

Quantum nonlocality for each pair in an ensemble

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If violations of Bell's inequality turn up in measurements on an ensemble of particle pairs, do they imply that each pair behaves nonlocally, or only the ensemble as a whole? We show that *each* pair in the ensemble behaves nonlocally when the particles are spins coupled in a singlet state. For spins in a nonsinglet state, however, a model in which some of the pairs behave locally reproduces quantum predictions.

In 1964, Bell [1] proved that quantum mechanics and local realism are incompatible. The incompatibility is statistical: no theory of local hidden variables can reproduce all the statistical predictions of quantum mechanics. But if measurements on an ensemble of particle pairs disprove local realism, it does not immediately follow that each pair in the ensemble behaves nonlocally. In 1989, Greenberger, Horne and Zeilinger [2] (GHZ) showed that if quantum predictions are correct, then every particle triplet in a particular ensemble of triplets must behave nonlocally. The GHZ result came as a surprise [3]. Yet, as we show in this Letter, the extension of Bell's inequality from ensembles to individual systems was implicit in Bell's original paper [1]. We make the extension explicit, for the case of an ensemble of pairs of spins coupled in a singlet state: if the predictions of quantum mechanics are correct, then nonlocality is a property of *each* pair in the ensemble, and not

only of the ensemble as a whole, or of just some of the pairs. For pairs in a nonsinglet state, however, the extension fails: nonlocality may be a property of just some of the pairs.

To clarify how nonlocality for an ensemble differs from nonlocality for individual systems, we recall Mermin's [4] application of Bell's inequality to pairs of photons in a singlet state. Let the polarization of each photon in a pair be measured along one of three axes, each axis perpendicular to the line of flight and making an angle $\frac{2}{3}\pi$ with the others. According to quantum mechanics, the photons' polarizations must agree whenever the same axis is chosen for both; when different axes are chosen, the polarizations agree in only $\frac{1}{4}$ of the cases. This prediction (consistent with experiment [5]) is incompatible with local realism. Any theory of local hidden variables yielding complete agreement when the axes coincide yields at least $\frac{1}{3}$ agreements when they are different. The numerical discrepancy proves that quantum theory and local realism cannot both be correct.

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But the discrepancy, while sufficient to refute local realism, does not refute the following hypothesis: perhaps there are two species of photons, which we shall call "local" and "nonlocal"; the former yield correlations fixed by local hidden variables, and so the observed violations of Bell's inequality are all due to the latter. Such a phenomenological model is analogous to the description of helium II as a mixture of "normal" and "superfluid" components. It follows from the statistical facts mentioned above that up to $\frac{3}{4}$ of the photons could be of the classical, "local" type. If the remaining "nonlocal" photons conspire to yield agreements whenever the axes coincide, and disagreements whenever the axes are different, then on the average there will be $\frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot 0 = \frac{1}{4}$ agreements when different axes are chosen, just as quantum theory predicts. So it appears that nonlocal correlations do not allow any conclusions about a given photon pair.

However, for other possible experiments quantum mechanics predicts results that further constrain the two-species model. If we consider photon polarizations along arbitrary axes, we can constrain the two-species model so severely as to rule it out. The proof is a simple application of the Clauser, Horne, Shimony and Holt [6] (CHSH) inequality. Suppose that there are two species of photons, local and nonlocal, and the quantum correlation function $E_Q(\hat{a}, \hat{b})$, for polarizations measured along axes \hat{a} and \hat{b} , results from a mixture of p of the former and $1-p$ of the latter:

$$E_Q(\hat{a}, \hat{b}) = pE_L(\hat{a}, \hat{b}) + (1-p)E_N(\hat{a}, \hat{b}). \quad (1)$$

