Chapter 5

Feedback control theory

Control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy used to compute corrective control action. When you go from your home to the university you use this principle continuously without being aware of it. To emphasize how effective feedback is, imagine you have to program a mobile robot, with no vision capability and therefore no feedback, to go open loop from your home to your SE380 classroom; the program has to include all motion instructions to the motors that drive the robot. The program would be unthinkably long, and in the end the robot would undoubtedly be way off target.

In this chapter we start to develop the basic theory and tools for feedback control analysis and design in the frequency domain. "Analysis" means you already have a controller and you want to study how good it is; "design" of course means you want to design a controller to meet certain specifications. The most fundamental specification is stability. Typically, good performance requires high-gain controllers, yet typically the feedback loop will become unstable if the gain is too high.

There are two main approaches to control analysis and design. The first, the one we're doing in this course, is the older, so-called "classical" approach in the frequency domain. Specifications are based on closed-loop gain, bandwidth, and stability margin. Design is done using Bode plots. The second approach, which is the subject of ECE488, is in the time domain and uses state-space models instead of transfer functions. Specifications may be based on closed-loop eigenvalues, that is, closed-loop poles. This second approach is known as the "state-space approach" or"modern control", although it dates from the 1960s and 1970s.

These two approaches are complementary. Classical control is appropriate

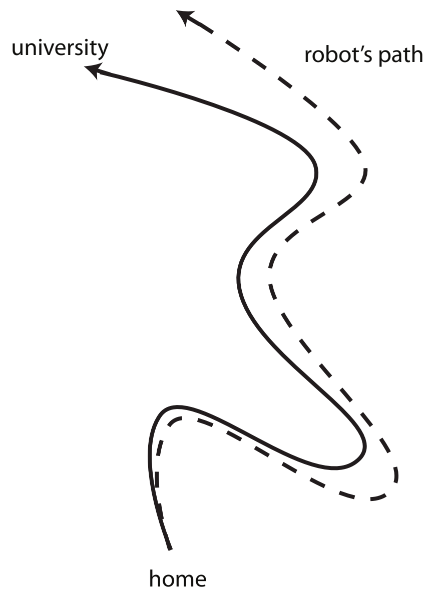

Figure 5.1: Getting to the university without feedback.

for a single-input, single-output plant, especially if it is open-loop stable. The state-space approach is appropriate for multi-input, multi-output plants; it is especially powerful in providing a methodical procedure to stabilize an unstable plant. Stability margin is very transparent in classical control and less so in the state-space approach. Of course, simulation must accompany any design approach. For example, in classical control you typically design for a desired bandwidth and stability margin; you test your design by simulation; you evaluate, and then perhaps modify the stability margin, redesign, and test again.

Beyond these two approaches is optimal control, where the controller is designed by minimizing a mathematical function. In this context classical control extends to $\mathcal{H}^\infty$ optimization and state-space control extends to Linear-quadratic Gaussian (LQG) control.

For all these techniques of analysis and design there are computer tools, the most popular being the Control System Toolbox of MATLAB.

5.1 Closing the Loop

As usual, we start with a series of examples to show how a typical control design might go. The first example deals with modeling.

Example 5.1.1. A favourite toy control problem is to get a cart to automatically balance a pendulum as shown in Figure 5.2. The natural state

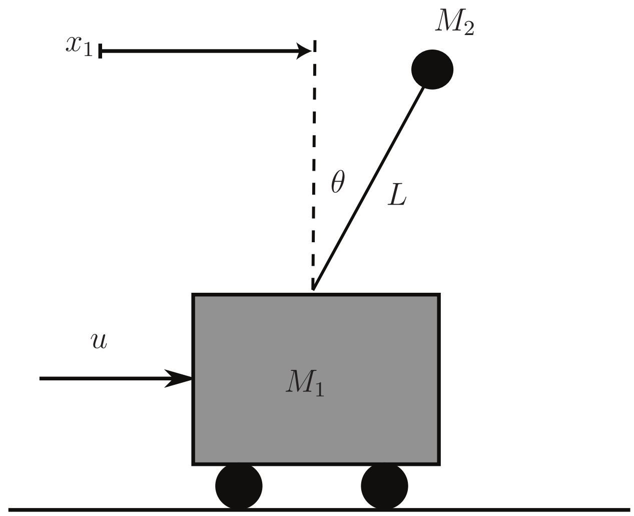

Figure 5.2: Pendulum on a cart.

is

$$x = (x_1, x_2, x_3, x_4) = (x_1, \theta, \dot{x}_1, \dot{\theta}).$$

Draw free body diagrams as shown in Figure 5.3. Newton's law for the ball in the horizontal direction is

$$M_2 \frac{d^2}{dt^2}(x_1 + L \sin\theta) = F_1 \sin\theta$$

and in the vertical direction is

$$M_2 \frac{d^2}{dt^2}(L - L\cos\theta) = M_2 g - F_1 \cos\theta$$

and for the cart is

$$M_1 \ddot{x}_1 = u - F_1 \sin\theta.$$

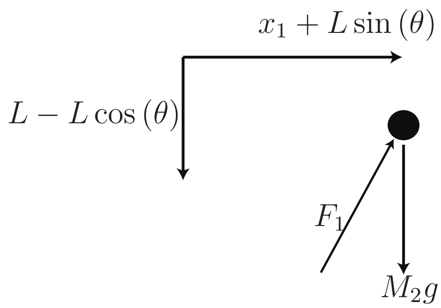

Figure 5.3: Free body diagram for the pendulum on a cart.

