CAYLEY GRAPHS AND REPRESENTATION THEORY

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1. Introduction

Cayley graphs, named after mathematician Arthur Cayley, are an important concept relating group theory and graph theory. Cayley graphs are frequently used to render the abstract structure of a group easily visible by way of representing this structure in graph form. Properties of a group G, such as its size or number of generators, become much easier to examine when G is rendered as a Cayley graph. Though also referred to as group diagrams, the definition of Cayley graphs is suggested by Cayley's theorem, which states that every group G is isomorphic to a subgroup of the symmetric group acting on G, $Sym(G)$.

Due to their ability to elegantly encode the algebraic structure of groups, Cayley graphs are particularly useful in the field of combinatorics [1]. Cayley graphs are also useful for studying the representation of finite groups; we can construct Cayley graphs to aid in the visualization of: $\mathbb{Z}_n$, dihedral groups, symmetric groups, alternating groups, direct and semidirect products, cyclic groups, etc. In relation to representation theory, Cayley graphs can aid in understanding algebraic structures of groups. We can study the properties of a Cayley graph for a particular group representation to find eigenvalues and decompose the representation into a set of unique irreps, both concepts that are explained in more detail in this paper. For more information regarding the properties of Cayley graphs and their many uses in various fields, refer to [1].

2. Graph Theory

Graphs, in the simplest form, provide mathematical models for a set of objects or ideas that are related in some way. With some thought, almost anything can be thought of and represented as a graph. For example, relationships between students in MATH 5559 can be depicted as graphs. Arrange the nine students in any manner and connect two students by a string if and only if they are in the same graduating class. This process would yield a unique graph, and we could create another graph with the same nine students simply by using a different criterion to relate the students, such as connecting two students by string if and only if they are of the same gender.

Graph theory seamlessly connects aspects of combinatorics and representation theory, allowing many propositions and theorems in graph theory to be proven by both combinatoric and representation theoretic methods simultaneously. Given the diverse nature of group theory, the field is rich in applications, and there are many avenues by which group theory can explain both everyday and supernatural phenomena.

2.1. **Basic definitions.** Here basic graph theory will be briefly explained, including the fundamental notions such as vertex and edge sets, adjacency, incidence, degree, and order, among other things. For purposes of this paper, we will consider graphs for finite groups only; information regarding representations for infinte groups and their respective Cayley graphs can be found in [1]. The following definitions are as described in the text [2].

A *multiset* is a collection of objects where items may appear in the collection more than once. The number of times that an object appears in the multiset is called the *multiplicity* of that element. If S is a multiset, then $|S|$ is the number of objects in S.

A *graph* is composed of two types of objects, a vertex set

$$V = \{v_1, v_2, v_3, \cdots, v_n\}$$

of elements called vertices and an edge multiset E of distinct pairs of vertices called *edges*. We denote the graph G with vertex set V and edge set E by

$$G = (V, E)$$

The number n of vertices in the set V is the order of the graph G, denoted $|G|$.

If $\alpha = \{v_1, v_2\}$ then we say that α joins v_1 and v_2 and that v_1 and v_2 are *adjacent* vertices, that is, v_1 and v_2 are connected by an edge $e \in E$. We also say that α is *incident* to v_1 and v_2, that is, α is an edge extending from vertices v_1 and v_2.

If $v \in V$, then the *degree* of v is the number, $\deg(v)$, of edges that are incident with v. For each general graph G, we assign a sequence of numbers that is the list of degrees of each vertex $v_i \in V \; \forall \, i \in \{1, \ldots, n\}$ in nonincreasing order

$$(d_1, d_2, \cdots, d_n), \quad d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$$

and call this the *degree sequence* of G.

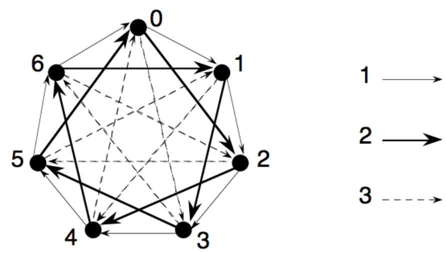

FIGURE 1. Cayley digraph $\overrightarrow{C}(\mathbb{Z}_7, \{1, 2, 3\})$

2.2. **Cayley graph construction.** Let G be a group with generating set Γ. Then the Cayley graph of G with respect to Γ, denoted by Cay(G, Γ), is constructed as follows.

