Ch 8 Nyquist Stability Criterion

We have two main methods for determining closed-loop stability

1) Routh - Hurwitz
2) Root locus

We now introduce a 3rd, the Nyquist criterion. It is a graphical technique involving the open-loop frequency response.

Why add another stability test?

Root locus:

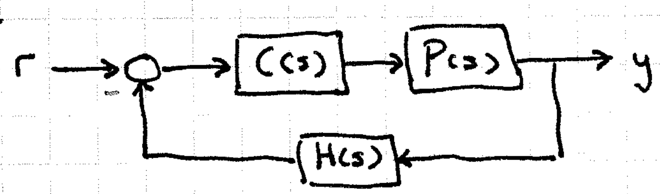
+ can see effect of parameter changes
+ can see effect of adding poles / zeros
- can't handle time delays
- can't handle uncertain Bode plot models

Nyquist

+ gives freq. domain info like bandwidth
+ can handle time delays
+ easily handles uncertainty, i.e. stability margins
- hard to draw by hand. (MATLAB 'nyquist' fragile.)

Conclusion: The two approaches are complimentary

Note: Both root locus and Nyquist work with open-loop data

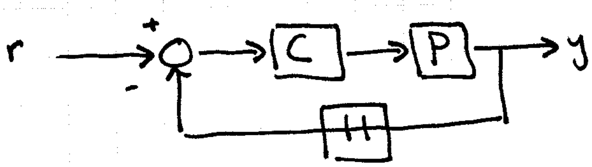


$$L(s) = C(s)P(s)H(s)$$

but they tell us about closed-loop system performance.

8.1 Principle of the argument

- the Nyquist criterion is a test for feedback stability
- it is based on the principle of the argument from complex function theory
- the principle of the argument involves two things:
  1) a curve in the complex plane
  2) a transfer function.



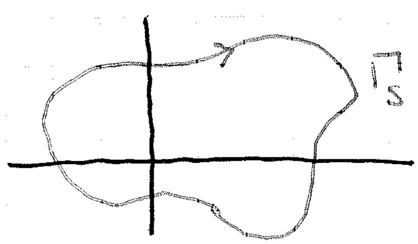
$$\frac{Y}{R} = \frac{PC}{1+PCH}$$

Define $\quad G(s) := 1 + PCH = 1 + L(s)$.

Treat $G(s)$ as a complex-valued function of a complex variable

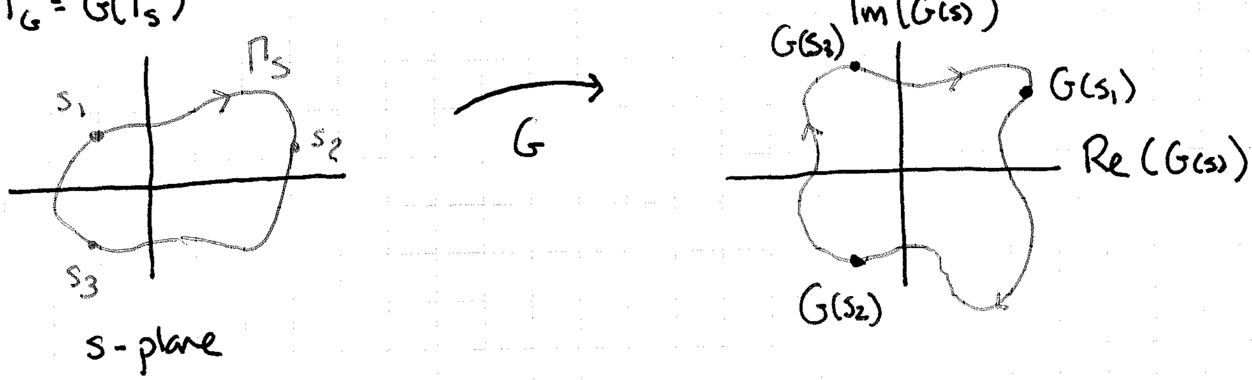
$$G:\ \mathbb{C} \to \mathbb{C}$$
$$s \mapsto 1 + L(s).$$

1) Consider a closed path (or curve or contour) in the s-plane with no self-intersections and negative, i.e., clockwise (CW) orientation. Name the path Γ_s

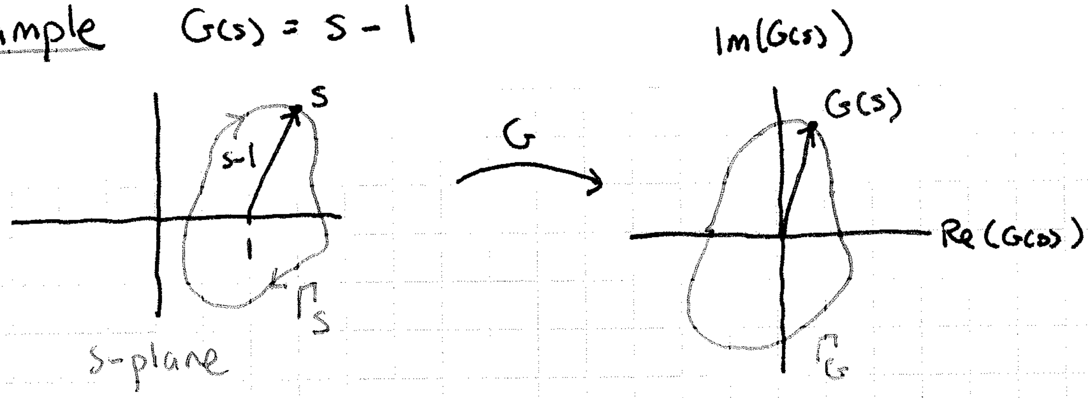


s-plane

2) If Γ_s does not pass through any poles ~~zeros~~ of $G(s)$ then the image of Γ_s will also be a closed curve.
Let $\Gamma_G = G(\Gamma_s)$

