Spheroidal harmonic expressions for the derivatives of the gravitational potential

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In this work the spheroidal harmonic expressions for the derivatives of the gravitational potential up to third-order are considered, including the oblate and prolate spheroidal cases.

I: oblate spheroidal harmonic expressions

Abstract

The oblate spheroidal harmonics can be used to describe the gravitational field of the celestial bodies with oblate surface shape. The gravity vector and its higher-order derivatives are the vital observed quantities in gravimetric measurement. The oblate spheroidal harmonic expressions for the derivatives of the gravitational potential up to third-order is presented in this work. The non-singular and the regular forms are both discussed. The derivatives of the potential can be written in linear combinations of the products including the associated Legendre functions of the first and second kinds and the trigonometric function in the oblate spheroidal harmonics, and specific expressions at the poles are given independent of the longitude coordinate. The non-singular expressions of the derivatives of the potential are suitable for arbitrary external observation points outside the Brillouin oblate spheroid without the problem of singularity, and are still valid near or at the poles. The regular expressions are more efficient for the observation points not near the poles although they have worse digital precision near the poles and are incalculable at the poles. The numerical experiments analyzing the case of the tested uniform prism show the correctness and the good precision of the algorithms of this work.

Keywords: Oblate spheroidal harmonics, Derivatives of the gravitational potential, Global reference frame, Local reference frame, Non-singular expressions

1 Introduction

The gravity vector and its derivatives are increasingly applied in geophysics and geodesy as well as in planetary science. The first- and second-order derivatives of the gravitational potential can be measured by existing instruments. The third-order derivatives of the gravitational potential can be observed by new sensors, and have also been investigated in the literature (Cunningham 1970; Šprlák and Novák 2015, 2017). The spherical harmonic function is a useful mathematical tool for the precise description of the gravitational field of 3D celestial bodies (Hofmann-Wellenhof and Moritz 2005). On account of the non-spherical shape of the Earth and other celestial bodies, compared with the spherical harmonics the spheroidal harmonics including the oblate and prolate harmonic functions may be more suitable for the gravitational field modelling (Hobson 1931; Bölling and Grafarend 2005; Hofmann-Wellenhof and Moritz 2005; Fukushima 2014; Hu and Jekeli 2015; Novák and Šprlák 2018; Šprlák et al. 2020). The spheroidal harmonic expansions of the gravitational field also converge faster than the spherical harmonic expansions for most observation point outside the bodies. For the planets and some other celestial bodies in the solar system, their surface is close to the oblate spheroid, and their gravitational field can be well described by the oblate spheroidal harmonics. When the spherical and the spheroidal harmonic coefficients of the potential model are known, the derivatives of the potential are expected to be computed at an observation point. The computations of the gravitational field (up to second-order derivatives) using the oblate spheroidal harmonics in the local reference frame centered at the observation point are considered in Koop (1993).

Since the regular spherical harmonic expansions of the derivatives of the gravitational potential contain the reciprocal factor of the sine function of the colatitude coordinate, they may be singular at the poles and incalculable for the observation point near the poles. In order to make the expansions be suitable for the poles, the non-singular spherical harmonic expressions of the gravitational vector and high-order derivatives in the global reference frame centered at a fixed point or the local reference frame are presented (Ilk 1983; Petrovskaya and Vershkov 2006, 2008, 2010; Eshagh 2008; Hamáčková et al. 2016). For the global reference frame, the three Cartesian coordinate axes are also fixed. The oblate spheroidal harmonic expansions of the derivatives of the potential have analogous problem of singularity. Petrovskaya and Vershkov (2014, 2015, 2016) and Vershkov and Petrovskaya (2016) discussed the non-singular oblate spheroidal harmonic expressions of the gravitational vector and tensor in the local north-oriented reference frame. The derivatives of the potential in the global reference frame can be computed by converting the local gravitational gradient components into the global components. One may be interested in the direct non-singular expressions for the oblate spheroidal harmonic expansions of the derivatives in the global reference frame without the aid of the local reference frame.

This work focuses on the oblate spheroidal harmonic expressions for the derivatives of the gravitational potential of celestial bodies. In Sect. 2, we derive the non-singular relations of the oblate spheroidal harmonic expansion and its derivatives in the global reference frame. The expressions of the first-, second- and third-order derivatives of the potential are given in Sect. 3. The computations of the derivatives of the potential for the observation point at the poles are explicitly presented in Sect. 4. The oblate spheroidal harmonic expansions of the derivatives of the potential in the local north-oriented reference frame using tensor analysis are discussed in Sect. 5. We implement the numerical experiments for the algorithms of this work in Sect. 6 and draw some conclusions in Sect. 7. The regular expressions of the derivatives of the potential are also discussed in the Appendix, which may be singular at the poles.

2 Relations of the oblate spheroidal harmonic expansion and its derivatives

2.1 Expansion of the gravitational potential

The oblate spheroidal coordinates (u, ϑ, λ) are based on the global reference frame, and their relations to the (global) Cartesian coordinates (x, y, z) are: $x = v \sin\vartheta \cos\lambda$, $y = v \sin\vartheta \sin\lambda$, and $z = u \cos\vartheta$ (Hobson 1931, pp. 421; Hofmann-Wellenhof and

Moritz 2005, pp. 36), where u and v are the semi-minor and the semi-major axes of the confocal oblate spheroid $x^2/v^2 + y^2/v^2 + z^2/u^2 = 1$ ($v = \sqrt{u^2 + E^2}$ with the linear eccentricity E), ϑ is the colatitude and λ is the longitude. From Hobson (1931, pp. 431-433) and Hofmann-Wellenhof and Moritz (2005, pp. 39), the external gravitational potential V in the body-fixed frame can be expressed as the oblate spheroidal harmonic expansion:

$$V = \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{Q_{n,m}(\mathbb{i}u/E)}{Q_{n,m}(\mathbb{i}b/E)} \overline{P}_{n,m}(\cos\vartheta) \Big(\overline{C}_{n,m} \cos(m\lambda) + \overline{S}_{n,m} \sin(m\lambda) \Big), \tag{1}$$

where $\mu = GM$ is the gravitational constant of the body, a, b are the semi-major and semi-minor axes of the reference oblate spheroid ($E = \sqrt{a^2 - b^2}$), $\overline{P}_{n,m}(\cos\vartheta)$ is the fully normalized associated Legendre function of the first kind, $Q_{n,m}(\mathbb{i}u/E)$ is the associated Legendre function of the second kind with the imaginary unit $\mathbb{i}$, $\overline{C}_{n,m}$ and $\overline{S}_{n,m}$ are the normalized oblate spheroidal harmonic coefficients. The relations of the harmonic coefficients $C_{n,m}$, $S_{n,m}$ and the Legendre function $P_{n,m}(\cos\vartheta)$ and their normalized forms are: $\overline{C}_{n,m} = C_{n,m}/N_{n,m}$, $\overline{S}_{n,m} = S_{n,m}/N_{n,m}$, and $\overline{P}_{n,m}(\cos\vartheta) = N_{n,m}P_{n,m}(\cos\vartheta)$ with $N_{n,m} = \sqrt{(2-\delta_{0,m})(2n+1)(n-m)!/(n+m)!}$, where $\delta_{0,m}$ is the Kronecker symbol. The oblate spheroidal harmonics is truncated to a certain degree and order (d/o) N in numerical computations.

The recursive algorithm to compute the Legendre function $Q_{n,m}(\mathbb{i}u/E)$ with the ratio $u/E > 1$ (for the Earth) is given in Fukushima (2013), and is also valid for $u/E < 1$ (for some celestial bodies) by the way of analytic continuation of hypergeometric functions. In order to make the modulus of the variable of the hypergeometric series in $Q_{n,m}(\mathbb{i}u/E)$ always less than 1, the Legendre function $Q_{n,m}(\mathbb{i}u/E)$ can be written as (Martinec and Grafarend 1997; Sebera et al. 2012)

$$Q_{n,m}(\mathbb{i}u/E) = \frac{(-1)^m}{\mathbb{i}^{n+1}} \frac{(n+m)!}{(2n+1)!!} \left(\frac{E}{v}\right)^{n+1} F_{n,m}(v/E), \tag{2}$$

where $F_{n,m}(v/E)$ denotes the following Gaussian hypergeometric series

$$F_{n,m}(v/E) = {}_2F_1\left(\frac{n-m+1}{2}, \frac{n+m+1}{2}; n+\frac{3}{2}; \frac{E^2}{v^2}\right). \tag{3}$$

Then, the ratio of $Q_{n,m}(\mathbb{i}u/E)$ to $Q_{n,m}(\mathbb{i}b/E)$ in Eq. (1) can be computed by

$$\frac{Q_{n,m}(\mathbb{i}u/E)}{Q_{n,m}(\mathbb{i}b/E)} = \left(\frac{a}{v}\right)^{n+1} \frac{F_{n,m}(v/E)}{F_{n,m}(a/E)}. \tag{4}$$

Since $E^2/v^2 < 1$ for arbitrary $u > 0$, the hypergeometric series $F_{n,m}(v/E)$ always converges. Taking a similar approach to the computation of the hypergeometric series for the prolate spherical harmonic expansions of the gravitational field in Fukushima (2014) and according to the recursions of $Q_{n,m}(\mathbb{i}u/E)$ (Gradshteyn and Ryzhik 2007, pp. 965-966) and Eq. (2), we can also obtain the recursive algorithm for computing $F_{n,m}(v/E)$:

$$\begin{aligned}
F_{n,m}(v/E) &= \frac{u}{v} F_{n+1,m}(v/E) + \alpha_{n,m} \frac{E^2}{v^2} F_{n+2,m}(v/E), \\
F_{n,m}(v/E) &= \frac{u}{v} F_{n,m-1}(v/E) + \beta_{n,m} \frac{E^2}{v^2} F_{n+1,m-1}(v/E), \\
F_{n,m}(v/E) &= \gamma_{n,m} \frac{u}{v} F_{n,m-1}(v/E) + \eta_{n,m} F_{n,m-2}(v/E),
\end{aligned} \tag{5}$$

where the coefficients $\alpha_{n,m}$, $\beta_{n,m}$, $\gamma_{n,m}$ and $\eta_{n,m}$ are the same as the coefficients $a_{n,m}$, $b_{n,m}$, $d_{n,m}$ and $1 - d_{n,m}$ in Fukushima (2014), i.e.,

$$\begin{aligned}
\alpha_{n,m} &= \frac{(n-m+2)(n+m+2)}{(2n+3)(2n+5)}, \ \beta_{n,m} = \frac{n-m+2}{2n+3}, \\
\gamma_{n,m} &= \frac{2(m-1)}{n+m}, \ \eta_{n,m} = 1 - \gamma_{n,m} = \frac{n-m+2}{n+m}.
\end{aligned} \tag{6}$$

The computation sequence of the hypergeometric series $F_{n,m}(v/E)$ is the same as Fukushima (2014), and the initial values $F_{N-1,0}(v/E)$, $F_{N,0}(v/E)$, $F_{N,1}(v/E)$ can be computed by

$$F_{n,m}(v/E) = \left(\frac{2v}{u+v}\right)^{n+\frac{1}{2}} {}_2F_1\left(m + \frac{1}{2}, -m + \frac{1}{2}; n + \frac{3}{2}; \frac{E^2}{2v(u+v)}\right), \tag{7}$$

where the variable $E^2/(2v(u+v)) = (v-u)/(2v) < 1/2 < 1$, and the hypergeometric series on the right-hand side of the equation is solved by termwise accumulation. Eq. (7) is derived from the transformation formulas of Sections 9.131 and 9.133 in Gradshteyn and Ryzhik (2007, pp. 1008-1009), where the required solution of the equation $4\omega(1-\omega) = E^2/v^2$ is: $\omega = (v-u)/(2v)$ and another solution $\omega = (u+v)/(2v) > 1/2$ is eliminated for the satisfaction of the condition: $|\omega| \leqslant 1/2$. The series $F_{N-1,0}(v/E)$, $F_{N,0}(v/E)$, $F_{N,1}(v/E)$ converge fast with the d/o N increasing. When N is very large, the floating-point underflow or overflow problems may occur in computing the series $F_{n,m}(v/E)$ in Eqs. (5) and (7), and the X-number method (Fukushima 2012a) or the logarithm method (Reimond and Baur 2016) can be applied to compute these values and the ratio $Q_{n,m}(\mathbb{i}u/E)/Q_{n,m}(\mathbb{i}b/E)$, as well as the product of the ratio and $\overline{P}_{n,m}$ in the expressions of the potential and its derivatives.

2.2 Relations of the harmonic expansion and its derivatives

The derivative operators $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial \vartheta}$ and $\frac{\partial}{\partial \lambda}$ of the oblate spheroidal coordinates can be easily written in the derivative operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ of the Cartesian coordinates, i.e., the components of the covariant basis vectors $(\mathbf{e}_u, \mathbf{e}_\vartheta, \mathbf{e}_\lambda)$ of the oblate spheroidal coordinates, where the basis vectors $\mathbf{e}_u = \frac{\partial \mathbf{r}}{\partial u}$, $\mathbf{e}_\vartheta = \frac{\partial \mathbf{r}}{\partial \vartheta}$, $\mathbf{e}_\lambda = \frac{\partial \mathbf{r}}{\partial \lambda}$ with the position vector $\mathbf{r}$. Then, by solving their inverse solutions the following conversions from the derivative operators in the oblate spheroidal coordinates to the Cartesian coordinates can be obtained:

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{uv\sin\vartheta\cos\lambda}{L^2}\frac{\partial}{\partial u} + \frac{v\cos\vartheta\cos\lambda}{L^2}\frac{\partial}{\partial\vartheta} - \frac{\sin\lambda}{v\sin\vartheta}\frac{\partial}{\partial\lambda},\\
\frac{\partial}{\partial y} &= \frac{uv\sin\vartheta\sin\lambda}{L^2}\frac{\partial}{\partial u} + \frac{v\cos\vartheta\sin\lambda}{L^2}\frac{\partial}{\partial\vartheta} + \frac{\cos\lambda}{v\sin\vartheta}\frac{\partial}{\partial\lambda},\\
\frac{\partial}{\partial z} &= \frac{v^2\cos\vartheta}{L^2}\frac{\partial}{\partial u} - \frac{u\sin\vartheta}{L^2}\frac{\partial}{\partial\vartheta},
\end{aligned}
\tag{8}
$$

with $L = \sqrt{u^2 + E^2\cos^2\vartheta}$. When the linear eccentricity $E = 0$, the oblate spheroidal coordinates (u, ϑ, λ) become the spherical coordinates (r, θ, λ), and then Eq. (8) is the conversions from the derivative operators in the spherical coordinates to the Cartesian coordinates (Gasiorowicz 1974, pp. 168). The operators with respect to x- and y-coordinates in Eq. (8) contain the fraction $1/\sin\vartheta$, and then the derivatives of the oblate spheroidal harmonics may contain the fraction. The derivative of the Legendre function $P_{n,m}(\cos\vartheta)$ with respect to ϑ-coordinate also have the fraction. Therefore, $1/\sin\vartheta$ is the reason that the derivatives of the oblate spheroidal harmonics with respect to the three Cartesian coordinates are singular at the poles ($\sin\vartheta = 0$), and the derivatives of the spherical harmonics are analogous (the singular fraction $1/\sin\theta$ with $\sin\theta = 0$).

The following two variables are the products of the associated Legendre functions of the first and second kinds and the trigonometric function in the oblate spheroidal harmonics:

$$
\begin{aligned}
V_{n,n_1,n_2,m,m_2} &= Q_{n_1,m}P_{n_2,m_2}T_{n,m,m_2},\\
V'_{n,n_1,n_2,m,m_2} &= Q_{n_1,m}P_{n_2,m_2}T'_{n,m,m_2},
\end{aligned}
\tag{9}
$$

where n, n_1, n_2, m and m_2 are non-negative integers, the coordinate variables in the functions $V_{n,n_1,n_2,m,m_2}(u,\vartheta,\lambda)$, $V'_{n,n_1,n_2,m,m_2}(u,\vartheta,\lambda)$, $Q_{n_1,m}(\mathbb{i}u/E)$, $P_{n_2,m_2}(\cos\vartheta)$, $T_{n,m,m_2}(\lambda)$ and $T'_{n,m,m_2}(\lambda)$ are omitted for simplified representation, and

$$
\begin{aligned}
T_{n,m,m_2} &= C_{n,m}\cos(m_2\lambda) + S_{n,m}\sin(m_2\lambda),\\
T'_{n,m,m_2} &= -C_{n,m}\sin(m_2\lambda) + S_{n,m}\cos(m_2\lambda).
\end{aligned}
\tag{10}
$$

It is obvious that the gravitational potential can be written as

$$
V = \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\frac{V_{n,n,n,m,m}}{Q^b_{n,m}},
\tag{11}
$$

where $Q^b_{n,m}$ is the simplified form of $Q_{n,m}(\mathbb{i}b/E)$.

