Chapter 7: Lead, Lag, Lead-Lag and PID controlle design

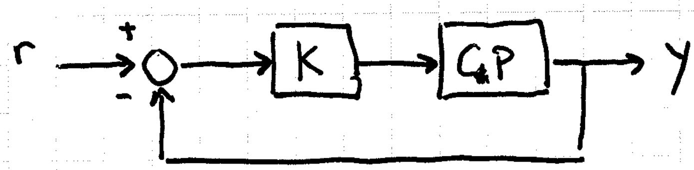
- This chapter is about "classical frequency domain design
- it is often natural to write specs in the time domain e.g. %OS, T_s, steady-state tracking error
- design usually involves trade-offs

  controller gains $\uparrow \Rightarrow$ s.s. error $\downarrow$, $T_p \downarrow$, %OS $\uparrow$

- Design philosophy: adjust gains to meet s.s. tracking specs, then design a dynamic compensator (a TF) to reduce OS with degrading steady-state performance.
- it turns out that, the best way to carry out the above program is by working, not in the time domain, but rather the s-plane or frequency domain (Bode plots)

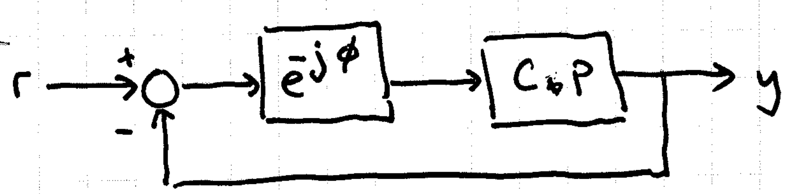
Remark: To really understand what we're doing we need to know about phase margin and gain margin

Gain margin



$$G_M := \max\{\bar{K} \geq 1 : \text{closed-loop stability for } K \in [1, \bar{K})\}$$

Phase margin

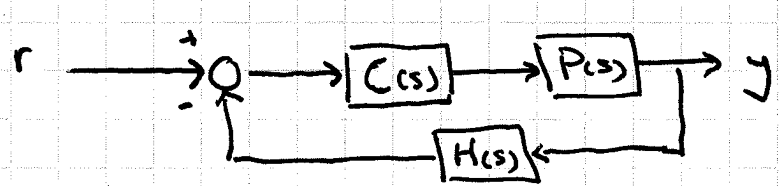


$$P_M := \max\{\bar{\phi} \geq 0 : \text{closed-loop stability for } \phi \in [0, \bar{\phi})\}$$

These are collectively called the stability margins of the contol system. They give a measure of system robustness i.e., how close the system is to being unstable.

Decent stability margins not only guarantee good stability robustness but also good transient behaviour. A system with poor stability margins is nearly unstable, so you may expect highly oscillatory and slowly decaying transients.

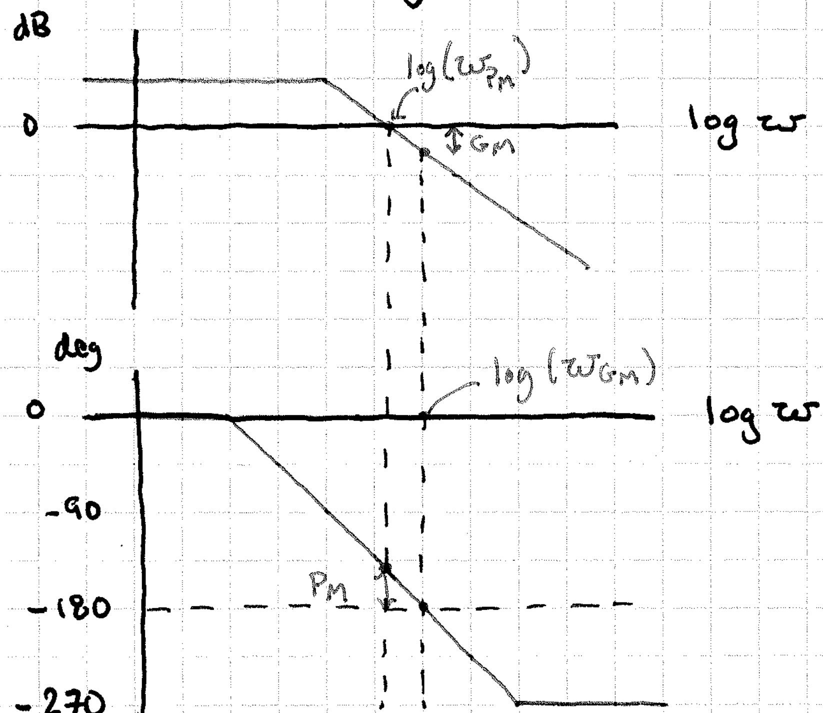
G_M, P_M are best understood using the Nyquist stability criterion. We won't see this until Ch 8. Instead we see how to read G_M, P_M off a Bode plot.



(Assume closed-loop stability)

Let $L(s) = C(s)P(s)H(s)$ (loop transfer function)

Draw Bode plot of $L(j\omega)$



ω_{GM} - phase crossover frequency (frequency where $\angle L(j\omega) = -\pi$)
- freq. at which G_M is measured

ω_{PM} - gain crossover frequency (freq. at which $20\log|L(j\omega)| = 0$)
- freq. at which P_M is measured.

