Robotics, Teaching & Learning

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Catmull-Rom splines

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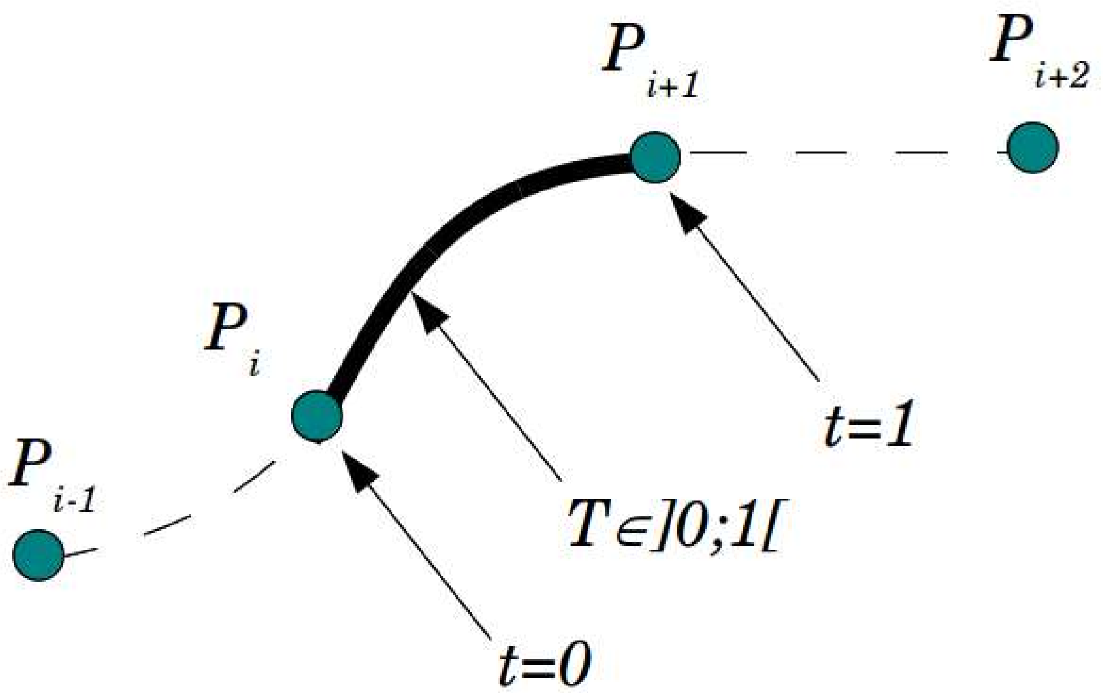
The Catmull-Rom splines is a method that approximate a set of points (named control points) with a smooth polynomial function that is piecewise-defined. One of the properties of the Catmull-Rom spline is that the curve will pass through all of the control points.

Equations

Two points on each side of the desired portion are required. In other words, points P_{i-1} and P_{i+2} are needed to calculate the spline between points P_i and P_{i+1}.



Given points P_{i-1}, P_i, P_{i+1} and P_{i+2}, the coordinates of a point P located between P_i and P_{i+1} are calculated as:

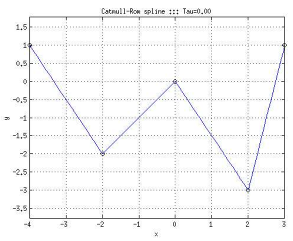
$$P = \frac{1}{2} . \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} . \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix} . \begin{bmatrix} P_{i-1} & P_i & P_{i+1} & P_{i+2} \end{bmatrix}^\top$$

General case : tension

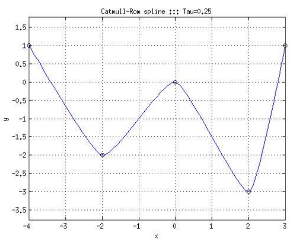
The previous equation is a particular case of the general geometry matrix given by the following equation:

$$P = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\tau & 0 & \tau & 0 \\ 2.\tau & \tau - 3 & 3 - 2\tau & -\tau \\ -\tau & 2 - \tau & \tau - 2 & \tau \end{bmatrix} . \begin{bmatrix} P_{i-1} & P_i & P_{i+1} & P_{i+2} \end{bmatrix}^\top$$

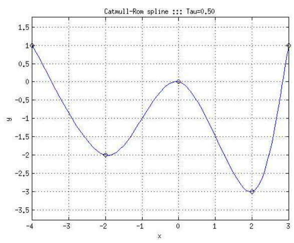
The parameter τ modify the tension of the curve. The following figure illustrates the influence of the parameter τ on the curve. Note that $\tau = \frac{1}{2}$ is commonly used (as in the particular case presented previously).



