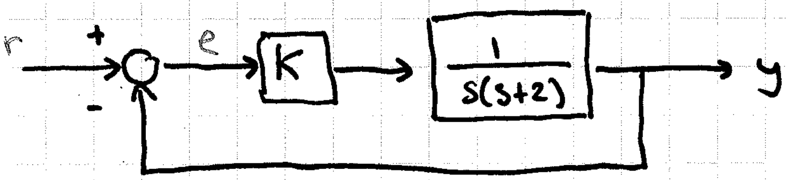
Chapter 6: Root locus methods

- closed-loop pole locations determine stability (directly) and performance (in a complicated way).
- Root locus methods show, graphically, how closed-loop poles change as a single real parameter, e.g. a controller gain, is changed.
- A root locus can be drawn by hand (sketched) and can be used for analysis & design. MATLAB: rltool.

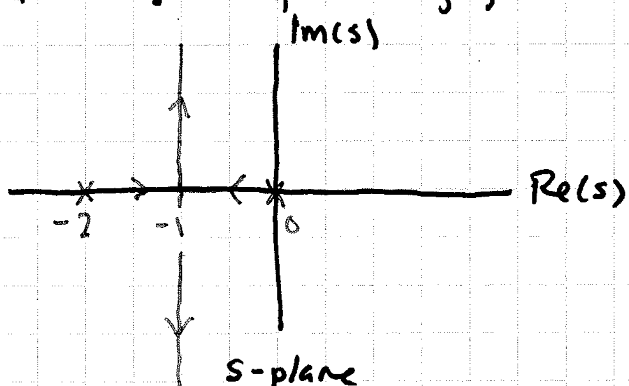
Example



$\pi(s) = s^2 + 2s + K$, closed-loop poles: $s = -1 \pm \sqrt{1-K}$

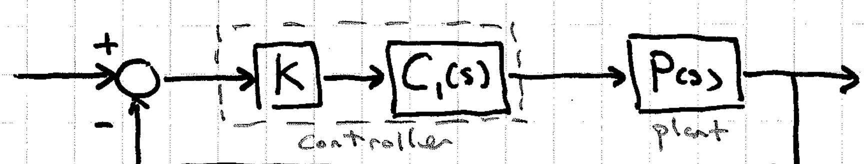
$K \in (0, 1]$: both real, system stable

$K > 1$: complex conjugate with -1 real part



Root locus of system.

6.1 Basic construction of root locii



$$\frac{Y(s)}{R(s)} = \frac{KC_1P}{1+KC_1P}$$

$$P(s)\,C_1(s) = \frac{N(s)}{D(s)}$$

The char. poly is $\Pi(s) = D(s) + K N(s)$

The roots of Π are the poles of the closed-loop system

$$\{s \in \mathbb{C} : \Pi(s) = 0\} = \{s \in \mathbb{C} : 1 + K C_1 P = 0\} = \{s \in \mathbb{C} : C_1 P = \tfrac{-1}{K}\}.$$

> The root locus shows how the roots of
> $$\Pi(s) = D(s) + K N(s)$$
> vary in the s-plane with respect to the scalar $K \in \mathbb{R}$

Notation: n = order of $D(s)$
m = order of $N(s)$

Assumptions (for now)

1. $n \geq m$ ($C_1 P$ is proper)
2. K varies from 0 to $+\infty$.
3. N and G are monic (if not, absorb into gain K).

Construction rules

Rule 1 The RL is symmetric about the real axis
→ follows from fact that Π has real coefficients

Rule 2 The RL has n branches
→ follows from the fact that n^{th} order poly has n roots.

Rule 3 The RL is a continuous function of K
→ follows from fact that roots of "Π" vary continuously as the coefficients vary.

Rule 4 The RL starts (when $K=0$) at roots of $D(s)$
→ $K=0$ corresponds to the "open-loop" case.

Rule 5 As $K \to \infty$, m branches of the root locus tend towards the roots of $N(s)$ (zeros of C.P)

The remaining $n-m$ branches tend to infinity along straight line asymptotes with angles

$$\phi_i = \frac{(2i+1)\pi}{n-m}, \quad i = 0, 1, \ldots, n-m-1$$

and a common intersection point on $\mathbb{R}$ at

$$\sigma = \frac{\sum \text{roots of } D(s) - \sum \text{roots of } N(s)}{n-m}.$$

Sketch of proof

$$\Pi(s) = D(s) + KN(s) = K\left[\frac{1}{K} D(s) + N(s)\right]$$

As K gets large, we intuitively see that the roots of N coincide with m of the roots of Π.

