**Theorem 1.** *For a well-behaved STL formula $\varphi$ with respect to a hybrid automaton $H$, if a signal $\sigma \in H$ satisfies a formula $\phi \in \mathrm{sub}(\varphi)$ at a time $t \geq 0$, (i.e., $\sigma, t \models_\tau \varphi$), then the formula is also satisfied over $[a, b)$, such that $0 \leq a \leq t < b$.*

*Proof.* Proof by structural induction. For the case $\varphi = p$, then

**Lemma 3.** *Given a signal $\sigma$, two successive intervals $J_1, J_2$, and clock valuations $\alpha_1, \alpha_2$, if the signal satisfies $Q$ and $Q'$ of a rule in $J_1$ and $J_2$, respectively, (i.e. $\sigma, J_1, \alpha_1 \models \bigwedge_{q \in Q} q \wedge \sigma, J_2, \alpha_2 \models \bigwedge_{q \in Q'} q$) then the signal also satisfies $P$ and $P'$ of the rule in $J_1$ and $J_2$, respectively, (i.e. $\sigma, J_1, \alpha_1 \models \bigwedge_{p \in P} p \wedge \sigma, J_2, \alpha_2 \models \bigwedge_{p \in P'} p$) under clock valuations $\alpha_1, \alpha_2$.*

$$(Id)\frac{P/P'}{Q/Q'}$$

*Proof.*
- ($\wedge$) By the assumption, $\sigma, \eta, A \models \varphi \wedge \sigma, \eta, A \models \psi$. Then, $\sigma, \eta, A \models \varphi \wedge \psi$ by definition. Therefore, a signal that satisfies $\varphi \sqcap \psi$ with $\eta$ and $A$ also satisfies $\varphi \wedge \psi$.
- ($\vee$) By the assumption, $\sigma, \eta, A \models \varphi \vee \sigma, \eta, A \models \psi$. Then, $\sigma, \eta, A \models \varphi \vee \psi$ by definition. Therefore, a signal that satisfies $\varphi \sqcup \psi$ with $\eta$ and $A$ also satisfies $\varphi \vee \psi$.
- ($\mathbf{U}$) Suppose $\sigma, \eta, A \models \varphi \sqcap \psi$. Then, $\sigma, \eta, A \models \varphi \wedge \psi$ by soundness of ($\wedge$). Therefore, $\sigma, J, \alpha \models \varphi \wedge \psi$ where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha$ is a clock valuation $A(0)$. For all point $t \in J$, there is a point $t' \in J$ such that $t' \geq t$. Therefore,

  $$\forall t \in J, \exists t' \geq t, t' \in J, \sigma, t', \alpha \models \psi \wedge \forall t'' \in [t, t'), \sigma, t'', \alpha \models \varphi$$

  because $t', t'' \in J$ and $\sigma, J, \alpha \models \varphi \wedge \psi$. So, $\sigma, \eta, A \models \varphi \, \mathbf{U} \, \psi$ is also satisfied by definition.

  Suppose $\sigma, \eta, A \models \varphi \sqcap \bigcirc(\varphi \, \mathbf{U} \, \psi)$. Then, by definition, $\sigma, J, \alpha \models \varphi \wedge \sigma, J', \alpha' \models \varphi \, \mathbf{U} \, \psi$ where $J$ and $J'$ are intervals $[\eta(0), \eta(1))$ and $[\eta(1), \eta(2))$, respectively, and $\alpha$ is $A(0)$ and $\alpha'$ is $A(1)$. Therefore, $\forall t \in J', \exists t' \geq t, \sigma, t', \alpha' \models \psi, \forall t'' \in [t, t'), \sigma, t''\alpha' \models \varphi$ by definition. Also, $\forall t \in J, \sigma, t, \alpha \models \varphi$ by assumption $\sigma, J, \alpha \models \varphi$. Since $\sigma, J', \alpha' \models \varphi \, \mathbf{U} \, \psi$,

  $$\exists t' \geq \eta(1), \sigma, t', \alpha' \models \psi, \forall t'' \in [\eta(1), t'), \sigma, t''\alpha' \models \varphi$$

  The interval $J \cup [\eta(1), t')$ is a continuous interval $[\eta(0), t')$. Therefore,

  $$\forall t \in J', \exists t' \geq t, \sigma, t', \alpha' \models \psi, \forall t'' \in [\eta(0), t'), \sigma, t'', \alpha' \models \varphi$$

  The point $t'$ is greater than or equal to all points in $J$. Therefore,

  $$\forall t \in J, \exists t' \geq t, \sigma, t', \alpha' \models \psi, \forall t'' \in [t, t'), \sigma, t'', \alpha', \models \varphi$$

  is also satisfied. So, $\sigma, \eta, A \models \varphi \, \mathbf{U} \, \psi$

– (**R**) Suppose $\sigma, \eta, A \models \varphi$. Then, $\forall t \in J, \forall t' > t, \exists t'' \in (t, t'), \sigma, t'', \alpha \models \varphi$ since $(t, t') \cap J \neq \emptyset$ and $\sigma, J, \alpha \models \varphi$ where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha$ is a clock valuation $A(0)$.
  Suppose $\sigma, \eta, A \models \psi \sqcap \bigcirc(\varphi \, \mathbf{R} \, \psi)$.
  Suppose $\sigma, \eta, A \models \psi \sqcap \bigcirc(*)$. Then, $\eta(1) = \tau$ since $\sigma, \eta, A \models \bigcirc(*)$. Therefore, $\forall t \in [\eta(0), \eta(1)), \forall t' > t, \sigma, t', A \models_\tau \psi$. since $\sigma, [\eta(0), \eta(1)), A \models_\tau \psi$ and $\eta(1) = \tau$.
– ($\Box$) By assumption, $\sigma, \eta, A \models \Box^{c}_{I} \varphi \sqcap c := 0$. By definition,

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models \Box_I^{[\inf(J), \tau)} \varphi \sqcap \alpha(c) = 0$$

  where $J$ is an interval $[\eta(0), \eta(1))$. The formula can be rewritten

$$\sigma, [\inf(J), \sup(J)) \models \Box_I \, \varphi$$

  because $\sup(J) - (\sup(J) - \inf(J)) = \inf(J)$ and $[\inf(J), \sup(J)) + I \subseteq [\inf(J), \tau)$. Therefore, $\sigma, \eta, A \models \Box_I \, \varphi$
– ($\Diamond$) By assumption, $\sigma, \eta, A \models \Diamond^{c}_{I} \varphi \sqcap c := 0$. By definition,

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models \Diamond_I^{[\inf(J), \tau)} \varphi \sqcap \alpha(c) = 0$$

  where $J$ is an interval $[\eta(0), \eta(1))$. The formula can be rewritten

$$\sigma, [\inf(J), \sup(J)) \models \Diamond_I \, \varphi$$

  because $\sup(J) - (\sup(J) - \inf(J)) = \inf(J)$ and $[\inf(J), \sup(J)) + I \subseteq [\inf(J), \tau)$. Therefore, $\sigma, \eta, A \models \Diamond_I \, \varphi$
– ($\Box$) Suppose $\sigma, \eta, A \models \bigcirc(c \leq \inf(I) \sqcap \Box^{c,c'}_{\Box_I} \varphi \sqcap c' := 0)$. Then,

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|)) \models_\tau \overset{[\inf(J'), \tau)}{\Box_I} \varphi \wedge \alpha'(c') = 0 \wedge \alpha'(c) \leq \inf(I)$$

  where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha' = A(1)$. The interval $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as

$$[\inf(J') - (\alpha'(c)), \inf(J'))$$

  since $|J'| = \sup(J') - \inf(J')$ and $\alpha'(c') = 0$. The interval can be rewritten as

$$[\sup(J) - (\alpha(c) + |J|), \sup(J))$$

  where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha = A(0)$ because $\alpha'(c) = \alpha(c) + |J|$. Also, $\inf(J) \leq \sup(J) - (\alpha(c) + |J|) + \inf(I)$ because $(\alpha(c) + |J|) = \alpha'(c) \leq \inf(I)$ and $\inf(J) \leq \sup(J)$ Therefore, $[\sup(J) - (\alpha(c) + |J|), \sup(J)) + I \subseteq [\inf(J), \tau)$. So,

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models_\tau \Box_I^{[\inf(J), \tau)} \varphi$$

By definition, $\sigma, \eta, A \models \Box^c_{\mid I}\varphi$ is also satisfied.