From Ilk (1983, pp. 114-121), Petrovskaya and Vershkov (2006) and Gradshteyn and Ryzhik (2007, pp. 965), we can get the following formulas for the derivatives of

$Q_{n,m}$, $P_{n,m}$ and the ratio of $P_{n,m}/\sin\vartheta$:

$$\begin{aligned} v^2\frac{\mathrm{d}Q_{n,m}}{\mathrm{d}u} &= -(n-m+1)\mathbb{i}EQ_{n+1,m} - (n+1)uQ_{n,m}, \\ \sin\vartheta\cos\vartheta\frac{\mathrm{d}P_{n,m}}{\mathrm{d}\vartheta} &= mP_{n,m} + (n+1)\sin^2\vartheta P_{n,m} - \sin\vartheta P_{n+1,m+1}, \\ \frac{mP_{n,m}}{\sin\vartheta} &= \frac{1}{2}\Big((n-m+1)(n-m+2)P_{n+1,m-1} + P_{n+1,m+1}\Big). \end{aligned} \tag{12}$$

where the third formula holds for $m \neq 0$. For the functions T_{n,m,m_2} and T'_{n,m,m_2} of the longitude coordinate, we can obtain the expressions of their derivatives

$$\begin{aligned} \frac{\mathrm{d}T_{n,m,m_2}}{\mathrm{d}\lambda} &= m_2T'_{n,m,m_2} \\ \frac{\mathrm{d}T'_{n,m,m_2}}{\mathrm{d}\lambda} &= -m_2T_{n,m,m_2} \end{aligned} \tag{13}$$

and the identities

$$\begin{aligned} T_{n,m,m_2+1} &= T_{n,m,m_2}\cos\lambda + T'_{n,m,m_2}\sin\lambda \\ T_{n,m,m_2-1} &= T_{n,m,m_2}\cos\lambda - T'_{n,m,m_2}\sin\lambda \\ T'_{n,m,m_2+1} &= T'_{n,m,m_2}\cos\lambda - T_{n,m,m_2}\sin\lambda \\ T'_{n,m,m_2-1} &= T'_{n,m,m_2}\cos\lambda + T_{n,m,m_2}\sin\lambda \end{aligned} \tag{14}$$

According to Eqs. (8), (9), (12), (13), (14) and the formula $L^2 = u^2 + E^2\cos^2\vartheta = v^2 - E^2\sin^2\vartheta$, the derivative of the function V_{n,n_1,n_2,m,m_2} with respect to the coordinate x is

$$\begin{aligned} \frac{\partial V_{n,n_1,n_2,m,m_2}}{\partial x} =& \frac{1}{L^2 v\sin\vartheta}\Bigg(\frac{\mathrm{d}Q_{n_1,m}}{\mathrm{d}u}P_{n_2,m_2}T_{n,m,m_2}uv^2\sin^2\vartheta\cos\lambda + Q_{n_1,m}\frac{\mathrm{d}P_{n_2,m_2}}{\mathrm{d}\vartheta}T_{n,m,m_2} \\ &\times v^2\sin\vartheta\cos\vartheta\cos\lambda - Q_{n_1,m}P_{n_2,m_2}\frac{\mathrm{d}T_{n,m,m_2}}{\mathrm{d}\lambda}(v^2 - E^2\sin^2\vartheta)\sin\lambda\Bigg) \\ =& \frac{\sin\vartheta}{L^2 v}\Bigg(\big[-(n_1-m+1)\mathbb{i}EuQ_{n_1+1,m} + ((n_1+1)E^2 - (n_1-n_2)v^2)Q_{n_1,m}\big] \\ &\times P_{n_2,m_2}T_{n,m,m_2}\cos\lambda + m_2Q_{n_1,m}P_{n_2,m_2}T'_{n,m,m_2}E^2\sin\lambda\Bigg) + \frac{v}{L^2}\Bigg(Q_{n_1,m}\frac{m_2P_{n_2,m_2}}{\sin\vartheta} \\ &\times (T_{n,m,m_2}\cos\lambda - T'_{n,m,m_2}\sin\lambda) - Q_{n_1,m}P_{n_2+1,m_2+1}T_{n,m,m_2}\cos\lambda\Bigg). \end{aligned} \tag{15}$$

Then, when $m_2 = 0$,

$$
\begin{aligned}
\frac{\partial V_{n,n_1,n_2,m,m_2}}{\partial x} =& \frac{\sin\vartheta}{L^2 v}\Bigg(\left[-(n_1-m+1)\mathbb{i}EuQ_{n_1+1,m} + ((n_1+1)E^2-(n_1-n_2)v^2)Q_{n_1,m}\right] \\
& \times P_{n_2,0}T_{n,m,0}\cos\lambda\Bigg) - \frac{v}{L^2}Q_{n_1,m}P_{n_2+1,1}T_{n,m,0}\cos\lambda.
\end{aligned}
\tag{16}
$$

and when $m_2 \geqslant 1$,

$$
\begin{aligned}
\frac{\partial V_{n,n_1,n_2,m,m_2}}{\partial x} =& \frac{\sin\vartheta}{L^2 v}\Bigg(\left[-(n_1-m+1)\mathbb{i}EuQ_{n_1+1,m} + ((n_1+1)E^2-(n_1-n_2)v^2)Q_{n_1,m}\right] \\
& \times P_{n_2,m_2}T_{n,m,m_2}\cos\lambda + m_2 Q_{n_1,m}P_{n_2,m_2}T'_{n,m,m_2}E^2\sin\lambda\Bigg) \\
& + \frac{v}{L^2}\Bigg(\frac{1}{2}(n_2-m_2+1)(n_2-m_2+2)Q_{n_1,m}P_{n_2+1,m_2-1}T_{n,m,m_2-1} \\
& - \frac{1}{2}Q_{n_1,m}P_{n_2+1,m_2+1}T_{n,m,m_2+1}\Bigg).
\end{aligned}
\tag{17}
$$

The right-hand sides of Eqs. (16) and (17) can be expressed in the forms of the functions V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2}. Hence, for $m_2 \geqslant 1$,

$$
\begin{aligned}
\partial_x V_{n,n_1,n_2,m,m_2} =& a^x_{n_1,m}\mathbb{i}V_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2}V_{n,n_1,n_2,m,m_2} + c^y_{m_2}V'_{n,n_1,n_2,m,m_2} \\
& + d_{n_2,m_2}V_{n,n_1,n_2+1,m,m_2-1} + e_{m_2}V_{n,n_1,n_2+1,m,m_2+1},
\end{aligned}
\tag{18}
$$

where $\partial_x V_{n,n_1,n_2,m,m_2}$ is the simplified form of the derivative of V_{n,n_1,n_2,m,m_2}, and

$$
\begin{aligned}
a^x_{n_1,m} &= -(n_1-m+1)\frac{Eu\sin\vartheta\cos\lambda}{L^2 v}, \\
b^x_{n_1,n_2} &= ((n_1+1)E^2-(n_1-n_2)v^2)\frac{\sin\vartheta\cos\lambda}{L^2 v}, \\
c^y_{m_2} &= m_2\frac{E^2\sin\vartheta\sin\lambda}{L^2 v}, \\
d_{n_2,m_2} &= (n_2-m_2+1)(n_2-m_2+2)\frac{v}{2L^2}, \\
e_{m_2} &= -(1+\delta_{0,m_2})\frac{v}{2L^2}.
\end{aligned}
\tag{19}
$$

For $m_2 = 0$, when $m = 0$, $T_{n,0,1} = T_{n,0,0} \cos\lambda$ due to $T_{n,0,0} = C_{n,0}$ and $T'_{n,0,0} = 0$ with $S_{n,0} = 0$, and then $\partial_x V_{n,n_1,n_2,m,m_2}$ becomes

$$\partial_x V_{n,n_1,n_2,0,0} = a^x_{n_1,0} \mathbb{i} V_{n,n_1+1,n_2,0,0} + b^x_{n_1,n_2} V_{n,n_1,n_2,0,0} + e_0 V_{n,n_1,n_2+1,0,1}, \tag{20}$$

where the expression of the coefficient e_{m_2} in Eq. (18) is also suitable for $m_2 = 0$ in this equation, and when $m \geqslant 1$,

$$\partial_x V_{n,n_1,n_2,m,0} = a^x_{n_1,m} \mathbb{i} V_{n,n_1+1,n_2,m,0} + b^x_{n_1,n_2} V_{n,n_1,n_2,m,0} + e_0 V^c_{n,n_1,n_2+1,m,1}, \tag{21}$$

where $V^c_{n,n_1,n_2,m,m_2} = Q_{n_1,m} P_{n_2,m_2} T^c_{n,m,m_2}$ with $T^c_{n,m,m_2} = C_{n,m} \cos(m_2\lambda)$, and $T^c_{n,m,1} = T_{n,m,0} \cos\lambda$ due to $T_{n,m,0} = C_{n,m}$. The expressions of the coefficients $c^y_{m_2}$ and e_{m_2} hold for $m_2 = 0$, while the integer $m_2 \geqslant 1$ for the coefficient d_{n_2,m_2}. Since the coefficients $a^x_{n_1,m}$, $b^x_{n_1,n_2}$, $c^y_{m_2}$, d_{n_2,m_2}, e_{m_2} have no the fraction $1/\sin\vartheta$ and other singular factors, the expression of $\partial_x V_{n,n_1,n_2,m,m_2}$ in Eq. (18) is non-singular for any observation point outside the reference oblate spheroid. In fact, the expression is computable for the point with $L \neq 0$, i.e., outside the singular circle line $\partial\mathcal{E}_e$: $\{(x,y,z)|x^2+y^2 = E^2, z = 0\}$.

Likewise, when $m_2 \geqslant 1$ the derivative of the function V_{n,n_1,n_2,m,m_2} with respect to y can be expressed as

$$\begin{aligned}\partial_y V_{n,n_1,n_2,m,m_2} =& a^y_{n_1,m} \mathbb{i} V_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2} V_{n,n_1,n_2,m,m_2} - c^x_{m_2} V'_{n,n_1,n_2,m,m_2} \\ &+ d_{n_2,m_2} V'_{n,n_1,n_2+1,m,m_2-1} - e_{m_2} V'_{n,n_1,n_2+1,m,m_2+1},\end{aligned} \tag{22}$$

where

$$\begin{aligned} a^y_{n_1,m} &= -(n_1 - m + 1)\frac{Eu\sin\vartheta\sin\lambda}{L^2 v} \\ b^y_{n_1,n_2} &= ((n_1+1)E^2 - (n_1 - n_2)v^2)\frac{\sin\vartheta\sin\lambda}{L^2 v} \\ c^x_{m_2} &= m_2 \frac{E^2 \sin\vartheta\cos\lambda}{L^2 v}. \end{aligned} \tag{23}$$

When $m_2 = 0$, we can also have

$$\begin{aligned} \partial_y V_{n,n_1,n_2,0,0} &= a^y_{n_1,0} \mathbb{i} V_{n,n_1+1,n_2,0,0} + b^y_{n_1,n_2} V_{n,n_1,n_2,0,0} - e_0 V'_{n,n_1,n_2+1,0,1}, \\ \partial_y V_{n,n_1,n_2,m,0} &= a^y_{n_1,m} \mathbb{i} V_{n,n_1+1,n_2,m,0} + b^y_{n_1,n_2} V_{n,n_1,n_2,m,0} - e_0 V'^c_{n,n_1,n_2+1,m,1}, \end{aligned} \tag{24}$$

for $m = 0$ and $m \geqslant 1$, respectively, where $V'^c_{n,n_1,n_2,m,m_2} = Q_{n_1,m} P_{n_2,m_2} T'^c_{n,m,m_2}$ with $T'^c_{n,m,m_2} = -C_{n,m} \sin(m_2\lambda)$, $T'_{n,0,1} = -T_{n,0,0} \sin\lambda$ due to $T'_{n,0,0} = 0$, and $T'^c_{n,m,1} = -T_{n,m,0} \sin\lambda$ due to $T_{n,m,0} = C_{n,m}$.

From Eqs. (8), (9), (12) and the identity for the product of $\sin\vartheta$ and the derivative of $P_{n,m}$ (Gradshteyn and Ryzhik 2007, pp. 965) :

$$\sin\vartheta\frac{\mathrm{d}P_{n,m}}{\mathrm{d}\vartheta} = -(n+1)\cos\vartheta P_{n,m} + (n-m+1)P_{n+1,m}, \tag{25}$$

the derivative of the function V_{n,n_1,n_2,m,m_2} with respect to z is

$$\partial_z V_{n,n_1,n_2,m,m_2} = p_{n_1,m}\mathbb{i}V_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2}V_{n,n_1,n_2,m,m_2} + t_{n_2,m_2}V_{n,n_1,n_2+1,m,m_2}, \tag{26}$$

where $m_2 = 0, 1, 2, \cdots$, and

$$\begin{aligned} p_{n_1,m} &= -(n_1-m+1)\frac{E\cos\vartheta}{L^2}, \\ q_{n_1,n_2} &= -(n_1-n_2)\frac{u\cos\vartheta}{L^2}, \\ t_{n_2,m_2} &= -(n_2-m_2+1)\frac{u}{L^2}. \end{aligned} \tag{27}$$

Following the derivations of $\partial_x V_{n,n_1,n_2,m,m_2}$, $\partial_y V_{n,n_1,n_2,m,m_2}$ and $\partial_z V_{n,n_1,n_2,m,m_2}$, the derivatives of the function V'_{n,n_1,n_2,m,m_2} with respect to the coordinates x, y, z can be given by

$$\begin{aligned} \partial_x V'_{n,n_1,n_2,m,m_2} &= a^x_{n_1,m}\mathbb{i}V'_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2}V'_{n,n_1,n_2,m,m_2} - c^y_{m_2}V_{n,n_1,n_2,m,m_2} \\ &\quad + d_{n_2,m_2}V'_{n,n_1,n_2+1,m,m_2-1} + e_{m_2}V'_{n,n_1,n_2+1,m,m_2+1}, \\ \partial_y V'_{n,n_1,n_2,m,m_2} &= a^y_{n_1,m}\mathbb{i}V'_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2}V'_{n,n_1,n_2,m,m_2} + c^x_{m_2}V_{n,n_1,n_2,m,m_2} \\ &\quad - d_{n_2,m_2}V_{n,n_1,n_2+1,m,m_2-1} + e_{m_2}V_{n,n_1,n_2+1,m,m_2+1}, \\ \partial_z V'_{n,n_1,n_2,m,m_2} &= p_{n_1,m}\mathbb{i}V'_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2}V'_{n,n_1,n_2,m,m_2} + t_{n_2,m_2}V'_{n,n_1,n_2+1,m,m_2}, \end{aligned} \tag{28}$$

where the integer $m_2 \geqslant 1$ for the expressions of $\partial_x V'_{n,n_1,n_2,m,m_2}$ and $\partial_y V'_{n,n_1,n_2,m,m_2}$. When the integer $m_2 = 0$,

$$\begin{aligned} \partial_x V'_{n,n_1,n_2,0,0} &= 0, \\ \partial_y V'_{n,n_1,n_2,0,0} &= 0, \end{aligned} \tag{29}$$

for $m = 0$, and

$$\begin{aligned} \partial_x V'_{n,n_1,n_2,m,0} &= a^x_{n_1,m}\mathbb{i}V'_{n,n_1+1,n_2,m,0} + b^x_{n_1,n_2}V'_{n,n_1,n_2,m,0} + e_0 V'^s_{n,n_1,n_2+1,m,1}, \\ \partial_y V'_{n,n_1,n_2,m,0} &= a^y_{n_1,m}\mathbb{i}V'_{n,n_1+1,n_2,m,0} + b^y_{n_1,n_2}V'_{n,n_1,n_2,m,0} + e_0 V^s_{n,n_1,n_2+1,m,1}, \end{aligned} \tag{30}$$

for $m \geqslant 1$, where the functions $V^s_{n,n_1,n_2,m,m_2} = Q_{n_1,m}P_{n_2,m_2}T^s_{n,m,m_2}$ with $T^s_{n,m,m_2} = S_{n,m}\sin(m_2\lambda)$, $V'^s_{n,n_1,n_2,m,m_2} = Q_{n_1,m}P_{n_2,m_2}T'^s_{n,m,m_2}$ with $T'^s_{n,m,m_2} = S_{n,m}\cos(m_2\lambda)$, $T'^s_{n,m,1} = T'_{n,m,0}\cos\lambda$ and $T^s_{n,m,1} = T'_{n,m,0}\sin\lambda$ due to $T'_{n,m,0} = S_{n,m}$.

Equations (18), (20), (21), (22), (24), (26), (28), (29) and (30) also hold for derivatives of the components V^c_{n,n_1,n_2,m,m_2}, V^s_{n,n_1,n_2,m,m_2}, V'^c_{n,n_1,n_2,m,m_2} and V'^s_{n,n_1,n_2,m,m_2} of the functions V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2}. The superscripts c or s of all the functions V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2} on both sides of Eqs. (18), (22), (26) and (28) are the same. Then, the third terms in the right-hand side of Eqs. (20), (21), (24) and (30) are eliminated when their superscripts are different from the left-hand side of the equations (the superscripts of the antiderivatives), where the terms $V_{n,n_1,n_2+1,0,1}$ and $V'_{n,n_1,n_2+1,0,1}$ contain the implied superscripts c, i.e. $V^s_{n,n_1,n_2+1,0,1} = V'^s_{n,n_1,n_2+1,0,1} = 0$.

2.3 Normalization of the relations

The normalized forms of V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2} can be defined by the normalizations of the Legendre functions and harmonic coefficients, and are given as

$$\begin{aligned}\overline{V}_{n,n_1,n_2,m,m_2} &= \frac{\mathbb{i}^{n_1-n} N_{n_2,m_2} V_{n,n_1,n_2,m,m_2}}{N_{n,m} Q^b_{n,m}} = \widehat{Q}_{n_1,n,m} \overline{P}_{n_2,m_2} \overline{T}_{n,m,m_2} \\ \overline{V}'_{n,n_1,n_2,m,m_2} &= \frac{\mathbb{i}^{n_1-n} N_{n_2,m_2} V'_{n,n_1,n_2,m,m_2}}{N_{n,m} Q^b_{n,m}} = \widehat{Q}_{n_1,n,m} \overline{P}_{n_2,m_2} \overline{T}'_{n,m,m_2}\end{aligned} \tag{31}$$

where the symbol $\widehat{Q}_{n_1,n,m}(u/E) = \mathbb{i}^{n_1-n} Q_{n_1,m}/Q^b_{n,m}$, the factor $\mathbb{i}^{n_1-n}$ is used to eliminate the imaginary term in $Q_{n_1,m}/Q^b_{n,m}$, and

$$\begin{aligned}\overline{T}_{n,m,m_2} &= \frac{T_{n,m,m_2}}{N_{n,m}} = \overline{C}_{n,m} \cos(m_2\lambda) + \overline{S}_{n,m} \sin(m_2\lambda), \\ \overline{T}'_{n,m,m_2} &= \frac{T'_{n,m,m_2}}{N_{n,m}} = -\overline{C}_{n,m} \sin(m_2\lambda) + \overline{S}_{n,m} \cos(m_2\lambda).\end{aligned} \tag{32}$$

Then, the gravitational potential can be written in the form of $\overline{V}_{n,n,n,m,m}$ as

$$V = \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \overline{V}_{n,n,n,m,m}. \tag{33}$$

From Eq. (2), the term $\widehat{Q}_{n_1,n,m}$ in Eq. (31) with extending to the third-order derivatives of the potential ($n_1 = n+1, n+2, n+3$) can be computed by

$$\begin{aligned}\widehat{Q}_{n+1,n,m} &= \frac{(n+m+1)E}{(2n+3)v} \frac{F_{n+1,m}}{F_{n,m}} \widehat{Q}_{n,n,m} \\ \widehat{Q}_{n+2,n,m} &= \frac{(n+m+2)E}{(2n+5)v} \frac{F_{n+2,m}}{F_{n+1,m}} \widehat{Q}_{n+1,n,m} \\ \widehat{Q}_{n+3,n,m} &= \frac{(n+m+3)E}{(2n+7)v} \frac{F_{n+3,m}}{F_{n+2,m}} \widehat{Q}_{n+2,n,m}\end{aligned} \tag{34}$$

where the computation of $\widehat{Q}_{n,n,m} = Q_{n,m}/Q^b_{n,m}$ has been given in Eq. (4).

