Alternative Approaches to Significance Testing with Weighted Means

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The creators of statistical processing software for the marketing research community have confronted them with a variety of approaches in dealing with significance testing relating to weighted sample means. Each of these approaches produces a different standard error of the weighted sample mean, and thus a different test statistic. The purpose of this note is to sort through these approaches, explain their bases in as nontechnical way as I can, and make some recommendations.

We start with a random sample of n observations, say $x_1, x_2, \ldots, x_n$, drawn from a population. For some reason we wish to weight each of these observations, with weights $w_1, w_2, \ldots, w_n$, and calculate a weighted mean $\overline{x}_w$ given by the expression

$$\overline{x}_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Finally, we wish to perform some statistical test of significance using this weighted mean (e.g., compare it to some hypothetical value μ_0, or compare it to another weighted mean $\overline{y}_w$).

As we learned in our Statistics courses, it is necessary to determine the standard error of $\overline{x}_w$ to perform such a test. In general the variance of the weighted mean is

$$Var(\overline{x}_w) = \frac{\sum_{i=1}^{n} w_i^2 Var(x_i)}{(\sum_{i=1}^{n} w_i)^2}$$

If each of the x=s has the same variance, σ^2, then this reduces to

$$Var(\overline{x}_w) = \frac{\sum_{i=1}^{n} w_i^2 \sigma^2}{(\sum_{i=1}^{n} w_i)^2} = \frac{\sigma^2}{e}$$

where the "effective sample size" e is given by

$$e = \frac{(\sum_{i=1}^{n} w_i)^2}{\sum_{i=1}^{n} w_i^2}$$

Thus the standard error of the weighted mean $\overline{x}_w$ is given by $\sigma / \sqrt{e}$. (This is the analogue to the formula for the standard error of the unweighted mean, namely $\sigma / \sqrt{n}$.)

The problem for the market researcher arises because three different software systems, WinCross, SPSS, and CfMC=s Mentor, take different tacks in estimating σ^2. SPSS uses the estimator

$$s_w^2 = \frac{\sum_{i=1}^{n} w_i (x_i - \overline{x}_w)^2}{\sum_{i=1}^{n} w_i - 1}$$

The rationale behind this estimator is that one should treat each weight w_i as the number of times the observations x_i is to be "replicated" in the sample, and calculate the estimate of the variance the way one would if one had "replicated" data. (The w_i are not necessarily integers; SPSS uses the noninteger values of w_i in this part of the computation.)

Unfortunately, s_w^2 is a biased estimate of σ^2. SPSS has been aware of this for sometime, but unfortunately has not made the appropriate correction. Moreover, instead of using s_w^2/e as its estimate of the variance of $\overline{x}_w$, SPSS calculates a "weighted sample size" c, as the sum of the weights

$$c = \sum_{i=1}^{n} w_i$$

and uses s_w^2/c as its estimate of the variance of $\overline{x}_w$, further contributing to the bias of its estimate of the variance of $\overline{x}_w$.

The Mentor software corrects for the bias in s_w^2 and uses as its estimate of σ^2 the estimator

$$s_c^2 = \frac{(\sum_{i=1}^{n} w_i)\sum_{i=1}^{n} w_i (x_i - \overline{x}_w)^2}{(\sum_{i=1}^{n} w_i)^2 - \sum_{i=1}^{n} w_i^2}$$

It also correctly divides s_w^2 by e to estimate the variance of $\overline{x}_w$ as s_c^2/e.

WinCross uses the usual unweighted estimate of the variance, σ^2, namely

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

in its software. It, too, is an unbiased estimate of σ^2, and so WinCross estimates the variance of $\overline{x}_w$ as s^2/e.