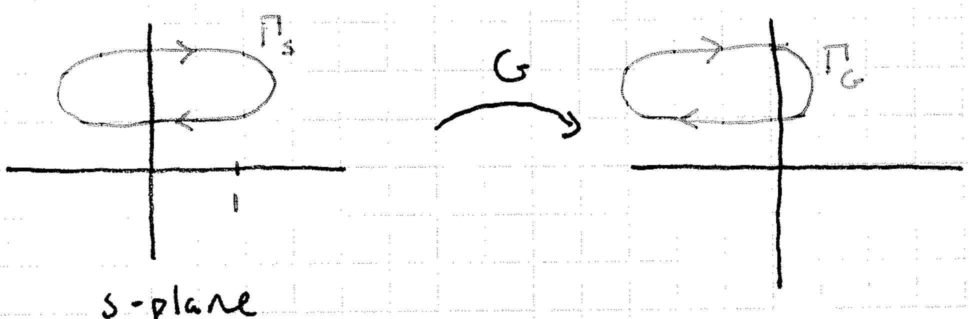


Example $G(s) = s - 1$



Note that Γ_G is just Γ_s shifted to the left by one.
Note that Γ_s encircles one zero of G and Γ_{sG} encircles the origin once, CW direction $\blacktriangle$

Example Same $G(s) = s-1$, different Γ_s

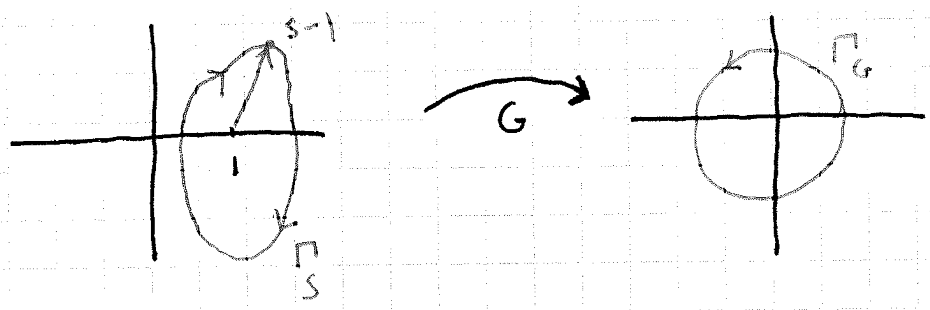


Now Γ_s encircles no zeros of G and Γ_G has no encirclements of the origin. $\blacktriangle$

Example $G(s) = \dfrac{1}{s-1}$

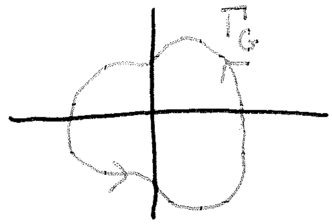
Note that the angle of $G(s)$ is the opposite of the angle of $s-1$

$$\angle G(s) = \angle 1 - \angle(s-1) = -\angle s-1$$



From this we get that if Γ_s encircles the pole at $s=1$ in the CW direction, then Γ_G encircles the origin once in the counterclockwise direction. (CCW).

Therefore Γ_G has $n-m$ counterclockwise encirclements of the origin.

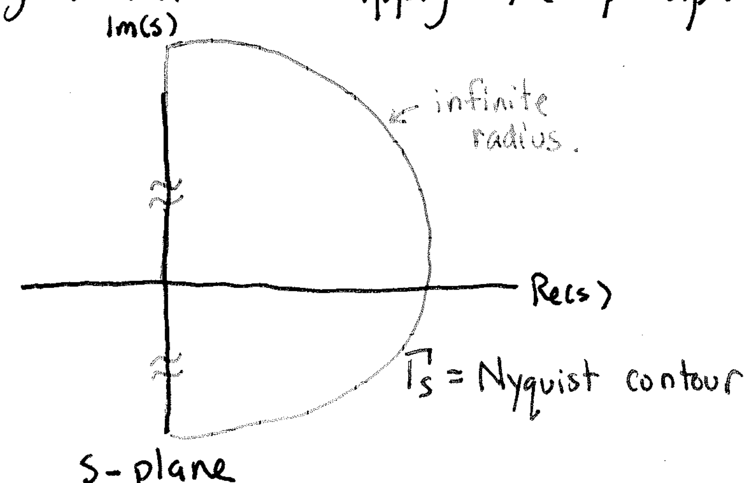


Q: What happens if we reverse the direction of Γ_s?

Adapting this idea to control systems

What we really need to know is: How many closed-loop poles are in $\mathbb{C}^+$?

Nyquist's bright idea: Choose Γ_s so that it encloses the unstable region and then apply the principle of the argument.