These are three equations in the four signals x_1, θ, u, F_1. Use

$$\frac{d^2}{dt^2}\sin\theta = \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta, \quad \frac{d^2}{dt^2}\cos\theta = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$$

to get

$$M_2\ddot{x}_1 + M_2L\ddot{\theta}\cos\theta - M_2L\dot{\theta}^2\sin\theta = F_1\sin\theta$$
$$M_2L\ddot{\theta}\sin\theta + M_2L\dot{\theta}^2\cos\theta = M_2g - F_1\cos\theta$$
$$M_1\ddot{x}_1 = u - F_1\sin\theta.$$

We can eliminate F_1: Add the first and the third to get

$$(M_1 + M_2)\ddot{x}_1 + M_2L\ddot{\theta}\cos\theta - M_2L\dot{\theta}^2\sin\theta = u;$$

multiply the first by $\cos\theta$, the second by $\sin\theta$, add, and cancel M_2 to get

$$\ddot{x}_1\cos\theta + L\ddot{\theta} - g\sin\theta = 0.$$

Solve the latter two equations for $\ddot{x}_1$ and $\ddot{\theta}$:

$$\begin{bmatrix} M_1 + M_2 & M_2L\cos\theta \\ \cos\theta & L \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + M_2L\dot{\theta}^2\sin\theta \\ g\sin\theta \end{bmatrix}.$$

Thus

$$\ddot{x}_1 = \frac{u + M_2L\dot{\theta}^2\sin\theta - M_2g\sin\theta\cos\theta}{M_1 + M_2\sin^2\theta}$$
$$\ddot{\theta} = \frac{-u\cos\theta - M_2L\dot{\theta}^2\sin\theta\cos\theta + (M_1 + M_2)g\sin\theta}{L(M_1 + M_2\sin^2\theta)}.$$

In terms of state variables we have

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{u + M_2 L x_4^2 \sin x_2 - M_2 g \sin x_2 \cos x_2}{M_1 + M_2 \sin^2 x_2} \\
\dot{x}_4 &= \frac{-u \cos x_2 - M_2 L x_4^2 \sin x_2 \cos x_2 + (M_1 + M_2) g \sin x_2}{L(M_1 + M_2 \sin^2 x_2)}.
\end{aligned}
$$

Again, these have the form

$$\dot{x} = f(x, u).$$

We might take the output to be

$$y = \begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h(x).$$

The system is highly nonlinear; as you would expect, it can be approximated by a linear system for $|\theta|$ small enough, say less than 5°.

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In order to apply the analysis and design tools of this course, we need an LTI plant. In the next example we linearize the pendulum on the cart.

Example 5.1.2. We continue the cart-pendulum. An equilibrium point

$$x_0 = (x_{10}, x_{20}, x_{30}, x_{40}), \quad u_0$$

satisfies $f(x_0, u_0) = 0$, i.e.,

$$x_{30} = 0$$

$$x_{40} = 0$$

$$u_0 + M_2 L x_{40}^2 \sin x_{20} - M_2 g \sin x_{20} \cos x_{20} = 0$$

$$-u_0 \cos x_{20} - M_2 L x_{40}^2 \sin x_{20} \cos x_{20} + (M_1 + M_2) g \sin x_{20} = 0.$$

Multiply the third equation by $\cos x_{20}$ and add to the fourth: We get in sequence

$$-M_2 g \sin x_{20} \cos^2 x_{20} + (M_1 + M_2) g \sin x_{20} = 0$$

$$(\sin x_{20})(M_1 + M_2 \sin^2 x_{20}) = 0$$

$$\sin x_{20} = 0$$

$$x_{20} = 0 \text{ or } \pi.$$

Thus the equilibrium points are described by

$$x_0 = (\text{arbitrary}, 0 \ \text{ or } \pi, 0, 0), \quad u_0 = 0.$$

We have to choose $x_{20} = 0$ (pendulum up) or $x_{20} = \pi$ (pendulum down). Let's take $x_{20} = 0$. Then the Jacobians compute to

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{M_2}{M_1}g & 0 & 0 \\ 0 & \frac{M_1+M_2}{M_1}\frac{g}{L} & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ -\frac{1}{LM_1} \end{bmatrix}.$$

The above is our general method of linearizing we introduced in Section 3.4. In this particular example, there's a faster way, which is to approximate $\sin\theta = \theta, \cos\theta = 1$ in the original equations:

$$M_2 \frac{d^2}{dt^2}(x_1 + L\theta) = F_1\theta$$

$$0 = M_2 g - F_1$$

$$M_1 \ddot{x}_1 = u - F_1\theta.$$

These equations are already linear and lead to the above A and B.

△

Now we close the loop on the cart-pendulum system.

