

In loving memory of Itamar Pitowsky

“...forasmuch as thou knowest how we are to encamp in the wilderness, and thou mayest be to us instead of eyes” (Num. 10, 31).

A.E.

HOW MANY MAXWELL’S DEMONS, AND WHERE?

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Maxwell’s demon has been conceived as a tool for challenging the law of entropy increase. Several resolutions of the paradox have been proposed, making it clear that the demon does not violate the second law of thermodynamics. Nevertheless, since recent experiments come close to realizing some variants of Maxwell’s demon, it is interesting to revisit it. In this article we first address two questions, left unnoticed despite many years of intensive study: i) on which side of the door should the demon be located when the door is shut? and ii) how is kinetic energy exchanged between the two compartments due to the demon’s sorting? We propose a simple setting which is more realistic than the current versions, in which the demon monitors and accesses both sides of the partition, so as to enable the sorting task. Next we study the impact of this sorting on the molecular kinetic energy exchanges. We show that the temperature difference between compartments grows till the cold part of the gas approaches 0 K. We then emphasize that this setting yields to the familiar resolution of the paradox. In the last part we derive the expression of the average rate of energy flow between the two compartments of the system, based on the new setting proposed.

Key words: Maxwell’s demon; “temperature demon,” “pressure demon,” entropy, second law.

Maxwell’s [1] demon is a famous paradox, the various resolutions of which have provided new insights into the nature of entropy and information. Yet some basic details of the demon’s alleged operation remain unclear. Below we address these still-open issues.

1. The Paradox

For the sake of clarity we begin with a simple and precise protocol for the demon’s operation. Let a closed box be divided into two halves by a compartment that has a small hole in it. The box is full with a gas at temperature T_o . As the gas’s molecules freely move between the box’s two halves through the hole, its entropy is maximal. Now consider a tiny demon positioned next to the hole. It measures the molecules’ velocities. It then determines the initial root mean square velocity, $s_0 = (\langle v^2 \rangle)^{1/2}$, as the threshold. Every time a fast molecule, $v^2 > s_0^2$, approaches the hole from the left, the

demon lets it pass to the right side, whereas every slow molecule, $v^2 < s_0^2$, is denied passage by closing the door. Conversely, the demon allows slow molecules to pass from the right to the left side and denies such passage to fast molecules. In time, fast molecules will accumulate on the right side and slow molecules on the left, with the resulting temperatures $T_r > T_o > T_l$.

The apparent paradox is clear. Entropy has been lowered in defiance of the Second Law: The energy required to open and close the slit can be made negligible, therefore operating the slit does not disperse energy and hence does not increase entropy outside of the box [2] [3].

Two species of the demon are known. The above demon, which Maxwell first had in mind, is a “temperature demon”. Later he considered also a “less intelligent” one, named “pressure demon,” whose task is simpler, namely, to concentrate all the gas in one side of the box [4]. Its protocol in the latter case is therefore simpler: Let all molecules pass from one side to the other, never *vice versa* (see [1] for Maxwell’s own references to both species). In what follows we discuss the commoner “temperature demon.”

The paradox importance stems from its being deeply rooted in some of thermodynamics’ basic principles. It also bears on some fundamental issues in biology and nanotechnology [5].

2. The Resolutions

Equally famous is the paradox's resolution, due to Szilard [6] and Brillouin [7]: The acquisition of the information needed for the sorting operation has its cost in energy (e.g., warranting an additional light source), which increases entropy more than that decreased by the sorting.

Landauer [8] and Bennett [9] have made the resolution more precise, proving that the principal energetic cost goes to the erasure stage: In order for the demon to perform continuously, it must erase one bit of information before registering a new one. It is this erasure that takes the basic energetic toll.

For the discussion’s completion we also mention a dissenting view that the above resolutions to the paradox are not satisfactory, hence the demon may violate the second law nonetheless [10].

3. How many Demons, and Where?

Surprisingly, in all the discussions of Maxwell's demon a seemingly trivial question has been left unaddressed: *On which side of the door should the demon wait during the time interval it is shut?*

For the pressure demon the answer is simple: He must always be positioned within the half of the box that has to be emptied, waiting for *any* molecule that approaches the hole at whatever velocity, in order to open the door and let it pass to the other side. But for the commoner, temperature demon, the question is trickier (Fig.1): In order to maintain an equal number of molecules on both sides of the partition, the demon must have access to both of them. Yet the door must be shut every time a molecule is about to make an inappropriate passage. From which side, then, should the demon shut it?