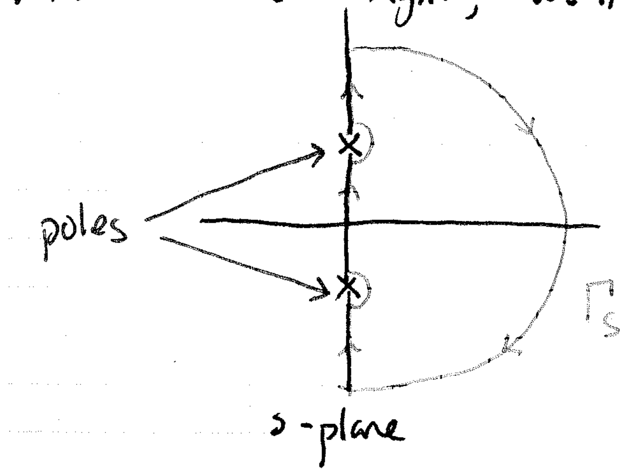
If we choose Γ_s = Nyquist contour then Γ_G is called the Nyquist plot of $G(s)$.

If $G(s)$ has no poles or zeros on Γ_s, then the Nyquist plot encircles the origin $n-m$ times CCW, where

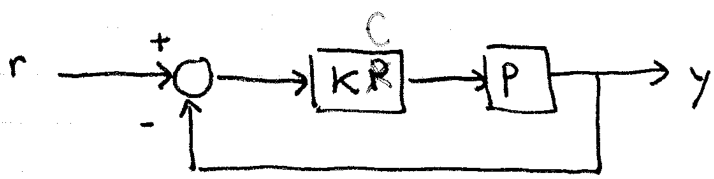
$n = \#$ of poles of G in $\mathbb{C}^+$

$m = \#$ of zeros of G in $\mathbb{C}^-$

In our application, if $G(s)$ has poles on the imaginary axis, we have to ident Γ_s around them. You can indent to the left or the right; we'll always indent to the right.



8.2 Nyquist stability criterion



Assumptions

1. P and C are proper, at least one them strictly proper.
2. The product PC has no unstable pole-zero cancellations. We have to assume this because the Nyquist criterion does not test for it and such cancellations would make the feedback system unstable
3. $K \neq 0$

The closed-loop TF from R to Y is

$$\frac{Y(s)}{R(s)} = \frac{KCP}{1+KCP}, \quad \text{let } G(s) = 1 + KCP$$

Since we've assumed no unstable pole-zero cancellations, feedback stability is equivalent to

$$\boxed{G(s) \text{ has no zeros on } \mathrm{Re}(s) \geq 0}$$

Logic

Closed-loop TF $\quad \dfrac{KPC}{G}$

feedback stability $\Leftrightarrow$ $\dfrac{KPC}{G}$ has no RHP poles

$\Leftrightarrow$ G has no RHP zeros

Observation

n = # of poles of G enclosed by Γ_s = # of poles of CP enclosed by Γ_s

$=: P_{\text{open-loop-bad-poles}}$.

m = # of zeros of G enclosed by Γ_s = # of zeros of $1+KCP$ enclosed by Γ_s
= # of closed loop poles enclosed by Γ_s

$=: P_{\text{closed-loop-bad-poles}}$

Nyquist criterion

1. Pick Γ_s as the Nyquist contour avoiding poles of CP
2. Draw Nyquist plot $\Gamma_G = G(\Gamma_s)$.
3. From plot, observe the number, N, of CCW encirclements by Γ_G
4. By Cauchy's principle of the argument

$$\underbrace{N}_{\text{known from plot}} = \underbrace{P_{\text{open-loop-bad-poles}}}_{\text{known from expression for } CP} - \underbrace{P_{\text{closed-loop-bad-poles}}}_{\text{unknown}}$$

5. The closed-loop system is feedback stable iff $P_{\text{closed-loop-bad-poles}} = 0$, i.e.,

$$\boxed{N = P_{\text{open-loop-bad-poles}}}$$

Remark To make plotting easier, instead of using the map $G = 1 + KCP$ and counting CCW encirclements of the origin, we'll use

$$L(s) = C(s)P(s)$$

and note that $L(s) = \frac{G(s)}{K} - \frac{1}{K}$

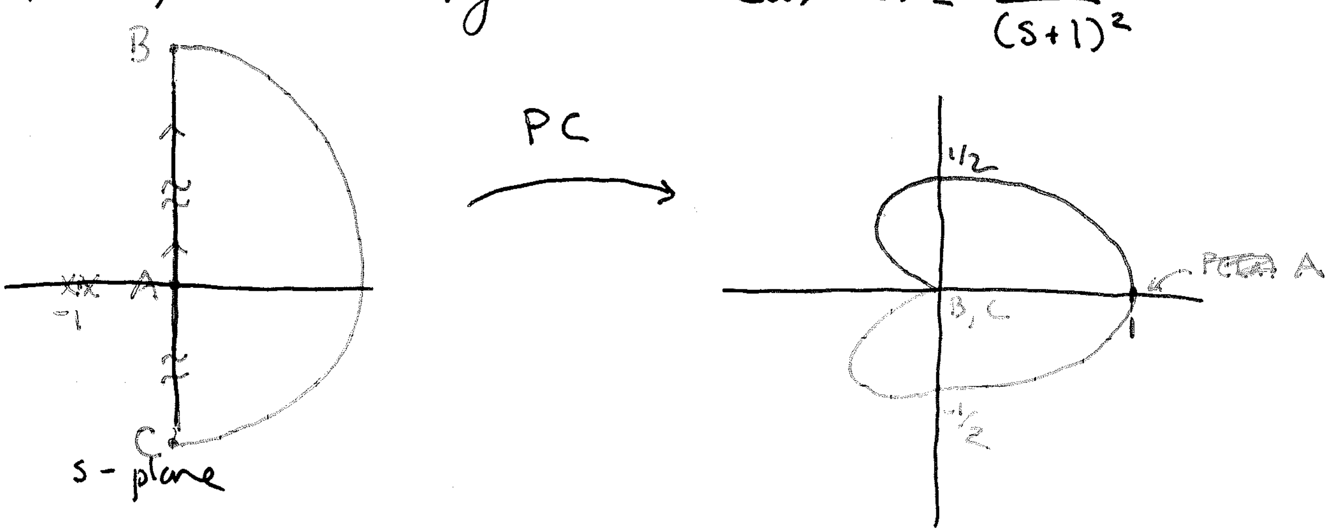
$\Rightarrow \Gamma_L$ is the same shape as Γ_G except it is scaled by K and shifted by $\frac{1}{K}$ to the left.

