CE 6305 Computer Arithmetic
Home Work 6

1. We have a module, M4, that performs the multiply and add function $m = a \times b + d$, for 4 bit quantities a, b, d, producing an 8 bit carry complete result. Show how to connect multiple M4 units to produce a 16 bit by 16 bit multiply and add unit.

   Answer:
   Answer

   Consider the product of 2 degree 3 polynomials:

   $$(ax^3 + bx^2 + cx + d) \times (px^3 + qx^2 + rx + s)$$

   We get 7 terms in the result and they are stacked as follows:

   ```
               33333333
           4444444422222222
       555555553333333311111111
   66666666444444442222222200000000
       555555553333333311111111
           4444444422222222
               33333333
   ```

   There are 16 multipliers, each producing an 8 bit result. The diagram shows how the 16 results are shifted ready to be added. Each 8 bit result is denoted by the power of 2 of the shift that applies to that block (you don't have to draw it that way, it's just easy to remember).

2. A $gb \times gb$ bit multiplier can easily be constructed from $b \times b$ bit multiplier modules. In such designs, each $b \times b$ bit module outputs a $2b$ bit carry complete result. These results are called partial products in this setting.

   (a) For this $gb \times gb$ bit multiplier, express the height of the remaining partial product matrix as a function of g.

   Answer

   The number of partial products is $2g - 1$.

   (b) Generalize the result of part (a) to a $gb \times hb$ multiplier built from $b \times b$ modules and give the solution for a $6b \times 4b$ multiplier.

Answer:

For a $gb \times hb$ multiplier, if $g > h$, the height is $2h$ partial products.

Let's consider an example, a $6b \times 4b$ multiplier.

We can understand the solution by expressing the two operands as polynomials of variable b and multiplying them:

$$(pb^5 + qb^4 + rb^3 + sb^2 + tb + v) \times (wb^3 + xb^2 + yb + z)$$
$$= p^8(pw) + p^7(px + qw) + p^6(py + qx + rw) + b^5(pz + qy + rx + sw)$$
$$+b^4(qz{+}ry{+}sx{+}tw){+}b^3(rz{+}sy{+}tx{+}vw){+}b^2(sz{+}ty{+}vx){+}b(tz{+}vx){+}vz$$

Clearly the four modules that compute the b^3 term and the four that compute the b^2 term overlap (since each of these modules has $2b$ outputs) to produce the worst case tree height of 8 modules.

Although there seems to be a contradiction between the answers for parts (a) and (b), there isn't. Consider the tree heights for the following multipliers:

g	h	height
4	1	2
4	2	4
4	3	6
4	4	7
4	5	8
4	6	8

3. Give conditions on the relative values of z and d so that overflow is avoided during integer division.

Answer:
Initially z has $2k$ bits and d has k bits. The quotient must fit into k bits. If $z = U \times 2^k + V$ where U and V are the two k-bit registers, $d > U$ guarantees that the quotient occupies no more than k bits.

For example, if $k = 4$, $z = 0101\,0101$ and $d = 0101$ the result of the division is 0001 0000, remainder 0101. The quotient is too big to fit into 4 bits. If $d = 0110$, one larger than U, the result is 1110 remainder 0001.

4. Show how the division 0110110100÷10011 is performed (a 10 bit dividend is divided by a 5 bit divisor giving a 5 bit quotient and a 5 bit remainder). Represent the integers as binary fractions: 00.0110110100÷0.10011 and show all the steps.

   (a) when the quotient digit set is $\{-1, 1\}$ and $s^{(j)} \in [-d, d)$ as in Figure 1 in the notes on high radix division.