We note in passing that this question has been indirectly dealt with by Leff and Rex's [2] review, which shows that some authors depicted the demon as located on one side of the door, while others depicted it as operating from outside the box, on which the reviewers themselves comment: "placing a temperature-demon outside the gas is questionable because of the need for thermal isolation" (p. 7). Yet, surprisingly, the authors did not indicate their own opinion concerning this essential point.

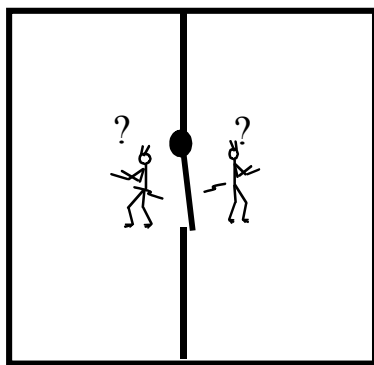


Fig 1.

Two possibilities, then, come to mind:

1. *One demon moving back and forth between the two compartments.* As the demon is supposed to be material, made of at least one atom and possessing at least one degree of freedom. This switching between compartments would therefore give rise to an additional thermal energy transfer. In other words, it will thermalize the gas, hence this procedure will fail.
2. *Two demons, each located at another side of the door.* The inefficiency of this method is even more obvious. Two demons must work in accordance with one

another, so as to avoid, for example, the case in which one opens the door at the same moment that the other has to shut it. For this propose, they need to communicate between them, necessitating again the energy investment for the information acquisition invoked in Szilard's [11] solution.

In what follows we present still another mode of operation involving a single demon, and show that it leads to formulae different from those of Leff [14] which are based on the selective effusion process (see Section 7 below). Our proposal is based on the following reasoning. Rather than performing bulk measurements, it is more practical to measure the velocity of a single isolated molecule within a small protected volume accessible to the demon.

Having indicated a single demon, we now deal with the “which side” problem. Let the two compartments be not in direct communication, but rather connected through a small intermediate chamber with two doors, one opening to the cold side and the other to the hot side (See Fig. 2).

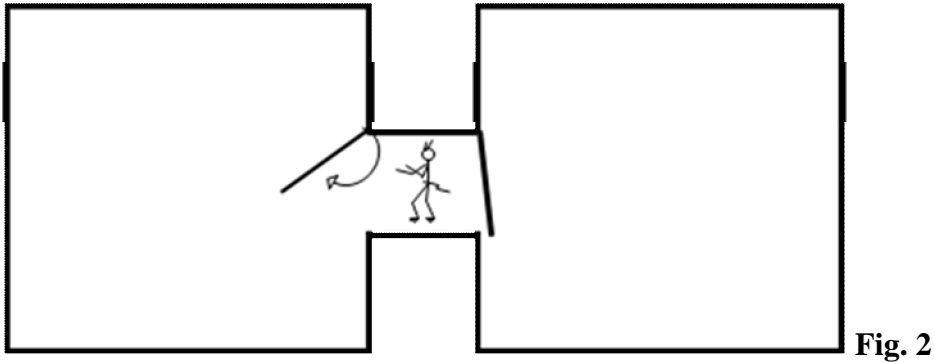


Fig. 2

Calling $n=N/V$ the density of molecules in the compartments, the volume of the connecting chamber should be $1/n$, so that it contains on average one molecule. The working cycle of the demon, starting from a state in which the door on the hot (right) side is open and the other door is closed, is as follows:

1. Close the door on the hot side.
2. Make sure a single molecule is trapped.
3. Measure its velocity.
4. If the velocity is above the threshold, open the door on the cold side, wait for some preset time Δt_0 (to let the molecule go back to where it came from), and

return to step 1. Otherwise open the other door to realize a transfer and wait some preset time Δt_0 .

5. Close the door on the cold side.
6. Make sure a single molecule is trapped.
7. Measure its velocity.
8. If the velocity is below the threshold, open the door on the hot side, wait for some preset time Δt_0 (to let the molecule go back to where it comes from), and return to step 5. Otherwise open the other door to realize a transfer (from the hot to the cold compartment) and wait some preset time Δt_0 .