Example 5.1.3. Consider the linearized cart-pendulum from the previous example. Take $M_1 = 1$ kg, $M_2 = 2$ kg, $L = 1$ m, $g = 9.8$ m/s^2. Then the state model is

$$\dot{x} = Ax + Bu, \ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Let's suppose we designate the cart position as the only output: $y = x_1$. Then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The transfer function from u to y is

$$P(s) = \frac{s^2 - 9.8}{s^2(s^2 - 29.4)}.$$

The poles and zeros of $P(s)$ are shown in Figure 5.4. Having three poles

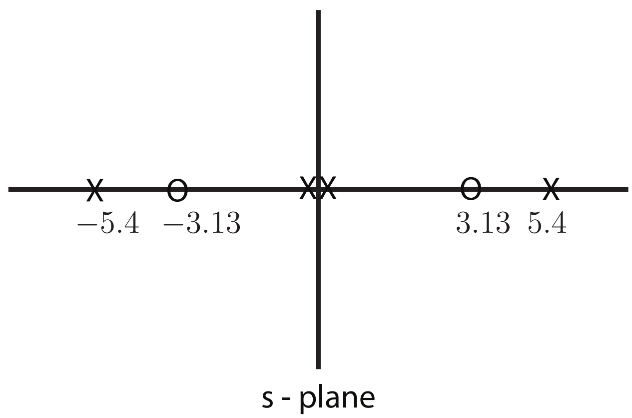

Figure 5.4: Pole zero map for the linearized pendulum on a cart.

in $\mathrm{Re}\, s \geq 0$, the plant is quite unstable. The right half-plane zero doesn't contribute to the degree of instability, but, as we shall see, it does make the plant quite difficult to control. The block diagram of the plant by itself is shown in Figure 5.5.

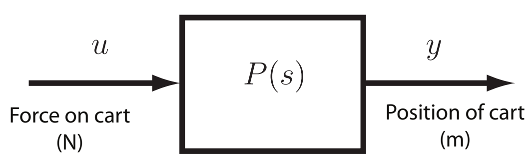

Figure 5.5: Block diagram of the linearized pendulum on a cart.

Let us try to stabilize the plant by feeding back the cart position, y, comparing it to a reference r, and setting the error $r - y$ as the controller input as shown in Figure 5.6. Here $C(s)$ is the transfer function of the

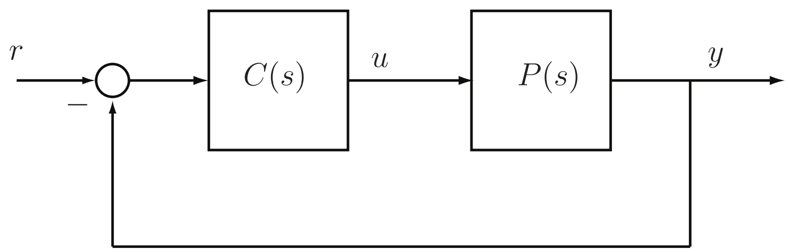

Figure 5.6: Control architecture for cart-pendulum example.

controller to be designed. One controller that does in fact stabilize is

$$C(s) = \frac{10395s^3 + 54126s^2 - 13375s - 6687}{s^4 + 32s^3 + 477s^2 - 5870s - 22170}.$$

The controller itself, $C(s)$, is unstable, as is $P(s)$. But when the controller and plant are connected in feedback, the system is stable. If the pendulum starts to fall, the controller causes the cart to move, in the right direction, to make the pendulum tend to come vertical again. You're invited to simulate the closed-loop system; for example, let r be the input shown in Figure 5.7. This reference signal corresponds to a command that the cart move right 0.1

Figure 5.7: Reference signal for Example 5.1.3.

m for 5 seconds, then return to its original position. Figure 5.8 is a plot of the cart position x_1 versus t. The cart moves rather wildly as it tries to balance the pendulum—it's not a good controller design—but it does stabilize.

We mentioned that our controller $C(s)$ is open-loop unstable. It can be proved (it's beyond the scope of this course) that **every** controller that stabilizes this $P(s)$ is itself unstable.

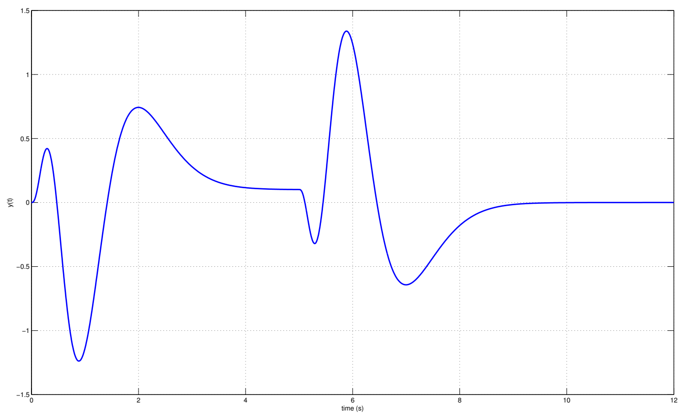

Figure 5.8: Cart position versus time for the reference input in Figure 5.7.

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Our objective in this section is to define what it means for the feedback system in Figure 5.9 to be stable.