Substituting Eq. (31) into Eqs. (18), (22), (26) and (28), we obtain the normalized relations for the derivatives of V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2}:

$$\begin{aligned}
\partial_x \overline{V}_{n,n_1,n_2,m,m_2} =& a^x_{n_1,m}\overline{V}_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2}\overline{V}_{n,n_1,n_2,m,m_2} + c^y_{m_2}\overline{V}'_{n,n_1,n_2,m,m_2} \\
&+ \overline{d}_{n_2,m_2}\overline{V}_{n,n_1,n_2+1,m,m_2-1} + \overline{e}_{n_2,m_2}\overline{V}_{n,n_1,n_2+1,m,m_2+1} \\
\partial_y \overline{V}_{n,n_1,n_2,m,m_2} =& a^y_{n_1,m}\overline{V}_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2}\overline{V}_{n,n_1,n_2,m,m_2} - c^x_{m_2}\overline{V}'_{n,n_1,n_2,m,m_2} \\
&+ \overline{d}_{n_2,m_2}\overline{V}'_{n,n_1,n_2+1,m,m_2-1} - \overline{e}_{n_2,m_2}\overline{V}'_{n,n_1,n_2+1,m,m_2+1} \\
\partial_z \overline{V}_{n,n_1,n_2,m,m_2} =& p_{n_1,m}\overline{V}_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2}\overline{V}_{n,n_1,n_2,m,m_2} + \bar{t}_{n_2,m_2}\overline{V}_{n,n_1,n_2+1,m,m_2}
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
\partial_x \overline{V}'_{n,n_1,n_2,m,m_2} =& a^x_{n_1,m}\overline{V}'_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2}\overline{V}'_{n,n_1,n_2,m,m_2} - c^y_{m_2}\overline{V}_{n,n_1,n_2,m,m_2} \\
&+ \overline{d}_{n_2,m_2}\overline{V}'_{n,n_1,n_2+1,m,m_2-1} + \overline{e}_{n_2,m_2}\overline{V}'_{n,n_1,n_2+1,m,m_2+1} \\
\partial_y \overline{V}'_{n,n_1,n_2,m,m_2} =& a^y_{n_1,m}\overline{V}'_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2}\overline{V}'_{n,n_1,n_2,m,m_2} + c^x_{m_2}\overline{V}_{n,n_1,n_2,m,m_2} \\
&- \overline{d}_{n_2,m_2}\overline{V}_{n,n_1,n_2+1,m,m_2-1} + \overline{e}_{n_2,m_2}\overline{V}_{n,n_1,n_2+1,m,m_2+1} \\
\partial_z \overline{V}'_{n,n_1,n_2,m,m_2} =& p_{n_1,m}\overline{V}'_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2}\overline{V}'_{n,n_1,n_2,m,m_2} + \bar{t}_{n_2,m_2}\overline{V}'_{n,n_1,n_2+1,m,m_2}
\end{aligned} \tag{36}$$

where the integer $m_2 \geqslant 1$ for the derivatives with respect to the coordinates x, y, and $\overline{d}_{n_2,m_2}$, $\overline{e}_{n_2,m_2}$, $\bar{t}_{n_2,m_2}$ are the normalization of the coefficients d_{n_2,m_2}, e_{m_2}, t_{n_2,m_2}, i.e.,

$$\begin{aligned}
\overline{d}_{n_2,m_2} &= \frac{N_{n_2,m_2}d_{n_2,m_2}}{N_{n_2+1,m_2-1}} = w^d_{n_2,m_2}\frac{v}{L^2} \\
\overline{e}_{n_2,m_2} &= \frac{N_{n_2,m_2}e_{m_2}}{N_{n_2+1,m_2+1}} = -w^e_{n_2,m_2}\frac{v}{L^2} \\
\bar{t}_{n_2,m_2} &= \frac{N_{n_2,m_2}t_{n_2,m_2}}{N_{n_2+1,m_2}} = -w^t_{n_2,m_2}\frac{u}{L^2}
\end{aligned} \tag{37}$$

with the notations

$$\begin{aligned}
w^d_{n_2,m_2} &= \frac{1}{2}\sqrt{\frac{(2-\delta_{0,m_2})(2n_2+1)(n_2-m_2+1)(n_2-m_2+2)}{(2-\delta_{0,m_2-1})(2n_2+3)}} \\
w^e_{n_2,m_2} &= \sqrt{\frac{(2n_2+1)(n_2+m_2+1)(n_2+m_2+2)}{(2-\delta_{0,m_2})(2-\delta_{0,m_2+1})(2n_2+3)}} \\
w^t_{n_2,m_2} &= \sqrt{\frac{(2n_2+1)(n_2-m_2+1)(n_2+m_2+1)}{2n_2+3}}
\end{aligned} \tag{38}$$

When $m_2 = 0$, from Eqs. (20), (21), (24), (29), (30) and (31) the normalized forms of the derivatives with respect to x, y are

$$\begin{aligned}
&\partial_x \overline{V}_{n,n_1,n_2,0,0} = a^x_{n_1,0}\overline{V}_{n,n_1+1,n_2,0,0} + b^x_{n_1,n_2}\overline{V}_{n,n_1,n_2,0,0} + \overline{e}_{n_2,0}\overline{V}_{n,n_1,n_2+1,0,1} \\
&\partial_y \overline{V}_{n,n_1,n_2,0,0} = a^y_{n_1,0}\overline{V}_{n,n_1+1,n_2,0,0} + b^y_{n_1,n_2}\overline{V}_{n,n_1,n_2,0,0} - \overline{e}_{n_2,0}\overline{V}'_{n,n_1,n_2+1,0,1} \\
&\partial_x \overline{V}'_{n,n_1,n_2,0,0} = 0 \\
&\partial_y \overline{V}'_{n,n_1,n_2,0,0} = 0
\end{aligned} \tag{39}$$

for $m = 0$, and

$$\begin{aligned}
&\partial_x \overline{V}_{n,n_1,n_2,m,0} = a^x_{n_1,m}\overline{V}_{n,n_1+1,n_2,m,0} + b^x_{n_1,n_2}\overline{V}_{n,n_1,n_2,m,0} + \overline{e}_{n_2,0}\overline{V}^c_{n,n_1,n_2+1,m,1} \\
&\partial_y \overline{V}_{n,n_1,n_2,m,0} = a^y_{n_1,m}\overline{V}_{n,n_1+1,n_2,m,0} + b^y_{n_1,n_2}\overline{V}_{n,n_1,n_2,m,0} - \overline{e}_{n_2,0}\overline{V}'^c_{n,n_1,n_2+1,m,1} \\
&\partial_x \overline{V}'_{n,n_1,n_2,m,0} = a^x_{n_1,m}\overline{V}'_{n,n_1+1,n_2,m,0} + b^x_{n_1,n_2}\overline{V}'_{n,n_1,n_2,m,0} + \overline{e}_{n_2,0}\overline{V}'^s_{n,n_1,n_2+1,m,1} \\
&\partial_y \overline{V}'_{n,n_1,n_2,m,0} = a^y_{n_1,m}\overline{V}'_{n,n_1+1,n_2,m,0} + b^y_{n_1,n_2}\overline{V}'_{n,n_1,n_2,m,0} + \overline{e}_{n_2,0}\overline{V}^s_{n,n_1,n_2+1,m,1}
\end{aligned} \tag{40}$$

for $m \geqslant 1$. Equations (31), (35), (36), (39) and (40) also hold for the c- or s-component of the functions V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2}. The coefficients $\overline{d}_{n_2,m_2}$ and $\overline{e}_{n_2,m_2}$ with getting rid of the sign and number parts are the same, i.e., the function v/L^2, and then the computations of their derivatives are the equivalent. When the integers in the subscripts of the coefficient $a^x_{n_1,m}$, $a^y_{n_1,m}$, $c^x_{m_2}$, $c^y_{m_2}$, $\overline{d}_{n_2,m_2}$, $\overline{e}_{n_2,m_2}$, $p_{n_1,m}$, q_{n_1,n_2} and $\overline{t}_{n_2,m_2}$ is changing, we just need change the number parts of the derivatives of the coefficient without changing the function part. The linear relations (35), (36), (39) and (40) do not contain singular factors, and then are all non-singular for arbitrary external observation point outside the Brillouin oblate spheroid. These inferences can be applied for the arbitrary-order derivatives.

2.4 Meanings of the superscripts of the coefficients of the relations

We now discuss the meanings of the superscripts x and y in Eqs. (19) and (23). Assuming the coefficients $a^x_{n_1,m}$ and $a^y_{n_1,m}$ are the derivatives of a function $a_{n_1,m}$ with respect to x- and y-coordinate, from Eq. (8) we have

$$\partial_u a_{n_1,m} = -(n_1 - m + 1)\frac{E}{v^2}. \tag{41}$$

Hence,

$$a_{n_1,m} = -\int (n_1 - m + 1)\frac{E}{v^2}\mathrm{d}u = -(n_1 - m + 1)\arctan\left(\frac{u}{E}\right), \tag{42}$$

where there is no constant term in the antiderivative. The reverse is also true, i.e., the relations $\partial_x a_{n_1,m} = a^x_{n_1,m}$ and $\partial_y a_{n_1,m} = a^y_{n_1,m}$ both hold when $a_{n_1,m}$ takes the function in Eq. (42). Furthermore, the following relation also holds: $\partial_z a_{n_1,m} = p_{n_1,m}$, and then the symbol $a^z_{n_1,m}$ can be used to represent $p_{n_1,m}$ without affecting any results of this paper, i.e. $p_{n_1,m} = a^z_{n_1,m}$. Likewise, we assume

$$\begin{aligned} b_{n_1,n_2} &= \int \Big((n_1+1)\frac{E^2}{uv^2} - (n_1-n_2)\frac{1}{u}\Big)\mathrm{d}u = (n_2+1)\ln\Big(\frac{u}{E}\Big) - (n_1+1)\ln\Big(\frac{v}{E}\Big), \\ c_{m_2} &= \int m_2 \frac{E^2}{uv^2}\mathrm{d}u = m_2 \ln\Big(\frac{u}{v}\Big), \end{aligned} \tag{43}$$

where the quantity E in the denominators is used to make the logarithmic functions be dimensionless, and obtain the relations: $\partial_x b_{n_1,n_2} = b^x_{n_1,n_2}$, $\partial_y b_{n_1,n_2} = b^y_{n_1,n_2}$, $\partial_x c_{m_2} = c^x_{m_2}$, and $\partial_y c_{m_2} = c^y_{m_2}$. It is obvious that $a_{n_1,m}$, b_{n_1,n_2} and c_{m_2} are the functions of the u-coordinate independent of ϑ- and λ-coordinates. Therefore, $a^{x_{l_1}x_{l_2}\cdots x_{l_k}}_{n_1,m}$, $b^{x_{l_1}x_{l_2}\cdots x_{l_k}}_{n_1,n_2}$ or $c^{x_{l_1}x_{l_2}\cdots x_{l_k}}_{m_2}$ can be regarded as the kth-order derivative of $a_{n_1,m}$, b_{n_1,n_2} or c_{m_2}, where the numbers $l_1, l_2, \cdots, l_k = 1, 2, 3$ ($l_1 \neq 3$ for b_{n_1,n_2} and c_{m_2}), the coordinates (x_1, x_2, x_3) denote (x, y, z) and the Cartesian coordinate symbols are served as the superscripts to denote the derivatives to the coordinates, even if it is primitively deduced from the coefficients $a^x_{n_1,m}$, $a^y_{n_1,m}$, $a^z_{n_1,m}$, $b^x_{n_1,n_2}$, $b^y_{n_1,n_2}$, $c^x_{m_2}$ and $c^y_{m_2}$ in Eqs. (19), (23) and (27), and the quantity is identical when the sort of the superscript $x_{l_1}x_{l_2}\cdots x_{l_k}$ changes ($l_1 \neq 3$ in all sorts for b_{n_1,n_2} and c_{m_2}). This property is useful for the subsequent derivations of this paper.

3 Oblate spheroidal harmonic expressions of the derivatives

3.1 First-order derivatives

Substituting the derivative relations (33) into Eq. (35), we obtain the expressions of the first-order derivatives of the gravitational potential

$$\begin{aligned} V_x =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\Big(a^x_{n,m}\overline{V}_{n,n+1,n,m,m} + b^x_{n,n}\overline{V}_{n,n,n,m,m} + \overline{e}_{n,m}\overline{V}_{n,n,n+1,m,m+1}\Big) \\ &+ \frac{\mu}{a}\sum_{n=1}^{+\infty}\sum_{m=1}^{n}\Big(c^y_m\overline{V}'_{n,n,n,m,m} + \overline{d}_{n,m}\overline{V}_{n,n,n+1,m,m-1}\Big), \\ V_y =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\Big(a^y_{n,m}\overline{V}_{n,n+1,n,m,m} + b^y_{n,n}\overline{V}_{n,n,n,m,m} - \overline{e}_{n,m}\overline{V}'_{n,n,n+1,m,m+1}\Big) \\ &+ \frac{\mu}{a}\sum_{n=1}^{+\infty}\sum_{m=1}^{n}\Big(-c^x_m\overline{V}'_{n,n,n,m,m} + \overline{d}_{n,m}\overline{V}'_{n,n,n+1,m,m-1}\Big), \\ V_z =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\Big(p_{n,m}\overline{V}_{n,n+1,n,m,m} + \overline{t}_{n,m}\overline{V}_{n,n,n+1,m,m}\Big). \end{aligned} \tag{44}$$

where the coordinate symbols x, y, z are served as the subscripts of V to represent the derivatives to the coordinates, also applied for the high-order derivatives in Sects. 3.2 and 3.3 and the oblate spheroidal coordinates in Sect. 5.2.

3.2 Second-order derivatives

Equation (44) for the first-order derivatives can be rewritten as

$$V_{x_i} = \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \sum_{l=1}^{\ell_1} \mathcal{C}^{x_i}_{n,m,l} \overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}, \tag{45}$$

where the subscript $i = 1, 2, 3$, ℓ_1 is the number of the subitems in the derivatives ($\ell_1 = 5$ for V_x and V_y, and $\ell_1 = 2$ for V_z), $n_{1,l}$, $n_{2,l}$ and $m_{2,l}$ denote the integer subscripts n_1, n_2 and m_2 of the l-th term, $\mathcal{C}^{x_i}_{n,m,l}$ is the coefficient of the l-th term, and $\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}} = \overline{V}_{n,n_{1,l},n_{2,l},m,m_{2,l}}$ or $\overline{V}'_{n,n_{1,l},n_{2,l},m,m_{2,l}}$, as well as their c- or s-component. The indices n and m of some terms in Eq. (45) start from 1 instead of 0, i.e., the coefficients of these terms being eliminated ($\mathcal{C}^{x_i}_{n,m,l} = 0$) when $m = 0$. For the term $\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}$ in Eq. (45), applying the derivative relations (35) and (36) we get the second-order derivatives of the gravitational potential

$$\begin{aligned} V_{x_i x_j} &= \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \sum_{l=1}^{\ell_1} \left(\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}} \partial_{x_j} \mathcal{C}^{x_i}_{n,m,l} + \mathcal{C}^{x_i}_{n,m,l} \partial_{x_j} \overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}} \right) \\ &= \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \sum_{l=1}^{\ell_2} \mathcal{C}^{x_i x_j}_{n,m,l} \overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}, \end{aligned} \tag{46}$$

where the subscript $j = 1, 2, 3$. There are only six independent quantities (including V_{xx}, V_{xy}, V_{xz}, V_{yy}, V_{yz} and V_{zz}) due to the symmetry of the second-order derivatives. Then, V_{xx}, V_{xy} and V_{xz} can be regarded as the derivatives of V_x with respect to the three Cartesian coordinates, V_{yy} and V_{yz} the derivatives of V_y with respect to the y- and z-coordinates, and V_{zz} the derivative of V_z with respect to the z-coordinate.