A few remarks are in order:

- i. At any time, only one door is in the “open” position.
- ii. The protocol ensures that transfers occur in pairs: a transfer in one direction is necessarily followed by a transfer in the reverse direction.
- iii. There is a non-zero probability that zero, two or more molecules would get trapped simultaneously during steps 1 and 5. In that case, the door should be reopened to the same side for another try.
- iv. Having verified that exactly one molecule is present (steps 2 and 6), and after the velocity measurement, the door must remain open for a sufficiently long time Δt_0 to allow for the trapped molecule to go out and for another to get in. This interval Δt_0 is a parameter of the protocol and its value depends on the velocity of the slowest molecules to be processed.
- v. Neglecting the measurement times, the minimum time for the total cycle is thus equal to $2 \cdot \Delta t_0$. In any case the total cycle time is $k \cdot \Delta t_0$, with $k > 1$, when at least one of the transfers fails on the first trial.
- vi. Once a molecule has been trapped, there is no need to measure its position, nor to perform complicated computations of individual trajectories to compute the time at which the trapdoor should be opened.

This defines the framework of the derivation in section 7. The demon picks at random *one* molecule at each cycle, and its velocity is thus drawn from Maxwell-Boltzmann

(MB) distribution (divided by 2 because half the velocity space is involved) at the current temperature.

Von Neumann [12] suggested a superficially similar model, in terms of “windows” in a membrane separating two compartments, but he posits the success of his model (and its dissimilarity from Maxwell's demon, of which he explicitly states that it is not an example) precisely on the fact that it ignores which side of the membrane the molecules originate from, a point which, as we have shown above, is essential to the operation of Maxwell's demon.

4. Intermolecular Kinetic Energy Exchanges

Our next question concerns the details of the heat extraction performed by the demon. All texts conclude Maxwell's *gedankenexperiment* with demon's apparent completion of the molecular sorting task, proceeding to the paradox's resolution. It thus appears that the demon's only operation on the molecules is their placement in the appropriate compartment. O. Penrose [13] (who incidentally preceded Landauer [8] and Bennett [9] in pointing out the role of erasure in resolving Maxwell's paradox) gives what seems to be the common view of the demon's task, namely, "collecting all the fast molecules on one side of a diaphragm and the slow ones on the other" (p. 1994).

But is it the case? A pure sorting based on velocities would result in truncated Maxwell-Boltzmann distributions, the above-threshold temperature part of the gas now being in the hot compartment, and the below-threshold temperature part in the cold one.

Let us analyze the process in detail. In what follows we are interested only in energy transfers and do not need to worry about the exact working cycle of the demon. Consider again the above demon's box divided into two compartments, each containing N molecules of mass m , at the same initial temperature T_0 . For a (monoatomic) ideal gas, the internal energy is $U = N \cdot (1/2) \cdot m \cdot s_0^2$ (N times the average kinetic energy). As a function of temperature T_0 : $U = (3/2) N k T_0$. For convenience, we will henceforth refer to the right compartment as the “hot” and to the left one as the “cold” compartment.

The demon executes the following protocol: It first lets one molecule with velocity v , such that $v^2 < s_0^2$, to go from the hot to the cold compartment. The removal of this

slower-than-average molecule results in a small increase of temperature of the hot side. The demon's next step is to let a molecule go from the cold to the hot side, in order to recover an equal number of molecules on both sides. This time it chooses a molecule that satisfies $v^2 > s_0^2$. This contributes to a slight decrease of the temperature of the cold side. The time between the two transfers is long enough for the system to relax toward statistical equilibrium, so that the demon always draws molecules from Maxwell-Boltzmann distributions (the one associated to the compartment considered). The variables T_{Hi} and T_{Ci} represent the temperature of these compartments, respectively, at step number i . By convention $i=0$ represents the initial step where $T_{H0}=T_{C0}=T_0$. An odd value of i is associated with a state with $N+1$ molecules in the cold compartment, and $N-1$ in the hot one. An even value is associated to a state with N molecules in each compartment.