$P_M = \pi + \angle L(j\omega_{PM})$ $\qquad$ $G_M = -20\log|L(j\omega_{GM})|$

Standard Design Problem : Given an LTI plant $P(s)$ design a controller $C(s)$ to achieve desired specs.

common specs:
- closed-loop stability
- time-domain specs: %OS, T_s, tracking error
- f.d. specs : BW, P_M, G_M
- desired closed-loop pole locations

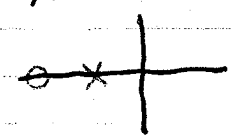
In this chapter we will focus on

1) Closed-loop stability 2) Steady-state tracking error due to step and ramp inputs

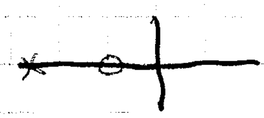
3) % OS spec (converted into a P_M spec)

and we focus on three classical controller types

(a) ~~lead~~ lag $C(s) = K_c \dfrac{s+z}{s+p}$, $z > p > 0$



(b) lead $C(s) = K_c \dfrac{s+z}{s+p}$, $p > z > 0$



(c) lead-lag: $C(s) = K_c \underbrace{\dfrac{s+z_1}{s+p_1}}_{\text{lag}} \underbrace{\dfrac{s+z_2}{s+p_2}}_{\text{lead}}$ $\quad z_1 > p_1$, $p_2 > z_2$

- These controllers are simple but very useful in practice
- the design techniques are not completely algorithmic and take place in the f.d.

Remark : the approaches we discuss work well for "nice plants"
 i) stable (or, at worst, unstable poles at $s=0$)
 ii) only one crossover frequency

Remark on SS spec

(a) $P(s)$ has no pole at $s=0$
⇒ s.s. spec is on the error due to step inputs
⇒ Need $\left| \frac{1}{1+P(0)C(0)} \right| \leq e_{ss}^{max}$

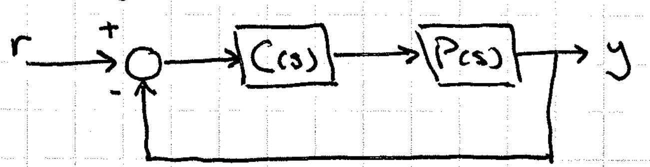
⇒ gives constraint on DC gain of controller $C(s)$.

(b) $P(s)$ has one pole at $s=0$
⇒ s.s. spec is on the error due to ramp inputs
⇒ Need $\left| \frac{1}{sP(s)C(s)\big|_{s=0}} \right| \leq e_{ss}^{max}$

⇒ gives constraint on DC gain of controller.

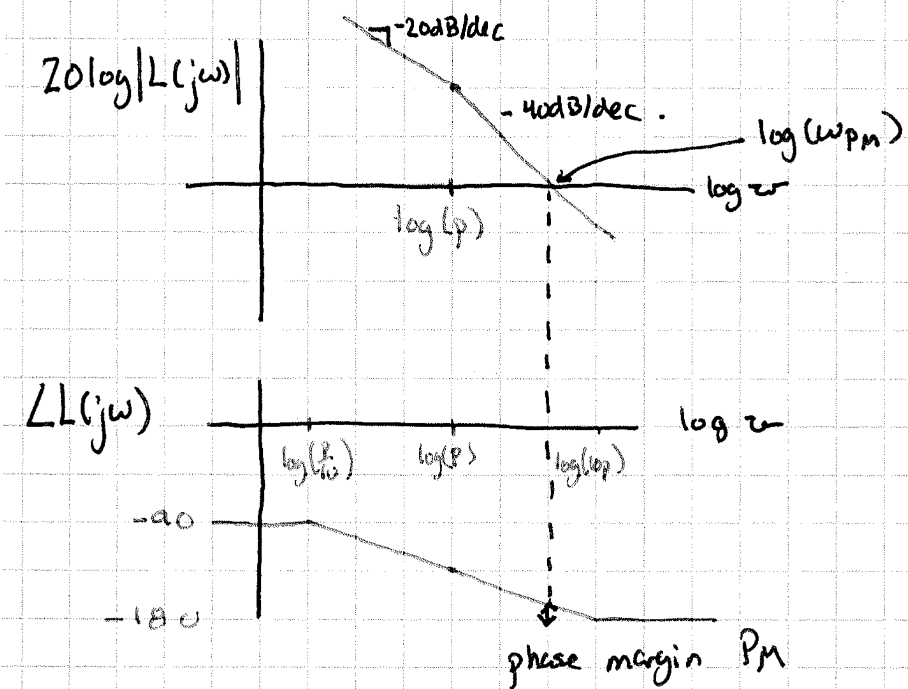
7.1 Converting overshoot to frequency domain specs

Example



$P(s) = \frac{K_m}{s(s+p)}$ DC MOTOR $\qquad C(s) = K_c$

$$L(s) = C(s)P(s) = \frac{K_c K_m}{s(s+1)} \qquad p > 0$$



$$\frac{Y(s)}{R(s)} = T_{ry}(s) = \frac{K_c K_p}{s^2 + ps + K_cK_p} \neq \cancel{\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} = \frac{K}{s^2+ps+K}$$

- closed-loop stability $\Leftrightarrow K = K_c K_p > 0$
- $\omega_n := \sqrt{K} \qquad \zeta := \frac{p}{2\sqrt{K}} \Rightarrow T_{ry}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- Assume $\zeta \in (0,1)$, $\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$
- We want a relationship between P_M and $\%OS$.

<u>Step 1</u> Find ω_{PM}

$$|L(j\omega_{PM})| = 1 \iff |L(j\omega_{PM})|^2 = 1$$

$$\iff \frac{\omega_n^4}{\omega_{PM}^2(\omega_{PM}^2 + 2\zeta\omega_n^2)} = 1$$

$$\iff \omega_{PM}^4 + 4\zeta^2\omega_n^2\omega_{PM}^2 - \omega_n^4 = 0$$

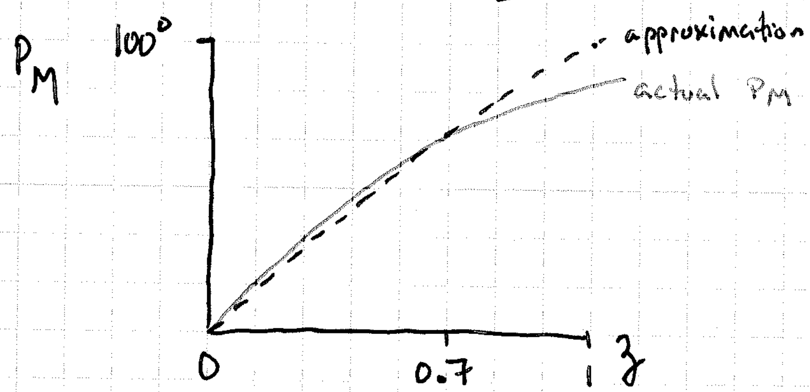
$$\omega_{PM} = \omega_n\left[(1+4\zeta^4)^{1/2} - 2\zeta^2\right]^{1/2}$$

