Meta-learning Implicit Neural Representation for Sparse Time Series Functional Data Analysis

Bofan Chen
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
cbfcbf.byron@gmail.com

Abstract

Sparse time series data is everywhere in our daily life. However, it is hard to analyze them by traditional statistical methods. Here, we propose MetaINR, a method to learning the implicit neural representation (INR) of time series given sparse and irregular obsevations. Based on MetaINR, estimations of mean function, covariance operator, and principal components can be implemented easily. The method introduce the meta-learning to the functional data analysis (FDA) framework, which greatly outperforms the traditional baseline.

1 Introduction

Sparse time series data is common in daily life, such as medical data measured at different times for patients or height data collected at various stages of adolescence. However, the sparsity and irregularity of such data make traditional statistical methods hard to apply directly. Additionally, the consideration of continuity and differentiability properties in most time series data pose challenges for multivariable statistical analysis.

Recent research suggests that meta-learning exhibits great advantages in handling sparse data. It possesses the capability to identify shared patterns among related tasks, facilitating few-shot learning or even one-shot learning with extremely limited training data. Moreover, there has been substantial progress in statistical analysis utilizing dense functional data to address continuous time series issues. These findings inspire us to leverage meta-learning techniques to transform sparse time series data into dense formats. Subsequently, we can extend the established functional data analysis methodologies tailored for dense data to analyze sparse time series datasets effectively.

In this paper, we introduce a novel approach, MetaINR, which combines meta-learning and functional data analysis to address the estimation challenges posed by sparse time series data. Through this method, we use meta-learning to realize the implicit neural representation (INR) to reconstruct the time series function for each sample based on sparse time series observations. Furthermore, we can estimate the mean, covariance, and principal components of the distribution behind these samples. The advantange of MetaINR can be summarized as follows:

- Efficient and accurate reconstruction of time series samples with sparse observations by leveraging information from the entire dataset
- Effective estimation of mean function, covariance operator and principal components for sparse time series dataset
- Independence from any initial assumptions about the distribution of time series samples
- Preservation of fine details in the samples and easy calculation for their derivatives through the SIREN-based INR

Therefore, MetaINR method provides a new solution for analyzing sparse time series data.

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2 Related works

2.1 Functional Data Analysis (FDA)

[TODO: add references]

Multivariate data versus functional data Traditional multivariate statistics take finite-dimensional vectors as the objects of study to analyze their statistical properties, such as mean vectors and covariance matrices. This approach is appropriate and is easy to compute when the dimension of the statistical objects is relatively small. However, if the dimension of the space is increased to infinite, for example, in L^2 space, traditional methods face certain challenges. Besides the expensive computational cost, in such functional spaces, people's focus is not only on the values of vectors in a particular dimension but also on properties such as continuity, smoothness and derivatives. Additionally, as the space grows to infinite dimensions, many useful tools in multivariate statistics, such as probability density functions, will lose their meaning. It is the differences in statistical properties and people's focus between infinite-dimensional and finite-dimensional spaces, that have led to widespread attention to functional data analysis (FDA) among researchers. In FDA, scientists treat functions as basic elements for analysis and have developed various methods to estimate the statistical properties for a functional distribution, such as mean function, covariance operator and functional principal components.

Sparse time series data Time series is a typical object that needs to be addressed in an infinite-dimensional function space. This is because time series data collected in reality often exhibit continuity, and, in certain situations, people may focus on topics such as derivatives and time alignment (registration). Many previous time series models were limited to multivariate statistical methods, such as Vector AutoRegression (VAR)[1], AutoRegressive Integrated Moving Average model (ARIMA) [2], Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) [3], to name a few. From the perspective of function approximation, these multivariate-statistics-based studies can indeed reflect certain properties of time series. However, in many cases, this approach is challenging to generalize, as is the case with sparse time series data. When the observation frequency of time series is low and irregular, sparse time series data is formed. We can represent the sparse dataset of n samples as

$$D = \{(t_{ij}, y_{ij})_{j=1}^{J}\}_{i=1}^{n}$$

where $y_{ij} = x_i(t_{ij}), i = 1, \cdots, n, j = 1, \cdots, J_i$ and J_i is the total observations of the i-th sample. This type of data is particularly common in the medical field, as the frequency of medical visits varies from person to person, with irregular patterns. Due to irregularity and sparsity, this form of data cannot be analyzed using traditional multivariate statistical methods or dense-data-based functional data analysis (FDA). Instead, it requires the introduction of new methods.