For the other $n-m$ branches, note that

$$\frac{N(s)}{D(s)} = \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} = \frac{s^m - (\sum z_i)s^{m-1} + \cdots}{s^n - (\sum p_i)s^{n-1} + \cdots}$$

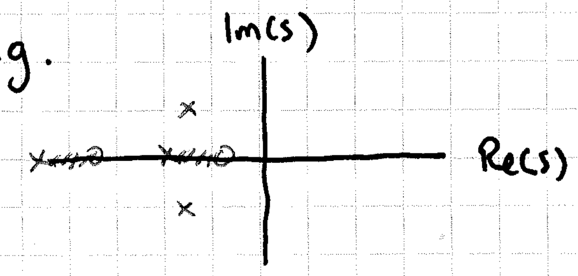
$$= \frac{1}{s^{n-m} - (\sum p_i - \sum z_i)s^{n-m-1} + \cdots}$$

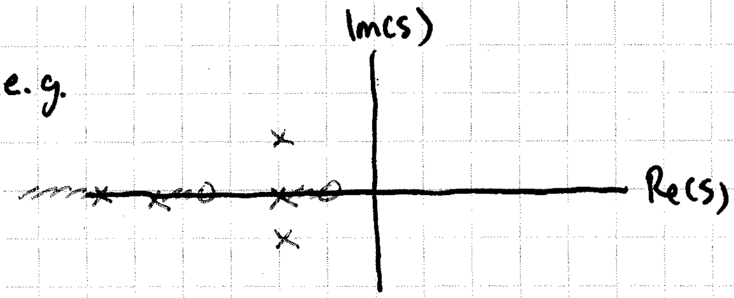
Compare this expression with

$$F(s) = \frac{1}{(s-\sigma)^{n-m}} = \frac{1}{s^{n-m} - (n-m)s^{n-m+1} + \cdots}$$

As $|s|$ becomes large, $\frac{N}{D}$ behaves like $F(s)$. $F(s)$ has the asymptotes discussed above.

Rule 6 A point s on the real axis is on the root locus if and only if the total # of ~~points~~ poles & zeros of N/D to the right of s is odd.

e.g. 

e.g. 

Note: (i) this is called the "yes-no-yes" rule for the real axis because of the "on-off" nature of it.
(ii) If you have repeated roots/zeros you count multiplicities.

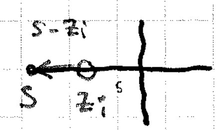
Proof $\Pi = 0 \iff$

$$K\frac{N}{D} = -1 \iff K \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = -1$$

$$\Rightarrow \measuredangle \sum_{i=1}^{m}(s-z_i) - \sum_{i=1}^{n}(s-p_i) = \angle -1/K \overset{K>0}{=} \text{odd multiple of } \pi$$

If s is real:

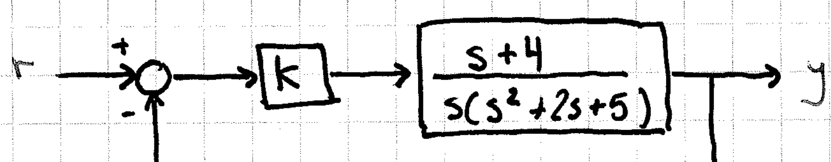
$$\angle s - z_i = \begin{cases} \pi & z_i \text{ to the right of } s \\ 0 & \text{else} \end{cases}$$



$$-\angle s - p_i = \begin{cases} -\pi & p_i \text{ to the right of } s. \\ 0 & \text{else.} \end{cases}$$

So a $s \in \mathbb{R}$ is part of the root locus for some $K>0$ iff the number of terms in $\sum(s-z_i) - \sum(s-p_i)$ with non-zero phase is odd. ∎

