Cheat Sheet

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Part I

Definitions

6 The Riemann-Stieltjes Integral

Definition 6.1 (Partition)**.** A partition P of $[a, b]$ is:

$$P = \{x_0 = a, \cdots, x_n = b\}$$

such that $x_{i-1} \leq x_i$.

Definition 6.2 (Upper Riemann Integral)**.**

$$\overline{\int_a^b} f\, dx = \inf_P U(P, f),$$

over all partitions P of $[a, b]$.

Definition 6.2 (Riemann-Stieltjes Integral)**.** Suppose

1. $\alpha\colon [a, b] \to \mathbb{R}$ is monotonically increasing.
2. $f\colon [a, b] \to \mathbb{R}$.

We say f is Riemann-Stieltjes integrable on $[a, b]$ (i.e. $f \in \mathcal{R}_\alpha[a, b]$) if:

$$\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha.$$

Definition 6.3 (Refinement)**.** Let P_1, P_2, P^* be partitions of $[a, b]$.

1. P^* is a refinement of P_1 if $P_1 \subset P^*$.
2. P^* is the common refinement of P_1 and P_2 if $P^* = P_1 \cup P_2$.

Definition 6.14 (Step Function)**.**

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{else.} \end{cases}$$

Definition 6.23 (Improper Integral)**.** Suppose $f\colon [a, b] \to \mathbb{R} \in \mathcal{R}_\alpha[a, b]\ \forall b > a$. The improper integral is defined as:

$$\int_a^\infty f(x) dx = \lim_{b \to \infty} \int_a^b f(x) dx.$$

7 Sequences and Series of Functions

Definition 7.7 (Uniform Convergence)**.** For functions $f_n\colon E \to \mathbb{R}$, A sequence of functions $\{f_n\}$ converges uniformly to f if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall x \in E,\ n > N, |f_n(x) - f(x)| < \varepsilon.$$

Definition 7.9 (Uniform Convergence for Series)**.** The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly if for partial sums

$$S_n = \sum_{k=1}^{n} f_k(x),$$

$\{S_n\}$ converges uniformly.

Definition 7.14 (Supremum Norm)**.** Let $f \in \mathcal{C}(X)$, where $\mathcal{C}(X) = \{f\colon X \to \mathbb{R}\colon f \text{ is bounded and continuous on } X\}$. The supremum (or infinity) norm of f is

$$\|f\| = \sup\{f(x)\colon x \in X\}.$$

Definition 7.19 (Pointwise Bounded)**.** A sequence of functions $\{f_n\}$ (with $f_n\colon E \to \mathbb{R}$) is pointwise bounded if $\exists \phi\colon E \to \mathbb{R}$ s.t. $|f_n(x)| < \phi(x)\ \forall x \in E$, $n \in \mathbb{N}$.

Definition 7.20 (Uniform Bounded)**.** $\{f_n\}$ is uniform bounded if the sequence is pointwise bounded for some $\phi(x) = M\ \forall x \in E$.

Definition 7.22 (Equicontinuity)**.** A set of functions $\mathcal{F}$ (with $f\colon E \to \mathbb{R}$ for $f \in \mathcal{F}$) is equicontinuous on E if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall x, y \in E,\ f \in \mathcal{F}, d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon$$

Definition 7.28 (Algebra)**.** An algebra $\mathcal{A}$ satisfies: $\forall f, g \in \mathcal{A}$,

1. $f + g \in \mathcal{A}$,
2. $cf \in \mathcal{A}$ for $c \in \mathbb{R}$,
3. $fg \in \mathcal{A}$.

A set of functions which are closed under addition, scalar multiplication, and regular multiplication.

Definition 7.29 (Uniform Closure)**.**

$$\overline{\mathcal{A}} = \{f\colon E \to \mathbb{R} \mid \exists \{f_n\} \subset \mathcal{A} \text{ s.t. } f_n \to f \text{ uniformly.}\}$$

The set of functions that an algebra can uniformly approximate. Or the set of limit points of an algebra w.r.t. the supremum norm.

