Assignment 1

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Question 1

(a) (i) In order to compute the probability of interest, we first count the number of hands containing exactly one pair. We'll start by considering values of the cards. There are $\binom{13}{1}$ ways to choose the value of a, and there are $\binom{12}{3}$ ways to choose the value of b, c, and d. Next we consider the possible suits that the cards can be. For cards with the value a, we must choose 2 suits, so there are $\binom{4}{2}$ ways to choose the suits of a. Then for each of the cards with values b, c, and d, there are $\binom{4}{1}$ ways to choose the suit of the card.

Next, count the total number of poker hands by choosing any 5 of the 52 cards. Hence, the probability of interest (say p) is:

$$p = \frac{1}{\binom{52}{5}} \binom{13}{1} \binom{12}{3} \binom{4}{2} \binom{4}{1}^3 \approx 0.4226.$$

(ii) Similar to before, count the number of hands containing exactly two pairs. First start by considering values. There are $\binom{13}{2}$ ways to choose the values of a and b, and thus there are $\binom{11}{1}$ ways to choose the value of c. Next, let's look at suits. For each of a and b, there are $\binom{4}{2}$ ways to choose the suits of the cards, and there are $\binom{4}{1}$ ways to choose the suit of the card with value of c. Similar to above, there are $\binom{52}{2}$ total poker hands. Hence, the probability of interest p is:

$$p = \frac{1}{\binom{52}{5}} \binom{13}{2} \binom{11}{1} \binom{4}{2}^2 \binom{4}{1} \approx 0.04754.$$

(b) (i) We'll determine the probability of interest by considering hands where order matters (e.g. $abcde \neq bacde$ even if $a = b$). First count the number of hands with exactly one pair. There are 6 ways to choose a, and thus there are $\binom{5}{3}$ ways to choose the values b, c, and d. Since we differentiate hands by order, there are $\frac{5!}{2!}$ ways to order a hand given the values of a, b, c, and d (the 2! in the denominator removing duplicates caused by the two a's). Next we count the total number of poker dice hands, which is to set $abcde$ with any value for each of the die, which is 6^5.

Therefore the probability in question p is:

$$p = \frac{1}{6^5} 6 \binom{5}{3} \frac{5!}{2!} = \frac{25}{54} \approx 0.4630.$$

(ii) Again, we consider hands with different order to be different. First count the number of hands with exactly two pairs. There are $\binom{6}{2}$ ways to choose the values of a and b, and thus there are 4 ways to choose the value of c. Since we differentiate hands by order, there are $\frac{5!}{2!2!}$ ways to order a hand given the values of a, b, c.

Therefore the probability in question p is:

$$p = \frac{1}{6^5} \binom{6}{2} (4) \frac{5!}{2!2!} = \frac{25}{108} \approx 0.2315.$$

Question 2

Proof. We will count the number of pairs (A, B) where $A, B \subset S$ s.t.

$$A \subset B.$$

First, let's consider B. Since we can either include or exclude each element of S in a subset, there are 2^n possible subsets of S, and in particular there are 2^n possible assignments for B. Furthermore, for $k \in \mathbb{Z}$ s.t. $0 \leq k \leq n$, the number of subsets B that have size k is $\binom{n}{k}$.

Next, we count the number of subsets A that satisfy $A \subset B$ for a given B with size k. Since there are 2^k subsets of B, the number of subset pairs (A, B) that satisfy $A \subset B$ is:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = (2+1)^n = 3^n. \qquad \text{(Binomial Theorem)}$$

We then count the total number of subset pairs (A, B). From above, there are 2^n possible assignments to A and 2^n possible assignments to B. Hence the total number of subset pairs (A, B) is

$$(2^n)^2 = 4^n.$$

Consider the probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = 2^S \times 2^S$, $\mathcal{F} = 2^\Omega$, and P the uniform measure with respect to Ω (given). From class,

$$\begin{aligned} P(A \subset B) &= P(\{(A, B) \in \Omega \colon A \subset B\}) \\ &= \frac{|\{(A, B) \in \Omega \colon A \subset B\}|}{|\Omega|} \\ &= \frac{3^n}{4^n} \\ &= \left(\frac{3}{4}\right)^n. \end{aligned}$$

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Question 3

(a) *Proof.* $\mathcal{F}$ is an algebra if the following three conditions hold:

(i) $\varnothing \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,

(iii) $A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$.