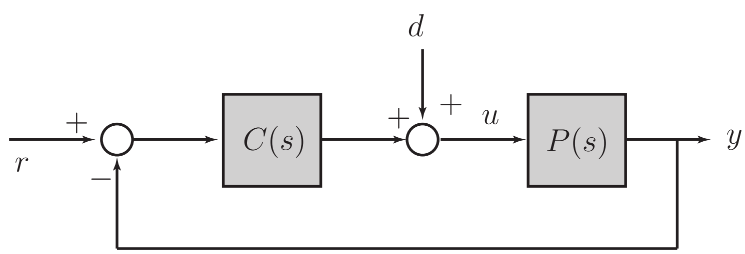

Figure 5.9: Feedback system.

The signals and systems in Figure 5.9 are

systems	$P(s)$, plant transfer function
	$C(s)$, controller transfer function
signals	$r(t)$, reference (or command) input
	$e(t)$, tracking error
	$d(t)$, disturbance
	$u(t)$, plant input
	$y(t)$, plant output.

We shall **assume throughout** that $P(s)$, $C(s)$ are rational, $C(s)$ is proper, and $P(s)$ is strictly proper.

5.1.1 Feedback Stability

Let's find all of the transfer functions from the independent signals $r(t)$ and $d(t)$ to the dependent signals $e(t)$, $u(t)$ and $y(t)$. We use our systematic method of doing block diagram reduction. First, write the equations for the outputs of the summing junctions:

$$\begin{aligned} E &= R - PU \\ U &= D + CE. \end{aligned}$$

Assemble into a vector equation:

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}.$$

In view of our standing assumptions (P strictly proper, C proper), the determinant of

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}$$

is not identically zero (why?). Thus we can solve for E, U:

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} R \\ D \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+PC} & \dfrac{-P}{1+PC} \\ \dfrac{C}{1+PC} & \dfrac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}.$$

The output is given by

$$Y = PU = \frac{PC}{1+PC} R + \frac{P}{1+PC} D.$$

We just derived the following closed-loop transfer functions:

$$\begin{aligned} &R \text{ to } E: \frac{1}{1+PC}, \qquad && R \text{ to } U: \frac{C}{1+PC}, \qquad && R \text{ to } Y: \frac{PC}{1+PC} \\ &D \text{ to } E: \frac{-P}{1+PC}, \qquad && D \text{ to } U: \frac{1}{1+PC}, \qquad && D \text{ to } Y: \frac{P}{1+PC}. \end{aligned}$$

Definition 5.1.1. The feedback system in Figure 5.9 is feedback stable provided e, u, and y are bounded signals whenever r and d are bounded signals; briefly, the system from (r, d) to (e, u, y) is BIBO stable.

Feedback stability is equivalent to saying that the 6 transfer functions from (r, d) to (e, u, y) are stable, in the sense that all poles are in $\operatorname{Re} s < 0$. Since, whenever r and e are bounded, so is $y = r - e$, it suffices to look at the 4 transfer functions from (r, d) to (e, u), namely,

$$\begin{bmatrix} \dfrac{1}{1+PC} & \dfrac{-P}{1+PC} \\ \dfrac{C}{1+PC} & \dfrac{1}{1+PC} \end{bmatrix}.$$

Example 5.1.4.

$$P(s) = \frac{1}{s^2 - 1}, \ C(s) = \frac{s-1}{s+1}$$

The 4 transfer functions are

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} \dfrac{(s+1)^2}{s^2+2s+2} & \dfrac{s+1}{(s-1)(s^2+2s+2)} \\ & \\ \dfrac{(s+1)(s-1)}{s^2+2s+2} & \dfrac{(s+1)^2}{s^2+2s+2} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}.$$

Three of these are stable; the one from D to E is not. Consequently, the feedback system is **not feedback stable**. This is in spite of the fact that a bounded r produces a bounded y. Notice that the problem here is that C cancels an unstable pole of P. As we'll see, that isn't allowed.

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Example 5.1.5.

$$P(s) = \frac{1}{s-1}, \ C(s) = K$$

The feedback system is input-output stable if and only if $K > 1$ (check).

$\triangle$

We now provide two ways to test feedback stability. Write

$$P = \frac{N_p}{D_p},\ C = \frac{N_c}{D_c}.$$

We assume $(N_p,\ D_p)$ are coprime, i.e., have no common factors, and $(N_c,\ D_c)$ are coprime too.

Definition 5.1.2. The characteristic polynomial of the feedback system is defined to be

$$\pi(s) := N_p N_c + D_p D_c.$$

The characteristic polynomial of the feedback system is the denominator of the four transfer functions from (r, d) to (e, u), i.e.,

$$\begin{bmatrix} \frac{1}{1+PC} & \frac{-P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} = \frac{1}{N_p N_c + D_p D_c} \begin{bmatrix} D_p D_c & -N_p D_c \\ N_c D_p & D_p D_c \end{bmatrix}. \tag{5.1}$$

Example 5.1.6.

$$P(s) = \frac{1}{s^2 - 1},\ C(s) = \frac{s-1}{s+1}.$$

We have seen that with the plant and controller combination the system is not feedback stable because the transfer function from d to e is unstable. Notice the unstable pole-zero cancellation. The characteristic polynomial is

$$\pi(s) = s - 1 + (s^2 - 1)(s + 1) = (s - 1)(s^2 + 2s + 2).$$

This has a right half-plane root.