<u>Step 2</u>

$$P_M = 180° + \angle L(j\omega_{PM})$$

$$= 180° + \left[-90 - \text{atan}\left(\frac{\omega_{PM}}{p}\right)\right]$$

$$= \text{atan}\left[2\zeta\left[(1+4\zeta^4)^{1/2} - 2\zeta^2\right]^{-1/2}\right]$$



$P_M \approx 100\zeta$ for $\zeta \in (0, 0.7)$. △

<u>Conclusion</u> If $C(s)P(s) = \dfrac{K}{s(s+p)}$ $p > 0$ or

$$C(s)P(s) \approx \frac{K}{s(s+p)}$$

and

$$\zeta = \frac{p}{2\sqrt{K}} \in (0, \overset{1}{\cancel{0.7}})$$

then the phase margin of the loop transfer function $L(s) = C(s)P(s)$ is

$$P_M = \text{atan}\left[2\zeta\left[(1+4\zeta^4)^{1/2} - 2\zeta^2\right]^{-1/2}\right]$$

$$\approx 100\zeta \quad \text{if} \quad \zeta \in (0, 0.7)$$

while the overshoot of the closed-loop system step response is

$$\%OS = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right).$$

> We can design for a desireable overshoot in the step response by choosing $C(s)$ so that $C(s)P(s)$ has a desireable phase margin.

Bandwidth

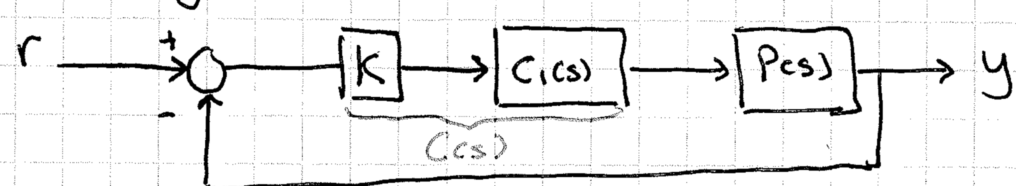
Recall from Ch 1, we can estimate the closed-loop bandwidth by looking at the Bode plot of the open-loop system

$$\omega_{BW} \approx \omega_{PM} \qquad (\text{usually } \omega_{PM} < \omega_{BW} < \omega_{GM}).$$

ω_{BW}: closed-loop BW

ω_{PM}: open-loop gain crossover frequency.

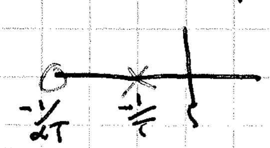
7.3 Lag compensation



$$\boxed{C(s) = K\,C_1(s) = K\,\frac{\alpha T s + 1}{T s + 1}}$$

$0 < \alpha < 1$
$T > 0$
$K > 0$

Remark:
- $C(s)$ has DC gain equal to K
- $C_1(s)$ has unity DC gain
- $C(s)$ has a pole at $s = \frac{-1}{T}$ and a zero at $s = \frac{-1}{\alpha T}$



At the start of the chapter we wrote a lag controller as

$$K_c \frac{s+z}{s+p} \quad \text{with} \quad z > p > 0$$

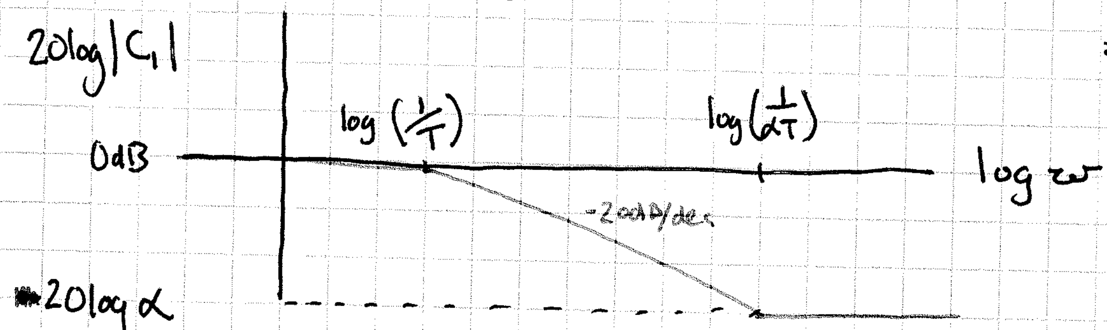
This is same as the form used here, except writing it in this new form makes design more systematic.

$$C(s) = K_c \frac{s+z}{s+p} = \frac{K_c z}{p} \frac{\frac{s}{z}+1}{\frac{s}{p}+1} =: K \frac{\alpha T s + 1}{T s + 1}.$$

