{curl}, \Omega_2)$, and λ, μ are tangential vector valued fields on the interface.

Overpenalization of gradient fields

The natural energy norm is

$$\|u\|^2 = \mu^{-1}\|\operatorname{curl} u\|_{L_2}^2 + |\kappa|\,\|u\|_{L_2}^2$$

For gradient fields $u = \nabla\phi$, this norms scales as

$$\|\nabla\phi\|^2 = O(\kappa)$$

This is small for small frequencies/conductivities.

The norm for the Nitsche method is

$$|||(u,\lambda)|||^2 = \sum_{i=1}^{2}\left\{\mu^{-1}\|\operatorname{curl} u\|_{\Omega_i}^2 + \kappa\|u\|_{\Omega_i}^2 + \alpha\mu^{-1}\|(u-\lambda)\times n\|_{\Gamma}^2\right\}$$

But, the last term of this norm does not scale with κ for gradient fields.

Thus, the penalty term $\|u-\lambda\|$ leads to an overpenalization of the jump for gradient fields.

Scalar potential at the boundary

Goal: Want to replace the continuity condition

$$(u_i - \lambda) \times n_i = 0 \qquad i = 1, 2$$

by

$$\begin{aligned} (u_i - \nabla\phi_i - \lambda) \times n_i &= 0 \\ \phi_i - \phi_\Gamma &= 0 \end{aligned}$$

with arbitrary scalar fields $\phi_1 = \phi_2 = \phi_\Gamma$ on the boundary.

This allows to scale the penalty terms for gradients and rotations differently.

Variational formulation

$$\sum_{i=1}^{2} \left\{ \int_{\Omega_i} \{\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \kappa u v\} + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} u \, [(v - \mu) \times n] + \int_{\partial\Omega_i} \mu^{-1} \operatorname{curl} v \, [(u - \nabla\phi - \lambda) \times n] + \alpha \int_{\partial\Omega_i} \mu^{-1} [(u - \nabla\phi - \lambda) \times n][(v - \nabla\psi - \mu) \times n] + \alpha \int_{\partial\Omega_i} \kappa(\phi - \phi_\Gamma)(\psi - \psi_\Gamma) \right\} = \int_\Omega j v$$

A boundary identity

Testing the weak form with $v = \nabla \psi$ gives

$$\int_{\Omega_i} \kappa u \cdot \nabla \psi + \int_{\partial \Omega_i} \mu^{-1} \operatorname{curl} u \cdot (\nabla \psi \times n) = \int_{\Omega} j \cdot \nabla \psi$$

Taking the divergence in the strong form, and integrating by parts leads to

$$\int_{\Omega_i} \operatorname{div}(\kappa u)\, \psi = \int_{\Omega_i} \operatorname{div} j \, \psi$$

$$-\int_{\Omega_i} \kappa u \cdot \nabla \psi + \int_{\partial \Omega_i} \kappa u_n \psi = -\int_{\Omega_i} j \cdot \nabla \psi + \int_{\partial \Omega_i} j_n \psi$$

Adding up leads to the boundary relation

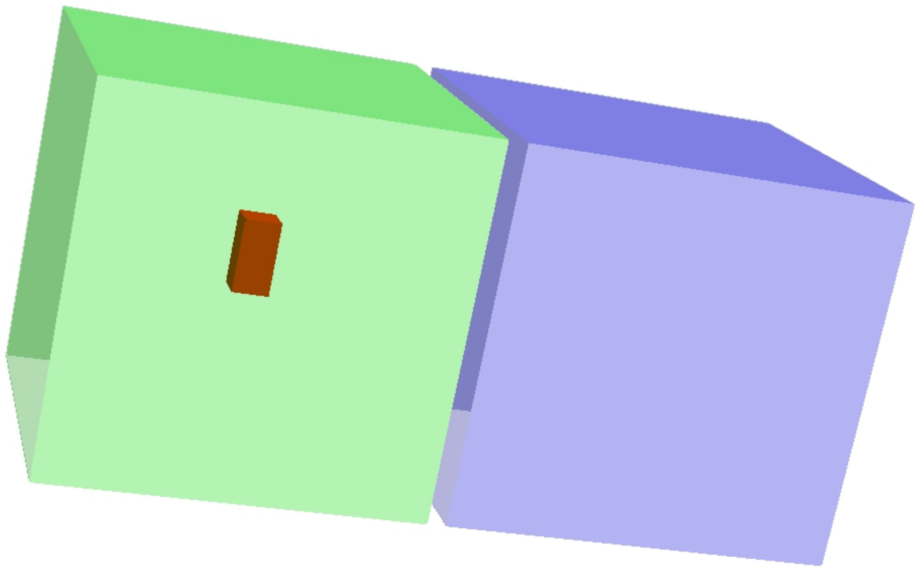
$$\int_{\partial \Omega_i} \mu^{-1} \operatorname{curl} u \cdot (\nabla \psi \times n) + \int_{\partial \Omega_i} \kappa u_n \psi = \int_{\partial \Omega_i} j_n \psi$$

