Verified non-recursive calculation of Beneš networks applied to Classic McEliece

Wrenna Robson[1] and Samuel Kelly[2]

[1] University of Bristol, Bristol, UK, wrenna.robson@bristol.ac.uk
[2] University of Bristol, Bristol, UK, sam.j.kelly.2020@bristol.ac.uk

Abstract. The Beneš network can be utilised to apply a single permutation to different inputs repeatedly. We present novel generalisations of Bernstein's [Ber20] formulae for the control bits of a Beneš network and from them derive an iterative control bit setting algorithm. We provide verified proofs of our formulae and prototype a a provably correct implementation in the Lean language and theorem prover. We develop and evaluate portable and vectorised implementations of our algorithm in the C programming language. Our implementation utilising Intel's Advanced Vector eXtensions 2 feature reduces execution latency by 25% compared to the equivalent implementation in the libmceliece software library.

Keywords: Computer aided cryptography, formal methods, efficient software implementation, permutation networks

1 Introduction

The secure and efficient implementation of permuting and sorting is a fundamental challenge in contexts such as privacy-preserving cloud computing [EGT10, OSF+16, DDCO17], the mitigation of side-channels Trusted Execution Environments (TEEs) [HOW22, SJG23], Fully Homomorphic Encryption (FHE) [GHS12, HKC+21], Multi-Party Computation (MPC) [HEK12, MS13, KS14], and post-quantum cryptography [BBC+20, ABC+22a].

Existing literature primarily focuses on the application of a single secret permutation to a single input. In this work, we instead focus on the question of how one securely and efficiently applies a single secret permutation to multiple different inputs. This is required, for example, in the decapsulation algorithm for the Classic McEliece Key Encapsulation Mechanism (KEM). Addressing this challenge prompted [BCS13], [Cho17], and [Ber20] to shed new light upon the theories of sorting networks [Bat68, Knu98] and permutation networks [Ben65, Lei14] and their high-assurance cryptographic implementation.

We motivate our work through the implementation challenges faced by the Classic McEliece team. Each decapsulation requires one to apply a secret permutation π to a q-bit string. Note that the obvious approach of using π to index into the string could leak information about the secret through the cache. They initially solve this problem in [BCS13] by adapting a common technique for obliviously applying a single permutation to a single input: the decorate-sort-undecorate idiom (see Subsection 2.1) with Batcher's odd-even sorting network [Bat68]. Briefly, if one records the $(m^2 + m + 4)2^{m-1} - 1$ comparator swaps applied by this sorting network to the permutation on 2^{m+1} elements π^{-1}, then 'replaying' this sorting network, with its comparators swapped for binary switches, is equivalent to applying π. Since a binary switch, or conditional flip, can be implemented in constant-time using only 4 bit operations, this is a simple constant-time solution. However, this solution is not optimal: all permutations can be realised through fewer binary switches.

The Beneš network [Ben64, Ben65, Ben75], or rearrangeable permutation network, is an arrangement of binary switches, connected by transmission links, that can be configured to realise arbitrary permutations to the input data passed through it. A binary switch has two inputs and two outputs. If the *control bit* assosciated with a given switch is set to 1, the positions of the two inputs are swapped and output. If the control bit is 0, the inputs are left in-place and output. The Beneš network realising a permutation on 2^{m+1} elements has $2m+1$ layers of 2^m switches. For the Classic McEliece parameter set where 2^{13} are permuted, this reduces the binary switches required to apply the permutation by two-thirds compared to the method described above. The Classic McEliece team specify the uniform random ordering of the field $\mathbb{F}_q$ as a sequence of control bits for the 'in-place' Beneš network [ABC+22b].

Utilising the Beneš network in decapsulation, however, adds the calculation of the control bits to the key generation phase. This is a complicated and relatively expensive computation with a large design space of algorithms and implementation characteristics to choose from. Bernstein implements the Nassimi-Sahni [NS82] fast parallel variant of the Waksman-Stone canonical 'looping' algorithm [Wak68]. This implementation is adopted in the libmceliece software implementation of Classic McEliece [Cho25]. However, as Bernstein notes, *"it takes effort to see why [some] algorithms work"* and why some do, or do not, give the same control bits as each other [Ber25b]. The 'looping' algorithm is recursive: one sets the control bits of the outermost layers so that the inner layers act seperately on two half-size disjoint permutations. Bernstein verifies this step with the HOL Light proof assistant [Har96] by providing formulae for the control bits of the outer layers [Ber20]. He proves that these formulae, when applied to an arbitrary permutation, do give outer layer permutations which when factored out leave a parity-preserving permutation.

By factoring this parity-preserving permutation into two permutations on odd and even inputs, Bernstein can proceed inductively. However, the Classic McEliece specification requires the control bits to be stored and applied following the in-place representation of the Beneš network (see Subsection 1.1). Therefore, the calculated control bits must be reordered to match this in-place form. There is a gap here. Bernstein does not prove that the specification's in-place application of the reordered control bits will indeed realise the permutation from which those control bits were calculated. Whilst a careful reader can check the reordering is valid and that the reference implementation of it is correct, this disconnect produces an algorithm that requires the delicate passing of position tracking variables between recursive calls.

The recursive construction also hampers the algorithm's implementation in the Single-Instruction Multiple-Data (SIMD) paradigm. The libmceliece software library provides an implementation designed for higher performance on Intel and AMD CPUs by utilising the Advanced Vector eXtensions 2 (AVX2) feature. The use of AVX2 accelerates several primitives in the Classic McEliece KEM, such as matrix and vector operations in $\mathbb{F}_q$. However, in the calculation of the control bits, the majority of recursive subcalls act on permutations with data sizes smaller than the width of AVX2's 256-bit ymm registers, resulting in suboptimal usage of the available hardware.

Contributions. In this paper, we further investigate the calculation of the control bits for the Beneš network. We present novel formulae for the control bits of a Beneš network, and use them to derive an iterative switch setting algorithm. We prove our formulae correct, and prototype a provably correct implementation in the Lean programming language and theorem prover [MU21]. We develop and evaluate portable and AVX2-enabled implementations of our algorithm. The latter reduces execution latency by 25% compared to the equivalent AVX2-enabled implementation in libmceliece version 20250507.

We build on prior work [Rob25] presenting the verified formulae as part of a wider monograph on the challenges in producing verified high-assurance cryptography. We

show that the promising initial conception of the algorithm presented there does result in a highly-performant software implementation. Further, we replicate Bernstein's proofs from [Ber20] with the Lean theorem prover and extend them: in our verification we prove that indeed the control bits he produces when interpreted correspond to the same permutation. Our algorithm uses a notion of 'interlacing' sorting networks and we provide a simple argument to evidence the validity of rearranging sorting networks to leverage SIMD instructions.

Structure. In Subsection 1.1, we provide an overview of Beneš networks. In Subsection 2.1, we cover preliminary material. In Subsection 2.2, we detail the recursive calculation of control bits. In Subsection 2.3, we present our novel iterative formulae and algorithm. In Section 3, we present our proofs and discuss their verification in Lean. In Section 4, we describe the implementation of our iterative algorithm and evaluate its performance. All associated material (e.g., source code relating to all hardware and software implementations) is available at `https://github.com/bristolcrypto/controlbits/` under the open source MIT license.

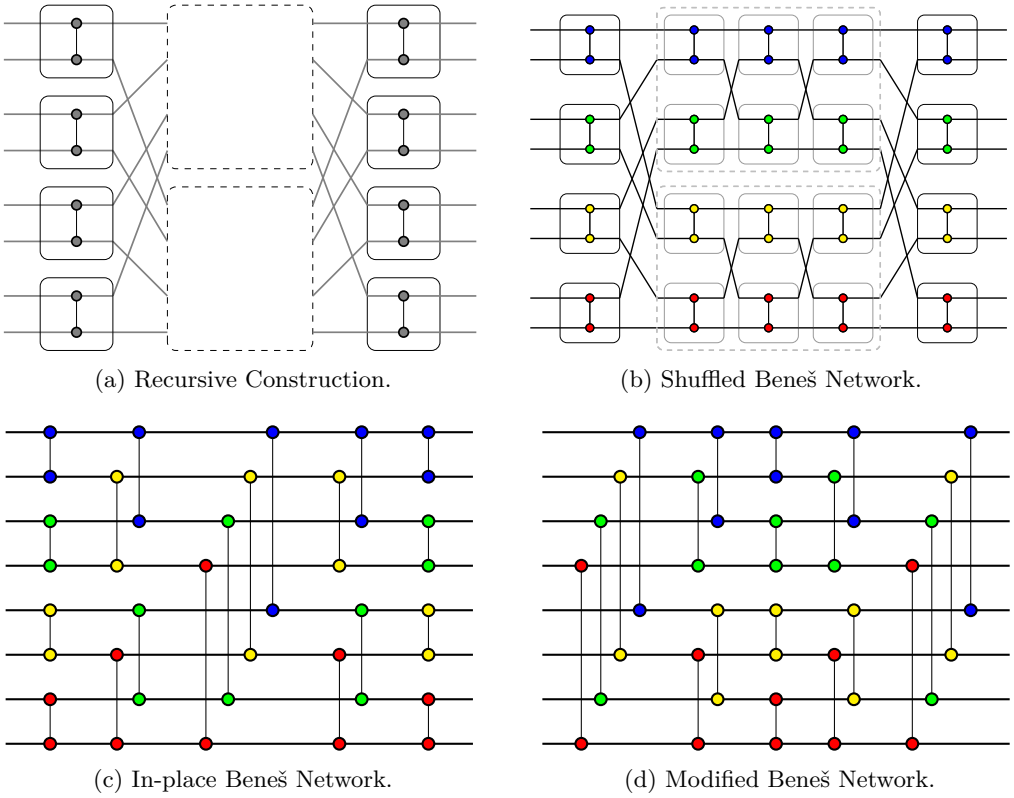
1.1 Beneš Networks

History. The first rearrangeable permutation networks were presented by Beneš [Ben64, Ben65, Ben75]. Later work by Waksman [Wak68], showed that a single switch could be spared at every step of the recursive construction of a Beneš permutation network; networks with this optimisation are often called Waksman networks. Chang and Melham show that Beneš networks can be extended to permute non-power of two sized lists [CM97], with Beauquier and Darrot showing the same result for Waksman networks specifically [BD99].

The canonical algorithm for calculating the control bits is the 'looping' algorithm of Waksman-Stone [Wak68]. In this work, Waksman presents a constructive proof, which he accredits to Stone, that the Beneš network is capable of applying any permutation on 2^{m+1} elements. Stone's proof leads directly to the aforementioned algorithm. The approach is to calculate the outermost stages of switches (via a 'looping' procedure) such that, after applying these two outer stages, the inner permutation of size 2^{m+1} is partitioned into two permutations of size 2^m; one then continues recursively. Fast parallel variants of this algorithm were introduced by Lev-Pippenger-Valiant [LPV81] in 1981, Nassimi-Sahni [NS82] in 1982, and Lee-Liew [LL96] in 1996. These parallel algorithms exploit a common idea, allowing them to compute the control bits of an outer layer in a number of steps logarithmic to the input permutation's size. Bernstein verifies the correctness of Nassimi-Sahni's algorithm and shows that it can be implemented obliviously using sorting networks [Ber20]. Our algorithm is an iterative variant of this 'looping' algorithm.

