Chapter 1

Introduction

Many shock wave phenomena are not amenable to analytical methods, for they involve complex shock-on-shock interactions whose behaviour is highly non-linear. Consequently, our understanding of such phenomena has come largely from experimentation, a process that started with Mach in 1878. But with the advent of the computer it has become possible to simulate shock wave phenomena numerically. Interest in computational shock hydrodynamics was kindled by von Neumann[69] in 1944, and has burgeoned dramatically since the mid 1970s. The main reasons for this growth are simple: numerical simulations are attractive as replacements for experiments that are difficult, dangerous or expensive; certain phenomena can only be studied numerically; simulations substantially reduce the time and cost required for engineering design.

During the development of computational shock hydrodynamics, two underlying strategies have emerged that allow complicated flows to be resolved clearly. One, employ a numerical scheme of inherently high resolution, usually a second-order Godunov-type scheme[29]. Two, locally refine the computational mesh so as to improve the resolution of the numerical solution in regions of interest. It has been demonstrated by Berger & Collela[8] that a combination of both strategies is necessary if a solution of very high resolution is sought. The present study combines Roe's flux-difference splitting scheme[54] with an Adaptive Mesh Refinement algorithm (AMR) developed from the ideas of Berger[7]. The result being a general purpose scheme which requires only modest computing power to resolve flows that contain complex shock-on-shock interactions.

Before proceeding much further, it is thought provoking to note some novel applications of shock waves thereby introducing shock hydrodynamics as a subject worthy of study. This brief interlude is followed by some background material which places the AMR algorithm in context. Finally, in preparation for subsequent chapters, we present an overview of the AMR algorithm together with an outline for the aims and contents of this thesis.

1.1 Motivations For Studying Shock Wave Phenomena

Shock wave phenomena are an integral part of our industrialized society, and although popular imagery focuses on the negative aspects, for example, nuclear Armageddon or noise pollution caused by sonic boom, shock waves have benefited mankind in numerous ways. An illuminating and comprehensive account of these benefits has been given by Glass in his classic book, SHOCK WAVES & MAN[27]. So, here it is sufficient to note just a few novel applications of shock waves which belie the notion that explosives are synonymous with death and destruction.

One such application from the medical world is the disintegration of kidney and gall bladder stones by bombardment with shock waves[59]. Shock waves are produced at the focus of an ellipsoidal mirror by some form of high energy discharge mechanism which could be a spark, a laser pulse or a micro-explosion. The energy from a single discharge will be focused by the mirror at its second focus in the form of a powerful implosion wave. If this second focus is located on a stone, the implosion waves generated by a succession of such discharges prove sufficient to break up the stone into many fragments which are then small enough to pass harmlessly from the body. This technique has not yet been perfected, for it is expensive and slight tissue damage can be sustained. Nevertheless, it has been used successfully on many tens of thousands of patients and in severe cases it has undoubtedly prolonged life.

Again in contradiction with the perceived nature of shock waves, many novel applications are to be found in manufacturing processes[42]. Explosive welding and forming have been used extensively since the early 1960s[50], and recent applications include the consolidation of metal and ceramic powders using the hydrodynamic compaction resulting from controlled explosions. This has proved to be the most reliable method for the bulk synthesis of exotic materials such as superconductors[43]. In all probability, the development of room temperature superconductors would trigger another industrial revolution. The importance of shock-consolidation manufacturing processes to this revolution could match the importance of explosives to *the* Industrial Revolution; without the use of explosives to help build roads, tunnels and canals *etc.* society might not have become industrialized!

Although the above examples introduce shock hydrodynamics as a subject worthy of study they should not be viewed as mainstream applications. In reality, the majority of research into shock wave phenomena is conducted by the aerodynamics community, and it is to this community that this thesis is principally directed. However, it is beyond the scope of this thesis to give a discussion that would do full justice to the numerous motivations behind this aerodynamic related research, hence our decision to present just a few of the more novel motivations for studying shock hydrodynamics.