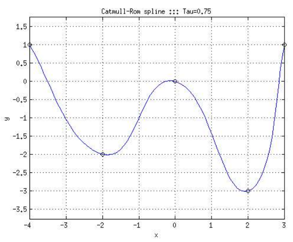
$\tau = 0$



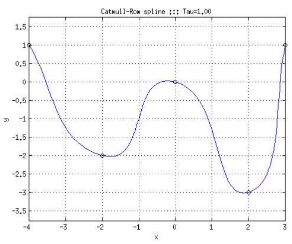
$\tau = 0.25$



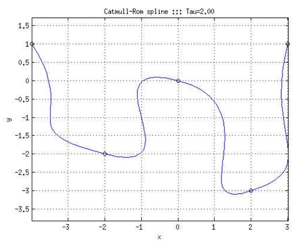
$\tau = 0.5$



$\tau = 0.75$



$\tau = 1$

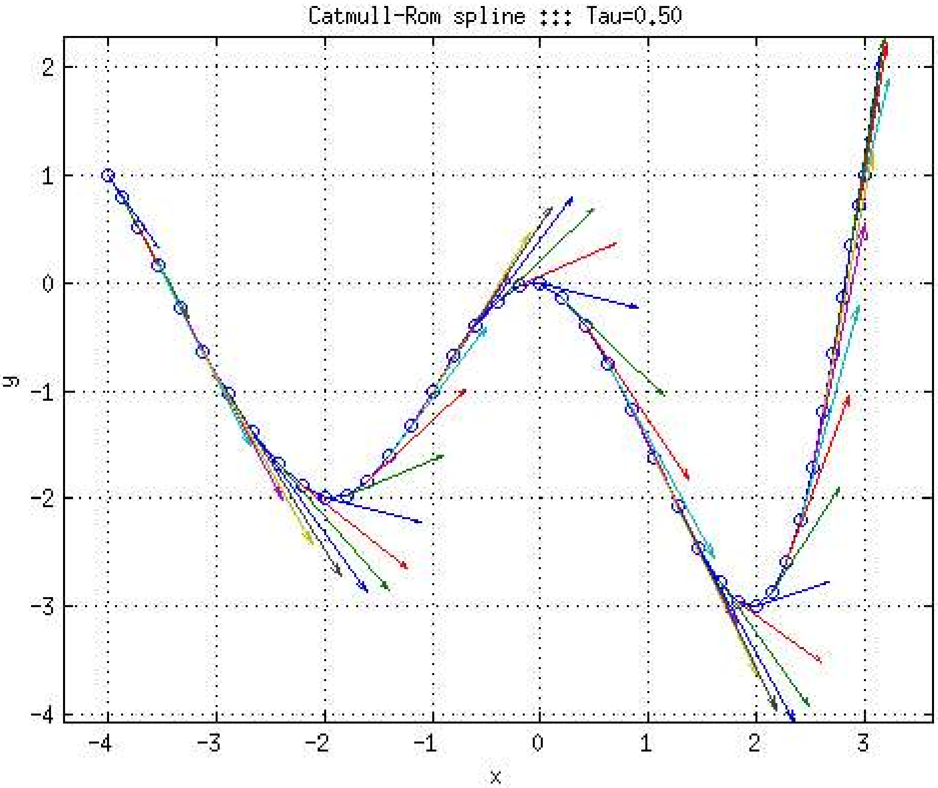


$\tau = 2$

Derivative

As the spline is C^1 continuous it is possible to compute the derivative for any value of t. Moreover, as the definition of the spline is a polynomial, it is quite trivial to compute the derivative at a given point:

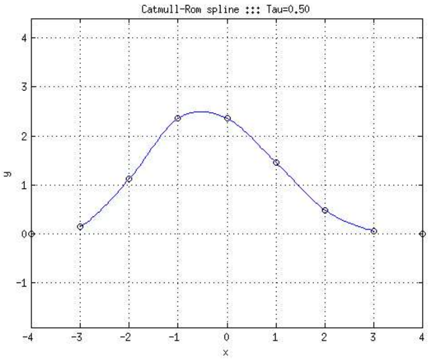
$$P = \begin{bmatrix} 0 & 1 & 2.t & 3.t^2 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\tau & 0 & \tau & 0 \\ 2.\tau & \tau - 3 & 3 - 2\tau & -\tau \\ -\tau & 2 - \tau & \tau - 2 & \tau \end{bmatrix} . \begin{bmatrix} P_{i-1} & P_i & P_{i+1} & P_{i+2} \end{bmatrix}^\top$$

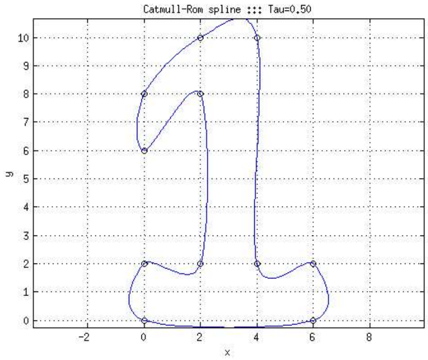


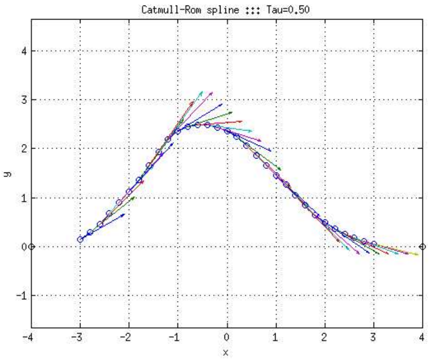
Properties

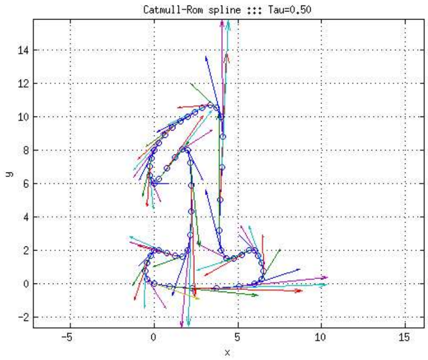
The spline passes through all of the control points.
The spline is C^1 continuous.
The spline is not C^2 continuous.
The spline does not lie within the convex hull of their control points

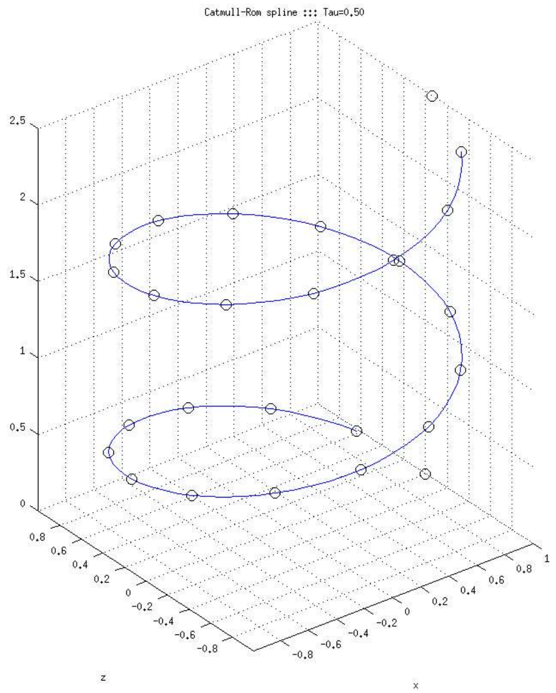
Examples


Catmull-Rom spline ::: Tau=0.50
x
y


Catmull-Rom spline ::: Tau=0.50
x
y


Catmull-Rom spline ::: Tau=0.50
x
y


Catmull-Rom spline ::: Tau=0.50
x
y


Catmull-Rom spline ::: Tau=0.50
z
x