A different approach to calculating the control bits was introduced by Lee [Lee85, Lee87]. This approach distinct from the others as it does not calculate pairs of outer layers. Instead, it iteratively sets the control bits a single layer at a time from left to right. The construction divides the layers into two halves. The control bits in the first half are set so that the control bits in the second half can be set by a trivial switch setting algorithm. This algorithm was recently used as a basis for oblivious sorting and shuffling in the online/offline model [SJG23].

Visual representations of Beneš networks. Algorithms for calculating and storing control bits are often framed inside a particular visual representation of the Beneš network. Many of these representations are rearrangements of each other. We depict the common representations in Figure 1. The inputs follow the transmission links which pass from left to right. A switch between two transmission links is represented by a vertical line with



(a) Recursive Construction.

(b) Shuffled Beneš Network.

(c) In-place Beneš Network.

(d) Modified Beneš Network.

Figure 1: Representations of Beneš Networks with $n = 8$ inputs.

circles at its end points. The colours on the switches are to help visualise the rearrangements. We use the terms 'in-place' and 'shuffled' to match Bernstein's usage in [Ber20] and 'modified' to match the usage in [SJG23]. The recursive construction of the Beneš network for the looping procedure is depicted in Figure 1a. Continuing the recursion inside the dashed rectangles gives rise to Figure 1b. The in-place and modified representations, depicted in Figure 1c and Figure 1d respectively, can be realised by 'straightening out' the transmission links of the shuffled representation. To obtain the in-place representation, fix the positions of the outer switches of Figure 1b and align the end points of the outermost switches of the recursive halves to them. To obtain the modified representation, fix the positions of the middle-most switches of Figure 1b and align the end points of their adjacent switches to them.

The control bits are typically stored in an order that corresponds with a given visual representation. For each of the representations, one stores the control bits ordered in layers from left to right and within layers from top to bottom. The Classic McEliece specification stores the control bits according to the in-place representation. This complicates the recursive libmceliece implementation as it calculates the control bits according to a depth-first traversal of the decomposed halves of the shuffled representation, whilst simultaneously storing them in their in-place representation's order.

From an implementation perspective, there is no need to strictly follow any given visual representation of the Beneš Network. One is free to rearrange both the memory addresses of the elements to be compared and the order of application of conditional swaps (within each layer). When implementing the Beneš network in hardware, one may wish to minimise the wire length and choose the shuffled Beneš network. In software, the in-place and modified representations are chosen to remove the perfect shuffle transmission links

between layers. The representation chosen determines the order of memory accesses and has a large impact on performance. For example, Sasy et al. find their implementation of Lee's algorithm with the modified representation is *'one to two orders of magnitude faster'* than the implementation of Nassimi-Sahni's algorithm in libmceliece with the in-place representation. Yet, when the same algorithm is ran with the modified representation, the difference is *'negligible'* [SJG23, Sec. 7.1]. Further, Chou's optimised software implementation of control bit application swaps between the in-place and modified representations in order to always apply 128 conditional swaps in parallel [Cho17].

2 Formulae for the in-place Beneš network

2.1 Preliminaries

Notation. We denote the exclusive-or, or XOR, operation with $\oplus$. Let $n \in \mathbb{N}$ be a nonnegative integer and $\mathbb{N}_{<n}$ denote the set of nonnegative integers less than n. Let V, U be arrays of length n. We denote the i-th entry of an array V by $\mathsf{V}[i]$. We use the following slice notation $\mathsf{V}[l : u : s]$ to denote the collection of elements in V of the form $\mathsf{V}[l + i \cdot s]$ for $0 \leq i < u/s$. We define the array $\min(\mathsf{U}, \mathsf{V})$ such that $\min(\mathsf{U}, \mathsf{V})[i] = \min(\mathsf{U}[i], \mathsf{V}[i])$. The array given by concatenating arrays V and U is given by $\mathsf{V} \| \mathsf{U}$. We denote by S_n the group of permutations on $\mathbb{N}$ which are fixed for all inputs $x \geq n$. Equivalently, this is the symmetric group on n elements $\{0, \ldots, n-1\}$. Transpositions are cycles that swap two elements, denoted (i, j). We denote the commutator of two group elements g, h as follows: $[g, h] := g \cdot h \cdot g^{-1} \cdot h^{-1}$. For a permutation $\pi \in S_n$, we define $\mathsf{Vec}(\pi)$ to be the array of length n such that $\mathsf{Vec}(\pi)[i] = \pi(i)$. If an array A represents a permutation, i.e., $\mathsf{A} = \mathsf{Vec}(\pi)$ for some π, then we define the reverse conversion $\pi = \mathsf{Perm}(\mathsf{A})$ We denote the identity permutation by $\mathbb{I}_n \in S_n$ and its corresponding array $\mathsf{id}_n = \mathsf{Vec}(\mathbb{I}_n)$. An involution $g \in S_n$ satisfies $g^2 = \mathbb{I}_n$. Finally, given a permutation $\pi \in S_n$ acting on an element x, we denote the minimum member of the cycle of π containing x as $\mathrm{cyclemin}(\pi)(x) = \min\{\pi^k(x) \mid k \in \mathbb{Z}\}$. Observe that for any π, $\mathrm{cyclemin}(\pi)(\pi(x)) = \mathrm{cyclemin}(\pi)(x)$.

Bit Operations. Given a nonnegative integer x, we access the k-th bit of x by the function $\mathrm{bit}_k : \mathbb{N} \to \{0, 1\}$. The least significant bit is given by $\mathrm{bit}_0(x)$ — this is the parity bit of x.

Definition 1. We define the following operations (here given arithmetically rather than as bit operations):

$$\mathrm{insertbit}_k(b, x) := 2^{k+1} \left\lfloor \frac{x}{2^k} \right\rfloor + \left(b \cdot 2^k\right) + \left(x \bmod 2^k\right).$$
$$\mathrm{removebit}_k(x) := 2^k \left\lfloor \frac{x}{2^{k+1}} \right\rfloor + \left(x \bmod 2^k\right).$$

The function $\mathrm{insertbit}_k(b, x) : \{0, 1\} \times \mathbb{N} \to \mathbb{N}$ inserts the bit b into bit position k of x, leaving the k-th least significant bits in-place and left shifting the most significant bits above the inserted bit. Conversely, the function $\mathrm{removebit}_k : \mathbb{N} \to \mathbb{N}$ removes the k-th bit of an integer, shifting the higher bits down. Observe that for all x, $\mathrm{insertbit}_k(\mathrm{bit}_k(x), \mathrm{removebit}_k(x)) = x$.

Definition 2. We define $\mathrm{flip}_k : \mathbb{N} \to \mathbb{N}$ as the map defined by $\mathrm{flip}_k(x) := x \oplus 2^k$: the involutive permutation that flips the k-th bit of its input.

Note that for all x, $\mathrm{flip}_k(x) = \mathrm{insertbit}_k(1 \oplus \mathrm{bit}_k(x), \mathrm{removebit}_k(x))$ and that $\mathrm{flip}_k(x) \equiv x \bmod 2^k$. As such, when $k \leq m$, $\mathrm{flip}_k \in S_{2^{m+1}}$. In this case we can view it it as the product of 2^m transpositions of the form $(\mathrm{insertbit}_k(0, j), \mathrm{insertbit}_k(1, j))$ for $0 \leq j < 2^m$.

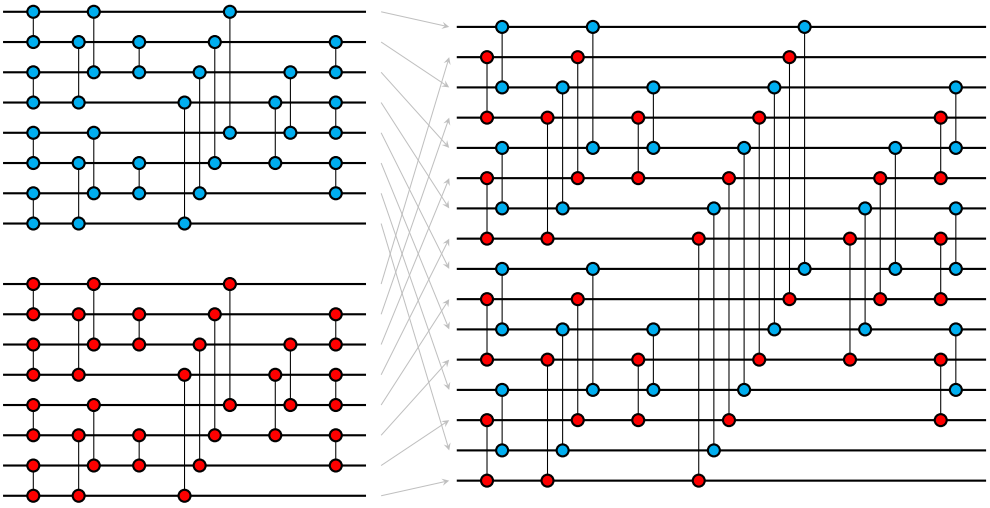


Figure 2: Interlacing two instances of Batcher's Odd-Even Merge sorting network on $n = 8$ elements.

Definition 3. We define $\mathrm{cflip}_k : \{0,1\} \times \mathbb{N} \to \mathbb{N}$ as the map given by $\mathrm{cflip}_k(b, x) := x \oplus b \cdot 2^k$. This is a conditional version of flip_k.

Note that for fixed b, $\mathrm{cflip}_k(b, \cdot) = (\mathrm{flip}_k)^b$ — it is also a permutation. We have $\mathrm{cflip}_k(b, x) = \mathrm{insertbit}_k(b \oplus \mathrm{bit}_k(x), \mathrm{removebit}_k(x))$ and $\mathrm{bit}_k(\mathrm{cflip}_k(b, x)) = b \oplus \mathrm{bit}_k(x)$.

Oblivious Application of Permutations. Given an array A and permutation π, the permuted array whose i-th entry is $\mathsf{A}[\pi(i)]$ is denoted by $\mathsf{A} \triangleleft \pi$. In Classic McEliece the permutation is secret, thus, to avoid the risk of leaking information about π through side-channels such as the cache, we never compute $\mathsf{A} \triangleleft \pi$ by using π to index into A. We instead apply permutations obliviously. For the particular case where π is a conditional transposition $(i, j)^b$, where $b \in \{0, 1\}$ is secret but i and j are not, we define the primitive $\textsc{oFlip}(\mathsf{A}, i, j, b)$. Using a short sequence of bit operations or instructions such as Intel's `cmovg`, we can obliviously swap elements $\mathsf{A}[i]$ and $\mathsf{A}[j]$ just when the bit $b = 1$. Therefore, we have $\textsc{oFlip}(\mathsf{A}, i, j, b) = \mathsf{A} \triangleleft (i, j)^b$. This primitive is utilised to apply the Beneš network.

For an arbitrary permutation π, we can calculate $\mathsf{A} \triangleleft \pi^{-1}$ with the decorate-sort-undecorate idiom and a data-oblivious sorting algorithm (for which, in this paper, we use sorting networks). The process is as follows: From A, form the decorated array B of tuples, whose i-th member is $(\mathsf{Vec}(\pi)[i], \mathsf{A}[i])$. Sort B in-place lexicographically. Undecorate B, returning the array whose i-th element is equal to the right element of the i-th tuple of B which is $\mathsf{A} \triangleleft \pi^{-1}$. We can calculate $\mathsf{A} \triangleleft \pi$ by first applying this process with $\mathsf{A} = \mathsf{id}$, calculating $\mathsf{id} \triangleleft \pi^{-1} = \mathsf{Vec}(\pi^{-1})$ and, then, using this to calculate $\mathsf{A} \triangleleft (\pi^{-1})^{-1} = \mathsf{A} \triangleleft \pi$.