1.2 Background Material

We now present some background material for the two principal strands that run through our study, namely high resolution schemes and adaptive mesh schemes. This material is fairly broad, nevertheless it is sufficient to place the AMR algorithm in context. In subsequent chapters more specific background information will be given as and when required.

1.2.1 High Resolution Schemes

The simulation of shock hydrodynamic flows is not straightforward. Indeed, early simulations which integrated the Euler equations using standard finite-difference techniques were not successful, spurious oscillations were invariably observed in the vicinity of shock waves. The main cause of numerical difficulty lies in the fact that the Euler equations can admit discontinuous solutions. The failure of these early simulations led many workers to believe that numerical schemes which attempted to capture shock waves by applying the same basic numerical process throughout the flow domain were ill-founded. They argued that one could not expect to find a legitimate solution by solving differential equations where the true solution is not differentiable. Subsequently, this notion that *shock-capturing* schemes are ill-founded has been refuted by the realization that the equations of fluid motion may be cast in integral rather than differential form. Nevertheless the basic numerical difficulties remain.

Stated simply, the numerical challenge is to devise schemes which can capture discontinuous flow features without introducing non-physical oscillations into the numerical solution. But, using a discrete computational grid it is not possible to represent a perfect discontinuity unless it lies along the boundary between two mesh cells. Therefore, implicit in this challenge is that the discontinuity be smeared over no more than 1–2 mesh cells. Roe's method belongs to a class of *shock-capturing* schemes which can meet this challenge for one-dimensional shocks. These schemes are often given the label *high resolution* to distinguish them from older schemes which smear discontinuities over a significantly larger number of cells. Here we outline the basic principles behind *shock-capturing* schemes.

Because of the actions of viscosity and heat conduction real fluids do not admit genuine discontinuities. Instead, there are thin regions of very steep gradient. For example, a shock wave at sea level is roughly $10^{-7}m$ in thickness. Consequently, no practical computational mesh would be sufficiently fine to resolve the internal structure of a shock wave[1]. Therefore in the vicinity of shock waves a numerical scheme that integrates the Navier-Stokes equations faces the same difficulties as one that integrates the Euler equations.

[1] Excepting very low density flows where shock waves can be quite thick; the earth is partly enveloped by a bow shock whose thickness is measured in thousands of miles!

However, as early as 1950, von Neumann & Richtmeyer[70] argued that it should be possible to artificially increase the physical coefficients of viscosity and heat conduction so as to smear shocks to a width resolvable by practical grids without significantly changing the overall flow. Thus the numerical solution would become smooth and effective difference schemes could then be designed in a systematic fashion. During the intervening years much research has been done to find ways of representing shock waves which are more accurate, more convenient and more elegant.

Amongst the oldest, and yet still the most popular *shock-capturing* schemes are the so-called *artificial-viscosity* methods which are typified by the work of Jameson *et al.*[33] and Rizzi & Eriksson[51]. These methods have evolved directly from von Neumann & Richtmeyer's original proposal for modelling shocks. In essence, to a high-order scheme which produces oscillations near discontinuities some extra terms are added. In smooth regions of the flow these additional terms are negligible, and so they do not compromise the accuracy of the high-order scheme, but near discontinuities they introduce substantial dissipation. This dissipation smears the numerical representation of the discontinuity such that it becomes a monotone profile. In general these dissipative terms are not very sophisticated, for example, the dissipation suggested by Jameson *et al.*[33] is based upon second differencing the pressure field. Consequently *artificial-viscosity* methods often have to be tuned to a particular application, hence the plethora of dissipation models presented in the literature. Furthermore, the amount of dissipation is usually overdone, so shocks can be smeared over several mesh cells and contact discontinuities may not be resolved at all. Nevertheless, if the dissipation terms are chosen carefully then acceptable results can be obtained. Indeed, *artificial-viscosity* methods are very popular for the computation of steady transonic flows in both two and three dimensions. This popularity stems from the relative ease with which *artificial-viscosity* methods may be implemented and their small operation count per mesh cell. However, *artificial-viscosity* methods have not proved to be satisfactory for simulations that involve rapid transient motion.