Here $E_L(\hat{a}, \hat{b})$ and $E_N(\hat{a}, \hat{b})$ represent the correlation functions for the local and nonlocal photons, respectively. Since $E_L(\hat{a}, \hat{b})$ represents the correlation function for local photons, it must satisfy the CHSH inequality, which holds in any theory compatible with local realism:

$$\begin{aligned} -2 \leq E_L(\hat{a}, \hat{b}) + E_L(\hat{a}, \hat{b}') \\ - E_L(\hat{a}', \hat{b}) + E_L(\hat{a}', \hat{b}') \leq 2, \end{aligned} \quad (2)$$

where \hat{a}, \hat{a}' refer to measurements on one photon and \hat{b}, \hat{b}' refer to measurements on the other. In addition, $E_L(\hat{a}, \hat{b})$ is required to satisfy three other conditions:

- (i) $E_L(\hat{a}, \hat{b})$ approaches 1 as \hat{a} approaches \hat{b} ;

- (ii) for some choice of axes \hat{a} and \hat{b} we have $E_L(\hat{a}, \hat{b}) \neq 1$; and

- (iii) $E_L(\hat{a}, \hat{b})$ is a function of $\hat{a} \cdot \hat{b}$ only.

These conditions are needed because we are considering an ensemble of pairs in a singlet state. For a singlet state, quantum mechanics predicts perfect correlation between particles in a pair when the axes \hat{a} and \hat{b} are parallel. If $E_L(\hat{a}, \hat{b})$ did not obey condition (i), our two-species theory would predict less than perfect correlation and so would not reproduce quantum predictions. In the same way, (ii) follows from the quantum prediction of perfect anti-correlation between orthogonal polarization states. Condition (iii) follows from the rotational invariance of quantum correlations in a singlet state. Strictly speaking, this rotational invariance could be a property of $E_Q(\hat{a}, \hat{b})$ alone, and not of $E_L(\hat{a}, \hat{b})$ and $E_N(\hat{a}, \hat{b})$ separately. But measurements made in the laboratory depend only on the angle between \hat{a} and \hat{b} , and so the measurements could be repeated and averaged over directions \hat{a} and \hat{b} with $\hat{a} \cdot \hat{b}$ held fixed. For measurements averaged over all directions and orientations, the correlation functions $E_L(\hat{a}, \hat{b})$ and $E_N(\hat{a}, \hat{b})$ are effectively rotationally invariant. In what follows, therefore, we refer to such averaged measurements and assume that the correlation functions $E_Q(\hat{a}, \hat{b})$, $E_L(\hat{a}, \hat{b})$ and $E_N(\hat{a}, \hat{b})$ all depend on the angle between \hat{a} and \hat{b} only. We can take them to be symmetric in \hat{a} and \hat{b} for the same reason.

Returning to eq. (2), let us choose axes \hat{a} and \hat{b} to be parallel, so that $E_L(\hat{a}, \hat{b}) = 1$, and define $E_L(\hat{a}, \hat{b}') = E_L(\theta_1)$, $E_L(\hat{a}', \hat{b}) = E_L(\theta_2)$, $E_L(\hat{a}', \hat{b}') = E_L(\theta_1 + \theta_2)$ where all the axes lie in a plane and θ_1 and θ_2 are positive. We assume that the correlation function $E_L(\theta)$ has an expansion in θ for small, but positive, values of the angle. $E_L(\theta)$ need not be analytic at $\theta = 0$, but it is easy to see that $E_L(\theta)$ cannot be constant in any neighborhood of $\theta = 0$. For suppose $E_L(\theta_1) = 1 = E_L(\theta_2)$ for all positive θ_1 and θ_2 in some neighborhood. Substituting into the CHSH inequality, we find that $E_L(\theta_1 + \theta_2) \geq 1$, and since correlations cannot exceed 1, we have $E_L(\theta_1 + \theta_2) = 1$. By induction we arrive at the conclusion that $E_L(\theta) = 1$ for all θ , contradicting condition (ii) above.

