

## 4 Path Equivariance

Classical group-equivariant neural networks (G-NNs) [9] enforce point-wise constraints  $F(g \cdot x) = \rho(g) \cdot F(x)$ , respecting symmetries in data. However, this formulation treats data manifolds as collections of isolated points rather than connected geometric objects, and cannot express how networks should behave as data varies continuously.

We introduce Path-Equivariant Networks (PENs), which replace point-wise with path-wise constraints: as we traverse continuous paths through the data manifold, network outputs must transform coherently via transport maps.

We formalize manifold paths (Section 4.1), define path-equivariance via transport maps (Section 4.2), analyze the pose/content decomposition (Section 4.3) and provide classical equivariance recovery (Section 4.4). This path-centric perspective reimagines equivariant learning: networks that respect not just how points transform, but how the manifold itself is organized.

In this section, we use the term "network" interchangeably with "function".

### 4.1 Manifold Path

In this section, we introduce the concept of "manifold paths" induced by Lie group (Definition 2.35) actions on manifolds (Definition 2.29). Since manifolds are inherently topological spaces, manifold paths inherit connectivity properties, enabling us to study how data points are connected through continuous deformations. Also, since these paths originate from group elements, they inherit group properties, particularly composition and the ability to factor transformations. This concept forms the theoretical foundation for our path equivariant networks.

**Definition 4.1** (*G*-space and *G*-manifold). *Let  $G$  be a Lie group. A  $G$ -space, denoted as  $(X, \rho_X)$ , is a topology space  $X$  together with a continuous action*

$$\rho_X : G \times X \rightarrow X$$

*satisfying  $e \cdot x = x$  and  $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ .*

*Similarly, a  $G$ -manifold, denoted as  $(M, \rho_M)$ , is a manifold  $M$  together with a smooth action*

$$\rho_M : G \times M \rightarrow M$$

*satisfying  $e \cdot m = m$  and  $(g_1 g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$ .*

*In general, we denote the action  $\rho : G \times X \rightarrow X$  as  $G \curvearrowright X$ .*

**Definition 4.2** (*G*-equivariant map and *G*-equivariant learner). *Given a  $G$ -space  $(X, \rho_X)$  and a  $G$ -manifold  $(M, \rho_M)$ , a map  $f : X \rightarrow M$  is  $G$ -equivariant map ( $G$ -map) if*

$$f(g \cdot x) = g \cdot f(x) \quad \forall g \in G, x \in X.$$

*Equivalently, this diagram*

$$\begin{array}{ccc}
G \times X & \xrightarrow{\rho_X} & X \\
id_G \times f \downarrow & & \downarrow f \\
G \times M & \xrightarrow{\rho_M} & M
\end{array}$$

commutes.

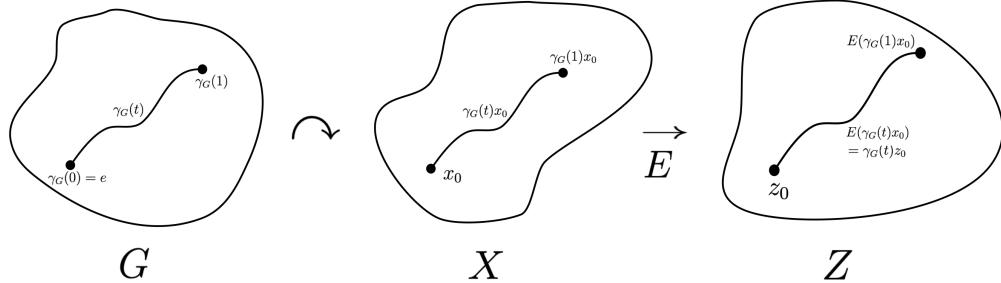
We call a  $G$ -map machine learning algorithm an  $G$ -equivariant learner.

The framework of  $G$  equivariant learners (Definition 4.2) generalizes classical manifold learning via autoencoders. In standard autoencoders, an encoder  $E : X \rightarrow \mathbb{R}^d$  maps data to a flat Euclidean latent space. That leads to a fundamental limitation:  $\mathbb{R}^d$  as latent space assumes the data manifold is topologically trivial (homeomorphic to an open subset of  $\mathbb{R}^d$ ). Further, the latent representation is generally unidentifiable [20]. By contrast, requiring  $E : X \rightarrow Z$  to be  $G$ -equivariant with respect to a  $G$ -manifold  $Z$  (such as  $Z = G$  or a homogeneous space  $Z = G/H$ ) imposes that any valid reparametrization  $h$ , i.e.,  $Z \rightarrow Z$  must satisfy  $h(g \cdot z) = g \cdot h(z)$  for all  $g \in G$ :  $h$  must be a  $G$ -equivariant automorphism of  $Z$ . This constraint dramatically reduces the equivalence class of solutions: when  $Z = G$  with the left-multiplication action, such automorphisms correspond to right translations  $h(z) = z \cdot g_0$  for fixed  $g_0 \in G$ , providing identifiability up to a global group element rather than an arbitrary nonlinear distortion.

**Definition 4.3** (Manifold path). *Let  $G$  be a Lie group,  $X$  be a topological  $G$ -space,  $Z$  be a  $G$ -manifold,  $E : X \rightarrow Z$  is a  $G$ -equivariant learner. Fix  $x_0 \in X$  and set  $z_0 := E(x_0) \in Z$ . For any continuous path  $\gamma_G : [0, 1] \rightarrow G$  with  $\gamma_G(0) = e$ , define*

$$\gamma_Z(t) := \gamma_G(t) \cdot z_0 = E(\gamma_G(t) \cdot x_0) \quad t \in [0, 1].$$

We call  $\gamma_Z$  a manifold path in  $Z$  (see Figure 1).



**Figure 1: Manifold path.** Group  $G$  act on  $X$ ,  $E(\gamma)$  is a path in  $Z$  induced from  $X$ .

The manifold path construction (Definition 4.3) makes this geometric content explicit. Given a path  $\gamma_G : [0, 1] \rightarrow G$  in the structure group, equivariance ensures

that the induced path  $\gamma_Z(t) = E(\gamma_G(t) \cdot z_0)$  in latent space coincides with the group orbit  $\gamma_G(t) \cdot z_0$ . This means the encoder does not merely lower data dimension, but represents the  $G$ -action. The choice of  $Z$  as a  $G$ -manifold thus serves as an inductive bias encoding our geometric assumptions about the data.

**Lemma 4.4.** *Manifold-paths are topological paths (Definition 2.23).*

*Proof.* The map  $\gamma_Z : [0, 1] \rightarrow Z$  is continuous, hence every manifold-path is an ordinary topological path.  $\square$

**Proposition 4.5** (Orbit reachability). *Let  $G^0$  denote the identity component (Definition 2.36) of  $G$ . For any  $z_0 \in Z$ ,*

$$\{\gamma_Z(1)\} = G^0 \cdot z_0.$$

*In particular, the orbit  $G^0 \cdot z_0$  is path-connected in the topological sense.*

*Proof.* Here we prove (1)  $\{\gamma_Z(1)\} = G^0 \cdot z_0$  and (2)  $G^0 \cdot z_0$  is path-connected.

(1) To show  $\{\gamma_Z(1)\} = G^0 \cdot z_0$ , we need to prove both  $\{\gamma_Z(1)\} \subseteq G^0 \cdot z_0$  and  $G^0 \cdot z_0 \subseteq \{\gamma_Z(1)\}$ .

- Prove  $\{\gamma_Z(1)\} \subseteq G^0 \cdot z_0$ .

Let  $\gamma_Z$  be any manifold-path. By Definition 4.3:

$$\gamma_Z(t) := \gamma_G(t) \cdot z_0,$$

where  $\gamma_G : [0, 1] \rightarrow G$  is continuous with  $\gamma_G(0) = e$ .