In an attempt to overcome the problems associated with *artificial-viscosity* schemes, *linear-hybridized* schemes have been developed. Such schemes employ two difference methods. A high-order method which is accurate in smooth regions of the flow, but badly behaved near discontinuities, is blended with a low-order scheme which can yield monotone profiles for discontinuities because it has a large dissipative truncation error. In essence, the low-order method is used to distinguish between those features of the high-order solution which are believable and those which are not. For example, Boris & Book[11] compute both low-order and high-order numerical fluxes. These two fluxes are then linearly combined so as to provide a single flux with which to update the flow solution. In smooth regions of the flow this single flux is taken exclusively from the high-order flux. Near discontinuities the high-order flux is effectively corrected by the low-order flux such that the updated flow solution remains monotone[2]. The broad aim of *linear-hybridized* schemes is to limit the width of a numerical shock to the width given by the sharply rising part

[2]Hence the alternative name for *linear-hybridized* schemes, Flux-Corrected Transport (FCT) schemes.

of the high-order solution without suffering the attendant post-shock oscillations. Under favourable conditions this can be achieved, and so *linear-hybridized* schemes generally perform much better than *artificial-viscosity* schemes. However, a number of problems remain. The resolution of a *linear-hybridized* scheme is effectively limited by that of the low-order scheme. Consequently, subtle interactions between discontinuities may not be detected unless a very fine computational mesh is used. Moreover, the process by which the low-order and high-order fluxes are combined owes little to the non-linear processes associated with real flow discontinuities. Therefore, although monotone, the numerical solution can often exhibit unphysical behaviour. Multi-dimensional schemes built up from one-dimensional operators can often exhibit *staircasing*[72]; shocks which run oblique to the mesh become over-steepened in the direction of the one-dimensional operators.

The main tenet upon which both *artificial-viscosity* and *linear-hybridized* methods are built is the assumption that the flow solution is smooth, for such schemes are based on Taylor series expansions. But at a discontinuity this tenet is inappropriate, hence the need to force a well-behaved solution by introducing either large amounts of dissipation or unphysical monotonicity constraints. A radically different approach was first suggested by Godunov[29]. He assumed that the flow solution is represented by a series of piecewise constant states. Thus the numerical representation closely approximates the true solution near discontinuities. The numerical flow solution is then evolved by considering the non-linear interactions between these piecewise constant states. Viewed in isolation, each pair of neighbouring states constitute a Riemann problem. The non-linear interactions arising from a Riemann problem may be computed exactly. The results from these separate Riemann problems are then averaged so as to find the updated flow solution. Because it mimics much of the relevant physics, Godunov's scheme results in an accurate and well-behaved treatment of shock waves. But the scheme is only first-order accurate and in practice it proves to be very diffusive. However, many high-order *Riemann-based* schemes have been developed which achieve excellent resolution without compromising the robustness of Godunov's scheme. In essence, such schemes either assume a higher-order interpolatory function for the solution within each mesh cell[16, 66], or they post-process the Riemann solutions before averaging to find the updated flow solution[54, 64].

Woodward & Collela[72] have made a comprehensive assessment of these different methodologies, and they suggest that the order of merit when judged in terms of robustness and quality of the numerical solution is, 1^{st} — *Riemann-based* schemes, 2^{nd} — *linear-hybridized* schemes, and 3^{rd} — *artificial-viscosity* schemes. The superiority of *Riemann-based* schemes being most marked for transient flows that contain interactions between strong shock waves. However, if the three methodologies are judged in terms of the computational expense required to update a single mesh cell then this order of merit must be reversed. Note, the Riemann problem does not have a closed-form solution, each separate case must be solved iteratively. Hence, *Riemann-based* schemes are more expensive than schemes based on standard finite-difference operators. So, it does not necessarily follow that *Riemann-based* schemes are the most suitable numerical method for