According to Eq. (8), the derivatives $\partial_{x_j} \mathcal{C}^{x_i}_{\underline{n},m,l}$ in Eq. (46), i.e. the derivatives of the coefficients $a^x_{n,m}$, $b^x_{n,n}$, c^y_m, $a^y_{n,m}$, $b^y_{n,n}$, c^x_m, $\overline{d}_{n,m}$, $\overline{e}_{n,m}$, $p_{n,m}$, and $\overline{t}_{n,m}$ with respect to the Cartesian coordinates, can be solved. The coordinate symbols x, y and z are added on the superscripts of these coefficients to represent corresponding derivatives, and we do the same in the following of this paper. From Sect. 2.4, the derivatives for $a_{n_1,m}$, b_{n_1,n_2} or c_{m_2} with same combinations of superscripts are identical (i.e., the superscripts have same numbers of x, y and z and their sorts may be different), and then some derivatives are not needed to be computed repeatedly, e.g. $c^{yx}_m = c^{xy}_m$. Now the expressions of the derivatives can be obtained, as listed in Table 1, where the

Table 1 Expressions of the derivatives $\partial_{x_j}\mathcal{C}^{x_i}_{n,m,l}$ $(1 \leqslant i \leqslant j \leqslant 3)$ in Eq. (46)

$a^{xx}_{n,m} = -(n-m+1)f^0_2 u + (n-m+1)f^a_2 \cos^2\lambda$	$a^{yz}_{n,m} = (n-m+1)f^{az}_2 \sin\lambda$
$b^{xx}_{n,n} = (n+1)f^0_2 E - (n+1)f^{bc}_2 \cos^2\lambda$	$b^{yz}_{n,n} = -(n+1)f^{bcz}_2 \sin\lambda$
$c^{xy}_{m} = -mf^{bc}_2 \sin\lambda\cos\lambda$	$c^{xz}_{m} = -mf^{bcz}_2 \cos\lambda$
$a^{xy}_{n,m} = (n-m+1)f^a_2 \sin\lambda\cos\lambda$	$p^{z}_{n,m} = (n-m+1)f^{pz}_2$
$b^{xy}_{n,n} = -(n+1)f^{bc}_2 \sin\lambda\cos\lambda$	$\bar{t}^{z}_{n,m} = w^t_{n,m} f^{tz}_2$
$c^{yy}_{m} = mf^0_2 E - mf^{bc}_2 \sin^2\lambda$	$\bar{d}^{x}_{n,m} = -w^d_{n,m} f^{de}_2 \cos\lambda$
$a^{xz}_{n,m} = (n-m+1)f^{az}_2 \cos\lambda$	$\bar{e}^{x}_{n,m} = w^e_{n,m} f^{de}_2 \cos\lambda$
$b^{xz}_{n,n} = -(n+1)f^{bcz}_2 \cos\lambda$	$\bar{d}^{y}_{n,m} = -w^d_{n,m} f^{de}_2 \sin\lambda$
$c^{yz}_{m} = -mf^{bcz}_2 \sin\lambda$	$\bar{e}^{y}_{n,m} = w^e_{n,m} f^{de}_2 \sin\lambda$
$a^{yy}_{n,m} = -(n-m+1)f^0_2 u + (n-m+1)f^a_2 \sin^2\lambda$	$\bar{d}^{z}_{n,m} = -w^d_{n,m} f^{dez}_2$
$b^{yy}_{n,n} = (n+1)f^0_2 E - (n+1)f^{bc}_2 \sin^2\lambda$	$\bar{e}^{z}_{n,m} = w^e_{n,m} f^{dez}_2$

notations f_2 represent

$$\begin{aligned}
&f^0_2 = \frac{E}{L^2 v^2},\; f^a_2 = \frac{Eu\sin^2\vartheta}{L^6 v^2}\Big(3u^2v^2 - 3E^2v^2 + (v^2+2E^2)E^2\sin^2\vartheta\Big),\\
&f^{bc}_2 = \frac{E^2\sin^2\vartheta}{L^6 v^2}\Big(4u^2v^2 - 2E^2v^2 + 2E^4\sin^2\vartheta\Big),\; f^{de}_2 = \frac{\sin\vartheta}{L^6}\Big(u^2v^2 - 2E^2v^2 + (3v^2-E^2)E^2\sin^2\vartheta\Big),\\
&f^{az}_2 = \frac{Ev\sin\vartheta\cos\vartheta}{L^6}\Big(3v^2 - 4E^2 + E^2\sin^2\vartheta\Big),\; f^{bcz}_2 = \frac{4E^2uv\sin\vartheta\cos\vartheta}{L^6},\; f^{dez}_2 = \frac{uv\cos\vartheta}{L^6}\Big(v^2 + 3E^2\sin^2\vartheta\Big),\\
&f^{pz}_2 = \frac{Eu}{L^6}\Big(2v^2 - (3v^2-2E^2)\sin^2\vartheta - E^2\sin^4\vartheta\Big),\; f^{tz}_2 = \frac{\cos\vartheta}{L^6}\Big(u^2v^2 - E^2v^2 + (3v^2-2E^2)E^2\sin^2\vartheta\Big).
\end{aligned} \tag{47}$$

The derivatives $a^{xx}_{n,m}$, $b^{xx}_{n,n}$, c^{yy}_{m}, $a^{yy}_{n,m}$ and $b^{yy}_{n,n}$ contain two parts: $f'(u,\vartheta)$, a function of the coordinates u and ϑ, and $f''(u,\vartheta,\lambda)$, a function of the coordinates u, ϑ and λ. In order to get third-order derivatives, the quantities in Table 1 may be differentiated again. Since $f'(u,\vartheta)$ is independent of the longitude coordinate λ, the derivatives of $f'(u,\vartheta)$ with respect to the three Cartesian coordinates do not contain the fraction $1/\sin\vartheta$. The function $f''(u,\vartheta,\lambda)$ and the quantities in Table 1 except $a^{xx}_{n,m}$, $b^{xx}_{n,n}$, c^{yy}_{m}, $a^{yy}_{n,m}$, $b^{yy}_{n,n}$ contain at least one $\sin\vartheta$ factor, and then their derivatives with respect to the three Cartesian coordinates also have no the factor $1/\sin\vartheta$. Now the second-order derivatives $V_{x_ix_j}$ in Eq. (46) can be specifically expressed, and their results are given in Table 2, where the expressions of the coefficients $\mathcal{C}^{x_ix_j}_{n,m,l}$ are listed in Table 3 ($\ell_2 = 16$ for V_{xx} and V_{yy}, $\ell_2 = 5$ for V_{zz}, $\ell_2 = 18$ for V_{xy}, and $\ell_2 = 14$ for V_{xz} and V_{yz}). For some terms in Eq. (46), their indices n and m start from 1 or 2, i.e. their coefficients $\mathcal{C}^{x_i}_{n,m,l} = 0$ with $m = 0$ or $m < 2$.

Table 2 Expressions of the derivatives $V_{x_i x_j}$ $(1 \leqslant i \leqslant j \leqslant 3)$

$$
\begin{aligned}
V_{xx} = \tfrac{\mu}{a} \textstyle\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \big(& a_{n,m,4}^{xx} \overline{V}_{n,n+2,n,m,m} + a_{n,m,3}^{xx} \overline{V}'_{n,n+1,n,m,m} + a_{n,m,2}^{xx} \overline{V}_{n,n+1,n,m,m} \\
& + a_{n,m,1}^{xx} \overline{V}'_{n,n,n,m,m} + a_{n,m,0}^{xx} \overline{V}_{n,n,n,m,m} + \bar{b}_{n,m,5}^{xx} \overline{V}_{n,n+1,n+1,m,m+1} \\
& + \bar{b}_{n,m,4}^{xx} \overline{V}'_{n,n,n+1,m,m+1} + \bar{b}_{n,m,3}^{xx} \overline{V}_{n,n,n+1,m,m+1} + \bar{d}_{n,m,2}^{xx} \overline{V}_{n,n,n+2,m,m} \\
& + \bar{e}_{n,m,0}^{xx} \overline{V}_{n,n,n+2,m,m+2} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=1}^{+\infty} \sum_{m=1}^{n} \big(& \bar{b}_{n,m,2}^{xx} \overline{V}_{n,n+1,n+1,m,m-1} + \bar{b}_{n,m,1}^{xx} \overline{V}'_{n,n,n+1,m,m-1} + \bar{b}_{n,m,0}^{xx} \overline{V}_{n,n,n+1,m,m-1} \\
& + \bar{d}_{n,m,1}^{xx} \overline{V}^{c}_{n,n,n+2,m,m} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=2}^{+\infty} \sum_{m=2}^{n} \big(& \bar{d}_{n,m,0}^{xx} \overline{V}_{n,n,n+2,m,m-2} + \bar{d}_{n,m,1}^{xx} \overline{V}^{s}_{n,n,n+2,m,m} \big) \\
V_{yy} = \tfrac{\mu}{a} \textstyle\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \big(& a_{n,m,4}^{yy} \overline{V}_{n,n+2,n,m,m} + a_{n,m,3}^{yy} \overline{V}'_{n,n+1,n,m,m} + a_{n,m,2}^{yy} \overline{V}_{n,n+1,n,m,m} \\
& + a_{n,m,1}^{yy} \overline{V}'_{n,n,n,m,m} + a_{n,m,0}^{yy} \overline{V}_{n,n,n,m,m} + \bar{b}_{n,m,5}^{yy} \overline{V}'_{n,n+1,n+1,m,m+1} \\
& + \bar{b}_{n,m,4}^{yy} \overline{V}'_{n,n,n+1,m,m+1} + \bar{b}_{n,m,3}^{yy} \overline{V}_{n,n,n+1,m,m+1} + \bar{d}_{n,m,2}^{yy} \overline{V}_{n,n,n+2,m,m} \\
& + \bar{e}_{n,m,0}^{yy} \overline{V}_{n,n,n+2,m,m+2} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=1}^{+\infty} \sum_{m=1}^{n} \big(& \bar{b}_{n,m,2}^{yy} \overline{V}'_{n,n+1,n+1,m,m-1} + \bar{b}_{n,m,1}^{yy} \overline{V}'_{n,n,n+1,m,m-1} + \bar{b}_{n,m,0}^{yy} \overline{V}_{n,n,n+1,m,m-1} \\
& + \bar{d}_{n,m,1}^{yy} \overline{V}^{s}_{n,n,n+2,m,m} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=2}^{+\infty} \sum_{m=2}^{n} \big(& \bar{d}_{n,m,0}^{yy} \overline{V}_{n,n,n+2,m,m-2} + \bar{d}_{n,m,1}^{yy} \overline{V}^{c}_{n,n,n+2,m,m} \big) \\
V_{zz} = \tfrac{\mu}{a} \textstyle\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \big(& a_{n,m,1}^{zz} \overline{V}_{n,n+2,n,m,m} + a_{n,m,0}^{zz} \overline{V}_{n,n+1,n,m,m} + \bar{d}_{n,m,2}^{zz} \overline{V}_{n,n+1,n+1,m,m} \\
& + \bar{d}_{n,m,1}^{zz} \overline{V}_{n,n,n+2,m,m} + \bar{d}_{n,m,0}^{zz} \overline{V}_{n,n,n+1,m,m} \big) \\
V_{xy} = \tfrac{\mu}{a} \textstyle\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \big(& a_{n,m,4}^{xy} \overline{V}_{n,n+2,n,m,m} + a_{n,m,3}^{xy} \overline{V}'_{n,n+1,n,m,m} + a_{n,m,2}^{xy} \overline{V}_{n,n+1,n,m,m} \\
& + a_{n,m,1}^{xy} \overline{V}'_{n,n,n,m,m} + a_{n,m,0}^{xy} \overline{V}_{n,n,n,m,m} + \bar{b}_{n,m,7}^{xy} \overline{V}'_{n,n+1,n+1,m,m+1} \\
& + \bar{b}_{n,m,6}^{xy} \overline{V}_{n,n+1,n+1,m,m+1} + \bar{b}_{n,m,5}^{xy} \overline{V}'_{n,n,n+1,m,m+1} + \bar{b}_{n,m,4}^{xy} \overline{V}_{n,n,n+1,m,m+1} \\
& + \bar{d}_{n,m,2}^{xy} \overline{V}'_{n,n,n+2,m,m} + \bar{e}_{n,m,0}^{xy} \overline{V}'_{n,n,n+2,m,m+2} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=1}^{+\infty} \sum_{m=1}^{n} \big(& \bar{b}_{n,m,3}^{xy} \overline{V}'_{n,n+1,n+1,m,m-1} + \bar{b}_{n,m,2}^{xy} \overline{V}_{n,n+1,n+1,m,m-1} + \bar{b}_{n,m,1}^{xy} \overline{V}'_{n,n,n+1,m,m-1} \\
& + \bar{b}_{n,m,0}^{xy} \overline{V}_{n,n,n+1,m,m-1} + \bar{d}_{n,m,1}^{xy} \overline{V}'^{c}_{n,n,n+2,m,m} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=2}^{+\infty} \sum_{m=2}^{n} \big(& \bar{d}_{n,m,0}^{xy} \overline{V}'_{n,n,n+2,m,m-2} + \bar{d}_{n,m,1}^{xy} \overline{V}'^{s}_{n,n,n+2,m,m} \big) \\
V_{xz} = \tfrac{\mu}{a} \textstyle\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \big(& a_{n,m,4}^{xz} \overline{V}_{n,n+2,n,m,m} + a_{n,m,3}^{xz} \overline{V}'_{n,n+1,n,m,m} + a_{n,m,2}^{xz} \overline{V}_{n,n+1,n,m,m} \\
& + a_{n,m,1}^{xz} \overline{V}'_{n,n,n,m,m} + a_{n,m,0}^{xz} \overline{V}_{n,n,n,m,m} + \bar{b}_{n,m,2}^{xz} \overline{V}_{n,n+1,n+1,m,m} \\
& + \bar{b}_{n,m,1}^{xz} \overline{V}'_{n,n,n+1,m,m} + \bar{b}_{n,m,0}^{xz} \overline{V}_{n,n,n+1,m,m} + \bar{d}_{n,m,5}^{xz} \overline{V}_{n,n+1,n+1,m,m+1} \\
& + \bar{d}_{n,m,4}^{xz} \overline{V}_{n,n,n+2,m,m+1} + \bar{d}_{n,m,3}^{xz} \overline{V}_{n,n,n+1,m,m+1} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=1}^{+\infty} \sum_{m=1}^{n} \big(& \bar{d}_{n,m,2}^{xz} \overline{V}_{n,n+1,n+1,m,m-1} + \bar{d}_{n,m,1}^{xz} \overline{V}_{n,n,n+2,m,m-1} + \bar{d}_{n,m,0}^{xz} \overline{V}_{n,n,n+1,m,m-1} \big) \\
V_{yz} = \tfrac{\mu}{a} \textstyle\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \big(& a_{n,m,4}^{yz} \overline{V}_{n,n+2,n,m,m} + a_{n,m,3}^{yz} \overline{V}'_{n,n+1,n,m,m} + a_{n,m,2}^{yz} \overline{V}_{n,n+1,n,m,m} \\
& + a_{n,m,1}^{yz} \overline{V}'_{n,n,n,m,m} + a_{n,m,0}^{yz} \overline{V}_{n,n,n,m,m} + \bar{b}_{n,m,2}^{yz} \overline{V}_{n,n+1,n+1,m,m} \\
& + \bar{b}_{n,m,1}^{yz} \overline{V}'_{n,n,n+1,m,m} + \bar{b}_{n,m,0}^{yz} \overline{V}_{n,n,n+1,m,m} + \bar{d}_{n,m,5}^{yz} \overline{V}'_{n,n+1,n+1,m,m+1} \\
& + \bar{d}_{n,m,4}^{yz} \overline{V}'_{n,n,n+2,m,m+1} + \bar{d}_{n,m,3}^{yz} \overline{V}'_{n,n,n+1,m,m+1} \big) \\
+ \tfrac{\mu}{a} \textstyle\sum_{n=1}^{+\infty} \sum_{m=1}^{n} \big(& \bar{d}_{n,m,2}^{yz} \overline{V}'_{n,n+1,n+1,m,m-1} + \bar{d}_{n,m,1}^{yz} \overline{V}'_{n,n,n+2,m,m-1} + \bar{d}_{n,m,0}^{yz} \overline{V}'_{n,n,n+1,m,m-1} \big)
\end{aligned}
$$

Table 3 Expressions of the coefficients $\mathcal{C}^{x_i x_j}_{n,m,l}$ for the derivatives $V_{x_i x_j}$ $(1 \leqslant i \leqslant j \leqslant 3)$