Definition 7.30 (Separates points)**.** An algebra $\mathcal{A}$ on X separates points if

$$\forall x, y \in X \text{ s.t. } x \neq y, \exists f \in \mathcal{A} \text{ s.t. } f(x) \neq f(y).$$

(We can choose functions from the algebra to make an injection)

Definition 7.31 (Vanishes at no point)**.** An algebra $\mathcal{A}$ on X vanishes at no point if

$$\forall x \in X\, \exists f \in \mathcal{A} \text{ s.t. } f(x) \neq 0.$$

(Every point is non-zero at least for one function)

8 Power Series, etc.

Definition 8.1 (Exponential function)**.** For $z \in \mathbb{C}$,

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Note that we can choose any z since the series has infinite radius of convergence.

Definition 8.2 (Logarithm)**.** Define $\log\colon (0, \infty) \to \mathbb{R}$ such that

$$\log\left(\exp(x)\right) = x \qquad \text{(For } x \in \mathbb{R}\text{)}$$

and

$$\exp\left(\log(y)\right) = y. \qquad \text{(For } y > 0\text{)}$$

Definition 8.3 (Inner product of functions)**.** For $f, g\colon [a, b] \to \mathbb{C}$ where $f, g \in \mathcal{R}[a, b]$, the inner product is defined as

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

Note that a metric defined with the inner product is valid, but only on $\mathcal{C}([a, b])$.

Definition 8.4 (Orthogonality of functions)**.** Normal definition of orthogonality w.r.t. our definition of the inner product between functions.

A set of functions is orthogonal if all pairs of distinct functions are orthogonal.

Definition 8.5 (Fourier basis)**.** $\{\phi_n\}$ is the Fourier basis where

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

Definition 8.6 (Fourier coefficient)**.** For $f \in \mathcal{R}$, and an orthonormal basis $\{\phi_n\}$, the nth Fourier coefficient c_n is defined as:

$$c_n = \langle f, \phi_n \rangle.$$

Definition 8.7 (Fourier series)**.** The fourier series of $f \in \mathcal{R}$ is:

$$\sum_{n=-\infty}^{\infty} c_n \phi_n$$

Definition 8.8 (Inner product between sequences)**.** Let $\{c_n\}_{n\in\mathbb{Z}}$ and let $\{\gamma_n\}_{n\in\mathbb{Z}}$. Then

$$(c_n, y_n) = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma}_n$$

9 Functions of Multiple Variables

Definition 9.11 (Differentiability of multi-variable function.)**.** Let $E \subset \mathbb{R}^n$ be open, and let $f\colon E \to \mathbb{R}^m$ be differentiable on E. f is differentiable at x if $\exists$ a linear map $A\colon E \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} \to 0.$$

Definition 9.16 (Partial derivative)**.** For $f\colon \mathbb{R}^n \to \mathbb{R}^m$,

$$\frac{\partial f_j}{\partial x_i}(x) = \lim_{h \to \infty} \frac{f_j(x + e_i h) - f_j(x)}{h}.$$

Definition 9.17 (Gradient)**.** Applies to $f\colon \mathbb{E} \to \mathbb{R}$ where $E \subset \mathbb{R}^n$.

$$\nabla f(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) e_i.$$

Part II

Theorems

6 The Riemann-Stieltjes Integral

Theorem 6.1 (Riemann Integrability). *$f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if*

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx.$$

Theorem 6.4 (Refinement Improves Integral). *If P^* refines P, then*

$$L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha).$$

Theorem 6.5 (Upper RS Integral Bounds Lower RS Integral).

$$\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha.$$

Theorem 6.6 (Characterization of RS Integrability). *The following are equivalent:*

1. $f \in \mathcal{R}_\alpha[a,b]$,
2. $\forall \varepsilon > 0\ \exists P_\varepsilon$ *such that* $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$.

Theorem 6.8 (Continuous f Entails Integrability). *Suppose*

1. $f\colon [a,b] \to \mathbb{R} \in \mathcal{C}_0[a,b]$,
2. $\alpha\colon [a,b] \to \mathbb{R}$ *is monotonically increasing.*

Then $f \in \mathcal{R}_\alpha[a,b]$.

