Calculus - Single Variable Part1 -Exponential

1. Using Euler's formula, compute the product $e^{ix} * e^{iy}$.
   What is the real part?
   Remember that $i^2 = -1$

Sol:

$e^{ix} * e^{iy} = e^{i(x+y)} = cos(x + y) + i\, sin(x + y)$

Re{ $e^{ix} * e^{iy}$ } = $cos(x + y)$
= $cos(x)\, cos(y) - sin(x)\, sin(y)$

Real part of $e^{ix} * e^{iy}$ is $cos(x)\, cos(y) - sin(x)\, sin(y)$

2. Let n be an integer. Using Euler's formula we have

$$e^{inx} = \cos(nx) + i\sin(nx)$$

On the other hand, we also have

$$e^{inx} = e^{ixn} = [cos(x) + i\, sin(x)]^n$$

Putting both expressions together, we obtain the de Moivre's formula

$$\cos(nx) + i\sin(nx) = [cos(x) + i\, sin(x)]^n$$

Use the latter to find expressions for $sin(3x)$ in terms of $sin(x)$ and $cos(x)$.

Sol:

e^{i3x}
$= cos(3x) + i\, sin(3x)$
$=[cos(x) + i\, sin(x)]^3$
$= cos^3(x) + 3\, cos^2(x) i\, sin(x) - 3\, cos(x)\, sin^2(x) - i\sin^3(x)$
$=[\, cos^3(x)\ - 3\, cos(x)\, sin^2(x)\,] + i\,[\, 3\, cos^2(x)\, sin(x) - \sin^3(x)]$

Compare imaginary part to derive $sin(3x)$, then we have

$sin(3x)$ = $3\, cos^2(x)\, sin(x) - \sin^3(x)$
= $3\, sin(x)\, cos^2(x) - \sin^3(x)$
= $3\, sin(x)\ \ [1 - \sin^2 x] - \sin^3(x)$ $\quad$ $by\ the\ Pythagorean\ \ identity:\ cos^2 x + sin^2(x) = 1$
= $3\, sin(x) - 4\, sin^3(x)$