Our benchmarks and the drawbacks of traditional FDA method for sparse data We introduce the following two traditional solutions for sparse time series data as benchmarks in this paper:

- **Local polynomial regression + dense-data-based FDA (Pre-smoothing method)**: In this method, we first utilize local polynomial regression directly to recover samples with sparse observations, and then we apply dense-data-based methods to the recovered dense samples. However, this method is inherently unreliable due to the extremely sparse observations for individual samples in the dataset. This sparsity leads to unreliable local polynomial reconstructions, undermining the overall reliability of the method.
- **Principal Analysis by Conditional Expectation (PACE method)**: In this method, we start by employing local polynomial regression to estimate the mean function and variance kernel. Next, we calculate principal components based on these estimates. Finally, we recover samples by computing the scores associated with each principal component (see details in the appendix). Comparing to Pre-smoothing method, this approach leverages information from observations of other samples in the dataset during the sample recovery process, thereby enhancing the reliability of the reconstruction. However, it's important to note that PACE also comes with limitations, primarily because it heavily depends on the assumption that scores follow a normal distribution while recovering time series samples.

2.2 Machine learning for functional data

Unlike traditional methods, this paper introduces machine learning techniques to analyze sparse time series data. Specifically, this involves two aspects:

- We will use Implicit Neural Representation (INR) to recover and represent each sample.
- We will leverage meta-learning methods to effectively estimate the INR for each sample with sparse observations.

2.2.1 Implicit Neural Representation

Traditional grid-based representation versus implicit neural representation Traditionally, the representation of functional data $y(t)$ involves using a standard grid-based method, wherein a finite set of coordinates and corresponding function values are given as pairs (t_i, y_i). However, this approach has certain drawbacks: the original functional data in an infinite-dimensional space is discretized into a finite-dimensional space. This not only hampers operations like differentiation but also significantly impacts the resolution of the function within a smaller time interval. With the emergence of deep learning, it has been discovered that utilizing deep neural networks can be an effective way to represent functions implicitly. Implicit neural representation (INR) parameterize the functional data by deep neural networks:

$$y(t) = F_\phi(t),$$

where $F_\phi(\cdot)$ is a deep neural network parameterized by ϕ. This type of continuous representation not only proves to be more memory-efficient but also enables higher resolution and the handling of more complex operations on functional data.

SIREN architecture for time series and its advantages Many prior works have utilized ReLU-based multilayer perceptrons (MLP) as the neural network architecture for Implicit Neural Representations (INR). However, ReLU-based INR often lack the ability to capture fine details in signal and struggle to effectively express higher-order derivatives. In addressing this limitation, [4] introduced the SIREN architecture as an alternative to traditional ReLU-MLPs:

$$\Phi(\mathbf{x}) = \mathbf{W}_n \left(\phi_{n-1} \circ \phi_{n-2} \circ \ldots \circ \phi_0\right)(\mathbf{x}) + \mathbf{b}_n, \quad \mathbf{x}_i \mapsto \phi_i\left(\mathbf{x}_i\right) = \sin\left(\mathbf{W}_i \mathbf{x}_i + \mathbf{b}_i\right)$$

SIREN employs the sine function as the activation function, effectively modeling signals with fine details. A remarkable observation is that even after taking derivatives of a function represented using SIREN, it can still be expressed using the SIREN architecture. These properties of SIREN make it the preferred choice for INR. While previous research predominantly applied SIREN to represent images and videos, its application in time series representation has been limited. This paper explores the use of SIREN in the analysis of time series and demonstates its superiority.

2.2.2 Meta-learning for sparse data INR

Meta-learning framework Meta-learning was initially introduced to enhance the efficiency of machine learning, often referred to as "learning to learn". By employing the deep learning model $F_\phi(\cdot)$ to address similar tasks, the "shared characteristics" among these tasks can be leveraged to expedite the machine learning process. [5] To fomulate, given the observation dataset D^* of the target task and the observation dataset $D = \bigcup_{i=1}^n D_i$ of n similar tasks, we need to find a meta-function to effectively learn the parameters of the given task, namely $\phi^* = f_\theta(D^*)$, where θ is the meta-parameter of the meta-function $f_\theta(\cdot)$ that can be learned from D. Accordingly, the original target task can be solved as $F_{\phi^*}(\cdot)$. In fact, the representation of meta-function $f_\theta(\cdot)$ can be diverse. There are two main classes of methods: Parameterized Policy Gradient Methods (PPG) and Black Box Methods.