Suppose $\sigma, \eta, A \models (c \leq \inf(I) \sqcap \varphi \sqcap \blacksquare_{\mid I}\, \varphi \sqcap \bigcirc\, (c > \inf(I)))$. By definition,

$$\sigma, J, \alpha \models \varphi \wedge \sigma, [\sup(J), \sup(J)+\sup(I)) \models \varphi \wedge \alpha(c) \leq \inf(I) \wedge \alpha'(c) > \inf(I)$$

where $J$ is an interval $[\eta(0), \eta(1))$, $\alpha$ and $\alpha'$ are clock valuation $A(0)$ and $A(1)$, respectively. Then, $\sigma, [\inf(J), \sup(J)+\sup(I)) \models \varphi$ because $J \cup [\sup(J), \sup(J)+\sup(I)) = [\inf(J), \sup(J)+\sup(I))$. Therefore, $\sigma, [\sup(J) - (\alpha(c) + |J|) + \inf(I), \sup(J) + \sup(I)) \cap [\inf(J), \tau) \models \varphi$ because

$$[\sup(J)-(\alpha(c)+|J|)+\inf(I), \sup(J)+\sup(I)) \cap [\inf(J), \tau) \subseteq [\inf(J), \sup(J)+\sup(I)).$$

By definition, $\sigma, \eta, A \models \Box^c_{\mid I}\varphi$ is also satisfied.

Suppose $\sigma, \eta, A \models \bigcirc(c \leq \inf(I) \sqcap *)$. Then, $\sup(J) - (\alpha(c) + |J|) + \inf(I) \geq \sup(J)$ becuase $\alpha(c) + |J| = \alpha'(c) \leq \inf(I)$ where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha$ and $\alpha'$ are clock valuations $A(0)$ and $A(1)$, respectively. Therefore,

$$\sigma, [\sup(J) - (\alpha(c) + |J|) + \inf(I), \sup(J) + \sup(I)) \cap [\inf(J), \tau) \models_\tau \varphi$$

is trivially satisfied because $\sup(J) = \tau$.

– ($\Box_{\mid}\lfloor_{\Box}$) Suppose $\sigma, \eta, A \models \bigcirc(c \leq \inf(I) \sqcap \Diamond_{\mid}\lfloor^{c,c'}_{\Diamond_I}\varphi)$. Then,

$$\sigma, [\sup(J')-(\alpha'(c)+|J'|), \sup(J')-(\alpha'(c')+|J'|)) \models_\tau \Box_I^{[\inf(J'),\tau)} \varphi \wedge \alpha'(c) \leq \inf(I)$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha' = A(1)$. The interval $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as

$$[\inf(J') - (\alpha'(c)), \inf(J') - \alpha'(c'))$$

since $|J'| = \sup(J') - \inf(J')$. The interval can be rewritten as

$$[\sup(J) - (\alpha(c) + |J|), \sup(J) - (\alpha(c') + |J|))$$

where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha = A(0)$ because $\alpha'(c) = \alpha(c) + |J|$. Also, $\inf(J) \leq \sup(J) - (\alpha(c) + |J|) + \inf(I)$ because $(\alpha(c) + |J|) = \alpha'(c) \leq \inf(I)$ and $\inf(J) \leq \sup(J)$ Therefore, $[\sup(J)-(\alpha(c)+|J|), \sup(J)-(\alpha(c')+|J|)) + I \subseteq [\inf(J), \tau)$. So,

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J) - (\alpha(c') + |J|)) \models_\tau \Box_I^{[\inf(J),\tau)} \varphi$$

By definition, $\sigma, \eta, A \models \Box_{\mid}\lfloor^{c}_{\Box_I}\varphi$ is also satisfied.

Suppose $\sigma, \eta, A \models (c \leq \inf(I) \sqcap \varphi \sqcap \blacksquare_{\mid}\lfloor^{c'}_{\Box_I}\varphi \sqcap \bigcirc\, (c > \inf(I)))$. By definition,

$$\sigma, J, \alpha \models \varphi \wedge \sigma, [\sup(J), \sup(J)-(\alpha(c')+|J|)+\sup(I)) \models \varphi \wedge \alpha(c) \leq \inf(I) \wedge \alpha'(c) > \inf(I)$$

where $J$ is an interval $[\eta(0), \eta(1))$, $\alpha$ and $\alpha'$ are clock valuation $A(0)$ and $A(1)$, respectively. Then, $\sigma, [\inf(J), \sup(J) - (\alpha(c') + |J|) + \sup(I)) \models \varphi$ because $J \cup [\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I)) = [\inf(J), \sup(J) - (\alpha(c') + |J|) + \sup(I))$. Therefore,

$$\sigma, [\sup(J) - (\alpha(c) + |J|) + \inf(I), \sup(J) - (\alpha(c') + |J|) + \sup(I)) \cap [\inf(J), \tau) \models \varphi$$

By definition, $\sigma, \eta, A \models \square\lfloor^{c,c'}_{\square_I} \varphi$ is also satisfied.

Suppose $\sigma, \eta, A \models \bigcirc(c \leq \inf(I) \sqcap *)$. Then, $\sup(J) - (\alpha(c) + |J|) + \inf(I) \geq \sup(J)$ becuase $\alpha(c) + |J| = \alpha'(c) \leq \inf(I)$ where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha$ and $\alpha'$ are clock valuations $A(0)$ and $A(1)$, respectively. Therefore,

$$\sigma, [\sup(J) - (\alpha(c) + |J|) + \inf(I), \sup(J) - (\alpha(c') + |J|) + \sup(I)) \cap [\inf(J), \tau) \models_\tau \varphi$$

is trivially satisfied because $\sup(J) = \tau$.

– ($\blacksquare$) Suppose $\bigcirc(\varphi \sqcap \blacksquare\lfloor^{c'}_{\square_I} \varphi \sqcap c' := 0)$. Then, by definition,

$$\sigma, J' \models \varphi \wedge \sigma, [\sup(J'), \sup(J') - (\alpha'(c') + |J'|) + \sup(I)) \models \varphi \wedge \alpha'(c') = 0$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha'$ is a clock evaluation $A(1)$. Since $J' \cap [\sup(J'), \sup(J') - |J'| + \sup(I))$ is a continuous interval, $\sigma, [\inf(J'), \sup(J') - |J'| + \sup(I)) \models \varphi$ The interval $[\inf(J'), \sup(J') - |J'| + \sup(I))$ can be rewritten as $[\sup(J), \sup(J) + \sup(I))$ because $\sup(J') - |J'| = \inf(J')$ and $\inf(J') = \sup(J)$ where $J$ is an interval $[\eta(0), \eta(1))$. Therefore,

$$\sigma, [\sup(J), \sup(J) + \sup(I)) \models \varphi.$$

By definiton, $\sigma, \eta, A \models \blacksquare_I \varphi$.

Suppose $\bigcirc *$. Then, $\eta(1) = \tau$. Therefore, $\sigma, [\sup(J), \sup(J) + \sup(I)) \models_\tau \varphi$ is trivially satisfied becuase $[\sup(J), \sup(J) + \sup(I)) \cap [0, \tau) = \emptyset$ where $J$ is an interval $[\eta(0), \eta(1))$.

– ($\blacksquare\lfloor_{\square}$) Suppose $\bigcirc(\varphi \sqcap \blacksquare\lfloor^{c'}_{\square_I} \varphi)$. Then, by definition,

$$\sigma, J' \models \varphi \wedge \sigma, [\sup(J'), \sup(J') - (\alpha'(c') + |J'|) + \sup(I)) \models \varphi$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha'$ is a clock evaluation $A(1)$. Since $J' \cap [\sup(J'), \sup(J') - (\alpha'(c') + |J'|) + \sup(I))$ is a continuous interval, $\sigma, [\inf(J'), \sup(J') - (\alpha'(c') + |J'|) + \sup(I)) \models \varphi$ The interval $[\inf(J'), \sup(J') - (\alpha'(c') + |J'|) + \sup(I))$ can be rewritten as $[\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I))$ because $\sup(J') - |J'| = \inf(J')$, $\inf(J') = \sup(J)$, and $\alpha'(c') = \alpha(c') + |J|$ where $J$ is an interval $[\eta(0), \eta(1))$. Therefore,

$$\sigma, [\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I)) \models \varphi.$$

By definiton, $\sigma, \eta, A \models \blacksquare\lfloor^{c'}_{\square_I} \varphi$.