Final variational formulation

$$
\sum_{i=1}^{2}\left\{\int_{\Omega_i}\{\mu^{-1}\operatorname{curl}u\cdot\operatorname{curl}v+\kappa uv\}+\int_{\partial\Omega_i}\mu^{-1}\operatorname{curl}u\,[(v-\nabla\psi-\mu)\times n]\right.
$$
$$
+\int_{\partial\Omega_i}\mu^{-1}\operatorname{curl}v\,[(u-\nabla\phi-\lambda)\times n]+\alpha\int_{\partial\Omega_i}\mu^{-1}[(u-\nabla\phi-\lambda)\times n][(v-\nabla\psi-\mu)\times n]
$$
$$
\left.-\int_{\partial\Omega_i}\kappa u_n(\psi-\psi_\Gamma)-\int_{\partial\Omega_i}\kappa v_n(\phi-\phi_\Gamma)+\alpha\int_{\partial\Omega_i}\kappa(\phi-\phi_\Gamma)(\psi-\psi_\Gamma)\right\}=
$$
$$
\sum_{i=1}^{2}\left\{\int_{\Omega_i}jv-\int_{\partial\Omega_i}j_n\psi\right\}
$$

- $u, v...$ $H(\mathrm{curl})$ conforming element basis functions on Ω_i
- $\phi, \psi...$ H^1 conforming element basis functions on $\Omega_i \cap \Gamma$
- $\lambda, \mu...$ tangential vector valued spline functions on Γ
- $\phi_\Gamma, \psi_\Gamma...$ scalar spline functions on Γ