At the endpoint  $t = 1$ , the path in  $Z$  is mapped to  $G$  via  $z_0$ :

$$\gamma_Z(1) = \gamma_G(1) \cdot z_0.$$

Since  $\gamma_G$  is a continuous path connecting  $e = \gamma_G(0)$  to  $\gamma_G(1)$ , both endpoints lie in the same connected component of  $G$ .

Since  $e \in G^0$  by Definition 4.3, we have:

$$\gamma_G(1) \in G^0.$$

Therefore:

$$\gamma_Z(1) = \gamma_G(1) \cdot z_0 \in G^0 \cdot z_0.$$

This proves  $\{\gamma_Z(1)\} \subseteq G^0 \cdot z_0$ .

- Prove  $G^0 \cdot z_0 \subseteq \{\gamma_Z(1)\}$ .

Let  $\forall g \in G^0$ , we need to show that  $g \cdot z_0$  can be written as manifold path  $\gamma_Z(1)$ .

Since  $g \in G^0$  and  $G^0$  is path-connected, there exists a continuous path:

$$\gamma_G : [0, 1] \rightarrow G^0 \subseteq G \quad (\text{a})$$

such that  $\gamma_G(0) = e$  and  $\gamma_G(1) = g$ .

Interpolate  $t = 1$  to Equation a and Definition 4.3 we have

$$\gamma_Z(1) = \gamma_G(1) \cdot z_0 = g \cdot z_0.$$

We choose  $g \in G^0$  arbitrarily, therefore  $g \cdot z_0 \in \{\gamma_Z(1)\}$ .

This proves  $G^0 \cdot z_0 \subseteq \{\gamma_Z(1)\}$ .

(2) Next, we prove the orbit  $G^0 \cdot z_0$  is path-connected.

The orbit  $G^0 \cdot z_0$  is the continuous image of  $G^0$  under the map:

$$\begin{aligned} \phi : G^0 &\rightarrow Z \\ g &\mapsto g \cdot z_0. \end{aligned}$$

Since the Lie group action  $g$  is continuous in a manifold sense,  $\phi$  is continuous, also  $G^0$  is path-connected, by "continuous image of path-connected space is path-connected", the image  $\phi(G^0) = G^0 \cdot z_0$  is path-connected.

□

This shows us that the set of points we can reach by manifold-paths is exactly the orbit under the identity component, also the orbit is path-connected.

## 4.2 Path Equivalent (PE) Maps

In this section, we formalize the conceptions of path-equivariant (PE) maps and path-equivariant networks (PENs). The core idea is that PE maps preserve path structure: as we traverse a path in the input data manifold, the network output transforms coherently according to a transport map in the target symmetry group. We will show that this framework is a generalized idea of classical group-equivariant networks [9]: when paths lie within single orbits (we call them pose paths) and satisfy a endpoint condition (Definition 4.14), PE reduces to standard group equivariance  $F(g \cdot x) = \rho(g) \cdot F(x)$ .

### 4.2.1 Path Equivariant

**Definition 4.6** (Reparametrization and Concatenation of Paths). *If  $\gamma$  is a path and  $\phi : [0, 1] \rightarrow [0, 1]$  is continuous with  $\phi(0) = 0$  and  $\phi(1) = 1$ , then the reparametrized path is  $\gamma \circ \phi$ .*

*If  $\gamma_1, \gamma_2$  are two paths with  $\gamma_1(1) = \gamma_2(0)$  (end of first meets start of second), then the concatenation<sup>3</sup> of them is:*

$$\gamma_1 \parallel \gamma_2,$$

where

$$(\gamma_1 \parallel \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t - 1) & t \in [1/2, 1] \end{cases}.$$

**Definition 4.7** (Path System). *Let  $X$  be a topological space. A path system on  $X$  is a non-empty family  $\mathcal{P}$  of continuous curves  $\gamma : [0, 1] \rightarrow X$  that is closed under reparametrization and concatenation.*

By Definition 4.6, if  $\gamma_1, \gamma_2 \in \mathcal{P}$  and  $\gamma_1(1) = \gamma_2(0)$ , then  $\gamma_1 \parallel \gamma_2 \in \mathcal{P}$ .

**Proposition 4.8.** *Manifold paths (Definition 4.3) form a path system (Definition 4.7).*

*Proof.* We define the family of manifold paths using Definition 4.7:

$$\mathcal{P}_{\text{manifold}} := \{\gamma_G : [0, 1] \rightarrow G \mid \gamma_Z(t) = \gamma_G(t) \cdot z_0 \text{ for some } z_0 \in Z, \\ \gamma_G : [0, 1] \rightarrow G \text{ continuous with } \gamma_G(0) = e\}.$$

To prove these manifold paths form a path system, we need to show they are both (1) closed under reparametrization and (2) closed under concatenation.

(1) Closed under reparametrization.

Given a manifold-path  $\gamma_Z(t) = \gamma_G(t) \cdot z$  where  $\gamma_G : [0, 1] \rightarrow G$  is continuous with  $\gamma_G(0) = e$  and  $z_0 \in Z$  is the base point, and a reparametrization  $\phi : [0, 1] \rightarrow [0, 1]$  with  $\phi(0) = 0, \phi(1) = 1$ .

The reparametrized path is:

$$(\gamma_Z \circ \phi)(t).$$

By composition law of function and Definition 4.3:

$$(\gamma_Z \circ \phi)(t) = \gamma_Z(\phi(t)) = \gamma_G(\phi(t)) \cdot z_0.$$

To show that this reparametrized path still belongs to the path system, we define the reparametrized path as

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<sup>3</sup>Note the order of concatenation,  $\gamma_1 \parallel \gamma_2$  means first apply  $\gamma_1$ , then  $\gamma_2$ .

$$\tilde{\gamma}_G(t) := \gamma_G(\phi(t)).$$

Next, we check this reparametrized path satisfies Definition 4.7:

- $\tilde{\gamma}_G$  is continuous because it is composition of continuous functions.
- $\tilde{\gamma}_G$  satisfies initial condition:  $\tilde{\gamma}_G(0) = \gamma_G(\phi(0)) = \gamma_G(0) = e$ .

Therefore:

$$(\gamma_Z \circ \phi)(t) = \tilde{\gamma}_G(t) \cdot z_0.$$

This is a manifold-path with reparametrized path  $\tilde{\gamma}_G$  and base point  $z_0$ . Thus the manifold paths are closed under reparametrization.

(2) Closed under concatenation.

Consider two manifold-paths:

- $\gamma_{z_1}(t) = \gamma_{G,1}(t) \cdot z_1$  where  $\gamma_{G,1}(0) = e$ ,
- $\gamma_{z_2}(t) = \gamma_{G,2}(t) \cdot z_2$  where  $\gamma_{G,2}(0) = e$ .

With  $\gamma_{z_1}(1) = \gamma_{z_2}(0)$ . This means:

$$\gamma_{G,1}(1) \cdot z_1 = \gamma_{G,2}(0) \cdot z_2 = e \cdot z_2 = z_2.$$

Therefore,  $z_2 = \gamma_{G,1}(1) \cdot z_1$ .