Let us now show that, following the above simple protocol, the temperature difference $T_{Hi}-T_{Ci}$ grows monotonically. Let us assume that we are at step $i=2p$. We want to express $T_{H,2p+2}$ and $T_{C,2p+2}$ as functions of $T_{H,2p}$ and $T_{C,2p}$. In the transition from $i=2p$ to $i=2p+1$, the demon lets one molecule with $v_{2p+1}^2 < s_0^2$ from the hot compartment go to the cold one. Denoting by e_{2p+1} the kinetic energy of this molecule, we can write: $e_{2p+1} = (3/2).kT_0 - \epsilon_{2p+1}$, where $\epsilon_{2p+1} > 0$ and $(3/2).kT_0$ represents the average kinetic energy of the molecules at T_0 . The internal energies of the compartments after the transfer of this molecule are:

$$U_{H,2p+1} = U_{H,2p} - e_{2p+1}$$

$$U_{C,2p+1} = U_{C,2p} + e_{2p+1} .$$

In the next step, $i=2p+2=2(p+1)$, the demon transfers a molecule with kinetic energy e_{2p+2} higher than $(3/2).kT_0$ from the cold to the hot side: $e_{2p+2} = (3/2).kT_0 + \epsilon_{2p+2}$, where $\epsilon_{2p+2} > 0$. Conservation of energy gives:

$$U_{H,2p+2} = U_{H,2p+1} + e_{2p+2} = U_{H,2p} - e_{2p+1} + e_{2p+2}$$

$$U_{C,2p+2} = U_{C,2p+1} - e_{2p+2} = U_{C,2p} + e_{2p+1} - e_{2p+2}$$

Using the expressions for the e_i 's, and for the U 's in terms of the T 's, we get:

$$T_{H,2(p+1)} = T_{H,2p} + 2 (\epsilon_{2p+1} + \epsilon_{2(p+1)}) / (3Nk)$$

$$T_{C,2(p+1)} = T_{C,2p} - 2 (\varepsilon_{2p+1} + \varepsilon_{2(p+1)}) / (3Nk).$$

From these expressions one deduces that indeed, when one considers only pairs of consecutive transfers (i.e., even steps), T_H monotonically increases and T_C monotonically decreases (since ε_{i+1} and ε_{i+2} are both strictly positive). This was not obvious because a molecule with $v_{2p+1}^2 < s_0^2$ leaving the hot side contributes to increase $T_{H,2p+1}$, but may well also increase the *cold* side, $T_{C,2p+1}$, when v_{2p+1}^2 is larger than the average squared velocity in the cold compartment at the current temperature $T_{C,2p+1}$. It turns out, however, that the increase in $T_{H,2p+1}$ is larger than the increase in $T_{C,2p+1}$, hence the difference $T_{H,2p+1} - T_{C,2p+1}$ is monotonically increasing. The same is true for a transfer in the reverse direction, from cold to hot: The criterion $v^2 > s_0^2$ insures that T_C will decrease, but the temperature T_H might decrease too, the important point being that the temperature difference is growing nevertheless.

Two remarks can be made at this point. First, the arithmetic average $\langle T \rangle_{2(p+1)}$ of $T_{H,2(p+1)}$ and $T_{C,2(p+1)}$ is equal to the average $\langle T \rangle_{2p}$. From this we deduce, by recurrence: $\langle T \rangle_{2p} = T_0$. Second, the monotonicity property of T_C and T_H holds true if the protocol compares the squared velocities to $(3k_B T_s)/m$, where T_s is any temperature within the interval $[T_{C,2p}, T_{H,2p}]$. The only requirement is that the *same* temperature T_s should be chosen (in $[T_{C,2p}, T_{H,2p}]$) for each pair of consecutive transfers (hot to cold, and cold to hot, the reference states being those in which there are exactly N molecules in each compartment).

The entropy in the transition from step $2p$ to step $2(p+1)$ can be calculated from the well-known formula for the change in entropy of an ideal gas, when the volume and number of molecules remain the same, and only the temperature changes:

$$S_{H,2(p+1)} - S_{H,2p} = (3/2) Nk \ln(T_{H,2(p+1)}/T_{H,2p})$$

$$S_{C,2(p+1)} - S_{C,2p} = (3/2) Nk \ln(T_{C,2(p+1)}/T_{C,2p})$$

Using the fact that $\ln(y) = y - 1$ when y is close to 1, we get:

$$S_{H,2(p+1)} - S_{H,2p} = (3/2) Nk (T_{H,2(p+1)} - T_{H,2p}) / T_{H,2p}$$

$$S_{C,2(p+1)} - S_{C,2p} = (3/2) Nk (T_{C,2(p+1)} - T_{C,2p}) / T_{C,2p}$$

Then, using the expressions we obtained above for the temperature variations:

$$S_{H,2(p+1)} - S_{H,2p} = (\varepsilon_{2p+1} + \varepsilon_{2(p+1)}) / T_{H,2p}$$

$$S_{C,2(p+1)} - S_{C,2p} = -(\varepsilon_{2p+1} + \varepsilon_{2(p+1)}) / T_{C,2p}$$

Finally, the total variation of entropy for both compartments is:

$$S_{Total,2(p+1)} - S_{Total,2p} = -(\varepsilon_{2p+1} + \varepsilon_{2(p+1)}) (T_{H,2p} - T_{C,2p}) / (T_{H,2p} T_{C,2p})$$

It can thus be verified that the entropy variation is negative, as expected (since ε_{2p+1} and $\varepsilon_{2(p+1)}$ are strictly positive quantities).