For certain highly structured permutations π, whose structure is parameterised by a nonnegative integer k, we below define *'interlaced'* sorting networks and define an oblivious permutation primitive $\textsc{oSort}(\mathsf{A}, \mathsf{Vec}(\pi), k)$ which calculates $\mathsf{A} \triangleleft \pi^{-1}$. The decorate-sort-undecorate idiom is preferred for constant-time sampling of permutations in Classic McEliece, PERK [ABB+24], and [Beu20] as it is much more efficient than constant-time Fisher-Yates shuffling (see [BCMP24]).

Interlaced sorting networks. Sorting networks are the default practical choice for implementing data-oblivious sorting. They are fixed, data-independent sequences of compare-and-exchange modules, or comparators, designed to sort a fixed number of inputs [Bat68,

Algorithm 1 Obliviously apply one layer of the in-place Beneš network.

1: **function** oLayer(A: 2^{m+1} element array, C: array of 2^m control bits, k: gap size)
2: **for** $j = 0$ to $2^m - 1$ inclusive **do**
3: $\mathsf{A} \leftarrow \text{oFlip}(\mathsf{A}, \text{insertbit}_k(0, j), \text{insertbit}_k(1, j), \mathsf{C}[j])$
4: **end for**
5: **return** A
6: **end function**

Algorithm 2 Apply the control bits of an in-place Beneš network.

1: **function** cbapply(A: 2^{m+1} element array to be permuted, C: array of $(2m+1)2^m$ control bits)
2: **for** $j = 0$ to $2m + 1$ inclusive **do**
3: $k \leftarrow \min(j,\ 2m - j)$
4: $\mathsf{A} \leftarrow \text{oLayer}(\mathsf{A},\ \mathsf{C}[j \cdot 2^m : (j+1) \cdot 2^m : 1],\ k)$
5: **end for**
6: **return** A
7: **end function**

Knu98]. The Bitonic [Bat68], Odd-Even Merge [Bat68], and Pairwise [Par92] constructions all require asympotically $\mathcal{O}(n \log^2 n)$ comparators, and remain the most practical for efficient implementation [Ebb15]. Bitonic is favoured in SIMD implementations for its regular structure [Bra17, BGK21, Ber24].

We describe in Section 2 that the input to the k-th iteration of our iterative algorithm is a permutation $\pi \in S_{2^{m+1}}$ satisfying, $\pi(x) \equiv x \mod 2^k$ for all x. This is a highly structured permutation: π is independently permuting 2^k interlaced subsets of size 2^{m+1-k}. Therefore, when utilising the decorate-sort-undecorate idiom to apply this permutation, we do not need to apply a full sorting network on 2^{m+1} elements as many of the comparators are redundant. We can instead utilise a comparator network created from 2^k interlaced sorting networks on 2^{m+1-k} elements. A visualisation of an interlaced sorting network for any array A on 16 elements satisfying $\mathsf{A}[x] \equiv x \mod 2$ is given in Figure 2.

For $\pi \in S_{2^{m+1}}$, we define $\text{oSort}(\mathsf{A}, \mathsf{Vec}(\pi), k)$ to denote the primitive that performs this interlaced decorate-sort-undecorate process. This performs 2^k simultaneous sorts of size 2^{m+1-k}: when $\pi(x) \equiv x \mod 2^k$ for all x, this indeed performs the decorate-sort-undecorate procedure as described above, calculating $\mathsf{A} \triangleleft \pi^{-1}$.

The layers of the in-place Beneš Network. The Classic McEliece specification applies the control bits in the in-place Beneš network representation [ABC+22a]. For a size 2^{m+1} array A with control bits C, we sequentially apply $\text{oFlip}(\mathsf{A}, \text{insertbit}_k(0, j), \text{insertbit}_k(1, j), \mathsf{C}[j])$ for $0 \le j < 2^m$. Algorithm 1 defines this operation. Note that we can also view this as a family of permutations applied to the indicies: the product of (conditional) transpositions $(\text{insertbit}_k(0, j), \text{insertbit}_k(1, j))^{C[j]}$. We call this permutation family layer_k: it is involutive, because all of the transpositions commute with one another (as insertbit_k is injective, there are no overlapping values) and as such when applied twice they all cancel to the identity.

Definition 4. We define the permutation family $\text{layer}_k : \{0,1\}^{2^m} \to S_{2^{m+1}}$ as

$$\text{layer}_k(\mathsf{C}) := \prod_{j=0}^{2^m-1} (\text{insertbit}_k(0, j), \text{insertbit}_k(1, j))^{C[j]}.$$

We then have $\text{oLayer}(\mathsf{A}, \mathsf{C}, k) = \mathsf{A} \triangleleft \text{layer}_k(\mathsf{C})$: oLayer contains precisely the sequence

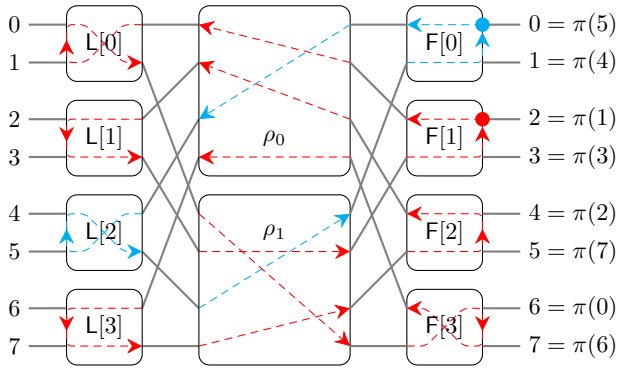


Figure 3: Example Calculation of the 'Looping' Algorithm.

of conditional transpositions that defines layer_k. It is convenient to have a closed form for the application of layer_k to a particular value.

Lemma 1. *For all* x, $\text{layer}_k(\mathsf{C})(x) = \text{cflip}_k\left(\mathsf{C}[\text{removebit}_k(x)], x\right)$

Proof. In the product of permutations in the definition of $layer_k$, each transposition appears exactly once. We have $x = \text{insertbit}_k(\text{bit}_k(x), \text{removebit}_k(x))$: as such it is exactly the transposition with $j = \text{removebit}_k(x)$ that x appears in. In this cases, the j-th bit of x is flipped just when $\mathsf{C}[\text{removebit}_k(x)]$ is set: i.e. the result is $\text{insertbit}_k(\mathsf{C}[\text{removebit}_k(x)] \oplus \text{bit}_k(x), \text{removebit}_k(x))$. This is exactly $\text{cflip}_k\left(\mathsf{C}[\text{removebit}_k(x)], x\right)$. □

The in-place Beneš network on 2^{m+1} elements is the product of $2m+1$ such layer applications, each layer represented by a array of 2^m bits, thus to represent the entire network requires an array of size $(2m+1) \cdot 2^m$. Algorithm 2 describes, given a size 2^{m+1} array A and an array of $(2m+1) \cdot 2^m$ control bits C, how to right-apply the permutation represented by the control bits C to the permutation A. This is done by sequentially applying the layers contained in C. This can be viewed as a permutation on the indices.

2.2 Recursive calculation of control bits

One of our primary conditions is a careful generalisation of the proofs contained in [Ber20], which the author himself describes as "*4 pages of definition-theorem-proof in brutalist style*" [Ber25b]. We favour a similarly brutalist style, as it naturally translates from our verified Lean implementation. However, to aid intuition, in this subsection we motivate this formal approach with a brief description of the algorithms of Waksman-Stone [Wak68] and Nassimi-Sahni [NS82], on which Bernstein's work is based. These algorithms choose the control bits for the outermost layers, denoted by arrays F and L, in order to factor our permutation $\pi \in S_{2^{m+1}}$ into the outer layers and a parity-preserving inner permutation ρ, i.e $\pi = \text{layer}_0(\mathsf{F}) \cdot \rho \cdot \text{layer}_0(\mathsf{L})$.

The 'looping' algorithm. The proof presented in [Wak68] leads directly to the 'looping' algorithm for determining the outer layers. To demonstrate this procedure, consider an example using the following permutation $\pi \in S_8$, given by $\pi = (0, 6, 7, 5)(1, 2, 4)$. Figure 3 is a standard depiction of this procedure. For the network to realise π, we must connect each input x on the left with its output $\pi(x)$ on the right. We are allowed to set the switches in the outermost layers and form arbitrary connections within ρ_0 and ρ_1. We begin the 'looping' procedure at output $0 = \pi(5)$ and arbitrarily set $\mathsf{F}[0] := 0$, i.e., set the switch straight by Waksman's switch saving scheme. We must then connect output 0 to its corresponding input $\pi^{-1}(0) = 5$, requiring us to pass through ρ_0 and then set $\mathsf{L}[2] := 1$.

This, in turn, connects input 4 to ρ_1 and we finish the loop at the switch we began. The blue path represents this loop. We now begin another loop, depicted in red, at the least unset output $2 = \pi(1)$, setting switch $\mathsf{F}[1] := 0$ again by Waksman's switch saving scheme. We continue looping backwards and forwards until the loop is complete. If not all of the control bits are set, we begin another loop at the least unset switch. Two proofs of the validity of this approach can be found in [Wak68, Lei14].

Cycle minima calculation Nassimi and Sahni observe that this approach of following cycles sequentially is parallelisable [NS82]. Instead of starting the loops at the least unset control bit and completing the entire cycle before continuing on, one can start the loop from an arbitrary output and record the minimum value of the cycle. The switch at this value must have been set to zero by Waksman's switch saving scheme. Nassimi and Sahni show that one can set all the control bits in terms of the parity of these cycle minima. Whilst calculating every cycle multiple times appears wasteful, the cycles are sufficiently structured so that, in each cycle, we need only take the minima over a logarithmic number of cycle indices to obtain the minima over the entire cycle. The parallelisation results in an algorithm that is natural to implement in constant-time. Calculating every cycle in parallel requires one to simply apply the permutation represented by an array to itself. This is done with the decorate-sort-undecorate idiom. Conversely, the naive sequential following of loops accesses memory dependent on secrets. This is a challenge faced in the constant-time implementation of Lee's switch setting algorithm. This algorithm has a similar loop-like process but does not benefit from the logarithmic speed up when similarly parallelised. A 'scramble-then-compute' approach to implementing the sequential following of loops in constant-time is presented in [SJG23].

We calculate $\mathrm{cyclemin}([\pi, \mathrm{flip}_k])$ using the algorithm FastCycleMin(π, k), taken in this form from Bernstein but based on Nassimi-Sahni's work. (Note that FastCycleMin calculates the cycle minima associated with a commutator, not the original permutation).

Recursive algorithm description. Algorithm 3 describes the recursive algorithm utilised by the Classic McEliece implementers, deriving from Bernstein's formulae as we describe below, for calculating the control bits for the in-place Beneš network. Observe that the recursive control flow requires a non-trivial usage of variables `pos` and `step` to ensure the control bits are stored in their in-place representation's positions. The first call to cbrecursion must set `pos` to 0 and `step` to 1. The two recursive calls operate on overlapping regions of memory (though there are no actual race conditions).