$\Rightarrow$ Count CCW encirclements of $-\frac{1}{K}$:

i.e. # CCW encirclements of 0 by Γ_G = # CCW encirclements of $-\frac{1}{K}$ by the Γ_L.

Example

Find range of K for feedback stability when $C(s)P(s) = \frac{1}{(s+1)^2}$



Remark: $P_{open\_loop\_bad\_poles} = 0$

Segment A to B: here $s = j\omega$, $\omega \geq 0$ so

$$C(s)P(s)\Big|_{AB} = \frac{1}{(j\omega+1)^2} = \frac{1-\omega^2}{(1-\omega^2)^2+(2\omega)^2} + j\frac{-2\omega}{(1-\omega^2)^2+(2\omega)^2}$$

As ω goes from $0 \to \infty$

$Re(CP(j\omega))$ goes from $1 \to \underset{(\omega=1)}{0} \to$ negative $\to 0$

$Im(CP(j\omega))$ goes from $0 \to$ negative $\to 0$.

Furthermore as $\omega \to +\infty$ $P(j\omega) \to \frac{1}{(j\omega)^2} = \frac{-1}{\omega^2}$

Segment B to C: infinite radius and, since CP is strictly proper, gets mapped to the origin.

Segment C to A: This segment is the complex conjugate of the segment A to B. This means that the image of C to A is the complex of the image from A to B

Why? Since PC has real coefficients $PC(\bar{s}) = \overline{PC(s)}$.

Stability Analysis

Since $P_{open-loop-bad-poles} = 0$, the system is stable iff

$$N = 0$$

where N is the # of CCW encirclements of $-1/K$.

This gives the conditions

$$\frac{-1}{K} < 0 \quad \text{or} \quad \frac{-1}{K} > 1$$

$$\Leftrightarrow \quad K > 0 \quad \text{or} \quad -1 < K < 0.$$

$$\text{i.e.} \quad K > -1, \quad \underline{K \neq 0}.$$

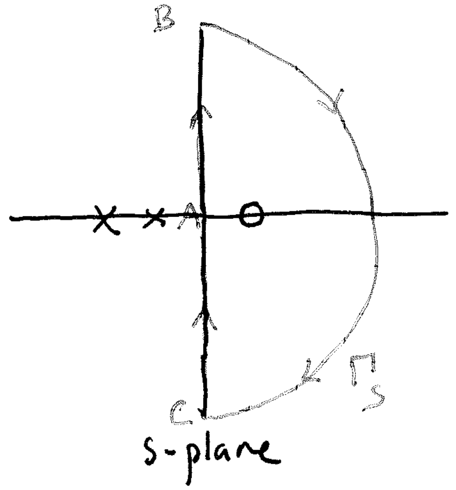
ruled out by our initial assumption but we can see that $K=0$ works.

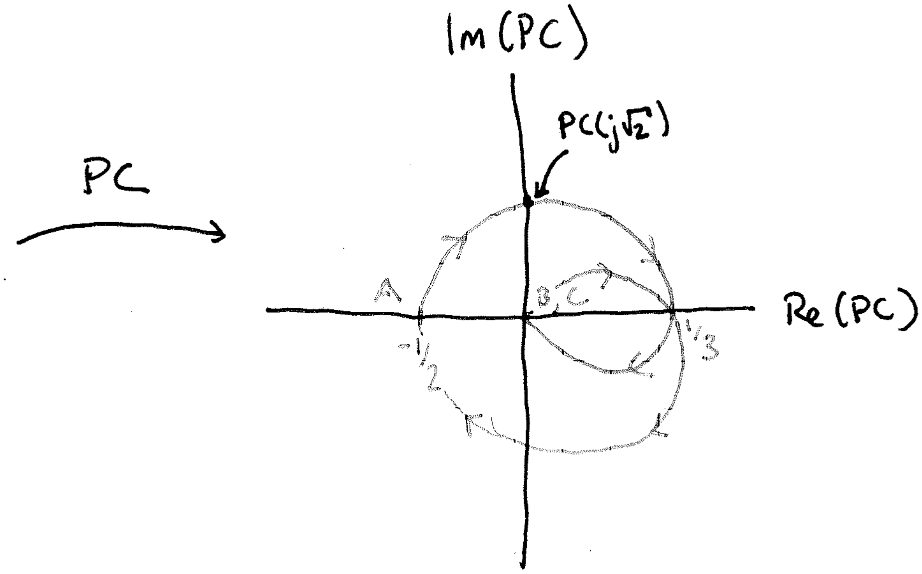
Therefore: $\boxed{K > -1}$

You can confirm this using RH on $\pi(s) = (s+1)^2 + K$ $\blacktriangle$

Example

$$C(s)P(s) = \frac{s-1}{(s+1)(s+2)}$$





$P_{\text{open-loop-bad-poles}} = 0$

A to B: $s = j\omega$, ω from 0 to $+\infty$.