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Theorem 5.1.3. *The feedback system is feedback stable if and only if the characteristic polynomial has no roots in* $\mathrm{Re}\, s \geq 0$.

Proof. ($\Leftarrow$ Sufficiency) If $N_p N_c + D_p D_c$ has no roots in $\mathrm{Re}\, s \geq 0$, then the four transfer functions on the left-hand side of (5.1) have no poles in $\mathrm{Re}\, s \geq 0$, and hence they are stable.
($\Rightarrow$ Necessity) Conversely, assume the feedback system is stable, that is, the four transfer functions on the left-hand side of (5.1) are stable. To conclude that $N_p N_c + D_p D_c$ has no roots in $\mathrm{Re}\, s \geq 0$, we must show that

the polynomial $N_pN_c + D_pD_c$ does not have a common factor with all four numerators in (5.1), namely, D_pD_c, N_pD_c, N_cD_p. That is, we must show that the four polynomials

$$N_pN_c + D_pD_c,\ D_pD_c,\ N_pD_c,\ N_cD_p$$

do not have a common root. This part is left for you. $\square$

Definition 5.1.4. The plant $P(s)$ and controller $C(s)$ have a pole-zero cancellation if there exists a $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} N_p(\lambda) = D_c(\lambda) = 0 &\qquad \text{(plant zero at controller pole)} \\ D_p(\lambda) = N_c(\lambda) = 0 &\qquad \text{(plant pole at controller zero).} \end{aligned}$$

It is called an unstable pole-zero cancellation if $\operatorname{Re}(\lambda) \geq 0$.

Corollary 5.1.5. *If there is an unstable pole-zero cancellation then the feedback system is not feedback stable.*

Proof. If there is an unstable pole-zero cancellation at $\lambda \in j\mathbb{R} \cup \mathbb{C}^+$, then

$$\begin{aligned} \pi(\lambda) &= N_P(\lambda)N_c(\lambda) + D_P(\lambda)D_c(\lambda) \\ &= 0. \end{aligned}$$

Thus π has a root in $j\mathbb{R} \cup \mathbb{C}^+$ so that by Theorem 5.1.3 the system is not feedback stable. $\square$

A second way to test feedback stability is as follows.

Theorem 5.1.6. *The feedback system is input-output stable if and only if*

1. *The transfer function $1 + PC$ has no zeros in $\operatorname{Re} s \geq 0$, and*
2. *the product PC has no unstable pole-zero cancellations.*

(Proof left to you.)

Example 5.1.7.

$$P(s) = \frac{1}{s^2 - 1},\ C(s) = \frac{s-1}{s+1}$$

Check that 1) holds but 2) does not.

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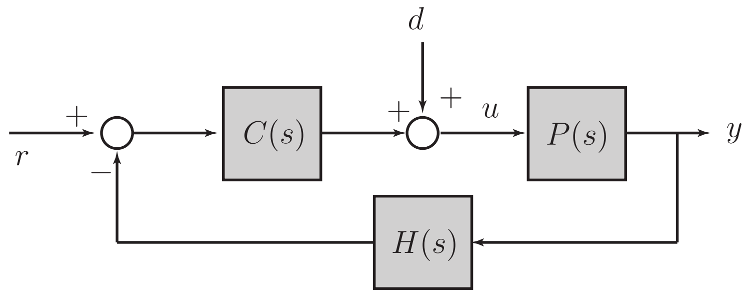

Figure 5.10: Non-unity feedback system.

Remark 5.1.7. Occasionally we will have a non-unity feedback system as shown in Figure 5.10. This is the case when we model the dynamics of the sensor that provides the feedback as a transfer function $H(s)$. If we write

$$P = \frac{N_p}{D_p}, \; C = \frac{N_c}{D_c}, \quad H = \frac{N_h}{D_h}$$

then you can check that the characteristic polynomial becomes

$$\pi(s) = N_c N_p N_h + D_c D_p D_h.$$

In this case we have feedback stability if and only if all the roots of π are in $\mathbb{C}^-$.

5.2 The Routh-Hurwitz Criterion

In practice, one checks feedback stability using MATLAB to calculate the the roots of the characteristic polynomial. However, if some of the coefficients of $\pi(s)$ are not fixed, e.g., a controller gain, then we can't solve for the roots numerically using MATLAB. The Routh-Hurwitz criterion provides a test to check if the roots of $\pi(s)$ are in $\mathbb{C}^-$ without actually finding them.

Consider a general characteristic polynomial

$$\pi(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0, \quad a_i \in \mathbb{R}. \tag{5.2}$$

Let's say $\pi(s)$ is stable if all its roots have $\operatorname{Re} s < 0$. The Routh-Hurwitz criterion is an algebraic test for $\pi(s)$ to be stable, without having to calculate the roots.

We make the observation that a necessary, but not sufficient condition for $\pi(s)$ to be stable is that all the coefficients a_i have the same sign.

