MATH 201 Differential Equations – University of Alberta
Winter 2016 – Labs – Carlos Contreras
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Lab 0: Integration techniques

1. Find the given integral to compute

$$e^{\int \frac{-4}{x} dx}.$$

2. Find

$$\int \frac{x}{\sqrt{1+x}} dx.$$

3. Find the following integrals

$$\int_{-\pi/2}^{\pi/2} \cosh(ix) dx, \qquad \int_{-\pi/2}^{\pi/2} \sinh(ix) dx.$$

4. Find the following integrals

$$\int x^3 \cos(n\pi x) dx, \qquad \int x^3 \sin(n\pi x) dx.$$

5. For constants a and b, find the integral

$$\int_0^{\infty} \sin(bt) e^{-at} dt.$$

6. Show that

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

Solutions

1. Find the given integral to compute

$$e^{\int \frac{-4}{x}dx}.$$

Solution
Integrating and applying logarithm properties

$$\int \frac{-4}{x}dx = -4\ln|x| + C = \ln|x|^{-4} + C = \ln|x^{-4}| + C.$$

Thus

$$\boxed{e^{\int \frac{-4}{x}dx} = e^{\ln|x^{-4}|+C} = Kx^{-4}},$$

for some constant K.

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2. Find

$$\int \frac{x}{\sqrt{1+x}}dx.$$

Solution
Using substitution $t = x + 1$, we have

$$\int \frac{x}{\sqrt{1+x}}dx = \int \frac{t-1}{\sqrt{t}}dt = \int t^{1/2}dt - \int t^{-1/2}dt = \frac{2}{3}t^{3/2} - 2t^{1/2} + C,$$

Subtituting back to x

$$\boxed{\int \frac{x}{\sqrt{1+x}}dx = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C}.$$

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3. Find the following integrals

$$\int_{-\pi/2}^{\pi/2} \cosh(ix)dx, \qquad \int_{-\pi/2}^{\pi/2} \sinh(ix)dx.$$

Solution
Recall that

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos(x),$$
$$\sin(ix) = \frac{e^{ix} - e^{-ix}}{2} = i\sin(x).$$

Thus,

$$\int_{-\pi/2}^{\pi/2} \cosh(ix)dx = \int_{-\pi/2}^{\pi/2} \cos(x)dx = 2\int_{0}^{\pi/2} \cos(x)dx = 2\,\sin x|_0^{\pi/2} = 2.$$

and

$$\int_{-\pi/2}^{\pi/2} \sinh(ix)dx = i\int_{-\pi/2}^{\pi/2} \sin(x)dx = 0.$$

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4. Find the following integrals

$$\int x^3\cos(n\pi x)dx, \qquad \int x^3\sin(n\pi x)dx.$$

Solution

Consider the first integral. We need to integrate by parts three time, so the following table becomes handy.

sign	derivative	integral
		$\cos(n\pi x)$
$+$	x^3	$\frac{1}{n\pi}\sin(n\pi x)$
$-$	$3x^2$	$-\left(\frac{1}{n\pi}\right)^2\cos(n\pi x)$
$+$	$6x$	$-\left(\frac{1}{n\pi}\right)^3\sin(n\pi x)$
$-$	6	$\left(\frac{1}{n\pi}\right)^4\cos(n\pi x)$
$+$	0	$\left(\frac{1}{n\pi}\right)^5\sin(n\pi x)$

There is no need to compute the last row when you now one term becomes zero. From there is easy to compute all terms of the integral at once by multiplying each row.

$$\boxed{\int x^3\cos(n\pi x)dx = x^3\frac{1}{n\pi}\sin(n\pi x) + 3x^2\left(\frac{1}{n\pi}\right)^2\cos(n\pi x) - 6x\left(\frac{1}{n\pi}\right)^3\sin(n\pi x) - 6\left(\frac{1}{n\pi}\right)^4\cos(n\pi x)}.$$

Consider now the second integral. The correspondent integration by parts table is

sign	derivative	integral
		$\sin(n\pi x)$
$+$	x^3	$-\frac{1}{n\pi}\cos(n\pi x)$
$-$	$3x^2$	$-\left(\frac{1}{n\pi}\right)^2\sin(n\pi x)$
$+$	$6x$	$\left(\frac{1}{n\pi}\right)^3\cos(n\pi x)$
$-$	6	$\left(\frac{1}{n\pi}\right)^4\sin(n\pi x)$

Thus,

$$\boxed{\int x^3\sin(n\pi x)dx = -x^3\frac{1}{n\pi}\cos(n\pi x) + 3x^2\left(\frac{1}{n\pi}\right)^2\sin(n\pi x) + 6x\left(\frac{1}{n\pi}\right)^3\cos(n\pi x) - 6\left(\frac{1}{n\pi}\right)^4\sin(n\pi x)}.$$

If you want to step further, you can compute both integrals at the same time by computing the single integral

$$\int x^3e^{in\pi x}dx = \int x^3\cos(n\pi x)dx + i\int x^3\sin(n\pi x)dx.$$

and then separating real and complex part. That is, using the table

sign	derivative	integral
		$e^{in\pi x}$
$+$	x^3	$-i\left(\frac{1}{n\pi}\right)e^{in\pi x}$
$-$	$3x^2$	$-\left(\frac{1}{n\pi}\right)^2 e^{in\pi x}$
$+$	$6x$	$i\left(\frac{1}{n\pi}\right)^3 e^{in\pi x}$
$-$	6	$\left(\frac{1}{n\pi}\right)^4 e^{in\pi x}$

Try it!

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5. For constants a and b, find the integral

$$\int_0^\infty \sin(bt)e^{-at}dt.$$

Solution
If we integrate by [arts twice we arrive to the same integral where we can isolate it.

$$\int_0^\infty \sin(bt)e^{-at}dt = \left[-\frac{1}{a}\sin(bt)e^{-at}\right]_0^\infty + \left[-\frac{b}{a^2}\cos(bt)e^{-at}\right]_0^\infty - \frac{a^2}{b^2}\int_0^\infty \sin(bt)e^{-at}dt.$$

Recall that

$$\lim_{t\to\infty}\cos(bt)e^{-at} = \lim_{t\to\infty}\sin(bt)e^{-at} = 0,$$

then

$$\left[-\frac{1}{a}\sin(bt)e^{-at}\right]_0^\infty = 0, \qquad \left[-\frac{b}{a^2}\cos(bt)e^{-at}\right]_0^\infty = \frac{b}{a^2}.$$

Denoting $I = \int_0^\infty \sin(bt)e^{-at}dt$ and multiplying by a^2, we get

$$a^2 I = b - b^2 I \quad \Rightarrow \quad I = \frac{b}{a^2+b^2} \quad \Rightarrow \quad \boxed{\int_0^\infty \sin(bt)e^{-at}dt = \frac{b}{a^2+b^2}}.$$

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6. Show that

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Solution
Using substitution $x = a\tan t$, $(dx = a\sec^2 t dt)$

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2 t dt}{a^2\sec^2 t} = \frac{1}{a}t + C = \boxed{\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}.$$

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