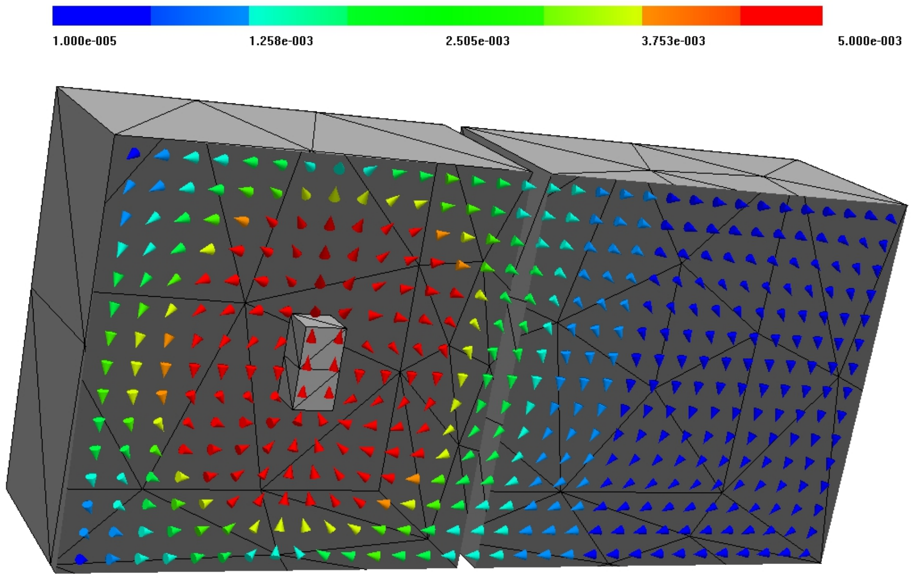
Magnetostatics



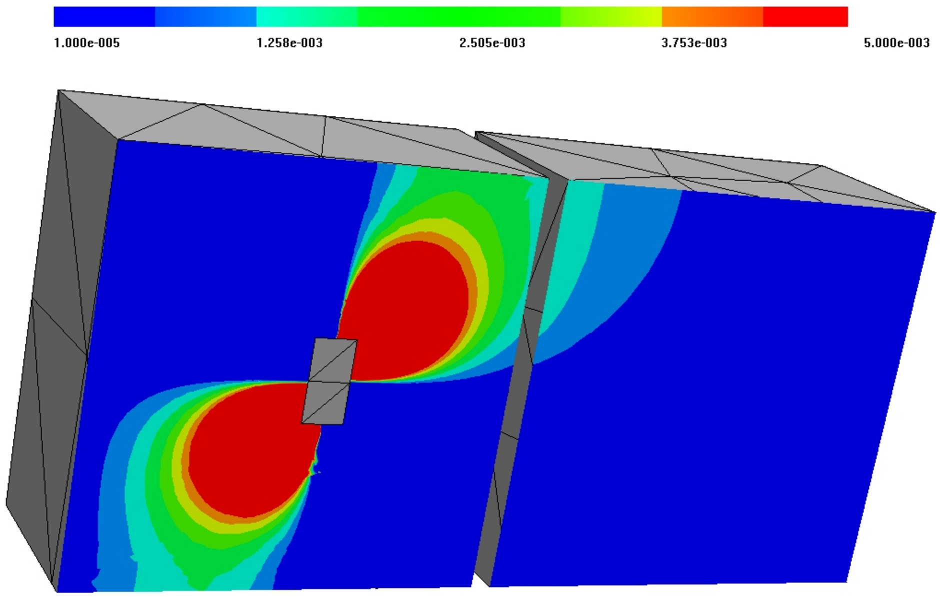
Permanent magnet (red) with two domains (green and blue)