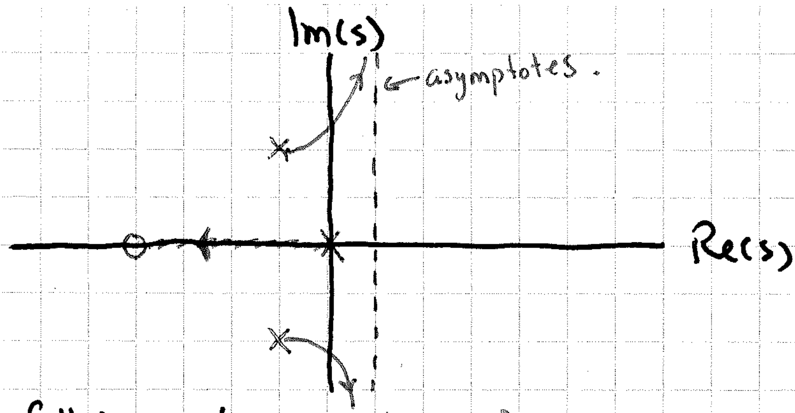
Example



Show how closed-loop poles vary as a function of $K \geq 0$.

$N(s) = s+4$ $\qquad D(s) = s(s+1+j2)(s+1-j2)$

$m = 1$ $\qquad n = 3$



O - zeros of N/D
× - poles of N/D.

(i) fill in real axis (rule 6)

(ii) asymptotes: $\phi_0 = \frac{\pi}{n-m} = \pi/2 \qquad \phi_1 = \frac{3\pi}{n-m} \approx -\frac{\pi}{2}$

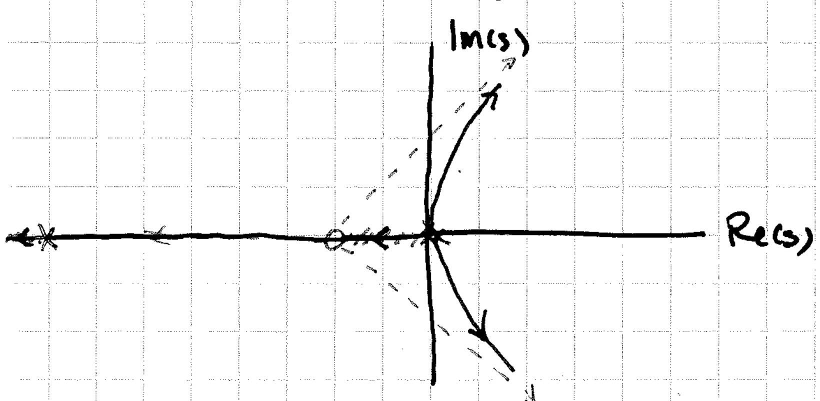
$$\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(0 - 1 + j2 - 1 - j2) - (-4)}{3-1}$$

$$= 1$$

(iii) draw rough root locus. Δ

<u>Example</u> $\pi(s) = s^3(s+4) + K(s+1) \Rightarrow m = 1 \quad \{-1\}$

$=: D(s) + K\,N(s) \qquad n = 4 \quad \{0, 0, 0, -4\}$



$$\phi_0 = \frac{\pi}{4-1} = \pi/3$$

$$\phi_1 = \frac{3\pi}{3} = \pi$$

$$\phi_2 = \frac{5\pi}{3} \approx -\pi/3$$

$$\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(-4) - (-1)}{4-1} = -1$$

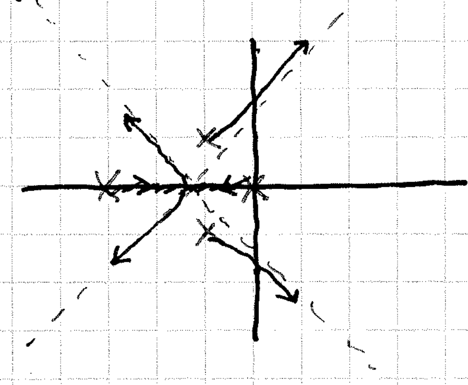
Δ

6.2 More rules

Rule 7: The points where the root locus crosses the imaginary axis can be found using the Routh-Hurwitz test.

Example $\pi(s) = s^4 + 5s^3 + 8s^2 + 6s + K =: N(s) + K D(s)$
$= s(s+3)(s^2+2s+2) + K$

s^4 :	1	8	K
s^3 :	5	6	0
s^2 :	$34/5$	K	0
s^1 :	$\dfrac{\frac{204}{5} - 5K}{34/5}$	0	0
s^0 :	K	0	0



- We get a zero in 1st col. of an odd row (s^1) when

$$\frac{204}{5} - 5K = 0 \iff K = \frac{204}{25} > 0$$

- Form the polynomial corresponding to the even row above (s^2)