Suppose $\bigcirc *$. Then, $\eta(1) = \tau$. Therefore, $\sigma, [\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I)) \models_\tau \varphi$ is trivially satisfied becuase $[\sup(J), \sup(J) + \sup(I)) \cap [0, \tau) = \emptyset$ where $J$ is an interval $[\eta(0), \eta(1))$ and $\alpha$ is a clock valuatoin $A(0)$.
Suppose $\bigcirc(c' \geq \sup(I))$. Then, $\alpha'(c') \geq \sup(I)$ where $\alpha'$ is a clock valuationa $A(1)$. The interval $[\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I))$ can be rewritten as $[\sup(J), \sup(J) - \alpha'(c') + \sup(I))$. Also, $\sup(J) \geq \sup(J) - \alpha'(c') + \sup(I)$ since $\alpha'(c') \geq \sup(I)$. Therefore, $\sigma, [\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I)) \models_\tau \varphi$ is trivially satisfied since

$$[\sup(J), \sup(J) - (\alpha(c') + |J|) + \sup(I)) = \emptyset$$

– $(\underset{|}{\Diamond})$ Suppose $\sigma, \eta, A \models c < \sup(I) \sqcap \bigcirc(\underset{|}{\Diamond}{}^{c,c'}_{\Diamond_I} \varphi \sqcap c' := 0)$. Then,

$$\alpha(c) < \sup(I) \wedge \sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|)) \models_\tau \overset{[\inf(J'),\tau)}{\Diamond_I} \varphi \wedge \alpha'(c') = 0$$

where $J'$ is an interval $[\eta(1), \eta(2))$, $\alpha$ and $alpha'$ are clock valuations $A(0)$ and $A(1)$, respectively. The interval $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as

$$[\inf(J') - (\alpha'(c)), \inf(J'))$$

since $|J'| = \sup(J') - \inf(J')$ and $\alpha'(c') = 0$. The interval can be rewritten as

$$[\sup(J) - (\alpha(c) + |J|), \sup(J))$$

where $J$ is an interval $[\eta(0), \eta(1))$ because $\alpha'(c) = \alpha(c) + |J|$. So,

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models_\tau \Diamond_I^{[\inf(J),\tau)} \varphi$$

By definition, $\sigma, \eta, A \models \underset{|}{\Diamond}{}^c_I \varphi$ is also satisfied.
Suppose $\sigma, \eta, A \models (c \leq \sup(I) \sqcap \varphi \sqcap \underset{|}{\blacklozenge}{}_I \varphi \sqcap \bigcirc (c > \inf(I)))$. By definition,

$$\sigma, J, \alpha \models \varphi \wedge \sigma, [\sup(J) - \inf(I), \sup(J)) \models \Diamond_I^{[\inf(J),\tau)} \varphi \wedge \alpha(c) \leq \sup(I) \wedge \alpha'(c) > \inf(I)$$

where $J$ is an interval $[\eta(0), \eta(1))$, $\alpha$ and $\alpha'$ are clock valuation $A(0)$ and $A(1)$, respectively. Then, $\sigma, J - I \models \Diamond_I^{[\inf(J),\tau)} \varphi$ by lemma 2. The interval $(J - I) \cup [\sup(J) - \inf(I), \sup(J))$ is continuous. Therefore,

$$\sigma, (\inf(J) - \sup(I), \sup(J)) \models \Diamond_I^{[\inf(J),\tau)} \varphi\,.$$

Since $\alpha(c) \leq \sup(I)$ by assumption, $\sup(J) - (\sup(I) + |J|) \leq \sup(J) - (\alpha(c) + |J|)$. Also, $\sup(J) - (\alpha(c) + |J|) < \sup(J) - \inf(I)$ because $\alpha'(c) > \inf(I)$ and $\alpha'(c) = \alpha(c) + |J|$. Therefore,

$$[\sup(J) - (\alpha(c) + |J|), \sup(J)) \subset [\inf(J) - \sup(I), \sup(J))$$

because $\inf(J) = \sup(J) - |J|$. So, $\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models \Diamond_I^{[\inf(J),\tau)} \varphi$ By definition, $\sigma, \eta, A \models \underset{|}{\Diamond}{}^c_I \varphi$ is also satisfied.

– $(\Diamond|_{\Diamond})$ Suppose $\sigma, \eta, A \models c < \sup(I) \sqcap \bigcirc(\Diamond|^{c,c'}_{\Diamond_I} \varphi)$. Then,

$$\alpha(c) < \sup(I) \wedge \sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|)) \models_\tau \Diamond_I^{[\inf(J'),\tau)} \varphi$$

where $J'$ is an interval $[\eta(1), \eta(2))$, $\alpha$ and $alpha'$ are clock valuations $A(0)$ and $A(1)$, respectively. The interval $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as

$$[\inf(J') - \alpha'(c), \inf(J') - \alpha'(c'))$$

since $|J'| = \sup(J') - \inf(J')$. The interval can be rewritten as

$$[\sup(J) - (\alpha(c) + |J|), \sup(J) - (\alpha(c') + |J|))$$

where $J$ is an interval $[\eta(0), \eta(1))$ because $\alpha'(c) = \alpha(c) + |J|$. So,

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J) - (\alpha(c') + |J|)) \models_\tau \Diamond_I^{[\inf(J),\tau)} \varphi$$

By definition, $\sigma, \eta, A \models \Diamond|^{c,c'}_{\Diamond_I} \varphi$ is also satisfied.

Suppose $\sigma, \eta, A \models (c \le \sup(I) \sqcap \varphi \sqcap \blacklozenge|^{c'}_{\Diamond_I} \varphi \sqcap \bigcirc\,(c > \inf(I)))$. By definition,

$$\sigma, J, \alpha \models \varphi \wedge \sigma, [\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|)) \models \Diamond_I^{[\inf(J),\tau)} \varphi \wedge \alpha(c) \le \sup(I) \wedge \alpha'(c) > \inf(I)$$

where $J$ is an interval $[\eta(0), \eta(1))$, $\alpha$ and $\alpha'$ are clock valuation $A(0)$ and $A(1)$, respectively. Then, $\sigma, J - I \models \Diamond_I^{[\inf(J),\tau)} \varphi$ by lemma 2. The interval $(J - I) \cup [\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|))$ is continuous. Therefore,

$$\sigma, (\inf(J) - \sup(I), \sup(J) - (\alpha(c') + |J|)) \models \Diamond_I^{[\inf(J),\tau)} \varphi\,.$$

Since $\alpha(c) \le \sup(I)$ by assumption, $\sup(J) - (\sup(I) + |J|) \le \sup(J) - (\alpha(c) + |J|)$. Also, $\sup(J) - (\alpha(c) + |J|) < \sup(J) - \inf(I)$ because $\alpha'(c) > \inf(I)$ and $\alpha'(c) = \alpha(c) + |J|$. Therefore,

$$[\sup(J) - (\alpha(c) + |J|), \sup(J) - (\alpha(c') + |J|)) \subset [\inf(J) - \sup(I), \sup(J) - (\alpha(c') + |J|))$$

because $\inf(J) = \sup(J) - |J|$. So, $\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J) - (\alpha(c') + |J|)) \models \Diamond_I^{[\inf(J),\tau)} \varphi$ By definition, $\sigma, \eta, A \models \Diamond|^{c,c'}_{\Diamond_I} \varphi$ is also satisfied.

