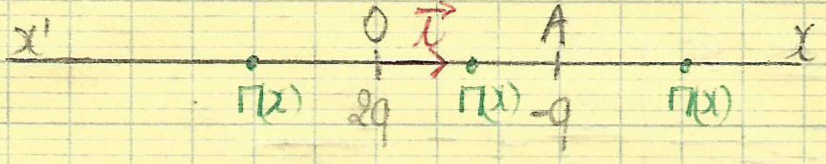
Travaux dirigés électrostatique TD 1

Exo 1 : Sur un axe $x'ox$ sont placées deux charges ponctuelles $+2q$ et $-q$ respectivement au point $O(0)$ et $A(a)$; $a > 0$.

1- Calculer le champ créé en un point $M(x)$

2- Calculer le potentiel électrostatique créé en un point $M(x)$

3- Vérifier que $E = -\text{grad}(V)$



1) Calcul du champ créé en $M(x)$

$$\vec{E}(M) = \vec{E}_O(M) + E_A(M)$$

$$- \; E_O(M) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{OM^2}\vec{U}_{OM} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{OM^2}\frac{\vec{OM}}{OM}$$

$$- \; E_A(M) = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{AM^2}\vec{U}_{AM} = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{AM^2}\frac{\vec{AM}}{AM}$$

$$E(M) = \frac{1}{4\pi\varepsilon_0}\left(\frac{2q}{OM^2}\times\frac{OM}{OM} + \frac{(-q)}{AM^2}\times\frac{AM}{AM}\right)$$

$$OM = |x-0| = |x| \begin{cases} x & \text{si } x > 0 \\ -x & \text{si } x < 0 \end{cases} \quad \Big| \quad AM = |x-a| = \begin{cases} x-a & \text{si } x > 0 \\ -x+a & \text{si } x < 0 \end{cases}$$

$$\boxed{\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{|x|^2}\times\frac{\vec{OM}}{OM} - \frac{1}{|x-a|^2}\times\frac{\vec{AM}}{AM}\right)}$$

1er Cas : si $x < 0$

$$\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{(-x)^2}(-\vec{i}) - \frac{1}{(x-a)^2}(-\vec{i})\right)$$

$$\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} - \frac{1}{(x-a)^2}\right)(-\vec{i}) = \boxed{-\frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} - \frac{1}{(x-a)^2}\right)\vec{i}}$$

2ᵉ Cas : $0 < x < a$

$$\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2}\vec{i} - \frac{1}{(-x+a)^2}(-\vec{i})\right)$$

$$\boxed{\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} + \frac{1}{(x-a)^2}\right)\vec{i}}$$

3ᵉ Cas : si $x > a$

$$\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2}\vec{i} - \frac{1}{(x-a)^2}\vec{i}\right)$$

$$\boxed{\vec{E}(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} - \frac{1}{(x-a)^2}\right)\vec{i}}$$

2) Calcul du potentiel électrostatique créé en $M(x)$

$$V(M) = V_O(M) + V_A(M)$$

– $V_O(M) = \frac{2q}{4\pi\varepsilon_0}\frac{1}{OM}$

– $V_A(M) = \frac{-q}{4\pi\varepsilon_0}\frac{1}{AM}$

$$V(M) = \frac{2q}{4\pi\varepsilon_0}\frac{1}{OM} - \frac{q}{4\pi\varepsilon_0}\frac{1}{AM}$$

$$V(M) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{OM} - \frac{1}{AM}\right)$$

$$\boxed{V(M) = V(x) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{|x|} - \frac{1}{|x-a|}\right)}$$

1ᵉʳ Cas : Si $x < 0$

$$V(x) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{(-x)} - \frac{1}{(-x+a)}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x-a} - \frac{2}{x}\right)$$

$$\boxed{V(x) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x-a} - \frac{2}{x}\right)}$$

2^e Cas : $0 < x < a$

$$V(x) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x} - \frac{1}{(-x+a)}\right) = \boxed{\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x-a} + \frac{2}{x}\right) = V(x)}$$

3^e Cas : si $x > a$

$$V(x) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x} - \frac{1}{x-a}\right) = \boxed{-\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x-a} - \frac{2}{x}\right) = V(x)}$$

3) Vérification

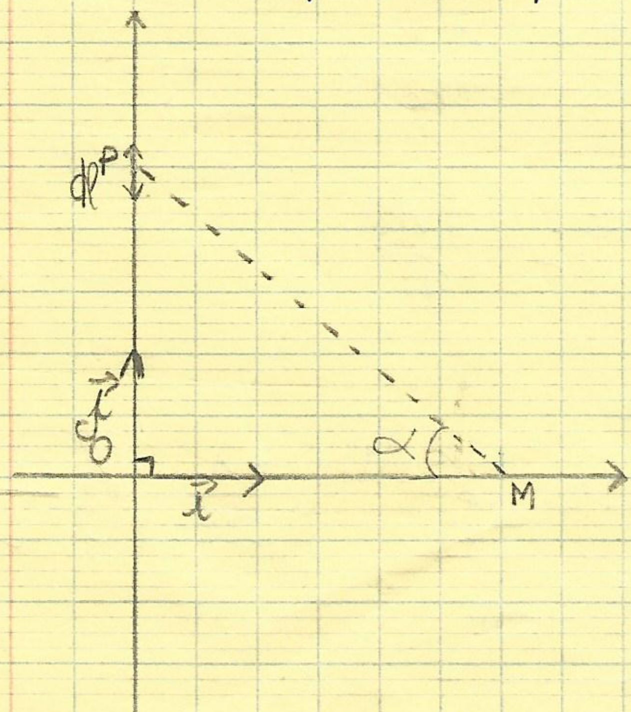
$$E(x) = -\frac{dV}{dx} \qquad V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x-a} - \frac{2}{x}\right)$$

$$\frac{dV}{dx} = \frac{d}{dx}\left[\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x-a} - \frac{2}{x}\right)\right] = \frac{q}{4\pi\varepsilon_0}\frac{d}{dx}\left[\frac{1}{x-a} - \frac{2}{x}\right]$$

$$\frac{dV}{dx} = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{(x-a)^2} + \frac{2}{x^2}\right)$$

$$\boxed{\frac{-dV}{dx} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{2}{x^2} - \frac{1}{(x-a)^2}\right) = E(x)}$$

Exo 2 : 1) Calcul du champ électrostatique à partir de $d\vec{E}$ en un point $M(z)$



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{PM^2}\frac{\vec{PM}}{PM}$$

Avec $dq = \lambda dl$

$$\frac{\vec{PM}}{PM} = \frac{\vec{PO} + \vec{OM}}{PM} = \frac{\vec{PO}}{PM} + \frac{\vec{OM}}{PM}$$

$$\frac{\vec{PM}}{PM} = \frac{\vec{PO}}{PO}\frac{PO}{PM} + \frac{\vec{OM}}{OM}\frac{OM}{PM}$$

$$\frac{\vec{PM}}{PM} = -\vec{j}\sin\alpha + \vec{i}\cos\alpha$$

$$dQ = \lambda\, dl \quad l = on\tan\alpha \Rightarrow dl = \frac{on}{\cos^2\alpha}\, d\alpha$$

$$\Rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda\, on}{\cos^2\alpha}\, d\alpha \times \frac{\cos^2\alpha}{on^2}\left(-\vec{j}\sin\alpha + \vec{i}\cos\alpha\right)$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda}{on}\, d\alpha\left(-\vec{j}\sin\alpha + \vec{i}\cos\alpha\right)$$

$$\vec{E} = \int_{\pi/2}^{-\pi/2} \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda}{on}\, d\alpha\left(-j\sin\alpha + i\cos\alpha\right)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda}{on}\int_{\pi/2}^{-\pi/2}\left(-j\sin\alpha + \vec{i}\cos\alpha\right) d\alpha$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 \times on}\left[\cos\alpha\right]_{\pi/2}^{\pi/2}\vec{j} + \left[\sin\alpha\right]_{\pi/2}^{-\pi/2}$$

$$\vec{E} = \frac{1 \times \lambda}{4\pi\varepsilon_0 \times on} \times \left(1-(-1)\right)\vec{i}$$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0\, on}\,\vec{i}}$$