Proof. Factor the polynomial as

$$\pi(s) = a_n(s - p_1)(s - p_2)\cdots(s - p_n).$$

For the real roots we have $s-p$ where $-p > 0$ since π is stable by assumption. For the complex conjugate roots we expand

$$(s - p)(s - p^\star) = s^2 - (p + p^\star)s + pp^\star$$

and note that $-(p + p^\star) > 0$ and $pp^\star > 0$ since π is stable by assumption. Therefore, when we expand $\pi(s)$, every coefficient will have the same sign as a_n. □

Example 5.2.1.

$$\begin{array}{ll} s^4 + 3s^3 - 2s^2 + 5s + 6 & \text{(a bad root)} \\ s^3 + 4s + 6 & \text{(a bad root)} \\ s^3 + 5s^2 + 9s + 1 & \text{(don't know).} \end{array}$$

△

Main result

Consider the general characteristic polynomial (5.2). Using π we generate the Routh array.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s^1 & \ell_1 & \ell_2 & 0 & 0 \\ s^0 & m_1 & 0 & 0 & 0 \end{array}$$

where

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \quad b_2 = \frac{a_{n-3}a_{n-4} - a_{n-2}a_{n-5}}{a_{n-3}}, \ldots,$$

and

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1}b_2}{b_1}, \quad c_2 = \frac{b_2 a_{n-5} - a_{n-3}b_3}{b_2}, \ldots.$$

1. We continue this pattern along each row until you end up with zeros.

2. Terminate the algorithm if you end up with a zero in the first column.

Routh-Hurwitz Stability Criterion

1. $\pi(s)$ is stable if and only if all elements in the first column of the Routh table have the same sign. If the algorithm terminates early, $\pi(s)$ is not stable and there is a bad root.

2. If there are no zeros in the first column then

   (a) The number of bad roots equals the number of sign changes in the first column of the Rough table.

   (b) There are no roots on the imaginary axis.

Remark 5.2.1. The spacial cases when the algorithm terminates can also be analyzed.

Example 5.2.2.

$$\pi(s) = a_2 s^2 + a_1 s + a_0, \qquad a_2 \neq 0.$$

$$\begin{array}{c|ccc} s^2 & a_2 & a_0 & 0 \\ s^1 & a_1 & 0 & 0 \\ s^0 & \frac{a_1 a_0}{a_1} = a_0 & 0 & 0 \end{array}$$

We conclude that all the roots of π are in $\mathbb{C}^-$ if and only if a_0, a_1, a_2 have the same sign.

$\triangle$

Example 5.2.3.

$$\pi(s) = 2s^4 + s^3 + 3s^2 + 5s + 10.$$

$$\begin{array}{c|cccc}
s^4 & 2 & 3 & 10 & 0 \\
s^3 & 1 & 5 & 0 & 0 \\
s^2 & \frac{3-10}{1} = -7 & 10 & 0 & 0 \\
s^1 & \frac{-35-10}{-7} = \frac{45}{7} & 0 & 0 & 0 \\
s^0 & 10 & 0 & 0 & 0
\end{array}$$

There are two sign changes in the first column so $\pi(s)$ has two roots in $\mathbb{C}^+$.

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Example 5.2.4.

$$\pi(s) = s^3 + s^2 + s + 1.$$

$$\begin{array}{c|ccc}
s^3 & 1 & 1 & 0 \\
s^2 & 1 & 1 & 0 \\
s^1 & 0 & &
\end{array}$$

The algorithm terminates early and we conclude that π has at least one bad root.

$\triangle$

Example 5.2.5. Consider the unity feedback system in Figure 5.11 with

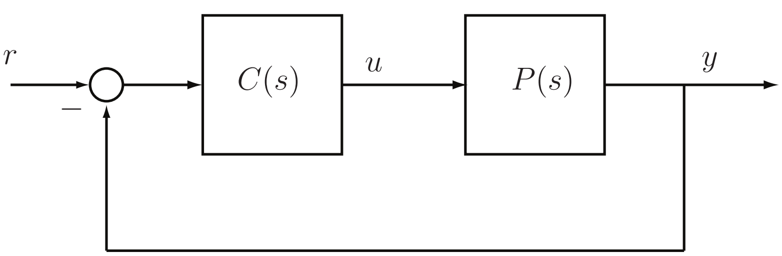

Figure 5.11: System for Example 5.2.5.

$$P(s) = \frac{1}{s^4 + 6s^3 + 11s^2 + 6s}, \qquad C(s) = K.$$

Find all $K \in \mathbb{R}$ such that the feedback system is stable. From Section 5.1.1 we know that the system is feedback stable if and only if all the roots of $\pi(s)$ are in $\mathbb{C}^-$. In this case

$$\pi(s) = N_p N_c + D_p D_c = s^4 + 6s^3 + 11s^2 + 6s + K.$$

We now apply the Routh-Hurwitz stability criterion

$$\begin{array}{c|cccc}
s^4 & 1 & 11 & K & 0 \\
s^3 & 6 & 6 & 0 & 0 \\
s^2 & 10 & K & 0 & 0 \\
s^1 & \frac{3}{5}(10-K) & 0 & 0 & 0 \\
s^0 & K & 0 & 0 & 0
\end{array}$$