Assume that

$$A, B \in \mathcal{F} \implies A \setminus B \in \mathcal{F}.$$

First we'll show that (ii) holds. Let $A \in \mathcal{F}$. Since $\Omega \in \mathcal{F}$ as well, the given assumption tells us that $\Omega \setminus A = A^c = \mathcal{F}$.

Note that this implies that (i) holds as well, since $\Omega^c = \varnothing$.

Finally, we'll show that (iii) holds. Note that showing (iii) is sufficient for the proof as finite unions follow from pairwise unions via induction (mentioned in class). Let $A, B \in \mathcal{F}$ and suppose that $A \cup B \notin \mathcal{F}$. Observe that

$$\begin{aligned}
&A^c \cup A = \Omega \\
&\implies (A^c \setminus B) \cup (A \cup B) = \Omega \\
&\implies A \cup B = \Omega \setminus (A^c \setminus B).
\end{aligned}$$

By the contrapositive of the assumption,

$$A \cup B = \Omega \setminus (A^c \setminus B) \notin \mathcal{F} \implies \text{either } \Omega \notin \mathcal{F} \text{ or } (A^c \setminus B) \notin \mathcal{F},$$

but we know this not to be true, since $\Omega \in \mathcal{F}$ is given, and $A^c \setminus B \in \mathcal{F}$ by (ii) and the assumption. So (iii) holds by contradiction. As all conditions hold, we conclude that $\mathcal{F}$ is an algebra. ■

(b) Let $\Omega = [0, 5]$ and consider

$$\mathcal{F} = \{\varnothing, [0, 5], [1, 3], [2, 4], [0, 1] \cup [3, 5], [0, 2] \cup [4, 5]\}$$

on Ω. We claim that while $\mathcal{F}$ is closed under complements and finite disjoint unions, $\mathcal{F}$ is not an algebra on Ω.

Proof. First we'll verify that $\mathcal{F}$ is closed under complements. This can be easily verified since

- $\varnothing^c = [0, 5]$,
- $[1, 3]^c = [0, 1] \cup [3, 5]$,
- $[2, 4]^c = [0, 2] \cup [4, 5]$.

Next we can verify that $\mathcal{F}$ is closed under finite disjoint unions by verifying that $\forall I \in \mathcal{F}$ where $I \neq \varnothing$, all other $J \in \mathcal{F}$ s.t. $I \neq J$ satisfies $I \cap J = \varnothing$ only if $J = \varnothing$ or $I = J^c$. If $I = \varnothing$, the only J satisfying the above condition is $[0, 5]$. In particular, this means no additional intervals need to be added to $\mathcal{F}$, and is thus closed under pairwise disjoint unions and hence is also closed under finite disjoint unions.

But note that $[1, 4] = [1, 3] \cup [2, 4] \notin \mathcal{F}$, so $\mathcal{F}$ is not closed under finite unions. Therefore we can conclude that $\mathcal{F}$ is not an algebra.

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Question 4

(a) Let $E = \{E_\alpha \colon \alpha \in A\}$ where $E \neq \varnothing$.

Proof. Let

$$\sigma(E) = \bigcap_\beta \mathcal{F}_\beta$$

where $\mathcal{F}_\beta$ is a σ-algebra s.t. $E \subset \mathcal{F}_\beta$.