Theorem 6.9 (Continuous α Entails Integrability). *Suppose*

1. f *is monotonically increasing.*
2. $\alpha \in \mathcal{C}_0[a,b]$,

Then $f \in \mathcal{R}_\alpha[a,b]$.

Theorem 6.10 (Continuous f or α Entails Integrability). *Let $S = \{s_1, \cdots, s_k\} \subset [a,b]$ be finitely many points.*

1. f *is continuous except for at* $s_i \in S$.
2. α *is monotonically increasing, and continuous at each* s_i.

Then $f \in \mathcal{R}_\alpha[a,b]$.

Theorem 6.11 (Continuous Mapping On Function Preserves Integrability)**.** *Suppose*

1. $f \in \mathcal{R}_\alpha[a,b]$,
2. $m \leq f(x) \leq M$ *for* $a \leq x \leq b$,
3. $\phi\colon [m,M] \to \mathbb{R} \in \mathcal{C}_0$,

Then $\phi \circ f \in \mathcal{R}_\alpha[a,b]$.

Theorem 6.12 (Linearity and Related Properties)**.** *Suppose* $f, f_1, f_2 \in \mathcal{R}_\alpha[a,b]$.

1. *Linearity of* f.
   (a) $f_1 + f_2 \in \mathcal{R}_\alpha[a,b]$.
   (b) $cf \in \mathcal{R}_\alpha[a,b]$ *(for* $c \in \mathcal{R}$*).*
2. *Integral preserves order.*
   - *If* $f_1 \leq f_2$ *on* $[a,b]$, *then*

$$\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$$

3. *Limits of integration can be split.*
   - $f \in \mathcal{R}_\alpha[a,c]$ *and* $f \in \mathcal{R}_\alpha[c,b]$. *Particularly,*

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha. \qquad \text{(Where } c \in (a,b))$$

4. *Upper bound on integral.*

$$\left| \int_a^b f\, d\alpha \right| \leq M(\alpha(b) - \alpha(a)) \qquad \text{(Where } M = \sup\{f(x)\colon a \leq x \leq b\})$$

5. *Linearity of* α.
   - *Suppose* $f \in \mathcal{R}_{\alpha_1}$ *and* $f \in \mathcal{R}_{\alpha_2}$.
   - *Then* $f \in \mathcal{R}_{\alpha_1+\alpha_2}$ *and*

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2.$$

Theorem 6.13 (Triangle inequality)**.** *For $f \in \mathcal{R}_\alpha[a,b]$,*

$$\left| \int_a^b f\, d\alpha \right| \leq \int_a^b |f| d\alpha.$$

Theorem 6.14 (Product of two integrable functions is integrable)**.** *If $f, g \in \mathcal{R}_\alpha[a,b]$, then $fg \in \mathcal{R}_\alpha[a,b]$.*

Theorem 6.15 (Integrator step at continuous point extracts value)**.** *Suppose*

1. *f is bounded,*
2. *f is continuous,*
3. *$s \in (a,b)$,*
4. *$\alpha(x) = I(x-s)$.*

Then,

$$\int_a^b f\, d\alpha = f(s).$$

Theorem 6.16 (Infinite step integrator)**.** *Suppose*

1. *$\alpha(x) = \sum_{i=1}^{\infty} c_i I(x - s_i)$*
   - *Where $c_n \geq 0$ such that $\sum_{i=1}^{\infty} c_i < \infty$,*
   - *and $\{s_i\}$ are distinct points,*
2. *f is continuous.*

Then

$$\int_a^b f\, d\alpha = \sum_{i=1}^{\infty} c_i f(s_i).$$

Theorem 6.17 (Explicit Form of Integrator)**.** *Suppose*

1. *$\alpha \in C_1[a,b]$ and monotone.*
2. *$\alpha' \in \mathcal{R}_\alpha[a,b]$.*

Then,

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x) dx$$

Theorem 6.19 (Change of Variables). *Suppose*

1. ϕ *is*
   (a) *Strictly increasing on* $[a, b]$,
   (b) *Continuous on* $[a, b]$,
   (c) *Onto on* $[a, b]$.
2. $f \in \mathcal{R}_\alpha[a, b]$.

