Math 418/544 Assignment 1 September 9, 2022

Dr. G. Slade

This assignment is due in Canvas at 9:59 a.m. on Friday, September 16.
Please read the submission instructions on Canvas.
Late assignments are not accepted.

1. In this problem, explain how you are counting—do not just write down an answer without explanation.
   (a) Compute the probability that a poker hand[1] contains:
      i. one pair ($aabcd$ with a, b, c, d distinct face values; answer: 0.4226)
      ii. two pairs ($aabbc$ with a, b, c distinct face values; answer: 0.04754).
   (b) Poker dice[2] is played by simultaneously rolling 5 dice. Compute the probabilities of the following outcomes:
      i. one pair ($aabcd$ with a, b, c, d distinct numbers; answer: 0.4630)
      ii. two pairs ($aabbc$ with a, b, c distinct numbers; answer: 0.2315).

2. Let $S = \{1, 2, \ldots, n\}$ and suppose that a pair of subsets (A, B) of S is chosen uniformly at random from all possible pairs of subsets. More precisely, the probability of choosing any specific pair is 2^{-2n}. Show that
$$P(A \subset B) = \left(\frac{3}{4}\right)^n.$$

3. Let Ω be a nonempty set and suppose that $\mathcal{F}$ is a collection of subsets of Ω such that $\Omega \in \mathcal{F} \subset 2^\Omega$.
   (a) Prove that $\mathcal{F}$ is an algebra if $A, B \in \mathcal{F}$ implies that $A \setminus B = A \cap B^c \in \mathcal{F}$.
   (For an algebra the condition of closure under countable unions in the definition of a σ-algebra is replaced by closure under finite unions.)
   (b) Suppose that $\mathcal{F}$ is closed under complements and finite *disjoint* unions. Show that $\mathcal{F}$ need not be an algebra.

4. (a) Given an arbitrary nonempty collection of sets $\{E_\alpha : \alpha \in A\}$, prove that there is a smallest σ-algebra that contains every E_α. This σ-algebra is called the σ-algebra *generated* by $\{E_\alpha : \alpha \in A\}$.
   (b) Suppose that we have σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$. Show (by counterexample) that the union $\mathcal{F}_1 \cup \mathcal{F}_2$ need not be a σ-algebra.

5. Let $\mathcal{F}$ be the σ-algebra generated by an arbitrary nonempty collection of sets $\{E_\alpha : \alpha \in A\}$. Prove that for each $E \in \mathcal{F}$, there exists a countable subcollection $\{E_{\alpha_j} : j = 1, 2, \ldots\}$ (depending on E) such that E already belongs to the σ-algebra generated by this subcollection.
   Hint: Consider the class of all sets with the asserted property and show that it is a σ-algebra containing each E_α.

Recommended problems. Each assignment will include additional recommended problems, which are not to be handed in for marking. The following problems from Rosenthal are recommended but are not to be handed in:
2.7.2, 2.7.4, 2.7.6.
For solutions to even-numbered problems see: `http://www.probability.ca/jeff/grprobbook.html`.

[1] A poker hand consists of five cards drawn from a deck of 52 cards. The cards have 13 distinct values $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$, each in four suits called Hearts, Diamonds, Clubs, Spades.

[2] Each die is a cube with six sides labelled 1,2,3,4,5,6.