– $(\blacklozenge|)$ Suppose $\bigcirc(\varphi \sqcap \blacklozenge|^{c'}_{\Diamond_I} \varphi \sqcap c' := 0)$. Then, by definition,

$$\sigma, J' \models \varphi \wedge \sigma, [\sup(J') - \inf(I), \sup(J') - (\alpha'(c') + |J'|)) \models \Diamond_I^{[\inf(J),\tau)} \varphi \wedge \alpha'(c') = 0$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha'$ is a clock evaluation $A(1)$. Then, $\sigma, J' - I \models \Diamond_I^{[\inf(J'),\tau)} \varphi$ by lemma 2. The interval $J' - I \cap [\sup(J') - \inf(I), \sup(J') - (\alpha'(c') + |J'|))$ in continuous. Therefore,

$$\sigma, [\inf(J') - \sup(I), \sup(J') - (\alpha'(c') + |J'|)) \models \Diamond_I^{[\inf(J),\tau)} \varphi\,.$$

The interval $[\inf(J') - \sup(I), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as $[\sup(J) - \sup(I), \sup(J))$ because $\inf(J') = \sup(J)$, $\alpha'(c') = 0$, and $\sup(J') - |J'| = \inf(J')$. Since $\inf(I) < \sup(I)$,

$$[\sup(J) - \inf(I), \sup(J)) \subseteq [\sup(J) - \sup(I), \sup(J)).$$

Therefore, $\sigma, [\sup(J) - \inf(I), \sup(J)) \models \Diamond_I^{[\inf(J),\tau)}\varphi$. By definition, $\sigma, \eta, A \models \overset{\blacklozenge}{|}_{I}\varphi$.

Suppose $\bigcirc(\overset{\Diamond}{|}\,|^{c,c'}_{\Diamond_I}\varphi \sqcap c := \inf(I) \sqcap c' := 0)$. Then, by definition,

$$\sigma, [\sup(J') - (a'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|)) \models \Diamond_I^{[\inf(J),\tau)}\varphi \wedge \alpha'(c) = \inf(I) \wedge \alpha'(c') = 0$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha'$ is a clock evaluation $A(1)$. The interval $[\sup(J') - (a'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as

$$[\inf(J') - \inf(I), \inf(J'))$$

since $\sup(J') - |J'| = \inf(J')$, and $\alpha'(c) = \inf(I) \wedge \alpha'(c') = 0$ by assumption. Therefore, $\sigma, [\sup(J) - \inf(I), \sup(J)) \models \Diamond_I^{[\inf(J),\tau)}\varphi$ because $J$ is an intervla $[\eta(0), \eta(1))$. By definition, $\sigma, \eta, A \models \overset{\blacklozenge}{|}_{I}\varphi$ is also satisfied

– ($\overset{\blacklozenge}{|}\,|_{\Diamond}$) Suppose $\bigcirc(\varphi \sqcap \overset{\blacklozenge}{|}\,|^{c'}_{\Diamond_I}\varphi)$. Then, by definition,

$$\sigma, J' \models \varphi \wedge \sigma, [\sup(J') - \inf(I), \sup(J') - (\alpha'(c') + |J'|)) \models \Diamond_I^{[\inf(J),\tau)}\varphi$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha'$ is a clock evaluation $A(1)$. Then, $\sigma, J' - I \models \Diamond_I^{[\inf(J'),\tau)}\varphi$ by lemma 2. The interval $J' - I \cap [\sup(J') - \inf(I), \sup(J') - (\alpha'(c') + |J'|))$ in continuous. Therefore,

$$\sigma, [\inf(J') - \sup(I), \sup(J') - (\alpha'(c') + |J'|)) \models \Diamond_I^{[\inf(J),\tau)}\varphi\,.$$

The interval $[\inf(J') - \sup(I), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as $[\sup(J) - \sup(I), \sup(J) - (\alpha(c') + |J|))$ because $\inf(J') = \sup(J)$, $\alpha'(c') = \alpha(c') + |J|$, and $\sup(J') - |J'| = \inf(J')$. Since $\inf(I) < \sup(I)$,

$$[\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|)) \subseteq [\sup(J) - \sup(I), \sup(J) - (\alpha(c') + |J|)).$$

Therefore, $\sigma, [\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|)) \models \Diamond_I^{[\inf(J),\tau)}\varphi$. By definition, $\sigma, \eta, A \models \overset{\blacklozenge}{|}\,|^{c'}_{\Diamond_I}\varphi$.

Suppose $\bigcirc(\overset{\Diamond}{|}\,|^{c,c'}_{\Diamond_I}\varphi \sqcap c := \inf(I))$. Then, by definition,

$$\sigma, [\sup(J') - (a'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|)) \models \Diamond_I^{[\inf(J),\tau)}\varphi \wedge \alpha'(c) = \inf(I)$$

where $J'$ is an interval $[\eta(1), \eta(2))$ and $\alpha'$ is a clock evaluation $A(1)$. The interval $[\sup(J') - (a'(c) + |J'|), \sup(J') - (\alpha'(c') + |J'|))$ can be rewritten as

$$[\inf(J') - \inf(I), \inf(J') - \alpha'(c'))$$

since $\sup(J') - |J'| = \inf(J')$, and $\alpha'(c) = \inf(I)$ by assumption. Therefore, $\sigma, [\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|)) \models \Diamond_I^{[\inf(J),\tau)}\varphi$ because $J$ is an intervla $[\eta(0), \eta(1))$ and $\alpha(c') = \alpha(c) + |J|$. By definition, $\sigma, \eta, A \models \blacklozenge|^{c'}_{\Diamond_I}\varphi$ is also satisfied

**Lemma 4.** *Given a signal $\sigma$, two successive intervals $J_1, J_2$, and clock valuations $\alpha_1, \alpha_2$, if the signal satisfies satisfies $P$ and $P'$ of a rule in $J_1$ and $J_2$, respectively, (i.e. $\sigma, J_1, \alpha_1 \models_\tau \bigwedge_{p\in P} p \wedge \sigma, J_2, \alpha_2 \models_\tau \bigwedge_{p\in P'} p$) and all subformulas of the formula in $P \cup P'$ are stable, then the signal also satisfy $Q$ and $Q'$ of the rule in $J_1$ and $J_2$, respectively, (i.e. $\sigma, J_1, \alpha_1 \models_\tau \bigwedge_{q\in Q} q \wedge \sigma, J_2, \alpha_2 \models_\tau \bigwedge_{q\in Q'} q$) when the rule has the form,*

$$(Id)\frac{P/P'}{Q/Q'}$$

*Proof.*
- ($\wedge$) By the assumption, $\sigma, \eta, A \models \varphi \wedge \psi$. Then, $\sigma, \eta, A \models \varphi \wedge \sigma, \eta, A \models \psi$ by definition. Therefore, a signal that satisfies $\varphi \sqcap \psi$ with $\eta$ and $A$.
- ($\vee$) By the assumption, $\sigma, \eta, A \models \varphi \vee \psi$. By the stability condition, $\sigma, \eta, A \models \varphi \vee \sigma, \eta, A \models \psi$. Then, $\sigma, \eta, A \models \varphi \sqcup \psi$ by definition.
- ($\mathbf{U}$) By definition, $\forall t \in [\eta(0), \eta(1)), \exists t' \geq t, t' < \tau, \sigma, t' \models \psi \wedge \forall t'' \in (t, t'), \sigma, t'' \models \varphi$. There are two cases where $t'$ can be located: 1. $t' \in [eta(0), \eta(1))$ and 2. $t' \geq \eta(1)$. For the first case, $\sigma, [\eta(0), \eta(1)), A \models \varphi \sqcap \psi$ by the stability condition. For the second case, $\varphi$ is satisfied in $[\eta(0), \eta(1))$, by definition. Also, $\sigma, \eta, A \models \bigcirc(\varphi\, \mathbf{U}\, \psi)$ is also satisfied because there is a point $t' \geq eta(1)$ such that $\sigma, t' \models \psi \wedge \sigma, (t, t') \models \varphi$.
- ($\mathbf{R}$) By definition, $\forall t \in [\eta(0), \eta(1)), \forall t' \geq t, t' < \tau, \sigma, t' \models \psi \vee \exists t'' \in (t, t'), \sigma, t'' \models \varphi$. There are three possible cases as follows: 1. $t'' \in [\eta(0), \eta(1))$, 2. $t''$ does not exists 3. $t'' \geq \eta(1)$. For the first case, if $\sigma, t'', A \models \varphi$, then $\sigma, [\eta(0), \eta(1)), A \models \varphi$ by the stability condition. For the second case, $\sigma, [\eta(0), \eta(1)), A \models \varphi\, \mathbf{R}\, \psi$ iff $\sigma, [\eta(0), \tau), A \models \psi$. It can be represented as $\psi \sqcap \bigcirc(*)$. For the third case,

$$\sigma, [\eta(0), \eta(1)), A \models \psi \wedge \forall t \in [\eta(1), \eta(2)), \forall t' \geq t, t' < \tau, \sigma, t' \models \psi \vee \exists t'' \in (t, t'), \sigma, t'' \models \varphi$$

  is satisfied. Therefore, $\sigma, \eta, A \models \psi \sqcap \bigcirc(\varphi\, \mathbf{R}\, \psi)$.
- ($\Box$) By assumption, $\sigma, \eta, A \models \Box_I\, \varphi$. By definition,

$$\sigma, [\eta(0), \eta(1)) \models \Box_I\, \varphi$$

  The formula can be rewritten

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models \Box_I^{[\inf(J),\tau)}\varphi \sqcap \alpha(c) = 0$$

  because $\sup(J) - (\sup(J) - \inf(J)) = \inf(J)$ and $[\inf(J), \sup(J)) + I \subseteq [\inf(J), \tau)$ where $J$ is an interval $[\eta(0), \eta(1))$.. Therefore, $\sigma, \eta, A \models \Box|^{c}_{I}\varphi \sqcap c := 0$.