We need all entries in the first column to be positive. From the s^1 row we get the constraint $K < 10$. From the s^0 row we get the constraint $K > 0$. Therefore the system is feedback stable provided

$$K \in (0, 10).$$

$\triangle$

5.3 Introduction to PID control

The majority of controllers in industry are proportional-integral-derivative (PID) controllers. They are popular because they are easy to understand and tune, often provide adequate performance, and the structure of the controller does not depend on the plant model. The block diagram of a PID controller is shown in Figure 5.12. The transfer function of a PID controller is given by

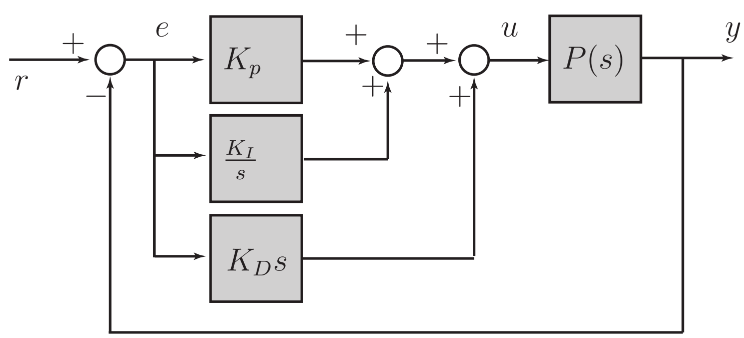

Figure 5.12: Basic PID control structure.

$$C(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_p s + K_I}{s}. \tag{5.3}$$

The corresponding differential equation in the time domain is

$$u(t) = K_p e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt} \tag{5.4}$$

where $e(t) = r(t) - y(t)$ as usual. The proportional, integral and derivative gains K, K_I and K_d are sometimes expressed as

$$K_p = K, \qquad K_D = KT_D, \qquad K_I = \frac{K}{T_I}$$

with T_I called the integral time and T_D called the derivative time. With these definitions the PID transfer function becomes

$$C(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right).$$

Remark 5.3.1. The basic PID controller is improper which makes it difficult to implement. More importantly, it often fails to provide closed-loop stability. Hence it is normally "rolled off" as

$$C(s) = K_p + \frac{K_I}{s} + K_D \frac{s}{\tau s + 1}, \qquad \tau > 0, \text{ small}$$

or the whole controller is rolled off as

$$C(s) = \frac{K_D s^2 + K_p s + K_I}{s(\tau s + 1)}, \qquad \tau > 0, \text{ small.}$$

5.4 Steady-state performance

For any control system, closed-loop or feedback stability is essential; good performance is desirable. There are two types of performance measures

1. Transient behaviour: depends in a complicated way on the location of the closed-loop poles and zeros. If the closed-loop system is dominated by a first or second order system, then we can use the results of Chapter 4.

2. Steady-state performance : This refers to specifications like steady-state tracking, stead-state disturbance rejection.

5.4.1 Steady-state tracking

Cruise control in a car regulates the speed to a prescribed set point. What is the principle underlying its operation? The answer lies in the final value theorem (FVT).

Example 5.4.1. Consider the unity feedback system in Figure 5.13 with

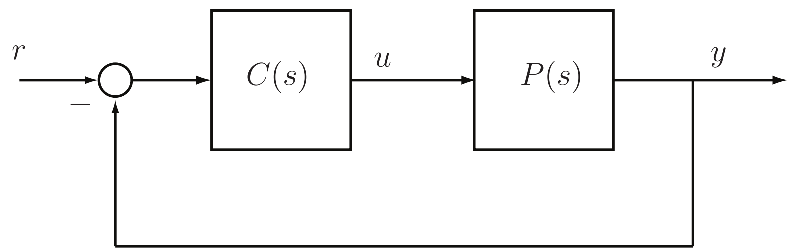

Figure 5.13: System for Example 5.4.1.

$$P(s) = \frac{1}{s+1}, \qquad C(s) = \frac{1}{s}.$$

Let r be a constant, $r(t) = r_0$. Then we have

$$\begin{aligned} E(s) &= \frac{1}{1+P(s)C(s)} R(s) \\ &= \frac{s(s+1)}{s^2+s+1} \frac{r_0}{s} \\ &= \frac{s+1}{s^2+s+1} r_0 \end{aligned}$$

The FVT applies to $E(s)$, and $e(t) \to 0$ as $t \to \infty$. Thus the feedback system provides **perfect** asymptotic tracking of step reference signals.

How it works: $C(s)$ contains an internal model of $R(s)$ (i.e., an integrator); closing the loop creates a **zero** from $R(s)$ to $E(s)$ exactly to cancel the unstable pole of $R(s)$. (This isn't an illegal pole-zero cancellation.)

$\triangle$

More generally, for a unity feedback control system let

$$\frac{E}{R} = T_{re}(s) = \frac{1}{1+CP} = \frac{D_p D_c}{N_p N_c + D_p D_c}.$$

Suppose that $R(s) = \frac{r_0}{s}$. If we **assume** that the controller $C(s)$ provides feedback stability then $\pi(s)$ has all its roots in $\mathbb{C}^-$ and the final-value theorem applies. We get

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+CP}\frac{r_0}{s} = \frac{1}{1+C(0)P(0)}.$$

Therefore $e_{ss} = 0$ if and only if $P(0)C(0) = \infty$. This means that $P(s)C(s)$ has at least one pole at $s = 0$, i.e., at least one integrator.