2) Déduisons

On pose $OM = r \quad \Rightarrow \vec{E} = \dfrac{\lambda}{2\pi\varepsilon_0 \times r}\,\vec{i}$

$\vec{E} = -\operatorname{grad} V(r)$

$\vec{E}$ est à une dimension $E = -\dfrac{dV(r)}{dr}$

$$\Rightarrow dV(r) = -E \times dr$$

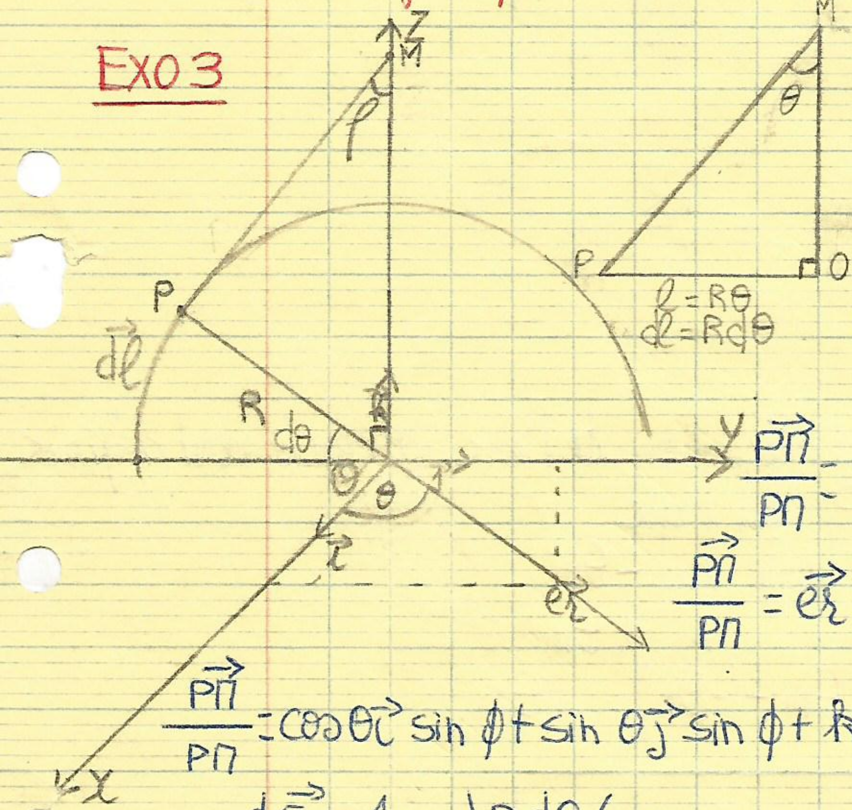
$$V(r) = \int -E \times dr$$

$$V(r) = -\int \frac{\lambda}{2\pi\varepsilon_0 r}\, dr \Rightarrow V(r) = \frac{-\lambda}{2\pi\varepsilon_0 r}\int \frac{dr}{r}$$

$$\boxed{V(r) = \frac{-\lambda}{2\pi\varepsilon_0}\ln r + cte}$$

Remarque : On peut considérer différentes géométries
- un fil infinie de densité $+\lambda$ suivant Oz et $-\lambda$ suivant Oz' (deux demi droite)
- un segment de longueur
- un fil infini incliné d'un angle φ
- un fil infini de densité $-\lambda$

Exo 3



1) Calculons le champ électrostatique $\vec{E}_{tot}(z)$ au point $M(z)$ sur $z'Oz \perp$ au plan (xOy).

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{PM^2} \times \frac{\vec{PM}}{PM}$$

$$\frac{\vec{PM}}{PM} = \frac{\vec{PO}+\vec{OM}}{PM} = \frac{\vec{PO}}{PM} \times \frac{PO}{PO} + \frac{\vec{OM}}{PM} \times \frac{OM}{OM}$$

$$\frac{\vec{PM}}{PM} = \vec{e_r}\sin\phi + \vec{k}\cos\phi \qquad \text{or } \vec{e_r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\frac{\vec{PM}}{PM} = \cos\theta\,\vec{i}\sin\phi + \sin\theta\,\vec{j}\sin\phi + \vec{k}\cos\phi$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,d\theta}{PM^2}\left(\cos\theta\sin\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\phi\,\vec{k}\right)$$

$$PM^2 = PO^2 + OM^2 = R^2 + z^2 \ ; \ \cos\phi = \frac{z}{\sqrt{R^2+z^2}} \ ; \ \sin\phi = \frac{R}{\sqrt{R^2+z^2}}$$

$$\Rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{(R^2+z^2)}\left(\frac{R}{\sqrt{R^2+z^2}}\cos\theta\,\vec{i} + \frac{R}{\sqrt{R^2+z^2}}\sin\theta\,\vec{j} + \frac{z}{\sqrt{R^2+z^2}}\vec{k}\right)$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R^2}{(R^2+z^2)^{3/2}}\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \frac{z}{R}\vec{k}\right)d\theta$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R^2}{(R^2+z^2)}\int\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \frac{z}{R}\vec{k}\right)d\theta$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R^2}{(R^2+z^2)^{3/2}} [\sin\theta]_{-\theta_m}^{\theta_m} \vec{i} + [-\cos\theta]_{-\theta_m}^{\theta_m} \vec{j} + \frac{z}{R} [\theta]_{-\theta_m}^{\theta_m} \vec{k}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R^2}{(R^2+z^2)^{3/2}} [\sin\theta_m - \sin(-\theta_m)] \vec{i} + [\cos\theta_m - \cos(-\theta_m)] \vec{j} + \frac{z}{R} (\theta_m + \theta_m)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R^2}{(R^2+z^2)^{3/2}} (2\sin\theta_m) \vec{i} + \frac{2z}{R} \theta_m \vec{k}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda R^2}{(R^2+z^2)^{3/2}} \left(2\sin\theta_m \vec{i} + \frac{2z}{R} \theta_m \vec{k}\right)$$

Remarque Les bornes d'intégration de l'arc de cercle sont définies par le déplacement de P sur l'arc

