# 4 A Category Theory Perspective

## 4.1 Activation Functions

As we introduced in section 3, non-linear activation functions break symmetry structure in Linear Network in an invariant group view. This section we explore how non-linear activation functions change Linear network in categorical view.

**Definition 4.1** (Neural networks category)**.** We formalize neural networks as a category **NNet**:

Objects: Vector spaces representing layers

$$\mathrm{Ob}(\mathbf{NNet}) = \{\mathbb{R}^n : n \in \mathbb{N}\}.$$

Morphisms: Parametrized transformations between layers

$$\mathrm{Hom}\left(\mathbb{R}^m, \mathbb{R}^n\right) = \{f : \mathbb{R}^m \to \mathbb{R}^n : f \in \mathcal{F}\}.$$

where $\mathcal{F}$ denotes the parameter space.

Composition: Standard function composition

$$\left(f \circ g\right)(x) = f\left(g(x)\right).$$

**Definition 4.2** (Linear neural networks category)**.** Consider the subcategory **LinNet** of **NNet** where morphisms are restricted to linear maps:

$$\mathrm{Hom}_{\mathrm{Lin}}\left(\mathbb{R}^m, \mathbb{R}^n\right) = \left\{x \mapsto Wx : W \in \mathbb{R}^{n \times m}\right\}.$$

The category **LinNet** is equivalent to the category of matrices **Mat** and finite-dimensional real vector spaces **FinVect**.

TODO: **LinNet** is a category enriched over a monoidal category. Non-linear breaks monoidal, symmetry and closure.

## 4.2 Categorical Explanation of Depth Efficiency

This categorical framework provides insight into the “depth efficiency” phenomenon in network: Depth provides exponential expressivity ([cite]).

Linear Networks: In the category **LinNet**, consider a depth-$L$ network:

$$\mathbb{R}^{n_0} \xrightarrow{W_1} \mathbb{R}^{n_1} \xrightarrow{W_2} \cdots \xrightarrow{W_L} \mathbb{R}^{n_L}.$$

Due to closure, this composition collapses to a single transformation:

$$W_L \circ W_{L-1} \circ \cdots \circ W_1 = W \in \mathbb{R}^{n_L \times n_0}.$$

The morphism space at depth $L$ remains:

$$\mathrm{Hom}_{\mathrm{Lin}}^{L}\left(\mathbb{R}^{n_0}, \mathbb{R}^{n_L}\right) \cong \mathbb{R}^{n_L \times n_0},$$

with dimension $n_L \cdot n_0$ independent of depth.

Nonlinear Networks: With activations, the depth-$L$ network becomes:

$$\mathbb{R}^{n_0} \xrightarrow{\sigma \circ W_1} \mathbb{R}^{n_1} \xrightarrow{\sigma \circ W_2} \cdots \xrightarrow{\sigma \circ W_L} \mathbb{R}^{n_L}.$$

The lack of closure prevents collapse. The morphism space

$$\mathrm{Hom}_{\mathrm{NL}}^{L}\left(\mathbb{R}^{n_0}, \mathbb{R}^{n_L}\right)$$

has complexity scaling as $\mathcal{O}\left(k^L\right)$, growing exponentially with depth.

**Remark (No Free Lunch) [cite]** While depth provides exponential expressivity, the learnable subset might not grow exponentially. The categorical structure reveals what is representable, not necessarily what is learnable.



## 4.3 Categorical Framework for Equivariant Neural Networks

In this section, we reformulate equivariant neural networks using category theory, providing a general language for the principle that "an equivariant map preserves structure." Thus different types of symmetries (groups, monoids, categories) can be treated uniformly through the lens of functors and natural transformations.

For a monoid (or group) $S$, denote $\mathbf{B}S$ as one-object category with endomorphism $S$.

**Definition 4.3** ($S$-object)**.** Let $\mathbf{C}$ be a category and $S$ be a monoid. An $S$-object in $\mathbf{C}$ is a functor:

$$F : \mathbf{B}S \longrightarrow \mathbf{C},$$

where $F$ maps the unique object $* \in \mathbf{B}S$ to some object $F(*) \in \mathbf{C}$; maps each morphism $s : * \to *$ in $\mathbf{B}S$ to a morphism $F(s) : F(*) \to F(*)$ in $\mathbf{C}$

An $S$-object is an object in C equipped with an $S$-action. The category of all $S$-objects in $\mathbf{C}$ is the functor category $[\mathbf{B}S, \mathbf{C}]$. Intuitively, $S$ specifies the "shape of constraints" inside $\mathbf{C}$, an $S$-object is a diagram of that shape inside $\mathbf{C}$.

**Definition 4.4** ($S$-map)**.** Given two $S$-objects $F, E \in [\mathbf{B}S, \mathbf{C}]$, an $S$-map is a natural transformation

$$\alpha : F \Rightarrow E.$$

Explicitly, $\alpha$ consists of a morphism $\alpha_* : F(*) \to E(*)$ in $\mathbf{C}$ satisfying

$$\alpha_* \circ F(s) = E(s) \circ \alpha_*.$$

Equivalently, the following diagram :

$$\begin{tikzcd}
F(*) \arrow[r, "\alpha_*"] \arrow[d, "F(s)"'] & E(*) \arrow[d, "E(s)"] \\
F(*) \arrow[r, "\alpha_*"'] & E(*)
\end{tikzcd}$$

commute for all $s \in S$.

**Definition 4.5** ($S$-equivariant learner)**.** An $S$-equivariant learner is an $S$-map in a machine learning context:

- Input is data equipped with $S$-symmetry (an $S$-object $F$).
- Output respects this symmetry (an $S$-object $E$).
- Model is a natural transformation $\alpha : F \Rightarrow E$.