Magnetic flux

magnetic flux



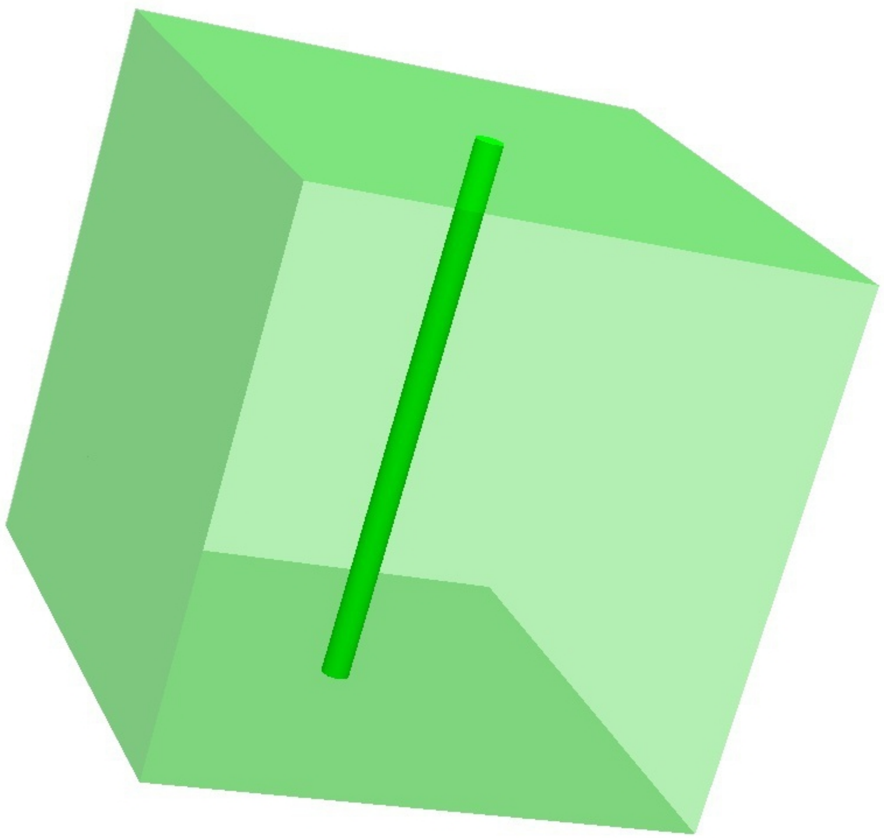
x-component of magnetic flux



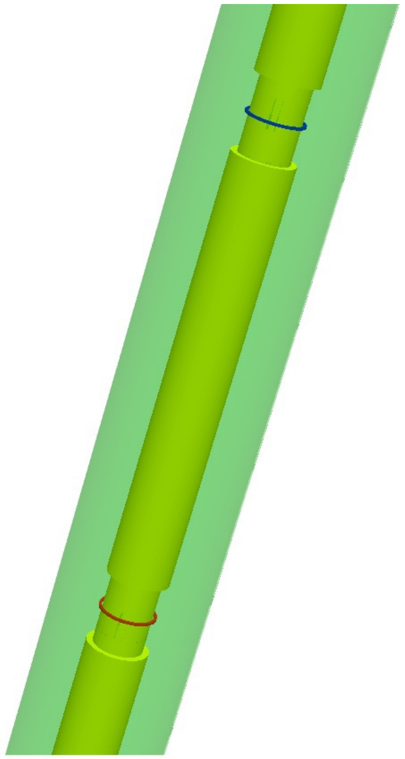
Finite element order $p = 4$.