$a^{xx}_{n,m,4} = a^x_{n,m} a^x_{n+1,m}$	$\overline{b}^{xy}_{n,m,7} = -a^x_{n,m} \overline{e}_{n,m}$
$a^{xx}_{n,m,3} = 2a^x_{n,m} c^y_m$	$\overline{b}^{xy}_{n,m,6} = a^y_{n,m} \overline{e}_{n,m}$
$a^{xx}_{n,m,2} = a^{xx}_{n,m} + a^x_{n,m}(b^x_{n+1,n} + b^x_{n,n})$	$\overline{b}^{xy}_{n,m,5} = -(b^x_{n,n} + c^x_{m+1})\overline{e}_{n,m}$
$a^{xx}_{n,m,1} = c^{xy}_m + 2b^x_{n,n} c^y_m$	$\overline{b}^{xy}_{n,m,4} = \overline{e}^y_{n,m} + (b^y_{n,n+1} + c^y_m)\overline{e}_{n,m}$
$a^{xx}_{n,m,0} = b^{xx}_{n,n} + (b^x_{n,n})^2 - (c^y_m)^2$	$\overline{b}^{xy}_{n,m,3} = a^x_{n,m} \overline{d}_{n,m}$
$\overline{b}^{xx}_{n,m,5} = 2a^x_{n,m} \overline{e}_{n,m}$	$\overline{b}^{xy}_{n,m,2} = a^y_{n,m} \overline{d}_{n,m}$
$\overline{b}^{xx}_{n,m,4} = (c^y_m + c^y_{m+1})\overline{e}_{n,m}$	$\overline{b}^{xy}_{n,m,1} = (b^x_{n,n} - c^x_{m-1})\overline{d}_{n,m}$
$\overline{b}^{xx}_{n,m,3} = \overline{e}^x_{n,m} + (b^x_{n,n} + b^x_{n,n+1})\overline{e}_{n,m}$	$\overline{b}^{xy}_{n,m,0} = \overline{d}^y_{n,m} + (b^y_{n,n+1} - c^y_m)\overline{d}_{n,m}$
$\overline{b}^{xx}_{n,m,2} = 2a^x_{n,m} \overline{d}_{n,m}$	$\overline{d}^{xy}_{n,m,2} = \overline{d}_{n+1,m+1} \overline{e}_{n,m}$
$\overline{b}^{xx}_{n,m,1} = (c^y_{m-1} + c^y_m)\overline{d}_{n,m}$	$\overline{d}^{xy}_{n,m,1} = -\overline{d}_{n,m} \overline{e}_{n+1,m-1}$
$\overline{b}^{xx}_{n,m,0} = \overline{d}^x_{n,m} + (b^x_{n,n} + b^x_{n,n+1})\overline{d}_{n,m}$	$\overline{d}^{xy}_{n,m,0} = \overline{d}_{n,m} \overline{d}_{n+1,m-1}$
$\overline{d}^{xx}_{n,m,2} = \overline{d}_{n+1,m+1} \overline{e}_{n,m}$	$\overline{e}^{xy}_{n,m,0} = -\overline{e}_{n,m} \overline{e}_{n+1,m+1}$
$\overline{d}^{xx}_{n,m,1} = \overline{d}_{n,m} \overline{e}_{n+1,m-1}$	$a^{xz}_{n,m,4} = a^x_{n,m} p_{n+1,m}$
$\overline{d}^{xx}_{n,m,0} = \overline{d}_{n,m} \overline{d}_{n+1,m-1}$	$a^{xz}_{n,m,3} = c^y_m p_{n,m}$
$\overline{e}^{xx}_{n,m,0} = \overline{e}_{n,m} \overline{e}_{n+1,m+1}$	$a^{xz}_{n,m,2} = a^{xz}_{n,m} + a^x_{n,m} q_{n+1,n} + b^x_{n,n} p_{n,m}$
$a^{yy}_{n,m,4} = a^y_{n,m} a^y_{n+1,m}$	$a^{xz}_{n,m,1} = c^{yz}_m$
$a^{yy}_{n,m,3} = -2a^y_{n,m} c^x_m$	$a^{xz}_{n,m,0} = b^{xz}_{n,n}$
$a^{yy}_{n,m,2} = a^{yy}_{n,m} + a^y_{n,m}(b^y_{n+1,n} + b^y_{n,n})$	$\overline{b}^{xz}_{n,m,2} = a^x_{n,m} \overline{t}_{n,m}$
$a^{yy}_{n,m,1} = -c^{xy}_m - 2b^y_{n,n} c^x_m$	$\overline{b}^{xz}_{n,m,1} = c^y_m \overline{t}_{n,m}$
$a^{yy}_{n,m,0} = b^{yy}_{n,n} + (b^y_{n,n})^2 - (c^x_m)^2$	$\overline{b}^{xz}_{n,m,0} = b^x_{n,n} \overline{t}_{n,m}$
$\overline{b}^{yy}_{n,m,5} = -2a^y_{n,m} \overline{e}_{n,m}$	$\overline{d}^{xz}_{n,m,5} = \overline{e}_{n,m} p_{n,m}$
$\overline{b}^{yy}_{n,m,4} = -\overline{e}^y_{n,m} - (b^y_{n,n} + b^y_{n,n+1})\overline{e}_{n,m}$	$\overline{d}^{xz}_{n,m,4} = \overline{e}_{n,m} \overline{t}_{n+1,m+1}$
$\overline{b}^{yy}_{n,m,3} = -(c^x_m + c^x_{m+1})\overline{e}_{n,m}$	$\overline{d}^{xz}_{n,m,3} = \overline{e}^z_{n,m} + \overline{e}_{n,m} q_{n,n+1}$
$\overline{b}^{yy}_{n,m,2} = 2a^y_{n,m} \overline{d}_{n,m}$	$\overline{d}^{xz}_{n,m,2} = \overline{d}_{n,m} p_{n,m}$
$\overline{b}^{yy}_{n,m,1} = \overline{d}^y_{n,m} + (b^y_{n,n} + b^y_{n,n+1})\overline{d}_{n,m}$	$\overline{d}^{xz}_{n,m,1} = \overline{d}_{n,m} \overline{t}_{n+1,m-1}$
$\overline{b}^{yy}_{n,m,0} = (c^x_{m-1} + c^x_m)\overline{d}_{n,m}$	$\overline{d}^{xz}_{n,m,0} = \overline{d}^z_{n,m} + \overline{d}_{n,m} q_{n,n+1}$
$\overline{d}^{yy}_{n,m,2} = \overline{d}_{n+1,m+1} \overline{e}_{n,m}$	$a^{yz}_{n,m,4} = a^y_{n,m} p_{n+1,m}$
$\overline{d}^{yy}_{n,m,1} = \overline{d}_{n,m} \overline{e}_{n+1,m-1}$	$a^{yz}_{n,m,3} = -c^x_m p_{n,m}$
$\overline{d}^{yy}_{n,m,0} = -\overline{d}_{n,m} \overline{d}_{n+1,m-1}$	$a^{yz}_{n,m,2} = a^{yz}_{n,m} + a^y_{n,m} q_{n+1,n} + b^y_{n,n} p_{n,m}$
$\overline{e}^{yy}_{n,m,0} = -\overline{e}_{n,m} \overline{e}_{n+1,m+1}$	$a^{yz}_{n,m,1} = -c^{xz}_m$
$a^{zz}_{n,m,1} = p_{n,m} p_{n+1,m}$	$a^{yz}_{n,m,0} = b^{yz}_{n,n}$
$a^{zz}_{n,m,0} = p^z_{n,m} + p_{n,m} q_{n+1,n}$	$\overline{b}^{yz}_{n,m,2} = a^y_{n,m} \overline{t}_{n,m}$
$\overline{d}^{zz}_{n,m,2} = 2p_{n,m} \overline{t}_{n,m}$	$\overline{b}^{yz}_{n,m,1} = -c^x_m \overline{t}_{n,m}$
$\overline{d}^{zz}_{n,m,1} = \overline{t}_{n,m} \overline{t}_{n+1,m}$	$\overline{b}^{yz}_{n,m,0} = b^y_{n,n} \overline{t}_{n,m}$
$\overline{d}^{zz}_{n,m,0} = \overline{t}^z_{n,m} + q_{n,n+1} \overline{t}_{n,m}$	$\overline{d}^{yz}_{n,m,5} = -\overline{e}_m p_{n,m}$
$a^{xy}_{n,m,4} = a^x_{n,m} a^y_{n+1,m}$	$\overline{d}^{yz}_{n,m,4} = -\overline{e}_m \overline{t}_{n+1,m+1}$
$a^{xy}_{n,m,3} = a^y_{n,m} c^y_m - a^x_{n,m} c^x_m$	$\overline{d}^{yz}_{n,m,3} = -\overline{e}^z_m - \overline{e}_m q_{n,n+1}$
$a^{xy}_{n,m,2} = a^{xy}_{n,m} + a^x_{n,m} b^y_{n+1,n} + a^y_{n,m} b^x_{n,n}$	$\overline{d}^{yz}_{n,m,2} = \overline{d}_{n,m} p_{n,m}$
$a^{xy}_{n,m,1} = c^{yy}_m - b^x_{n,n} c^x_m + b^y_{n,n} c^y_m$	$\overline{d}^{yz}_{n,m,1} = \overline{d}_{n,m} \overline{t}_{n+1,m-1}$
$a^{xy}_{n,m,0} = b^{xy}_{n,n} + b^x_{n,n} b^y_{n,n} + c^x_m c^y_m$	$\overline{d}^{yz}_{n,m,0} = \overline{d}^z_{n,m} + \overline{d}_{n,m} q_{n,n+1}$

3.3 Third-order derivatives

Substituting Eq. (46) into the derivative relations (35) and (36), we can also obtain the expressions of the third-order derivatives of the gravitational potential

$$
\begin{aligned}
V_{x_i x_j x_k} =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_2}\left(\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}\partial_{x_k}\mathcal{C}^{x_i x_j}_{n,m,l} + \mathcal{C}^{x_i x_j}_{n,m,l}\partial_{x_k}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}\right) \\
=& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_3}\mathcal{C}^{x_i x_j x_k}_{n,m,l}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}},
\end{aligned}
\tag{48}
$$

where the subscript $k = 1, 2, 3$. Likewise, due to the symmetry of the third-order derivatives, there are only ten independent quantities including V_{xxx}, V_{xxy}, V_{xxz}, V_{xyy}, V_{xyz}, V_{xzz}, V_{yyy}, V_{yyz}, V_{yzz} and V_{zzz}. Then, V_{xxx}, V_{xxy} and V_{xxz} can be regarded as the derivatives of V_{xx} with respect to the three Cartesian coordinates, V_{xyy}, V_{yyy} and V_{yyz} the derivatives of V_{yy} with respect to the three Cartesian coordinates, V_{xzz}, V_{yzz} and V_{zzz} the derivatives of V_{zz} with respect to the three Cartesian coordinates, and V_{xyz} the derivative of V_{xy} with respect to z-coordinate.

The second formula of Eq. (48) must be solved when the expressions of the fourth- and higher-order derivatives of the potential are required. In this paper, the derivatives of the potential are computed up to third-order, and then the first formula of Eq. (48) is sufficient. The expressions of the terms $\partial_{x_k}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}$ have been given in (35), (36), (39) and (40), and the terms $\partial_{x_k}\mathcal{C}^{x_i x_j}_{n,m,l}$ can be solved from Eq. (8) and Tables 1, 2 and 3. In order to obtain the expressions of $\partial_{x_k}\mathcal{C}^{x_i x_j}_{n,m,l}$, we first take the derivatives of the quantities in Table 1, and give the results in Tables 4 and 5.

For the third-order derivatives of the potential, we also need know the expressions of the quantities $b^{xx}_{n+1,n}$, $b^{xy}_{n+1,n}$, $b^{xz}_{n+1,n}$, $b^{yy}_{n+1,n}$, $b^{yz}_{n+1,n}$, $b^{xx}_{n,n+1}$, $b^{xy}_{n,n+1}$, $b^{xz}_{n,n+1}$, $b^{yy}_{n,n+1}$, $b^{yz}_{n,n+1}$, c^{xx}_m, $p^x_{n,m}$, $p^y_{n,m}$, $\bar{t}^x_{n,m}$, $\bar{t}^y_{n,m}$, $q^x_{n+1,n}$, $q^y_{n+1,n}$, $q^z_{n+1,n}$, $q^x_{n,n+1}$, $q^y_{n,n+1}$ and $q^z_{n,n+1}$. Taking the derivative of c^x_m with respect to x-coordinate, we have

$$c^{xx}_m = m f^0_2 E - m f^{bc}_2 \cos^2\lambda. \tag{49}$$

The coefficients $b^x_{n_1,n_2}$ and $b^y_{n_1,n_2}$ (including $b^x_{n+1,n}$, $b^y_{n+1,n}$, $b^x_{n,n+1}$ and $b^y_{n,n+1}$) in Eqs. (19) and (23) can be written as

$$b^x_{n_1,n_2} = b^x_{n_1,n_1} + (n_1 - n_2)b^x_0,\ b^y_{n_1,n_2} = b^y_{n_1,n_1} + (n_1 - n_2)b^y_0, \tag{50}$$

with $b^x_0 = -(v\sin\vartheta\cos\lambda)/L^2$ and $b^y_0 = -(v\sin\vartheta\sin\lambda)/L^2$. Taking the same way as Sect. 2.4, we can obtain the antiderivative $b_0 = -\int(1/u)\mathrm{d}u = -\ln(u/E)$ which satisfies the equations: $\partial_x b_0 = b^x_0$ and $\partial_y b_0 = b^y_0$, and rewrite Eq. (50) as a more general form:

$$b_{n_1,n_2} = b_{n_1,n_1} + (n_1 - n_2)b_0 \tag{51}$$

Table 4 Expressions of the derivatives of the quantities in Table 1 with respect to the Cartesian coordinates

$a_{n,m}^{xxx} = -3(n-m+1)f_3^{a0}\cos\lambda - (n-m+1)f_3^{a}\cos^3\lambda$
$b_{n,n}^{xxx} = 3(n+1)f_3^{bc0}\cos\lambda + (n+1)f_3^{bc}\cos^3\lambda$
$c_{m}^{xxy} = mf_3^{bc0}\sin\lambda + mf_3^{bc}\cos^2\lambda\sin\lambda$
$a_{n,m}^{xxy} = -(n-m+1)f_3^{a0}\sin\lambda - (n-m+1)f_3^{a}\cos^2\lambda\sin\lambda$
$b_{n,n}^{xxy} = (n+1)f_3^{bc0}\sin\lambda + (n+1)f_3^{bc}\cos^2\lambda\sin\lambda$
$c_{m}^{xyy} = mf_3^{bc0}\cos\lambda + mf_3^{bc}\sin^2\lambda\cos\lambda$
$a_{n,m}^{xxz} = -(n-m+1)f_3^{az0} - (n-m+1)f_3^{az}\cos^2\lambda$
$b_{n,n}^{xxz} = -(n+1)f_3^{bcz0} - (n+1)f_3^{bcz}\cos^2\lambda$
$c_{m}^{xyz} = -mf_3^{bcz}\sin\lambda\cos\lambda$
$a_{n,m}^{xyy} = -(n-m+1)f_3^{a0}\cos\lambda - (n-m+1)f_3^{a}\sin^2\lambda\cos\lambda$
$b_{n,n}^{xyy} = (n+1)f_3^{bc0}\cos\lambda + (n+1)f_3^{bc}\sin^2\lambda\cos\lambda$
$a_{n,m}^{xyz} = -(n-m+1)f_3^{az}\sin\lambda\cos\lambda$
$b_{n,n}^{xyz} = -(n+1)f_3^{bcz}\sin\lambda\cos\lambda$
$c_{m}^{yyz} = -mf_3^{bcz0} - mf_3^{bcz}\sin^2\lambda$
$a_{n,m}^{yyy} = -3(n-m+1)f_3^{a0}\sin\lambda - (n-m+1)f_3^{a}\sin^3\lambda$
$b_{n,n}^{yyy} = 3(n+1)f_3^{bc0}\sin\lambda + (n+1)f_3^{bc}\sin^3\lambda$
$a_{n,m}^{yyz} = -(n-m+1)f_3^{az0} - (n-m+1)f_3^{az}\sin^2\lambda$
$b_{n,n}^{yyz} = -(n+1)f_3^{bcz0} - (n+1)f_3^{bcz}\sin^2\lambda$
$\bar{d}_{n,m}^{xx} = w_{n,m}^{d}(f_3^{de0} + f_3^{de}\cos^2\lambda)$
$\bar{e}_{n,m}^{xx} = -w_{n,m}^{e}(f_3^{de0} + f_3^{de}\cos^2\lambda)$
$\bar{d}_{n,m}^{xy} = w_{n,m}^{d}f_3^{de}\sin\lambda\cos\lambda$
$\bar{e}_{n,m}^{xy} = -w_{n,m}^{e}f_3^{de}\sin\lambda\cos\lambda$
$\bar{d}_{n,m}^{xz} = -w_{n,m}^{d}f_3^{dez}\cos\lambda$
$\bar{e}_{n,m}^{xz} = w_{n,m}^{e}f_3^{dez}\cos\lambda$
$\bar{d}_{n,m}^{yy} = w_{n,m}^{d}(f_3^{de0} + f_3^{de}\sin^2\lambda)$
$\bar{e}_{n,m}^{yy} = -w_{n,m}^{e}(f_3^{de0} + f_3^{de}\sin^2\lambda)$
$\bar{d}_{n,m}^{yz} = -w_{n,m}^{d}f_3^{dez}\sin\lambda$
$\bar{e}_{n,m}^{yz} = w_{n,m}^{e}f_3^{dez}\sin\lambda$
$p_{n,m}^{xz} = (n-m+1)f_3^{p}\cos\lambda$
$p_{n,m}^{yz} = (n-m+1)f_3^{p}\sin\lambda$
$p_{n,m}^{zz} = (n-m+1)f_3^{pz}$
$\bar{t}_{n,m}^{xz} = -w_{n,m}^{t}f_3^{t}\cos\lambda$
$\bar{t}_{n,m}^{yz} = -w_{n,m}^{t}f_3^{t}\sin\lambda$
$\bar{t}_{n,m}^{zz} = w_{n,m}^{t}f_3^{tz}$

Table 5 Expressions of the notations f_3 in Table 4

$f_3^{a0} = \frac{Eu\sin\vartheta}{L^6 v^3}\big(3v^2(2E^2-v^2) - E^2(2E^2+v^2)\sin^2\vartheta\big)$
$f_3^{bc0} = \frac{E^2\sin\vartheta}{L^6 v^3}\big(2v^2(3E^2-2v^2)-2E^4\sin^2\vartheta\big)$
$f_3^{de0} = \frac{1}{L^6 v}\big(v^2(3E^2-v^2)+E^2(E^2-3v^2)\sin^2\vartheta\big)$
$f_3^{a} = \frac{Eu\sin^3\vartheta}{L^{10}v^3}\big(5v^4(16E^4-16E^2v^2+3v^4)-10E^2v^2(4E^4+2E^2v^2-3v^4)\sin^2\vartheta + E^4(8E^4+4E^2v^2+3v^4)\sin^4\vartheta\big)$
$f_3^{bc} = \frac{E^2\sin^3\vartheta}{L^{10}v^3}\big(8v^4(10E^4-12E^2v^2+3v^4)-8E^2v^2(5E^4-3v^4)\sin^2\vartheta+8E^8\sin^4\vartheta\big)$
$f_3^{de} = \frac{\sin^2\vartheta}{L^{10}v}\big(v^4(35E^4-30E^2v^2+3v^4)+2v^2E^2(7E^4-30E^2v^2+15v^4)\sin^2\vartheta - E^4(E^4+6E^2v^2-15v^4)\sin^4\vartheta\big)$
$f_3^{az0} = \frac{E\cos\vartheta}{L^6}\big(4E^2-3v^2-E^2\sin^2\vartheta\big)$
$f_3^{bcz0} = \frac{4E^2u\cos\vartheta}{L^6}$
$f_3^{az} = \frac{3Ev^2\sin^2\vartheta\cos\vartheta}{L^{10}}\big((16E^4-20E^2v^2+5v^4)-2E^2(6E^2-5v^2)\sin^2\vartheta+E^4\sin^4\vartheta\big)$
$f_3^{bcz} = \frac{24E^2uv^2\sin^2\vartheta\cos\vartheta}{L^{10}}\big(2E^2-v^2-E^2\sin^2\vartheta\big)$
$f_3^{dez} = \frac{3u\sin\vartheta\cos\vartheta}{L^{10}}\big(v^4(5E^2-v^2)+10E^2v^2(E^2-v^2)\sin^2\vartheta+E^4(E^2-5v^2)\sin^4\vartheta\big)$
$f_3^{p} = \frac{3Euv\sin\vartheta}{L^{10}}\big(4v^2(2E^2-v^2)+(8E^4-20E^2v^2+5v^4)\sin^2\vartheta-2E^2(4E^2-5v^2)\sin^4\vartheta+E^4\sin^6\vartheta\big)$
$f_3^{t} = \frac{3v\sin\vartheta\cos\vartheta}{L^{10}}\big(v^2(8E^4-8E^2v^2+v^4)+2E^2(4E^4-10E^2v^2+5v^4)\sin^2\vartheta-E^4(4E^2-5v^2)\sin^4\vartheta\big)$
$f_3^{pz} = \frac{E\cos\vartheta}{L^{10}}\big(2v^4(4E^2-3v^2)+v^2(32E^4-50E^2v^2+15v^4)\sin^2\vartheta+2E^2(4E^4-19E^2v^2+15v^4)\sin^4\vartheta - E^4(2E^2-3v^2)\sin^6\vartheta\big)$
$f_3^{tz} = \frac{u}{L^{10}}\big(2v^4(4E^2-v^2)+v^2(32E^4-38E^2v^2+3v^4)\sin^2\vartheta+2E^2(4E^4-25E^2v^2+15v^4)\sin^4\vartheta - 3E^4(2E^2-5v^2)\sin^6\vartheta\big)$

