Assigned: Thursday Sept. 27, 2018
Due: Tuesday Oct. 23, 2018

MATH 666: Homework 2
Fall 2018

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

2D Cartesian FEM:

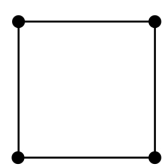
1. Consider the 2D Poisson equation:

$$\begin{aligned} \textbf{PDE}: &\quad -\nabla\cdot\nabla u = f(x,y) \quad \text{in} \quad \Omega = [-1,1]\times[-1,1], \\ \textbf{BC}: &\quad u + \nabla u\cdot\hat{\mathbf{n}} = g \quad \text{on} \quad \partial\Omega. \end{aligned}$$

(a) Recast this problem as a variational problem. Clearly state the test and trial function spaces.

(b) Show that the variational problem has a unique solution by showing that it meets all of the criteria of the Lax-Milgram theorem.

(c) Develop a finite element method for this problem based on square (not triangular) elements of the form:



On each element, the solution on the four corners is interpolated with the following interpolant:

$$s(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy.$$

Write out the basis functions and explicitly write out the linear system that arises from your FEM (you can just assume that the grid spacing in x and y are the same and uniform).

(d) Use this method to solve the 2D Poisson equation with

$$f(x,y) = \pi^2 e^{\sin(\pi x)\sin(\pi y)}\left\{2\sin(\pi x)\sin(\pi y) + 2\cos^2(\pi x)\cos^2(\pi y) - \cos^2(\pi x) - \cos^2(\pi y)\right\},$$

$$g = \begin{cases} 1+\pi\sin(\pi y) & \text{if} \quad x=-1 \quad \text{where} \quad \hat{\mathbf{n}} = (-1,0), \\ 1-\pi\sin(\pi y) & \text{if} \quad x=1 \quad \text{where} \quad \hat{\mathbf{n}} = (1,0), \\ 1+\pi\sin(\pi x) & \text{if} \quad y=-1 \quad \text{where} \quad \hat{\mathbf{n}} = (0,-1), \\ 1-\pi\sin(\pi x) & \text{if} \quad y=1 \quad \text{where} \quad \hat{\mathbf{n}} = (0,1). \end{cases}$$

Create a convergence table showing both $\|\cdot\|_V$ and $\|\cdot\|_{L^2(\Omega)}$ errors. What order of convergence do you observe? In this the case the exact solution is

$$u(x, y) = e^{\sin(\pi x)\sin(\pi y)}.$$

HINT: use the `pcg` command in MATLAB to invert the linear system.

2D Unstructured FEM:

2. Dowload the 2D mesh generation code from the course website:

   `http://www.public.iastate.edu/~rossmani/math666/Meshgen.tar.gz`

   Make sure that you can get the provided example to work in MATLAB.

   Next, use the MATLAB code to generate a mesh for the following geometry: a star-shaped region with a circular hole. The boundaries of this region are zero contours of the following functions:

$$f_{\text{outer}}(r, \theta) = r - 0.75 - 0.25\sin(5\theta),$$
$$f_{\text{inner}}(r, \theta) = (0.25)^2 - x^2 - y^2.$$

   NOTE: in order to get a good mesh, you may have to do apply one or both of the following tricks:

   (a) You may need to fix a few points on the boundary (especially points near "corners"). You can do this by adding values to the `fix` vector.

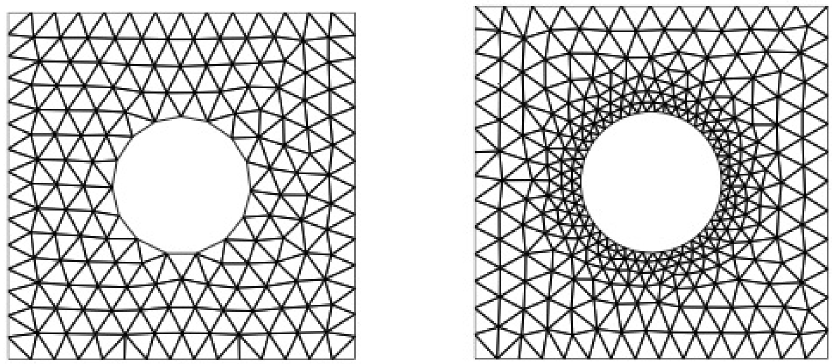
   (b) You may need to use a slightly non-uniform grid. At the momment the code uses `huniform`, but you can replace it with a function `fh.m` that takes as input the vector `[x; y]` and returns a number. The grid spacing will be reduced in regions where `fh.m` is small and increased where `fh.m` is large. Below is an example of a square with a circular hole.

   Example #1 (uniform grid spacing):

```
>> fd=inline(diff(drectangle(p,-1,1,-1,1),dcircle(p,0,0,0.4)));
>> pfix=[-1,-1;-1,1;1,-1;1,1];
>> [p,t]=distmesh2d(fd,@huniform,0.05,[-1,-1;1,1],pfix);
```

   Example #2 (non-uniform grid spacing):

```
>> fd=inline(diff(drectangle(p,-1,1,-1,1),dcircle(p,0,0,0.4)));
>> pfix=[-1,-1;-1,1;1,-1;1,1];
>> fh=inline(min(4*sqrt(sum(p.^2,2))-1,2));
>> [p,t]=distmesh2d(fd,fh,0.05,[-1,-1;1,1],pfix);
```



3. Implement the cG(1) method discussed in class.

4. Consider the following Poisson equation on the domain created in Problem #2:

$$\begin{aligned} \textbf{PDE}: &\quad -u_{,x,x} - u_{,y,y} = 1 \quad \text{in} \quad \Omega, \\ \textbf{BCs}: &\quad u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

   Construct a sufficiently resolved grid and solve the cG(1) finite element variational problem.