Then,

$$\int_a^b f\, d\alpha = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} (f \circ \phi)\, d(\alpha \circ \phi).$$

Theorem 6.20 (Continuity and Differentiability of Antiderivative). *Let* $f \in \mathcal{R}_\alpha[a, b]$ *with* $x \in [a, b]$. *If we define*

$$F(x) = \int_a^x f(t)dt,$$

then

1. F *is continuous on* $[a, b]$,
2. *For* x_0 *where* f *is continuous at* x_0, $F'(x_0) = f(x_0)$.

Theorem 6.21 (Fundamental Theorem of Calculus). *If there exists* F *such that:*

1. F *is differentiable on* $[a, b]$
2. $F' = f$,

$$F(b) - F(a) = \int_a^b f(x)dx$$

Theorem 6.22 (Integration By Parts). *For differentiable* F, G *such that* $F' = f$ *and* $G' = g$,

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

7 Sequences and Series of Functions

Theorem 7.8 (Cauchy Criterion for Uniform Convergence)**.** *$\{f_n\}$ converges uniformly with the Cauchy Criterion, i.e.*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > m > N, x \in E, |f_m(x) - f_n(x)| < \varepsilon.$$

Theorem 7.10 (Weierstrass M Test)**.** *If $|f_n(x)| \leq M_n$ for all $n > N$ and $x \in E$, where $\sum_{n=1}^{\infty} M_n$ is convergent. then $\{f_n\}$ converges uniformly.*

Theorem 7.11 (Interchange of Limits with Uniform Convergence)**.**

1. *Suppose X is a metric space where $E \subseteq X$ and $f_n \colon E \to \mathbb{R}$ **converges uniformly** to $f \colon E \to \mathbb{R}$.*
2. *Also suppose that x is a limit point of E.*
3. *Finally, suppose $\lim_{t \to x} f_n(t)$ exists $\forall n$.*

Then the limits can be interchanged, i.e.

$$\lim_{n \to \infty} \lim_{t \to x} f_n(x) = \lim_{t \to x} \lim_{n \to \infty} f_n(t)$$

Theorem 7.12 (Continuity preserved by uniform convergence)**.** *If*

1. *$f_n \colon E \to \mathbb{R}$ is continuous,*
2. *$f_n \to f$ uniformly,*

then f is continuous on E.

Theorem 7.13 (Uniform convergence on a compact set)**.** *If K is compact and*

1. *$f_n \colon K \to \mathbb{R}$ is continuous,*
2. *$f_n \to f$ pointwise on K,*
3. *f is continuous,*
4. *$f_{n+1}(x) \leq f_n(x)$ for $x \in K$,*

then $f_n \to f$ uniformly on K.

In other words, on a compact domain, a decreasing sequence of continuous functions which converge to a continuous function converges uniformly.

Theorem 7.15 ($\mathcal{C}(X)$ is a complete metric space)**.**

Theorem 7.16 (Integrability is preserved by uniform convergence). *Suppose*

1. $f_n \in \mathcal{R}[a, b]$,
2. $f_n \to f$ *uniformly,*

Then $f \in \mathcal{R}[a, b]$*, and in particular,*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

Corollary is that this works for when a series of functions converges uniformly.

Theorem 7.17 (Differentiability is preserved by uniform convergence). *Suppose*

1. f_n *is differentiable on* $[a, b]$,
2. $f_n \to f$ *uniformly for some* f
3. $\frac{d}{dx} f_n(x_0) \to \frac{d}{dx} f(x_0)$ *for some* $x_0 \in [a, b]$ *(anchor)*

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

Theorem 7.18 (Existence of continuous but nowhere differentiable function). $\exists$ *continuous* $f : \mathbb{R} \to \mathbb{R}$ *s.t.* $f'(x)$ *does not exist* $\forall x \in \mathbb{R}$.

- *Idea is that you keep adding spikes of increasing period, have the heights follow a geometric series so it converges.*
- *Then the* f_n*'s converge uniformly by* M *test.*

Theorem 7.23 (Selection Theorem). *Suppose*

1. E *is countable,*
2. $f_n : E \to \mathbb{C}$ *is pointwise bounded*

Then $\exists\{f_{n_k}\}$ *that is pointwise convergent.*

Bounded implies convergent subsequence applied to entire function.