$\Rightarrow p = 1/T \qquad K = \frac{K_c z}{p}$

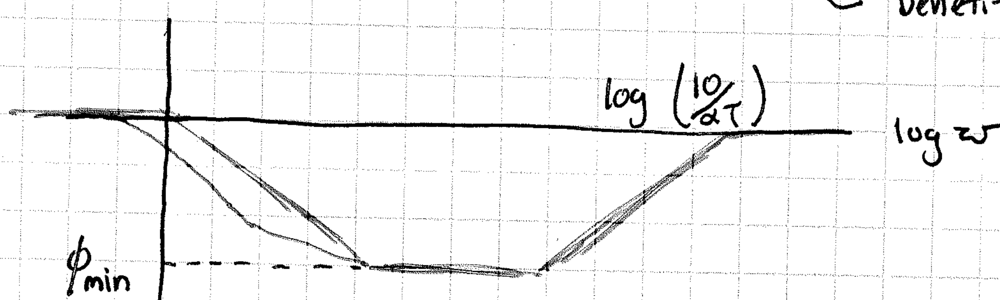
$z = \frac{1}{\alpha T}$

Bode plot of a Lag compensator

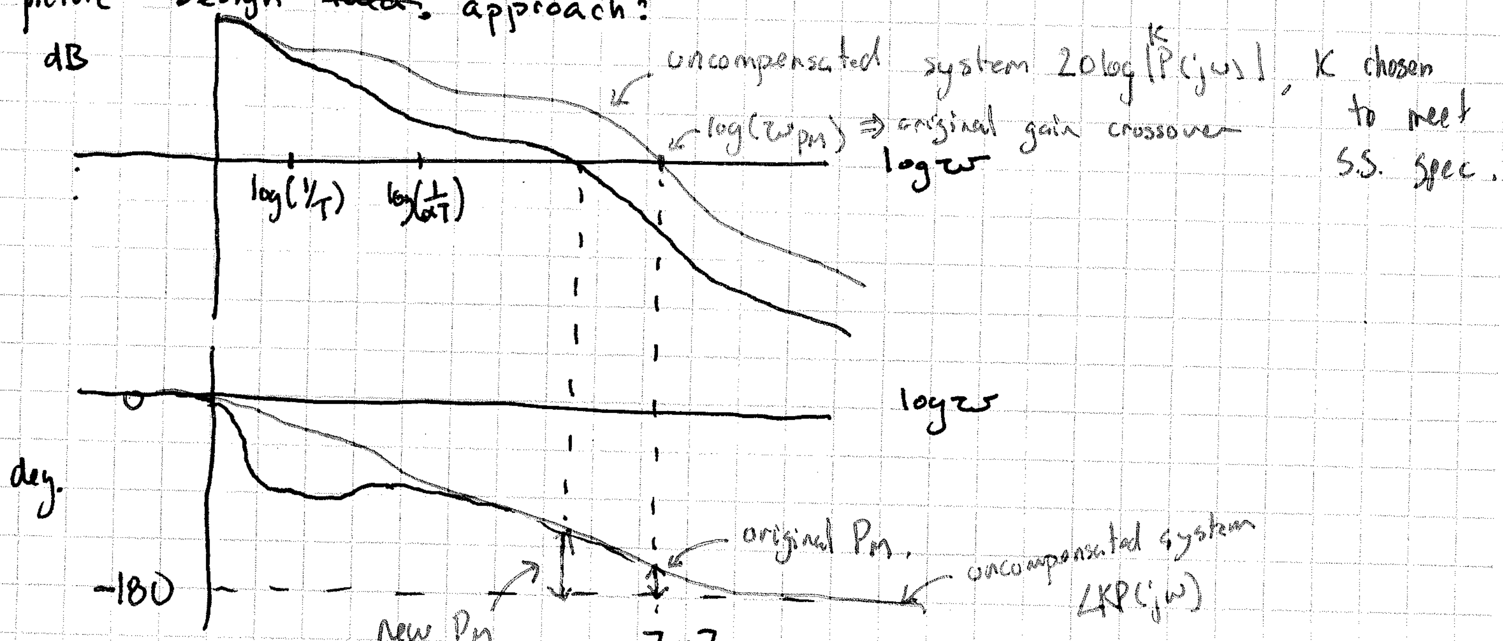


$$\boxed{\begin{aligned} &\log\left(\tfrac{1}{\alpha T}\right) - \log\left(\tfrac{1}{T}\right) \\ &= \log\left(\tfrac{1}{\alpha}\right) \times -20 \\ &\Rightarrow 20 \log|\alpha|. \end{aligned}}$$

benefit: gain reduction without phase lag.



~~A picture~~ Design ~~idea~~: approach:



Example

$P(s) = \frac{1}{s(s+2)}$ Specs: 1. $r(t)$ = unit ramp, steady-state error $\leq 5\%$
2. $P_M = 45°$ (for adequate damping)

Step 1: Choose K to get spec 1 $\quad C(s) = K\frac{\alpha T s + 1}{\alpha s + 1}$

$E(s) = \frac{1}{1+P(s)C(s)} R(s)$, $R(s) = \frac{1}{s^2}$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+P(s)C(s)}\frac{1}{s} = \frac{1}{sPC}\Big|_{s=0} = \frac{2}{K} \leq 0.05$$

FVT, assumes $C(s)$ stabilizes loop

$\Rightarrow K \geq \frac{2}{0.05} = 40.$, Pick $\boxed{K=40}$

Step 2 We now draw the Bode plot for $KP(s) = \frac{40}{s(s+2)}$.
From the figure we get that $\omega_{PM} = 6$ rad/s and $P_M = 18°$. To increase P_M while preserving spec 1, we'll design a lag compensator $C_1(s)$. We want $P_M \approx 45°$.

We'll try for a P_M of $50°$, a bit more than $45°$, to compensate for the straight line approximation in the Bode plot of $C_1(s)$.