From the relations (8), the derivatives of b_0^x and b_0^y with respect to the Cartesian coordinates are

$$\begin{aligned} &b_0^{xx} = f_2^{b0} \cos^2 \lambda - f_1^0, \ b_0^{xy} = f_2^{b0} \sin \lambda \cos \lambda, \ b_0^{xz} = f_2^{bz0} \cos \lambda, \\ &b_0^{yy} = f_2^{b0} \sin^2 \lambda - f_1^0, \ b_0^{yz} = f_2^{bz0} \sin \lambda, \end{aligned} \tag{52}$$

with $f_1^0 = 1/L^2$ and

$$f_2^{b0} = \frac{2v^2 \sin^2 \vartheta}{L^6} \Big(v^2 - 2E^2 + E^2 \sin^2 \vartheta \Big), \ f_2^{bz0} = \frac{2uv \sin \vartheta \cos \vartheta}{L^6} \Big(v^2 + E^2 \sin^2 \vartheta \Big). \tag{53}$$

From Eqs. (50)-(52), the derivatives of b_{n_1,n_2}^x and b_{n_1,n_2}^y are

$$\begin{aligned} &b_{n_1,n_2}^{xx} = b_{n_1,n_1}^{xx} + (n_1 - n_2) b_0^{xx}, \ b_{n_1,n_2}^{xy} = b_{n_1,n_1}^{xy} + (n_1 - n_2) b_0^{xy}, \\ &b_{n_1,n_2}^{xz} = b_{n_1,n_1}^{xz} + (n_1 - n_2) b_0^{xz}, \ b_{n_1,n_2}^{yy} = b_{n_1,n_1}^{yy} + (n_1 - n_2) b_0^{yy}, \\ &b_{n_1,n_2}^{yz} = b_{n_1,n_1}^{yz} + (n_1 - n_2) b_0^{yz}, \end{aligned} \tag{54}$$

where the expressions of the quantities $b_{n,n}^{xx}$, $b_{n,n}^{xy}$, $b_{n,n}^{xz}$, $b_{n,n}^{yy}$ and $b_{n,n}^{yz}$ have been given in Table 1. Substituting the expressions of $p_{n,m}$ and $\bar{t}_{n,m}$ into Eq. (8) and from Sect. 2.4, we have

$$\begin{aligned} &p_{n,m}^x = a_{n,m}^{xz} = (n - m + 1) f_2^p \cos \lambda, \ p_{n,m}^y = a_{n,m}^{yz} = (n - m + 1) f_2^p \sin \lambda, \\ &\bar{t}_{n,m}^x = w_{n,m}^t f_2^t \cos \lambda, \ \bar{t}_{n,m}^y = w_{n,m}^t f_2^t \sin \lambda, \end{aligned} \tag{55}$$

with

$$f_2^p = f_2^{az}, \ f_2^t = \frac{uv \sin \vartheta}{L^6} \Big(v^2 - 4E^2 + 3E^2 \sin^2 \vartheta \Big). \tag{56}$$

The coefficient q_{n_1,n_2} (including $q_{n+1,n}$, $q_{n,n+1}$) in Eq. (27) can be expressed as

$$q_{n_1,n_2} = -(n_1 - n_2) f_1^q \tag{57}$$

with $f_1^q = (u \cos \vartheta)/L^2$, and the derivatives of q_{n_1,n_2} are

$$q_{n_1,n_2}^x = (n_1 - n_2) f_2^q \cos \lambda, \ q_{n_1,n_2}^y = (n_1 - n_2) f_2^q \sin \lambda, \ q_{n_1,n_2}^z = (n_1 - n_2) f_2^{qz} \tag{58}$$

where

$$\begin{aligned} &f_2^q = \frac{2uv \sin \vartheta \cos \vartheta}{L^6} \Big(v^2 - 2E^2 + E^2 \sin^2 \vartheta \Big), \\ &f_2^{qz} = \frac{1}{L^6} \Big(u^2 v^2 - E^2 v^2 - 2(u^2 v^2 - 2E^2 v^2 + E^4) \sin^2 \vartheta + E^2 (E^2 - 2v^2) \sin^4 \vartheta \Big). \end{aligned} \tag{59}$$

Table 6 Expressions of $\partial_{x_k}\mathcal{C}^{x_i x_j}_{n,m,l}$ $(i=j=1,\ k=1,2,3)$ for the derivatives V_{xxx}, V_{xxy} and V_{xxz}

$$a^{xxx}_{n,m,4} = 2a^{xx}_{n,m}a^{x}_{n+1,m}$$
$$a^{xxx}_{n,m,3} = 2(a^{xx}_{n,m}c^{y}_{m} + a^{x}_{n,m}c^{xy}_{m})$$
$$a^{xxx}_{n,m,2} = a^{xxx}_{n,m} + a^{xx}_{n,m}(b^{x}_{n+1,n} + b^{x}_{n,n}) + a^{x}_{n,m}(b^{xx}_{n+1,n} + b^{xx}_{n,n})$$
$$a^{xxx}_{n,m,1} = c^{xxy}_{m} + 2b^{xx}_{n,n}c^{y}_{m} + 2b^{x}_{n,n}c^{xy}_{m}$$
$$a^{xxx}_{n,m,0} = b^{xxx}_{n,n} + 2b^{x}_{n,n}b^{xx}_{n,n} - 2c^{y}_{m}c^{xy}_{m}$$
$$\bar{b}^{xxx}_{n,m,5} = 2(a^{xx}_{n,m}\bar{e}_{n,m} + a^{x}_{n,m}\bar{e}^{x}_{n,m})$$
$$\bar{b}^{xxx}_{n,m,4} = (c^{xy}_{m} + c^{xy}_{m+1})\bar{e}_{n,m} + (c^{y}_{m} + c^{y}_{m+1})\bar{e}^{x}_{n,m}$$
$$\bar{b}^{xxx}_{n,m,3} = \bar{e}^{xx}_{n,m} + (b^{xx}_{n,n} + b^{xx}_{n,n+1})\bar{e}_{n,m} + (b^{x}_{n,n} + b^{x}_{n,n+1})\bar{e}^{x}_{n,m}$$
$$\bar{b}^{xxx}_{n,m,2} = 2(a^{xx}_{n,m}\bar{d}_{n,m} + a^{x}_{n,m}\bar{d}^{x}_{n,m})$$
$$\bar{b}^{xxx}_{n,m,1} = (c^{xy}_{m-1} + c^{xy}_{m})\bar{d}_{n,m} + (c^{y}_{m-1} + c^{y}_{m})\bar{d}^{x}_{n,m}$$
$$\bar{b}^{xxx}_{n,m,0} = \bar{d}^{xx}_{n,m} + (b^{xx}_{n,n} + b^{xx}_{n,n+1})\bar{d}_{n,m} + (b^{x}_{n,n} + b^{x}_{n,n+1})\bar{d}^{x}_{n,m}$$
$$\bar{d}^{xxx}_{n,m,2} = 2\bar{d}^{x}_{n+1,m+1}\bar{e}_{n,m},\ \ \bar{d}^{xxx}_{n,m,1} = 2\bar{d}^{x}_{n,m}\bar{e}_{n+1,m-1},\ \ \bar{d}^{xxx}_{n,m,0} = 2\bar{d}^{x}_{n,m}\bar{d}_{n+1,m-1}$$
$$\bar{e}^{xxx}_{n,m,0} = 2\bar{e}^{x}_{n,m}\bar{e}_{n+1,m+1}$$
$$a^{xxy}_{n,m,4} = 2a^{xy}_{n,m}a^{x}_{n+1,m}$$
$$a^{xxy}_{n,m,3} = 2(a^{xy}_{n,m}c^{y}_{m} + a^{x}_{n,m}c^{yy}_{m})$$
$$a^{xxy}_{n,m,2} = a^{xxy}_{n,m} + a^{xy}_{n,m}(b^{x}_{n+1,n} + b^{x}_{n,n}) + a^{x}_{n,m}(b^{xy}_{n+1,n} + b^{xy}_{n,n})$$
$$a^{xxy}_{n,m,1} = c^{xyy}_{m} + 2b^{xy}_{n,n}c^{y}_{m} + 2b^{x}_{n,n}c^{yy}_{m}$$
$$a^{xxy}_{n,m,0} = b^{xxy}_{n,n} + 2b^{x}_{n,n}b^{xy}_{n,n} - 2c^{y}_{m}c^{yy}_{m}$$
$$\bar{b}^{xxy}_{n,m,5} = 2(a^{xy}_{n,m}\bar{e}_{m} + a^{x}_{n,m}\bar{e}^{y}_{m})$$
$$\bar{b}^{xxy}_{n,m,4} = (c^{yy}_{m} + c^{yy}_{m+1})\bar{e}_{m} + (c^{y}_{m} + c^{y}_{m+1})\bar{e}^{y}_{m}$$
$$\bar{b}^{xxy}_{n,m,3} = \bar{e}^{xy}_{m} + (b^{xy}_{n,n} + b^{xy}_{n,n+1})\bar{e}_{m} + (b^{x}_{n,n} + b^{x}_{n,n+1})\bar{e}^{y}_{m}$$
$$\bar{b}^{xxy}_{n,m,2} = 2(a^{xy}_{n,m}\bar{d}_{n,m} + a^{x}_{n,m}\bar{d}^{y}_{n,m})$$
$$\bar{b}^{xxy}_{n,m,1} = (c^{yy}_{m-1} + c^{yy}_{m})\bar{d}_{n,m} + (c^{y}_{m-1} + c^{y}_{m})\bar{d}^{y}_{n,m}$$
$$\bar{b}^{xxy}_{n,m,0} = \bar{d}^{xy}_{n,m} + (b^{xy}_{n,n} + b^{xy}_{n,n+1})\bar{d}_{n,m} + (b^{x}_{n,n} + b^{x}_{n,n+1})\bar{d}^{y}_{n,m}$$
$$\bar{d}^{xxy}_{n,m,2} = 2\bar{d}^{y}_{n+1,m+1}\bar{e}_{n,m},\ \ \bar{d}^{xxy}_{n,m,1} = 2\bar{d}^{y}_{n,m}\bar{e}_{n+1,m-1},\ \ \bar{d}^{xxy}_{n,m,0} = 2\bar{d}^{y}_{n,m}\bar{d}_{n+1,m-1}$$
$$\bar{e}^{xxy}_{n,m,0} = 2\bar{e}^{y}_{n,m}\bar{e}_{n+1,m+1}$$
$$a^{xxz}_{n,m,4} = 2a^{xz}_{n,m}a^{x}_{n+1,m}$$
$$a^{xxz}_{n,m,3} = 2(a^{xz}_{n,m}c^{y}_{m} + a^{x}_{n,m}c^{yz}_{m})$$
$$a^{xxz}_{n,m,2} = a^{xxz}_{n,m} + a^{xz}_{n,m}(b^{x}_{n+1,n} + b^{x}_{n,n}) + a^{x}_{n,m}(b^{xz}_{n+1,n} + b^{xz}_{n,n})$$
$$a^{xxz}_{n,m,1} = c^{xyz}_{m} + 2(b^{xz}_{n,n}c^{y}_{m} + b^{x}_{n,n}c^{yz}_{m})$$
$$a^{xxz}_{n,m,0} = b^{xxz}_{n,n} + 2b^{x}_{n,n}b^{xz}_{n,n} - 2c^{y}_{m}c^{yz}_{m}$$
$$\bar{b}^{xxz}_{n,m,5} = 2(a^{xz}_{n,m}\bar{e}_{m} + a^{x}_{n,m}\bar{e}^{z}_{m})$$
$$\bar{b}^{xxz}_{n,m,4} = (c^{yz}_{m} + c^{yz}_{m+1})\bar{e}_{m} + (c^{y}_{m} + c^{y}_{m+1})\bar{e}^{z}_{m}$$
$$\bar{b}^{xxz}_{n,m,3} = \bar{e}^{xz}_{m} + (b^{xz}_{n,n} + b^{xz}_{n,n+1})\bar{e}_{m} + (b^{x}_{n,n} + b^{x}_{n,n+1})\bar{e}^{z}_{m}$$
$$\bar{b}^{xxz}_{n,m,2} = 2(a^{xz}_{n,m}\bar{d}_{n,m} + a^{x}_{n,m}\bar{d}^{z}_{n,m})$$
$$\bar{b}^{xxz}_{n,m,1} = (c^{yz}_{m-1} + c^{yz}_{m})\bar{d}_{n,m} + (c^{y}_{m-1} + c^{y}_{m})\bar{d}^{z}_{n,m}$$
$$\bar{b}^{xxz}_{n,m,0} = \bar{d}^{xz}_{n,m} + (b^{xz}_{n,n} + b^{xz}_{n,n+1})\bar{d}_{n,m} + (b^{x}_{n,n} + b^{x}_{n,n+1})\bar{d}^{z}_{n,m}$$
$$\bar{d}^{xxz}_{n,m,2} = 2\bar{d}^{z}_{n+1,m+1}\bar{e}_{n,m},\ \ \bar{d}^{xxz}_{n,m,1} = 2\bar{d}^{z}_{n,m}\bar{e}_{n+1,m-1},\ \ \bar{d}^{xxz}_{n,m,0} = 2\bar{d}^{z}_{n,m}\bar{d}_{n+1,m-1}$$
$$\bar{e}^{xxz}_{n,m,0} = 2\bar{e}^{z}_{n,m}\bar{e}_{n+1,m+1}$$

Table 7 Expressions of $\partial_{x_k}\mathcal{C}^{x_ix_j}_{n,m,l}$ $(i=j=2,\ k=1,2,3)$ for the derivatives V_{xyy}, V_{yyy} and V_{yyz}