Theorem 7.24 (Uniform convergence implies Equicontinuity on a compact domain). *Suppose*

1. K *is compact,*
2. $f_n : K \to \mathbb{R}$ *is continuous,*
3. $f_n \to f$ *uniformly,*

then $\{f_n\}$ *is equicontinuous.*

Theorem 7.25 (Arzelà-Ascoli Theorem)**.** *Suppose*

1. *K is compact,*
2. *$f_n \colon K \to \mathbb{R}$ where $\{f_n\}$ is pointwise bounded and equicontinuous,*

then

1. *$\{f_n\}$ is uniformly bounded,*
2. *$\{f_n\}$ has a uniformly convergent subsequence $\{f_{n_k}\}$.*

An equicontinuous and pointwise bounded sequence of functions on a compact domain has a uniformly continuous subsequence (and is also uniformly bounded).

Theorem 7.26 (Weierstrass Theorem)**.** *Suppose $f \colon [a, b] \to \mathbb{R}$ is continuous. Then $\exists$ polynomials $\{P_n\}$ s.t.*

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n \geq N \implies \|P_n - f\| < \varepsilon.$$

(Any continuous function can be uniformly approximated by a sequence of polynomials on a closed interval.)

Theorem 7.27 (Uniform Closure of an Algebra is an Algebra)**.**

Theorem 7.31 (We can find a function that goes through two points)**.** *Let $\mathcal{A}$ be an algebra that separates points and vanishes at no point. Then*

$$\forall x, y \in \mathcal{K}\ (x \neq y), \forall c, d \in \mathbb{R}, \exists f \in \mathcal{A} \text{ s.t. } f(x) = c \text{ and } f(y) = d.$$

Theorem 7.32 (Stone-Weierstrass Theorem)**.** *The uniform closure of any algebra of continuous functions on a compact set K which:*

1. *Separates points,*
2. *Vanishes at no points,*

is $\mathcal{C}(K)$. The complex case also requires the algebra to be self-adjoint ($f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$).

(optimality of $\mathcal{C}(K)$)

8 Power Series

Theorem 8.1 (Convergent power series converges uniformly). *If $f(x) = \sum c_n x^n$ converges on $|x| < R$, then $f_n \colon [-R+\varepsilon, R-\varepsilon]$ converges uniformly $\forall \varepsilon > 0$ (partial sum).*

Furthermore, the derivative is well-defined for $|x| < R$,

$$f'(x) = \sum n c_n x^{n-1}.$$

Theorem 8.2 (Abel's Theorem). *Suppose*

1. *$\sum c_n$ converges,*
2. *$f(x) = \sum c_n x^n$ has radius of convergence 1.*

Then

$$\lim_{x \to 1^-} f(x) = \sum c_n.$$

(Consistent behavior as we approach radius of convergence).

Theorem 8.3 (Interchange of summation order). *If*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty,$$

then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Theorem 8.4 (Taylor Series). *If $\sum c_n x^n$ converges on $|x| < R$, then $\forall |a| < R$,*

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

at least *on $|x - a| < R - |a|$.*

(We can center a Taylor series within the radius of convergence).

Theorem 8.5 (Series on limiting sequence uniquely identifies sequence.). *Let $\sum a_n x^n$ and $\sum b_n x^n$ have the same radius of convergence $\geq R$. Also suppose that $D \subset (-R, R)$ has a limit point in $(-R, R)$. If $\sum a_n x^n = \sum b_n x^n$ $\forall x \in D$, then $a_n = b_n$.*

Theorem 8.8 (Fundamental Theorem of Algebra)**.** *All complex polynomials have a root.*

Theorem 8.11 (Optimality of partial Fourier series coefficients)**.** *Suppose $f \in \mathcal{R}[a,b]$ and $\{\phi_n\}$ is orthonormal. Then let $S_n = \sum_{m=1}^{n} c_m \phi_m$ and let $T_n = \sum_{m=1}^{n} a_m \phi_m$. Then*

$$\|f - S_n\|_2 \leq \|f - T_n\|_2$$

$\forall n$. Equality only holds if $a_m = c_m$ $\forall m$. Note this is w.r.t. L2 norm (defined with inner product), and not infinity norm.