all types of flow problem. Indeed, many workers would vehemently argue against the use of *Riemann-based* schemes for computing solutions to steady flow problems which contain only weak shock waves. Our interest lies in simulating transient flows that contain subtle shock-on-shock interactions. Therefore we have thought it necessary to adopt a *Riemann-based* strategy. In particular, we have chosen to use Roe's method because of its balance between computational expense, robustness and numerical resolution. Note, Roe's method avoids much of the computational expense associated with Godunov's method by computing approximate instead of exact solutions to Riemann problems.

In this section it has not been our aim to present a comprehensive review of numerical methods applied to shock hydrodynamics. Indeed, given the many comprehensive reviews that have appeared in the literature[40, 54, 72] such an act would be foolish. Instead, we have tried to introduce some of the ideas behind a large class of numerical methods of which Roe's scheme is one of the more successful. However, we do not wish to leave the impression that all schemes conform to a *shock-capturing* methodology. In reality, there are many schemes[1, 39] which are the antithesis of *shock-capturing*. These schemes track flow discontinuities, and in the vicinity of a discontinuity the solution procedure switches from solving the equations of fluid motion to solving the appropriate jump relationships. Thus shock waves and slip lines are fitted rather than captured. Nevertheless, *shock-fitting* and *shock-capturing* are similar in the sense that they both employ a continuum flow model. But again it would be wrong to suggest that all schemes employ such a model. For example, significant progress has been made in the direct molecular modelling of shock wave phenomena using monte-carlo techniques[61]. In conclusion we note, computational fluid dynamics has not yet reached the stage where it is possible to state categorically that one methodology is superior to all others.

1.2.2 Adaptive Mesh Schemes

In general, high-order methods require far fewer mesh cells to achieve a given accuracy of solution than do low-order schemes. So, despite their larger operation count, high-order methods often prove to be computationally more efficient than low-order schemes. But they can only yield accurate results for length scales that are larger than a few mesh cells; interpolation, no matter how high the order, cannot reproduce unresolved variations within a single mesh cell. Consequently, for many types of problem, a limit exists beyond which it is no longer advantageous to improve the order of accuracy of the numerical solution procedure, and further improvements can only be gained by improving the resolution of the computational grid. For shock hydrodynamic flows this limit is quite low, the majority of successful *shock-capturing* schemes are only second-order accurate. Note, near discontinuities the notional truncation error will be secondary to errors caused by numerical diffusion and dispersion. Therefore, the order of accuracy for a *shock-capturing* scheme is not necessarily a good indication of either its absolute accuracy or its resolving power. Nevertheless, it is an incontrovertible fact that the accuracy of many

simulations is limited by the resolution of the computational grid; the computational grid must be able to resolve all the physical length scales pertinent to the flow. Unfortunately, for many problems the pertinent length scales are so disparate that it is not practical to employ a uniform mesh whose spacing has been chosen so as to resolve the smallest length scale. The following example illustrates the absurdity of attempting to use a uniform mesh for problems which contain disparate physical length scales.