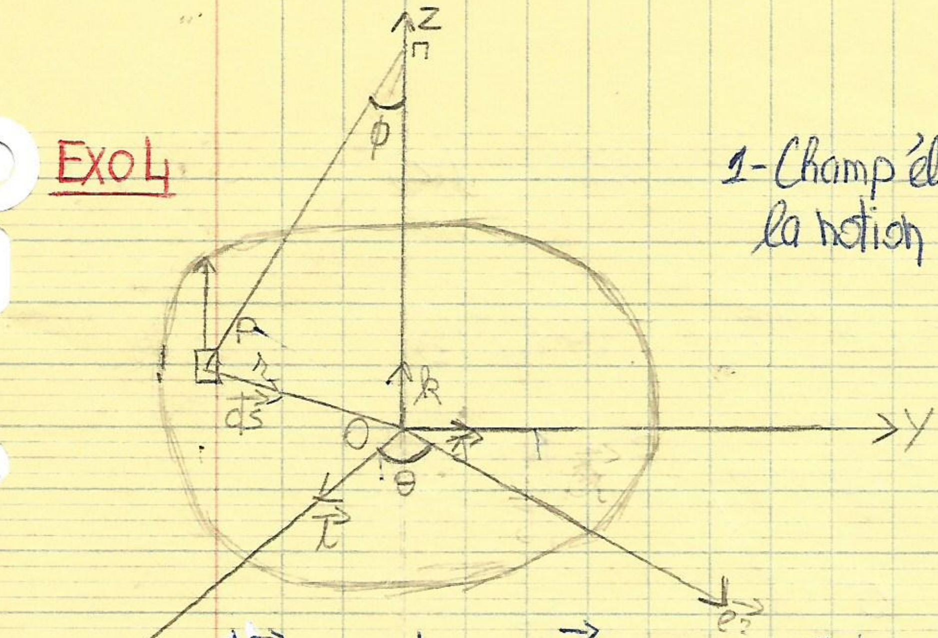
- lorsque P se trouve en A, l'angle limite vaut $-\theta_m$
- " P " " " B, " " " $+\theta_m$
- de A à B, l'angle total balayé est $\theta_m - (-\theta_m) = 2\theta_m$.

$$\boxed{\vec{E} = \frac{\lambda R^2}{2\pi\varepsilon_0 (R^2+z^2)^{3/2}} \sin\theta_m \vec{i} + \frac{z}{R} \theta_m \vec{k}}$$

En déduire le champ

$$\vec{E} = \frac{1}{2\pi\varepsilon_0}$$

Exo 4



1- Champ électrostatique $\vec{E}(z)$ à l'aide de la notion d'angle solide.

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{PM^2}\times\frac{\vec{PM}}{PM} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, ds}{PM^2}\times\frac{\vec{PM}}{PM}$$

$$d\Omega = \frac{\vec{dS}}{PM^2}\cdot\frac{\vec{PM}}{PM} = \frac{ds}{PM^2}\cdot\frac{\vec{PM}}{PM}\vec{k}.\vec{k} = \frac{ds}{PM^2}\cdot\frac{\vec{PM}}{PM}\vec{k}$$

donc $d\vec{E} = \frac{\sigma}{4\pi\varepsilon_0}d\Omega.\vec{k} \Rightarrow \vec{E} = \frac{\sigma}{4\pi\varepsilon_0}d\Omega.\vec{k}$

et $d\Omega = \sin\phi\, d\phi\, d\theta$

$$\Omega = \int_0^{\theta_m}\sin\phi\, d\phi \times \int_0^{2\pi}d\theta = [-\cos\phi]_0^{\theta_m}\times[\theta]_0^{2\pi} = (+\cos\theta_m - 1)\,2\pi$$

$$\Omega = 2\pi(1-\cos\phi)$$

$$E = \frac{\sigma}{4\pi\varepsilon_0}\times 2\pi(1-\phi_m)\vec{k}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}(1-\cos\phi_m)\vec{k} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\vec{k}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\vec{k}$$

$$\vec{E} = \vec{E}(z) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\vec{k} = E(z)\cdot\vec{k}$$

$$\boxed{E(z) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)}$$

2- Déduisons le potentiel électrostatique $V(z)$

$$E(z) = -\frac{dV(z)}{dz} \Rightarrow V(z) = -\int E(z)\,dz$$

$$V(z) = -\int \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) dz + K$$

$$V(z) = \frac{-\sigma z}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} \int \frac{z}{\sqrt{R^2+z^2}}\, dz$$

$$V(z) = \frac{-\sigma}{2\varepsilon_0} z + \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2}\right)$$

$$\boxed{V(z) = \frac{\sigma}{2\varepsilon_0}\left(-z + \sqrt{R^2+z^2}\right)}$$

3- Continuité de V et discontinuité de E en $z=0$.

$$\vec{E}(z=0^-) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{\text{"}0^-\text{"}}{\sqrt{R^2+\text{"}0^-\text{"}}}\right)(-\vec{k}) = -\frac{\sigma}{2\varepsilon_0}\vec{k}$$

$$\vec{E}(z=0^+) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{\text{"}0^+\text{"}}{\sqrt{R^2+\text{"}0^+\text{"}}}\right)+\vec{k} = \frac{\sigma}{2\varepsilon_0}\vec{k}$$

$\vec{E}$ est discontinue en $(z=0)$

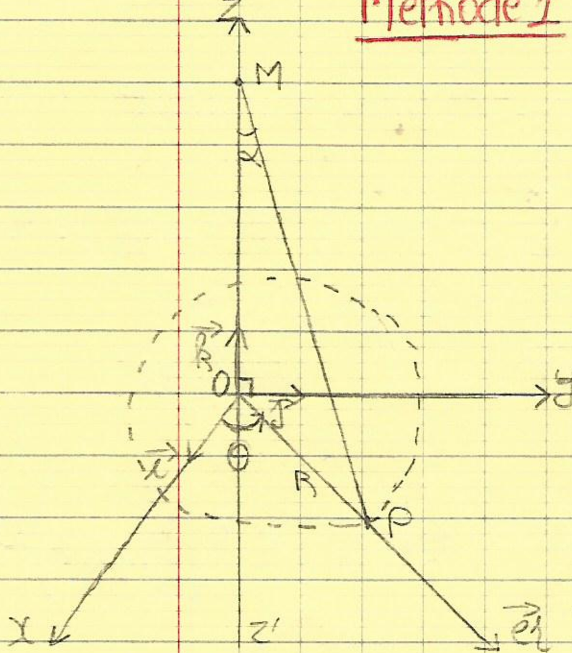
$$V(z=0^-) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+\text{"}0^-\text{"}} - \text{"}0^-\text{"}\right) = \frac{\sigma}{2\varepsilon_0} R$$

$$V(z=0^+) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+\text{"}0^+\text{"}} - \text{"}0^+\text{"}\right) = \frac{\sigma}{2\varepsilon_0} R$$