LWD-Tool

borehole with soil

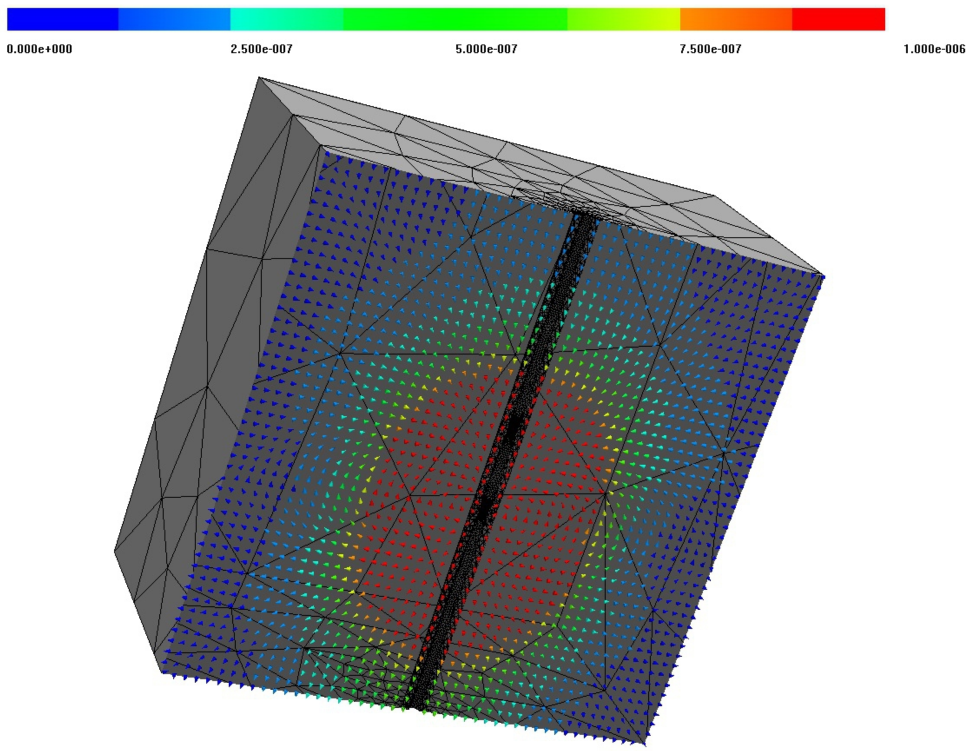


tool with antennas

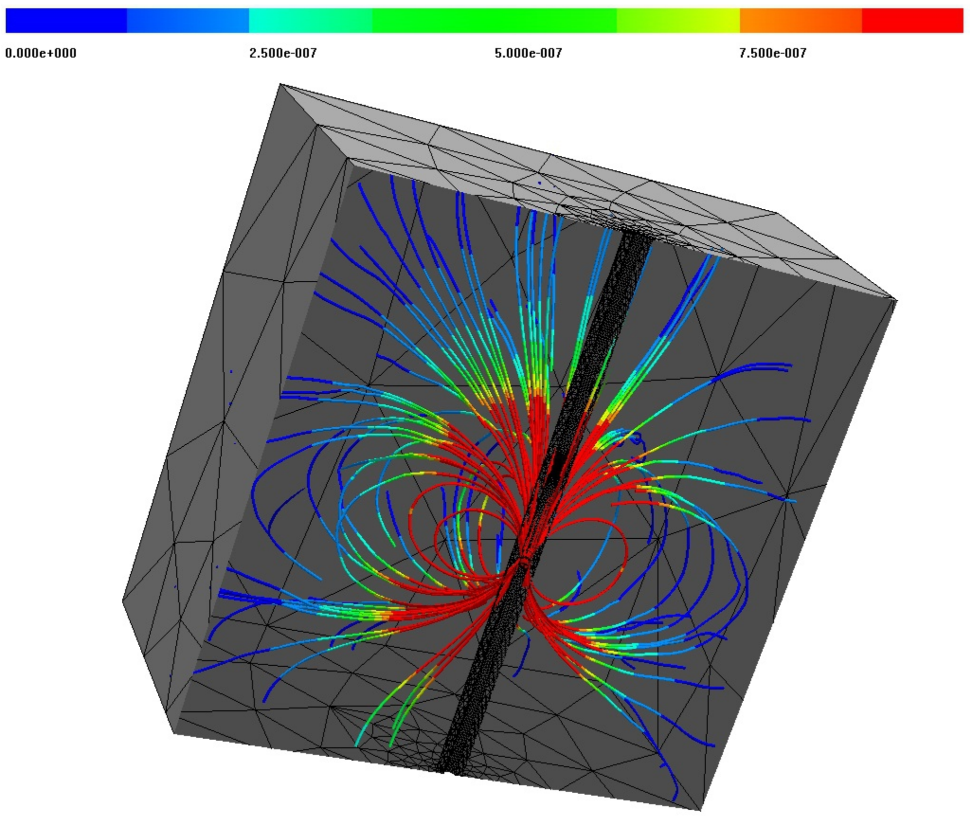


LWD-Tool

B-field



B-field, field-lines:



LWD-Tool

Numerical results for first order elements:

frequency [kHz]	standard, rec volt [nV] 185 810 dofs	Nitsche, rec volt [nV] 195 383 dofs
20	$25.44 - i\,18.38$	$25.43 - i\,18.37$
100	$71.68 - i\,197.5$	$71.65 - i\,197.3$
400	$124.9 - i\,963.0$	$124.9 - i\,962.3$
2000	$-635.9 - i\,5295$	$-634.8 - i\,5255$

Numerical results for second order elements:

frequency [kHz]	standard, rec volt [nV] 733 881 dofs	Nitsche, rec volt [nV] 736 939 dofs
20	$24.99 - i\,18.47$	$24.98 - i\,18.47$
100	$70.25 - i\,196.7$	$70.23 - i\,196.7$
400	$121.7 - i\,957.9$	$121.7 - i\,957.7$
2000	$-648.1 - i\,5256$	$-647.9 - i\,5255$

Conclusions and Ongoing Work

We have

- Hybrid Nitsche-type mortaring for scalar equation
- Stable transmission conditions for low frequency Maxwell's equations
- Interface fields discretized by smooth B-spline spaces for simple numerical integration

We work on

- Error estimators and adaptivity for spline space and numerical integration
- Iterative solvers, in particular BDDC domain decomposition methods

The methods are implemented within Netgen/NGSolve software