Construction of concatenation according to Definition 4.6:

$$(\gamma_{z_1} \| \gamma_{z_2})(t) = \begin{cases} \gamma_{G,1}(2t) \cdot z_1 & t \in [0, 1/2] \\ \gamma_{G,2}(2t - 1) \cdot z_2 & t \in [1/2, 1] \end{cases}.$$

Substituting  $z_2 = \gamma_{G,1}(1) \cdot z_1$ :

$$= \begin{cases} \gamma_{G,1}(2t) \cdot z_1 & t \in [0, 1/2] \\ \gamma_{G,2}(2t - 1) \cdot (\gamma_{G,1}(1) \cdot z_1) & t \in [1/2, 1] \end{cases}.$$

Using group associativity:

$$= \begin{cases} \gamma_{G,1}(2t) \cdot z_1 & t \in [0, 1/2] \\ ((\gamma_{G,2}(2t - 1) \cdot \gamma_{G,1}(1)) \cdot z_1) & t \in [1/2, 1] \end{cases}.$$

Similarly, to show that this reparametrized path still belongs to the path system, we define the concatenated path as

$$\tilde{\gamma}_G(t) := \begin{cases} \gamma_{G,1}(2t) & t \in [0, 1/2] \\ \gamma_{G,2}(2t - 1)\gamma_{G,1}(1) & t \in [1/2, 1] \end{cases}.$$

Next, we check that this concatenated path satisfies Definition 4.7:

- $\tilde{\gamma}_G$  satisfies initial condition:  $\tilde{\gamma}_G(0) = \gamma_{G,1}(0) = e$ .
- $\tilde{\gamma}_G$  is continuous at  $t = 1/2$ :

$$\gamma_{G,2}(0)\gamma_{G,1}(1) = e \cdot \gamma_{G,1}(1) = \gamma_{G,1}(1).$$

We arbitrarily choose paths  $\gamma_{z_1}(t)$  and  $\gamma_{z_2}(t)$ , thus  $\tilde{\gamma}_G$  is piecewise continuous with matching at boundary.

Therefore

$$(\gamma_{z_1} \parallel \gamma_{z_2})(t) = \tilde{\gamma}_G(t) \cdot z_1.$$

This is a manifold-path with concatenated path  $\tilde{\gamma}_G$  and base point  $z_1$ . Thus, the manifold paths are closed under concatenation.

□

**Definition 4.9** (Path Equivariance (PE) Map). *Let  $X$  be a topological space with path system (Definition 4.7)  $\mathcal{P}$ ,  $A$  be a Lie group,  $Z$  be a manifold. Consider a (usually not discrete) Lie group action  $A \curvearrowright Z$ . A continuous map  $F : X \rightarrow Z$  is path equivariant with respect to  $\mathcal{P}$  if for every  $\gamma \in \mathcal{P}$ , there exists a continuous transport  $a_\gamma : [0, 1] \rightarrow A$  with  $a_\gamma(0) = e$  such that*

$$F(\gamma(t)) = a_\gamma(t) \cdot F(\gamma(0)) \quad \forall t \in [0, 1].$$

*A neural network with PE property is called a path equivariant network (PEN).*

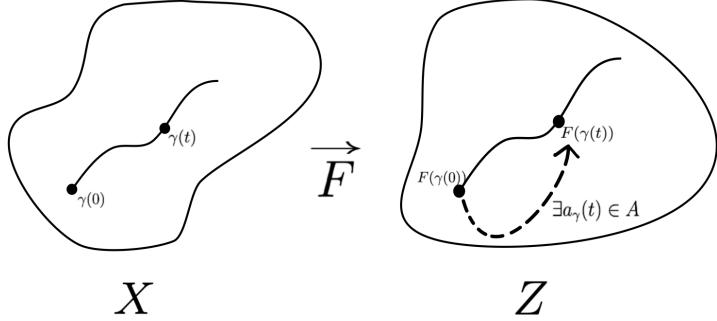
A visual explanation can be found in Figure 2.

While  $F : X \rightarrow Z$  acts point-wise, it induces a mapping on paths: each path  $\gamma \in X$  corresponds to the path  $F \circ \gamma \in Z$ .

**Definition 4.10** (Induced path). *Given path  $\gamma : [0, 1] \rightarrow X$ , define the induced path:*

$$\begin{aligned} (F_*)(\gamma) &:= F \circ \gamma : [0, 1] \rightarrow Z \\ (F_*)(\gamma)(t) &= F(\gamma(t)), \end{aligned}$$

*where  $F_*$  maps paths in  $X$  to paths in  $Z$ .*



**Figure 2: Visual explanation for path equivariant map.** There exists a continuous transport  $a_\gamma$  that preserves path  $\gamma$  in  $X$  to path  $F(\gamma)$  in  $Z$ .

Compared to the  $G$ -equivariant learner  $E$  defined earlier (Definition 4.2), the path-equivariant map  $F$  provides more flexibility. We introduce a new Lie group  $A$  acting on the output space  $Z$ , which need not coincide with the input symmetry group  $G$  acting on  $X$ , or even,  $X$  need not possess any group structure at all. This difference enables practical meanings: for example, when  $X$  contains images with  $G = SE(2)$  symmetries and  $Z$  contains features with  $A = SO(2)$  symmetries, the map  $F$  becomes rotation-equivariant yet translation-invariant, which means objects at different positions may be detected as identical, but their detection varies with rotation.

**Definition 4.11** (Group Path System). *Let  $X$  be a topological  $G$ -space. Define the group path system in  $X$ :*

$$\mathcal{P} := \{\gamma_X : [0, 1] \rightarrow X \mid \gamma_X(t) = \gamma_G(t) \cdot x \text{ for some } x \in X, \\ \gamma_G : [0, 1] \rightarrow G \text{ continuous with } \gamma_G(0) = e\}.$$

*A group path in  $X$  is a path in  $\mathcal{P}$ . That is, the group path system  $\mathcal{P}$  consists of all group paths in  $X$ .*

**Proposition 4.12** (Manifold Paths Satisfy Path Equivariance). *Let  $G$  be a Lie group,  $X$  be a topological  $G$ -space, and  $Z$  be a  $G$ -manifold. Let  $E : X \rightarrow Z$  be a  $G$ -equivariant map, i.e.,*

$$E(g \cdot x) = g \cdot E(x) \quad \forall g \in G, x \in X.$$

*The map  $E : X \rightarrow Z$  is path-equivariant (Definition 4.9) with respect to the group path system  $\mathcal{P}$  in  $X$  (Definition 4.11) by taking  $A = G$  as the acting group on  $Z$ .*

*Proof.* To show that  $E : X \rightarrow Z$  is path-equivariant (Definition 4.9) with respect to  $\mathcal{P}$ , we need to prove that there exists a continuous transport  $a_{\gamma_X} : [0, 1] \rightarrow G$  with  $a_{\gamma_X}(0) = e$  such that:

$$E(\gamma_X(t)) = a_{\gamma_X}(t) \cdot E(\gamma_X(0)) \quad \forall t \in [0, 1]. \tag{a}$$

1. Define the transport  $a_{\gamma_X}$ .

Let  $\gamma_X \in \mathcal{P}$  be arbitrary. By definition of  $\mathcal{P}$ , there exist:

- A point  $x_0 \in X$ .
- A continuous path  $\gamma_G : [0, 1] \rightarrow G$  with  $\gamma_G(0) = e$ .

Such that:

$$\gamma_X(t) = \gamma_G(t) \cdot x_0 \quad \forall t \in [0, 1]. \quad (\text{b})$$

Define the transport:

$$a_{\gamma_X}(t) := \gamma_G(t) \in G. \quad (\text{c})$$

2. Verify the transport we defined satisfies properties in Definition 4.9.

- Verify continuity:  
 $a_{\gamma_X} : [0, 1] \rightarrow G$  is continuous since  $\gamma_G$  is continuous.
- Verify initial condition:

$$a_{\gamma_X}(0) = \gamma_G(0) = e.$$

- Verify condition a:

$$E(\gamma_X(t)) = a_{\gamma_X}(t) \cdot E(\gamma_X(0)) \quad \forall t \in [0, 1].$$

Compute the left-hand side using Equation b:

$$E(\gamma_X(t)) = E(\gamma_G(t) \cdot x_0).$$

Since  $E$  is  $G$ -equivariant (Definition 4.2), we have:

$$E(\gamma_G(t) \cdot x_0) = \gamma_G(t) \cdot E(x_0). \quad (\text{d})$$

Compute the right-hand side:

$$a_{\gamma_X}(t) \cdot E(\gamma_X(0)) = \gamma_G(t) \cdot E(\gamma_G(0) \cdot x_0).$$

Using  $\gamma_G(0) = e$  Equation b and Equation c:

$$= \gamma_G(t) \cdot E(e \cdot x_0) = \gamma_G(t) \cdot E(x_0). \quad (\text{e})$$

Combine Equation d and Equation e, we have:

$$E(\gamma_X(t)) = \gamma_G(t) \cdot E(x_0) = a_{\gamma_X}(t) \cdot E(\gamma_X(0)).$$

This holds for all  $t \in [0, 1]$ .