$$C(s)P(s)\Big|_{AB} = CP(j\omega) = \frac{4\omega^2-2}{(\omega^2+1)(\omega^2+4)} + j\,\frac{5\omega-\omega^3}{(\omega^2+1)(\omega^2+4)}$$

Find ω where $\operatorname{Im} CP = 0$, $\omega = 0, +\sqrt{5}, -\sqrt{5}$

	$\omega = 0$	$\omega = \sqrt{5}$	$\omega \to \infty$
Re CP	$-\frac{1}{2}$	$\frac{1}{3}$	0
Im CP	0	0	0

+

B to C: Mapped to origin

C to D: Complex conjugate of A to B

Stability analysis Need $N = P_{\text{open-loop-bad-poles}} = 0$

Interval of $\mathbb{R}$:	$(-\infty, -\frac{1}{2})$	$(-\frac{1}{2}, 0)$	$(0, \frac{1}{3})$	$(\frac{1}{3}, +\infty)$
N	0	-1	-2	0

So we need

$$\frac{-1}{K} \in (-\infty, -\tfrac{1}{2}) \quad \text{or} \quad \frac{-1}{K} \in (\tfrac{1}{3}, +\infty)$$

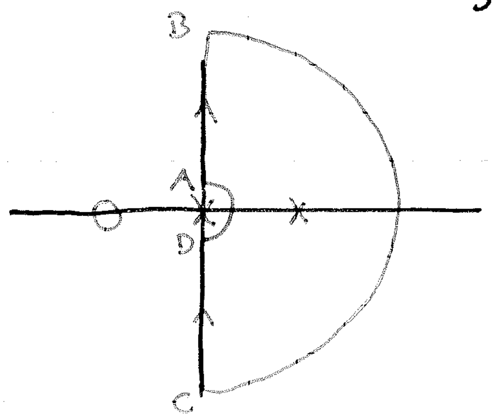
$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$0 < K < 2 \qquad\qquad -3 < K < 0$$

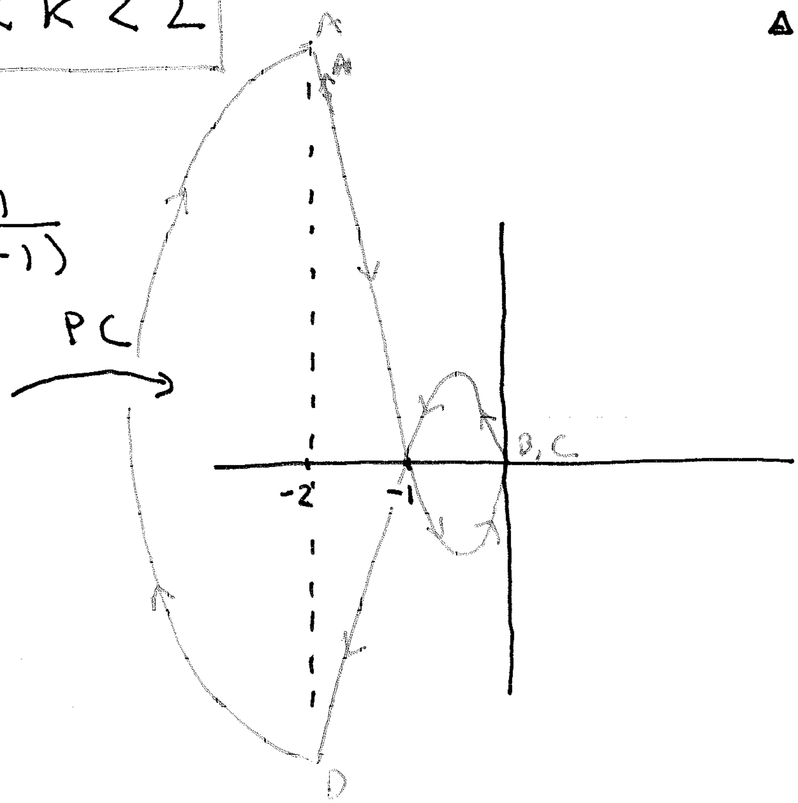
$$\Downarrow$$

Answer: $\boxed{-3 < K < 2}$ Δ

<u>Example</u> $P C(s) = \dfrac{s+1}{s(s-1)}$



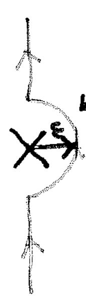
S-plane



<u>Notes</u>: pole at $s=0$, we indent to the right.

$P_{open\_loop\_bad\_poles} = 1$

Zoom in on A to D



indent with radius ε
$0 < \varepsilon << 1$.

<u>A to B:</u> $s = j\omega$, ω from ε to $+\infty$.

$$C(s)P(s)\Big|_{AB} = \frac{-2}{\omega^2+1} + j\,\frac{1-\omega^2}{\omega(\omega^2+1)}$$

$\omega = \varepsilon$

$CP \approx -2 + j\infty$

	$\omega = \varepsilon$	$\omega = 1$	$\omega \to \infty$
Re CP	-2	-1	0
Im CP	$+\infty$	0	0

$\searrow + \nearrow \qquad \searrow - \nearrow$

B to C: mapped to origin

C to D: complex conjugate of A to B

D to A: $s = \varepsilon e^{j\theta}$, $\theta \in [-\pi/2, \pi/2]$

$$PC\Big|_{DA} = \frac{\varepsilon e^{j\theta} + 1}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta} - 1)} \approx \frac{-1}{\varepsilon e^{j\theta}} = \cancel{-\varepsilon e^{-j\theta} = \varepsilon e^{j(\pi-\theta)}}$$

$$\frac{-1}{\varepsilon e^{j\theta}} = -\frac{1}{\varepsilon} e^{-j\theta} = \frac{1}{\varepsilon} e^{j(\pi-\theta)}$$