We'll show that $\sigma(E)$ is indeed a σ-algebra, since all $\mathcal{F}_\beta$ are also σ-algebras. First $\varnothing \in \sigma(E)$, since $\varnothing \in \mathcal{F}_\beta$ $\forall \beta$. Similar logic shows that $\sigma(E)$ is closed under complements and countable unions, as follows. Let $B \in \sigma(E)$. We can be sure that $B^c \in \sigma(E)$, since $B^c \in \mathcal{F}_\beta$ $\forall \beta$. Let $B_1, B_2, \cdots$ be a countable number of sets. We can also be sure that $\bigcup_{n=1}^\infty \in \sigma(E)$ since $\bigcup_{n=1}^\infty \in \mathcal{F}_\beta$ $\forall \beta$. In particular, $\sigma(E)$ is a σ-algebra.

It is then trivial that $\sigma(E)$ is the smallest of all σ-algebras $\mathcal{F}_\beta$ s.t. $E_\alpha \in \mathcal{F}_\beta$ $\forall \alpha \in A$ by definition of intersection.

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(b) *Proof.* Consider $A = \{0\}$ and $B = \{1\}$. Let $\mathcal{F}_A = \{\varnothing, A\}$ and let $\mathcal{F}_B = \{\varnothing, B\}$.

We can easily verify that $\mathcal{F}_A$ and $\mathcal{F}_B$ are indeed σ-algebras on A and B respectively. First $\varnothing \in \mathcal{F}_A$, $A^c = \varnothing \in \mathcal{F}_A$, and $A \cup \varnothing = A$, so $\mathcal{F}_A$ is a σ-algebra on A. Similarly, $\mathcal{F}_B$ is a σ-algebra on B. But $\mathcal{F}_A \cup \mathcal{F}_B$ is not a σ-algebra on any set, since it is not closed under countable unions. In particular $A \cup B \notin \mathcal{F}_A \cup \mathcal{F}_B$.

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Question 5

Proof. Let $S = \{E_\alpha : \alpha \in A\}$. Following the hint, let $\mathcal{M}$ be the set of all sets E that satisfy the property that $\exists$ a countable $S_E := \{E_{\alpha_j} : j = 1, 2, \cdots\}$ s.t. $E \in \sigma(S_E)$, where $\sigma(S_E)$ is the σ-algebra generated by S_E.

We'll first show that $\mathcal{M}$ is a σ-algebra. Consider $E = \varnothing$ and $S_E = \varnothing$, which generates the σ-algebra $\sigma(S_E) = \{\varnothing\}$. In particular, $E \in \sigma(S_E)$ so $\varnothing \in \mathcal{M}$ by definition of $\mathcal{M}$.

Next let's check that $\mathcal{M}$ satisfies closure under complements. Let $E \in \mathcal{M}$. By definition of $\mathcal{M}$, $\exists S_E$ that is countable subcollection of S where $E \in \sigma(S_E)$. But since $\sigma(S_E)$ is a σ-algebra, $E^c \in \sigma(S_E)$. In particular, this means that for E^c, S_E satisfies the asserted property, so $E^c \in \mathcal{M}$.

Now we'll verify that $\mathcal{M}$ is closed under countable unions. Let $E^1, E^2, \cdots \in \mathcal{M}$ be a countable list of sets. For each $n = 1, 2, \cdots$, $\exists S_{E^n}$ which is a countable subcollection of S s.t. $E^n \in \sigma(S_{E^n})$. Consider the σ-algebra generated by the countable unions of each S_{E^n}:

$$\sigma\left(\bigcup_{i=1}^{\infty} S_{E^i}\right).$$

Since $E^n \in \sigma(S_{E^n})$, $E^n \in \sigma\left(\bigcup_{i=1}^{\infty} S_{E^i}\right)$ for each n as $S_{E^n} \subset \bigcup_{i=1}^{\infty} S_{E^i}$. Hence

$$\bigcup_{n=1}^{\infty} E^n \in \sigma\left(\bigcup_{i=1}^{\infty} S_{E^i}\right). \qquad \text{(Closure under countable unions)}$$

In particular $\mathcal{M}$ is a σ-algebra that satisfies the asserted property. Since $\mathcal{F}$ is the generated σ-algebra, $\mathcal{F} \subset \mathcal{M}$ so the asserted property also applies to $\mathcal{F}$ as required.

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