Since  $\gamma_X \in \mathcal{P}$  was arbitrary, we have shown that for every path in  $\mathcal{P}$ , there exists a continuous transport satisfying the Definition 4.9.

Therefore,  $E$  is path-equivariant with respect to  $\mathcal{P}$ .

□

**Corollary 4.13** (Manifold Paths are PE Paths). *For any fixed  $x_0 \in X$ , let  $z_0 := E(x_0)$ . Then for any continuous group path  $\gamma_G : [0, 1] \rightarrow G$  with  $\gamma_G(0) = e$ , the manifold-path:*

$$\gamma_Z(t) := E(\gamma_G(t) \cdot x_0)$$

satisfies:

$$\gamma_Z(t) = \gamma_G(t) \cdot z_0 = \gamma_G(t) \cdot \gamma_Z(0).$$

*Proof.* This follows directly from the proof above by setting  $\gamma_X(t) = \gamma_G(t) \cdot x_0$  and noting that:

$$\gamma_Z(t) = E(\gamma_X(t)) = \gamma_G(t) \cdot E(x_0) = \gamma_G(t) \cdot z_0.$$

□

The commutative diagram between  $G, X, Z$  in manifold path equivariance:

$$\begin{array}{ccc}
G : & & \gamma_G(t) \in G \\
& \swarrow \text{DRAFT} \searrow & \downarrow \\
X : & x_0 \xrightarrow{\gamma_X(t)} \gamma_G(t)x_0 & \\
& \downarrow E & \downarrow E \\
Z : & z_0 \xrightarrow{\gamma_Z(t)} E(\gamma_G(t)x_0) = \gamma_G(t)z_0 &
\end{array}$$

#### 4.2.2 From Path Equivariance to Classical Group Equivariance

To recover classical group-equivariant neural networks [9] from path-equivariance (PEN), we must preserve the group structure from  $X$  to  $Z$ . In the input space  $X$ , group elements  $g \in G^0$  act on points. We can represent each group element  $g$  as the endpoint of a path starting from the identity (Corollary 2.37):

$$g = \gamma(1)$$

where  $\gamma : [0, 1] \rightarrow G$  with  $\gamma(0) = e$

A path-equivariant map  $F : X \rightarrow Z$  induces (Definition 4.10) a mapping on paths:

$$F_* : \gamma \mapsto a_\gamma$$

where  $\gamma$  is a path in  $X$  (induced by group path in  $G$ ), and  $a_\gamma$  is the corresponding transport path in  $A$ .

For  $F$  to preserve the group structure, elements in  $Z$  must be determined by the corresponding group elements in  $X$ , not by the specific path taken. This means if two paths in  $G$  have the same endpoint:

$$\gamma_1(1) = \gamma_2(1) = g.$$

Then their induced transports must also have the same endpoint:

$$a_{\gamma_1}(1) = a_{\gamma_2}(1)$$

This gives us the definition of "endpoint condition".

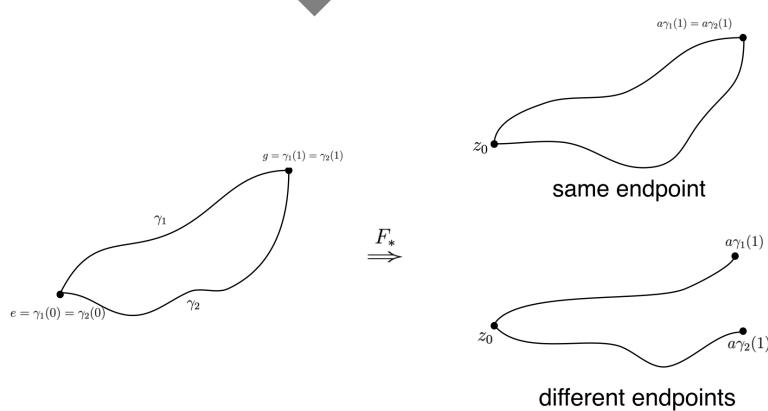
**Definition 4.14** (Endpoint Condition). *Two group paths  $\gamma_1, \gamma_2 : [0, 1] \rightarrow G^0$  with the same endpoints ( $\gamma_1(0) = \gamma_2(0) = e$  and  $\gamma_1(1) = \gamma_2(1) = g$ ) satisfy the endpoint condition if their induced transports coincide:  $a_{\gamma_1}(1) = a_{\gamma_2}(1)$  (see Figure 3).*

We call maps that satisfy the endpoint condition "endpoint maps".

**Definition 4.15** (Endpoint maps). *Under the endpoint condition, we can define the endpoint map  $\rho : G^0 \rightarrow A$  by*

$$\rho(g) := a_\gamma(1)$$

where  $\gamma$  is any path in  $G^0$  from  $e$  to  $g$ ,  $G^0$  is identity component.



**Figure 3: Endpoint condition.** The induced map  $F_*$  preserves the endpoints of a path in  $G$ .

The endpoint condition ensures  $\rho$  is independent of the choice of path from  $e$  to  $g$ , all paths with the same endpoints give the same value. So  $\rho$  is well-defined.

**Proposition 4.16.**  $\rho$  is a group homomorphism.

*Proof.* 1. Prove  $\rho(e) = e$ .

Take the constant path  $\gamma(t) = e$  from  $e$  to  $e$ . By definition,  $\rho(e) = a_\gamma(1) = e$ .

2. Prove  $\rho(g_1g_2) = (\rho g_1)(\rho g_2)$ .

In this step, we need to prove  $\rho$  maps groups to its corresponding paths, which means  $\rho(g_i) = a_{\gamma_i}(1)$ , and  $\rho$  preserves group operation, which means  $(\rho g_1)(\rho g_2) = (a_{\gamma_1}(1))(a_{\gamma_2}(1))$ .

- $\rho$  maps groups to their corresponding paths.

For  $\forall x \in X$ , path  $x \mapsto (g_1g_2)x$  can be decomposed to

$$x \mapsto g_2x \mapsto g_1(g_2x)$$

by

$$\gamma_1 : e \mapsto g_1, \quad \gamma_2 : e \mapsto g_2$$

where  $(g_1g_2)(x) = g_1(g_2x)$ .

Write the path  $x \mapsto g_1g_2(x)$  as  $\gamma$ :

$$\gamma(t)(x) = \begin{cases} \gamma_2(2t)x & t \in [0, \frac{1}{2}] \\ \gamma_1(2t - 1)(g_2x) & t \in [\frac{1}{2}, 1] \end{cases} \quad (a)$$

Using Definition 4.6 to concatenate  $\gamma_1$  and  $\gamma_2$ :

$$= (\gamma_2 \parallel \gamma_1)(t)(x) \quad t \in [0, 1]$$

Using Definition 4.15:

$$\rho(g_1g_2) = a_\gamma(1).$$

By Proposition 4.12:

$$\rho(g_1g_2) = a_\gamma(1) = a_{g_2 \cdot \gamma_1}(1) \cdot a_{\gamma_2}(1)x = (\gamma_2 \parallel (g_2\gamma_1))x$$

represents path  $\gamma : x \xrightarrow{\gamma_2} g_2x \xrightarrow{\gamma_1} g_1(g_2x)$ ;

$$\rho(g_1) = a_{\gamma_1}(1)$$

represents path  $\gamma_1 : x \mapsto g_1x$ ;

$$\rho(g_2) = a_{\gamma_2}(1)$$

represents path  $\gamma_2 : x \mapsto g_2x$ .

- $\rho$  preserves group operation.