$\Rightarrow$ as θ goes from $-\pi/2$ to $\pi/2$, PC traces out a semi-circle of radius $\frac{1}{\varepsilon} \approx \infty$ and the angle goes from $-\pi/2$ (pointing down) to $\pi/2$ (pointing up)

Stability analysis

Need $N = 1$, From plot

Interval of $\mathbb{R}$:	$(-\infty, -1)$	$(-1, 0)$	$(0, +\infty)$
N:	-1	1	0

$\Rightarrow$ we need $-\frac{1}{K} \in (-1, 0) \Rightarrow \boxed{K > 1}$. $\blacktriangle$

8.3 Stability margins re-visited

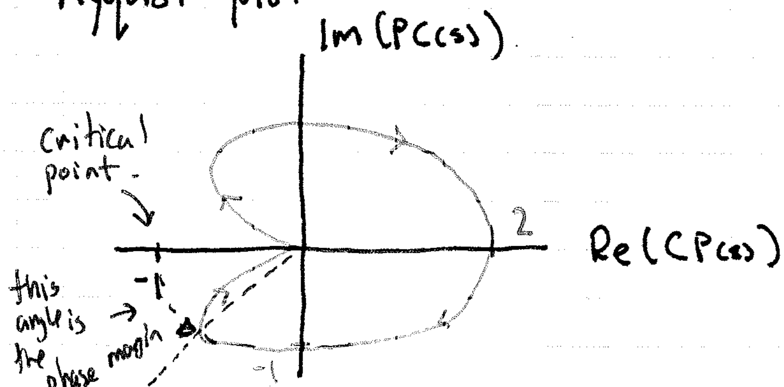
If a feedback system is stable, how stable is it?

This depends entirely on our plant model, how we got it, and what uncertainty there is about the model. In the frequency domain, uncertainty is naturally measured in terms of magnitude and phase as functions of frequency.

We've seen how to do this using Bode plots, but it turns out Nyquist plots are most revealing.

Example $\quad C(s) = 2 \qquad P(s) = \dfrac{1}{(s+1)^2}$

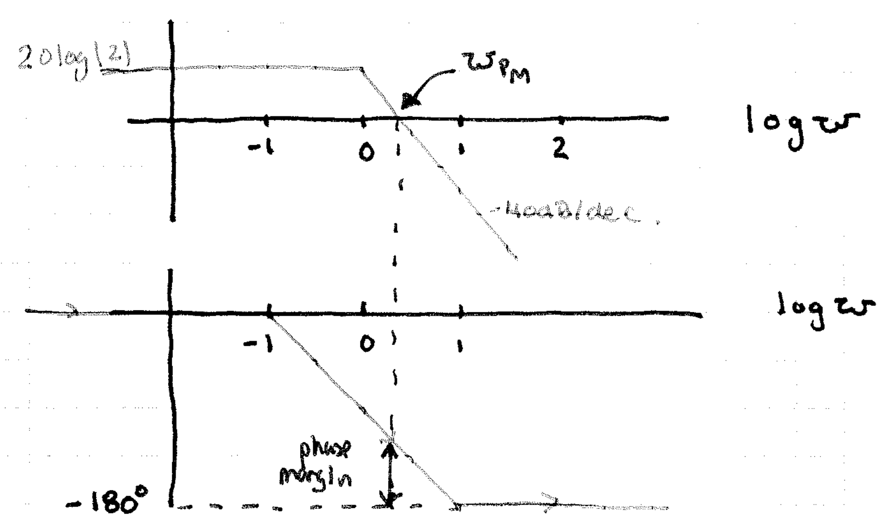
Nyquist plot



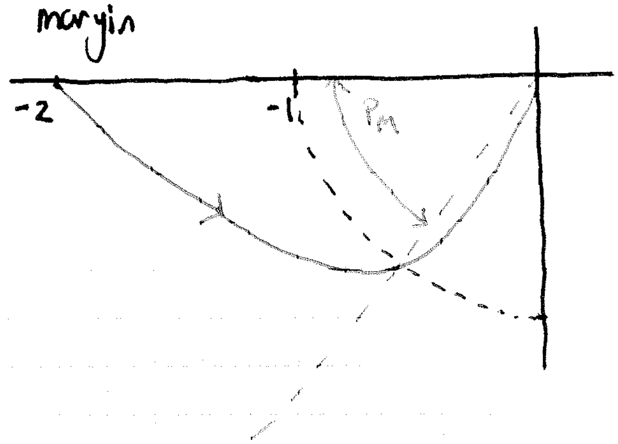
No encirclements of critical point ⇒ feedback stability

The phase margin is related to the distance from the critical point -1 to the point where the plot crosses the unit circle.

Bode plot of $PC = \dfrac{2}{(j\omega+1)^2}$



Phase margin



If $C(s) = 2K$, then K can be reduced until $-\frac{1}{K} = -2$,

i.e. $\min K = 1/2$ $(-6\,\text{dB})$.

If we'd used MATLAB it would say that we have a negative GM and we might interpret this as being unstable. ~~In fact~~ Wrong!

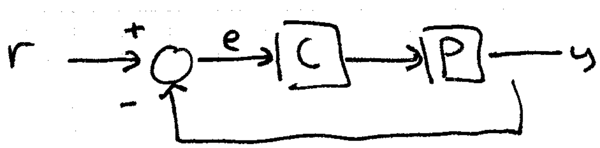
Conclusion: we need Nyquist plots for the correct interpretation of stability margins.