Parameterized Policy Gradient Methods and MAML The PPG method use an iterative algorithm to represent f_θ, which has an innerloop of the form:

$$\phi_{j+1}^* = \phi_j^* - \alpha_\theta \nabla_{\phi_j^*} \hat{L}_\theta \left(D^*, F_{\phi_j^*}\right),$$

with the initial value $\phi_0^* = \phi_0^*(\theta)$. This is very intuitive because it is similar to the traditional gradient decent method to optimize. The only difference is that the initial value, loss function and gradient step should be carefully trained by the observations of other tasks D in this scenario. One important

example of PPG method is Model-Agnostic Meta-Learning (MAML), where $\phi_0^*(\theta) = \theta$ and $\hat{L}_\theta$ can be any loss function independent of θ. MAML can be regarded as learning an initial value for a task distribution, which retains the gradient decent framework of traditional optimization algorithms.

Black box methods Since we only care about the target function $F_\phi(\cdot)$ rather than the value of parameters ϕ, an alternative way to treat $f_\theta(\cdot)$ is to see it as a black box. Based on this idea, we can rewrite the target function:

$$F_\phi(t) = F_{f_\theta(D^*)}(t) = g_\theta(t, D^*)$$

Here we do not need to explicitly specify the value of ϕ, and treat the meta-learning task as another "big" task, which adds all the sparse sample observations into input. From this perspective, some existing deep learning models for learning differential equations, such as Neural ODE [6] and Neural Laplace [7], can actually be classified as Black box methods. Instead of treating INR for a single time series as a task, these models consider learning the commonalities among a group of time series as a task, where t and D^* are all their input.

Advantages of MAML for sparse INR tasks For black box methods, because of the sequential properties of input observations, sequential model architectures such as RNN are always used in g_θ to encode, which unavoidably change the initial architecture F_ϕ. In contrast, PPG methods use the meta-parameters θ to find the task parameters ϕ explicitly, which maintain the initial network architecture. In this sense, black box methods have inductive bias from data while PPG methods have inductive bias in network architecture. In the context of time series Implicit Neural Representation (INR), selecting an SIREN-based appropriate network architecture is crucial for modeling complicated signals and their derivatives. Only PPG methods like MAML can learn the "shared characteristics" among samples while preserving the network architecture. Hence, in this paper, we opt for MAML as our meta-learning method for the INR task.

3 Method

Overview of MetaINR Given sparse observations of n similar time series, the goal of FDA for sparse data is to reconstruct each sample as accurately as possible and identify the principal components (PC) of the distribution behind these time series. As discussed in the previous section, traditional FDA approaches either do not work well with sparse data or rely on many assumptions. Here, we choose an alternative way by using the method of meta-learning, which involves three main steps:

- Learning the meta-model: Use SIREN architecture and MAML algorithm to learn the meta-model for the INR of given time series.
- Recovery of samples: Calculate the SIREN-based INR efficiently for each time series sample based on the mata-model.
- Functional data estimation: Apply dense-data-based traditional FDA methods to the MetaINR of each sample to estimate the mean function, covariance kernel and principal components effectively.

We now discuss each step.

Learning the meta-model The tasks here are realizing the implicit neural representation of samples by the SIREN architecture $F_\phi(t)$. Given the sparse observations of n similiar time series $D = \{(t_{ij}, y_{ij})_{j=1}^{J_i}\}_{i=1}^n$, we use MAML algorithm 1 to learning the meta-model of these tasks, whose initial value of parameters are denoted by ϕ_0.

Recovery of samples Based on this meta-model, we can use gradient decent algorithm 2 to calculate the optimal parameters ϕ_i^* for the sample i and hence obtain the INR $F_{\phi_i^*}(t)$.

Functional data estimation Based on the estimated INR $\hat{y}_i(t) = F_{\phi_i^*}(t)$, we subsequently estimate the mean function $\hat{\mu}(t)$ and covariance kernel $\hat{c}(t, s)$ as follows:

$$\hat{\mu}(t) = \frac{1}{n}\sum_{i=1}^{n} \hat{y}_i(t)$$

$$\hat{c}(t, s) = \frac{1}{n}\sum_{i=1}^{n} [\hat{y}_i(t)) - \hat{\mu}(t)][\hat{y}_i(s) - \hat{\mu}(s)]$$