From previous proof we observe that both  $\gamma_1$  and  $\gamma_2$  start from  $x$ , while  $\gamma$  sequentially combines paths  $x \mapsto g_2x$  and  $g_2x \mapsto g_1(g_2x)$ . We need to show that transport  $a$  is independent of starting point, i.e.,  $\forall \gamma_g : [0, 1] \rightarrow G$  with  $\gamma_g(0) = e, \gamma_g(1) = g$ , the transport along  $\gamma_g(t) \cdot x$  (start at  $\forall x \in X$ ) gives the same  $a\gamma_g(1) := \rho(g)$ .

By Definition 4.9:

$$F(\gamma_g(t)x) = a_{\gamma_g}(t) \cdot F(x)$$

where  $a_{\gamma_g}(t) \in A, F(x) \in Z, a_{\gamma_g}$  only determined by path  $\gamma_g$  not  $x$ .

So  $g_2x \xrightarrow{\gamma_1} g_1(g_2x)$  and  $x \xrightarrow{\gamma_1} g_1x$  be transported to same path in  $[0, 1] \rightarrow A$  by  $a$ . We can combine  $\rho(g_1) = a_{\gamma_1}(1)$  and  $\rho(g_2) = a_{\gamma_2}(1)$  to  $a_{g_2 \cdot \gamma_1}(1) \cdot a_{\gamma_2}(1)x = (\gamma_2 \parallel (g_2\gamma_1))x$ .

Therefore  $\rho(g_1g_2) = (\rho g_1)(\rho g_2)$  is homomorphism.

3. Prove  $\rho(g^{-1}) = \rho(g)^{-1}$ .

If  $\gamma$  is a path from  $e$  to  $g$ , then by Equation a  $\gamma^{-1}(t) = \gamma(1-t) = \gamma(t)^{-1}$  is a path from  $e$  to  $g^{-1}$ , or equivalently, from  $g$  to  $e$ .

By Proposition 4.12,  $a_\gamma$  preserves paths structure, so the reversed transport:

$$a_{\gamma^{-1}}(1) = a_\gamma(1)^{-1}.$$

Therefore  $\rho(g^{-1}) = \rho(g)^{-1}$ .

□

**Proposition 4.17** (Reduction to Classical Group Equivariance on  $G^0$ ). *Let  $G \curvearrowright X$  and  $A \curvearrowright Z$ ,  $F : X \rightarrow Z$  be path-equivariant with respect to the group path system (Definition 4.11)  $\mathcal{P} = \{g(t) \cdot x : g(0) = e\}$ .*

*Assume the endpoint condition (Definition 4.14): For all paths  $\gamma_1, \gamma_2$  from  $e$  to  $g$ :*

$$a_{\gamma_1}(1) = a_{\gamma_2}(1).$$

*Then:*

$$F(g \cdot x) = \rho(g) \cdot F(x) \quad \forall g \in G^0, x \in X$$

*where  $\rho : G^0 \rightarrow A$  is the endpoint map.*

*This is exactly the classical group-equivariance law on  $G^0$ .*

*Proof.* Fix  $g \in G^0$  and  $x \in X$ . Since  $G^0$  is path-connected, there exists a continuous path  $\gamma : [0, 1] \rightarrow G^0$  with  $\gamma(0) = e$  and  $\gamma(1) = g$ .

Consider the group path (Definition 4.11) in  $X$ :

$$\gamma_X(t) := \gamma(t) \cdot x.$$

By definition:  $\gamma_X(0) = e \cdot x = x$  and  $\gamma_X(1) = g \cdot x$ .

Since  $F$  is path-equivariant, there exists transport  $a_\gamma : [0, 1] \rightarrow A$  with  $a_\gamma(0) = e$  such that:

$$F(\gamma_X(t)) = a_\gamma(t) \cdot F(\gamma_X(0)).$$

At  $t = 1$ :

$$F(g \cdot x) = F(\gamma_X(1)) = a_\gamma(1) \cdot F(x).$$

By the endpoint condition and definition of  $\rho$ :

$$a_\gamma(1) = \rho(g).$$

Therefore:

$$F(g \cdot x) = \rho(g) \cdot F(x).$$

This holds for all  $g \in G^0$  and  $x \in X$ . □

This proposition establishes that if  $F$  is path-equivariant and satisfies the endpoint condition, then:

1. Preservation of group structure: The transport map  $\rho : G^0 \rightarrow A$  becomes a continuous homomorphism, preserving the group structure:  $G^0 \ni g_1 \xrightarrow{\rho} \rho(g_1) \in A$ .
2. Recovery of classical equivariance:  $F$  reduces to a classical group-equivariant map:  $F(g \cdot x) = \rho(g) \cdot F(x) \quad \forall g \in G^0, x \in X$ .

From a geometric viewpoint, the endpoint condition has a natural interpretation when  $G^0$  is simply-connected. In this case, any two paths with the same endpoints are homotopic (Definition 2.24). Consequently, if the transport is homotopy invariant, meaning homotopic paths (Definition 2.26) induce the same transport, then the endpoint condition is automatically satisfied. This geometric insight suggests an intermediate framework between full path-equivariance and classical group equivariance.

**Definition 4.18** (Homotopy-equivariant network). *Under the condition Path Equivariance Map (Definition 4.9), if two homotopic paths have same group representation via a network, we call this network a homotopy-equivariant network.*

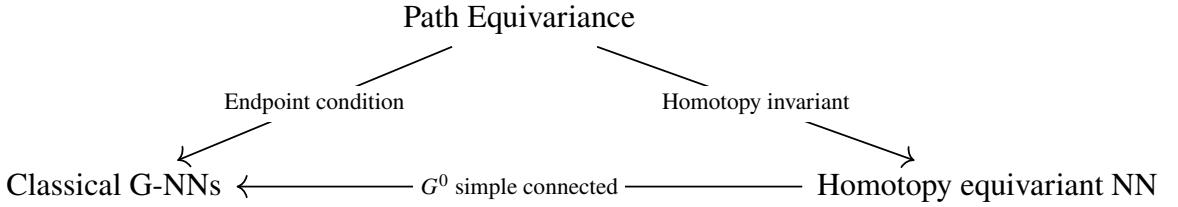
Formally, a homotopy-equivariant network satisfies:

$$\gamma_1 \simeq \gamma_2 \implies a_{\gamma_1}(1) = a_{\gamma_2}(1).$$

where  $\gamma$  is the path,  $\simeq$  means homotopic,  $a$  is the continuous transport in Definition 4.9.

In a homotopy-equivariant network, paths in the same homotopy class induce the same output transformation. This property respects the topological structure of the group manifold while being weaker than classical group equivariance, which requires all paths to the same endpoint to give the same transport, regardless of homotopy class.

The relationship between these frameworks forms a hierarchy:



When  $G^0$  is not simply-connected, the relationship becomes more nuanced:

- Paths between distinct points ( $e \neq g$ ): Paths can still satisfy the endpoint condition (Definition 4.14), but paths connecting these endpoints are not necessarily homotopic. For example, the annulus (see Figure 4). These homotopy classes represent the number of topologically distinct ways to reach the group element  $g$  from the identity. A homotopy equivariant network may distinguish between these different topological routes, while classical group equivariance would treat all paths to  $g$  identically, regardless of their homotopy class.
- Closed loops ( $e = g$ ): When paths are closed loops, the endpoint-based group representation becomes uninformative: since  $\gamma(0) = \gamma(1) = e$ . Instead, the loop itself carries the homotopic information, for example, the torus (see Figure 4<sup>4</sup>). The distinct homotopy classes of closed loops based at the identity are labeled by fundamental group (Definition 2.28)  $\pi_1(G^0, e)$ . These classes measure the topology of the group manifold rather than identifying individual group elements, revealing global geometric features that are invisible to local algebraic structure.