   Answer:

z = 0.01000101, d = 0.10011

z	0 01101	10100	
d	0 10011		
$-d$	1 01101		
$s^{(0)}$	0 01101	10100	
$2s^{(0)}$	0 11011	01000	≥ 0 so set $q_{-1} = 1$
$+(-d)$	1 01101		subtract
$s^{(1)}$	0 01000	01000	
$2s^{(1)}$	0 10000	10000	> 0 so set $q_{-2} = 1$
$+(-d)$	1 01101		subtract
s^2	1 11101	10000	
$2s^{(2)}$	1 11011	00000	< 0 set $q_{-3} = -1$
$+(d)$	0 10011		
$s^{(3)}$	0 01110	00000	
$2s^{(3)}$	0 11100	00000	> 0 set $q_{-4} = 1$
$+(-d)$	1 01101		subtract
$s^{(4)}$	0 01001	00000	
$2s^{(4)}$	0 10010	00000	> 0 so set $q_{-5} = 1$
$+(-d)$	1 01101		subtract
$s^{(5)}$	1 11111	00000	< 0 add back, reduce q by 1
$+(d)$	0 10011		
s	0 10010	00000	rem = 18_{10}
q	0 11T10		BSD value
q	0 10110		q = 22_{10}

(b) when the quotient digit set is $\{-1, 0, 1\}$ and $s^{(j)} \in [-d, d)$ as in Figure 2 in the notes on high radix division.

Answer:

z = 0.01000101, d = 0.10011

z	0 01101	10100	$z \in [-1/2, 1/2)$, no normalization necessary
d	0 10011		$d \in [1/2, 1)$, no normalization necessary
$-d$	1 01101		
$s^{(0)}$	0 01101	10100	
$2s^{(0)}$	0 11011	01000	$> d$ set $q_{-1} = 1$
$+(-d)$	1 01101		
$s^{(1)}$	0 01000	01000	
$2s^{(1)}$	0 10000	10000	$< d$ set $q_{-2} = 0$
s^2	0 10000	10000	
$2s^{(2)}$	1 00001	00000	$> d$ set $q_{-3} = 1$
$+(-d)$	1 01101		
$s^{(3)}$	0 01110	00000	
$2s^{(3)}$	0 11100	00000	$> d$ set $q_{-4} = 1$
$+(-d)$	1 01101		
$s^{(4)}$	0 01001	00000	
$2s^{(4)}$	0 10010	00000	$< d$ so set $q_{-5} = 0$
s	0 10010	00000	rem = 18_{10}
q	0 10110		q = 22_{10}

(c) when $d \in [1/2, 1)$, the quotient digit set is $\{-1, 0, 1\}$ and $s^{(j)} \in [-1/2, 1/2)$ as in Figure 3 in the notes on high radix division.

Answer:

z = 0.01000101, d = 0.10011

z	0.01101	10100	$z \in [-1/2, 1/2)$, no normalization necessary
d	0.10011		$d \in [1/2, 1)$, no normalization necessary
$-d$	1.01101		
$s^{(0)}$	0.01101	10100	
$2s^{(0)}$	0.11011	01000	> 0.5 set $q_{-1} = 1$
$+(-d)$	1.01101		
$s^{(1)}$	0.01000	01000	
$2s^{(1)}$	0.10000	10000	> -0.5 set $q_{-2} = 1$
$+(-d)$	1.01101		
$s^{(2)}$	1.11101	10000	
$2s^{(2)}$	1.11011	00000	$-0.5 < 2s^{(2)} < 0.5$ set $q_{-3} = 0$
$s^{(3)}$	1.11011	00000	
$2s^{(3)}$	1.10110	00000	$-0.5 < 2s^{(3)} < 0.5$ set $q_{-4} = 0$
$s^{(4)}$	1.10110	00000	
$2s^{(4)}$	1.01100	00000	$2s^{(4)} < -0.5$ set $q_{-4} = -1$
$+d$	0.10011		
$s^{(5)}$	1.11111	00000	Add, reduce q by 1
s	0.10010	00000	rem = 18_{10}
q	0.1100T-1		
q	0.10010		q = 22_{10}

The add, $s^{(5)} + d$ is necessary to make the remainder sign the same as the sign of the dividend.