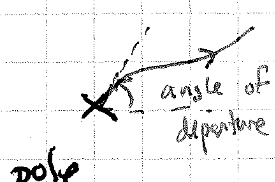
$$A(s) = {}^{34}/_{5}\, s^2 + {}^{204}/_{25}$$

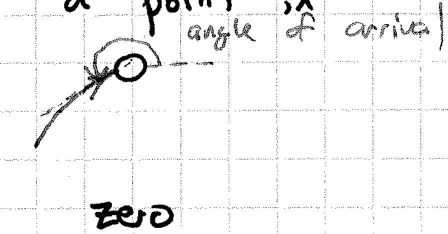
roots of A: $\quad s = \pm j\,1.095$

- You can check: roots of $A \subset$ roots of π

so root locus crosses $j\mathbb{R}$ when $K = 204/25$. $\triangle$

Rule 8: Angles of departure (from poles) and arrival (to zeros): represents the angle of the tangent to the locus near a point in the s-plane.





Use the angle criteria we derived before

$$\angle s-z_1 + \cdots + \angle s-z_m - \angle s-p_1 - \cdots - \angle s-p_n = (2k+1)\pi$$

Let θ_1 be the angle of a point on the RL $(k>0)$ but near $s=z_1$, then

$$\theta_1 + \angle s-z_2 + \cdots + \angle s-z_m - \angle s-p_1 - \cdots - \angle s-p_n = (2k+1)\pi$$

since the point in question is "far" from the othe poles and zeros, we can approximate them. Similarly for poles.

<u>Example</u> $\pi(s) = D(s) + KN(s) = (s+3)(s^2+2s+2) + K(s+2)$

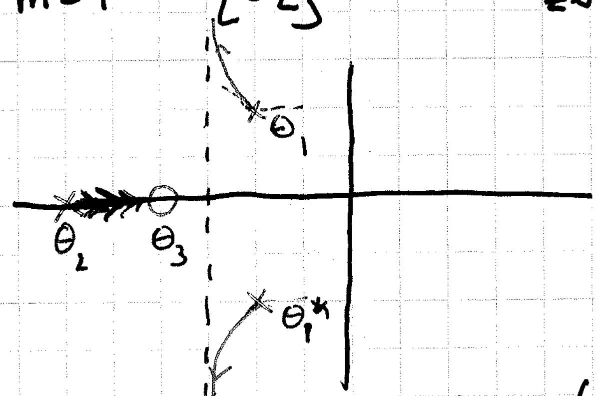
$n=3$ $\{-3, -1\pm j\}$ poles

$m=1$ $\{-2\}$ zeros

You can check

$\sigma = -3/2$

$\phi_0 = \pi/2$, $\phi_1 = -\pi/2$



Find angle of departures / arrival

$$\underset{\theta_3}{\angle(s+2)} - \underset{\theta_2}{\angle(s+3)} - \underset{\theta_1}{\angle(s+1-j)} - \underset{\theta_1^*}{\angle(s+1+j)}$$

@ $s=-3$ (a pole $\Rightarrow$ departure angle)

$$\pi - \theta_2 - \theta_1 - \theta_1^* = (2k+1)\pi$$

$$\Rightarrow \theta_2 = -2k\pi \approx 0°.$$

@ $s=-2$ (a zero $\Rightarrow$ arrival angle)

$$\theta_3 - 0 - \theta_1 - \theta_1^* = (2k+1)\pi$$

$$\Rightarrow \theta_3 \approx \pi$$

@ $s = -1 + j$ (pole $\Rightarrow$ departure angle)

$$\theta_3 - \theta_2 - \theta_1 - \theta_1^* = \pi(1+2k)$$

$$\operatorname{atan}\left[\frac{1}{1}\right] - \operatorname{atan}\left[\frac{1}{2}\right] - \theta_1 - \frac{\pi}{2} = \pi(2k+1)$$