(1) The vertices of Cay(G, Γ) are the elements of G.
(2) Two vertices $v, w \in G$ are adjacent if and only if there exists $\gamma \in \Gamma$ such that $v = w\gamma$, that is, $vw^{-1} \in \Gamma$.
(3) The multiplicity of the edge $\{v, w\}$ in edge multiset E is the multiplicity of vw^{-1} in Γ.

This definition of a Cayley graph follows the definition outlined in [1].

Remark 2.1. For simplicity, we define construction of a Cayley graph, Cay(G, Γ), in terms of a *generating* set Γ. If Γ does not generate the group, however, (1) does not hold. In the case that the elements $\{\gamma_1, \gamma_2, \cdots, \gamma_m\} \in \Gamma$ do not generate G, the vertices of Cay(G, Γ) will be only the elements which can be generated by some combination of $\gamma_i \in \Gamma \ \forall \ i \in \{1, \cdots, m\}$ under the group operation.

Figure 1, provided in [3] shows construction of the Cayley digraph for $G = \mathbb{Z}_7$ with generating set $\Gamma = \{1, 2, 3\}$. For better understanding, a digraph is depicted, which allows the reader to see where each element $\gamma \in \Gamma$ is sent. Cay($\mathbb{Z}_7, \{1, 2, 3\}$)) is the underlying graph of $\overrightarrow{C}(\mathbb{Z}_7, \{1, 2, 3\})$, that is, the graph containing undirected edges and in which all edge fibers are identical.

3. ADJACENCY OPERATORS

The concept of an adjacency operator is an important concept to establish for the remaining parts of the paper. Adjacency operators are critical in understanding the connection between representation theory and Cayley graphs.

3.1. **Establishing a vector space.** We borrow from [1] the complex vector space $L^2(S)$ to aid in our understanding of the adjacency operator and its relation to the adjacency matrix.

Definition. Let S be a finite set. Define the complex vector space $L^2(S)$ by

$$L^2(S) = \{f : S \to \mathbb{C}\}$$

. Let $f, g \in L^2(S)$ and $\sigma \in \mathbb{C}$. Addition in $L^2(S)$ is given by $(f + g)(x) = f(x) + g(x)$ and scalar multiplication in $L^2(S)$ is given by $(\sigma f)(x) = \sigma f(x)$. Standard inner product and norm are given by

$$\langle f, g \rangle = \sum_{x \in S} f(x)\overline{g(x)} \ \ and \ \ ||f|| = \sqrt{\langle f, f \rangle}$$

We now use this definition to define the *standard basis* for $L^2(S)$. Consider the finite set $S = x_1, x_2, \ldots, x_n$. Let $\Omega = \{\omega_{x_1}, \omega_{x_2}, \cdots, \omega_{x_n}\} \subset L^2(S)$, where $\omega_{x_i}(x_j) = 1$ if $i = j$ and $\omega_{x_i}(x_j) = 0$ if $i \neq j$.

If $f \in L^2(S)$, we have

$$f(x) = f(x_1)\omega_{x_1}(x) + \cdots + f(x_n)\omega_{x_n}(x)$$

thus Ω spans the vector space $L^2(S)$. This, accompanied with the observation that ω_{x_i} are mutually orthogonal and therefore linearly independent, yields that Ω is a basis for $L^2(S)$. We call this the *standard basis* for $L^2(S)$, and we can see that the dimension of $L^2(S)$ as a vector space over $\mathbb{C}$ is n.

3.2. **Adjacency matrices.** Let X be a graph with vertex set V ordered as $v_1, v_2, \cdots, v_n$. We define the *adjacency matrix*, A, of graph X to be the matrix where

$$A_{i,j} = \{number\ of\ edges\ incident\ to\ both\ v_i\ and\ v_j\}$$

Remark 3.1. For any vertices $v, w \in X$ we denote $A_{v,w}$ as the number of edges connecting v and w. We define this notation so that we may refer to an entry of the adjacency matrix without requiring the vertex set V to be ordered. Note that $A_{v,w} = A_{w,v}$, thus A is symmetric.