– ($\Diamond$) By assumption, $\sigma, \eta, A \models \Diamond_I \varphi$. By definition,

$$\sigma, [\inf(J), \sup(J)) \models \Diamond_I \varphi.$$

The formula can be rewritten

$$\sigma, [\sup(J) - (\alpha(c) + |J|), \sup(J)) \models \Diamond_I^{[\inf(J), \tau)} \varphi \sqcap \alpha(c) = 0$$

because $\sup(J) - (\sup(J) - \inf(J)) = \inf(J)$ and $[\inf(J), \sup(J)) + I \subseteq [\inf(J), \tau)$ where $J$ is an interval $[\eta(0), \eta(1))$. Therefore, $\sigma, \eta, A \models \Diamond_I^c \varphi \sqcap c := 0$.

– ($\Box$) Suppose $\sigma, \eta, A \models c \leq \inf(I)$. Then, there are three possible case: 1. $\sigma, \eta^1, A^1 \models c \leq \inf(I)$ and $\eta(1) < \tau$. 2. $\sigma, \eta^1, A^1 \models c \leq \inf(I)$ and $\eta(1) = \tau$. 3. $\sigma, \eta^1, A^1 \models c > \inf(I)$ For the first case, By definiton $\Box_I^c \varphi$ can be rewritten as $\bigcirc(\Box\lfloor_{\Box_I}^{c,c'} \varphi \sqcap c' := 0)$. For the second case, it can be represented as $\sigma, \eta, A \models \bigcirc(c \leq \inf(I) \sqcap *)$ For the third case, $\inf(J) \leq \sup(J) - (\alpha(c) + |J|) + \inf(I) \leq \sup(J)$ because $\alpha(c) \leq \inf(I)$, $\alpha(c) + |J| \geq \inf(I)$, and $\sup(J) = \inf(J) + |J|$. $\sigma, \sup(J) - (\alpha(c) + |J|) + \inf(I), A \models \varphi$ by assumption. By the stability condition, $\sigma, [\eta(0), \eta(1)), \models \varphi$. Also, $\sigma, [eta(1), \sup(J) + \sup(I)) \models \varphi$. Bydefinition, $\sigma, eta, A \models \blacksquare_I \varphi$.

– ($\Box\lfloor_{\Box}$) Suppose $\sigma, \eta, A \models c \leq \inf(I)$. Similar the proof procedure of *upopenalways*

– ($\blacksquare$)

– ($\blacksquare\lfloor_{\Box}$) By definition, $\sigma, [\sup(J), \sup(J) - (\alpha(c') + |J|), \sup(I)) \models \varphi$, where $J$ is an interval $[\eta(0), \eta(1))$. There are three possible cases: 1. $\sup(J) = \tau$ 2. $\sup(J) - (\alpha(c') + |J|), \sup(I) \leq \sup(J)$ 3. otherwise. The first case can be represented as $*$. For the second case, $\alpha(c') + |J| \geq \sup(I)$. The formula can be rewritten as $\bigcirc(\alpha(c') \geq \sup(I))$. For the third case, the interval $[\sup(J), \sup(J) - (\alpha(c') + |J|), \sup(I))$ can be devided $[\inf(J'), \sup(J')) \sqcap [\sup(J'), \sup(J') - (\alpha'(c') + |J'|))$ because $\sup(J) = \sup(J') - |J'|$ and $\alpha'(c') = \alpha(c') + |J|$ where $J'$ is an interval $[\eta(1), \eta(2))$. Then,

$$\sigma, [\inf(J'), \sup(J')), A \models \varphi \wedge \sigma, [\sup(J'), \sup(J') - (\alpha'(c') + |J'|)), A \models \varphi.$$

. The formula can be rewrriten as

$$\sigma, \eta, A \models \bigcirc(\varphi \sqcap \blacksquare\lfloor_{\Box_I}^{c'} \varphi).$$

– ($\blacklozenge$) By definition, $\sigma, [\sup(J) - \inf(I), \sup(J)), A \models \Diamond_I^{[\inf(J), \tau]} \varphi$ where $J$ is an interval $[\eta(0), \eta(1))$. There are two possible cases: 1. $\sigma, eta^1, A^1 \models \varphi$ and 2. otherwise. For the first case, $\sigma, [\inf(J') - \sup(I), \sup(J') - \inf(I)), A^1 \models \Diamond_I^{[\inf(J'), \tau]} \varphi$ by lemma 2 where $J'$ is an interval $[\eta(1), \eta(2))$. Since $\inf(J') =$

$\sup(J)$ and $\inf(I) < \sup(I)$, $\inf(J') - \sup(I) < \sup(J) - \inf(I) < \sup(J') - \inf(I)$. Therefore, we can devide interval $[\sup(J) - \inf(I), \sup(J))$ into $[\sup(J) - \inf(I), \sup(J') - \inf(I))$ and $[\sup(J') - \inf(I), \sup(J))$. Thus,

$$\sigma, [\sup(J') - \inf(I), \sup(J)), \models \Diamond_I^{[\inf(J),\tau]} \varphi\,.$$

By definition, the formula can be rewrriten as

$$\sigma, \eta, A \models \bigcirc( \blacklozenge\!\rfloor_{\Diamond_I}^{c'} \varphi \sqcap c' := 0)$$

because $\inf(J), \inf(J') < \sup(J') - inf(I) + \inf(I)$.

For the second case, $\sigma, [\sup(J) - \inf(I), \sup(J)), A \models \Diamond_I^{[\inf(J),\tau]} \varphi$ can be rewritten as

$$\sigma, [\sup(J') - (\inf(I) + |J'|), \sup(J') - |J'|), A \models \Diamond_I^{[\inf(J'),\tau]} \varphi$$

because $\sup(J') - (\inf(I) + |J'|) + \inf(I) \geq \inf(J')$.

- ($\blacklozenge\!\rfloor_{\Diamond}$) By definition, $\sigma, [\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|)) \models \Diamond_I^{[\inf(J),\tau)} \varphi$.