From the Bode plot of $KP(s)$ we have that

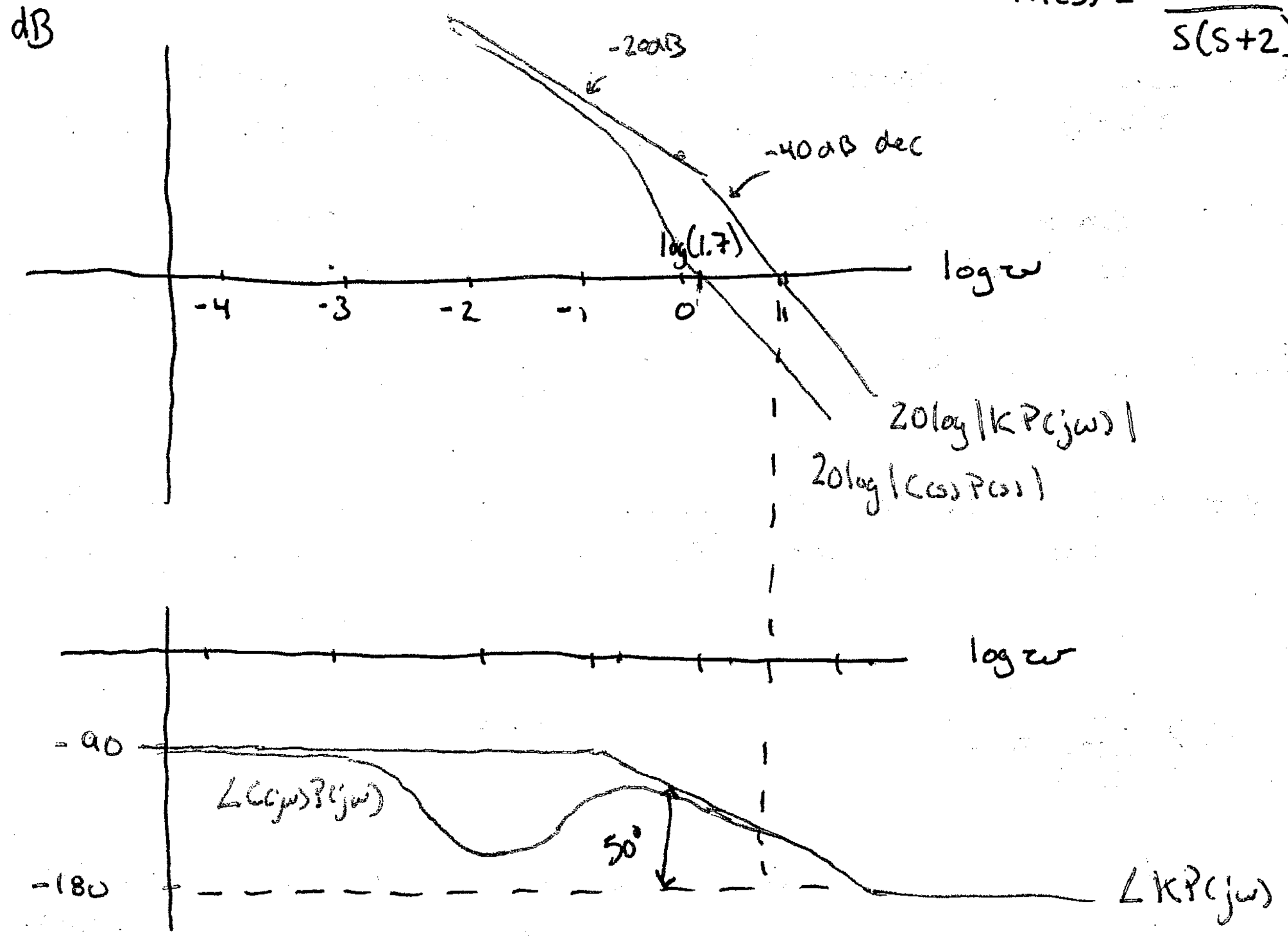
$180° + \angle KP(j\omega) = 50$ when $\omega = 1.7$.

So we would like the new ω_{PM} to be 1.7.

From the Bode plot we see that $20\log|KP(j\omega)| = 19$dB at $\omega = 1.7$. $\left(20\log\frac{40}{2} - 20\log|j\omega| - 20\log\left|\frac{j\omega}{2}+1\right| = 19\right)$

Therefore we want to reduce the gain at $\omega = 1.7$ by 19dB

$$KP(s) = \frac{40}{s(s+2)}$$



without changing the phase.

i.e. $20\log|C_1(j\omega)|_{\omega=1.7} = -19\text{dB} = 20\log\alpha$

$$\Rightarrow |C_1(j\omega)|_{1.7} = \frac{1}{8.96} = 0.111 = \alpha.$$

~~We put the pole~~

Set $\frac{10}{\alpha T} = 1.7 \Rightarrow T = 52.7.$

We now plot the Bode plot for the fully compensated system $KC_1(s)P(s)$. From the plot we read

$P_M = 44.6°$ (close enough!)

$$\therefore \boxed{C(s) = K\frac{\alpha Ts+1}{Ts+1}, \quad K=40, \alpha=0.111, T=52.7.}$$ ▲

<u>Algorithm</u> (lag) $\quad C(s) = KC_1(s) = K\frac{\alpha Ts+1}{sT+1} \quad K>0, T>0, 0<\alpha<1.$

① Use FVT to find K

② Draw Bode plot of $KP(s)$

③ If P_M spec isn't met already, find ω_{PM}' s.t.

$$180 + \angle KP(j\omega_{PM}') = P_M^{\text{desired}} + \delta \quad (\delta \approx 5°).$$

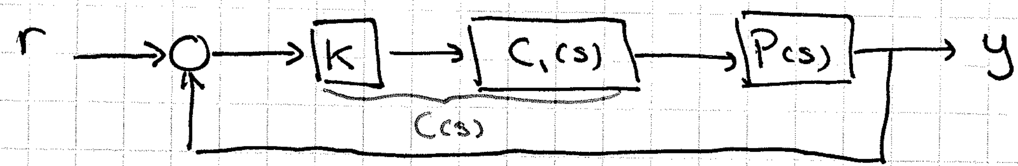
↑ takes care of approx.

④ Set $\alpha = \frac{1}{|KP(j\omega_{PM}')|}$ ← shift the gain down at ω_{PM}'

⑤ Set $\frac{10}{\alpha T} \leq \omega_{PM}'$ ← to ensure phase isn't affected near ω_{PM}'.