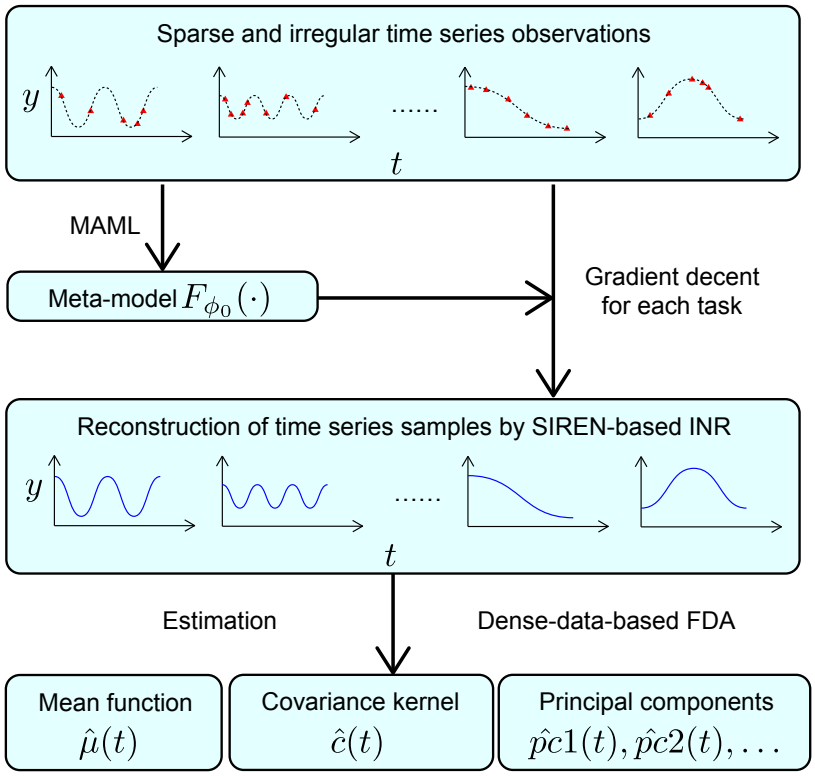

Figure 1: **MetaINR workflow.** Given the sparse and irregular observations from several samples of an functional distribution, MAML algorithm is implemented to learning the optimal initial value ϕ_0 for SIREN architecture F. The meta-model F_{ϕ_0} contain the "shared characteristics" from the distribution, hence the recovery of samples can be implemented efficiently and accurately. The dense-data-based FDA methods then can be applied effectively on the INRs of samples, such as the estimation of mean function, covariance kernel and principal components.

According to the covariance operator, we can further estimate its principal components $\hat{\phi}_k$ by calculate the eigenfunctions of $\hat{c}(t, s)$.

Advantages of MetaINR Inspired by traditional methods, MetaINR utilizes meta-learning to complete implicit neural representation for samples with only sparse observations, enabling them to be compatible with various FDA methods based on dense data. It is worth noting that, we treat meta-learning here not merely as an efficient learning method for sparse training sets but as a means to learn commonalities among a large number of tasks. Compared to the Pre-smoothing method in Section 2, a more robust estimation is achieved with MetaINR's sample recovery, as it leverages all the information from the entire dataset. Furthermore, unlike the PACE method in Section 2, MetaINR imposes no constraints on the distribution of samples, making it more versatile. In contrast to other learning-based black box method, MetaINR preserve the SIREN-based INR architecture, enabling more detailed representation and easy differentiation of each sample. The overall comparison of these methods are shown in table 1.

4 Experiments

We evaluate our method on the synthetic data, with benchmark estimations provided by Pre-smoothing method and the PACE method. Our results clearly demonstrate that MetaINR is able to effectively recover the samples from sparse obervations. The estimations provided by MetaINR significantly outperforms the conventional baseline. Moreover, we extended our analysis to a real-world medical dataset to validate the robustness and applicability of our approach in practical scenarios.

Algorithm 1 Model-Agnostic Meta-Learning for Time Series Implicit Neural Representation

Input: $D = \{(t_{ij}, y_{ij})_{j=1}^{J_i}\}_{i=1}^{n}$: the sparse observations for tasks
Input: α, β : step size hyperparameters
  randomly initialize θ
  while not done **do**
    for all i **do**
      $\theta^0 \leftarrow \theta$
      for $q = 0$ to $Q - 1$ **do**
        Evaluate $\nabla_{\theta^q} \mathcal{L}_i \left(F_{\theta^q}\right)$ using support set $D_i^{\text{support}} = (t_{ij}, y_{ij})_{j=1}^{\lfloor 0.5 * J_i \rfloor}$
        Compute adapted parameters with gradient descent: $\theta_i^{q+1} = \theta_i^q - \alpha \nabla_{\theta_i^q} \mathcal{L}_i \left(F_{\theta_i^q}\right)$
      end for
      Compute the loss $\mathcal{L}_i \left(F_{\theta_i^Q}\right)$ in sample i using query set $D_i^{\text{query}} = (t_{ij}, y_{ij})_{j=\lfloor 0.5 * J_i \rfloor}^{J_i}$
    end for
    Compute the total loss $\mathcal{L} = \sum_{i=1}^{n} \mathcal{L}_i \left(F_{\theta_i^Q}\right)$
    Update meta-parameters $\theta \leftarrow \theta - \beta \nabla_\theta \mathcal{L}$
  end while
  $\phi_0 \leftarrow \theta$
Output: the meta-parameters ϕ_0