In conclusion of this part, with the protocol considered, the temperatures of the two compartments are growing apart monotonically, and the average of the two temperatures stays equal to T_0 (only at even steps $i=2p$, strictly speaking).

How far can the demon go? It is clear that, when the internal energy $U_{C,i} = N \cdot (m/2) \cdot \langle v^2 \rangle_i$ becomes smaller than the threshold value $(m/2) \cdot s_0^2$, the process stops: Even if a fluctuation concentrates the total kinetic energy of the N molecules into one of them, this molecule would not be selected by the demon. The lower bound for $T_{C,i}$ is $T_{C,min} = T_0/N$, and is thus very close to $0K$ (N being of the order of the Avogadro constant).

However, with reference to our above remark on monotonicity, it is still conceivable to go beyond that limit by decreasing the temperature T_s used in the selection criterion, instead of keeping it constant at T_0 during the process. Of course the durations involved now become unphysically long well before reaching T_0/N . In other words, given infinite time, and neglecting the fact that the gas would eventually experience a phase transition, the demon could get very close to $0K$.

5. The Paradox Made More Precise

Let us summarize the above analysis. The demon monitors not only velocity differences between one molecule and another, but also variations in the velocity of the same molecule over time. Thus, *a molecule that was denied passage at one instant may be allowed to pass at the next, and a molecule that has passed from one side to the other may make the passage back and forth, in accordance with the changing velocities it acquires due to collisions*. What the demon actually accumulates in the hot compartment, therefore, is not merely fast molecules, but kinetic energy itself.

Gradually, the cold and hot portions of the gas would deviate further and further from the initial temperature T_0 , decreasing the likelihood of velocity fluctuations large

enough to enable molecules crossing sides. The demon will now have to expend another resource, namely, time [14].

One could think of optimizing the durations by devising a meta-protocol for T_s decrease (requiring an external control device), as explained at the end of section 4, but the time between two successive transfers would nevertheless remain extremely long. The non trivial result is that, in theory, with this protocol Maxwell's demon could extract almost all the energy from one compartment, and the temperature would approach $0K$.

6. Reaffirming the Thermodynamic Resolution

We next show how our refined analysis of the demon's feat is subject to the same restrictions that hold for the familiar one, namely, that the second law of thermodynamics is not violated.

In the standard experiment, these restrictions say that in order to transfer molecules between the hot and cold compartments, the demon must measure their velocities,[6,7] and further erase each measurement's result so as to enable further measurements,[8,9] for which it must pay with energy.

How do these restrictions apply to our version? A possible source of confusion lies in the use of the word *sorting* itself, which implies that the sorted objects have permanent attributes. We have stressed, however, that “fast” and “slow” are not permanent but temporary attributes of molecules. The demon's task is therefore different from sorting, say, ^{16}O and ^{17}O isotopes, whose properties remains unchanged. Therefore, as the demon is likely to measure the changing velocities of the same molecule several times, that means that it has to carry out a larger number of measurements than that envisaged in the common version described in section 4. In view of the fact that the present protocol allows the demon to achieve much higher energy difference, it perfectly makes sense that it has to carry out a much larger number of measurements, for which the thermodynamic price in energy is naturally higher.

Notice also that, as the probability for molecules to deviate from their ensemble average velocity decreases with the increase of temperature difference between

compartments, the demon must wait for longer and longer periods, eventually for an infinite time.

7. Analysis of the Rate of Energy Transfer for a Variant of the Improved Protocol

Leff [14] presented a derivation of the rate of energy transfer between the two compartments, which is based on the kinetic theory of the effusion process. In Leff's picture the demon measures the velocity of the molecules by emitting a photon in one of the box's two compartments and measuring its scattering upon collision with a gas molecule. This scheme may give rise to some errors, because a photon does not necessarily interact with the nearest molecule. In addition, the molecule's *position* is not measured by this method, making it difficult to open the trapdoor during just the time needed to let the right molecule go through. In other words, bulk measurements are not practical. Moreover, Leff does not mention the question addressed in section 3 concerning the demon's position on whither side of the door.