How does one choose between the two unbiased estimates of σ^2, WinCross's s^2 and Mentor's s_c^2? By selecting the one that most closely estimates σ^2, that is, the one whose sampling distribution has the smallest variance. One can derive mathematically that the variance of s^2 is $2\sigma^4/(n-1)$ and that the variance of s_c^2 is

$$Var(s_c^2) = 2\sigma^4 \frac{\sum_{i=1}^{n} w_i^2 (\sum_{i=1}^{n} w_i)^2 - 2\sum_{i=1}^{n} w_i^3 \sum_{i=1}^{n} w_i + (\sum_{i=1}^{n} w_i^2)^2}{(\sum_{i=1}^{n} w_i)^4 - 2\sum_{i=1}^{n} w_i^2 (\sum_{i=1}^{n} w_i)^2 + (\sum_{i=1}^{n} w_i^2)^2}$$

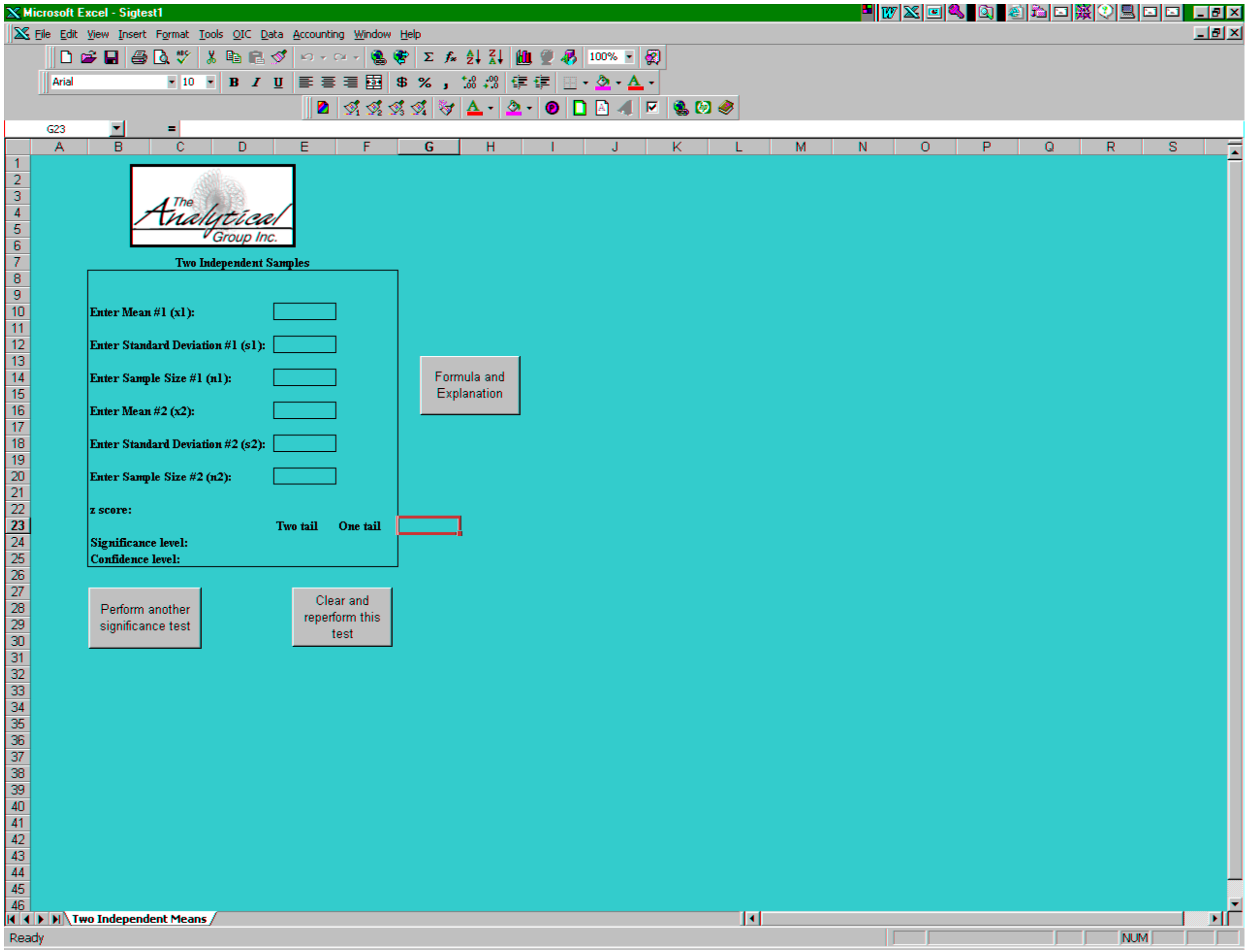
The derivation of these results, and the proof that $Var(s^2) \leq Var(s_c^2)$, so that WinCross's estimator is the preferred estimator of the variance of $\overline{x}_w$, can be found on The Analytical Group, Inc. website:

http://www.analyticalgroup.com/download/WEIGHTED MEAN.pdf

The following example illustrates how discrepant these two estimators will be. I selected as weights 100 random numbers from a uniform distribution between 0 and 1. These weights, along with their squares and cubes, are given on The Analytical Group, Inc. website. The variance of s^2, excluding the factor $2\sigma^4$, is 1/99= 0.010101. The variance of s_c^2, again excluding the factor $2\sigma^4$, is 0.014756. Thus use of s_c^2 will produce an estimate of the variance of $\overline{x}_w$ which is 1.46 times the variance produced using s^2 A simulation experiment using 1,000 replicates of 100 observations drawn from a single population and these weights, illustrating the three approaches and their resulting estimates of the variance of $\overline{x}_w$, can be found on The Analytical Group, Inc. website:

http://www.analyticalgroup.com/download/simulation.pdf

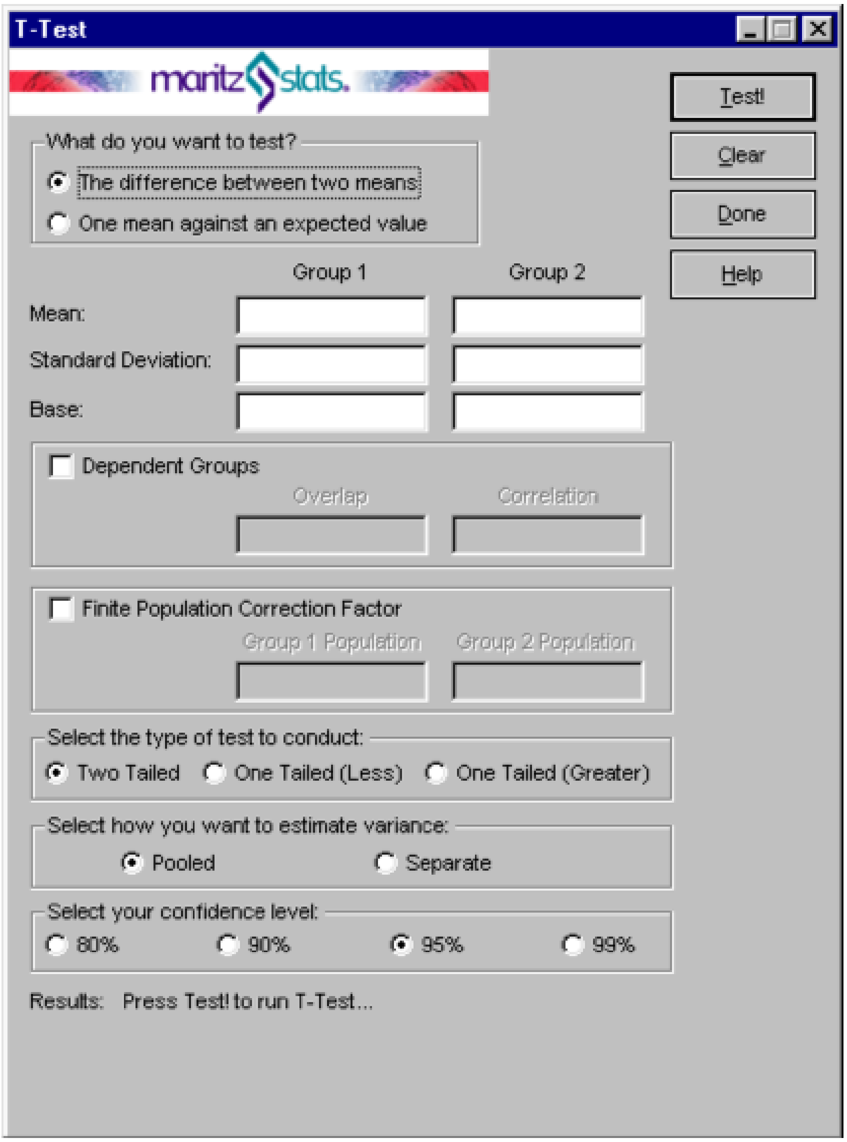
How does one use these statistics in standard statistical testing templates? Here is the T-Test template provided by The Analytical Group. To obtain a copy of this program visit: http://www.analyticalgroup.com/download/sigtest.xls