As mentioned, Bernstein defines formulae representing the control bit settings as they are given by Nassimi-Sahni, from which this algorithm is derived. Specifically, if $m = 0$ there is just a single control bit which can be read off from the parity of the first element of $\mathsf{Vec}(\pi)$. For $m > 0$, Bernstein then provides formulae for the control bits of the two outer layers F and L. He defines $\mathsf{F}[j] := \mathrm{cyclemin}([\pi, \mathrm{flip}_0])(2j) \mod 2$ and $\mathsf{L}[j] := \mathrm{layer}_0(\mathsf{F})(\pi(2j)) \mod 2$. These formulae are derived from the 'looping' procedure of Waksman-Stone by representing it as the commutator element $[\pi, \mathrm{flip}_0]$. (Note this represents following the loops clockwise, instead of anticlockwise as in Figure 3 above).

He then defines a permutaion $\rho = \mathrm{layer}_0(\mathsf{F}) \cdot \pi \cdot \mathrm{layer}_0(\mathsf{L})$ which he proves is parity-preserving. He is then able to define $\rho_0, \rho_1 \in S_{2^m}$ such that they satisfy $\rho(2x) = 2\rho_0(x)$ and $\rho(2x+1) = 2\rho_1(x)+1$. Proceeding inductively, he can calculate control bits for these permutations. The resulting control bits are not in the order required for in-place application, due to this recursive construction, and they must be reassembled via interlacing (or, equivalently, the two recursive steps must operate on overlapping regions of memory).

Algorithm 3 Recursively calculate the control bits for an in-place Beneš network.

Require: Array P such that $\mathsf{P}[x] = \pi(x)$ for some $\pi \in S_{2^{m+1}}$. A global array C of size $(2m+1) \cdot 2^m$ for the control bits.

1: **function** CBRECURSION(P, C, $\mathtt{pos}$, $\mathtt{step}$)
2: **if** $m = 0$ **then** ▷ Base case
3: $\mathsf{C}[\mathtt{pos}] \leftarrow \mathsf{P}[0]$
4: **return**
5: **end if**
6: $\mathsf{C}_{\min} \leftarrow$ FASTCYCLEMIN($\mathsf{Perm}(\mathsf{P})$, 0)
7: **for** $j = 0$ to $2^m - 1$ inclusive **do**
8: $\mathsf{F}[j] \leftarrow \mathrm{bit}_0(\mathsf{C}_{\min}[2j])$ ▷ $\mathsf{F}[j] = \mathrm{cyclemin}([\pi, \mathrm{flip}_0])(2j) \mod 2$.
9: **end for**
10: $\mathsf{P} \leftarrow$ OSORT($\mathrm{layer}_0(\mathsf{F})$, OSORT($\mathsf{id}_n$, P, 0), 0) ▷ Set $\mathsf{P}[x] = \mathrm{layer}_0(\mathsf{F})(\pi(x))$
11: **for** $j = 0$ to $2^m - 1$ inclusive **do**
12: $\mathsf{L}[j] \leftarrow \mathrm{bit}_0(\mathsf{P}[2j])$ ▷ $\mathsf{L}[j] = \mathrm{layer}_0(\mathsf{F})(\pi(x)) \mod 2$.
13: **end for**
14: $\mathsf{C}[\mathtt{pos} : \mathtt{pos} + 2^m \cdot \mathtt{step} : \mathtt{step}] \leftarrow \mathsf{F}$
15: $\mathsf{C}[\mathtt{pos} + m \cdot 2^{m+1} \cdot \mathtt{step} : \mathtt{pos} + (2m+1) \cdot 2^m \cdot \mathtt{step} : \mathtt{step}] \leftarrow \mathsf{L}$
16: $\mathsf{P} \leftarrow$ OLAYER($\mathsf{P}, \mathsf{L}, 0$) ▷ Set $\mathsf{P}[x] = \mathrm{layer}_0(\mathsf{F})(\pi(\mathrm{layer}_0(\mathsf{L})(x)))$
17: **for** $j = 0$ to $2^m - 1$ inclusive **do**
18: $\mathsf{P}_0[j],\ \mathsf{P}_1[j] \leftarrow \left\lfloor \frac{\mathsf{P}[2j]}{2} \right\rfloor, \left\lfloor \frac{\mathsf{P}[2j+1]}{2} \right\rfloor$
19: **end for**
20: CBRECURSION(P_0, C, $\mathtt{pos} + 2^m \cdot \mathtt{step}$, $2 \cdot \mathtt{step}$)
21: CBRECURSION(P_1, C, $\mathtt{pos} + (2^m + 1) \cdot \mathtt{step}$, $2 \cdot \mathtt{step}$)
22: **end function**

2.3 Iterative calculation of control bits

The first iteration of the loop of our iterative Algorithm 4 calculates the same outer layers and parity-preserving permutation using the same steps as Algorithm 3. The sole difference is that we choose not to decompose the parity-preserving permutation. Instead, we leave it in-place and adapt the calculations for the outer layers. Consider the decomposition map giving P_0 and P_1 in line 18 of Algorithm 3. When passed into the recursive subcalls, we study the cycle minima of their commutators with flip_0. Yet, if we never decomposed them, we could equivalently consider the cycle minima of the commutator of P and flip_1. Similarly, the sorting networks required in the subcalls to apply OSORT to P_0 and P_1 can be interlaced, as in Figure 2, to give a comparator network that sorts P. Combining the two cycle minima calculations into one also implies we calculate all of the second layer's control bits before calculating any control bits for deeper layers. Therefore, we can apply the layers represented by these control bits, thus factorising our parity-preserving permutation again. In the same fashion that P_0 and P_1 will become parity preserving permutations, our in-place P, after being factorised, will satisfy $\mathsf{P}[x] \equiv x \mod 4$.

Our iterative algorithm is the natural extension of these observations. Upon completing the k-th iteration, we obtain a factorised permutation in array P satisfying $\mathsf{P}[x] \equiv x \mod 2^k$. Algorithm 4 describes the function CBITERATIVE. Definition 5 below gives our formulae for the control bits, which correspond to the algorithm. The theorems and corollaries that follow in Section 3 are our main results which show the correctness of these formulae. Note that Bernstein's formulae and ours agree when $k = 0$ — the distinction is between what happens next.

There are three primary differences between Algorithm 3 and Algorithm 4. Firstly and

Algorithm 4 Iteratively calculate the control bits for an in-place Beneš network.

Require: Array P such that $\mathsf{P}[x] = \pi(x)$ for some $\pi \in S_{2^{m+1}}$.

function CBITERATIVE(P)
  Initialise array C of size $(2m+1) \cdot 2^m$ for storing control bits.
  for $k = 0$ to $m-1$ inclusive **do**
    $\mathsf{C}_{\min} \leftarrow$ FASTCYCLEMIN($\mathsf{Perm}\,(P)$, k)
    for $j = 0$ to $2^m - 1$ inclusive **do**
      $\mathsf{F}[j] \leftarrow \mathrm{bit}_k(\mathsf{C}_{\min}[\mathrm{insertbit}_k(0, j)])$
    end for
    $\mathsf{P} \leftarrow$ OSORT($\mathrm{layer}_k(\mathsf{F}), \text{OSORT}(\mathsf{id}_n, \mathsf{P}, k), k$)
    for $j = 0$ to $2^m - 1$ inclusive **do**
      $\mathsf{L}[j] \leftarrow \mathrm{bit}_k(F \cdot \pi(\mathrm{insertbit}_k(0, j)))$
    end for
    $\mathsf{C}[k \cdot 2^m : (k+1) \cdot 2^m : 1] \leftarrow \mathsf{F}$
    $\mathsf{C}[(2m-k) \cdot 2^m : (2m-k+1) \cdot 2^m : 1] \leftarrow \mathsf{L}$
    $\mathsf{P} \leftarrow$ OLAYER($\mathsf{P}, \mathsf{L}, k$)
  end for
  for $j = 0$ to $2^m - 1$ inclusive **do**
    $\mathsf{L}[j] \leftarrow \mathrm{bit}_m(\mathsf{P}[j])$
  end for
  $\mathsf{C}[m \cdot 2^m : (m+1) \cdot 2^m : 1] \leftarrow \mathsf{L}$
  return C
end function

primarily, instead of proceeding by decomposing our permutation into two smaller permutations, we modify our permutation in-place and change the parameter of the operation we perform upon it. Consequently, the call to FASTCYCLEMIN takes its second parameter depending on k: we discuss this in Subsection 3.2, but essentially we need to account for the fact that as our permutations are increasingly structured, we need to do less work in calculating cycle minima. Thirdly, the control bits are in our algorithm calculated and stored in their natural order, removing the need for variables `pos` and `step` or indeed any kind of global array. Writing happens only to adjacent slices of memory.

At the end of the k-th iteration, P then stores $\mathrm{middle}_k(\pi)$ as defined below. Instead of an m-th iteration, however, we can actually read off the final layer of control bits from P. This corresponds to the base case of Bernstein's recursion in which he similarly can just read off a result.

The proofs of these theorems and corollaries are presented in Section 3. The analogy of Corollary 3 for CBRECURSION is exactly that which Bernstein does not prove in his work. We do have a proof of this, but we do not present this proof in this paper. However, it is in our Lean formalisation which extends Bernstein's work. We do not provide an explicit proof that CBITERATIVE($\mathsf{Vec}\,(\pi)$) calculates the same control bits that CBRECURSION($\mathsf{Vec}\,(\pi)\,, \mathsf{C}, 0, 1$) sets C to, but the careful reader can verify for themselves that they do. Note that, for instance, statement *(ii)* of Theorem 1 exactly corresponds via interlacing to Bernstein's property that the first bit of each layer he calculates is 0.

Definition 5. Let k, m be nonnegative integers and $\pi \in S_{2^{m+1}}$ be an arbitrary permutation. We define $\mathsf{F}_k(\pi), \mathsf{L}_k(\pi), \mathrm{first}_k(\pi), \mathrm{last}_k(\pi) \in \{0,1\}^{2^m}$ and $\rho_k(\pi), \mathrm{middle}_k(\pi) \in S_{2^{m+1}}$ as follows:

(i) $\mathsf{F}_k(\pi)[i] := \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])(\mathrm{insertbit}_k(0, i))\right)$.

(ii) $\mathsf{L}_k(\pi)[i] := \mathrm{bit}_k\left(\mathrm{layer}_k\left(\mathsf{F}_k(\pi)\right)\left(\pi\left(\mathrm{insertbit}_k(0, i)\right)\right)\right)$.

(iii) $\rho_k(\pi) := \mathrm{layer}_k\left(\mathsf{F}_k(\pi)\right) \cdot \pi \cdot \mathrm{layer}_k\left(\mathsf{L}_k(\pi)\right)$.

(iv) $\mathrm{middle}_0(\pi) := \pi$ and $\mathrm{middle}_{k+1}(\pi) := \rho_k(\mathrm{middle}_k(\pi))$.

(v) $\mathrm{first}_k(\pi) := \mathsf{F}_k(\mathrm{middle}_k(\pi))$.

(vi) $\mathrm{last}_k(\pi) := \mathsf{L}_k(\mathrm{middle}_k(\pi))$.

Theorem 1. *Let k, m be nonnegative integers and $\pi \in S_{2^{m+1}}$ be an arbitrary permutation. Then $\pi = \mathrm{layer}_k(\mathsf{F}_k(\pi)) \cdot \rho_k(\pi) \cdot \mathrm{layer}_k(\mathsf{L}_k(\pi))$. Moreover if, for all x, $\pi(x) \equiv x \mod 2^k$, then:*

(i) The first 2^k bits in F_k are 0.

(ii) $\rho_k(\pi)(x) \equiv x \mod 2^{k+1}$.