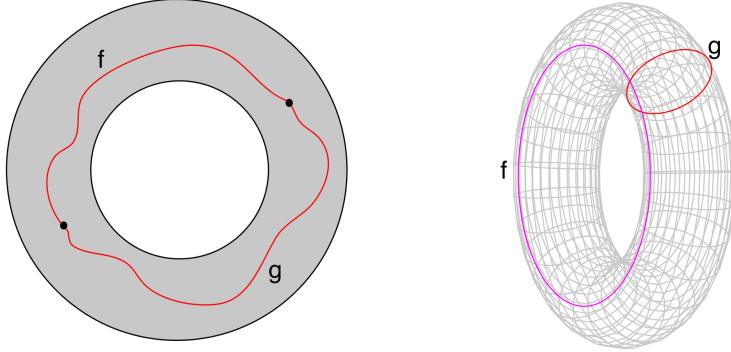
**Remark 4.19** (Connectivity constraint). *If Lie group  $G$  is disconnected, i.e., it can be written as:*

$$G = G^0 \sqcup G_1 \sqcup G_2 \sqcup \dots$$

*where  $G^0$  is the identity component,  $G_1, G_2, \dots$  are the other connected components,  $\sqcup$  denotes the disjoint union, and we cannot reach any other component from  $e$  (or  $G^0$ ) by a continuous path. In this case, we might need to specify the discrete component group  $\pi_0(G) = G/G^0$ . Or, we can define the content space  $U = X/G$ , which is not a group. We will discuss this idea further in Section 4.3.*

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<sup>4</sup>Torus image from <https://en.wikipedia.org/wiki/Torus>



**Figure 4: Examples of different homotopy classes.** Different homotopy classes  $[f] \neq [g]$  for paths between distinct points (left) and closed loops (right).

#### 4.2.3 From Homotopy to Homology

While homotopy groups  $\pi_k$  study continuous deformations, homology groups  $H_k(X)$  provide a more algebraic and computable approach, measuring "holes" of all dimensions [46]. This has driven Topological Data Analysis (TDA), which extracts robust topological features via persistent homology. Recent neural architectures leverage TDA through topology-preserving autoencoders [42], persistence-based layers [4], and topological regularization [51].

Intuitively (or counterintuitively), although both talk about groups, G-NNs and TDA appear to capture different aspects: G-NNs encode how data transforms under symmetries (extrinsic), while TDA captures the topology of data manifolds (intrinsic). However, we know that homology is also a group  $H_k(X)$  which is discrete and abelian, and previously (Section 3.3) we analyzed that regularization can be thought of as a soft group constraint via a loss term. We conjecture that deeper connections exist between G-NNs and TDA.

### 4.3 Content Space and Pose

In this section, we explore PEN’s expressibility beyond group structures. The key insight is that paths need not be a group. We define the pose that forms a group  $G$ , and the content spaces inhabit the quotient manifold  $U = X/G$  with no group structure. By defining path systems (Definition 4.7) on both pose and content spaces, PENs unify algebraic structure preservation (group equivariance on pose paths within orbits) with geometric structure preservation (smoothness on content paths between orbits). This framework interpolates between classical G-NNs (pose only, symmetry) and unconstrained smooth maps (content spaces, no symmetry).

**Definition 4.20** (Content Space and Pose). *Let a Lie group  $G$  act smoothly on a data manifold  $X$ . The action partitions  $X$  into orbits:*

$$Gx = \{g \cdot x : g \in G\}.$$

Define the content space as the quotient (orbit) space:

$$U := X/G = \{Gx : x \in X\}$$

We call  $G$  a pose on content  $U$ .

**Remark 4.21.**  $U$  is not a group in general.

The content space  $U$  is the collection of all orbits, each point  $u \in U$  represents an entire orbit  $O_x = \{g \cdot x : g \in G\}$  in  $X$ . Intuitive thinking, content space is invariant to transformations, it captures the intrinsic identity, meaning "what the data is". For example, in face recognition tasks, data  $X$  is all face images with various rotations,  $G = SO(2)$  acts by rotating images,  $U$  means face identities corresponds to different person, each  $u \in U$  represents one person's face,  $U$  can be transformed by morphing (see Figure 5).



**Figure 5: An example of non-group transform  $U$ .** Different  $U$  means different faces (under morphing transformation),  $G = SO(2)$  as "pose" of each faces.

**Definition 4.22** (Projection Map in Content Space). Define the projection map:

$$\begin{aligned}\pi : X &\rightarrow U \\ x &\mapsto O_x\end{aligned}$$

where  $O_x$  means the orbits containing  $x$ .

For example, in face recognition tasks,  $\pi(x)$  means one specific person  $x$ , regardless of one's position.

**Theorem 4.23.** [41, Theorem 1.21]

If the action  $G$  is free (Definition 2.41) and proper (Definition 2.42), the projection  $\pi : X \rightarrow U$  is a submersion,  $X$  is a principal  $G$ -bundle (Definition 2.45) over  $U$ , the fibers are the orbits (see Figure 6).

In this case,  $X$  is total space,  $U = X/G$  is base space,  $\pi : X \rightarrow U$  is the projection sends  $x$  to its orbit,  $\pi^{-1}(u) \subset X$  is the fiber over  $u$ ,  $c : U \rightarrow X$  is a section picks one point from each fiber. If we stack all orbits together we reconstruct  $X = \bigsqcup_{u \in U} \pi^{-1}(u)$ .

According to principal bundle properties:

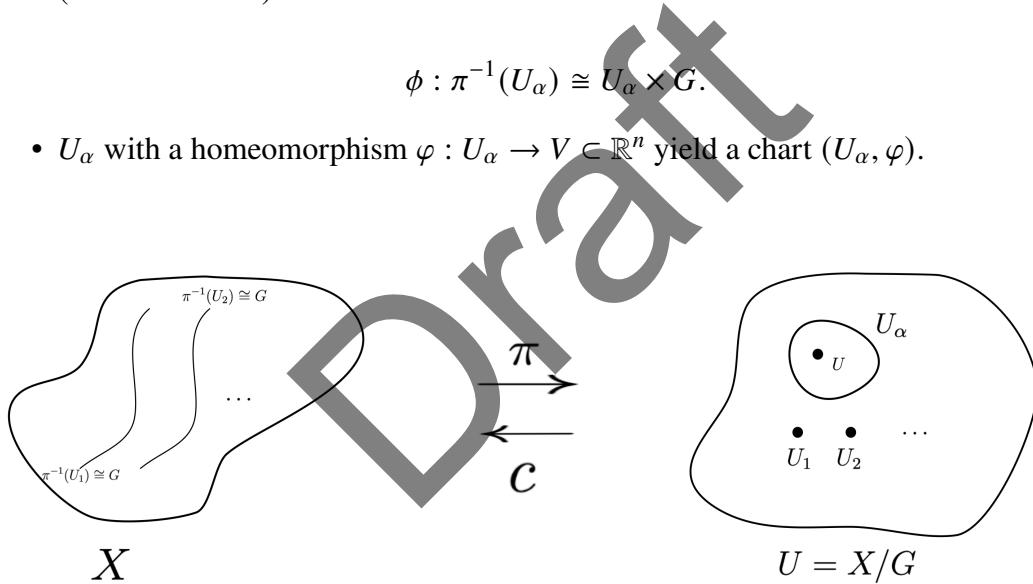
- Each fiber is a copy of  $G$ :

$$\pi^{-1}(u) \cong G.$$

- Around each  $u \in U$ , there exists a neighborhood  $U_\alpha$  and a diffeomorphism (Definition 2.34):

$$\phi : \pi^{-1}(U_\alpha) \cong U_\alpha \times G.$$

- $U_\alpha$  with a homeomorphism  $\varphi : U_\alpha \rightarrow V \subset \mathbb{R}^n$  yield a chart  $(U_\alpha, \varphi)$ .



**Figure 6:**  $X$  is a principal  $G$ -bundle over  $U$ . Fibers in bundle are the orbits in group, projection  $\pi$  sends  $x$  to its orbit,  $c : U_\alpha \rightarrow X$  of  $\pi$  on a chart  $U_\alpha$ .