These problems are avoided by our setting, where the demon first *isolates* a molecule and then measures its velocity. In what follows this setting enables us also to compute the limit temperatures reached in infinite time and present a new derivation of the rate of energy transfer using the setting presented in section 3 with a single demon residing within an intermediate chamber.

We now consider a variant of the above setting: The demon uses the current temperatures T_C and T_H instead of T_0 . It lets a molecule go from the hot to the cold compartment if $e_{i+1} < (3/2)kT_H$, and in the next step lets a molecule from the cold to the hot compartment if $e_{i+2} > (3/2)kT_C$. In other words, there are now two different threshold values, and the selection criteria are now less stringent. This results in a larger number of transfers realized per time unit, and in the loss of the monotonicity property. At a given temperature T the Maxwell-Boltzmann (MB) velocity distribution is:

$$f(\mathbf{v}) = (m/2\pi kT)^{3/2} \exp(-mv^2/2kT).$$

The distribution of the norm v of the velocity is: $F(v) = 4\pi v^2 f(\mathbf{v})$ where the factor 4π results from the integration of the surface element $\sin\theta d\theta d\phi$ (spherical coordinates) over a sphere. In our case the demon selects molecules in a half space and this factor

should be replaced by 2π . The root mean square velocity at temperature T is denoted ρ_T (equal to $(3kT/m)^{1/2}$).

Starting from a state where the temperature is T_0 in both compartments, the temperatures will grow apart steadily during the first steps. The system will thus reach a state where the two F distributions are significantly separated. Consequently, a molecule from the hot side which passes the test might well heat up the cold side (if its kinetic energy is between $(3/2)kT_C$ and $(3/2)kT_H$). In the long run, the system will reach a point where the balance of the energy transfers is 0 when averaged over time. To make this more precise, let us consider a series of measurement cycles. We assume the measurement time itself is negligible, and only take into account the time Δt_0 during which the door remains open (see section 3). Every Δt_0 a molecule is trapped in the intermediate chamber and its velocity measured. If the molecule does not meet the above criterion, it is sent back to where it came from, and the cycle is repeated until a molecule passes the test. Thus each transfer is characterized by the number of trials n which have been made, and by the kinetic energy e of the selected molecule, these two quantities being both stochastic variables. Starting from step $i=2p$, a pair of consecutive $H-C$ and $C-H$ transfers is described by $((n_{2p+1}, e_{2p+1}), (n_{2(p+1)}, e_{2(p+1)}))$, meaning that the transfer to $2p+1$ took $n_{2p+1} \cdot \Delta t_0$ unit of time and the transfer to $2(p+1)$, in the reverse direction, took $n_{2(p+1)} \cdot \Delta t_0$. Each pair of transfers changes the total energy of the compartments, and consequently their temperature, by an infinitesimal amount, so that a very large number of transfers is required to obtain a significant displacement of the F_{TC} and F_{TH} distributions. Now let us assume that the demon has made a large number M of pairs of consecutive transfers over a period of time. M is chosen to be large enough for the averages of n_{2p} and e_{2p} to make sense, and small enough to consider that the F_{TC} and F_{TH} distributions are constant. The variation of temperature δT resulting from a molecule transfer is of the order T/N , where T is the temperature of the compartment considered and N the number of molecules. If N is taken as being the Avogadro number, then M can be fairly large before having a significant impact on the MB distributions. The average over M transfer pairs of the kinetic energy flowing from the cold to the hot compartment is $\Delta e_{CH} = \langle e_{2(p+1)} - e_{2p+1} \rangle$, which can be written $\langle e \rangle_{CH} - \langle e \rangle_{HC}$ because the draws are independent. It should be understood that $\langle e \rangle_{HC}$ is the arithmetic average over the odd indices, corresponding to transitions from hot to cold, and similarly for $\langle e \rangle_{CH}$.