We therefore write

$$E_L(\hat{a}, \hat{b}') = 1 - c\theta_1^\kappa + \dots,$$

$$E_L(\hat{a}', \hat{b}') = 1 - c\theta_2^\kappa + \dots, \quad (3)$$

$$E_L(\hat{a}', \hat{b}) = 1 - c(\theta_1 + \theta_2)^\kappa + \dots,$$

where $c > 0$ since correlations cannot exceed 1, and κ is the lowest exponent in the expansion of $E_L(\theta)$ (the dots indicate higher-order terms in the angles). Substituting everything back into the CHSH inequality, we obtain

$$c(\theta_1 + \theta_2)^\kappa - c\theta_1^\kappa - c\theta_2^\kappa + \dots \leq 0, \quad (4)$$

which can be satisfied for small angles only if $\kappa \leq 1$. This result was noted by Bell in his original paper [1]. (Bell actually stated that $E_L(\theta)$ has a term linear in θ for small θ .)

It now follows from this result alone that no mixture of local and nonlocal photons could reproduce the actual quantum correlation function, given by $E_Q(\theta) = \cos 2\theta$. The fraction of local photons is p . Since the local correlation function $E_L(\theta)$ decreases too rapidly for small θ , the nonlocal photons must offset the decrease; but their correlation function $E_N(\theta)$ cannot exceed 1 either, and so we have

$$pE_L(\theta) + (1-p)1 \geq pE_L(\theta) + (1-p)E_N(\theta) \\ = E_Q(\theta), \quad (5)$$

or

$$p(1 - c\theta^\kappa + \dots) + (1-p) \geq 1 - 2\theta^2 + \dots, \quad (6)$$

with c positive and $\kappa \leq 1$. Clearly, whatever the magnitude of c , this inequality is violated for θ sufficiently small, unless p vanishes. Thus, in principle, a correlation experiment with small angles could detect any admixture of local photons! The nonlocality inherent in violations of Bell's inequality is therefore not only a statistical effect, it holds for any pair of photons (or other particles) prepared in a singlet state, if quantum predictions are correct.

Does this result extend to particle pairs in entangled states other than the singlet state? For convenience we will discuss entangled states of two electrons although for photons (or any particle with two spin states) the argument is essentially the same. An entangled state is one that cannot be written as a product of localized wave functions, and a general entangled state of two electrons can be written as a sum of just two terms

$$\Psi = \alpha|\uparrow\rangle|\uparrow\rangle + \beta|\downarrow\rangle|\downarrow\rangle, \quad (7)$$

with α and β real and $\alpha \geq \beta > 0$. This simple form is obtained by defining the spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ with reference to different coordinate systems for the two particles. (When we compare measurements between the two particles, we must refer to these different systems of coordinates.) It has been shown [7,8] that for any entangled state, and for a prescribed set of measurements, quantum correlations violate the CHSH inequality. Can we once again implicate *all* particle pairs in these violations? The answer is, we cannot. The general entangled state of eq. (7) lacks the rotational invariance of the singlet state, and so does its quantum correlation function. Condition (i) above, that $E_L(\hat{a}, \hat{b}) \rightarrow 1$ for $\hat{a} \rightarrow \hat{b}$, now holds only when \hat{a} and \hat{b} both approach $|\uparrow\rangle$ or $|\downarrow\rangle$ (in the respective coordinate systems of each particle), and we cannot impose condition (iii), that $E_L(\hat{a}, \hat{b})$ should depend only on $\hat{a} \cdot \hat{b}$. These complications are sufficient to spoil the result obtained for the singlet case. Indeed, we can construct a model for a mixture of local and nonlocal electrons which reproduces all the predictions of quantum mechanics.