DELAY - - - DELAY

Recap: Phase margin is related to the distance from the crit pt to the Nyquist plot along the unit circle.

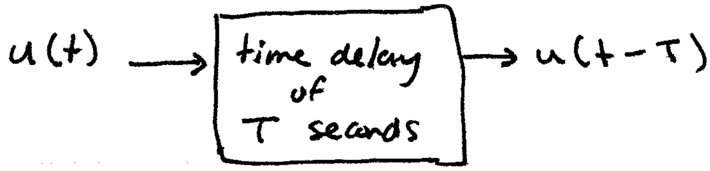
Gain margin is related to the distance from crit pt to the Nyquist plot along real axis.

More generally, it makes sense to define the stability margin as the distance from crit pt to the closest point on Nyquist plot.



$$S(s) := \frac{1}{1 + C(s)P(s)}$$ (TF from R to E)

Remark: Phase margin & time delay



$$\mathcal{L}\{u(t)\} = U(s)$$

$$\mathcal{L}\{u(t-T)\} = e^{-sT}\,U(s)$$

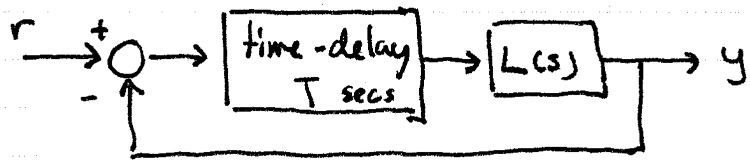
Bode plot of time-delay $e^{-sT}\big|_{s=j\omega}$.

$$|e^{-j\omega T}| = 1$$

$$\angle e^{-j\omega T} = -\omega T$$

time-delay in a control loop does not affect magnitude, does affect phase.

Now consider a system



~~with~~ Suppose that when $T=0$ (no delay), system has phase margin

$$P_M \text{ degrees at } \omega_{PM}$$

$\Rightarrow$ system will go unstable if we reduce the phase by $-P_M$ degrees at $\omega = \omega_{PM}$

$\Rightarrow$ the maximum delay we can handle is given by

$$-\omega_{PM}\, T_{max} = -P_M \iff \boxed{T_{max} = \frac{P_M}{\omega_{PM}}}$$

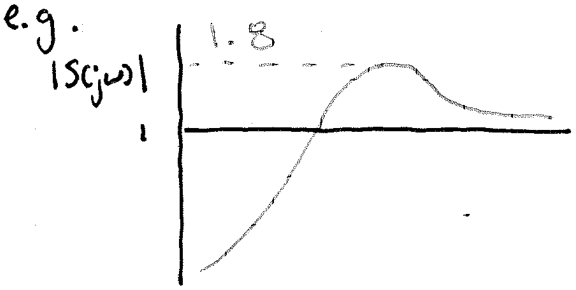
Assume feedback system is stable.

$$\min_{\omega} \left| -1 - P(j\omega)C(j\omega) \right| = \min_{\omega} \left| 1 + P(j\omega)C(j\omega) \right|$$

$$= \left[\max_{\omega} \left| S(j\omega) \right| \right]^{-1}$$

= reciprocal of peak magnitude on Bode plot of S.

e.g.



$\Rightarrow$ stability margin $= \dfrac{1}{1.8} = 0.56$.

8.4 Loop shaping

The TF $S(s) := \dfrac{1}{1 + C(s)P(s)}$ is the sensitivity function.

Reason for name:

$$T := \frac{PC}{1+PC} \quad \text{(TF from } R \text{ to } Y\text{)}$$

Relative change in T due to changes in P

$$\lim_{\Delta P \to 0} \frac{\Delta T / T}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{\Delta T}{\Delta P} \frac{P}{T}$$

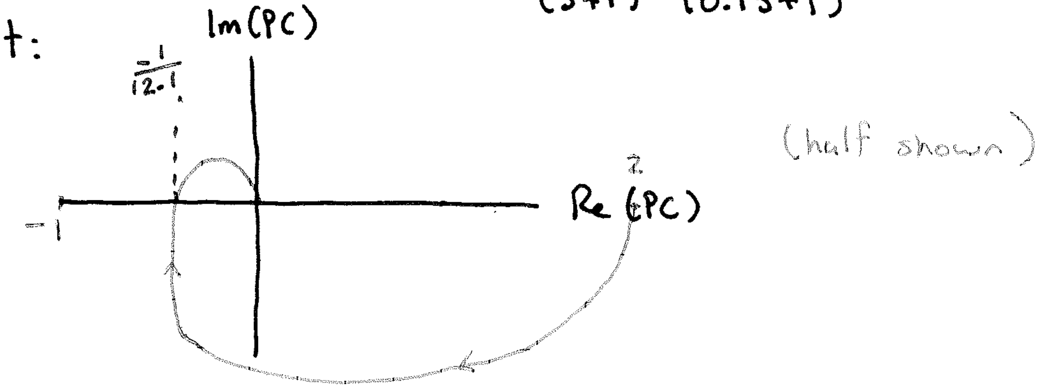
$$= \frac{dT}{dP} \frac{P}{T}$$

$$= S.$$

So S is a measure of the sensitivity of the closed-loop TF to variations in plant TF.