The average time taken by the M transfer pairs is equal to $(\langle n \rangle_{CH} + \langle n \rangle_{HC}) \Delta t_0$. Finally the rate κ of energy transfer from C to H is

$$\kappa = \frac{1}{2} \cdot (m / \Delta t_0) \cdot (\langle v^2 \rangle_{CH} - \langle v^2 \rangle_{HC}) / (\langle n \rangle_{CH} + \langle n \rangle_{HC}).$$

We assume next that the above arithmetic averages over M independent draws, M being large, can be approximated by ensemble averages. To express these, we define $I_C(\rho, \alpha)$ and $I_H(\rho, \alpha)$ as the integrals of $v^4 \exp(-\alpha v^2)$ from ρ to $+\infty$, and from 0 to ρ , respectively. Similarly $J_C(\rho, \alpha)$ and $J_H(\rho, \alpha)$ are the integrals of $v^2 \exp(-\alpha v^2)$ from ρ to $+\infty$, and from 0 to ρ , respectively. Thus $\langle v^2 \rangle_{HC} = I_H(\rho_{TH}, \alpha_{TH}) / J_H(\rho_{TH}, \alpha_{TH})$, as it samples from the cold portion of the hot distribution, and similarly $\langle v^2 \rangle_{CH} = I_C(\rho_{TC}, \alpha_{TC}) / J_C(\rho_{TC}, \alpha_{TC})$, as it samples from the hot portion of the cold side's distribution, with $\alpha_{TH} = m / (2kT_H)$ and $\alpha_{TC} = m / (2kT_C)$.

The stationary regime is reached when the average $\langle v^2 \rangle_{HC}$ of molecules drawn from the hot compartment is equal to the average $\langle v^2 \rangle_{CH}$ of the ones drawn from the cold compartment, which can be rewritten as: $\langle v^2 \rangle_{HC} = \langle v^2 \rangle_{CH}$.

The integrals to compute are well-known and the results are expressed in term of the Gaussian error function $\text{erf}(x)$ and $\text{erfc}(x) = 1 - \text{erf}(x)$:

$$J_H(\rho, \alpha) = \sqrt{\pi/4} \alpha^{-3/2} \text{erf}(\rho\sqrt{\alpha}) + (1/2) (\rho/\alpha) \exp(-\alpha\rho^2)$$

$$J_C(\rho, \alpha) = \sqrt{\pi/4} \alpha^{-3/2} \text{erfc}(\rho\sqrt{\alpha}) - (1/2) (\rho/\alpha) \exp(-\alpha\rho^2)$$

$$I_H(\rho, \alpha) = (3/8) \sqrt{\pi} \alpha^{-5/2} \text{erf}(\rho\sqrt{\alpha}) - (1/2) (3/2 + \alpha\rho^2) (\rho/\alpha^2) \exp(-\alpha\rho^2)$$

$$I_C(\rho, \alpha) = (3/8) \sqrt{\pi} \alpha^{-5/2} \text{erfc}(\rho\sqrt{\alpha}) + (1/2) (3/2 + \alpha\rho^2) (\rho/\alpha^2) \exp(-\alpha\rho^2)$$

Considering that $\alpha_T = m / 2kT$ and $\rho_T = (3kT/m)^{1/2}$, a simplification occurs: $\alpha\rho^2 = 3/2$.

$$\langle v^2 \rangle_{HC} = (kT_H / m) \cdot [(3\sqrt{\pi} \text{erf}((3/2)^{1/2}) - 6\sqrt{6} \exp(-3/2)) / (\sqrt{\pi} \text{erf}((3/2)^{1/2}) - \sqrt{6} \exp(-3/2))]$$

$$\langle v^2 \rangle_{CH} = (kT_C / m) \cdot [(3\sqrt{\pi} \text{erfc}((3/2)^{1/2}) + 6\sqrt{6} \exp(-3/2)) / (\sqrt{\pi} \text{erfc}((3/2)^{1/2}) + \sqrt{6} \exp(-3/2))]$$

A remarkable result ensues: The condition for a stationary state reduces to a linear relationship between T_C and T_H . Numerically: $T_H = 3.63 T_C$. Conservation of energy entails $T_C + T_H = 2T_0$, so that:

$$T_H = 1.568T_0, T_C = 0.432T_0$$

With $T_0 = 298K$ (room temperature), for instance, we get $T_H = 467.3K$ (close to the melting point of Lithium) and $T_C = 128.7K$ (close to the condensation point of NO)!