Theorem 2. *Let k, m be nonnegative integers and $\pi \in S_{2^{m+1}}$ be an arbitrary permutation. Then:*

(i) $\mathrm{middle}_k(\pi) = \mathrm{layer}_k(\mathrm{first}_k(\pi)) \cdot \mathrm{middle}_{k+1}(\pi) \cdot \mathrm{layer}_k(\mathrm{last}_k(\pi))$.

(ii) The first 2^k bits of $\mathrm{first}_k(\pi)$ are zero. In particular, when $m \leq k$, every entry of $\mathrm{first}_k(\pi)$ is zero.

(iii) $\mathrm{middle}_k(\pi)(x) \equiv x \mod 2^k$. *In particular,* $\mathrm{middle}_{m+1} = \mathbb{I}_{2^{m+1}}$.

Corollary 1.

$$\pi = \mathrm{layer}_0(\mathrm{first}_0(\pi)) \cdots \mathrm{layer}_{m-1}(\mathrm{first}_{m-1}(\pi)) \cdot \mathrm{layer}_m(\mathrm{last}_m(\pi)) \cdots \mathrm{layer}_0(\mathrm{last}_0(\pi)).$$

Corollary 2.

$$\textsc{cbiterative}(\mathsf{Vec}\,(\pi)) = \mathrm{first}_0(\pi) \| \cdots \| \mathrm{first}_{m-2}(\pi) \| \mathrm{last}_{m-1}(\pi) \| \cdots \| \mathrm{last}_0(\pi).$$

Corollary 3.

$$\mathsf{A} \triangleleft \pi = \textsc{cbapply}(\mathsf{A}, \textsc{cbiterative}(\pi)).$$

3 Proofs and Verification

In this section, we generalise Bernstein's definitions and proofs in [Ber20] by generalising his results and formulae. It is only implicit in Bernstein's work that the control bits that result from this inductive process can be correctly re-interpreted by Algorithm 2.

Some care is needed in the proofs, and this has motivated our careful setting up of the relevant definitions in the previous section. Certain inferences which are natural and obvious in Bernstein's work — where he considers only the $k = 0$ case — are still analogously true in our generalisation, but are less clearly true to the casual observer. To support our work, alongside these pen and paper proofs we have produced a formalisation of them in the Lean theorem prover, as Bernstein formalised his own proofs. We present below our written proofs, generalised from Bernstein's results.

3.1 Results on the commutator

Let m be a nonnegative integer, and $\pi \in S_{2^{m+1}}$ an arbitrary permutation. We consider the commutator $[\pi, \mathrm{flip}_k] = \pi \cdot \mathrm{flip}_k \cdot \pi^{-1} \cdot \mathrm{flip}_k$. (Recall that flip_k is an involution and so is its own inverse). Note that $[\pi, \mathrm{flip}_k] \in S_{2^{m+1}}$, as it is the product of elements of $S_{2^{m+1}}$. Further, if π is such that, for all x, $\pi(x) \equiv x \mod 2^k$, then we also have $[\pi, \mathrm{flip}_k](x) \equiv x \mod 2^k$ for all x. As flip_k is not only an involution but also one without any fixed points, we can derive some useful lemmas using elementary properties of the commutator (mirroring Bernstein's approach: all of our proofs are generalisations of his).

Lemma 2. *Suppose $\pi \in S_{2^{m+1}}$ for nonnegative m. For all integers t, we have*

$$\mathrm{flip}_k \cdot [\pi, \mathrm{flip}_k]^t = [\pi, \mathrm{flip}_k]^{-t} \cdot \mathrm{flip}_k.$$

In other words, applying the commutator t times and then flip_k, is equivalent to applying flip_k and then applying the inverse of the commutator t times.

Proof. Observe that without loss of generality we may take $t \geq 0$ (if it is negative we can simply multiply both sides of the equation by flip_k on the left and right and move to a positive case). As such, we can induct on t. Clearly when $t = 0$ the statement holds. Suppose in the inductive case that t is of the form $s+1$, where s is a nonnegative integer. Then we have:

$$\begin{aligned}
\mathrm{flip}_k \cdot [\pi, \mathrm{flip}_k]^{s+1} &= \mathrm{flip}_k \cdot [\pi, \mathrm{flip}_k]^s \cdot [\pi, \mathrm{flip}_k] \cdot \mathrm{flip}_k \cdot \mathrm{flip}_k \\
&= [\pi, \mathrm{flip}_k]^{-s} \cdot \mathrm{flip}_k \cdot [\pi, \mathrm{flip}_k] \cdot \mathrm{flip}_k \cdot \mathrm{flip}_k \\
&= [\pi, \mathrm{flip}_k]^{-s} \cdot \mathrm{flip}_k \cdot \pi \cdot \mathrm{flip}_k \cdot \pi^{-1} \cdot \mathrm{flip}_k \cdot \mathrm{flip}_k \cdot \mathrm{flip}_k \\
&= [\pi, \mathrm{flip}_k]^{-s} \cdot \mathrm{flip}_k \cdot \pi \cdot \mathrm{flip}_k \cdot \pi^{-1} \cdot \mathrm{flip}_k \\
&= [\pi, \mathrm{flip}_k]^{-s} \cdot [\pi, \mathrm{flip}_k]^{-1} \cdot \mathrm{flip}_k \\
&= [\pi, \mathrm{flip}_k]^{-(s+1)} \cdot \mathrm{flip}_k.
\end{aligned}$$

□

Lemma 3. *Suppose $\pi \in S_{2^{m+1}}$ for nonnegative m. Let $i, j \in \mathbb{Z}$. Then for all x,*

$$[\pi, \mathrm{flip}_k]^i(x) \neq \mathrm{flip}_k([\pi, \mathrm{flip}_k]^j(x)) = [\pi, \mathrm{flip}_k]^{-j}(\mathrm{flip}_k(x)).$$

In other words, for any nonnegative integer $x < 2^m$, the cycles of x and $\mathrm{flip}_k(x)$ under the commutator are disjoint.

Proof. The equality in our statement holds by Lemma 2, so we need only prove the inequality. Fix x. Suppose for a contradiction there exists some i, j which in fact satisfy $[\pi, \mathrm{flip}_k]^i(x) = \mathrm{flip}_k \cdot [\pi, \mathrm{flip}_k]^j(x)$. We can multiply both sides by flip_k to reverse the roles of i and j, so without loss of generality we may take $i \geq j$. We choose such a i, j so that $i - j$ is minimal. If $i = j$ then we would immediately have x as a fixed point of flip_k, which is impossible. So $i \geq j + 1$ Suppose $i = j + 1$. Then we would have:

$$\begin{aligned}
\pi^{-1}([\pi, \mathrm{flip}_k]^{j+1}(x)) &= \pi^{-1}([\pi, \mathrm{flip}_k]([\pi, \mathrm{flip}_k]^j(x))) \\
&= \mathrm{flip}_k(\pi^{-1}(\mathrm{flip}_k([\pi, \mathrm{flip}_k]^j(x)))) \\
&= \mathrm{flip}_k(\pi^{-1}([\pi, \mathrm{flip}_k]^{j+1}(x))).
\end{aligned}$$

But, again, flip_k has no fixed points, so this cannot be so. We must have $i \geq j + 2$. Then:

$$\begin{aligned}
[\pi, \mathrm{flip}_k]^{i-1}(x) &= [\pi, \mathrm{flip}_k]^{-1}([\pi, \mathrm{flip}_k]^i(x)) \\
&= [\pi, \mathrm{flip}_k]^{-1}(\mathrm{flip}_k([\pi, \mathrm{flip}_k]^j(x))) \\
&= \mathrm{flip}_k([\pi, \mathrm{flip}_k]^{j+1}(x)).
\end{aligned}$$

So $(i-1, j+1)$ are also such a pair. We have $i \geq j + 2$, so $i - 1 \geq j + 1$. However, $(i-1) - (j+1) = (i-j) - 2$. This contradicts the minimality of $i - j$. □

Lemma 4. *Suppose $\pi \in S_{2^{m+1}}$ for nonnegative m and $x < 2^{m+1}$. Suppose $S \subseteq \mathbb{N}_{<2^{m+1}}$ satisfies: (1) For all $y \in S$, $\mathrm{flip}_k(y) \in S$, and (2) For all $i \in \mathbb{Z}$, $[\pi, \mathrm{flip}_k]^i(x) \in S$. Then, the period of x under the permutation $[\pi, \mathrm{flip}_k]$ is at most $\frac{|S|}{2}$.*

Proof. Fix x. We have $x, \text{flip}_k(x) \in S$. Let $S_0 = \{[\pi, \text{flip}_k]^i(x) \mid i \in \mathbb{Z}\}$ and $S_1 = \{[\pi, \text{flip}_k]^i(\text{flip}_k(x)) \mid i \in \mathbb{Z}\}$. We have $S_0 \subseteq S$ by assumption. We also have $S_1 \subseteq S$: $[\pi, \text{flip}_k]^i(\text{flip}_k(x)) = \text{flip}_k([\pi, \text{flip}_k]^{-i}(x))$ by Lemma 2, and as S is invariant under flip_k and contains S_0, it follows that it contains S_1. As such $|S_0 \cup S_1| \leq |S|$. We can also use this observation to note that flip_k is surjective and maps S_0 into S_1. As flip_k is globally injective, it will also be injective on this restriction. Thus, it defines a bijection between them and so $|S_0| = |S_1|$. Moreover, by Lemma 3, the cycles of x and $\text{flip}_k(x)$ are disjoint, i.e. S_0 and S_1 are disjoint. Thus, $|S_0 \cup S_1| = |S_0| + |S_1|$. It follows that $2 \cdot |S_0| = |S_0| + |S_1| = |S_0 \cup S_1| \leq |S|$, and hence the period of x, $|S_0|$, is at most $\frac{|S|}{2}$. □

Lemma 5. *Suppose $\pi \in S_{2^{m+1}}$ for nonnegative m. Suppose $k \leq m$ is such that for all x, we have $\pi(x) \equiv x \mod 2^k$. Then for any x, the period of x under the permutation $[\pi, \text{flip}_k]$ is at most 2^{m-k}.*

Proof. Clearly if $2^{m+1} \leq x$ or $x < 0$ then the period of x is 1, which is certainly at most 2^{m-k}. Hence, without loss of generality, assume $x < 2^{m+1}$. Let $S_x = \left\{y \mid 0 \leq y < 2^{m+1} \wedge y \equiv x \bmod 2^k\right\}$. In other words, S_x is the set of nonnegative integers which can be represented in $m+1$ bits and which share their bottom k bits with x.