$$
\begin{aligned}
&a^{yyx}_{n,m,4} = 2a^{xy}_{n,m}a^{y}_{n+1,m}\\
&a^{yyx}_{n,m,3} = -2(a^{xy}_{n,m}c^{x}_{m} + a^{y}_{n,m}c^{xx}_{m})\\
&a^{yyx}_{n,m,2} = a^{xyy}_{n,m} + a^{xy}_{n,m}(b^{y}_{n+1,n} + b^{y}_{n,n}) + a^{y}_{n,m}(b^{xy}_{n+1,n} + b^{xy}_{n,n})\\
&a^{yyx}_{n,m,1} = -c^{xxy}_{m} - 2b^{xy}_{n,n}c^{x}_{m} - 2b^{y}_{n,n}c^{xx}_{m}\\
&a^{yyx}_{n,m,0} = b^{xyy}_{n,n} + 2b^{y}_{n,n}b^{xy}_{n,n} - 2c^{x}_{m}c^{xx}_{m}\\
&\bar{b}^{yyx}_{n,m,5} = -2(a^{xy}_{n,m}\bar{e}_{n,m} + a^{y}_{n,m}\bar{e}^{x}_{n,m})\\
&\bar{b}^{yyx}_{n,m,4} = -\bar{e}^{xy}_{n,m} - (b^{xy}_{n,n} + b^{xy}_{n,n+1})\bar{e}_{n,m} - (b^{y}_{n,n} + b^{y}_{n,n+1})\bar{e}^{x}_{n,m}\\
&\bar{b}^{yyx}_{n,m,3} = -(c^{xx}_{m} + c^{xx}_{m+1})\bar{e}_{n,m} - (c^{x}_{m} + c^{x}_{m+1})\bar{e}^{x}_{n,m}\\
&\bar{b}^{yyx}_{n,m,2} = 2(a^{xy}_{n,m}\bar{d}_{n,m} + a^{y}_{n,m}\bar{d}^{x}_{n,m})\\
&\bar{b}^{yyx}_{n,m,1} = \bar{d}^{xy}_{n,m} + (b^{xy}_{n,n} + b^{xy}_{n,n+1})\bar{d}_{n,m} + (b^{y}_{n,n} + b^{y}_{n,n+1})\bar{d}^{x}_{n,m}\\
&\bar{b}^{yyx}_{n,m,0} = (c^{xx}_{m-1} + c^{xx}_{m})\bar{d}_{n,m} + (c^{x}_{m-1} + c^{x}_{m})\bar{d}^{x}_{n,m}\\
&\bar{d}^{yyx}_{n,m,2} = 2\bar{d}^{x}_{n+1,m+1}\bar{e}_{n,m},\ \ \bar{d}^{yyx}_{n,m,1} = 2\bar{d}^{x}_{n,m}\bar{e}_{n+1,m-1},\ \ \bar{d}^{yyx}_{n,m,0} = -2\bar{d}^{x}_{n,m}\bar{d}_{n+1,m-1}\\
&\bar{e}^{yyx}_{n,m,0} = -2\bar{e}^{x}_{n,m}\bar{e}_{n+1,m+1}\\
&a^{yyy}_{n,m,4} = 2a^{yy}_{n,m}a^{y}_{n+1,m}\\
&a^{yyy}_{n,m,3} = -2(a^{yy}_{n,m}c^{x}_{m} + a^{y}_{n,m}c^{xy}_{m})\\
&a^{yyy}_{n,m,2} = a^{yyy}_{n,m} + a^{yy}_{n,m}(b^{y}_{n+1,n} + b^{y}_{n,n}) + a^{y}_{n,m}(b^{yy}_{n+1,n} + b^{yy}_{n,n})\\
&a^{yyy}_{n,m,1} = -c^{xyy}_{m} - 2b^{yy}_{n,n}c^{x}_{m} - 2b^{y}_{n,n}c^{xy}_{m}\\
&a^{yyy}_{n,m,0} = b^{yyy}_{n,n} + 2b^{y}_{n,n}b^{yy}_{n,n} - 2c^{x}_{m}c^{xy}_{m}\\
&\bar{b}^{yyy}_{n,m,5} = -2(a^{yy}_{n,m}\bar{e}_{n,m} + a^{y}_{n,m}\bar{e}^{y}_{n,m})\\
&\bar{b}^{yyy}_{n,m,4} = -\bar{e}^{yy}_{n,m} - (b^{yy}_{n,n} + b^{yy}_{n,n+1})\bar{e}_{n,m} - (b^{y}_{n,n} + b^{y}_{n,n+1})\bar{e}^{y}_{n,m}\\
&\bar{b}^{yyy}_{n,m,3} = -(c^{xy}_{m} + c^{xy}_{m+1})\bar{e}_{n,m} - (c^{x}_{m} + c^{x}_{m+1})\bar{e}^{y}_{n,m}\\
&\bar{b}^{yyy}_{n,m,2} = 2(a^{yy}_{n,m}\bar{d}_{n,m} + a^{y}_{n,m}\bar{d}^{y}_{n,m})\\
&\bar{b}^{yyy}_{n,m,1} = \bar{d}^{yy}_{n,m} + (b^{yy}_{n,n} + b^{yy}_{n,n+1})\bar{d}_{n,m} + (b^{y}_{n,n} + b^{y}_{n,n+1})\bar{d}^{y}_{n,m}\\
&\bar{b}^{yyy}_{n,m,0} = (c^{xy}_{m-1} + c^{xy}_{m})\bar{d}_{n,m} + (c^{x}_{m-1} + c^{x}_{m})\bar{d}^{y}_{n,m}\\
&\bar{d}^{yyy}_{n,m,2} = 2\bar{d}^{y}_{n+1,m+1}\bar{e}_{n,m},\ \ \bar{d}^{yyy}_{n,m,1} = 2\bar{d}^{y}_{n,m}\bar{e}_{n+1,m-1},\ \ \bar{d}^{yyy}_{n,m,0} = -2\bar{d}^{y}_{n,m}\bar{d}_{n+1,m-1}\\
&\bar{e}^{yyy}_{n,m,0} = -2\bar{e}^{y}_{n,m}\bar{e}_{n+1,m+1}\\
&a^{yyz}_{n,m,4} = 2a^{yz}_{n,m}a^{y}_{n+1,m}\\
&a^{yyz}_{n,m,3} = -2(a^{yz}_{n,m}c^{x}_{m} + a^{y}_{n,m}c^{xz}_{m})\\
&a^{yyz}_{n,m,2} = a^{yyz}_{n,m} + a^{yz}_{n,m}(b^{y}_{n+1,n} + b^{y}_{n,n}) + a^{y}_{n,m}(b^{yz}_{n+1,n} + b^{yz}_{n,n})\\
&a^{yyz}_{n,m,1} = -c^{xyz}_{m} - 2b^{yz}_{n,n}c^{x}_{m} - 2b^{y}_{n,n}c^{xz}_{m}\\
&a^{yyz}_{n,m,0} = b^{yyz}_{n,n} + 2b^{y}_{n,n}b^{yz}_{n,n} - 2c^{x}_{m}c^{xz}_{m}\\
&\bar{b}^{yyz}_{n,m,5} = -2(a^{yz}_{n,m}\bar{e}_{n,m} + a^{y}_{n,m}\bar{e}^{z}_{n,m})\\
&\bar{b}^{yyz}_{n,m,4} = -\bar{e}^{yz}_{n,m} - (b^{yz}_{n,n} + b^{yz}_{n,n+1})\bar{e}_{n,m} - (b^{y}_{n,n} + b^{y}_{n,n+1})\bar{e}^{z}_{n,m}\\
&\bar{b}^{yyz}_{n,m,3} = -(c^{xz}_{m} + c^{xz}_{m+1})\bar{e}_{n,m} - (c^{x}_{m} + c^{x}_{m+1})\bar{e}^{z}_{n,m}\\
&\bar{b}^{yyz}_{n,m,2} = 2(a^{yz}_{n,m}\bar{d}_{n,m} + a^{y}_{n,m}\bar{d}^{z}_{n,m})\\
&\bar{b}^{yyz}_{n,m,1} = \bar{d}^{yz}_{n,m} + (b^{yz}_{n,n} + b^{yz}_{n,n+1})\bar{d}_{n,m} + (b^{y}_{n,n} + b^{y}_{n,n+1})\bar{d}^{z}_{n,m}\\
&\bar{b}^{yyz}_{n,m,0} = (c^{xz}_{m-1} + c^{xz}_{m})\bar{d}_{n,m} + (c^{x}_{m-1} + c^{x}_{m})\bar{d}^{z}_{n,m}\\
&\bar{d}^{yyz}_{n,m,2} = 2\bar{d}^{z}_{n+1,m+1}\bar{e}_{n,m},\ \ \bar{d}^{yyz}_{n,m,1} = 2\bar{d}^{z}_{n,m}\bar{e}_{n+1,m-1},\ \ \bar{d}^{yyz}_{n,m,0} = -2\bar{d}^{z}_{n,m}\bar{d}_{n+1,m-1}\\
&\bar{e}^{yyz}_{n,m,0} = -2\bar{e}^{z}_{n,m}\bar{e}_{n+1,m+1}
\end{aligned}
$$

Table 8 Expressions of $\partial_{x_k}\mathcal{C}^{x_i x_j}_{n,m,l}$ ($i=j=3$, $k=1,2,3$, or $i=1$, $j=2$, $k=3$) for the derivatives V_{xzz}, V_{yzz}, V_{zzz} and V_{xyz}

$$
\begin{aligned}
&a^{zzx}_{n,m,1} = 2p^{x}_{n,m}p_{n+1,m}\\
&a^{zzx}_{n,m,0} = p^{xz}_{n,m} + p^{x}_{n,m}q_{n+1,n} + p_{n,m}q^{x}_{n+1,n}\\
&\bar{d}^{zzx}_{n,m,2} = 2(p^{x}_{n,m}\bar{t}_{n,m} + p_{n,m}\bar{t}^{x}_{n,m})\\
&\bar{d}^{zzx}_{n,m,1} = 2\bar{t}^{x}_{n,m}\bar{t}_{n+1,m}\\
&\bar{d}^{zzx}_{n,m,0} = \bar{t}^{xz}_{n,m} + q^{x}_{n,n+1}\bar{t}_{n,m} + q_{n,n+1}\bar{t}^{x}_{n,m}\\
&a^{zzy}_{n,m,1} = 2p^{y}_{n,m}p_{n+1,m}\\
&a^{zzy}_{n,m,0} = p^{yz}_{n,m} + p^{y}_{n,m}q_{n+1,n} + p_{n,m}q^{y}_{n+1,n}\\
&\bar{d}^{zzy}_{n,m,2} = 2(p^{y}_{n,m}\bar{t}_{n,m} + p_{n,m}\bar{t}^{y}_{n,m})\\
&\bar{d}^{zzy}_{n,m,1} = 2\bar{t}^{y}_{n,m}\bar{t}_{n+1,m}\\
&\bar{d}^{zzy}_{n,m,0} = \bar{t}^{yz}_{n,m} + q^{y}_{n,n+1}\bar{t}_{n,m} + q_{n,n+1}\bar{t}^{y}_{n,m}\\
&a^{zzz}_{n,m,1} = 2p^{z}_{n,m}p_{n+1,m}\\
&a^{zzz}_{n,m,0} = p^{zz}_{n,m} + p^{z}_{n,m}q_{n+1,n} + p_{n,m}q^{z}_{n+1,n}\\
&\bar{d}^{zzz}_{n,m,2} = 2(p^{z}_{n,m}\bar{t}_{n,m} + p_{n,m}\bar{t}^{z}_{n,m})\\
&\bar{d}^{zzz}_{n,m,1} = 2\bar{t}^{z}_{n,m}\bar{t}_{n+1,m}\\
&\bar{d}^{zzz}_{n,m,0} = \bar{t}^{zz}_{n,m} + q^{z}_{n,n+1}\bar{t}_{n,m} + q_{n,n+1}\bar{t}^{z}_{n,m}\\
&a^{xyz}_{n,m,4} = a^{xz}_{n,m}a^{y}_{n+1,m} + a^{x}_{n,m}a^{yz}_{n+1,m}\\
&a^{xyz}_{n,m,3} = a^{yz}_{n,m}c^{y}_{m} + a^{y}_{n,m}c^{yz}_{m} - a^{xz}_{n,m}c^{x}_{m} - a^{x}_{n,m}c^{xz}_{m}\\
&a^{xyz}_{n,m,2} = a^{xyz}_{n,m} + a^{xz}_{n,m}b^{y}_{n+1,n} + a^{x}_{n,m}b^{yz}_{n+1,n} + a^{yz}_{n,m}b^{x}_{n,n} + a^{y}_{n,m}b^{xz}_{n,n}\\
&a^{xyz}_{n,m,1} = c^{yyz}_{m} - b^{xz}_{n,n}c^{x}_{m} - b^{x}_{n,n}c^{xz}_{m} + b^{yz}_{n,n}c^{y}_{m} + b^{y}_{n,n}c^{yz}_{m}\\
&a^{xyz}_{n,m,0} = b^{xyz}_{n,n} + b^{xz}_{n,n}b^{y}_{n,n} + b^{x}_{n,n}b^{yz}_{n,n} + c^{xz}_{m}c^{y}_{m} + c^{x}_{m}c^{yz}_{m}\\
&\bar{b}^{xyz}_{n,m,7} = -(a^{xz}_{n,m}\bar{e}_{m} + a^{x}_{n,m}\bar{e}^{z}_{m})\\
&\bar{b}^{xyz}_{n,m,6} = a^{yz}_{n,m}\bar{e}_{m} + a^{y}_{n,m}\bar{e}^{z}_{m}\\
&\bar{b}^{xyz}_{n,m,5} = -(b^{xz}_{n,n} + c^{xz}_{m+1})\bar{e}_{m} - (b^{x}_{n,n} + c^{x}_{m+1})\bar{e}^{z}_{m}\\
&\bar{b}^{xyz}_{n,m,4} = \bar{e}^{yz}_{m} + (b^{yz}_{n,n+1} + c^{yz}_{m})\bar{e}_{m} + (b^{y}_{n,n+1} + c^{y}_{m})\bar{e}^{z}_{m}\\
&\bar{b}^{xyz}_{n,m,3} = a^{xz}_{n,m}\bar{d}_{n,m} + a^{x}_{n,m}\bar{d}^{z}_{n,m}\\
&\bar{b}^{xyz}_{n,m,2} = a^{yz}_{n,m}\bar{d}_{n,m} + a^{y}_{n,m}\bar{d}^{z}_{n,m}\\
&\bar{b}^{xyz}_{n,m,1} = (b^{xz}_{n,n} - c^{xz}_{m-1})\bar{d}_{n,m} + (b^{x}_{n,n} - c^{x}_{m-1})\bar{d}^{z}_{n,m}\\
&\bar{b}^{xyz}_{n,m,0} = \bar{d}^{yz}_{n,m} + (b^{yz}_{n,n+1} - c^{yz}_{m})\bar{d}_{n,m} + (b^{y}_{n,n+1} - c^{y}_{m})\bar{d}^{z}_{n,m}\\
&\bar{d}^{xyz}_{n,m,2} = 2\bar{d}^{z}_{n+1,m+1}\bar{e}_{m}\\
&\bar{d}^{xyz}_{n,m,1} = -2\bar{d}^{z}_{n,m}\bar{e}_{m-1}\\
&\bar{d}^{xyz}_{n,m,0} = 2\bar{d}^{z}_{n,m}\bar{d}_{n+1,m-1}\\
&\bar{e}^{xyz}_{n,m,0} = -2\bar{e}^{z}_{m}\bar{e}_{m+1}
\end{aligned}
$$

Therefore, we can obtain all the expressions of $\partial_{x_k}\mathcal{C}^{x_ix_j}_{n,m,l}$ in Eq. (48), as listed in Tables 6, 7 and 8. The quantities in the right-hand side of the formulas of Tables 6, 7 and 8 are given in Tables 4 and 5, Eqs. (49)-(59) and previous sections of this paper. If it's not confusing with the second formula of Eq. (48), the symbol $\mathcal{C}^{x_ix_jx_k}_{n,m,l}$ also represents the derivative $\partial_{x_k}\mathcal{C}^{x_ix_j}_{n,m,l}$, i.e. $\mathcal{C}^{x_ix_jx_k}_{n,m,l} = \partial_{x_k}\mathcal{C}^{x_ix_j}_{n,m,l}$. Since the derivatives $\mathcal{C}^{x_ix_jx_k}_{n,m,l}$ in Tables 6, 7 and 8 with exchanging the positions of x_i, x_j and x_k may not be constant, or the meaning of the changed quantities is different, the sort of the superscript $x_ix_jx_k$ cannot be adjusted, such as the derivative of $a^{yy}_{n,m,4}$ with respect to x-coordinate being $a^{yyx}_{n,m,4}$ rather than $a^{xyy}_{n,m,4}$ (the derivative of $a^{xy}_{n,m,4}$ with respect to y-coordinate) or $a^{yxy}_{n,m,4}$. Now all the quantities in the first formula of Eq. (48) are known, and the third-order derivatives of the potential can be computed. The regular expressions of the derivatives of the potential up to third-order including the singular factor $\frac{1}{\sin\vartheta}$ are also discussed in the Appendix of this work.

4 Computation of the gravitational field for the observation point at the poles

When the observation point P at the poles (P_0), the longitude coordinate is discontinuous and the latitude coordinate $\vartheta_0 = 0$ or π, i.e. $\sin\vartheta_0 = 0$ ($\cos\vartheta_0 = \pm 1$). Since the potential and its arbitrary derivative f^V are continuous at the poles unless the observation point is also on the body's surface, i.e. the intersection point of the body and z-axis (only for the second- or higher-order derivative, see Kellogg (1929, pp. 156)), the gravitational quantity $f^V \to f^V_0 = \lim_{P\to P_0} f^V$ when the point reaches the poles gradually ($\sin\vartheta \to 0$), and then at the poles all the coefficients in the expressions of the derivatives of the gravitational potential containing the factor $\sin\vartheta$ (in the numerator) are eliminated. The coefficients $a^x_{n_1,m}$, $a^y_{n_1,m}$, $b^x_{n_1,n_2}$, $b^y_{n_1,n_2}$, $c^x_{m_2}$ and $c^y_{m_2}$ in Eqs. (35) and (36) are eliminated for the poles, and the nonzero coefficients are

$$\begin{aligned} &\overline{d}_{n_2,m_2} = w^d_{n_2,m_2} f^{de}_{10},\ \overline{e}_{n_2,m_2} = -w^e_{n_2,m_2} f^{de}_{10},\ p_{n_1,m} = -(n_1 - m + 1) f^p_{10}, \\ &q_{n_1,n_2} = -(n_1 - n_2) f^q_{10},\ \overline{t}_{n_2,m_2} = -w^t_{n_2,m_2} f^t_{10}, \end{aligned} \tag{60}$$

where the additional subscript 0 denotes the values at the poles, and the notations f_{10} are

$$f^0_{10} = \frac{1}{v^2},\ f^{de}_{10} = \frac{1}{v},\ f^p_{10} = \pm\frac{E}{v^2},\ f^q_{10} = \pm\frac{u}{v^2},\ f^t_{10} = \frac{u}{v^2}. \tag{61}$$

The coefficients in Eqs. (49)-(59) and Table 1 for the poles are eliminated except the following nonzero coefficients:

$$\begin{aligned}
a^{xx}_{n,m} &= -(n-m+1)f^0_{20}u,\ b^{xx}_{n,n} = (n+1)f^0_{20}E,\ c^{xx}_m = mf^0_{20}E,\\
a^{yy}_{n,m} &= -(n-m+1)f^0_{20}u,\ b^{yy}_{n,n} = (n+1)f^0_{20}E,\ c^{yy}_m = mf^0_{20}E,\\
b^{xx}_{n_1,n_2} &= (n_1+1)f^0_{20}E-(n_1-n_2)f^0_{10},\ b^{yy}_{n_1,n_2} = (n_1+1)f^0_{20}E-(n_1-n_2)f^0_{10},\\
\bar{d}^{z}_{n,m} &= -w^d_{n,m}f^{dez}_{20},\ \bar{e}^{z}_{n,m} = w^e_{n,m}f^{dez}_{20},\\
p^{z}_{n,m} &= (n-m+1)f^{pz}_{20},\ q^{z}_{n_1,n_2} = (n_1-n_2)f^{qz}_{20},\ \bar{t}^{z}_{n,m} = w^t_{n,m}f^{tz}_{20},
\end{aligned} \tag{62}$$

where the notations f_{20} are

$$f^0_{20} = \frac{E}{v^4},\ f^{dez}_{20} = \pm\frac{u}{v^3},\ f^{pz}_{20} = \frac{2Eu}{v^4},\ f^{qz}_{20} = \frac{u^2-E^2}{v^4},\ f^{tz}_{20} = \pm\frac{u^2-E^2}{v^4}, \tag{63}$$

and the coefficients in Table 4 for the poles are also eliminated except

$$\begin{aligned}
a^{xxz}_{n,m} &= -(n-m+1)f^{az0}_{30},\ b^{xxz}_{n,n} = -(n+1)f^{bcz0}_{30},\\
a^{yyz}_{n,m} &= -(n-m+1)f^{az0}_{30},\ b^{yyz}_{n,n} = -(n+1)f^{bcz0}_{30},\ c^{yyz}_m = -mf^{bcz0}_{30},\\
\bar{d}^{xx}_{n,m} &= w^d_{n,m}f^{dex}_{30},\ \bar{e}^{xx}_{n,m} = -w^e_{n,m}f^{dex}_{30},\ \bar{d}^{yy}_{n,m} = w^d_{n,m}f^{dey}_{30},\ \bar{e}^{yy}_{n,m} = -w^e_{n,m}f^{dey}_{30},\\
p^{zz}_{n,m} &= (n-m+1)f^{pz}_{30},\ \bar{t}^{zz}_{n,m} = w^t_{n,m}f^{tz}_{30},
\end{aligned} \tag{64}$$

where the notations f_{30} are

$$\begin{aligned}
f^{az0}_{30} &= \pm\frac{E(4E^2-3v^2)}{v^6},\ f^{bcz0}_{30} = \pm\frac{4E^2u}{v^6},\ f^{dex}_{30} = f^{dey}_{30} = \frac{3E^2-v^2}{v^5},\\
f^{pz}_{30} &= \pm\frac{2E(4E^2-3v^2)}{v^6},\ f^{tz}_{30} = \frac{2u(4E^2-v^2)}{v^6}.
\end{aligned} \tag{65}$$