V est continue

Ex05 1- Champ E à partir du potentiel

Méthode 1



$$dV(M) = \frac{1}{4\pi\varepsilon_0}\frac{dq}{PM} \quad \text{avec } dq = \lambda_0 dl \text{ et } dl = R d\theta$$

$$PM = \sqrt{R^2+z^2}$$

$$dV(M) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R}{\sqrt{z^2+R^2}}\,d\theta$$

$$V(M) = \int_0^{2\pi}\frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R}{\sqrt{z^2+R^2}}\,d\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R}{\sqrt{z^2+R^2}}\int_0^{2\pi}d\theta$$

$$V(M) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R}{\sqrt{z^2+R^2}}\,[\theta]_0^{2\pi} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R}{\sqrt{z^2+R^2}}\,(2\pi-0)$$

$$V(M) = \frac{1}{2\varepsilon_0}\frac{\lambda_0 R}{\sqrt{z^2+R^2}}$$

$$\begin{cases} V(0,0,z) = \dfrac{1}{2\varepsilon_0}\dfrac{\lambda_0 R}{\sqrt{z^2+R^2}} \\ V(0,0,-z) = \dfrac{1}{2\varepsilon_0}\dfrac{\lambda_0 R}{\sqrt{(-z)^2+R^2}} \end{cases} \Rightarrow V(0,0,z)$$

$$\vec{E}(M) = -\vec{\text{grad}}\,V(M) = -\frac{dV(M)}{dz}\vec{k} = -\frac{\lambda}{dz}\left(-\frac{1}{2\varepsilon_0}\frac{\lambda R}{\sqrt{z^2+R^2}}\right)$$

$$\vec{E} = -\frac{\lambda R}{2\varepsilon_0}\frac{d}{dz}\left(-\frac{1}{\sqrt{z^2+R^2}}\right) = -\frac{\lambda_0 R}{2\varepsilon_0}\left(\frac{\frac{-2z}{2\sqrt{z^2+R^2}}}{z^2+R^2}\right)\vec{k} \qquad \boxed{\vec{E}(M) = \frac{\lambda_0 R}{2\varepsilon_0}\frac{z}{(z^2+R^2)^{3/2}}\vec{k}}$$

$$\vec{E}(0,0,z) = \frac{\lambda_0 R}{2\varepsilon_0}\frac{z}{(z^2+R^2)^{3/2}}\vec{k}$$

$$\vec{E}(0,0,-z) = \frac{\lambda_0 R}{2\varepsilon_0}\frac{z}{(z^2+R^2)^{3/2}}\vec{k}$$

$\vec{E}$ est perpendiculaire au plan.

Méthode 2

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{P\Pi^2}\frac{\vec{P\Pi}}{P\Pi}$$ avec $dq = \lambda_0 dl$ et $dl = R d\theta$

$$\frac{\vec{P\Pi}}{P\Pi} = \frac{\vec{PO}+\vec{O\Pi}}{P\Pi} = \frac{PO}{P\Pi}+\frac{O\Pi}{P\Pi} = \frac{\vec{PO}}{PO}\times\frac{PO}{P\Pi}+\frac{\vec{O\Pi}}{O\Pi}\times\frac{O\Pi}{P\Pi} = -\vec{e_r}\sin\alpha + \vec{k}\cos\theta$$

$$\Rightarrow \vec{e_r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \quad \Rightarrow \frac{\vec{P\Pi}}{P\Pi} = -\sin\alpha(\cos\theta\,\vec{i}+\sin\theta\,\vec{j}) + \vec{k}\cos\alpha$$

$$P\Pi^2 = z^2+R^2 \;;\; \sin\alpha = \frac{R}{\sqrt{z^2+R^2}} \quad \text{et} \quad \cos\alpha = \frac{z}{\sqrt{z^2+R^2}}$$

$$d\vec{E} = \frac{\lambda_0}{4\pi\varepsilon_0}\frac{R}{z^2+R^2}\left(-\frac{R}{\sqrt{z^2+R^2}}(\cos\theta\,\vec{i}+\sin\theta\,\vec{j}) + \vec{k}\frac{z}{\sqrt{z^2+R^2}}\right)d\theta$$

$$d\vec{E} = \frac{\lambda_0 R}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\left(-\cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \frac{z}{R}\vec{k}\right)d\theta \,/\, \vec{E} = \frac{\lambda_0 R}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\int_0^{2\pi}\left(-\cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \frac{z}{R}\vec{k}\right)d\theta$$

$$\vec{E} = \frac{\lambda_0 R^2}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\left[-\sin\theta\,\vec{i}\right]_0^{2\pi} + \left[\cos\theta\,\vec{j}\right]_0^{2\pi} + \frac{z}{R}\left[\theta\right]_0^{2\pi}\vec{k}$$

$$\vec{E} = \frac{\lambda_0 R^2}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\left(\frac{z}{R}2\pi\right)\vec{k}$$

$$\boxed{\vec{E} = \frac{\lambda_0 R z}{2\varepsilon_0(z^2+R^2)^{3/2}}\vec{k}}$$

2) Potentiel et champ électrostatique en tout point Π de l'axe de boucle

Potentiel

$$\lambda(P) = \lambda_0\sin\theta \quad \text{où} \quad \theta(Ox, Oy)$$

$$d(V) = \frac{1}{4\pi\varepsilon_0}\frac{dq}{P\Pi}$$ où $\left.\begin{array}{l} dq = \lambda_0\sin\theta\, dl \\ dl = R\,d\theta \end{array}\right\} \Rightarrow dq = \lambda_0 R\sin\theta\, d\theta$

$$d(V) = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R \sin\theta\, d\theta}{P\Pi} \quad \text{où } P\Pi = \sqrt{z^2+R^2}$$

$$\Rightarrow d(V) = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R \sin\theta\, d\theta}{\sqrt{z^2+R^2}}$$

$$V = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R \sin\theta\, d\theta}{\sqrt{z^2+R^2}} = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R}{\sqrt{z^2+R^2}} \int_0^{2\pi} \sin\theta\, d\theta$$

$$V = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R}{\sqrt{z^2+R^2}}\,[-\cos\theta]_0^{2\pi} = \frac{\lambda_0 R}{4\pi\varepsilon_0\sqrt{z^2+R^2}}\,(-\cos 2\pi + \cos 0)$$

$\boxed{V = 0}$ Le potentiel est nul en tout point Π.

<u>Champ</u> $\vec{E}(\Pi)$

$$d\vec{E}(\Pi) = \frac{1}{4\pi\varepsilon_0}\,\frac{dq}{P\Pi^2}\,\frac{\vec{P\Pi}}{P\Pi} \qquad \text{or } \frac{\vec{P\Pi}}{P\Pi} = -\sin\alpha(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) + \vec{k}\cos\alpha$$

$$dq = \lambda_0 R \sin\theta\, d\theta.$$

$$d\vec{E}(\Pi) = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R \sin\theta\, d\theta}{(z^2+R^2)}\left[-\sin\alpha(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) + \vec{k}\cos\alpha\right]$$

$$d\vec{E}(\Pi) = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R \sin\theta\, d\theta}{z^2+R^2} \times \left(\sin\alpha\cos\theta\,\vec{i} + \sin\alpha\sin\theta\,\vec{j} - \vec{k}\cos\alpha\right)$$

$$\sin\alpha = \frac{R}{\sqrt{z^2+R^2}} \; ; \; \cos\alpha = \frac{z}{\sqrt{z^2+R^2}}$$

$$d\vec{E}(\Pi) = -\frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R \sin\theta\, d\theta \times R}{(z^2+R^2)^{3/2}} \times \left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j} - \frac{z}{R}\vec{k}\right)$$

$$\vec{E}(\Pi) = -\frac{1}{4\pi\varepsilon_0}\,\frac{\lambda_0 R^2}{(z^2+R^2)^{3/2}} \int_0^{2\pi} \left(\sin\theta\cos\theta\,\vec{i} + \sin^2\theta - \frac{z}{R}\sin\theta\,\vec{k}\right) d\theta$$