In order to obtain the full expression of the rate κ , we still need to compute $\langle n \rangle_{HC}$ and $\langle n \rangle_{CH}$. We denote by q_C the probability that $v^2 > \rho_C^2$ when a molecule is picked at random in the distribution F_{TC} . It is easy to show that the probability $pr(n_{2(p+1)} = a)$ for the number of trials to be equal to a is $(1 - q_C)^{a-1} q_C$. Consequently:

$$\langle n \rangle_{CH} = \sum a (1 - q_C)^{a-1} q_C = -q_C \frac{d}{dq_C} \left(\sum (1 - q_C)^a \right)$$

where the summations are from $a=1$ to infinity. From this we deduce $\langle n \rangle_{CH} = 1/q_C$. Since by definition $q_C = 2\pi(m/2\pi k T_C)^{3/2} J_C(\rho_{TC}, \alpha_{TC})$, this gives the expression of $\langle n \rangle_{CH}$ in term of an integral we have already computed. The same derivation would give for odd steps: $\langle n \rangle_{HC} = 1/q_H$, with $q_H = 2\pi(m/2\pi k T_H)^{3/2} J_H(\rho_{TH}, \alpha_{TH})$. Since the rate κ is expressed in term of $\langle v^2 \rangle_{HC}$, $\langle v^2 \rangle_{CH}$, $\langle n \rangle_{HC}$ and $\langle n \rangle_{CH}$, we now have a full expression of K as a function of the temperatures (T_H and T_C), Δt_0 and basic constants:

$$\kappa = \frac{1}{2\Delta t_0} \sqrt{\frac{m^5}{2\pi k^3}} \cdot \frac{J_H I_C - J_C I_H}{J_H T_C^{3/2} + J_C T_H^{3/2}}$$

Remember that Δt_0 , the time interval during the doors remain open, is a parameter of the protocol, as explained in section 3.

We now revisit Leff's [14] derivation of the flux generated by the demon and the rate of energy exchange between the compartments. The setting is different (selective effusion process), and the protocol is also slightly different from the one considered here: A molecule is simply allowed to enter into H , or C , if its kinetic energy is greater than $(3/2)kT_H$, or lower than $(3/2)kT_C$, respectively. The second difference is minor as it suffices to adapt the integration bounds in the above computation. The expressions obtained are different, and a comparison of the two approaches is interesting.

In Leff's derivation the action of the demon is viewed as a selective effusion process. Two comments can be made:

- i.* The validity of Leff's derivation is questionable. The demon manipulates one molecule at a time. Consequently, the energy transferred, and the time between two arrivals, should both be considered as stochastic variables. The rate of energy transfer should then be computed from the probability distribution of these variables. It is nevertheless possible that Leff's results are qualitatively correct if the fluxes and energy rates are interpreted as averages over a large number of transfers (a method we used explicitly in our derivation).
- ii.* In Leff's picture, the demon measures the velocity of the molecules by emitting a photon in one of the box's two compartments and measuring the scattered photon. This scheme would generate many errors because the photon would not necessarily interact with the closest molecule. In addition the molecule's position is not measured this way, and it is thus difficult to open the trapdoor during just the time needed to select the right molecule. In other words, bulk measurements are not practical.

In conclusion, two categories of protocols have been considered in this paper for Maxwell's demon operation. In the first category the demon uses a single threshold to select "hot" and "cold" molecules. Then, this single threshold can either stay fixed at the initial temperature, or be adjusted as the demon proceeds. We showed that the evolution of T_C and T_H is monotonous as long as the threshold temperature used at each step belongs to the interval $[T_C, T_H]$. This allows, in theory, to reach a temperature very close to $0K$. Next we studied a different kind of protocol in which the demon uses two different thresholds, one for each compartment. We showed that with this protocol the system reaches a stationary regime, and that is a linear relationship holds between the temperatures, involving only mathematical constants.

8. Summary

In this paper we presented a new, refined scheme for the operation mode of Maxwell's demon, in which the demon resides in a little intermediate chamber between the box's two compartments, and makes measurements on isolated molecules trapped within the chamber. In addition we have made the Maxwell demon paradox

more precise, stressing the fact the demon does not actually *sort* molecules on the basis of a fixed attribute, but transfers kinetic energy from one compartment to the other. In the last section we analyzed a variant of the demon protocol in which the molecule velocity is not compared to a fixed temperature. Using the scheme mentioned above, we proposed a new derivation of the rate of energy transfer, and of the stationary temperature reached. We then commented on a similar study proposed by Leff [12] and argued that our scheme is more realistic.

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