  If the interval $[\sup(J) - \inf(I), \sup(J) - (\alpha(c') + |J|))$ is empty, then the formula is satisfied. The interval is empty iff $(\alpha(c') + |J|) \geq \inf(I)$. It can be represented as

$$\sigma, \eta, A \models \bigcirc(c' \geq \inf(I)).$$

**Theorem 2.** *Given a signal $\sigma$, two successive intervals $J_1, J_2$, and clock valuations $\alpha_1, \alpha_2$, the signal satisfies the numerator of the reduction rules in Figure 12 and Figure 14 over $J_1$ and $J_2$, respectively, if and only if the signal the denominator of the rules in $J_1$ and $J_2$, respectively, under clock valuations $\alpha_1, \alpha_2$.*

*Proof.* – (r-$\boxdot$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c' := 0 \sqcap \Box\!\rfloor_{\Box_I}^{c,c'} \varphi \sqcap k := 0 \sqcap \Box_I^{k} \varphi)$$

also satisfies

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J') - |J'|), a' \models \Box_I^{[\inf(J'),\tau)} \varphi \wedge \sigma, [\sup(J') - |J'|, \sup(J')), a' \models \Box_I^{[\inf(J'),\tau)} \varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. The interval $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - |J'|) \cup [\sup(J') - |J'|, \sup(J'))$ can be rewritten as

$$[\sup(J') - (\alpha'(c) + |J'|), \sup(J'))$$

. Therefore, The formula can be rewritten as

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J')), a' \models \Box_I^{[\inf(J'),\tau)} \varphi\,.$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc( \Box_I^{c} \varphi).$$

– (r-$\underset{|}{\blacksquare}$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c' := 0 \sqcap \underset{|}{\blacksquare}\Big|^{c'}_{\Box_I}\varphi \sqcap k := 0 \sqcap \underset{|}{\Box}{}^{k}_{I}\varphi)$$

also satisfies

$$\sigma, [\sup(J'), \sup(J') - |J'| + \sup(I)), a' \models \varphi \wedge \sigma, [\sup(J') - |J'|, \sup(J')), a' \models \Box_I^{[\inf(J'),\tau)}\varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. By definition, $\sigma, [\sup(J') - |J'|, \sup(J')), a' \models \Box_I^{[\inf(J'),\tau)}\varphi$ iff

$$\sigma, [\sup(J') - |J'| + \inf(I), \sup(J') + \sup(I)], a' \models \varphi$$

because $\inf(I) \geq 0$ and $[\sup(J') - |J'|, \sup(J')) + I \subseteq [\inf(J'), \tau)$. Since $\inf(I) \geq \sup(I)$,

$$\sup(J') - |J'| + \inf(I) \geq \sup(J') - |J'| + \sup(I)$$

Therefore,

$$\sigma, [\sup(J'), \sup(J') + \sup(I)), a' \models \varphi$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc(\underset{|}{\blacksquare}{}_{I}\, c\,\varphi).$$

– (r-$\underset{|}{\Diamond}$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c' := 0 \sqcap \underset{|}{\Diamond}\Big|^{c,c'}_{\Diamond_I}\varphi \sqcap k := 0 \sqcap \underset{|}{\Diamond}{}^{k}_{I}\varphi)$$

also satisfies

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J') - |J'|), a' \models \Diamond_I^{[\inf(J'),\tau)}\varphi \wedge \sigma, [\sup(J') - |J'|, \sup(J')), a' \models \Diamond_I^{[\inf(J'),\tau)}\varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. The interval $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - |J'|) \cup [\sup(J') - |J'|, \sup(J'))$ can be rewritten as

$$[\sup(J') - (\alpha'(c) + |J'|), \sup(J'))$$

. Therefore, The formula can be rewritten as

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J')), a' \models \Diamond_I^{[\inf(J'),\tau)}\varphi\,.$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc(\underset{|}{\Diamond}{}^{c}_{I}\varphi).$$

– (r-$\underset{|}{\blacklozenge}$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c' := 0 \sqcap \underset{|}{\blacklozenge}\Big|^{c'}_{\Diamond_I} \varphi \sqcap k := 0 \sqcap \underset{|\,I}{\Diamond}^{k} \varphi)$$

also satisfies

$$\sigma, [\sup(J') - \inf(I), \sup(J') - |J'|), a' \models \Diamond_I^{[\inf(J), \tau)} \varphi \wedge \sigma, [\sup(J') - |J'|, \sup(J')), a' \models \Diamond_I^{[\inf(J'), \tau)} \varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. Since $[\sup(J') - \inf(I), \sup(J') - |J'|) \cap [\sup(J') - |J'|, \sup(J')) = [\sup(J') - \inf(I), \sup(J'))$,

$$\sigma, [\sup(J') - \inf(I), \sup(J')), a' \models \Diamond_I^{[\inf(J'), \tau)} \varphi$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc(\underset{|\,I}{\blacklozenge}\ c\, \varphi).$$

– (rs-$\underset{|}{\blacklozenge}$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c' := 0 \sqcap \underset{|}{\Diamond}\Big|^{c,c'}_{\Diamond_I} \varphi \sqcap k := 0 \sqcap \underset{|\,I}{\Diamond}^{k} \varphi)$$

also satisfies

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J') - |J'|), a' \models \Diamond_I^{[\inf(J), \tau)} \varphi \wedge \sigma, [\sup(J') - |J'|, \sup(J')), a' \models \Diamond_I^{[\inf(J'), \tau)} \varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. Since $[\sup(J') - (\alpha'(c) + |J'|), \sup(J') - |J'|) \cap [\sup(J') - |J'|, \sup(J')) = [\sup(J') - (\alpha'(c) + |J'|), \sup(J'))$,

$$\sigma, [\sup(J') - (\alpha'(c) + |J'|), \sup(J')), a' \models \Diamond_I^{[\inf(J'), \tau)} \varphi$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc(\underset{|\,I}{\Diamond}^{c} \varphi).$$

– (s-$\underset{|}{\blacklozenge}$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c := \inf(I) \sqcap c' := 0 \sqcap \underset{|}{\Diamond}\Big|^{c,c'}_{\Diamond_I} \varphi \sqcap \varphi)$$

also satisfies

$$\sigma, [\sup(J') - (\inf(I) + |J'|), \sup(J') - |J'|), \alpha' \models \Diamond_I^{[\inf(J), \tau)} \varphi \wedge \sigma, J', \alpha' \models \varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. By lemma 2,

$$\sigma, J', \alpha' \models \varphi \text{ iff } \sigma, J' - I, \alpha' \models \Diamond_I^{[\inf(J'),\tau)} \varphi\,.$$

Since $\inf(I) \leq \sup(I)$,

$$\inf(J') - \sup(I) = \sup(J') - |J'| - \sup(I) \leq \sup(J') - (\inf(I) + |J'|).$$

Also, $\sup(J') - (\inf(I) + |J'|) \leq \sup(J') - \inf(I)$. Therefore, $\sigma, J', \alpha' \models \varphi$ implies $\sigma, [\sup(J') - (\inf(I) + |J'|), \sup(J') - \inf(I)), \alpha' \models \Diamond_I^{[\inf(J),\tau)} \varphi$. Thus, we only need to check

$$\sigma, [\sup(J') - \inf(I), \sup(J') - |J'|), \alpha' \models \Diamond_I^{[\inf(J),\tau)} \varphi$$

to check

$$\sigma, \eta, A \models \bigcirc(c := \inf(I) \sqcap c' := 0 \sqcap \Diamond\!\!\mid\,\rfloor^{c,c'}_{\Diamond_I} \varphi).$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc(c' := 0 \sqcap \blacklozenge\!\!\mid\,\rfloor^{c'}_{\Diamond_I} \varphi \sqcap \varphi).$$

– (s-$\blacklozenge\!\!\mid\,\rfloor_{\Diamond}$) If a signal $\sigma$ that satisfies

$$\sigma, \eta, A \models \bigcirc(c := \inf(I) \sqcap \Diamond\!\!\mid\,\rfloor^{c,c'}_{\Diamond_I} \varphi \sqcap \varphi)$$

also satisfies

$$\sigma, [\sup(J') - (\inf(I) + |J'|), \sup(J') - (\alpha'(c') + |J'|)), \alpha' \models \Diamond_I^{[\inf(J),\tau)} \varphi \wedge \sigma, J', \alpha' \models \varphi$$

by definition where $J'$ is an intervla $[\eta(1), \eta(2))$ and $\alpha'$ is a clock valuation $A(1)$. By lemma 2,

$$\sigma, J', \alpha' \models \varphi \text{ iff } \sigma, J' - I, \alpha' \models \Diamond_I^{[\inf(J'),\tau)} \varphi\,.$$

Since $\inf(I) \leq \sup(I)$,

$$\inf(J') - \sup(I) = \sup(J') - |J'| - \sup(I) \leq \sup(J') - (\inf(I) + |J'|).$$

Also, $\sup(J') - (\inf(I) + |J'|) \leq \sup(J') - \inf(I)$. Therefore, $\sigma, J', \alpha' \models \varphi$ implies $\sigma, [\sup(J') - (\inf(I) + |J'|), \sup(J') - \inf(I)), \alpha' \models \Diamond_I^{[\inf(J),\tau)} \varphi$. Thus, we only need to check

$$\sigma, [\sup(J') - \inf(I), \sup(J') - (\alpha'(c') + |J'|)), \alpha' \models \Diamond_I^{[\inf(J),\tau)} \varphi$$

to check

$$\sigma, \eta, A \models \bigcirc(c := \inf(I) \sqcap \Diamond\!\!\mid\,\rfloor^{c,c'}_{\Diamond_I} \varphi).$$

By definition, the formula can be rewritten as

$$\sigma, \eta, A \models \bigcirc(\blacklozenge\!\!\mid\,\rfloor^{c'}_{\Diamond_I} \varphi \sqcap \varphi).$$

**Theorem 3.** *For a triple $P/P' \mid C$, if $P$ is not an empty set, then the relation $\dashrightarrow$ is always applicable and $P$ is always fully expanded. by applying a finite number of the relation $\dashrightarrow$ .*

*Proof.* To prove that, we define complexity for each interval predicates. According to the complexity, when a rule is applied to a formula, the sum of the complexity of the remaining formula that must be satisfied in the current interval always decreases. The complexity cannot be negative, so the number of times the rules can be applied is finite.