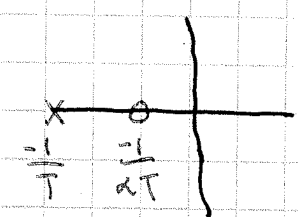
⑥ Check Bode plot of $KC_1(s)P(s)$.

7.4 Lead compensation



$$\boxed{C(s) = K C_1(s) = K \frac{\alpha T s + 1}{T s + 1} \qquad \alpha > 1, \; T > 0, \; K > 0}$$

Remark:
- $C(s)$ has DC gain K
- $C_1(s)$ has unity DC gain
- $C(s)$ has a pole at $s = \frac{-1}{T}$, zero at $s = \frac{-1}{\alpha T}$

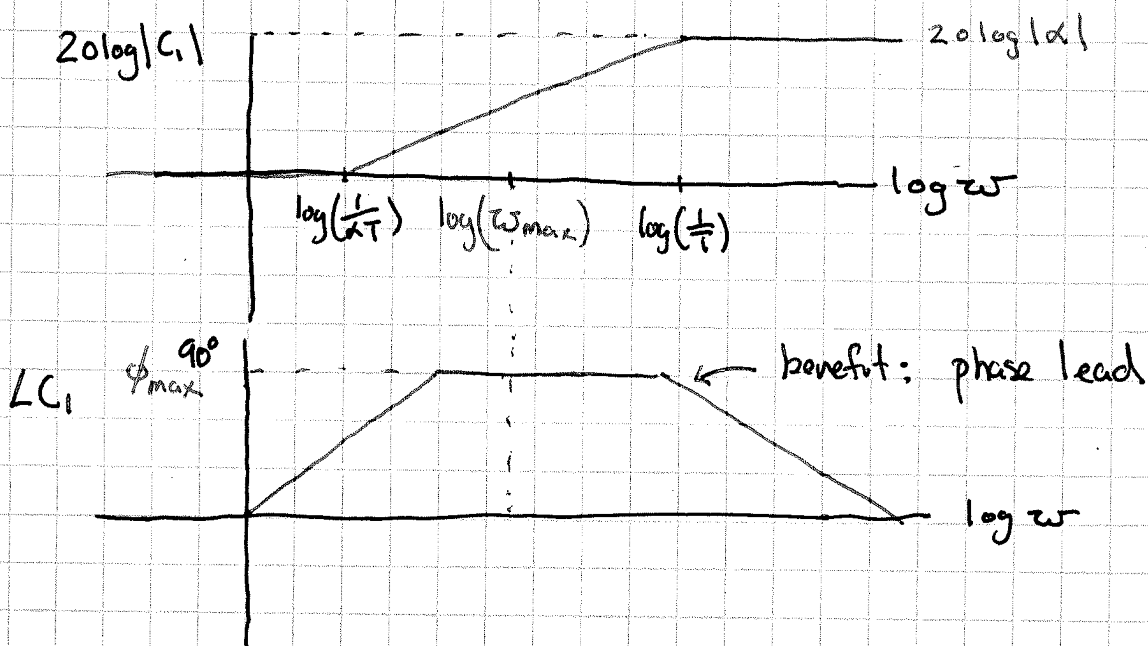


- Compare to form of lead controller from start of chapter:

$$C(s) = K_c \frac{s+z}{s+p} = K \frac{\alpha T s + 1}{T s + 1} \Leftrightarrow \quad K = K_c \frac{z}{p}$$

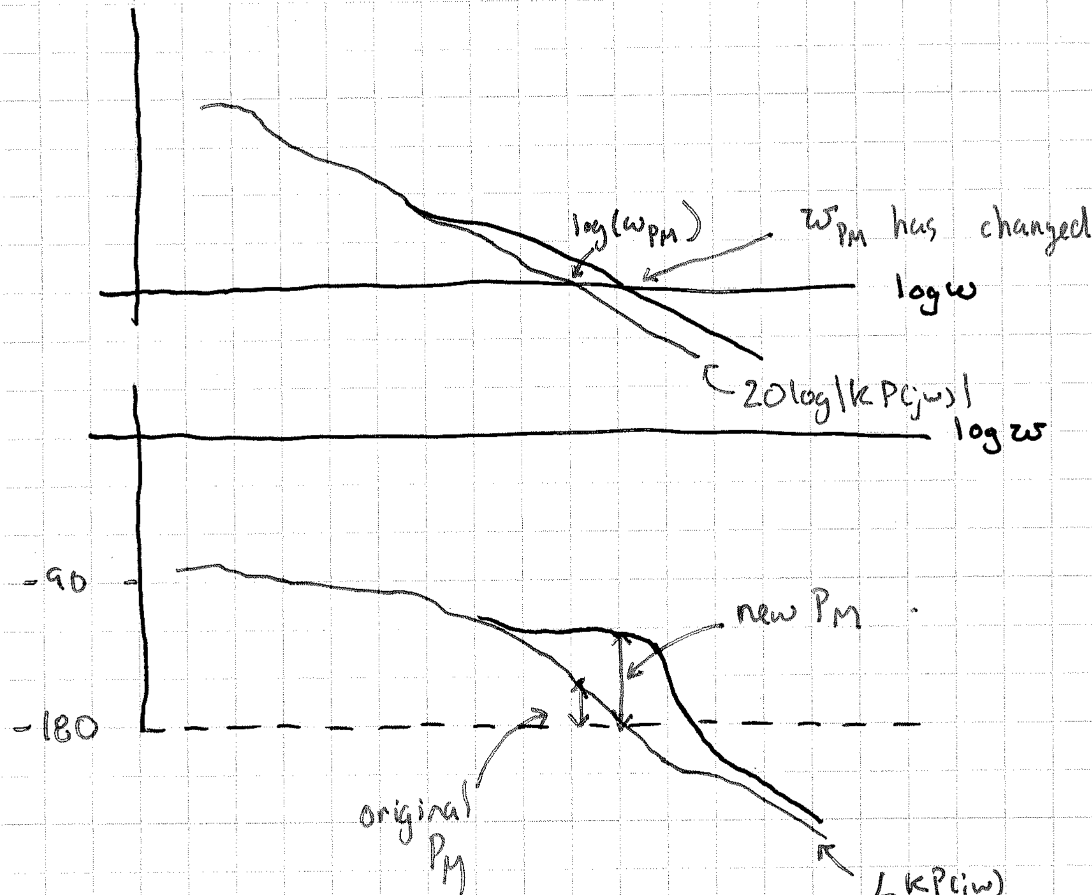
$$z = 1/\alpha T \qquad p = \frac{1}{T}.$$

