Assigned: Tuesday Nov. 6, 2018
Due: Tuesday Dec. 4, 2018

MATH 666: Homework 3
Fall 2018

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

1. Consider the Crank-Nicolson-cG(1) FEM for the diffusion equation. Find $u_h^n \in V_h^1$ such that

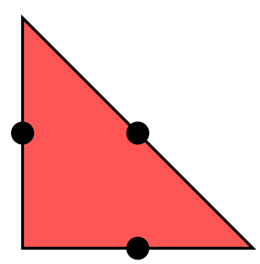
$$\left\langle \frac{u_h^n - u_h^{n-1}}{\Delta t}, \, v \right\rangle + \left\langle \nabla \left(\frac{u_h^n + u_h^{n-1}}{2} \right), \, v \right\rangle = \left\langle \frac{f^n + f^{n-1}}{2}, \, \nabla v \right\rangle \quad \forall \, v \in V_h^1, \quad \forall \, n = 1, 2, 3, \ldots,$$

$$\langle u_h^0, \, v \rangle = \langle u_0, \, v \rangle \, .$$

Following the convergence proof of the Backward-Euler-cG(1) FEM from class, prove that the Crank-Nicolson-cG(1) FEM satisfies:

$$\| u_h^n - u\left(t^n\right) \| = \mathcal{O}\left(\Delta t^2 + h^2\right).$$

2. Consider the non-conforming FEM for the Poisson equation using the linear Crouzeix-Raviart element discussed in class:



(a) Explicitly write out the stiffness matrix on a uniform Cartesian mesh (with triangles created by cutting each Cartesian square into two triangles) on $[0,1] \times [0,1]$.

(b) Use this to solve the Poisson equation on $[0,1] \times [0,1]$ with $u = 0$ on the boundary:

$$-u_{,x,x} - u_{,y,y} = 8\pi^2 \sin(2\pi x) \sin(2\pi y).$$

(c) Show a plot of the solution on a reasonable resolved mesh. NOTE: Before plotting, you may want to evaluate the solution in element centers.

(d) Do a numerical convergence study in the L^2 norm. For simplicity, you can just compute the errors based on errors at element centers. The exact solution is

$$u(x, y) = \sin(2\pi x) \sin(2\pi y).$$