Now, if $y \in S_x$, it is easy to see that $\text{flip}_k(y) \in S_x$. Moreover, for all i, $[\pi, \text{flip}_k]^i(x) \in S_x$: $[\pi, \text{flip}_k]$ is invariant on the bottom k bits, and is in $S_{2^{m+1}}$, so it preserves the defining property of S_x. Observe that $|S_x| = 2^{(m+1)-k}$, because there are $(m+1)-k$ "free" bits. It follows from Lemma 4 that the period of x is at most 2^{m-k} as required. □

Lemma 6. *Let $x, y \in \mathbb{Z}$. Suppose $x \equiv y \mod 2^k$, $x \leq \text{flip}_k(y)$, and $y \leq \text{flip}_k(x)$. Then either $y = x$ or $y = \text{flip}_k(x)$.*

Proof. As $x \equiv y \mod 2^k$, we have $y = x + 2^k \cdot a$ for some integer a. Without loss of generality, suppose $x \leq y$. Then $0 \leq a$, and $x \leq \text{flip}_k(x)$, so in fact $\text{flip}_k(x) = x + 2^k$. As such, $x + 2^k \cdot a \leq x + 2^k$, so $a \leq 1$. If $a = 0$, we have $y = x$. If $a = 1$, $y = \text{flip}_k(x)$. □

Lemma 7. *Suppose $\pi \in S_{2^{m+1}}$ for nonnegative m. Suppose $k \leq m$ is such that for all x, $\pi(x) \equiv x \mod 2^k$. Then it follows that for all $x \leq 2^{m+1}$,*

$$\text{cyclemin}([\pi, \text{flip}_k])(\text{flip}_k(x)) = \text{flip}_k(\text{cyclemin}([\pi, \text{flip}_k])(x)).$$

Proof. Let m, π, x be as in the theorem statement. Let j, s be integers such that:

$$\begin{aligned}\text{cyclemin}([\pi, \text{flip}_k])(x) &= [\pi, \text{flip}_k]^j(x).\\ \text{cyclemin}([\pi, \text{flip}_k])(\text{flip}_k(x)) &= [\pi, \text{flip}_k]^s(\text{flip}_k(x)).\end{aligned}$$

Using minimality, Lemma 2, Lemma 3, and the congruence property of π, we have:

$$\begin{aligned}[\pi, \text{flip}_k]^s(\text{flip}_k(x)) &\leq [\pi, \text{flip}_k]^{-j}(\text{flip}_k(x))\\ &= \text{flip}_k([\pi, \text{flip}_k]^j(x)).\\ [\pi, \text{flip}_k]^j(x) &\leq [\pi, \text{flip}_k]^{-s}(x)\\ &= \text{flip}_k([\pi, \text{flip}_k]^s(\text{flip}_k(x))).\\ [\pi, \text{flip}_k]^j(x) &\equiv [\pi, \text{flip}_k]^s(\text{flip}_k(x)) \mod 2^k.\\ [\pi, \text{flip}_k]^j(x) &\neq [\pi, \text{flip}_k]^s(\text{flip}_k(x)).\end{aligned}$$

The result follows by Lemma 6. □

Algorithm 5 Calculate cycle minima of $[\pi, \mathrm{flip}_k]$ where $\pi(x) \equiv x \mod 2^k$ for all x.

```
1: function FastCycleMin(π, k)
2:     C_min ← id_n                                  ▷ Initialise cyclemin array.
3:     P ← Vec(π · flip_k)                           ▷ Implemented as P[x] = π(x ⊕ 2^k).
4:     Q ← Vec(flip_k · π)                           ▷ Implemented as Q[x] = π(x) ⊕ 2^k.
5:     for i in 0, 1, ..., m − k − 1 do
6:         P, Q ← oSort(P, Q, k), oSort(Q, P, k)     ▷ P is [π, flip_k]^(2^i), Q is [π, flip_k]^(−2^i)
7:         CP ← oSort(C_min, Q, k)                   ▷ CP[x] = C_min[[π, flip_k]^(2^i)(x)]
8:         C_min ← min(C, CP)
9:     end for return C_min
10: end function
```

3.2 The cycle minima calculation

A naive calculation of the cycle minima is costly. However as Bernstein notes from Nassimi-Sahni, the operation can be performed in parallel. Note this corollary of Lemma 5.

Corollary 4. *Let $\pi \in S_{2^{m+1}}$ be such that $\pi(x) \equiv x \mod 2^k$ for all x. Then:*

$$\mathrm{cyclemin}([\pi, \mathrm{flip}_k]) = \min\{x, [\pi, \mathrm{flip}_k](x), [\pi, \mathrm{flip}_k]^2(x), \ldots, [\pi, \mathrm{flip}_k]^{2^{m-k}-1}(x)\}.$$

Proof. By Lemma 5, the period of any x under $[\pi, \mathrm{flip}_k]$ is at most 2^{m-k}. Thus the minimum defined by the right-hand side is indeed the minimum over the whole cycle. □

Theorem 3. *Let $\pi \in S_{2^{m+1}}$ be such that $\pi(x) \equiv x \mod 2^k$ for all x. Then Algorithm 5,* FastCycleMin*, has* $\mathrm{cyclemin}([\pi, \mathrm{flip}_k])(x) = \textsc{FastCycleMin}(\pi, k)[x]$ *for all $x < 2^{m+1}$.*

Proof. For every use of oSort, we will be dealing with a permutation that has the required congruence property (because π does), and so we may indeed conclude that the interlaced sorting network are correctly calculating $\mathsf{A} \triangleleft \rho$ for various A and ρ. We argue inductively via loop invariants that $\mathsf{C}_{\min}[x] = \min\{x, [\pi, \mathrm{flip}_k](x), [\pi, \mathrm{flip}_k]^2(x), \ldots, [\pi, \mathrm{flip}_k]^{2^i-1}(x)\}$ before the i-th iteration. Before the loop begins, $\mathsf{C}_{\min} \leftarrow \mathsf{id}_n$ (as required), $\mathsf{Perm}(\mathsf{P}) \leftarrow \pi \cdot \mathrm{flip}_k$, and $\mathsf{Perm}(\mathsf{Q}) \leftarrow \mathrm{flip}_k \cdot \pi$. On each iteration, we set $\mathsf{Perm}(\mathsf{P}) \leftarrow \mathsf{Perm}(\mathsf{P}) \cdot \mathsf{Perm}(\mathsf{Q})^{-1}$ and $\mathsf{Perm}(\mathsf{Q}) \leftarrow \mathsf{Perm}(\mathsf{Q}) \cdot \mathsf{Perm}(\mathsf{P})^{-1}$.

It follows that when $i = 0$, $\mathsf{Perm}(\mathsf{P})$ is set to $[\pi, \mathrm{flip}_k]$ and $\mathsf{Perm}(\mathsf{Q})$ is set to $[\pi, \mathrm{flip}_k]^{-1}$. Thereafter on the i-th iteration, $\mathsf{Perm}(\mathsf{P})$ is set to $[\pi, \mathrm{flip}_k]^{2^i}$ and $\mathsf{Perm}(\mathsf{Q})$ is set to $[\pi, \mathrm{flip}_k]^{-2^i}$. It then follows that CP is set such that $\mathsf{CP}[x] = \mathsf{C}_{\min}[[\pi, \mathrm{flip}_k]^{2^i}(x)]$, using the values of $\mathsf{C}_{\min}$ before the loop, and then $\mathsf{C}_{\min}$ is set to the minimum of its current values and the values in CP. Using the inductive hypothesis, this will be exactly the minimum of $\min\{x, \ldots, [\pi, \mathrm{flip}_k]^{2^i-1}(x)\}$ and $\min\{[\pi, \mathrm{flip}_k]^{2^i}(x), \ldots, [\pi, \mathrm{flip}_k]^{2^{i+1}-1}(x)\}$. This minimum is indeed equal to the cycle minima we require by Corollary 4. □

3.3 Proving our theorems and corollaries

Recall that for all i, $\mathrm{insertbit}_k(\mathrm{bit}_k(i), \mathrm{removebit}_k(i)) = i$. We use this, and the formalism we have set up, to prove two crucial lemmas without too much trouble.

Lemma 8. *Suppose $\pi \in S_{2^{m+1}}$, and for all x, $\pi(x) \equiv x \mod 2^k$. Then*

$$\mathrm{bit}_k\left(\mathrm{layer}_k(\mathsf{F}_k(\pi))(i)\right) = \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])(i)\right).$$

Proof.

$$\begin{aligned}\mathrm{bit}_k\left(\mathrm{layer}_k(\mathsf{F}_k(\pi))(i)\right) &= \mathrm{bit}_k\left(\mathrm{cflip}_k(\mathsf{F}_k(\pi)[\mathrm{removebit}_k(i)], i)\right)\\ &= \mathrm{bit}_k(i) \oplus \mathsf{F}_k(\pi)[\mathrm{removebit}_k(i)]\\ &= \mathrm{bit}_k(i) \oplus \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])[\mathrm{insertbit}_k(0, \mathrm{removebit}_k(i))]\right).\end{aligned}$$

If $\mathrm{bit}_k(i) = 0$, we are done. If $\mathrm{bit}_k(i) = 1$, then we use Lemma 7:

$$\begin{aligned}&= 1 \oplus \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])[\mathrm{flip}_k(i)]\right)\\ &= 1 \oplus \mathrm{bit}_k\left(\mathrm{flip}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])[i]\right)\right)\\ &= 1 \oplus 1 \oplus \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])[i]\right).\end{aligned}$$

□

Lemma 9. *Suppose $\pi \in S_{2^{m+1}}$, and for all x, $\pi(x) \equiv x \mod 2^k$. Then*

$$\mathrm{bit}_k\left(\mathrm{layer}_k(\mathsf{L}_k(\pi))(i)\right) = \mathrm{bit}_k\left(\left(\mathrm{layer}_k(\mathsf{F}_k(\pi))\right)\left(\pi\left(i\right)\right)\right).$$

Proof.

$$\begin{aligned}\mathrm{bit}_k\left(\mathrm{layer}_k(\mathsf{L}_k(\pi))(i)\right) &= \mathrm{bit}_k\left(\mathrm{cflip}_k(\mathsf{L}_k(\pi)[\mathrm{removebit}_k(i)], i)\right)\\ &= \mathrm{bit}_k(i) \oplus \mathsf{L}_k(\pi)[\mathrm{removebit}_k(i)]\\ &= \mathrm{bit}_k(i) \oplus \mathrm{bit}_k\left(\mathrm{layer}_k\left(\mathsf{F}_k(\pi)\right)\left(\pi\left(\mathrm{insertbit}_k(0, \mathrm{removebit}_k(i))\right)\right)\right).\end{aligned}$$

If $\mathrm{bit}_k(i) = 0$, we are done. If $\mathrm{bit}_k(i) = 1$, then we use Lemma 8, Lemma 7, the fact that $\mathrm{cyclemin}(\rho)(x) = \mathrm{cyclemin}(\rho)(\rho(x))$, and the the relation $[\pi, \mathrm{flip}_k] \cdot \mathrm{flip}_k \cdot \pi \cdot \mathrm{flip}_k = \pi$.

$$\begin{aligned}&= 1 \oplus \mathrm{bit}_k\left(\mathrm{layer}_k\left(\mathsf{F}_k(\pi)\right)\left(\pi\left(\mathrm{flip}_k(i)\right)\right)\right)\\ &= 1 \oplus \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])\left(\pi\left(\mathrm{flip}_k(i)\right)\right)\right)\\ &= \mathrm{bit}_k\left(\mathrm{flip}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])\left(\pi\left(\mathrm{flip}_k(i)\right)\right)\right)\right)\\ &= \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])\left(\mathrm{flip}_k\left(\pi\left(\mathrm{flip}_k(i)\right)\right)\right)\right)\\ &= \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])\left([\pi, \mathrm{flip}_k] \cdot \mathrm{flip}_k \cdot \pi \cdot \mathrm{flip}_k\right)(i)\right)\\ &= \mathrm{bit}_k\left(\mathrm{cyclemin}([\pi, \mathrm{flip}_k])\left(\pi\left(i\right)\right)\right)\\ &= \mathrm{bit}_k\left(\mathrm{layer}_k\left(\mathsf{F}_k(\pi)\right)\left(\pi\left(i\right)\right)\right).\end{aligned}$$

□

We are now able to prove our central claims.