$$\angle 1+j - \angle 2+j - \theta_1 - \angle 2j = \pi(2k+1)$$

$$\Rightarrow \theta_1 = -2\pi k \pm \pi + \pi/4 - \tan^{-1}(1/2) - \pi/2$$

$$= -2\pi k - 4.3906$$

$$\approx 360^\circ k - 251.6^\circ$$

$$\approx 108^\circ + 360^\circ k.$$

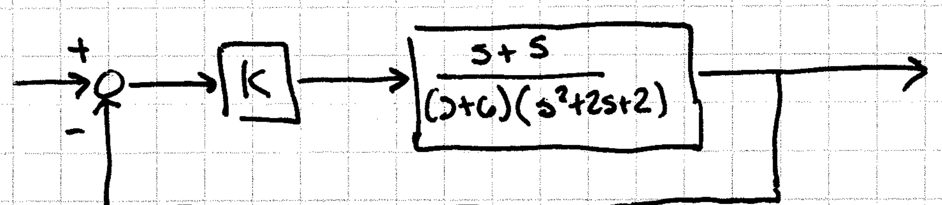
By symmetry, we now have θ_1^* too. Δ

Rule 9 Given a point s on the root locus, we can calculate the corresponding value of K

$$1 + K\frac{N(s)}{D(s)} = 0$$ ← every point on RL satisfies this equation for some value of K.

$$K = -\frac{D(s)}{N(s)}$$ ← this will always be a real number when s is on the RL.

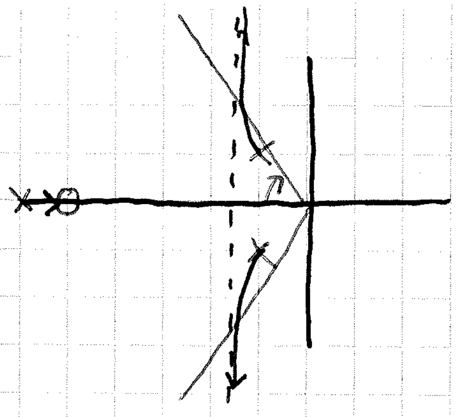
Example



$$T(s) = (s+6)(s^2+2s+2) + K(s+5)$$

Find the value of K that results in a closed-loop system that behaves dominantly like a 2nd order underdamped system w/ $\zeta = 0.3$

$n = 3$ $\sigma = -1.5$
$m = 1$



$\operatorname{acos}(\gamma) = \theta = 72.54°$

Desired point on RL

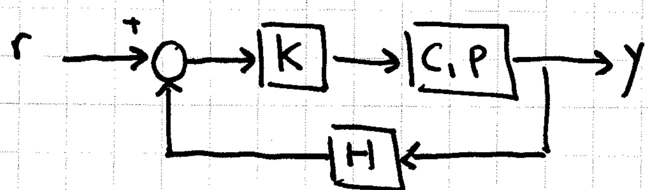
$s = -1.2 + j3.8$

$$K = -\frac{D(-1.2 + j3.8)}{N(-1.2 + j3.8)} \approx 15.36$$

△

6.3 Non-standard problem

If the sensor has dynamics then



Let $L(s) = C_1 P H = \frac{N}{D}$

Here, the char. poly is $\pi(s) = D(s) + K N(s)$.

In other words, the RL is exactly the same except that

$N = N_{C_1} N_P N_H$ and $D = D_{C_1} D_P D_H$.

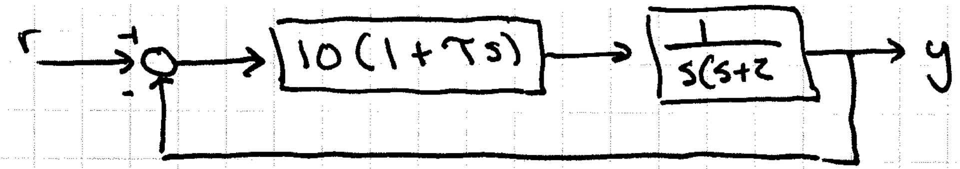
What happens when the parameter K does not appear as in the figure at the start of this chapter? In that case we can still write

$$\pi(s) = D(s) + K N(s)$$

but in this case

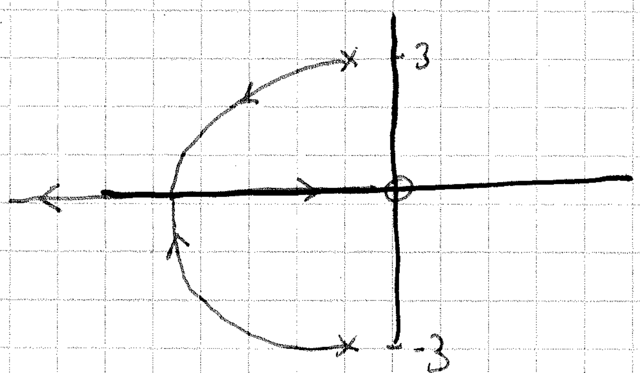
$D \neq D_{C_1} D_P \quad N \neq N_{C_1} N_P$.