Integral control is fundamental for perfect set-point tracking.

If $P(s)$ does not have a pole at $s = 0$ and we want perfect step tracking, it is common to choose

$$C(s) = \frac{1}{s}C_1(s)$$

so that $C(s)P(s)$ has an integrator. Then $C_1(s)$ is designed to provide feedback stability. The integrator adds more phase lag though and this can make the overall system harder to stabilize.

When CP does not have an integrator, the number $K_p := C(0)P(0)$ is called the position error constant. In this case the steady-state error for a step input is

$$e_{ss} = \frac{r_0}{1+K_p}.$$

Example 5.4.2. This time take

$$C(s) = \frac{1}{s},\ P(s) = \frac{2s+1}{s(s+1)}$$

and take r to be a ramp, $r(t) = r_0 t$. Then $R(s) = r_0/s^2$ and so

$$E(s) = \frac{s+1}{s^3+s^2+2s+1}r_0.$$

You can check using, say, the Routh-Hurwitz stability criterion, that this TF is BIBO stable and the FVT applies. Again $e(t) \to 0$; perfect tracking of a ramp. Here $C(s)$ and $P(s)$ together provide the internal model, a double integrator.

$\triangle$

More generally, suppose that $R(s) = \frac{r_0}{s^2}$ (ramp reference). If we **assume** that the controller $C(s)$ provides feedback stability then $\pi(s)$ has all its roots in $\mathbb{C}^-$ and the final-value theorem applies. We get

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+CP}\frac{r_0}{s^2} = \left.\frac{1}{sC(s)P(s)}\right|_{s=0}.$$

(i) If $P(s)C(s)$ has no poles at $s=0$ then e_{ss} is unbounded.

(ii) If $P(s)C(s)$ has one pole at $s=0$ then e_{ss} is finite and non-zero.

(ii) If $P(s)C(s)$ has two or more poles at $s=0$ then e_{ss} equals zero.

When CP has one or fewer poles at $s=0$, the number $K_v := sC(s)P(s)|_{s=0}$ is called the velocity error constant. In this case the steady-state error for a ramp input is

$$e_{ss} = \frac{r_0}{K_v}.$$

It is possible to generalize the above discussion.

Theorem 5.4.1 (Internal model principle)**.** *Assume that $P(s)$ is strictly proper, $C(s)$ is proper and the feedback system is stable. If $P(s)C(s)$ contains an internal model of the unstable part of $R(s)$, then perfect asymptotic tracking occurs, i.e., $e(t) \to 0$.*

Example 5.4.3. Let

$$P(s) = \frac{1}{s+1}.$$

We want the closed-loop system to be stable and to track the reference signal

$$r(t) = r_o \sin(t).$$

Then

$$R(s) = \frac{r_0}{s^2+1}.$$

The internal model principle suggests that we should pick a controller of the form

$$C(s) = \frac{1}{s^2+1}C_1(s).$$

That is, we embed an internal model of the unstable part of $R(s)$ in $C(s)$ and allow an extra factor $C_1(s)$ to achieve feedback stability. You can check that $C_1(s) = s$ works.

△

5.4.2 Steady-state disturbance rejection

A disturbance is an external input over which we have no control, e.g., turbulence acting on a plane, load torque on a car due to hills. Furthermore, in practive a control system has noise in every signal. Consider a system with an input disturbance d_i, an output disturbance d_0 and sensor noise $n(t)$ illustrated in Figure 5.14. We can once again use the final-value theorem to

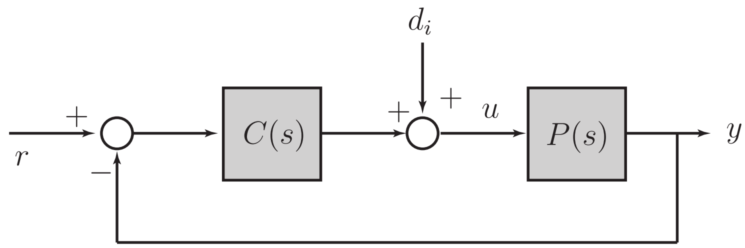

Figure 5.14: Feedback system with disturbances and noise.

check for the effect of certain types of disturbances and noise.

Example 5.4.4. Suppose that $r = d_i = n = 0$ in Figure 5.14 and that $d_o(t) = \mathbf{1}(t)$. Find conditions on $P(s)$ and $C(s)$ so that the steady-state effect of this disturbance on the output y is zero. One can check that the transfer function from d_o to y is

$$\frac{Y(s)}{D_o(s)} = T_{d_o y} = \frac{1}{1 + C(s)P(s)}.$$

If we have feedback stability, then the above TF is stable and the steady-state effect of d_o is

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{1}{1 + C(0)P(0)}.$$

Therefore the disturbance has no steady-state effect on the output if and only if $C(0)P(0) = \infty$, i.e., $C(s)P(s)$ has at least one integrator. We conclude that the output disturbance $d_o(t) = \mathbf{1}(t)$ has no effect on the steady-state output of this system as long as (i) the system is feedback stable and (ii) $C(s)P(s)$ has at least one pole at $s = 0$.

△