Bode plot of Lead compensator



$$\phi_{max} = \max \angle C_1(j\omega) = \angle C_1(j\omega_{max})$$

Design approach:



We need three design formulas

1. ω_{max}: the midpoint between $\frac{1}{\alpha T}$ and $\frac{1}{T}$ on the log scale

$$\log \omega_{max} = \frac{1}{2}\left(\log\left(\frac{1}{\alpha T}\right) + \log\left(\frac{1}{T}\right)\right)$$

$$= \frac{1}{2} \log \frac{1}{\alpha T^2}$$

$$= \log \frac{1}{T\sqrt{\alpha}}$$

$$\Rightarrow \boxed{\omega_{max} = \frac{1}{T\sqrt{\alpha}}}$$

2. The magnitude of $C_1(s)$ at ω_{max}. This is the midpoint between $20\log|1|$ and $20\log|\alpha|$

$$\log|C_1(j\omega_{max})| = \frac{1}{2}\left(\log|1| + \log\alpha\right)$$ (divided both sides by 20)

$$= \frac{1}{2}\log\alpha$$

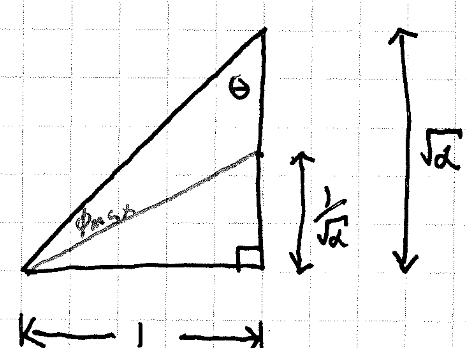
$$= \log\sqrt{\alpha}$$

$$\Rightarrow \boxed{|C_1(j\omega_{max})| = \sqrt{\alpha}}$$

3. ϕ_{max}: the angle of C_1 at ω_{max}

$$\phi_{max} = \angle C_1(j\omega_{max})$$

$$= \angle \frac{1+\sqrt{\alpha}\,j}{1+\frac{1}{\sqrt{\alpha}}\,j}$$



Sine law: $$\frac{\sin\theta}{\sqrt{1+1/\alpha}} = \frac{\sin\phi_{max}}{\sqrt{\alpha}-\frac{1}{\sqrt{\alpha}}}$$

but $\sin\theta = \dfrac{1}{\sqrt{1+\alpha}}$, so

$$\sin\phi_{max} = \left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{1+\alpha}\sqrt{1+1/\alpha}} = \frac{\alpha-1}{\alpha+1}$$

$$\Rightarrow \boxed{\phi_{max} = \sin^{-1}\left(\frac{\alpha-1}{\alpha+1}\right) \Leftrightarrow \alpha = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}}}$$

Example Let's re-do the previous example using a lead controller.

We once again choose $K=40$ to meet the tracking spec. Once again we draw the Bode plot of $KP(s)$ and find that $P_M = 18°$ at $\omega_{PM} = 6$ rad/s.

Step 1 We have to add at least $45 - 18 = 27°$ of phase. However, the lead controller will add gain at ω_{max}. ~~so we Usually this means we'll~~ and change ω_{PM}. This usually decreases phase so we add some margin

Let's add $27° + 10\% = 30°$

$$\alpha = \frac{1+\sin 30°}{1-\sin 30°} = 3.$$

Step 2 We want to make ω_{max} the new crossover frequency in order to get the max phase addition.

At ω_{max} we will increase the gain by

$20\log\sqrt{\alpha} = 4.77\,dB$.

From the Bode plot of $KP(s)$ we have that

$20\log|KP| = -4.77$ at $\omega = 8.4$ rad/s.

So we set

$$\omega_{max} = 8.4 \Rightarrow \frac{1}{T\sqrt{\alpha}} = 8.4 \Rightarrow T = 0.0687$$

With this controller, we get a P_M of $44°$ (check Bode plot of $C(s)P(s)$.)

$$\boxed{C(s) = K\,\frac{\alpha T s + 1}{Ts+1}, \quad K = 40, \ \alpha = 3, \ T = 0.0687}$$

In this case you can compare the responses of the lead & lag controllers. They both have the correct P_M but the lead controller produces a faster step response, at the expense of a larger control signal!

Algorithm (lead) $\quad C(s) = K\,\frac{\alpha T s+1}{\alpha T+1} \quad K>0 \ T>0 \ \alpha>1$

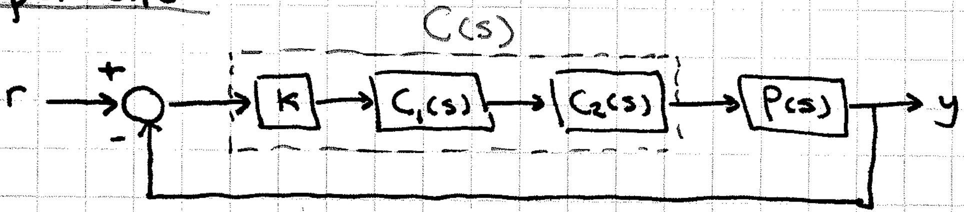
1. Use FVT to get K
2. Draw Bode plot of $KP(s)$
3. Find ω_{PM} and P_M
4. Let $\phi_{max} = P_M^{desired} - P_M + \delta$

   ↑ takes into account change in ω_{PM}
5. Set $\alpha = \frac{1+\sin\phi_{max}}{1-\sin\phi_{max}}$
6. Find frequency ω_{PM}' at which $20\log|KP(j\omega)| = -20\log\sqrt{\alpha}$

   Set $\omega_{max} = \omega_{PM}' = \frac{1}{T\sqrt{\alpha}}$
7. Check Bode plot of $C(s)P(s) = KC_1(s)P(s)$.

