Assigned: Thursday Sept. 6, 2018
Due: Thursday Sept. 27, 2018

MATH 666: Homework 1
Fall 2018

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Two-point boundary value problem

1. Consider the following boundary value problem with periodic boundary conditions:

$$\begin{aligned} \textbf{ODE}: &\quad -u'' + q(x)\,u = f(x), \\ \textbf{BC1}: &\quad u(-\pi) = u(\pi), \\ \textbf{BC2}: &\quad u'(-\pi) = u'(\pi), \end{aligned}$$

where $q > 0\ \forall x \in [-\pi, \pi]$.

(a) Recast this equation into a variational problem, stating the trial and test function spaces.

(b) Interpret the variational problem as an energy minimization problem, clearly stating the energy functional. Prove that the variational problem and the energy minimization problems are equivalent.

(c) Derive a Galerkin orthogonality expression for this problem and deduce from this an appropriate *energy norm*.

(d) Develop a cG(1) method for this problem, clearly stating the appropriate spaces, trial, test, and basis functions.

(e) Show that the cG(1) FEM solution gives the optimal solution in this *energy norm*. Use this to prove an *a priori* error estimate in the *energy norm*.

(f) Prove an *a priori* error estimate in the L_2-norm.

(g) Numerically implement the cG(1) method on a uniform mesh with the following functions

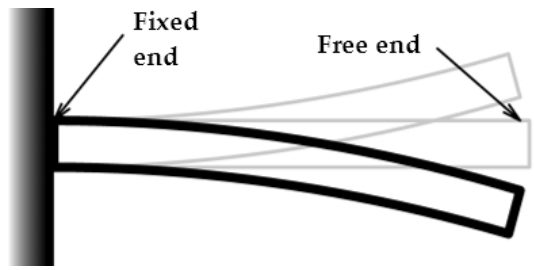
$$q(x) = 3-\sin(x)-\sin(2x)-\cos(x), \quad f(x) = 2e^{\sin(x)}e^{\cos(x)}, \quad u(x) = e^{\sin(x)}e^{\cos(x)}.$$

Numerically determine the convergence rate in both the *energy norm* and the L_2-norm by creating an appropriate convergence table.

Beam equation

2. Consider the Beam equation from mechanics with boundary conditions that model a *cantilever* beam:

$$\begin{aligned} \textbf{PDE}: &\quad u^{(iv)} = f(x) \quad x \in (0,1)\,, \\ \textbf{BC}: &\quad u(0)=0, \quad u'(0)=0, \quad u''(1)=0, \quad u'''(1)=0. \end{aligned}$$



   (a) Recast this problem as a variational problem. Clearly state the test and trial function spaces.
   (b) Interpret the variational problem as an energy minimization problem, clearly stating the energy functional. Prove that the variational problem and the energy minimization problems are equivalent.
   (c) Develop a cG(3) finite element method for this problem.
   (d) Prove an *a priori* error estimate for this method in the *energy norm*:

$$\|e\|_E = \left\{ \int_0^1 \left(e''\right)^2 \, dx \right\}^{1/2}.$$

   (e) Prove an *a priori* error estimate for this method in the L_2 norm:

$$\|e\|_{L_2} = \left\{ \int_0^1 (e)^2 \, dx \right\}^{1/2}.$$

   (f) Numerically implement the cG(3) method on a uniform mesh with the following functions

$$\begin{aligned} f(x) &= p_1(x)\, \cos(2\pi x) + p_2(x)\, \sin(2\pi x), \\ p_1(x) &= -64\pi(18 - 6\pi^4 x^2 + 18x^3\pi^2 + 2\pi^4 x - 36\pi^2 x^2 - 27x + 12\pi^2 x + 4\pi^4 x^3 + 3\pi^2), \\ p_2(x) &= 216 - 816\pi^2 - 96\pi^4 + 32\pi^6 x^2 - 384\pi^4 x^3 - 64\pi^6 x^3 - 288\pi^4 x^2 \\ &\quad + 3456\pi^2 x + 576\pi^4 x - 2592\pi^2 x^2 + 144\pi^4 x^4 + 32\pi^6 x^4, \\ u(x) &= \left((18+2\pi^2)x^2 + (-24-4\pi^2)x^3 + (9+2\pi^2)x^4\right) \sin(2\pi x). \end{aligned}$$

   Numerically determine the convergence rate in both the *energy norm* and the L_2-norm by creating an appropriate convergence table.