3.3. **Defining the adjacency operator.** Suppose we have a graph X and ordered vertex set V, as before, with A being the adjacency matrix of X. Given $f \in L^2(X)$, we can think of f as a vector in $\mathbb{C}_\ltimes$.

We can then multiply

$$Af = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} A_{1,j} f(v_j) \\ \sum_{j=1}^{n} A_{2,j} f(v_j) \\ \vdots \\ \sum_{j=1}^{n} A_{n,j} f(v_j) \end{pmatrix}$$

In this way, we can think of A as a linear transformation from $L^2(X)$ to itself, given by the equation

$$(Af)(v) = \sum_{w \in V} A_{v,w} f(w)$$

A defined in this manner is the *adjacency operator* of X.

When looking at how adjacency operators act with regard to Cayley graphs of the form $X = \mathrm{Cay}(G, \Gamma)$, a simple formula arises, as noted in [1], which describes the action of an operator A on a function $f \in L^2(G)$. The action is given by

$$(Af)(x) = \sum_{\gamma \in \Gamma} f(x\gamma)$$

.

4. Finding Eigenvalues of Cayley Graphs

For a Cayley graph X on a group G whose set Γ is a union of conjugacy classes, the eigenvalues of X can be expressed as the sum of characters of irreducible representations.

4.1. **Decomposing adjacency operator into irreps.** Consider the Cayley graph $\mathrm{Cay}(G, \Gamma)$, with adjacency operator A. Let $f \in L^2 G$ and let R be the right regular representation of G. From above, we have

$$(*) \qquad (Af)(x) = \sum_{\gamma \in \Gamma} f(x\gamma) = \sum_{\gamma \in \Gamma} (R(\gamma) f)(x)$$

The equation $(*)$ forms a link between Cayley graphs and representation theory that allows us to discover information about the eigenvalues of $\mathrm{Cay}(G, \Gamma)$. The following proposition is of key note.

Proposition 4.1. Let G be a finite group, $\Gamma \subset G$, and A be the adjacency operator of $\mathrm{Cay}(\Gamma, G)$. If $\rho_i, \cdots, \rho_k$ partition the set of unique matrix irreps of G, then

$$A \cong d_1 M_{\rho_1} \oplus d_2 M_{\rho_2} \oplus \cdots \oplus d_k M_{\rho_k}$$

where d_i is the dimension of ρ_i and $M_\rho = \sum\limits_{\gamma\in\Gamma} \rho(\gamma)$

Proof. Let R be the right regular representation of G. Using equation $(*)$ and knowledge of properties of the right regular representation, namely that the regular representation decomposes into a direct sum of irreducible representations, we have

$$A = \sum_{\gamma\in\Gamma} R(\gamma) \cong \sum_{\gamma\in\Gamma} d_1\rho_1(\gamma) \oplus \cdots \oplus d_k\rho_k(\gamma)$$

4.2. **Eigenvalues as sums of irreducible characters.**

Proposition 4.2. Let G be a finite group and Γ a subset of G such that $g\Gamma g^{-1} = \{g\gamma g^{-1} \mid \gamma \in \Gamma\} = \Gamma$ for all $g \in G$, that is, where Γ is the union of conjugacy classes. Let $X = \mathrm{Cay}(G, \Gamma)$ and A be the adjacency operator of X.

As before, let $\rho_i, \cdots, \rho_k$ be the complete set of inequivalent irreps of G. Let χ_i be the character of ρ_i and d_i the dimension of ρ_i. Then the eigenvalues of the adjacency operator A are given by

$$\mu_i = \frac{1}{d_i}\sum_{\gamma\in\Gamma} \chi_i(\gamma), \quad i = 1, \cdots, r,$$

where each eigenvalue μ_i occurs with multiplicity d_i^2.