7.5 Lead - Lag compensator

Same specs as before



- closed-loop stability
- $|e_{ss}| < e_{ss}^{max}$
- $P_M \geqslant P_M^{min}$.

$$C_1(s) = \frac{\alpha_1 T_1 s + 1}{T_1 s + 1} \qquad C_2(s) = \frac{\alpha_2 T_2 s + 1}{T_2 s + 1}$$

$0 < \alpha_1 < 1$ $\qquad$ $\alpha_2 > 1$

$T_1 > 0$ $\qquad$ $T_2 > 0$

lag $\qquad$ lead

- Lead & Lag on their own each have weaknesses

  Phase-lag : slows system down

  Phase-lead : u may be large.

- Motivation: use a combo to hopefully obtain a compromise.

Algorithm (Lead-Lag)

① Divide P_M^{min} into two, roughly equal, parts $\quad P_M^{min} = P_{M_1} + P_{M_2}$

② Design the lag controller to obtain P_{M_1}

③ Design the lead (for the partially compensated system) to achieve P_M^{min}.

Q: Why lag first?

A: The lead tends to flatten the phase, which means that if a lag were then designed, the crossover frequency would be small (sluggish).

Example $P(s) = \dfrac{1}{s(s+1)(s+20)}$ Specs: $|e_{ss}| \leq 0.1$ $r(t) = t$
$P_M \geq 45°$

$C(s) = K\, C_1(s)\, C_2(s)$

C_1: lag, C_2: lead

Step 1: Pick K $e_{ss} = \dfrac{1}{[sP(s)]_{s=0}\, C(0)} = \dfrac{1}{\frac{1}{20}K} = \dfrac{20}{K} \leq 0.1$

$\Rightarrow$ Set $K = 200$

Step 2: Divide $P_M^{min} = 45$ into two

$P_{M_1} = 25$ $P_{M_2} = 20$

Step 3: Lag

(a) Bode plot of $KP(s)$

(b) $\omega'_{P_M} = 1.4$ rad/s (where $\angle KP(j\omega) + 180 = P_{M_1} + 5°$)

(c) $\alpha_1 = \dfrac{1}{|KP(j\omega'_{P_M})|} = \cancel{4}\ 0.25$

(d) $\dfrac{10}{\alpha_1 T_1} \leq \omega'_{P_M} \Rightarrow T_1 = 28.6$

$$\Rightarrow \boxed{C_1(s) = \frac{1 + 7.15 s}{1 + 28.6 s}}$$

The phase margin of $K\,C_1(s)\,P(s)$ is $P_{M_1} \approx 25°$

Step 4: Lead

(a) Bode plot of $K\,C_1(s)\,P(s)$

(b) $\omega_{P_M} = 1.4$ rad/s, $P_M \approx 25°$

(c) $\phi_{max} = 45 - 25 + \delta = 20 + \delta = 30$ (had to iterate)

(d) $\alpha_2 = \dfrac{1 + \sin\phi_{max}}{1 - \sin\phi_{max}} = 3$

(e) Find ω'_{PM} at which $20\log|KC_1P(j\omega'_{PM})| = -20\log\sqrt{\alpha}$

Set $\omega_{max} = \omega'_{PM} = 2 = \dfrac{1}{T\sqrt{\alpha}}$

$$\Rightarrow T = \frac{1}{2\sqrt{3}} = 0.29$$

$$\Rightarrow \boxed{C_2(s) = \frac{1 + 0.866s}{1 + 0.29s}}$$

$$C(s) = KC_1(s)C_2(s)$$

$$\boxed{= 200\,\frac{1 + 7.15s}{1 + 28.6s}\,\frac{1 + 0.866s}{1 + 0.29s}}$$

Δ

7.6 PID Control

Ideal PD $\quad C(s) = K_P + K_D s$

Implemented PD $\quad C(s) = \dfrac{K_P + K_D s}{1 + \tau s}$, τ small

$$= K_P\,\frac{1 + \frac{K_D}{K_P}s}{1 + \tau s}$$

Compare w/ lead/lag

$$K\,\frac{1 + \alpha Ts}{1 + Ts}$$

Since τ is small, $\quad \alpha T = \dfrac{K_D}{K_P} > T = \tau$

$\Rightarrow \alpha > 1 \Rightarrow$ <u>Lead controller</u>.

Ideal PI: $C(s) = K_p + \frac{K_I}{s}$

$$= K_I \frac{\frac{K_p}{K_I} s + 1}{s}$$

Approximation: $C(s) = K_I \frac{K_p/K_I\, s + 1}{s + p}$, p small

$$= \frac{K_I}{p} \frac{\frac{K_p}{K_I} s + 1}{\frac{s}{p} + 1}$$

Compare w/ lead/lag

$$K \frac{\alpha T s + 1}{T s + 1}$$

Since p is small: $T = 1/p > \alpha T = \frac{K_p}{K_I}$

$\Rightarrow$ $0 < \alpha < 1$

$\Rightarrow$ looks like a lag compensator.

Similarly, a lead-lag controller looks like a PID controller.

You can design "pure" PD and PI compensators in the frequency domain or using root locus. You can also do this in the s-plane using root locus methods.