**Definition 4.24** (Canonical Pose Map). A (local) canonical pose is a smooth section  $c : U_\alpha \rightarrow X$  of  $\pi$  on a chart  $U_\alpha$ .

For each  $u \in U_\alpha$ , the section chooses one specific representative from the fiber  $\pi^{-1}(u)$ . This choice is usually not unique; it serves as the reference for that chart. Together with pose action  $g$ , we can represent  $x \in X$ .

**Proposition 4.25.** Every sample  $x \in X$  can be uniquely decomposed as

$$x = g \cdot c(u), \quad u \in U_\alpha, g \in G.$$

The data point  $x$  can be obtained by applying the transformation  $g$  to the canonical pose  $c(u)$ .

*Proof.* For any  $x \in X$ , we need to prove its representation exists and is unique.

- Existence:

$u = \pi(x)$  is uniquely determined by orbit of  $x$ ,  $c(u)$  is determined by the section  $c$ ,  $G$  acts transitively on the fiber  $\pi^{-1}(u)$ , so  $\exists g$  such that  $x = g \cdot c(u)$ .

- Uniqueness:

Action  $G$  is free, so if  $g_1 \cdot c(u) = g_2 \cdot c(u)$ , then  $g_1 = g_2$ . Therefore, the pose  $g$  is uniquely determined.

□

**Remark 4.26.** The canonical pose  $c$  is locally defined on a chart  $U_\alpha$ , not all of  $U$ , because the global section  $c : U \rightarrow X$  may not exist.

**Lemma 4.27** (Union of orbits are path-connected). Let  $G^0$  be the identity component of  $G$ . Fix a chart  $U_\alpha \subset U$  with section  $c$  and let

$$C_\alpha := \{g \cdot c(u) : u \in U_\alpha, g \in G^0\} = \bigcup_{u \in U_\alpha} G^0 \cdot c(u).$$

If  $U_\alpha$  is path-connected, then  $C_\alpha$  is path-connected.

*Proof.* Consider the map

$$F : U_\alpha \times G^0 \rightarrow X$$

where  $F(u, g) = g \cdot c(u)$ , we prove  $F$  is a continuous map, and its pre-image is path-connected, thus its image is path-connected.

- The action  $(g, x) \mapsto g \cdot x$  is continuous, and  $c$  is a section thus continuous. So  $F$  is continuous.
- $U_\alpha$  is path-connected by hypothesis,  $G^0$  is path-connected by definition. So  $U_\alpha \times G^0$  is path-connected.
- Images of path-connected sets are path-connected under continuous maps; hence

$$C_\alpha = F(U_\alpha \times G^0)$$

is path-connected.

□

Intuitively, pose means "where"; it moves within an orbit. This movement can be represented by group-equivariant layers  $F(gx) = \rho(g)F(x)$ , which are exactly the classical G-NNs. Content means "what"; it moves across orbits, handles with smoothness across  $U$ , and has no group law to enforce.

**Definition 4.28** (Pose Path and Content Path). *Pose path (within orbit) is defined as:*

$$\gamma_{X,G}(t) := \gamma_G(t) \cdot c(u) \quad t \in [0, 1],$$

where  $\gamma_G : [0, 1] \rightarrow G$  is a continuous path in the group with  $\gamma_G(0) = e$ ,  $u \in U_\alpha$  is fixed with same content,  $c(u)$  is the canonical pose (fixed point in  $X$ ).

Content paths (between orbits) are defined as:

$$\gamma_{X,U}(t) := c(\gamma_U(t)) \quad t \in [0, 1],$$

where  $\gamma_U : [0, 1] \rightarrow U_\alpha$  is a continuous path in content space, pose is fixed at canonical pose  $c(\cdot)$ . (See Figure 7)

The combined path system of pose paths and content paths is defined as:

$$\mathcal{P} = \{\gamma_{X,G}(t) : u \in U_\alpha, \gamma_G(0) = e\} \cup \{\gamma_{X,U}(t) : \gamma_U \text{ continuous in } U_\alpha\}.$$

Geometric thinking, pose paths start at canonical pose  $\gamma_{X,G}(0) = e \cdot c(u) = c(u)$ , and stay in the same orbit  $\pi^{-1}(u)$ , explore different poses of the same content; content paths start at  $\gamma_{X,U}(0) = c(\gamma_U(0))$ , cross different orbits, and always pick the canonical representative from each orbit.

Recall the PE (Definition 4.9) requirement: For  $F : X \rightarrow Z$  with  $A \curvearrowright Z$ , and every  $\gamma \in \mathcal{P}$ :

$$\exists a_\gamma : [0, 1] \rightarrow A \text{ with } a_\gamma(0) = e$$

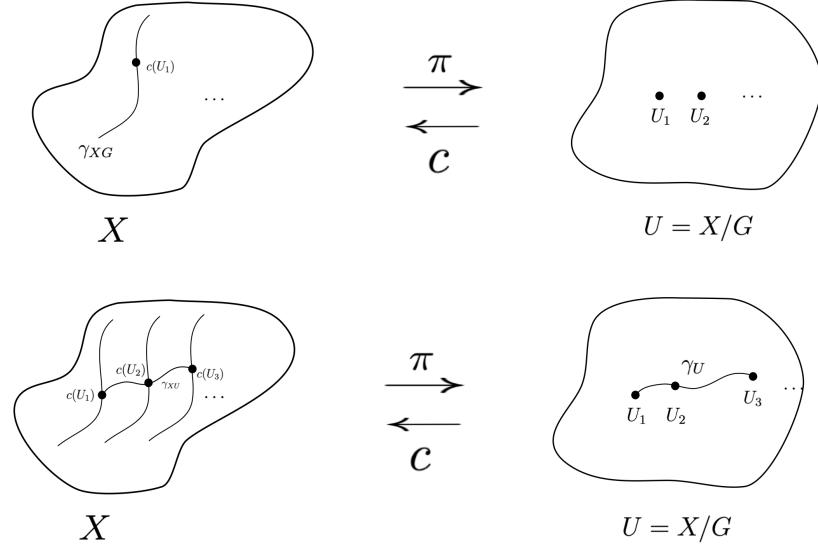
such that

$$F(\gamma(t)) = a_\gamma(t) \cdot F(\gamma(0)).$$

In the pose/content scenario, we geometrically abstract  $X$  into  $U = X/G$  space, but this space is generally not equipped with a natural group action (like  $A \curvearrowright Z$  in PE). In PE,  $Z$  is a new space where we want to encode the data; the group  $A$  that acts on  $Z$  is part of our design choice; ideally,  $Z$  should "represent"  $U$  in some sense.

## 4.4 Path Equivariance on Decomposed Spaces

We now analyse how path equivariance manifests differently on these two spaces. This analysis reveals a fundamental distinction: the group structure of pose enables strong equivariance guarantees via homomorphisms, while the manifold structure of content space permits only consistency constraints. By examining concrete cases in each space, we demonstrate how PENs flexibly encode different preservation requirements through appropriate choices of action groups and transport maps.



**Figure 7: Pose paths(up) and content paths(down).** Poses have group structures, while contents act as a transport map.

#### 4.4.1 Path Equivariance on Pose

The pose inherits its structure from the Lie group \$G\$, enabling two cases:

1. Pose-Invariant Encoder.

Let \$F : X \rightarrow Z\$, where \$Z\$ is a feature space representing content, we want \$F\$ to be constant on orbits (pose-invariant).