Corollaries:

- $\|S_n\|_2^2 \leq \|f\|_2^2$.

Theorem 8.12 (Changes in definition)**.** *Now*

1. *We only consider functions $f \colon [-\pi, \pi] \to \mathbb{C}$.*
2. *$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\bar{g}(x)dx$.*

Theorem 8.14 (Lipchitz continuity implies convergence of Fourier series)**.** *Suppose $x \in [-\pi, \pi]$, and $f \colon [-\pi, \pi] \to \mathbb{C}$. If $\exists \delta, M > 0$ s.t.*

$$|f(x+t) - f(x)| \leq M|t|$$

$\forall |t| < \delta$, then

$$\lim_{N\to\infty} S_N(x) = f(x).$$

Theorem 8.15 (Trig polynomial approximation)**.** *All 2π-periodic continuous functions can be uniformly approximated by a trig polynomial.*

Let $f \colon \mathbb{R} \to \mathbb{C}$ be 2π-periodic. Then $\forall \varepsilon > 0$, $\exists$ a trig polynomial in the form

$$p_N(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

s.t. $\|p_N - f\| < \varepsilon$.

Theorem 8.16 (Convergence of Fourier series in L2)**.** *For $f \in \mathcal{R}[-\pi, \pi] \to \mathbb{C}$,*

1. $\lim_{N\to\infty} \|f - S_N(f)\|_2 = 0$.
2. *Parseval relation: $\langle f, g\rangle = (c, \gamma)$.*
    - *Bessel's inequality follows from this: $\|f\|_2^2 = (c, c) = \sum_{n=-\infty}^{\infty} |c_n|^2$.*

9 Functions of Multiple Variables

Theorem 9.1 (Banach fixed-point/contraction mapping theorem)**.** *A contraction mapping theorem has a unique fixed point.*

Let (X, d) be a complete metric space. If the map $\varphi\colon X \to X$ satisfies $d(\varphi(x), \varphi(y)) \leq cd(x, y)$ for some $c < 1$ and $\forall x \in X$, then $\exists! x \in X$ s.t. $\varphi(x) = x$.

Theorem 9.12 (Uniqueness of derivative A)**.**

Theorem 9.15 (Chain rule)**.** *Let $f\colon E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$ is open, and let $g\colon V : \mathbb{R}^k$ where $V \subset \mathbb{R}^m$ is open s.t. $f(E) \subset V$. Let f be differentiable at x and let g be differentiable at $f(x)$. Then $g \circ f$ is differentiable at x, where*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Theorem 9.16 (Derivative linear map is defined by the Jacobian)**.** *Suppose $f\colon E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$ is open. If f is differentiable, then $\frac{\partial f_j}{\partial x_i}$ exists and $A =$ the Jacobian of f.*

Theorem 9.17 (Lipchitz continuity defined by bounded derivative)**.** *Suppose $f\colon E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$ is convex. Also suppose that f is differentiable and $\|f'(x)\| < M$ for some M and for all x. Then f is Lipchitz continuous on E, and in particular*

$$\|f(a) - f(b)\| \leq M|a - b|$$

$\forall a, b \in E$.

Theorem 9.18 (Inverse function theorem in one dimension)**.** *If $f\colon \mathbb{R} \to \mathbb{R}$ is strictly monotone and differentiable, then $g = f^{-1}$ is also differentiable, where $g'(f(x)) = \frac{1}{f'(x)}$. In particular, $f'(x) \neq 0\ \forall x$.*

Theorem 9.19 (Inverse Function Theorem (multiple variables))**.** *Let $f\colon E \to R^n$ where $E \subset \mathbb{R}^n$ is open. Also suppose f is continuously differentiable and $f'(x)$ is invertible $\forall x \in E$.*

Then $\exists$ an open $V \subset f(E)$ s.t. f is a bijection. If we define $g\colon V \to E$ to be the inverse, g is also continuously differentiable where $g'(f(x))f'(x) = I$.