So the learner preserves the symmetry structure by design.

**Proposition 4.6** (Closure)**.** *The composition of $S$-maps is again an $S$-map. In particular, $S$-equivariant learners are closed under composition.*

### 4.3.1 Categorial $G$-map

In the general $S$-map framework, $S$ can be any monoid or category. A particularly important case in machine learning is when $S = G$ is a Lie group, providing the symmetries of interest (rotations, translations, etc.).

First we places Lie groups within the categorical framework.

Let **Top** be topological spaces category, and **Man** be smooth manifolds category.

**Proposition 4.7.** *A Lie group is a group object in the category* **Man** *of smooth manifolds.*

This means a Lie group consists of a smooth manifold $G$ and smooth maps satisfying:

- Multiplication: $m : G \times G \to G, (g, h) \mapsto gh$;

- Unit: $e : \{*\} \to G, * \mapsto e$;

- Inverse: $i : G \to G, g \mapsto g^{-1}$.

**Definition 4.8** ($G$-space and $G$-manifold)**.** Let $G$ be a Lie group.

A $G$-space, denoted as $(X, \rho_X)$, is a topology space $X$ together with a continuous action

$$\rho_X : G \times X \to X$$

satisfying identity $e \cdot x = x$ and compatibility $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G, x \in X$.

Similarly, a $G$-manifold, denoted as $(M, \rho_M)$, is a manifold $M$ together with a smooth action

$$\rho_M : G \times M \to M$$

satisfying the same identity and compatibility conditions.

We write $\mathbf{Top}^G$ for the category of $G$-spaces with $G$-equivariant continuous maps as morphisms; $\mathbf{Man}^G$ for the category of $G$-manifolds with $G$-equivariant smooth maps as morphisms.

**Definition 4.9** ($G$-equivariant map)**.** Given a $G$-space $(X, \rho_X)$ and a $G$-manifold $(M, \rho_M)$, a map $f : X \to M$ is $G$-equivariant map ($G$-map) if

$$f(g \cdot x) = g \cdot f(x) \quad \forall g \in G, x \in X.$$

Equivalently, this diagram

$$\begin{tikzcd}
G \times X \arrow[r, "\rho_X"] \arrow[d, "id_G \times f"'] & X \arrow[d, "f"] \\
G \times M \arrow[r, "\rho_M"'] & M
\end{tikzcd}$$

commutes.

**Definition 4.10** ($G$-equivariant learner)**.** Similarly, a $G$-equivariant learner is a $G$-map in a machine learning context processing data equipped with $G$-symmetry while preserving this symmetry structure.

**Conjecture 4.11** (Categorical Equivalence)**.** *Let $G$ be a discrete group (equipped with discrete topology) and $\mathbf{B}G$ be the one-object category with endomorphisms $G$.*

*Then there is an isomorphism of categories:*

$$\mathbf{Top}^G \cong [\mathbf{B}G, \mathbf{Top}].$$

*Similarly for manifolds (with discrete $G$ ):*

$$\mathbf{Man}^G \cong [\mathbf{B}G, \mathbf{Man}].$$

*If $G$ is not discrete, the isomorphism of categories degrade to equivalence of categories:*

$$\mathbf{Top}^G \simeq [\mathbf{B}G, \mathbf{Top}].$$

**Proposition 4.12** (Equivariance as Naturality)**.** *Let $F : \mathbf{B}G \to \mathbf{Top}$ and $E : \mathbf{B}G \to \mathbf{Man}$ be functors, with $U : \mathbf{Man} \to \mathbf{Top}$ the forgetful functor.*

*A $G$-equivariant map $f : F(*) \to E(*)$ (from the $G$-space $F(*)$ to the underlying topological space of $G$-manifold $E(*)$) is precisely a natural transformation:*

$$\alpha : F \Rightarrow U \circ E$$

*The naturality condition for each $g \in \mathrm{Mor}(\mathbf{B}G)$ is $f \circ F(g) = UE(g) \circ f$. Equivalently the commutative diagram:*

$$\begin{tikzcd}
F(*) \arrow[r, "f"] \arrow[d, "F(g)"'] & UE(*) \arrow[d, "UE(g)"] \\
F(*) \arrow[r, "f"'] & UE(*)
\end{tikzcd}$$

*Reading this diagram element-wise:*

$$f(g \cdot x) = g \cdot f(x) \quad \forall g \in G, x \in F(*)$$

*This is exactly the $G$-equivariance condition.*

**Conjecture 4.13** (Total symmetry group)**.** *Real data normally carries more than one group structure, for example, most image recognition tasks satisfy both translation and rotation symmetry. But according to definition, a $G$-space, or a functor category $[\mathbf{B}G, \mathbf{Top}]$ specifies one group. We need to pick one "total" group $G_{tot}$ that captures all the symmetries. The construction of $G_{tot}$ depends on individual symmetries interaction. Suppose $X$ carries actions of groups $G_1, \ldots, G_k$.*

*- If the actions commute on $X$, they assemble to a group of the direct product*

$$G_{\text{tot}} = G_1 \times \cdots \times G_k.$$

*- If $H$ relabels the $N$-action (e.g., rotations acting on translations), we use a semidirect product $G_{tot} = N \rtimes_\varphi H$ with $\varphi : H \to \mathrm{Aut}(N)$.*

*- If there is no relations between each $G_i$, we use the free product $G_{tot} = *_i G_i$.*

*- If structure of interest is not a group, we can use method introduce in section 4.*