For pose paths (Definition 4.28):

$$\gamma_{X,G}(t) = \gamma_G(t) \cdot c(u)$$

\$F\$ behaves constantly on pose path \$\gamma\_G(t)\$:

$$F(\gamma_{X,G}(t)) = F(c(u)) = \text{constant}$$

In PE condition, this is equivalent to choose \$A = \{e\}\$ and transport \$a\_{\gamma\_{X,G}}(t) = e\$ for all \$t\$:

$$F(\gamma_{X,G}(t)) = e \cdot F(\gamma_{X,G}(0)) = F(c(u))$$

2. Pose-Equivariant Encoder.

For pose paths (Definition 4.28):

$$\gamma_{X,G}(t) = \gamma_G(t) \cdot c(u).$$

$F$  behaves:

$$F(\gamma_{X,G}(t)) = \rho(\gamma_G(t)) \cdot F(c(u))$$

where  $\rho : G \rightarrow A$  is a homomorphism.

This requires us to choose some  $A$  that can represent behaviors of  $G$ , for example same group  $A = G$ , then we can design  $A \curvearrowright Z$  and transport  $a_{\gamma_{X,G}}(t) = \rho(\gamma_G(t)) \in A$ , the PE condition in this case is

$$F(\gamma_G(t) \cdot c(u)) = \rho(\gamma_G(t)) \cdot F(c(u)).$$

Because the pose is inherent to the group  $G$ , we naturally have pose equivariance, as in Proposition 4.17.

**Proposition 4.29** (Reduce to classical pose equivariance). *Assume endpoint condition holds for pose paths: for all  $\gamma_{G,1}, \gamma_{G,2} : [0, 1] \rightarrow G$  with  $\gamma_{G,1}(0) = \gamma_{G,2}(0) = e$  and  $\gamma_{G,1}(1) = \gamma_{G,2}(1) = g$ :*

$$a_{\gamma_{X,G_1}}(1) = a_{\gamma_{X,G_2}}(1).$$

*Then there exists a continuous homomorphism  $\rho : G^0 \rightarrow A$  such that:*

$$F(g \cdot c(u)) = \rho(g) \cdot F(c(u)) \quad \forall g \in G^0, \forall u \in U_\alpha,$$

*which gives us classical pose equivariance. Moreover,  $a_{\gamma_{X,G}}(t) = \rho(\gamma_G(t))$  for any pose path.*

#### 4.4.2 Path Equivariance on Content space

The content space  $U = X/G$  lacks a group structure, thereby altering path equivariance. Since  $U$  is not a group, no homomorphism  $U \rightarrow A$  exists, and we cannot recover classical equivariance-type guarantees.

For content paths  $\gamma_{X,U}(t) = c(\gamma_U(t))$  where  $\gamma_U : [0, 1] \rightarrow U$ , we distinguish two cases:

1. Content-Invariant Encoder.

If  $F$  is constant across all content:

$$F(c(\gamma_U(t))) = F(c(\gamma_U(0))) = \text{constant}.$$

In PE, we choose  $A = \{e\}$  and  $a_{\gamma_{X,U}}(t) = e$ . This is rarely useful because it loses all content information.

## 2. Content-Consistent Encoder.

If  $F$  varies consistently as content changes, for content path  $\gamma_{X,U}(t) = c(\gamma_U(t))$ , PE condition:

$$F(c(\gamma_U(t))) = a_{\gamma_{X,U}}(t) \cdot F(c(\gamma_U(0)))$$

where  $a_{\gamma_{X,U}} : [0, 1] \rightarrow A$  with  $a_{\gamma_{X,U}}(0) = e$ .

**Remark 4.30.** In content space, no homomorphism from  $U$  to  $A$ , so the transportation  $a_{\gamma_{X,U}}(t)$  can depend on the specific path taken from  $u_0$  to  $u_1$ , not just endpoints. In other words, the paths matter in content space; different paths with the same endpoint remain different paths in latent space, which makes sense because the content identifies elements we want to recognise.

### 4.4.3 Soft and Hard constraint

The pose/content decomposition induces a fundamental dichotomy in how constraints are enforced:

- Hard constraints (pose): Group structure provides an equivalence relation where all points in an orbit  $\{g \cdot c(u) : g \in G\}$  represent the same content under different transformations. This equivalence enables weight sharing, for example, group-equivariant CNNs [9] and steerable filters [12], which generate the entire families of transformed filters from a single base filter, building exact equivariance  $F(g \cdot x) = \rho(g) \cdot F(x)$  directly into the architecture.
- Soft constraints (content): Manifold structure provides only geometric smoothness, not equivalence. Different contents  $c(u_1), c(u_2)$  are fundamentally distinct, precluding weight sharing (no systematic pattern relates processing of different contents). Instead, we desire smooth variation—nearby contents should map to nearby features. This is enforced through regularization (e.g.,  $\mathcal{L}_{\text{smooth}} = \int_U \|\nabla_u F\|^2$ ), which encourages but does not guarantee smoothness via loss function penalties during optimization.

Next, we will demonstrate how the smoothness constraint acts as regularization. If PE holds exactly:

$$F(c(\gamma_U(t))) = a_{\gamma_{X,U}}(t) \cdot F(c(\gamma_U(0))). \quad (4.1)$$

Differentiate both sides Equation 4.1 with respect to  $t$ :

$$\frac{d}{dt} F(c(\gamma_U(t))) = \frac{d}{dt} [a_{\gamma_{X,U}}(t) \cdot F(c(\gamma_U(0)))] . \quad (4.2)$$

Compute left side of Equation 4.2 (chain rule):

$$\frac{d}{dt} F(c(\gamma_U(t))) = dF_{c(\gamma_U(t))} \left( \frac{dc}{du} \cdot \frac{d\gamma_U}{dt} \right) = \nabla_u F(c(u))|_{u=\gamma_{U(t)}} \cdot \dot{\gamma}_U(t) . \quad (4.3)$$

Then for any content path  $\gamma_U(t)$  :

$$\left\| \frac{d}{dt} F(c(\gamma_U(t))) \right\| = \|\nabla_u F(c(u)) \cdot \dot{\gamma}_U(t)\| \leq \|\nabla_u F(c(u))\| \cdot \|\dot{\gamma}_U(t)\| \quad (4.4)$$

Compute right side of Equation 4.2:

$$\dot{a}_{\gamma_{X,U}}(t) \cdot F(c(\gamma_U(0))). \quad (4.5)$$

Combining Equation 4.5 and Equation 4.3 gives a differential constraint:

$$\nabla_u F(c(u)) \cdot \dot{\gamma}_U(t) = \dot{a}_{\gamma_{X,U}}(t) \cdot F(c(\gamma_U(0))). \quad (4.6)$$

Combine Equation 4.6 and Equation 4.4, for any content path  $\gamma_U(t)$  :

$$\|\dot{a}_{\gamma_{X,U}}(t) \cdot F(c(\gamma_U(0)))\| = \|\nabla_u F(c(u)) \cdot \dot{\gamma}_U(t)\| \leq \|\nabla_u F(c(u))\| \cdot \|\dot{\gamma}_U(t)\|. \quad (4.7)$$

Recall standard smoothness regularization in manifold learning (via auto-encoder):

$$\mathcal{L}_{\text{smooth}} = \int_{U_\alpha} \|\nabla_u F(c(u))\|^2 du,$$

which penalizes large gradients of  $F$  with respect to content  $u$ , thus encouraging  $F$  to vary smoothly across content space.

If  $\|\nabla_u F(c(u))\|$  is small, which means  $F(c(\gamma_U(t)))$  changes slowly with  $t$ , then  $\dot{a}_{\gamma_{X,U}}(t)$  is small, which means  $a_{\gamma_{X,U}}(t)$  varies slowly, gives a smoothly transport  $X \rightarrow Z$ .

Therefore, regularization ensures a special case of PE where the transport is smooth and bounded.

**Remark 4.31.** If a global section  $c$  does not exist, we need to cover  $U$  by charts  $\{U_\alpha\}$  with local sections  $c_\alpha$  and apply the construction per chart; on overlaps  $U_\alpha \cap U_{\alpha'}$ , the corresponding feature fields match through the induced gauge transform in  $A$ . More about multiple charts used in manifold learning is introduced in the chart autoencoder [54].