The usefulness of explosive materials stems from their ability to perform the rapid conversion of one form of energy into another form; a good solid explosive converts energy at a rate of the order 10^{10} watts per square centimetre of its detonation front[25]! Note, the leading part of a detonation front is a strong shock wave. When such a shock wave propagates through an explosive material, the material is compressed and so heats up. This raise in temperature triggers a chemical reaction which releases large amounts of energy in the form of heat. This energy release provides motive force for the shock, and a balance is reached such that the chemical reaction supports a nominally steady speed of shock propagation. So, the most easily measured characteristic of a detonation front is its speed of propagation. Traditionally, detonation speeds are determined from experiment. A cylindrical charge of explosive (known as a rate-stick) is ignited at one end. And the propagation speed is measured at the other end; it being assumed that the length of the stick is sufficient to allow the detonation front to reach its nominally steady speed. To perform a numerical simulation of such an experiment represents a formidable challenge. Since the chemical reaction drives the shock wave, the simulation must be able to resolve the chemical reaction zone. Results for model problems suggest that at least 10 mesh cells are required to resolve the reaction zone[17]. For certain types of explosive the reaction zone may be only 0.02 mm in thickness. So, within the reaction zone the mesh spacing should not be larger than 0.002 mm. Now, a rate-stick is typically 100 mm in length and 100 mm in diameter. Therefore, assuming the problem to be axisymmetric, the computational domain would be 50 mm by 100 mm. If this domain were discretized using a uniform mesh whose spacing were set to 0.002 mm, so as to be able to resolve the reaction zone no matter where it was positioned in the domain, there would be some 1.25×10^9 mesh cells. From the point of view of both numerical accuracy and stability, it is unlikely that the detonation front could be propagated by more than one mesh cell per time step. Consequently, it would take at least 5×10^4 time steps for the detonation to travel the full length of the rate-stick. Therefore, the total workload for the simulation would be of the order of 6.25×10^{13} cell updates. Such a calculation would be absurd. Note, even after taking the overly optimistic view that 10^6 mesh updates could be performed per second, the calculation would still take 723 days to run! Clearly, to make such a simulation viable, something other than a larger computer is required.

Adaptive mesh schemes attempt to match the local resolution of the computational grid to the requirements of the local flow solution. Thus very fine mesh cells are restricted to those regions where they are needed, and elsewhere the computational grid may be quite coarse. Such a strategy can dramatically reduce the computational effort required

to perform simulations of problems that contain disparate physical length scales. For example, returning to our detonation simulation. If the fine mesh cells were restricted to the vicinity of the reaction zone, only about 2.5×10^5 cells would be required to resolve the detonation front[3]. For an ideal adaptive mesh scheme, the computational effort required to perform the simulation would be the same as the effort required to integrate the cells in the vicinity of the detonation front; about 1.25×10^{10} cell updates. Therefore, whereas the uniform mesh simulation would take 723 days to run, the ideal adaptive mesh simulation would take just 208 minutes! Admittedly, a real scheme must fall short of this ideal scheme. But not necessarily by much; we estimate that the AMR algorithm might only require 2.53×10^{10} cell updates or 422 minutes to run.

Because the potential savings are so large, the adoption of almost any form of mesh adaption, no matter how naïve, will pay some dividend. Consequently, a wide variety of strategies have been utilized. Given this variety, it would serve no purpose to present an all encompassing review of adaptive mesh schemes. Indeed, many strategies have been developed with the sole aim of dealing with a specific flow; they serve their special purpose well, but lack generality. And other methods appear unsuited to shock hydrodynamic flows. Instead, it is more appropriate that we simply allude to some of the reasoning that has shaped the development of the AMR algorithm. To this end, we briefly touch on two adaptive mesh strategies that are frequently applied to compressible flow problems, highlighting their respective advantages and disadvantages.

Penalty methods (otherwise known as fixed-cost methods) employ a topologically fixed grid, the grid nodes being allowed to gravitate towards areas deemed to be of special importance. In this manner mesh cells are bunched so as to improve the resolution of the calculation. The limitation of penalty methods is all too obvious, hence their generic name. Since there are a fixed number of grid cells, increasing the resolution in one part of the computational domain must pay the penalty of decreasing the resolution in another part. Nevertheless, as Brakbill & Saltzmann[13] have demonstrated, penalty methods prove quite effective when only a limited number of features need resolving. Moreover, because they employ a fixed grid structure, such schemes are fairly simple to implement. But, there a number of insidious problems associated with penalty methods that can add considerably to the complexity of a practical code. First, most *shock-capturing* schemes only maintain their accuracy on smoothly varying, well-behaved grids — a penalty adapted grid invariably exhibits extreme distortion. Second, penalty methods are not inherently suited to energetic flow problems, the grid may become entangled leading to badly formed mesh cells. With care both these problems may be overcome, nevertheless penalty methods have not been shown to produce very high resolution solutions to problems that contain complex shock-on-shock interactions.