• $\int_0^{2\pi} \sin\theta\cos\theta \, d\theta = \int_0^{2\pi} \frac{\sin 2\theta}{2} \, d\theta = \frac{1}{2}\times\frac{1}{2}\left[\cos 2\theta\right]_0^{2\pi} = 0$

• $\int_0^{2\pi} \sin\theta \, d\theta = \left[-\cos\theta\right]_0^{2\pi} = 0$

• $\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0 \left(\frac{1-\cos 2\theta}{2}\right) d\theta = \frac{1}{2}\times 2\pi - \frac{1}{4}\left[\sin 2\theta\right]_0^{2\pi}$

$$\vec{E}(M) = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R^2}{(z^2+R^2)^{3/2}} \times \pi \, \vec{j} = \boxed{-\frac{\lambda_0 R^2}{4\varepsilon_0}\frac{1}{(z^2+R^2)^{3/2}}\vec{j} = \vec{E}(M)}$$

$\vec{E}(0,0,z) = -\frac{\lambda_0 R^2}{4\pi\varepsilon_0 (z^2+R^2)^{3/2}}\vec{j}$ Rq Le plan (xOy) est un plan de symétrie

$\vec{E}(0,0,-z) = -\frac{\lambda_0 R^2}{4\pi\varepsilon_0 (z^2+R^2)^{3/2}}\vec{j}$ Le champ est parallèle au plan de symétrie.

1er Cas • $\vec{E}(M) = -\vec{E}(M')$ ⇒ quand le champ est normal au plan de symétrie

2e Cas • $\vec{E}(M) = \vec{E}(M')$ ⇒ si le champ est parallèle au plan de symétrie

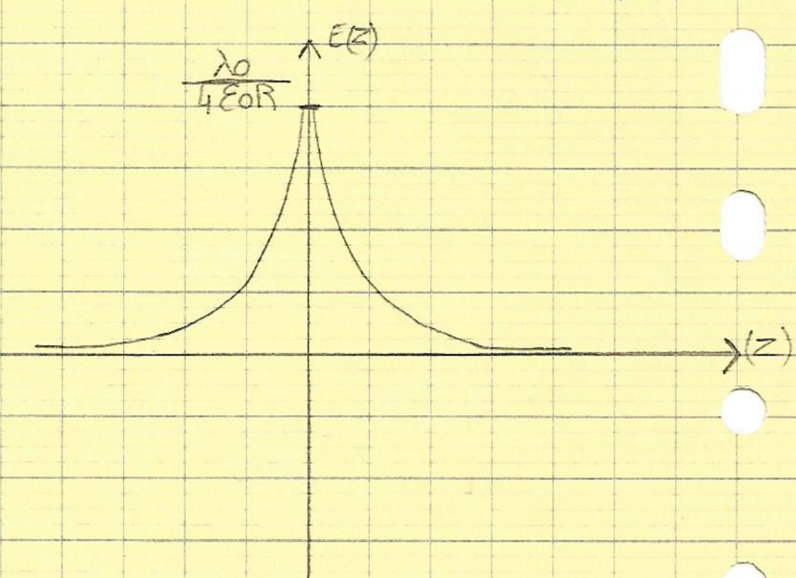
3)

$$\vec{E} = -\frac{\lambda_0 R^2}{4\varepsilon_0 (z^2+R^2)^{3/2}}\vec{j}$$

Pour $|z| \gg R$

$\|\vec{E}\| = \frac{\lambda_0 R^2}{4\varepsilon_0 (z^2)^{3/2}}$

$z \to \infty$ donc $E \to 0$

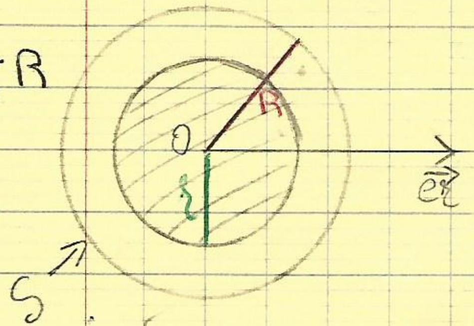


Pour $|z| \ll R$

$\|\vec{E}\| = \frac{\lambda_0 R^2}{4\varepsilon_0 (R^2+z^2)^{3/2}} = \frac{\lambda_0 R^2}{4\varepsilon_0 R^3} = \frac{\lambda_0}{4\varepsilon_0 R}$

Exo6 Champ et potentiel électrostatiques $E(r)$ et $V(r)$, en tout point $M(r)$ de son axe $z'Oz$

1er Cas : $r < R$



$S_G = 4\pi r^2$

$V_{charge\,S_G} = \rho \frac{4}{3} \pi r^3$

1) Champ $E_1(r)$ et Potentiel $V_1(r)$

Théorème de Gauss

$$E(r) \times S_G = \frac{\sum Q_{int}}{\varepsilon_0}$$

$$\Rightarrow E_1(r) \times S_G = \frac{\sum Q_{int}}{\varepsilon_0} = E_1(r) \times 4\pi r^2 = \rho \frac{4\pi}{3} r^3 \times \frac{1}{\varepsilon_0}$$

$$\boxed{E_1(r) = \rho \frac{r}{3\varepsilon_0}}$$

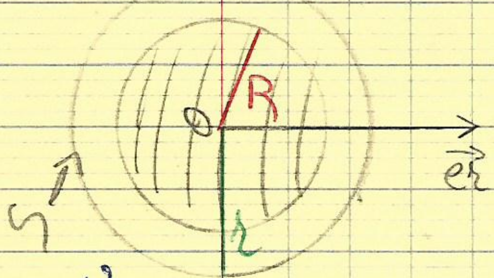
$$V_1(r) \Rightarrow E_1(r) = -\frac{dV(r)}{dr} \Rightarrow -dV_1(r) = E_1(r) \times dr$$

$$V_1(r) = -\int E_1(r) \times dr = -\int \rho \frac{r}{3\varepsilon_0}\, dr = -\frac{\rho}{3\varepsilon_0} \int r.dr$$

$$V_1(r) = -\frac{\rho}{3\varepsilon_0} \times \frac{1}{2} r^2 + K_1$$

$$\boxed{V_1(r) = -\frac{\rho}{6\varepsilon_0} r^2 + K_1}$$

2e Cas : $r > R$



$S_G = 4\pi r^2$

$V = \frac{4}{3}\pi R^3$

2) Champ $E_2(r)$ et Potentiel $V_2(r)$

$$E_2(r) \times S_G = \frac{\Sigma Q_{int}}{\varepsilon_0}$$

$$E_2(r) \times 4\pi r^2 = \rho \frac{4\pi R^3}{3} \times \frac{1}{\varepsilon_0}$$

$$\boxed{E_2(r) = \rho \frac{R^3}{3 r^2 \varepsilon_0}}$$

$$V_2(r) = -\frac{dV_2(r)}{dr} \Rightarrow -dV_2(r) = E_2(r).dr$$

$$V_2(r) = -\int E_2(r).dr = -\int \rho \frac{R^3}{3r^2 \varepsilon_0} dr = -\rho \frac{R^3}{3\varepsilon_0} \int \frac{dr}{r^2}$$

$$V_2(r) = -\rho \frac{R^3}{3\varepsilon_0}\left(-\frac{1}{r}\right)$$

$$\boxed{V_2(r) = \rho \frac{R^3}{3 r \varepsilon_0}} + k_2$$

Rq : Une charge ponctuelle Q crée en un point de l'espace M un champ et ce dernier sous la forme $E = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2}$