If one wanted to use this to compare weighted means, one need only fill in the weighted mean $\bar{x}_w$ in the box denoted **Enter Mean**, the unweighted standard deviation s in the box denoted **Enter Standard Deviation**, and the effective sample size e in the box denoted **Enter Sample Size** and you will have emulated the WinCross procedure for hypothesis testing using weighted means. (To emulate the Mentor procedure, use the bias-corrected weighted standard deviation s_c in the box denoted **Enter Standard Deviation** , and the effective sample size e in the box denoted **Enter Sample Size**. To emulate the SPSS procedure, use the biased weighted standard deviation s_w in the box denoted **Enter Standard Deviation**, and the weighted sample size c in the box denoted **Enter Sample Size**.)

And here is the T-Test template for using Maritz Stats. To obtain a copy of this program visit:

http://www.maritzresearch.com/contactformresearchstats.asp

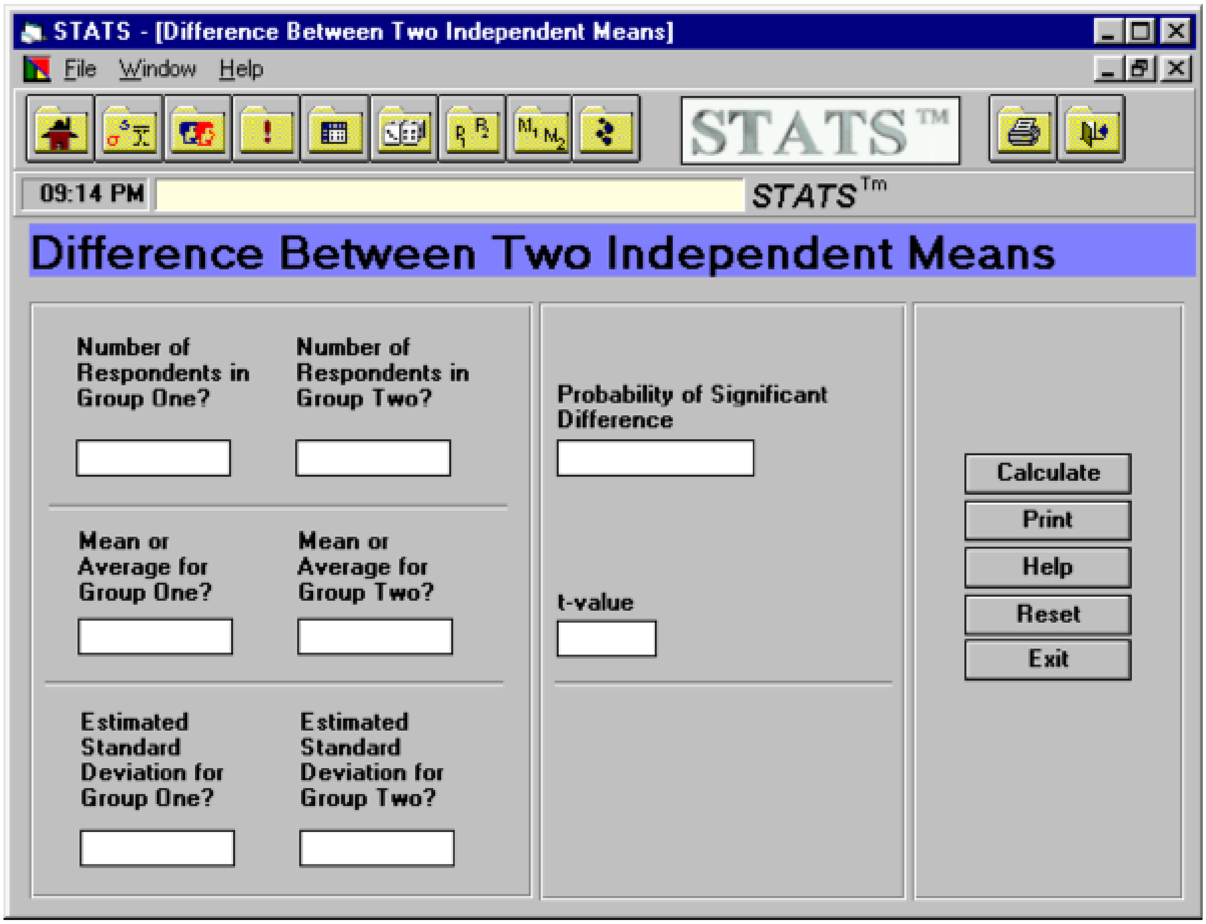


If one wanted to use this to compare weighted means, one need only fill in the weighted mean $\bar{x}_w$ in the box denoted **Mean:**, the unweighted standard deviation s in the box denoted **Standard Deviation:**, and the effective sample size e in the box denoted **Base:** and you will have emulated the WinCross procedure for hypothesis testing using weighted means. (To emulate the Mentor procedure, use the bias-corrected weighted standard deviation s_c in the box denoted **Standard Deviation:** , and the effective sample size e in the box denoted **Base:**. To emulate the SPSS procedure, use the biased weighted standard deviation s_w in the box denoted **Standard**