[3]The detonation front remains roughly planar throughout a rate-stick experiment. However, evidence suggests that the physical processes are not strictly one-dimensional[25], hence our consideration of an axisymmetric calculation.

To overcome the limitations associated with penalty methods, many workers have turned to schemes which employ unstructured grids; examples being, Löhner *et al.*[36] and Holmes & Lamson[31]. An unstructured grid is one that is formed from an arbitrary collection of grid nodes *via* some triangulation process, principally Delaunay triangulation[12]. Such grids are very flexible. Nodes may be added and deleted at will, thus allowing total control over the local resolution of the computational grid. This flexibility makes unstructured grids eminently suitable for discretizing domains which have irregularly shaped boundaries. However, there are several drawbacks associated with unstructured grids which become particularly troublesome for transient flow simulations: the process of re-gridding using Delaunay triangulation is computationally expensive; the addition (or the deletion) of a single node can change the triangulation dramatically, thus the numerical representation of a flow discontinuity is invariably perturbed between grid adaptions; the storage overheads associated with unstructured grids are considerably larger than those for structured grids.

An adaptive mesh strategy has to strike a balance between that which is desirable and that which is practicable. Therefore, given that individual circumstances dictate those desirable properties which can be compromised and those which cannot, no one adaptive mesh strategy will win out under all circumstances. Indeed, despite their undoubted popularity, neither penalty methods nor unstructured grid methods were adjudged to be suitable for our purposes. Nevertheless, such schemes exhibit both, many of the qualities which we wished to instil into our algorithm, and many of the disadvantages which we wished to avoid; hence the preceding descriptions. The following is a list of the desirable properties which we were not prepared to compromise.

- The scheme must be general purpose — *we want to apply it to a variety of problems.*
- The scheme must be well suited to transient simulations — *our main interest lies with simulating unsteady flow problems.*
- No unreasonable constraint should be placed on the method used to integrate the flow solution — *At some stage in the future, we hope to employ both a number of different shock-capturing schemes and a number of different flow models.*
- A mesh enrichment strategy must be used — *we do not want to compromise the resolution of the simulation.*
- The adaption process must be cheap — *our computer resources are meagre.*

After taking all things into consideration, we have opted for a strategy based on ideas proposed by Berger[7]. Apart from Berger's co-workers, this type of strategy has found little favour — an exception being Arney[3] — for it is perceived as being overly complicated. But this is unfortunate, because the basic approach is excellent. Note, unlike most adaptive mesh schemes, Berger's scheme has been shown to produce very high resolution

results[8, 4]. Although the foundations of our scheme lie with Berger's work, hence our decision to borrow the acronym AMR, we believe our approach is sufficiently fresh to open up the possibilities of this type of mesh adaption to a much larger audience. In particular, our scheme removes much of the complexity of Berger's scheme, without compromising the quality of the resultant simulations.

1.3 The AMR Algorithm — An Overview

The AMR algorithm is a general purpose numerical scheme for simulating shock hydrodynamic flows. The algorithm is particularly suited to unsteady flows, it requires only modest computing power to resolve complex shock-on-shock interactions. This computational efficiency stems form the algorithm's grid system; a hierarchical set of grids is used to discretize the flow domain. At the bottom of the hierarchy there is a coarse grid that delineates the computational domain. Additional grid tiers may be added so as to locally refine the computational domain. With each extra grid tier the effective mesh spacing becomes progressively finer thereby increasing the resolution of the simulation. The algorithm can automatically restrict these extra grid tiers to those regions where the flow activity is of particular interest, and so the computational grid can be made to adapt to the evolving flow solution. Thus it is possible to achieve the numerical resolution associated with very fine meshes without incurring the computational penalties of having to employ a fine mesh throughout the flow domain.