Une distribution de charge à symétrie sphérique produit à l'extérieur le champ qu'une charge ponctuelle placée en un point O donné.

$$\frac{Q}{4\pi\varepsilon_0} \times \frac{1}{r^2} \quad / \quad E(r) = \rho \frac{R^3}{3\varepsilon_0} \cdot \frac{1}{r^2} = \rho \frac{R^3 4\pi}{3\varepsilon_0} \cdot \frac{1}{r^2} \quad \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2}$$

$E(r) = \rho\, 4$

Le potentiel $V(r)$ est continu à la traversée d'une surface chargée

2. a) Calculons la force F_{AB}

force de Laplace : $d\vec{F} = I\,d\vec{l} \wedge \vec{B}$.

$$d\vec{F}_{AB} = I'\,d\vec{l} \wedge \vec{B}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi c}\vec{j}$$

$$d\vec{F}_{AB} = I'\,dl \wedge \frac{\mu_0 I}{2\pi c}\vec{j}$$

$$d\vec{F}_{AB} = I'\,dl \wedge \frac{\mu_0 I}{2\pi c}\vec{j}$$

$$= I'\,dl\,\vec{k} \wedge \frac{\mu_0 I}{2\pi c}\vec{j}$$

$$\frac{II'\mu_0}{c\,2\pi}\,dl\,\vec{k}\wedge\vec{j}$$

$$dF_{AB} = \frac{II'}{c}\frac{\mu_0}{2\pi}(-\vec{i})\int_A^B dl$$

$$\boxed{\vec{F}_{AB} = \frac{II'}{c}\frac{\mu_0}{2\pi}a(-\vec{i})}$$

Attraction du cadre sur son côté AB

b) Calcul de $\vec{F}_{CD}$

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{c+b}\vec{j}$$

$$d\vec{F}_{CB} = I\,d\vec{l}(-\vec{k}) \wedge \left(\frac{\mu_0 I}{2\pi}\frac{1}{c+b}\vec{j}\right)$$

$$d\vec{F}_{CB} = \frac{II'\mu_0}{2\pi}\frac{1}{c+b}\,dl\,\vec{i}$$

$$\vec{F}_{CD} = \frac{II'\mu_0}{2\pi(c+b)}\vec{i}\int dl$$

$$\boxed{\vec{F}_{CD} = \frac{II'\mu_0}{2\pi(c+b)}a\,\vec{i}}$$

Répulsion du cadre sur le côté CD

c) Calcul de $\vec{F}_{BC}$

$$\vec{B} = \frac{\mu_0 I}{2\pi x}\vec{j}$$

$$d\vec{F}_{BC} = I'dx(\vec{i}) \wedge \left(\frac{\mu_0}{2\pi}\frac{I}{x}\vec{j}\right)$$

$$= \frac{I'I}{2\pi x}\mu_0\,dx\,\vec{i}\wedge\vec{j}$$

$$= \frac{I'I}{2\pi x}\mu_0\,dx\,\vec{k}$$

$$= \frac{I'I}{2\pi x}\mu_0\,\vec{k}\int\frac{1}{x}dx$$

$$= \frac{I'I}{2\pi}\mu_0\,\vec{k}\,[\ln x]_c^{c+b}$$

$$\boxed{F_{BC} = \frac{I'I}{2\pi}\mu_0\,\vec{k}\,\ln\left(\frac{c+b}{c}\right)}$$

d) Calcul de $\vec{F}_{AD}$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{x}\vec{j}$$

$$d\vec{F}_{AD} = I'dx(\vec{i}) \wedge \left(\frac{\mu_0 I}{2\pi}\frac{1}{x}\right)\vec{j}$$

$$= \frac{II'}{2\pi x}\mu_0\,dx\,\vec{i}\wedge\vec{j}.$$

$$= \frac{II'}{2\pi x}\mu_0\,dx\,\vec{k} = \frac{II'\mu_0\vec{k}}{2\pi}\int_c^{c+b}\frac{1}{x}dx$$

$$= \frac{II'}{2\pi}\mu_0\,\vec{k}\int_c^{c+b}\frac{1}{x}dx$$

$$= \frac{II'}{2\pi}\mu_0\,\vec{k}\,[\ln x]_c^{c+b}$$

$$\boxed{F_{AB} = \frac{II'}{2\pi}\mu_0\,\vec{k}\,\ln\left(\frac{c+b}{c}\right)}$$

$$F_{DA} = -F_{AD}$$

2) Calcul de la résultante

$$\vec{F} = \vec{F}_{AB} + \vec{F}_{CD} + \vec{F}_{BC} + \vec{F}_{DA}$$

$$F = \frac{II'}{c}\frac{\mu_0}{2\pi} a(\vec{i}) + \frac{II'}{2\pi}\frac{\mu_0}{(c+b)} a(\vec{i})$$

$$+ \frac{II'}{2\pi}\mu_0 \vec{k} \ln\left(\frac{c+b}{c}\right) -$$

$$\frac{II'}{2\pi}\mu_0 \vec{k} \ln\left(\frac{c+b}{c}\right)$$

$$\vec{F} = \frac{\mu_0 II'}{2\pi} a \left[-\frac{1}{c} + \frac{1}{(c+b)}\right](\vec{i})$$

$$\vec{F} = \frac{\mu_0 II'}{2\pi} a \left[\frac{-\cancel{c}-b+c}{c(c+b)}\right](\vec{i})$$

$$\vec{F} = \frac{\mu_0 II'}{2\pi} a \left[\frac{-b}{c(c+b)}\right](\vec{i})$$

$$\boxed{\vec{F} = -\frac{\mu_0 II'}{2\pi} a \left(\frac{b}{c(c+b)}\right)}$$

La force résultante est attractive

3°) Calcul du flux magnétique

$$\Phi = \int \vec{B}\, d\vec{S}$$

$$\Rightarrow \begin{cases} B = \frac{\mu_0}{2\pi}\frac{I}{x}\vec{j} \\ d\vec{S} = a.dx\,\vec{j} \end{cases}$$

$$\Phi = \int_c^{c+b} \frac{\mu_0}{2\pi}\frac{I}{x} a\,dx$$

$$\Phi = \frac{\mu_0 I}{2\pi} a \int_c^{c+b} \frac{1}{x} dx$$

$$\Phi = \frac{\mu_0 I}{2\pi} a \left[\ln x\right]_c^{c+b}$$

$$\boxed{\Phi = \frac{\mu_0 I}{2\pi} a \ln\left(\frac{c+b}{c}\right)}$$

Evaluons k_1 et k_2

Le potentiel est nul à l'infinie pour $V_2(r)$

$r < R$, $r > R$

$V_1(R) = V_2(R)$

$R = cte$ / r = varie

Potentiel de référence

$r \to \infty$; $V_2(r \to \infty) + k_2 = 0$

$$\frac{\rho R^3}{3 r \varepsilon_0}\Big|_{r \to +\infty} + k_2 = 0 \Rightarrow k_2 = 0$$