Proof of Theorem 1. As $\mathrm{layer}_k(\mathsf{F}_k(\pi))$ and $\mathrm{layer}_k(\mathsf{L}_k(\pi))$ are both involutions, it immediately follows from the definition of ρ_k that indeed $\pi = \mathrm{layer}_k(\mathsf{F}_k(\pi)) \cdot \rho_k(\pi) \cdot \mathrm{layer}_k(\mathsf{L}_k(\pi))$. Now let us also assume that for all x, $\pi(x) \equiv x \mod 2^k$.

Suppose $i < 2^k$. $\mathrm{insertbit}_k(0, i) = i$ because inserting a 0 at the k-th bit doesn't actually change the value. $[\pi, \mathrm{flip}_k]$ can only send i to numbers which are congruent to it modulo 2^k, and i is the smallest such number, so the minimum of the cycle of i under $[\pi, \mathrm{flip}_k]$ is also i. The k-th bit of i is 0. It follows that $\mathsf{F}_k(\pi)[i] = 0$. This proves the first claim.

We now want to show that $\rho_k(\pi)(x) \equiv x \mod 2^{k+1}$ for all x. Simply by definition of ρ_k, and the congruence property of π, this congruence equation is true modulo 2^k, i.e. the bits below the k-th bit of both sides match. Hence it is sufficient to verify that $\mathrm{bit}_k\left(\rho_k(\pi)(x)\right) = \mathrm{bit}_k(x)$. We can see this by using Lemma 9.

$$\begin{aligned}\mathrm{bit}_k\left(\rho_k(\pi)(x)\right) &= \mathrm{bit}_k\left(\mathrm{layer}_k\left(\mathsf{F}_k(\pi)\right)\left(\pi\left(\mathrm{layer}_k(\mathsf{L}_k(\pi))\left(x\right)\right)\right)\right)\\ &= \mathrm{bit}_k\left(\mathrm{layer}_k\left(\mathsf{L}_k(\pi)\right)\left(\mathrm{layer}_k(\mathsf{L}_k(\pi))\left(x\right)\right)\right)\\ &= \mathrm{bit}_k\left(\mathrm{layer}_k(\mathsf{L}_k(\pi)) \cdot \mathrm{layer}_k(\mathsf{L}_k(\pi))\left(x\right)\right)\\ &= \mathrm{bit}_k(x).\end{aligned}$$

□

Proof of Theorem 2. This is a straightforward inductive application of Theorem 1. That $\text{middle}_k(\pi) = \text{layer}_k(\text{first}_k(\pi)) \cdot \text{middle}_{k+1}(\pi) \cdot \text{layer}_k(\text{last}_k(\pi))$ is just the first claim of Theorem 1. It is also certainly true that $\text{middle}_0(\pi)(x) \equiv x \mod 2^0$ (everything is equal modulo 1)! Proceeding inductively, assuming $\text{middle}_k(\pi)(x) \equiv x \mod 2^k$, we can apply the conditional conclusions of Theorem 1: because we have $\text{first}_k(\pi) = \mathsf{F}_k(\text{middle}_k(\pi))$, the first 2^k bits of $\text{first}_k(\pi)$ are zero, and because $\text{middle}_{k+1}(\pi) = \rho_k(\text{middle}_k(\pi))$, we have $\text{middle}_{k+1}(\pi)(x) \equiv x \mod 2^{k+1}$. The particular cases we note in the theorem are thereafter clearly true. □

Proof of Corollary 1. This corollary is true simply by using the facts that $\text{middle}_{m+1} = \mathbb{I}_{2^{m+1}}$, $\text{layer}_m(\text{first}_m(\pi)) = \mathbb{I}_{2^{m+1}}$, and $\text{middle}_k(\pi) = \text{layer}_k(\text{first}_k(\pi)) \cdot \text{middle}_{k+1}(\pi) \cdot \text{layer}_k(\text{last}_k(\pi))$, and expanding inductively from $\pi = \text{middle}_0(\pi)$. □

Proof of Corollary 2. We have Theorem 3, which justifies that we really are calculating the right cycle minima on each iteration of the loop. Similarly, the interlaced sorts will happen correctly because of the congruence properties we have shown. Our layers in each iteration are set more or less directly from the above formulae, so that on the i-th loop they are set to $\text{first}_i(\pi)$ and $\text{last}_i(\pi)$, and the value of the permutation each loop is set such that before the i-th loop, $\mathsf{Perm}(\mathsf{P}) = \text{middle}_k(\pi)$, and after the i-th loop it is $\mathsf{Perm}(\mathsf{P}) = \text{middle}_{k+1}(\pi)$. We do not bother performing the full loop when $i = k$, as we know that $\text{middle}_{m+1}(\pi)$ and $\text{first}_m(\pi)$ are trivial: moreover note that $\text{insertbit}_m(0, i)$ for $i < 2^m$, and so $\text{last}_m(\pi)[i] = \text{bit}_k(\text{middle}_m(\pi)(i))$, justifying the final layer that we set. □

Proof of Corollary 3. This proof is essentially straightforward: the definition of Algorithm 2 is one of applying the control bits in a long sequence in 2^m-sized slices, which is exactly what we have concatenated, and Algorithm 1 precisely applies the sequence of conditional transpositions that make up each application of layer_k. As such, it is essentially immediate from Corollary 1 and Corollary 2 that the claim is true. □

3.4 Formal verification in Lean

Lean is a programming language and interactive theorem prover [dMU21], based originally on the same foundation as the Rocq [BC04] theorem prover. Lean aspires to be a first-class programming language in its own right, rather than a theorem prover first with programming capabaility. Mathlib [mC20] is Lean's main mathematics library: it is comparable to Rocq's MathComp. In recent years Lean has enjoyed unusually high usage by leading mathematicians [ABLM20, Sch22, Tao24], which reflects the strong community that has built up around it. This relevance motivates our usage.

The primary message of our work — other than the proofs themselves — was a strong conclusion of the value of a careful, exploratory approach to formalisation. This contrasts with beelining to the final conclusion. One can often prove useful things this way: but it is useful and considerate both to others and also your future self to be more careful, more modular, and less monolithic. Employing this approach to our initial translation of Bernstein's results made extending them — even totally rebuilding our implementation and proofs from scratch with our new iterative approach — relatively easy: well-written and modular formal proofs are much easier to refactor and port.

Indeed, we believe there is no real distinction between "good programming" and "good (formal) theorem proving": both are forms of software and software design principles like the effective structuring APIs apply to both: moreover it makes much sense to design these simultaneously. Good verification leads to good software and vice versa. We advocate for cryptographic software to be built in similar ways and to develop verifications of new implementations alongside their design.

4 Implementation and Evaluation

Overview. We provide two implementations of Algorithm 4 in the C programming language. The implementation cbiterative is designed for portability, whereas the implementation cbiteravx demonstrates the vectorisability of the algorithm using instructions from Intel's AVX2 feature. We evaluate our implementations and compare their performance against cbrecursion: the implementation of Algorithm 3 in libmceliece version 20250507 [Cho25]. All the implementations take as input an array of 2^{m+1} 16-bit integers representing a permutation, and write, to a pre-allocated buffer, the $(2m+1) \cdot 2^m$ control bits for the in-place Beneš network realising said permutation. Both our implementations follow constant-time coding best practices: we do not (1) branch on secret values, (2) make memory accesses dependent on secret values, or (3) pass secret values as input to variable time instructions. In Classic McEliece, the permutation is secret and its size is public. In the portable implementation, we utilise the cryptoint library [Ber25a] to reduce the risk of the compiler optimising away our precautions. In cbiteravx, we utilise only instructions in the Intel Data-Operand Independent Timing (DOIT) list [Cor25]. We verify that our implementations satisfy constant-time best practices (1) and (2) using Valgrind's memcheck tool [NS07].

Intel AVX2. Intel's AVX2 provides 16 256-bit ymm registers for integer SIMD instructions. These registers offer divisions into lanes of size 128, 64, 32, 16, and 8 bits. A *swizzle* instruction is an intra-register permutation. Data movement across the two 128-bit lanes is both limited and relatively expensive. We identify instructions by their assembly mnemonic. We loosely term an algorithm SIMD-friendly if its scalar implementation is similar to its SIMD implementation. Algorithm 4 is SIMD-friendly; to utilise the ymm registers in the implementation of cbiteravx, we need only unroll the loops in the cbiterative Implementation. On the other hand, the deepest subcalls (which are the majority) of Algorithm 3 act on less data than a single ymm register will hold: this algorithm is not SIMD-friendly.

4.1 Implementation Strategy.

Permutations The permutation in Classic McEliece acts on 2^{13} elements and is initially represented by an array of 16-bit integers. The challenge of the implementation is to apply oSort as little as possible because sorting large arrays is expensive. Conversely, oFlip can be implemented with 3 XOR instructions and a mask or instructions such as `cmovg`. To implement oSort, we represent the tuple array as an array of 32-bit integers formed by concatenating the two 16-bit halves. Sorting these 32-bit integers is equivalent to sorting the tuples with the upper half as the key. Recall that to apply π to A on the right requires *two* sorts: one sort to obtain π^{-1}, and one to obtain $\mathsf{A}[\pi]$. The implementations of cbiterative and cbrecursion repeatedly utilise this primitive to calculate the cycle minima and to left-apply $\text{layer}_k(\mathsf{F}_k(\pi))$ to π. The C implementations require storage space for two temporary arrays of 2^{13} 32-bit integers for these cycle calculations. We call these A and B.

Storing the control bits. The control bits are stored as a bit array. Towards the base cases of cbrecursion, less than 8 bits are set. To set individual bits in a byte, Bernstein bit-shifts and XORs them into a zero-initialised array. The iterative algorithm can avoid these smaller cases and place a byte of control bits at a time by unrolling the loop. We found this to have an negligible performance effect in the scalar implementations, aside from avoiding the call to memset to zero-initialise the array. The cbiteravx implementation, however, sets 32 control bits per memory store, leading to noticeable performance gains.

Table 1: **Performance of portable implementations.** Latency (in median clock cycles) required to calculate control bits for a permutation of 2^{13} integers, taken over $n = 63$ samples, with an acceleration factor relative to cbrecursion with portable4. Both are compiled with `-O2` optimisations.

Sorting Implementation	Control Bit Implementation			
	cbrecursion		cbiterative	
portable4	257751980	1.00×	455080940	1.77×
iportable4	N/A		245809658	0.95×

Table 2: **Performance of AVX2-enabled implementations.** Latency (in median clock cycles) required to calculate control bits for a permutation of 2^{13} integers, taken over $n = 63$ samples, with an acceleration factor relative to cbrecursion with djbsort. Both are compiled with `-O2`, `-mavx2`, `-mtune=meteorlake` optimisations.

Sorting Implementation	Control Bit Implementation					
	cbrecursion		cbiterative		cbiteravx	
bitonic	47785198	1.27×	75137950	1.99×	71083334	1.88×
djbsort	37724474	1.00×	57897860	1.53×	53736906	1.42×
ibitonic	N/A		38637248	1.02×	34864186	0.92×
ipermsort	N/A		31981508	0.85×	28135818	0.75×

Vectorisation. The primary challenge in the cbiteravx implementation is extracting the control bits. After having calculated the cycle minima for iteration k in Algorithm 4, the i-th control bit (for the current layer) exists in the k-th bit of the $\mathrm{insertbit}_k(0, i)$-th element of B. We must extract these control bits out of our ymm registers and into thier condensed representation as an array of bits. This can be done with the movemask instructions `vpmovmsk{b, ps, pd}`. However, only `vpmovmskb`, which extracts the most significant bit of each byte in a ymm register into a 32-bit integer, is on the DOIT list. To extract 32 control bits per iteration, each loop must load 64 elements from B. Therefore, cbiteravx only takes as input permutations with more than 64 elements. Indexing in bits, the i-th control bit is in bit $k + 32 \cdot \mathrm{insertbit}_k(0, i)$ of B and must be placed in bit $8i + 7$ of a ymm register in order to obtain the control bits for the in-place Beneš network. For values of $k \leq 2$, this rearrangement requires special cases as the $\mathrm{insertbit}_k(0, i)$ addend changes their positions by awkward multiples of 32 bits.