Algorithm 2 Implicit Neural Representation Learning for Target Sample

Input: $D^* = (t_{ij}, y_{ij})_{j=1}^{J^*}$: the sparse observations for the target task
Input: α : step size hyperparameters
  initialize $\phi \leftarrow \phi_0$
  while not done **do**
    Evaluate $\nabla_\phi \mathcal{L} \left(F_\phi\right)$ using support set $D^* = (t_j^*, y_j^*)_{j=1}^{J^*}$
    Update parameters with gradient descent: $\phi = \phi - \alpha \nabla_\phi \mathcal{L} \left(F_\phi\right)$
  end while
  $\phi^* \leftarrow \phi$
Output: ϕ^*: the parameters of INR for the target model

4.1 Synthetic data

[IDEA: synthetic data generated by DE + evaluation principle]

Data generating process The ground truth of our synthetic data distribution is given by:

$$y(t) = Asin(t) + Bsin(2t), \quad t \in [0, 10],$$

where A and B are independent and follows the uniform distribution

$$A \sim Uniform[0, 1], \quad B \sim Uniform[0, 0.5].$$

We randomly draw $n = 100$ sample tasks $\{y_i(t)\}_{i=1}^{n}$ from this distribution. For each sample task, $J = 20$ observations randomly chosen in irregular time points. The overall observation dataset used

Method	Irregularity of observations	Sparsity of obervations	General distribution of samples	Preservation of architecture
Multivariable Statistics	✗	✗	✓	–
Pre-smoothing	✓	✗	✓	–
PACE	✓	✓	✗	–
Neural ODE	✓	✓	✗	✗
Neural Laplace	✓	✓	✓	✗
MetaINR	✓	✓	✓	✓

Table 1: Comparison of analysis methods for sparse time series data

for analysis can be represented as

$$D = \{(t_{ij}, y_{ij})_{j=1}^{J}\}_{i=1}^{n}.$$

Parameter setting [TODO]

Meta-learning the commonalities of time series We observed the capability of meta-learning to capture commonalities among a series of similar time-series data. As shown in Figure 2a, when providing observations solely from the latter segment ($t \in [5, 10]$) of a particular sample, meta-learning could extract information from data of other samples, learning the periodicity inherent in this distribution. In contrast, local polynomial regression relies solely on the data of an individual sample for estimation, resulting in less accurate predictions for the first half ($t \in [0, 5]$) of the time series. Additionally, meta-learning inherently learns information about the mean function of this distribution. By directly plugging the initial values learned by MAML into the SIREN architecture to form the meta-model, we observed a notable proximity between the meta-model and the means of these samples. As demonstrated in Figure 2b, the meta-learning process is accompanied by the meta-model gradually converging towards the mean. Aside from these two examples, [5] illustrates that MAML accelerates learning through feature reuse. This also clarifies why the meta-model encompasses various features of the dataset and MetaINR is a more reliable way for sample recovery than local polynomial regression.