On a par ailleurs $V_2(R) = V_1(R)$

$$\frac{\rho R^3}{3 R \varepsilon_0} = -\frac{\rho R^2}{6 \varepsilon_0} + k_1$$

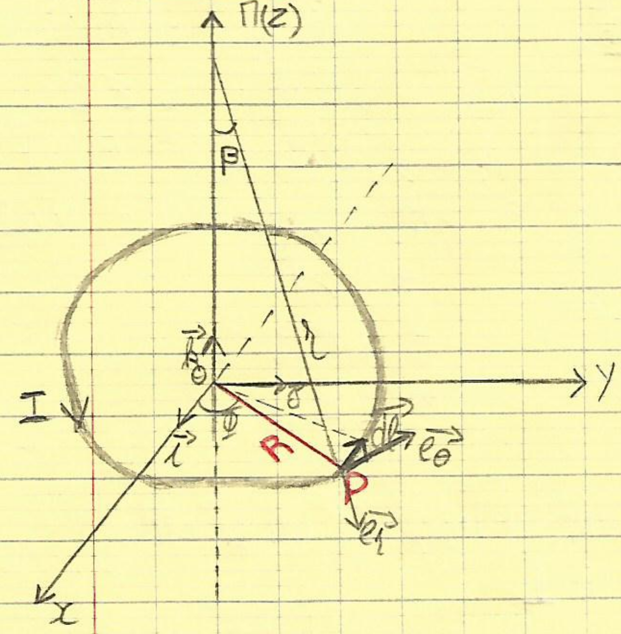
$$k_1 = \frac{\rho R^2}{3 \varepsilon_0} + \frac{\rho R^2}{6 \varepsilon_0} = \rho \frac{R^2}{3 \varepsilon_0}\left(1 + \frac{1}{2}\right)$$

$$\boxed{k_1 = \frac{\rho R^2}{2 \varepsilon_0}}$$

$$\boxed{V_1(r) = -\frac{\rho r^2}{6 \varepsilon_0} + \frac{\rho R^2}{2 \varepsilon_0} = \frac{\rho}{2 \varepsilon_0}\left(R^2 - \frac{r^2}{3}\right)}$$

TAF : Variation de E en fonction de r.

Exo 4



Calcul du champ magnétique $\vec{B}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\wedge\vec{r}}{r^2\, r} = \frac{\mu_0 I}{4\pi} d\vec{l}\wedge\frac{\vec{PM}}{PM}$$

$$\frac{\vec{PM}}{PM} = \frac{\vec{PO}+\vec{OM}}{PM} = \frac{PO}{PO}\times\frac{PO}{PM} + \frac{OM}{OM}\times\frac{OM}{PM}$$

$$\frac{\vec{PM}}{PM} = -\vec{e_r}\sin\beta + \vec{k}\cos\beta$$

$$d\vec{l} = R\,d\theta\,\vec{e_\theta}$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{R}{PM^2}\, d\theta\, \vec{e_\theta}\wedge\left(-\vec{e_r}\sin\beta + \vec{k}\cos\beta\right) = \frac{\mu_0 I}{4\pi}\frac{R\,d\theta}{PM^2}\left[\vec{k}\sin\beta + \vec{e_r}\cos\beta\right]$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi}\frac{R}{PM^2}\int(\vec{k}\sin\beta + \vec{e_r}\cos\beta)\,d\beta$$

$$\vec{B} = \frac{\mu_0 I R}{4\pi\, PM^2}\left[\vec{k}\sin\beta\right]_0^{2\pi} d\varphi + \cos\beta\int\underbrace{\vec{e_r}\,d\varphi}_{=0} = \frac{\mu_0}{4\pi}\frac{IR}{PM^2}\vec{k}\sin\beta\left[\varphi\right]_0^{2\pi}$$

$$\vec{B} = \frac{\mu_0}{2}\frac{IR}{PM^2}\sin\beta\,\vec{k}$$

et $PM^2 = r^2$ $\quad \vec{B} = \frac{\mu_0}{2}\frac{IR}{r^2}\sin\beta\,\vec{k}$

avec $r^2 = R^2 + z^2$.

$z \gg R \rightarrow \vec{B} = \frac{\mu_0 I}{2}\frac{R}{z^2+R^2}\frac{R}{r}\vec{k}$	$z \ll R \longrightarrow \vec{B} = \frac{\mu_0 I}{2}$
$r = \sqrt{R^2+z^2}$	
$\vec{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\vec{k} = \frac{\mu_0 I}{2}\frac{R^2}{z^3}\vec{k}$	$\vec{B} = \frac{\mu_0 I}{2}\frac{R^2}{R^3}\vec{k} = \frac{\mu_0 I}{2R}\vec{k}$

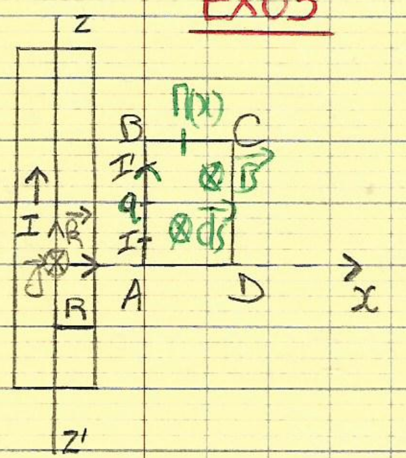
si $z = 0$ $\quad z \ll R$

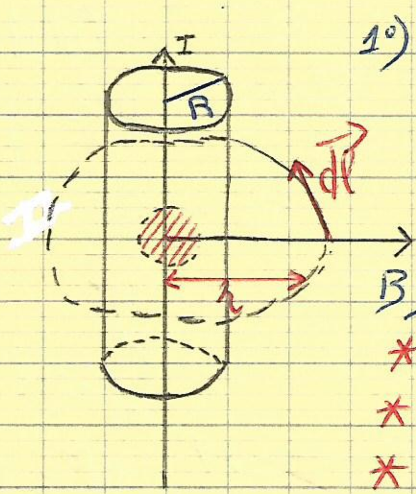
$z = \infty$ $\quad z \gg R$

$\vec{e_\theta}\wedge\vec{e_r} = \vec{k}$

$\vec{e_\theta}\wedge\vec{k} = \vec{e_r}$

Ex05





1°) Champ magnétique $B(r)$

Théorème d'Ampère

$\oint \vec{B}.\vec{dl} = \mu_0 \sum I$

$\oint \vec{B}.\vec{dl} = ?$ $\quad B\oint \vec{dl}$

$B\oint dl = B 2\pi r = \mu_0 I.$

* $r > R$
* $r < R$
* $r = r$

* 1er Cas : $r > R$ (le courant est intérieur au contour)