In the same loop that we extract the control bits, we also apply them to the identity to obtain $\mathrm{layer}_k(\mathsf{F})$ for the calculation of $\mathrm{layer}_k(\mathsf{F}) \cdot \pi$. After having calculated the cycle minima for iteration k, B is as-above and the i-th element of array A has the value i in its most significant half and the value $\pi^{-1}(i)$ in its least significant half. We must prepare A to contain π^{-1} in its most significant half and F in its least significant half for sorting. To swap halves, we use the `vpshufb` instruction. Applying the control bits is simple. We first copy each control bit at index $\mathrm{insertbit}_k(0, i)$ to their counterpart at $\mathrm{insertbit}_k(1, i)$. As the control bit is in the k-th bit of each 32-bit integer, we can apply it to the identity in A with a single XOR. Note that a ymm register holds 8 32-bit integers, thus for $k \leq 3$, both $\mathrm{insertbit}_k(0, i)$ and $\mathrm{insertbit}_k(1, i)$ are in the same register. This requires us to use specific swizzle instructions for the cases $k \leq 2$, again complicating the implementation and requiring a large amount of unrolling.

4.2 Results

We evaluate our implementations and compare their performance against the implementation of cbrecursion in libmceliece version 20250507 [Cho25]. As the performance of all three implementations is strongly determined by the underlying sorting network, we benchmark the three control bit implementations across six different sorting network implementations.

The first is portable4, a scalar implementation of Batcher's odd-even sorting network from the djbsort library [Ber24]. The second is bitonic, an AVX2 implementation of Batcher's Bitonic sorting network. The third is djbsort, which is essentially Batcher's Bitonic sorting network with its wires permuted for SIMD-friendliness. Each has an interlaced variant, named iportable4, ibitonic, and ipermsort respectively.

Experimental environment. All benchmarks are performed on an Intel (R) Core (TM) Ultra 7 165U CPU (Meteor Lake) running Ubuntu 24.04.3 LTS. We use clang version 18.1.3 and gcc version 13.3.0. We utilise the same compiler options as the libmceliece implementation. Table 1 presents the benchmarks for the portable implementations compiled with flags `-O2`, `-fwrapv`, and `-fPIC`. Table 2 presents the benchmarks for the AVX-enabled implementations compiled with extra flags for Intel intrinsics such as `-mavx2` and with optimisations such as `-mtune=meteorlake`. To reduce timing variations, we follow the guidance recommended on the Supercop website [BL24], i.e., we disable Turbo Boost, Hyper Threading, etc. The device has an L1D cache of size 352KiB L1I cache of size 640KiB, an L2 cache of size 6MiB and an L3 cache of size 12MiB.

We record the median clock cycles required to calculate control bits for an array of 2^{13} 16-bit integers (the standard parameters in Classic McEliece) using the read time-stamp counter (`rdtsc`) instruction. We calculate the median from 63 measurements. For $n = 63$, it is highly likely that the interquartile range was less than 1% of the median for every test. We compile and benchmark with both gcc and clang, we record and report the faster of the two. The implementations require several intitialisation steps, such as generating the permutation, temporary storage, and output storage. We initialise all these data structures, and then take all measurements in a tight loop. We then verify the correctness of the control bits and record the timings. We chose this experimental set-up to prioritise reproducibility, one should note that the key-generation phase of Classic McEliece requires only *one* call to the control bits implementation and thus the cache will be in a different state to our benchmarks.

4.3 Sorting Networks

The performance of all three implementations is strongly determined by the sorting network utilised. When using a scalar sorting network implementation, a simple sampling-based analysis with gprof [GKM82] suggested over 97% of execution time is spent sorting in both cbiterative and cbrecursion. The implementation of cbrecursion with djbsort is 6.8× faster than with portable4. The scalar portable4 is an implementation of Odd-Even Merge sort. This is chosen as it requires slightly fewer comparators than Batcher's Bitonic sort. The AVX2-enabled implementations utilise Bitonic, despite it requiring more comparisons than Odd-Even Merge, because it is easy to capture its incredibly regular structure with SIMD instructions [Ebb15].

Interlaced sorting networks. The recursive algorithms observe a natural advantage: each recursive subcall sorts array sizes descending in powers of two. Conversely, *every* call to the sorting network in the iterative implementations must sort an array of full size. Across all standard sorting networks, the iterative implementations are atleast 1.42× slower than cbrecursion. When the iterative algorithms utilise interlaced sorting networks, we observe an interesting effect: the recursive decomposition introduces performance increases *until* the number of elements to sort becomes too small to vectorise. The majority of calls from cbrecursion to djbsort are on array sizes that are too small for djbsort to vectorise. In contrast, the full-size arrays that the iterative variant sorts become increasingly structured after every iteration, requiring fewer comparators in increasingly SIMD-friendly locations. We find cbiteravx with ibitonic is 37% faster than cbrecursion with bitonic.

Permuting sorting networks. The instructions `vpminsd` and `vpmaxsd` apply 8 comparators in parallel *if* all the data elements we want to compare are at the same indices of two different registers. To apply the comparators of distance less than 8, which is the majority in Bitonic, we require heavy usage of swizzle instructions. However, one can reduce the amount of swizzles required by permuting the horizontal wires of the network to ensure the majority of comparators are between addresses a distance of 8 apart. We measure the performance effect of permuted sorting networks on control bit algorithms by providing non-permuted bitonic and ibitonic, and their respective permuted variants djbsort and ipermsort. We find cbrecursion is 27% slower with bitonic than with djbsort. We find cbiteravx is 24% slower than ibitonic than with ipermsort. Our most performant implementation, cbiteravx with ipermsort is 34% faster than cbrecursion with djbsort, which is the current implementation in libmceliece.

Interlaced adaptations and validity. To implement a scalar interlaced sorting network requires minimal adaptations to its standard variant. For example, obtaining iportable4 from portable4 requires altering the terminating clause of a single for loop. The vectorised implementations require considerably more care as the comparisons between addresses of length less than 8 require sequences of swizzle instructions.

Bernstein exerts considerable effort into verifying that djbsort [Ber24] is correct. He provides a verifier which takes the compiled library, symbolically executes and converts it into a sequence of compare-and-exchange operations with the angr tool [SWS+16]. He then uses a 2-way merging argument to verify that the extracted sequence is a sorting network. We do not use symbolic execution to map compiled code into a comparator network to be verified. However, we do have a logical argument that justifies our network choices. We utilise the notion of *equivalence up to wire relabelling*, which was originally used to reduce the search space in the study of minimal sorting network enumeration [BZ14, CCFFSK14, BCCF+17], to increase the SIMD-friendliness of sorting networks by permuting the comparators for comparisons of index widths 1, 2, and 4 (which are the most common and most expensive) to comparators of widths 2^m, 2^{m-1}, and 2^{m-2} (which are the least common and inexpensive). If one permutes the wires of a sorting network by a permutation π, then the composition of this permuted sorting network and subsequently permuting by π correctly sorts all input data. We provide a verified proof of this in the accompanying artifact. We constructed ipermsort with the goal of removing small-distance comparisons by choosing similar bit-swapping permutations and following this argument. When $k \geq 3$, there are no small sorts to remove. When $k = 0$, we use djbsort. When $k = 1, 2$, the k least significant bits are sorted and we construct permutations sending elements that differ in these bits to elements which differ in their most significant bits.

5 Conclusion

Future and related work. It is discussed in McBits [BCS13] whether one could remove the looping control bit calculation entirely. Instead, one would choose the control bits randomly to realise a random permutation. In 1999, Abe proposed the setting of each control bit via a coin flip for the construction of mixnets [Abe99]. This approach, however, leads to the Beneš network realising some permutations exponentially (in the number of elements to permute) more than others [AH01], thus giving the attacker the advantage. This idea is very natural; there is a large amount of research studying the generation of random permutations through random switch setting schemes on the Beneš [AH01, GTS14] and Butterfly network topologies [LP98, Mor05, CV14, Czu15]. However, the random sampling constructions on the Beneš network only consider the coin toss, whereas, the result of 'looping' algorithm is highly structured and non-uniform. For example, we showed that the k-th layer, with $0 \leq k < m$, has 2^k control bits set to zero. Further, for

small cases we observe that when one enters a uniformly random permutation into our control bits algorithm, the probability that the control bit j of layer i $\mathsf{F}[i,j]$ in the first half of the Beneš network ($0 \leq i < m-1$) is set is given by

$$\mathbb{P}(\mathsf{F}[i,j]=1) = \frac{\left\lfloor \frac{j}{2^i} \right\rfloor}{2\left\lfloor \frac{j}{2^i} \right\rfloor + 1}$$

and, for all the control bits in the middle layer through to the last layer, the probability of being set is $\frac{1}{2}$. We sampled control bits independently using the above probabilities and found the resulting network did not realise uniformly random permutations. However, our formulae and Bernstein's appear to exemplify the close relationship between the theory of sorting networks and group theory [Ben75, Yav87, AK89, LJD93]. The formulae of Section 3 present a bridge between control bit settings and the highly-structured cycles of commutators with involutions in the symmetric group. Perhaps, in the aim of generating random permutations, one could avoid choosing uniformly random control bits and, instead, generate random sequences of pairs modelling the commutators and study their minima.

The simplicity, and subsequent ability to prove correctness, of our algorithm suggests it could be used in other investigations of Beneš networks. In the MPC setting, Beneš networks are utilised for oblivious shuffling with a dishonest majority [KS14]. Smart and Talibi Alaoni show how to distribute single-party protocols into secret-shared ones and demonstrate this on the looping control bits algorithm [ST19].

Due to the nature of shuffling as a fundamental primitive in secure computation, there are many implementations of the Beneš network in many different contexts. We detail notable use-cases below. As our algorithm offers increased performance on SIMD enabled processors, it could be utilised in some of the following use-cases. One approach to obliviously sorting data on untrusted hardware, e.g. the cloud, is to obliviously shuffle it and then apply a non-data-oblivious sorting algorithm such as quicksort [HKI+13, DDCO17]. This is broadly called the 'scramble-then-compute' paradigm. It has been used in the construction of mixnets [AKTZ17] and an oblivious radix sort [HICT14]. To protect against cache side-channel attacks in TEEs [BMD+17, MIE17], Beneš networks are utilised in the client/server model for oblivious permutation and shuffling [HOW22, SJG23]. The Beneš network is utilised in recent adaptations of Goldreich and Ostrovsky's classical construction [Gol87, Ost90, GO96] of Oblivious RAM (ORAM) that focus on improving practical efficiency, such as the Onion ORAM protocol [DvDF+16, CCR19] and the work of Zahur et al. [ZWR+16] which shows the classical construction of ORAM is practically efficient for small-to-medium sizes of data despite being asymptotically worse than alternatives such as Circuit ORAM [WCS15].

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