Deviation:, and the weighted sample size c in the box denoted **Base:**.)

Finally here is the T-Test template for using Decision Analyst=s STATS. To obtain a copy of this program visit

http://www.decisionanalyst.com/download.asp



If one wanted to use this to compare weighted means, one need only fill in the weighted mean $\bar{x}_w$ in the box denoted **Mean or Average**, the unweighted standard deviation s in the box denoted **Estimated Standard Deviation**, and the effective sample size e in the box denoted **Number of Respondents** and you will have emulated the WinCross procedure for hypothesis testing using weighted means. (To emulate the Mentor procedure, use the bias-corrected weighted standard deviation s_c in the box denoted **Estimated Standard Deviation**, and the effective sample size e in the box denoted **Number of Respondents**. To emulate the SPSS procedure, use the biased weighted standard deviation s_w in the box denoted **Estimated Standard Deviation**, and the weighted sample size c in the box denoted **Number of Respondents**.)

Given both the bias in the SPSS estimate of σ^2 and its incorrect denominator in determining the standard error of $\overline{x}_w$, the probabilities calculated based on the t-statistic will be incorrect. The probabilities based on both the WinCross and Mentor statistics will be correct, but, because Mentor uses an estimate of the variance of $\overline{x}_w$ with a larger variance than that of the estimate used by WinCross, it is more likely that one will find fewer significant differences using the Mentor procedure than using the WinCross procedure.