$\sum_i I_i = I$ $\quad B\oint dl = B 2\pi r = \mu_0 I$

$\Rightarrow B = \dfrac{\mu_0 I}{2\pi r}$

* 2e Cas : $r < R$

$\sum_i I_i = I_{int}$ $\quad$ densité du courant

$J = \dfrac{I}{\pi R^2} \Rightarrow I = J\pi R^2$

$I_{int} = J\pi r^2$

$\dfrac{I_{int}}{I} = \dfrac{J\pi r^2}{J\pi R^2} = \dfrac{r^2}{R^2}$

$I_{int} = \dfrac{r^2}{R^2} I$

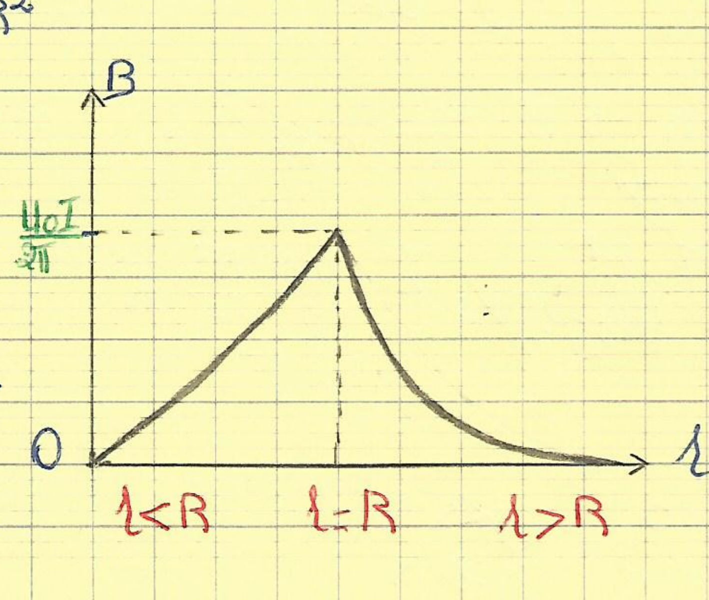
$B\oint dl = \mu_0 I_{int} = B 2\pi r$

$B = \dfrac{\mu_0 I_{int}}{2\pi r} = \dfrac{\mu_0 r^2/R^2 \, I}{2\pi r} = \dfrac{\mu_0 r I}{2\pi R^2}$

$B = \dfrac{\mu_0 r I}{2\pi R^2}$

* 3e Cas : $r = R$

$B(r = R) = \dfrac{\mu_0 I}{2\pi R}$



Par raison de symétrie, le champ magnétique de (D1) et (D2) est :

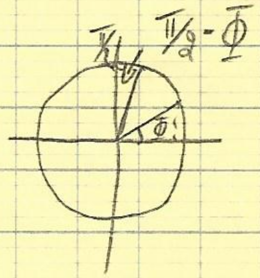
$\vec{B} = \vec{B_1} + \vec{B_2} = 2\vec{B_1} = 2\vec{B_2}$.

$$\vec{B} = 2 \times \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}(1-\cos\Phi)\vec{j}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{x\sin\Phi}(1-\cos\Phi)\vec{j}}$$

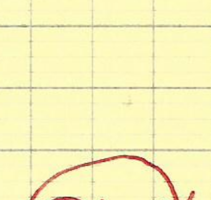
$$\Phi + \alpha + \frac{\pi}{2} = \pi \Rightarrow \alpha = \pi - \frac{\pi}{2} - \Phi$$

$$\alpha = \pi/2 - \Phi$$



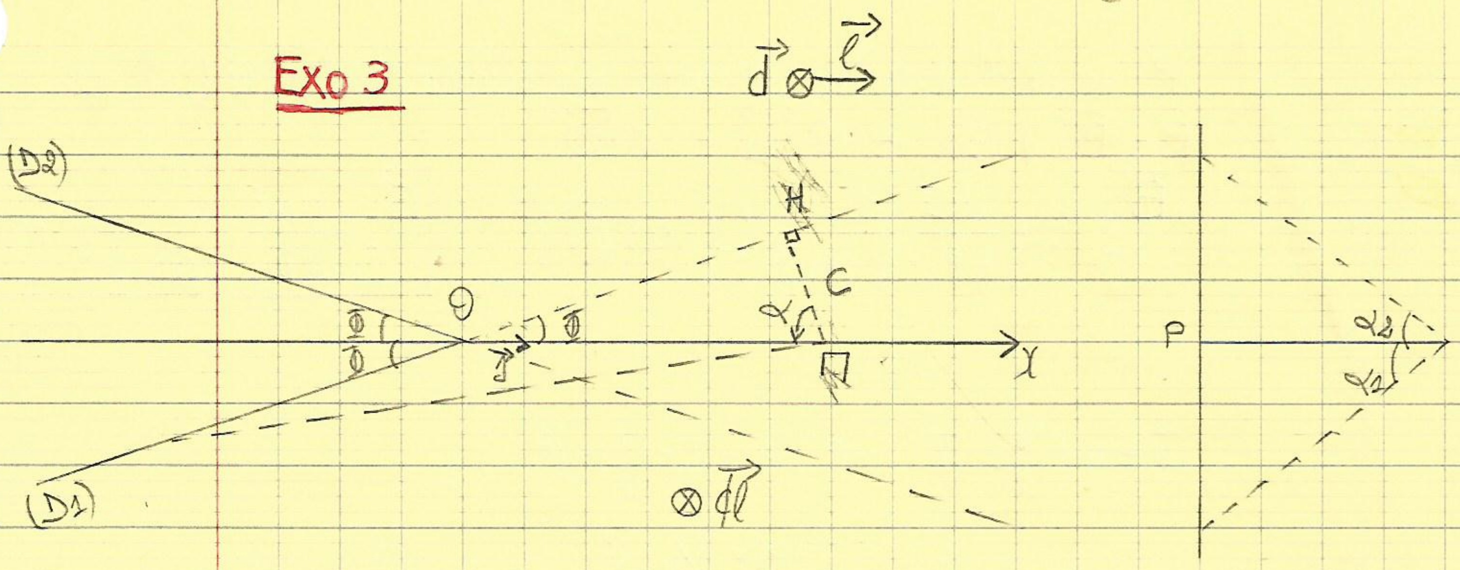
$$\boxed{\begin{array}{l}\cos^2 x + \sin^2 x = 1 \\ \cos^2 x - \sin^2 x = \cos 2x\end{array}}$$

$$\sin\left(\frac{\pi}{2} - \Phi\right) = \cos\Phi$$



$c = HM$

Exo 3



Calcul du champ magnétique $\vec{B_1}$ créé par (D1)

$$d\vec{B_1} = \frac{\mu_0 I}{4\pi}\frac{\vec{dl}\wedge\vec{r}}{r^2\, r} \longrightarrow dB = \frac{\mu_0 I}{4\pi}\cos\alpha\, d\alpha\, \vec{j}$$

Pour notre cas $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{1}{HM}\cos\alpha\, d\alpha\, \vec{j}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}\cos\alpha\, d\alpha\, \vec{j} \longrightarrow B_1 = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}\int\cos\alpha . d\alpha\, \vec{j}$$

$P = -\infty \rightarrow \alpha = -\pi/2$

$P = 0 \rightarrow \alpha = -\left(\frac{\pi}{2} - \Phi\right)$

$$B_1 = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}\int_{-\pi/2}^{-(\pi/2-\Phi)}\cos\alpha\, d\alpha\, \vec{j}$$

$$B_1 = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}\left[\sin\alpha\right]_{-\pi/2}^{-(\pi/2-\Phi)}\vec{j} = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}\left[-\sin(\pi/2-\Phi) - \sin(-\pi/2)\right]$$

$$B_1 = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}(-\cos\Phi + 1)\vec{j}$$

$$B_1 = \frac{\mu_0 I}{4\pi}\frac{1}{x\sin\Phi}(1-\cos\Phi)\vec{j} \qquad \vec{B_1} = \frac{\mu_0 I}{4\pi x}\tan\frac{\Phi}{2}\vec{j}$$

Exo + Donner les caractéristiques du champ électrostatique au centre du triangle ci-après