Calculus - Single Variable Part1 -Functions

1. What is the domain of the function $\sqrt{2x - x^3}$?

Sol:

The value inside square root must be positive or 0

=> $2x - x^3 \geq 0$

=> $-x(x^2 - 2) \geq 0$

=> $-x\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) \geq 0$

=> $x\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) \leq 0$

=> $x \in \left(-\infty, -\sqrt{2}\ \right]$ or $x \in \left(0, \sqrt{2}\ \right]$

2. What is the domain of the function $\frac{x-3}{x^2-4}\ln x$?

Sol:

The denominator part must be non-zero by the rule of division. (2-a)
In addition, x must be > 0 for logarithmic function $\ln x$. (2-b)

=> From (2-a), we have $x^2 - 4 \neq 0$

=> $(x - 2)(x + 2) \neq 0$

=> $x \neq 2$ and $x \neq -2$

=> From (2-b), we have x > 0

=> From the result of (2-a) and (2-b), we have $x \in (0,2) \cup (2, \infty)$

3. What is the domain of the function $\arcsin\frac{x-2}{3}$?

Sol:

Recall that the range of sin(x) is [-1, 1].
Therefore, the domain function inverse $\arcsin(x)$ is [-1, 1]

=> $-1 \leq \frac{x-2}{3} \leq 1$

=> we have $-1 \leq \frac{x-2}{3}$ on the left-hand side, and $\frac{x-2}{3} \leq 1$ on the right-hand side.

=> $-1 \leq x$ on the left-hand side, and $x \leq 5$ on the right-hand side.

=> Put them together, we get $x \in [-1, 5]$

4. What is the range of the function $-x^2 + 1$?

Sol:

f(x) = $-x^2 + 1$ <= 1, and f(x) goes lower when $x \neq 0$

=> $f(x) \in (-\infty, 1)$

5. What is range of the function $\ln(1+x^2)$?

Sol:

Recall the logarithmic function $\ln x$ is increasing function from negative infinity to positive infinity.

From $1+x^2$, we have minimize value 1, and $1+x^2$ goes upper when $x \neq 0$

=> minimal value of $\ln(1+x^2) = \ln(1) = 0$

=> Thus, we have $f(x) \in [0, \infty)$

6. What is the range of the function arctan(cos(x)) ?
   (i.e., the inverse of the tangent function with the parameter cos x)

Recall the domain of tan(x) is $x \in \{x | x \neq k\pi + \frac{\pi}{2}, k \in Z\}$.
Also, the range of cos(x) is $[-1, 1]$

=> With the rule of thumb, $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{-\pi}{4}\right) = -1$

=> Range of arctan(cos(x)) $\in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

7. If $f(x) = 4x^3 + 1$ and $g(x) = \sqrt{x+3}$, compute (f o g)(x) and (g o f)(x),

Sol:

(f o g)(x) = f(g(x)) = f($\sqrt{x+3}$) = $4*\left(\sqrt{x+3}\right)^3 + 1 = 4 * (x+3)^{\frac{3}{2}} + 1$

(g o f)(x) = g(f(x)) = g($4x^3 + 1$) = $\sqrt{(4x^3+1)+3} = \sqrt{4x^3+4} = \sqrt{4(x^3+1)}$

$= 2\sqrt{x^3+1} = 2 \cdot (x^3+1)^{\frac{1}{2}}$

8. What is the inverse function $f(x) = e^{2x}$?

Sol:

$y = e^{2x}$

$=> \ \ln y = \ln e^{2^x} = 2x$

$=> \ x = \frac{\ln y}{2} = \frac{1}{2}\ln y = \ln y^{\frac{1}{2}} = \ln\sqrt{y}$

$=> \ f^{-1}(x) = \frac{1}{2}\ln x = \ln\sqrt{x}$