The quantum probability function for the general entangled state of eq. (7) is

$$P_Q(\theta_1, \phi_1; \theta_2, \phi_2) = \alpha^2 \cos^2(\tfrac{1}{2}\theta_1) \cos^2(\tfrac{1}{2}\theta_2) \\ + \beta^2 \sin^2(\tfrac{1}{2}\theta_1) \sin^2(\tfrac{1}{2}\theta_2) + 2\alpha\beta \cos(\phi_1 + \phi_2) \\ \times \cos(\tfrac{1}{2}\theta_1) \cos(\tfrac{1}{2}\theta_2) \sin(\tfrac{1}{2}\theta_1) \sin(\tfrac{1}{2}\theta_2). \quad (8)$$

This function gives the probability for the two particles to both have spin up when one spin is measured along an axis fixed by angles θ_1, ϕ_1 and the other along an axis θ_2, ϕ_2 . The probability functions for other outcomes (such as both spins down) can be obtained from $P_Q(\theta_1, \phi_1; \theta_2, \phi_2)$, so without loss of generality we can consider only this function. A two-species model is viable if and only if it reproduces the quantum probability function $P_Q(\theta_1, \phi_1; \theta_2, \phi_2)$ as a sum of probability functions for local and nonlocal particles:

$$P_Q(\theta_1, \phi_1; \theta_2, \phi_2) = pP_L(\theta_1, \phi_1; \theta_2, \phi_2) \\ + (1-p)P_N(\theta_1, \phi_1; \theta_2, \phi_2). \quad (9)$$

The only constraints on $P_N(\theta_1, \phi_1; \theta_2, \phi_2)$ are that it should take values in the interval $[0, 1]$, and that the

probabilities for various possible results of a measurement sum up to one.

Now suppose that for local particles, the probability to have spin up along an axis θ , ϕ is $\rho(\theta)$, a function independent of ϕ . Let us (for $\alpha \geq \beta$) define $\rho(\theta)$ to be

$$\begin{aligned}\rho(\theta) &= 1, & 0 \leq \theta < \frac{1}{2}\pi, \\ &= \frac{1}{2}, & \theta = \frac{1}{2}\pi, \\ &= 0, & \frac{1}{2}\pi < \theta \leq \pi,\end{aligned}\quad (10)$$

so that the probability for a local pair of particles to have spin projections along θ_1, ϕ_1 and θ_2, ϕ_2 is $P_L(\theta_1, \phi_1; \theta_2, \phi_2) = \rho(\theta_1)\rho(\theta_2)$. The pairs of local particles then have only local correlations, as the two-species model requires. The remaining part of $P_Q(\theta_1, \phi_1; \theta_2, \phi_2)$ then represents the probability function for the nonlocal pairs of particles:

$$\begin{aligned}P_N(\theta_1, \phi_1; \theta_2, \phi_2) \\ = \frac{P_Q(\theta_1, \phi_1; \theta_2, \phi_2) - pP_L(\theta_1, \phi_1; \theta_2, \phi_2)}{1 - p}.\end{aligned}\quad (11)$$

The maximum allowed value for p is $p = \frac{1}{4}(\alpha - \beta)^2$, and it is straightforward to check that this value makes $P_N(\theta_1, \phi_1; \theta_2, \phi_2)$ obey the constraints mentioned above. The probability function for the nonlocal particles may seem artificial, but the two-species model is consistent. We have shown that as many as $\frac{1}{4}(\alpha - \beta)^2$ of the pairs could be local pairs governed by the local probability function $\rho(\theta_1)\rho(\theta_2)$.

As expected, the fraction of local pairs goes to zero when $\alpha = \beta$. This condition also maximizes the violation of the CHSH inequality. Are maximal violation [9] and the absence of local pairs related? We

conjecture that for a general entangled state of n systems, the proportion of "local" systems goes to zero just when the generalized CHSH inequality [8] is violated maximally.

It is quite likely that better models than the one presented here would allow a larger fraction of the particle pairs to be local when $\alpha \neq \beta$. The model offered here (which has some overlap with a model discussed by Bell [1]) is merely the simplest example how an ensemble of pairs yielding nonlocal correlations could contain some pairs following local plans. We have seen that an ensemble for a singlet state excludes such pairs. The simple model of this paper is adequate to point out this difference, as a step towards clarifying how and where quantum nonlocality operates. Elsewhere [10] we have discussed further implications of Bell's inequality.

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