Proof. Following the proof in [1], we decompose $L^2(G) = \oplus_{i=1}^r d_iV_i$ where $V_1, \cdots, V_r$ form a list of unique irreps of G and $R(\gamma)v_i = \rho(\gamma)v_i$ for all $v_i \in V_i$. That is, we have that the restriction of $R(\gamma)$ to V_i is ρ_i.

We use the fact that Γ is closed under conjugation to show that

$$AR(g) = \sum_{\gamma\in\Gamma} R(\gamma)R(g) = \sum_{\gamma\in\Gamma} R(\gamma g) = \sum_{\gamma\in\Gamma} R((g\gamma g^{-1})g) = \sum_{\gamma\in\Gamma} R(g\gamma) = R(g)A$$

for all $g \in G$.

Fix any i with $1 \leq i \leq r$. We know that $A = \sum_{\gamma\in\Gamma} R(\gamma)$ and $R : V_i \to V_i$, thus we have that $A : V_i \to V_i$. Picking a basis β for V_i gives $[R(g)]_\beta[A]_\beta = [A]_\beta[R(g)]_\beta$ for all $g \in G$. Because R restricted to V_i is the irreducible representation ρ_i, we must have $[A]_\beta = \mu_i I_{d_i}$. Here, I_{d_i} is the $d_i \times d_i$ identity matrix and $\mu_i \in \mathbb{C}$.

Look at the trace of the restriction of A to V_i, denoted $A|_{V_i}$ to get

$$d_i\mu_i = tr(A|_{V_i}) = tr\left(\sum_{\gamma\in\Gamma} \rho_i(\gamma)\right) = \sum_{\gamma\in\Gamma} \chi_i(\gamma)$$

. This gives us $\mu_i = \frac{1}{d_i}\sum\limits_{\gamma\in\Gamma} \chi_i(\gamma)$.

	$K_{()}$	$K_{(1,2)}$	$K_{(1,2,3)}$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

FIGURE 2. Character table for S_3

By repeating this process for each i, we can see

$$A \cong d_1(\mu_1 I_{d_1}) \oplus \cdots \oplus d_r(\mu_r I_{d_r})$$

. Thus, the eigenvalues of A are $\mu_1, \mu_2, \cdots, \mu_r$ where μ_i has multiplicity d_i^2 for $i = 1, \cdots, r$, what we wanted to prove. $\square$

5. Cayley Graph on S_3 as an Example

Consider the Cayley graph $\mathrm{Cay}(S_3, K_{(1,2)})$ on the symmetric group S_3 with subset $\Gamma = K_{(1,2)}$, where $K_{(1,2)} = \{(1,2), (2,3), (1,3)\}$, ie: the two-cycles in S3. Note that $|K_{(1,2)}| = 3$.

Figure 2 shows the character table for S_3, and we can use this information and the equation given in Proposition 4.2 to find the eigenvalues of A, the adjacency operator for $\mathrm{Cay}(S_3, K_{(1,2)})$. The eigenvalues

$$\mu_0 = \frac{1}{1}|K_{(1,2)}|\chi_0((1,2)) = \left(\frac{1}{1}\right)|3|(1) = 3$$

and

$$\mu_1 = \frac{1}{1}|K_{(1,2)}|\chi_1((1,2)) = \left(\frac{1}{1}\right)|3|(-1) = -3$$

occur with multiplicity $d_0^2 = d_1^2 = 1^2 = 1$, whereas

$$\mu_2 = \frac{1}{2}|K_{(1,2)}|\chi_2((1,2)) = \left(\frac{1}{2}\right)|3|(0) = 0$$

occurs with multiplicity $d_2^2 = 2^2 = 4$.

6. Implications

Learning about representations of finite groups through the lens of graph theory, particularly by studying Cayley graphs, provides an alternative way to think about representation theory of finite groups. Equipped with Propositions 4.1 and 4.2, we are able to bound eigenvalues by bounding the characters of the group.

7. References

[1] Krebs, Mike, and Anthony Shaheen. Expander Families and Cayley Graphs: A Beginner's Guide. New York: Oxford UP, 2011. Print.

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[3] Alspach, Brian. CS E6204 Lecture 6 Cayley Graphs, adapted. University of Regina, Canada. Online notes.