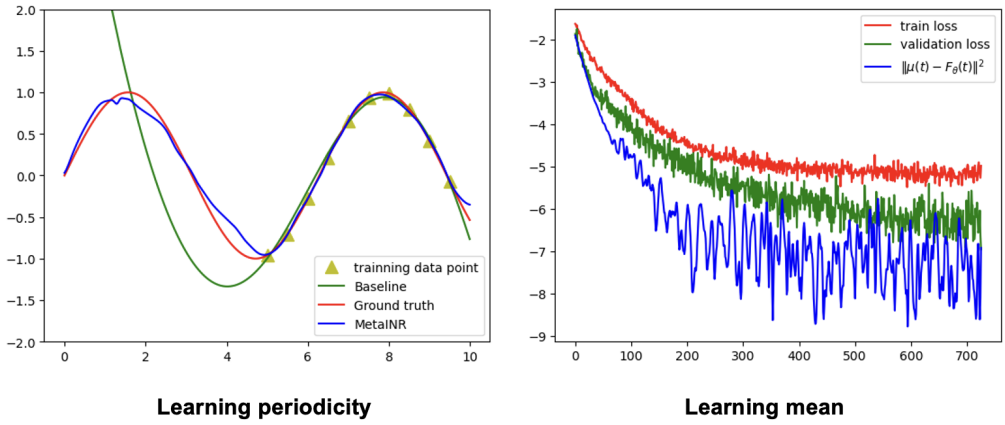

Figure 2: **Learning commonalities.** The figure illustrates two examples (periodicity & mean function) of the "shared characteristics" that MAML learned among these tasks. It is noticable that MetaINR is able to learn the periodicity between the interval $[0, 5]$ while local polynomial regression totally misses this information. It is also clear that the meta-model gradually approximates the true mean function during the training process.

Estimation of mean, covariance and principal components Based on MetaINR for each sample, we estimated the mean function, covariance kernel, and principal components of this synthetic dataset. As demonstrated in Figure 3, the estimators based on MetaINR for all three metrics significantly outperformed the benchmark. Particularly at the boundaries of the data, where t approaches 0 or 10, the baseline estimators often deviated substantially from the true values, while the estimators based on MetaINR remained relatively reliable.

4.2 Real-world data

Data description In real-world scenarios, we apply MetaINR to the longitudinal data on primary biliary cirrhosis (PBC). The data resulted from a Mayo Clinic trial that was conducted in 1974 to 1984, which is also described in [8] and [9]. They are available at http://lib.stat.cmu.edu/datasets/pbc or at a R package called "survival". The data contain various sparsely and irregularly sampled covariates for each of 312 patients. Among these medical measurements, serum bilirubin concentration is known to be elevated in the presence of chronic liver cirrhosis, such as PBC. Therefore, it is important to

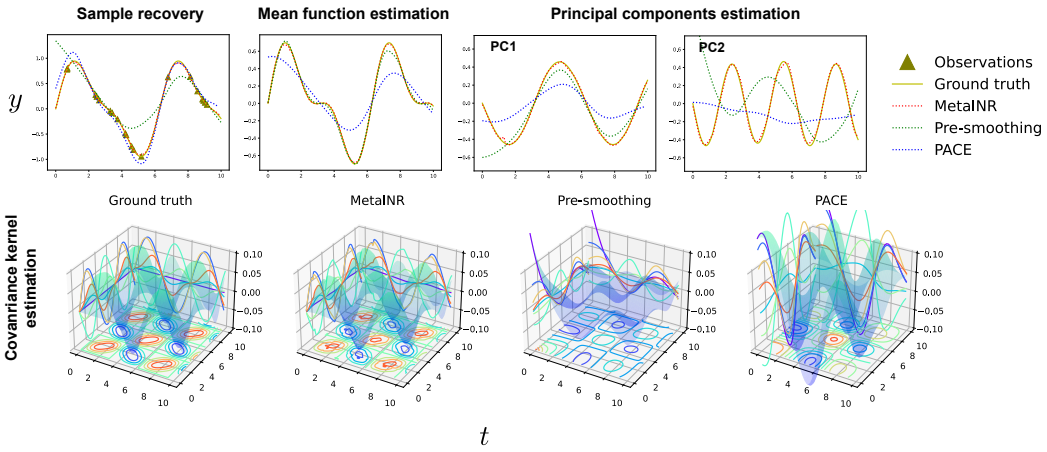

Figure 3: **Estimation for synthetic data.** The figure shows the results of the estimation of mean function, covariance kernel and principal components in synthetic data.

analyze the time series data of bilirubin measurements for PBC prediction. We extracte data from the dataset for the initial 910 days, analyzing bilirubin measurements (in mg/dl) for different patients during the pre-onset phase. Our goal is to characterize the pattern of this measurement over time through estimates of mean function, covariance kernel and principal components. In practice, we applied a logarithmic transformation to the bilirubin measurements and removed data for patients with a survival time less than 910 days. A total of 260 patients in the study satisfied these criteria and only 2.9 observations per sample on average.

Estimation of mean, covariance and principal components Figure 4 illustrates the estimation results of MetaINR on the real-world medical sparse functional dataset mentioned above. We observe that despite the minimal number of observations per sample, MetaINR can effectively recover the sample function. Regarding the estimation of statistical quantities, MetaINR demonstrates similar performance to the baseline in this specific real dataset. From the estimation results, we draw the following conclusions: Over the initial 910 days of observation, patients' serum bilirubin concentration initially experiences a slight decrease, followed by a sharp increase around 400 days (mean function). Apart from differences in "baseline values" which remain invariant with time (PC1), significant divergence in serum bilirubin concentration among patients emerges around 400 days (PC2).