**Theorem 4.** *For an STL formula $\varphi$, a signal $\sigma : [0, \tau) \to Q \times \mathbb{R}^l$, that is accepted by a tableau $G_\varphi$, satisfies $\varphi$ at time 0, i.e. $\sigma, 0 \models_\tau \varphi$ .*

*Proof.* For an accepting path $\pi = s_1, ..., s_n$, a signal $\sigma$, that is accepted by $\pi$, satisfies all propositions in $L(s_n)$ by definition. For a label $l \in L(s_n)$, the satisfaction of the label is decomposed satisfiabitliy of subformulas in the current interval and formulas in the next interval. However, there is no state corresponding the next inerval because $s_n$ is a final state. Therefore, the satisfaction of the label can be represented using only the satisfaction of subformulas in the current interval. If $l = \varphi \sqcap \psi$, then the signal satisfies $\varphi \wedge \psi$ by soundness of $(\sqcap)$. If $l = \varphi \sqcup \psi$, then the signal satisfies $\varphi \vee \psi$ by soundness of $(\sqcup)$. If $l = \varphi \,\mathbf{U}\, \psi$, then the $(\varphi \sqcap \psi) \in l$ because $s_n$ is a final state. By soundness of $(\mathbf{U})$, the signal satisfies $\varphi \,\mathbf{U}\, \psi$. If $l = \varphi \,\mathbf{R}\, \psi$, then $\varphi \sqcup (\psi \sqcap \bigcirc *) \in l$ because $s_n$ is a final state. By soundness of $(\mathbf{U})$, the signal satisfies $\varphi \,\mathbf{R}\, \psi$.

If $\Box_I \varphi \in L(s_n)$, then $\Box\!\!|^c_I \varphi \sqcap c := 0 \in L(s_n)$ by $(\Box)$. If $\Box\!\!|^c_I \varphi \in L(s_n)$, then

$$\bigcirc(c \leq \inf(I) \sqcap \Box\!\!|^{c,c'}_{\Box_I} \varphi \sqcap c' := 0) \sqcup (c \leq \inf(I) \sqcap \varphi \sqcap \blacksquare\!\!|_I\, \varphi \sqcap \bigcirc(c > \inf(I))) \sqcup \bigcirc(c \leq \inf(I) \sqcap *) \in L(s_n) \quad (1)$$

by the rule $(\Box\!\!|)$. By soundness of $(\Box\!\!|)$, the signal that satisfies (1) also satisfies $\Box\!\!|^c_I \varphi$. Also, by soundness of $(\Box)$, the signal satisfies $\Box_I \varphi$.

If $\Diamond_I \varphi \in L(s_n)$, $\Diamond\!\!|^c_I \varphi \sqcap c := 0 \in L(s_n)$ by $(\Diamond)$. If $\Diamond\!\!|^c_I \varphi \in L(s_n)$, then

$$(c < \sup(I) \sqcap \bigcirc(\Diamond\!\!|^{c,c'}_{\Diamond_I} \varphi \sqcap c' := 0)) \sqcup (c \leq \sup(I) \sqcap \varphi \sqcap \blacklozenge\!\!|_I\, \varphi \sqcap \bigcirc(c > \inf(I))) \in L(s_n) \quad (2)$$

by the rule $(\Diamond\!\!|)$. By soundness of $(\Diamond\!\!|)$, the signal that satisfies (2) also satisfies $\Diamond\!\!|^c_I \varphi$. Also, by soundness of $(\Diamond)$, the signal satisfies $\Diamond_I \varphi$.

For the labels in $L(s_{n-1})$, the satisfaction of the labels is decomposed into subformulas into the same level and formulas in the next level by the rules. By applying the rules repeatedly, the satisfaction of the subformulas can be finally expressed as the satisfaction of propositions at the same level and formulas at the next level. Based on the above proof, the signal $\sigma$ satisfies all formulas in the next level corresponding to $s_n$. The signal $\sigma$ also satisfies all propositions in the current level $s_{n-1}$ by definition. Therefore, the signal satisfies all labels in $L(s_{n-1})$. For the same reasons as above, the signal satisfies all labels in $Ls_1$. Therefore, the signal satisfies $\varphi$ because $\varphi \in L(s_1)$ by construction rules.

**Lemma 4.** *Given a signal $\sigma$, two successive intervals $J_1, J_2$, and clock valuations $\alpha_1, \alpha_2$, if the signal satisfies satisfies $P$ and $P'$ of a rule in $J_1$ and $J_2$, respectively, (i.e. $\sigma, J_1, \alpha_1 \models_\tau \bigwedge_{p \in P} p \wedge \sigma, J_2, \alpha_2 \models_\tau \bigwedge_{p \in P'} p$) and all subformulas of the formula in $P \cup P'$ are stable, then the signal also satisfy $Q$ and $Q'$ of the rule in $J_1$ and $J_2$, respectively, (i.e. $\sigma, J_1, \alpha_1 \models_\tau \bigwedge_{q \in Q} q \wedge \sigma, J_2, \alpha_2 \models_\tau \bigwedge_{q \in Q'} q$) when the rule has the form,*

$$(Id)\frac{P/P'}{Q/Q'}$$

*Proof.* For a signal $\sigma$ that satisfies $\varphi$, there is a partition $P = \{J_1, J_2, ..., J_N\}$ such that $\bigcup_{i=1}^{N} J_i = \mathrm{dom}(\sigma)$, $\sup(J_i) = \inf(J_{i+1})$, $J_i \cap J_{i+1} = \emptyset$, and all subformulas of $\varphi$ is stably true or false for each $J_i$ where $1 \leq i \leq N$. For the certain signal $\sigma$, the truth values of labels are determined for each $J_i$. Assume that $\varphi$ is true in an interval $J_i$. We prove that the satisfaction of $\varphi$ can only be expressed as the satisfaction of subformulas of $\varphi$ in $J_k$ for $i \leq k \leq N$. In the case of $\varphi = \varphi \wedge \psi$, then the signal $\sigma$ satisfies $\varphi$ in $J_i$ because the signal satisfies $\varphi$ by the assumption. Then, $\varphi \sqcap \psi$ are also satisfied in $J_i$ by the completeness of the rule $(\sqcap)$. Therefore, the satisfaction of $\varphi$ can only be expressed using its subformulas in $J_i$. A tableau $G_\varphi$ accepts the signal because the tableau accepts the signal satisfying $\varphi$ and $\psi$ by induction hypothesis. If $\varphi = \varphi \vee \psi$, then the signal satisfies $\varphi \sqcup \psi$ in $J_i$ by the completeness of the rule $(\sqcup)$. Therefore, the satisfaction of $\varphi$ can only be expressed using its subformulas in $J_i$. A tableau $G_\varphi$ accepts the signal by induction hypothesis. If $l = \varphi \,\mathbf{U}\, \psi$, then there are two possible cases (a) $\varphi$ and $\psi$ are true in $J_i$, (b) $\varphi$ and $\varphi \,\mathbf{U}\, \psi$ are true in $J_i$, and $J_{i+1}$, respectively. by the completeness of expansion rules. For the first case, the satisfaction of $l$ can only be expressed using its subformulas in $J_i$. For the second case, $\varphi$ and $\psi$ are satisfied in $J_k$ for $i < k \leq N$ and $\varphi$ is satisfied in $\bigcup_{j=i}^{k} J_j$ because the signal satisfies $l$ by the assumption. Then, $\varphi$ and $\psi$ are satisfied in $J_k$ before the timebound by definition. Therefore, the satisfaction of $l$ can only be expressed using its subformulas in $J_k$ for $i \leq k \leq N$. If $l = \varphi \,\mathbf{R}\, \psi$, then there are three possible cases (a) $\varphi$ are true in $J_i$, (b) $\psi$ are true in $J_i$ and $\sup(J_i) = \tau$, (c) $\psi$ and $\varphi \,\mathbf{R}\, \psi$ are true in $J_i$, and $J_{i+1}$, respectively. by the completeness of the rule $(\mathbf{R})$. For the first and second cases, the satisfaction of $l$ can only be expressed using its subformulas in $J_i$. For the last case, $\varphi$ is satisfied in $J_k$ and $\psi$ is satisfied in $\bigcup_{j=i}^{k-1} J_j$ for $i < k \leq N$ or $\psi$ is satisfied in $\bigcup_{j=i}^{N} J_j$ because the signal satisfies $l$ by the assumption. Therefore, the satisfaction of $l$ can only be expressed using its subformulas in $J_k$ for $i \leq k \leq N$.