Example



Note: This is a PD controller so it is improper. To implement it we'd roll it off as $\dfrac{10(1+\tau s)}{1+\varepsilon s}$, $\varepsilon > 0$ tiny.

Char poly: $\pi(s) = s(s+2) + 10(1+\tau s)$
$= (s^2+2s+10) + \underbrace{(10\tau)}_{K} s$



What happens if $\deg(D) > \deg(N)$?

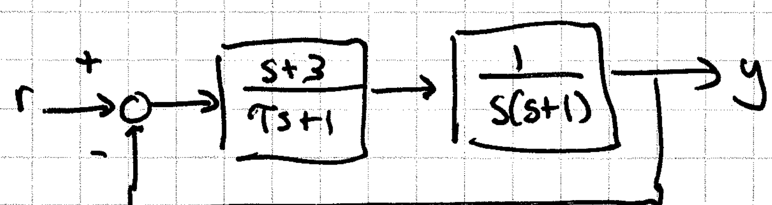
Write

$$\pi(s) = D(s) + K N(s) = 0$$

$$\iff \underbrace{N(s)}_{\hat{D}(s)} + \underbrace{\frac{1}{K}}_{\hat{K}} \underbrace{D(s)}_{\hat{N}(s)} = 0$$

Do the usual root locus on $\hat{\pi} = \hat{D} + \hat{K}\hat{N}$.

Example



$\pi(s) = s^2 + 2s + 3 + \tau\, s^2(s+1)$

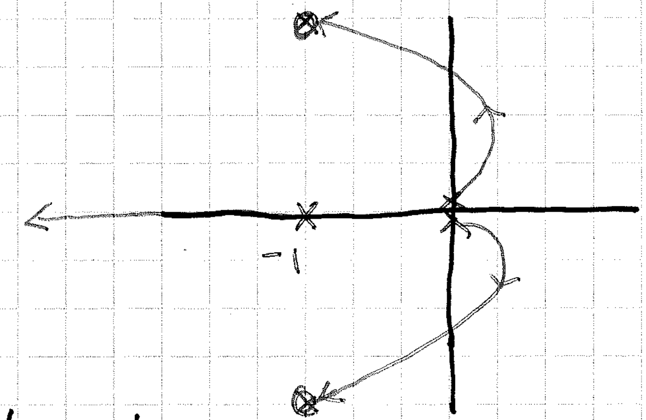
$n - m = -1 \Rightarrow$ use ~~the~~ approach above.

$\hat{\Pi}(s) = \hat{D}(s) + \hat{K}\,\hat{N}(s)$

$\hat{D}(s) = N(s) = s^2(s+1) \qquad \hat{n} = 3 \qquad \{0, 0, -1\}$

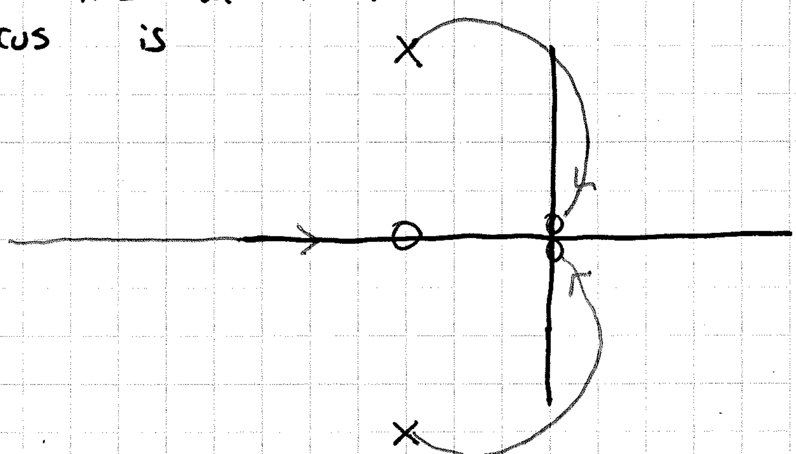
$\hat{N}(s) = D(s) = s^2 + 2s + 3 \qquad \hat{m} = 2 \qquad \{-1 \pm j\sqrt{2}\}$

$\hat{K} = \dfrac{1}{\tau}$



RL for $\hat{K} \geq 0$.

So the actual root locus is



RL for ~~1/K~~ $\tau \geq 0$