5 Conclusion and future work

In this paper, we have shown a method named MetaINR utilizing meta-learning to analyze sparse time series data. MetaINR enables the estimation of functional form samples with very few observation points by leveraging information from the entire dataset through meta-learning. The application of MetaINR makes functional data analysis (FDA) methods, which are typically used in dense datasets, feasible for sparse datasets. Therefore, MetaINR provides a novel tool for further analyzing sparse time series data.

In future research, we aim to extend the application of MetaINR to other functional data analysis methods that were initially designed for dense datasets, such as Functional Partial Least Squares Regression (FPLS), functional regression, registration and more. We can also explore other algorithms of meta-learning, such as SNAIL, and compare the performances between them.

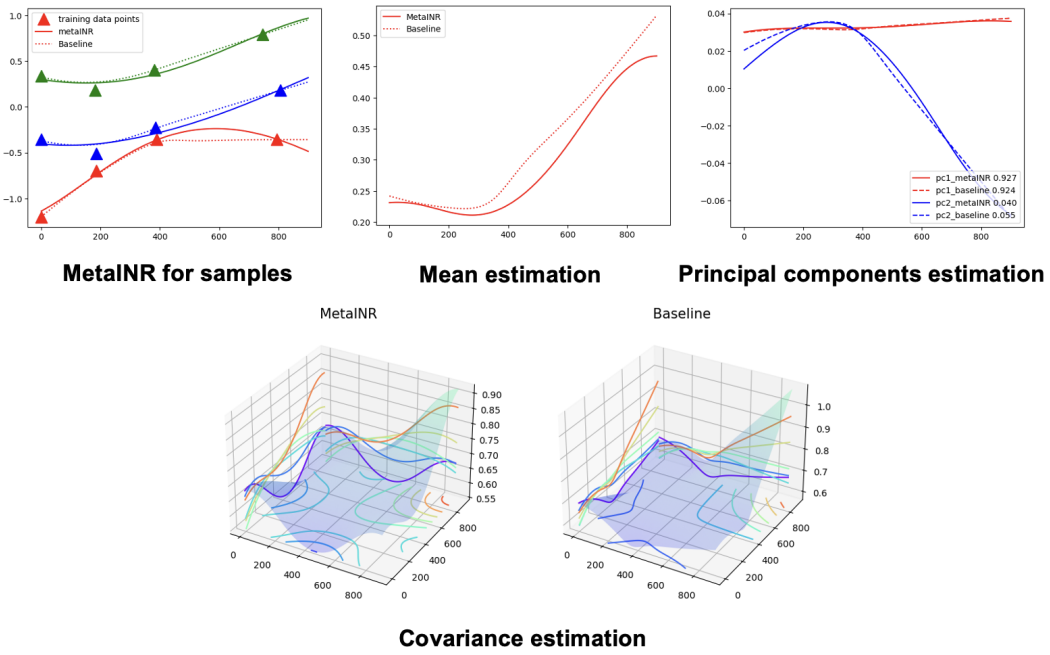

Figure 4: **Estimation for real-world data.**

6 Appendix

6.1 Traditional FDA method for sparse functional data: PACE

In functional data analysis, it is inappropriate to apply dense-data-based approaches directly on sparse dataset. However, PACE method gives a alternative solution.

Although local polynomial regression is not suitable for a single sample with sparse observations, it is proved that one can employ this method to estimate the mean function and covariance operator consistently. Specifically, given our sparse observations of dataset D, we can define the loss function to estimate mean function $\mu(t)$

$$L_t^{\mu}(\beta) = \sum_{i=1}^{n} \sum_{j=1}^{J_i} \left[y_{ij} - \sum_{r=0}^{R} \beta_r \left(t - t_{ij}\right)^r \right]^2 K\left(\frac{t - t_{ij}}{h}\right),$$

where $K(\cdot)$ is the kernel function and h is the bandwidth which shoud be carefully chosen based on data. By minimizing this function, we can get $\beta_t = (\beta_0(t), \cdots, \beta_R(t)) = arg\, min L_t^{\mu}(\beta)$. The estimator of mean function cam be defined as $\hat{\mu}(t) = \beta_0(t)$. Similarly, to estimate the covariance kernel $c(t, s)$, we define the loss function

$$L_{t,s}^{c}(\beta) = \sum_{i=1}^{n} \sum_{1 \le j,l, \le J_i} K_{t,s}(\frac{t - t_{ij}}{h}, \frac{s - t_{il}}{h}) \left[g(t_{ij}, t_{il}) - f\left(\beta, (t, s), (t_{ij}, t_{il})\right)\right]^2$$

where

$$g(t_{ij}, t_{il}) = (y_{ij} - \hat{\mu}(t_{ij}))(y_{il} - \hat{\mu}(t_{il}))$$
$$f\left(\beta, (t, s), (t_{ij}, t_{il})\right) = \beta_0 + \beta_{11}(t - t_{ij}) + \beta_{12}(s - t_{il})$$