Example $C(s) = 2 \quad P(s) = \dfrac{1}{(s+1)^2(0.1s+1)}$

Nyquist:

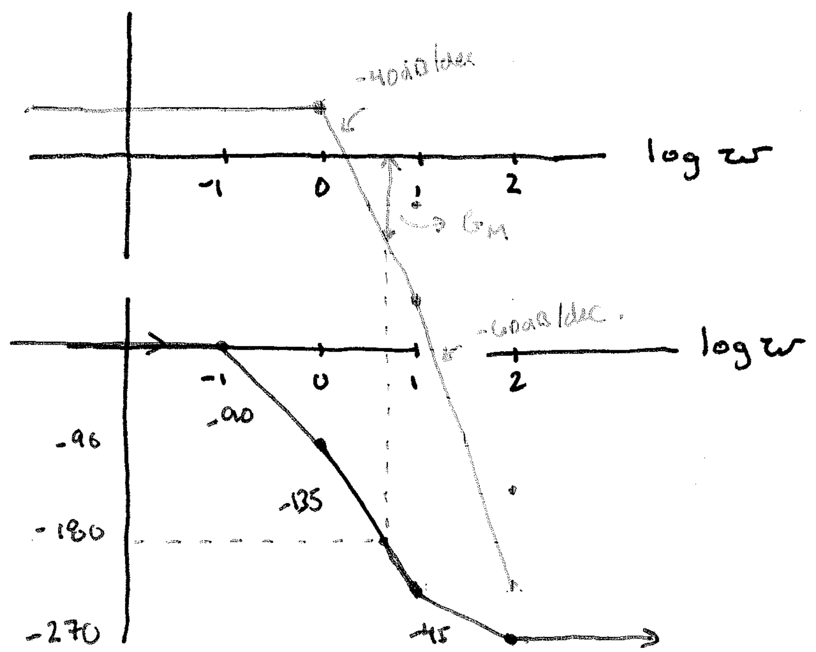


The feedback loop is stable for $C(s) = 2K$ if

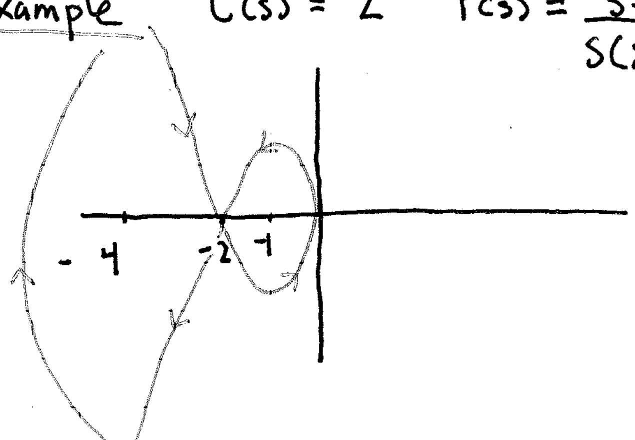
$\dfrac{-1}{K} < \dfrac{-1}{12.1} \Rightarrow$ we can increase K from 1 up to 12.1 (21.67 dB) before instability occurs.

21.67 dB is the gain margin of this system.

Bode plot



Example $C(s) = 2 \quad P(s) = \dfrac{s+1}{s(s-1)}$



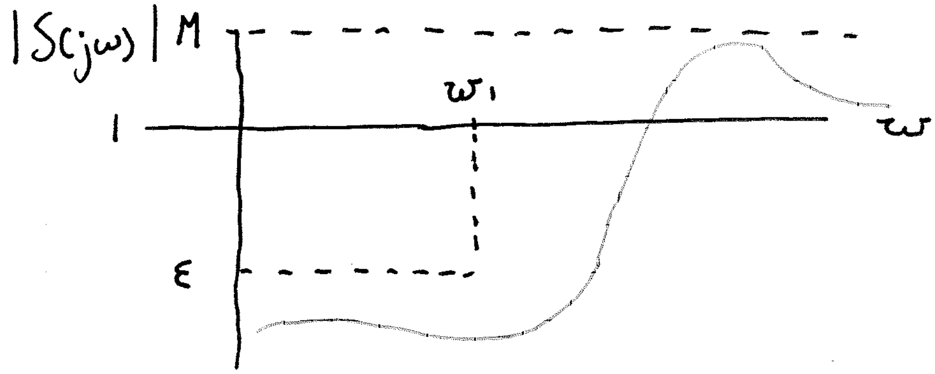
Critical point: -1
We need 1 ccw encirclement so the system is stable.

S is important for two reasons

1) It is the TF from R to E $\Rightarrow$ $|S(j\omega)|$ should be small over the range of frequencies of $r(t)$.

2) The peack magnitude of S is the reciprocal of the stability margin.

Thus, a typical desired magnitude plot of S is



Here

ω_1 - maximum frequency of $r(t)$
ε - maximum allowable tracking error $\varepsilon < 1$
M - maximum peak value of $|S(j\omega)|$

Why? If S has this shape and system is stable, then

(i) for $r(t) = \cos \omega t$, $\omega \leq \omega_1$, $|e(t)| \leq \varepsilon$ in steady-state
(ii) stability margin is at least $1/M$. Typical values for $M = 2$ or 3.

In these terms, the design problem can be stated as:

Given P, M, ε, ω_1; design C so that the feedback system is stable and

$$|S(j\omega)| \leq \varepsilon \quad \text{for} \quad \omega \leq \omega_1$$

$$|S(j\omega)| \leq M \quad \text{for all} \quad \omega.$$

Of course S is a nonlinear function of C, so in practice it is easier to design using the loop TF $L(s) = PC$ instead of $S = \dfrac{1}{1+L}$. If the loop gain is high then $|S| \approx \dfrac{1}{|L|}$.