If $l = \Box_I \varphi$, then there are three possible cases (a) $\bigcirc(c \leq \inf(I) \sqcap \underset{|}{\Box} \big|^{c,c'}_{\Box_I} \varphi \sqcap c' := 0)$ is true in $J_i$, (b) $(c \leq \inf(I) \sqcap \varphi \sqcap \underset{|}{\blacksquare}_{I} \varphi \sqcap \bigcirc(c > \inf(I)))$ is true in $J_i$, (c) $\bigcirc(c \leq \inf(I) \sqcap *)$ is true in $J_i$ by the completeness of rules $(\Box)$ and $(\underset{|}{\Box})$. For the first case, the signal satisfis one of the following: (1) $c \leq \inf(I) \sqcap \underset{|}{\blacksquare}\lfloor^{c'}_{\Box_I} \varphi \sqcap \bigcirc (c > \inf(I))$ and $\varphi$ is satisfied in $J_i$, (2) $c \leq \inf(I)$ and $\underset{|}{\Box}\lfloor^{c,c'}_{\Box_I} \varphi$ are true in $J_{i+1}$ (3) $\bigcirc(c \leq inf(I) \sqcap *)$ is true in $J_i$ by the completeness of the rule $(\underset{|}{\Box}\lfloor_{\Box})$. The signal that satisfies $l$ accepted by the tableau, because a set of signals that satisfy the above cases is accepted by the tableau for $l$ For the case $(b)$, one of the following is implied: (1) $\bigcirc(\varphi \sqcap \underset{|}{\blacksquare}\lfloor^{c'}_{\Box_I} \varphi \sqcap c' := 0)$ is true in $J_i$, (2) $\bigcirc *$ is true in $J_{i+1}$, by the completeness of the rule $(\underset{|}{\blacksquare})$.

If $l = \Diamond_I \varphi$, then there are two possible cases (a) $(c < \sup(I) \sqcap \bigcirc(\underset{|}{\Diamond}\lfloor^{c,c'}_{\Diamond_I} \varphi) \sqcap c' := 0)$ is true in $J_i$, (b) $(c \leq \sup(I) \sqcap \varphi \sqcap \underset{|_I}{\blacklozenge} \varphi \sqcap \bigcirc (c > \inf(I)))$ is true in $J_i$ by the completeness of rules $(\Diamond)$ and $(\underset{|}{\Diamond})$. For the first case, the signal that satisfies $l$ also satisfies one of the following: (1) $c < \sup(I \sqcap \bigcirc(\underset{|}{\Diamond}\lfloor^{c,c'}_{\Diamond_I} \varphi))$ is satisfied in $J_i$, (2) $c \leq \sup(I) \sqcap \varphi \sqcap \underset{|}{\blacklozenge}\lfloor^{c'}_{\Diamond_I} \varphi \sqcap \bigcirc (c > \inf(I))$ is true in $J_i$, by the completeness of the rule $(\underset{|}{\Diamond}\lfloor_{\Diamond})$. The signal that satisfies $l$ accepted by the tableau, because a set of signals that satisfy the above cases is accepted by the tableau for $l$ by construction.

**Theorem 6.** *Given a simplified tableau $G = (S, T, L, s_0, S_f)$, a signal $\sigma$ is accepted by $G$ iff $\sigma$ is accepted by $G \backslash \mathcal{R}_{pre,=}$ and $G \backslash \mathcal{R}_{post,=}$.*

*Proof.* For $s \in S$ and $s' \in [s]_{\mathcal{R}_{\sqsubseteq}}$, $(s, s') \in \mathcal{R}_{\sqsubseteq}$ by definition. $\Pi_1$ is a set of all paths that reaching to $s$, and $\Pi_2$ is a set of every path that reaching to $s'$. For every path $s^1_1, s^1_2, ..., s^1_n = s \in \Pi_1$, there is a corresponding path $s^2_1, s^2_2, ..., s^2_n = s' \in [s]_{\mathcal{R}_{\sqsubseteq}} \in \Pi_2$ in $G \backslash \mathcal{R}_{\text{pre}}$ such that $L(s^1_i) = L(s^2_i)$ for every $1 \leq i$ because $(s, s') \in \mathcal{R}_{\sqsubseteq}$. Therefore, a set of signals that satisfies a path $\pi_1 \in \Pi_1$ is equal to a set of signals that satisfies the corresponding path $\pi_2 \in \Pi_2$ by Lemma **??**. By symmetry, a set of signals that satisfies a path $\pi_2 \in \Pi_2$ is equal to a set of signals that satisfies the corresponding path $\pi_1 \in \Pi_1$. Thus, a signal $\sigma$ is accepted by $G$ iff $\sigma$ is accepted by $G \backslash \mathcal{R}_{\sqsubseteq}$.

For $s \in S$ and $s' \in [s]_{\mathcal{R}_{\subseteq}}$, $(s, s') \in \mathcal{R}_{\subseteq}$ by definition. $\Pi_1$ is a set of all paths that starting from $s$, and $\Pi_2$ is a set of every path that starting from $s'$. For every path $s = s^1_1, s^1_2, .... \in \Pi_1$, there is a corresponding path of the same length $[s]_{\mathcal{R}_{\subseteq}} \ni s' = s^2_1, s^2_2, .... \in \Pi_2$ in $G \backslash \mathcal{R}_{\text{post}}$ such that $L(s^1_i) = L(s^2_i)$ for every $1 \leq i$ because $(s, s') \in \mathcal{R}_{\subseteq}$. Therefore, a set of signals satisfies a path $\pi_1 \in \Pi_1$ is equal to a set of signals satisfies the corresponding path $\pi_2 \in \Pi_2$ by Lemma **??**. By symmetry, a set of signals that satisfies a path $\pi_2 \in \Pi_2$ is equal to a set of signals that satisfies the corresponding path $\pi_1 \in \Pi_1$. Thus, a signal $\sigma$ is accepted by $G$ iff $\sigma$ is accepted by $G \backslash \mathcal{R}_{\subseteq}$.

**Theorem 7.** *For an STL formula $\varphi$ and a hybrid automaton $H$, a trajectory of $H$ satisfies $\varphi$ iff a trajectory of $H_c$ can reach a state in* $\mathsf{Goal}$ *where $H_c$ and* $\mathsf{Goal}$ *can be obtained by composing $H$ and $G_\varphi$.*

*Proof.* To prove the theorem, we translate the tableau $G = (S, T, L, s_0, S_f)$ to a hybrid automaton. A set of modes of the translated automaton is the set of states in $G$ excluding $s_0$. A set of real-valued variables of the automaton is a set of variables in the formula labeled for each state of the tableau. In each mode of the automaton, the dynamics of clock variables are 1 and the dynamics of other variables are arbitrary functions that follow the invariant conditions of each mode. The set of signals accepted by the tableau if and only if the signal is a trajectory of the hybrid automaton. We can compose the translated automata and the give hybrid automaton $H$ using the exising technique [5]. The signals of the composed hybrid automata are also the signals of $H$ and are accepted by the tableau.