By minimization of $L_{t,s}^{c}(\beta)$, we obtain $(\beta_0(t), \beta_{11}(t), \beta_{12}(t)) = arg\, min L_{t,s}^{c}(\beta)$. Then the estimator of the covariance kernel is defined as $\hat{c}(t, s) = \beta_0(t, s)$. Under suitable conditions, including the fact that observations are random and well-spread, the estimators of mean function and covariance kernel are consistent.[10]

$$\sup_{t \in \mathcal{T}} |\hat{\mu}(t) - \mu(t)| = O_p\left(\frac{1}{\sqrt{n h_{\mu}}}\right)$$

$$\sup_{t,s\in\mathcal{T}} |\hat{c}(s,t) - c(s,t)| = O_p\left(\frac{1}{\sqrt{n}h_G^2}\right)$$

where h_μ and h_G are bandwidths for estimating $\hat{\mu}$ and $\hat{c}$ satisfying

$$h_\mu \to 0, nh_\mu^4 \to \infty, \text{ and } nh_\mu^6 < \infty;$$
$$h_G \to 0, nh_G^6 \to \infty, \text{ and } nh_G^8 < \infty.$$

Base on the estimated covariance kernel, we can derive the estimated eigenvalues $\hat{\lambda}$ and eigenfunctions $\hat{\phi}_k(t)$ which are also consistent, ***i.e.***

$$\left|\hat{\lambda}_k - \lambda_k\right| = O_p\left(\frac{1}{\sqrt{n}h_G^2}\right);$$

$$\sup_{t\in\mathcal{T}} \left|\hat{\phi}_k(t) - \phi_k(t)\right| = O_p\left(\frac{1}{\sqrt{n}h_G^2}\right), \quad k \in \mathcal{I}'.$$

Let

$$\tilde{x}_i = \left(x_i\left(t_{i,1}\right), \ldots, x_i\left(t_{i,J_i}\right)\right)^T,$$

$$\tilde{\mu}_i = \left(\widehat{\mu}\left(t_{i,1}\right), \ldots, \widehat{\mu}\left(t_{i,J_i}\right)\right)^T,$$

$$\tilde{\phi}_{i,k} = \left(\widehat{\phi}_k\left(t_{i,1}\right), \ldots, \widehat{\phi}_k\left(t_{i,J_i}\right)\right)^T.$$

Assume that $X_i(t) = \mu(t) + \sum_{k=1}^{\infty} a_{ik}\phi_k(t)$, with the $a_{ik} \sim N(0, \lambda_k)$, Principal Analysis by Conditional Expectation (PACE) reconstructs each sample and estimates their scores by $\widehat{a}_{ik} = \hat{\mathbb{E}}\left(a_{ik} \mid \tilde{x}_i\right)$. It can be shown that

$$\widehat{a}_{i,k} = \widehat{\lambda}_k \tilde{\phi}_{i,k}^T \Sigma_{x_i}^{-1}\left(\tilde{x}_i - \tilde{\mu}_i\right)$$

where $\left\{\Sigma_{x_i}\right\}_{j,l} = \widehat{c}\left(t_{ij}, t_{il}\right)$. The recovered sample can be represented by the first K components

$$\widehat{X}_i^K(t) = \widehat{\mu}(t) + \sum_{k=1}^{K} \widehat{a}_{i,k}\widehat{\phi}_k(t)$$

It can also be proved [10] that under suitable conditions

$$\lim_{n\to\infty} \hat{a}_{ik} = a_{ik} \text{ in probability,}$$

and for all $t \in \mathcal{T}$,

$$\lim_{K\to\infty} \lim_{n\to\infty} \widehat{X}_i^K(t) = X_i(t) \text{ in probability.}$$

6.2 Brief introduciton to SIREN

Data and code availability

PBC data used in this paper is available at `http://lib.stat.cmu.edu/datasets/pbc` , and it also can be found in a R package called "survival". All the experimental code in this paper is available at `https://github.com/cbfcbf/code_metaINR`.

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