The AMR algorithm refines in time as well as space. More, but smaller time steps are taken on fine grids than on coarse grids. The hierarchical nature of the grid system allows the different sized time steps to be interleaved such that the simulation remains time accurate. Therefore, in contrast to other mesh refinement schemes, the presence of a few extremely fine mesh cells in one part of the flow domain does not have an adverse affect on the rate at which the rest of the flow solution may be advanced. Note, this temporal refinement strategy should not be confused with the local time stepping strategy that is often used to improve the rate of convergence for steady state calculations. Here we are primarily concerned with computing time accurate solutions to unsteady flow problems. It is this combination of spatial and temporal refinement strategies that accounts for the efficiency of the AMR algorithm.

The AMR algorithm places no special constraints on the basic numerical method used to integrate the discretized flow solution. Each grid tier is formed from a collection of mesh patches, each patch being a quasi-rectangular set of cells which contain *cell-centred* projections of the flow solution. The algorithm contains machinery which allows a mesh patch to be integrated independently of any other mesh patch; the process which integrates a mesh patch never sees a mesh boundary. In principle, any *cell-centred* scheme developed for a single quasi-rectangular mesh may form the basis of the flow integration process. To date we have incorporated three different schemes to integrate the Euler equations, and

one scheme for detonation flows. However, for the majority of our calculations we have chosen to use Roe's flux-difference splitting scheme. Roe's scheme was chosen because of its inherent robustness and for the balance it strikes between resolution and computational expense.

1.3.1 Thesis Outline

The contents of this thesis reflects three broad aims. First, to explain the methodology and intricacies of the AMR algorithm. Second, to vindicate the algorithm's relative complexity. Third, to give sufficient details of our implementation so as to allay the apprehensions of any person who might wish to implement the algorithm.

Compared to non-adaptive schemes the AMR algorithm is undeniably complicated; it contains many elements which require careful coordination. However, by virtue of its modular design the algorithm is extremely robust. Indeed, through the judicious choice of properties for each module it has been possible to bestow certain properties on the algorithm as a whole, one of which is robustness. Since many of these modules are inextricably linked it is not possible to develop a purely sequential description for the AMR algorithm. Our description is split loosely by function and is spread over three chapters. Chapter 2 describes the hierarchical grid system and its associated data storage mechanism, chapter 3 describes the process that integrates the flow solution contained by a static set of grids, and chapter 4 describes the process that dynamically adapts the grid to the numerical flow solution. It is inevitable that certain features of the algorithm will not be clear until all three of these chapters have been digested. So for a first reading of this thesis we recommend that chapters 2–4 be read in quick succession without bothering too much about the detail. This cursory view will make it easier to digest the intricacies of the AMR algorithm during a second more careful reading before progressing on to chapter 5 which outlines Roe's method.

The effort required to implement the AMR algorithm is not inconsiderable. Therefore a sizeable proportion of this thesis is given over to demonstrating that the resultant computational benefits make this effort worthwhile. This process culminates with the simulations that are presented in chapter 6. These simulations yield results which are comparable in resolution to Schlieren photographs yet they were performed on a small desktop workstation. Despite this incentive, certain parts of the audience to which this thesis is directed may find the programming effort off-putting. So, we present several fragments of *pseudo-code* which accurately describe the intricacies of the AMR algorithm. The structure of our implementation follows directly from these code fragments. Therefore they should prove invaluable to any person who wishes to implement the ideas described in this thesis. But, it should be noted that these fragments cannot be cribbed verbatim so as to build a working code for they lack many mundane details such as variable typing and scoping declarations.

Finally, in chapter 7 we close this thesis by presenting, a summary of our work, a list of the conclusions that we have drawn, and a few suggestions as to how our work could be usefully extended.