When the point at the poles, the nonzero coefficients in Table 3 are

$$a^{xx}_{n,m,2} = a^{xx}_{n,m},\ a^{xx}_{n,m,0} = b^{xx}_{n,n},\ a^{yy}_{n,m,2} = a^{yy}_{n,m},\ a^{yy}_{n,m,0} = b^{yy}_{n,n},\ a^{xy}_{n,m,1} = c^{yy}_m, \tag{66}$$

and in Tables 6, 7 and 8 are

$$\begin{aligned}
&a^{xxz}_{n,m,2} = a^{xxz}_{n,m},\ a^{xxz}_{n,m,0} = b^{xxz}_{n,n},\ a^{yyz}_{n,m,2} = a^{yyz}_{n,m},\ a^{yyz}_{n,m,0} = b^{yyz}_{n,n},\ a^{xyz}_{n,m,1} = c^{yyz}_m,\\
&\bar{b}^{xxx}_{n,m,5} = 2a^{xx}_{n,m}\bar{e}_{n,m},\ \bar{b}^{xxx}_{n,m,3} = \bar{e}^{xx}_{n,m} + (b^{xx}_{n,n}+b^{xx}_{n,n+1})\bar{e}_{n,m},\ \bar{b}^{xxx}_{n,m,2} = 2a^{xx}_{n,m}\bar{d}_{n,m},\\
&\bar{b}^{xxx}_{n,m,0} = \bar{d}^{xx}_{n,m} + (b^{xx}_{n,n}+b^{xx}_{n,n+1})\bar{d}_{n,m},\ \bar{b}^{xxy}_{n,m,4} = (c^{yy}_m + c^{yy}_{m+1})\bar{e}_m,\\
&\bar{b}^{xxy}_{n,m,1} = (c^{yy}_{m-1}+c^{yy}_m)\bar{d}_{n,m},\ \bar{b}^{yyx}_{n,m,3} = -(c^{xx}_m+c^{xx}_{m+1})\bar{e}_{n,m},\ \bar{b}^{yyx}_{n,m,0} = (c^{xx}_{m-1}+c^{xx}_m)\bar{d}_{n,m},\\
&\bar{b}^{yyy}_{n,m,5} = -2a^{yy}_{n,m}\bar{e}_{n,m},\ \bar{b}^{yyy}_{n,m,4} = -\bar{e}^{yy}_{n,m} - (b^{yy}_{n,n}+b^{yy}_{n,n+1})\bar{e}_{n,m},\\
&\bar{b}^{yyy}_{n,m,2} = 2a^{yy}_{n,m}\bar{d}_{n,m},\ \bar{b}^{yyy}_{n,m,1} = \bar{d}^{yy}_{n,m} + (b^{yy}_{n,n}+b^{yy}_{n,n+1})\bar{d}_{n,m},
\end{aligned} \tag{67}$$

where the expressions of the coefficients $a^{zz}_{n,m,l}$ $(l=0,1)$, $\overline{d}^{x_ix_j}_{n,m,l}$ $(0 \leqslant l \leqslant 5)$ and $\overline{e}^{x_ix_j}_{n,m,0}$ in Table 3 and the coefficients $a^{zzz}_{n,m,l}$ $(l=0,1)$, $\overline{d}^{x_ix_jz}_{n,m,l}$ $(l=0,1,2)$ and $\overline{e}^{x_ix_jz}_{n,m,0}$ in Tables 6, 7 and 8 remain unchanged on account of the nonzero values of $\overline{d}_{n_2,m_2}$, $\overline{e}_{n_2,m_2}$, $p_{n_1,m}$, q_{n_1,n_2} and $\overline{t}_{n_2,m_2}$ and their nonzero derivatives, and all these nonzero coefficients can be computed by Eqs. (60)-(65).

When the point is located at the poles, the Legendre function $P_{n_2,m_2}(\cos\vartheta_0) = 0$ for $m_2 \geqslant 1$. Hence, for $m_2 \geqslant 1$ the functions $V_{n,n_1,n_2,m,m_2}(u,\vartheta_0) = V'_{n,n_1,n_2,m,m_2}(u,\vartheta_0) = 0$, and for $m_2 = 0$, $V_{n,n_1,n_2,m,0}(u,\vartheta_0) = Q_{n_1,m}P_{n_2,0}T_{n,m,0}$, $V'_{n,n_1,n_2,m,0}(u,\vartheta_0) = Q_{n_1,m}P_{n_2,0}T'_{n,m,0}$, i.e.,

$$\overline{V}_{n,n_1,n_2,m,0} = \widehat{Q}_{n_1,n,m}\overline{P}_{n_2}\overline{C}_{n,m},\ \ \overline{V}'_{n,n_1,n_2,m,0} = \widehat{Q}_{n_1,n,m}\overline{P}_{n_2}\overline{S}_{n,m}, \tag{68}$$

where $\overline{P}_{n_2}$ $(\overline{P}_{n_2,0})$ is the fully Legendre function of the first kind. When $m = m_2 = 0$, $T'_{n,0,0} = 0$ and $V'_{n,n_1,n_2,0,0} = \overline{V}'_{n,n_1,n_2,0,0} = 0$.

From above discussions and Eq. (44), the first-order derivatives of the potential at the poles are

$$\begin{aligned} V_x &= \frac{\mu}{a}\sum_{n=1}^{+\infty}\overline{d}_{n,1}\overline{V}_{n,n,n+1,1,0}, \\ V_y &= \frac{\mu}{a}\sum_{n=1}^{+\infty}\overline{d}_{n,1}\overline{V}'_{n,n,n+1,1,0}, \\ V_z &= \frac{\mu}{a}\sum_{n=0}^{+\infty}\Big(p_{n,0}\overline{V}_{n,n+1,n,0,0} + \overline{t}_{n,0}\overline{V}_{n,n,n+1,0,0}\Big), \end{aligned} \tag{69}$$

where the (nonzero) coefficients have been given in Eq. (60), and they have no the longitude coordinate. The second- and third-order derivatives for the observation point at the poles are listed in Tables 9 and 10, and the expressions of the coefficients and the derivatives with respect to z-coordinate are also known.

Table 9 Expressions of the second-oder derivatives of the gravitational potential at the poles

$V_{xx} = \frac{\mu}{a}\sum_{n=0}^{+\infty}\big(a^{xx}_{n,0,2}\overline{V}_{n,n+1,n,0,0} + a^{xx}_{n,0,0}\overline{V}_{n,n,n,0,0} + \overline{d}^{xx}_{n,0,2}\overline{V}_{n,n,n+2,0,0}\big)$ $+ \frac{\mu}{a}\sum_{n=2}^{+\infty}\overline{d}^{xx}_{n,2,0}\overline{V}_{n,n,n+2,2,0}$
$V_{yy} = \frac{\mu}{a}\sum_{n=0}^{+\infty}\big(a^{yy}_{n,0,2}\overline{V}_{n,n+1,n,0,0} + a^{yy}_{n,0,0}\overline{V}_{n,n,n,0,0} + \overline{d}^{yy}_{n,0,2}\overline{V}_{n,n,n+2,0,0}\big)$ $+ \frac{\mu}{a}\sum_{n=2}^{+\infty}\overline{d}^{yy}_{n,2,0}\overline{V}_{n,n,n+2,2,0}$
$V_{zz} = \frac{\mu}{a}\sum_{n=0}^{+\infty}\big(a^{zz}_{n,0,1}\overline{V}_{n,n+2,n,0,0} + a^{zz}_{n,0,0}\overline{V}_{n,n+1,n,0,0} + \overline{d}^{zz}_{n,0,2}\overline{V}_{n,n+1,n+1,0,0}$ $+ \overline{d}^{zz}_{n,0,1}\overline{V}_{n,n,n+2,0,0} + \overline{d}^{zz}_{n,0,0}\overline{V}_{n,n,n+1,0,0}\big)$
$V_{xy} = \frac{\mu}{a}\sum_{n=2}^{+\infty}\overline{d}^{xy}_{n,2,0}\overline{V}'_{n,n,n+2,2,0}$
$V_{xz} = \frac{\mu}{a}\sum_{n=1}^{+\infty}\big(\overline{d}^{xz}_{n,1,2}\overline{V}_{n,n+1,n+1,1,0} + \overline{d}^{xz}_{n,1,1}\overline{V}_{n,n,n+2,1,0}\big)$
$V_{yz} = \frac{\mu}{a}\sum_{n=1}^{+\infty}\big(\overline{d}^{yz}_{n,1,2}\overline{V}'_{n,n+1,n+1,1,0} + \overline{d}^{yz}_{n,1,1}\overline{V}'_{n,n,n+2,1,0}\big)$

Table 10 Expressions of the third-oder derivatives of the gravitational potential at the poles

$$
\begin{aligned}
V_{xxx} &= \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{d}_{n,1}(a^{xx}_{n,1,2}\overline{V}_{n,n+1,n+1,1,0} + a^{xx}_{n,1,0}\overline{V}_{n,n,n+1,1,0}) + (\bar{d}^{xx}_{n,1,2} + \bar{d}^{xx}_{n,1,1})\bar{d}_{n+2,1}\overline{V}_{n,n,n+3,1,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{b}^{xxx}_{n,1,2}\overline{V}_{n,n+1,n+1,1,0} + \bar{b}^{xxx}_{n,1,0}\overline{V}_{n,n,n+1,1,0}\big) + \tfrac{\mu}{a}\textstyle\sum_{n=3}^{+\infty}\bar{d}^{xx}_{n,3,0}\bar{d}_{n+2,1}\overline{V}_{n,n,n+3,3,0} \\
V_{xxy} &= \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{d}_{n,1}(a^{xx}_{n,1,2}\overline{V}'_{n,n+1,n+1,1,0} + a^{xx}_{n,1,0}\overline{V}'_{n,n,n+1,1,0}) + \bar{d}^{xx}_{n,1,2}\bar{d}_{n+2,1}\overline{V}'_{n,n,n+3,1,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\bar{b}^{xxy}_{n,1,1}\overline{V}'_{n,n,n+1,1,0} + \tfrac{\mu}{a}\textstyle\sum_{n=3}^{+\infty}\bar{d}^{xx}_{n,3,0}\bar{d}_{n+2,1}\overline{V}'_{n,n,n+3,3,0} \\
V_{xxz} &= \tfrac{\mu}{a}\textstyle\sum_{n=0}^{+\infty}\big(a^{xx}_{n,0,2}\partial_z\overline{V}_{n,n+1,n,0,0} + a^{xx}_{n,0,0}\partial_z\overline{V}_{n,n,n,0,0} + \bar{d}^{xx}_{n,0,2}\partial_z\overline{V}_{n,n,n+2,0,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=0}^{+\infty}\big(a^{xxz}_{n,0,2}\overline{V}_{n,n+1,n,0,0} + a^{xxz}_{n,0,0}\overline{V}_{n,n,n,0,0} + \bar{d}^{xxz}_{n,0,2}\overline{V}_{n,n,n+2,0,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=2}^{+\infty}\big(\bar{d}^{xx}_{n,2,0}\partial_z\overline{V}_{n,n,n+2,2,0} + \bar{d}^{xxz}_{n,2,0}\overline{V}_{n,n,n+2,2,0}\big) \\
V_{xyy} &= \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{d}_{n,1}(a^{yy}_{n,1,2}\overline{V}_{n,n+1,n+1,1,0} + a^{yy}_{n,1,0}\overline{V}_{n,n,n+1,1,0}) + \bar{d}^{yy}_{n,1,2}\bar{d}_{n+2,1}\overline{V}_{n,n,n+3,1,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\bar{b}^{yyx}_{n,1,0}\overline{V}_{n,n,n+1,1,0} + \tfrac{\mu}{a}\textstyle\sum_{n=3}^{+\infty}\bar{d}^{yy}_{n,3,0}\bar{d}_{n+2,1}\overline{V}_{n,n,n+3,3,0} \\
V_{yyy} &= \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{d}_{n,1}(a^{yy}_{n,1,2}\overline{V}'_{n,n+1,n+1,1,0} + a^{yy}_{n,1,0}\overline{V}'_{n,n,n+1,1,0}) + \bar{d}_{n+2,1}(\bar{d}^{yy}_{n,1,2} + \bar{d}^{yy}_{n,1,1})\overline{V}'_{n,n,n+3,1,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{b}^{yyy}_{n,1,2}\overline{V}'_{n,n+1,n+1,1,0} + \bar{b}^{yyy}_{n,1,1}\overline{V}'_{n,n,n+1,1,0}\big) + \tfrac{\mu}{a}\textstyle\sum_{n=3}^{+\infty}\big(\bar{d}^{yy}_{n,3,0}\bar{d}_{n+2,1}\overline{V}'_{n,n,n+3,3,0}\big) \\
V_{yyz} &= \tfrac{\mu}{a}\textstyle\sum_{n=0}^{+\infty}\big(a^{yy}_{n,0,2}\partial_z\overline{V}_{n,n+1,n,0,0} + a^{yy}_{n,0,0}\partial_z\overline{V}_{n,n,n,0,0} + \bar{d}^{yy}_{n,0,2}\partial_z\overline{V}_{n,n,n+2,0,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=0}^{+\infty}\big(a^{yyz}_{n,0,2}\overline{V}_{n,n+1,n,0,0} + a^{yyz}_{n,0,0}\overline{V}_{n,n,n,0,0} + \bar{d}^{yyz}_{n,0,2}\overline{V}_{n,n,n+2,0,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=2}^{+\infty}\big(\bar{d}^{yy}_{n,2,0}\partial_z\overline{V}_{n,n,n+2,2,0} + \bar{d}^{yyz}_{n,2,0}\overline{V}_{n,n,n+2,2,0}\big) \\
V_{xzz} &= \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{d}_{n,1}(a^{zz}_{n,1,1}\overline{V}_{n,n+2,n+1,1,0} + a^{zz}_{n,1,0}\overline{V}_{n,n+1,n+1,1,0}) \\
&\qquad\qquad + \bar{d}_{n+1,1}(\bar{d}^{zz}_{n,1,2}\overline{V}_{n,n+1,n+2,1,0} + \bar{d}^{zz}_{n,1,0}\overline{V}_{n,n,n+2,1,0}) + \bar{d}^{zz}_{n,1,1}\bar{d}_{n+2,1}\overline{V}_{n,n,n+3,1,0}\big) \\
V_{yzz} &= \tfrac{\mu}{a}\textstyle\sum_{n=1}^{+\infty}\big(\bar{d}_{n,1}(a^{zz}_{n,1,1}\overline{V}'_{n,n+2,n+1,1,0} + a^{zz}_{n,1,0}\overline{V}'_{n,n+1,n+1,1,0}) \\
&\qquad\qquad + \bar{d}_{n+1,1}(\bar{d}^{zz}_{n,1,2}\overline{V}'_{n,n+1,n+2,1,0} + \bar{d}^{zz}_{n,1,0}\overline{V}'_{n,n,n+2,1,0}) + \bar{d}^{zz}_{n,1,1}\bar{d}_{n+2,1}\overline{V}'_{n,n,n+3,1,0}\big) \\
V_{zzz} &= \tfrac{\mu}{a}\textstyle\sum_{n=0}^{+\infty}\big(a^{zz}_{n,0,1}\partial_z\overline{V}_{n,n+2,n,0,0} + a^{zz}_{n,0,0}\partial_z\overline{V}_{n,n+1,n,0,0} + \bar{d}^{zz}_{n,0,2}\partial_z\overline{V}_{n,n+1,n+1,0,0} \\
&\qquad\qquad + \bar{d}^{zz}_{n,0,1}\partial_z\overline{V}_{n,n,n+2,0,0} + \bar{d}^{zz}_{n,0,0}\partial_z\overline{V}_{n,n,n+1,0,0}\big) \\
&\quad + \tfrac{\mu}{a}\textstyle\sum_{n=0}^{+\infty}\big(a^{zzz}_{n,0,1}\overline{V}_{n,n+2,n,0,0} + a^{zzz}_{n,0,0}\overline{V}_{n,n+1,n,0,0} + \bar{d}^{zzz}_{n,0,2}\overline{V}_{n,n+1,n+1,0,0} \\
&\qquad\qquad + \bar{d}^{zzz}_{n,0,1}\overline{V}_{n,n,n+2,0,0} + \bar{d}^{zzz}_{n,0,0}\overline{V}_{n,n,n+1,0,0}\big) \\
V_{xyz} &= \tfrac{\mu}{a}\textstyle\sum_{n=2}^{+\infty}\Big(\bar{d}^{xy}_{n,2,0}\partial_z\overline{V}'_{n,n,n+2,2,0} + \bar{d}^{xyz}_{n,2,0}\overline{V}'_{n,n,n+2,2,0}\Big)
\end{aligned}
$$

5 Oblate spheroidal harmonic expressions in the local north-oriented reference frame

5.1 Conversions between the derivatives of the potential in the global and the local north-oriented reference frames

The local north-oriented reference frame located on the observation point is a right-handed North-West-Up system (Koop 1993, pp. 183). The local North-West-Up and the local East-North-Up systems are usually adopted in geodesy (Moritz 1971 pp. 9; Reed 1973, pp. 24; Koop 1993, pp. 183; Casotto and Fantino 2009). We denote $\mathbf{r}$ as the position vector in the global reference frame, $\mathbf{r}_P$ as the position vector of the observation point, $\mathbf{r}^*$ as the position vector in the local north-oriented reference frame, (x^*, y^*, z^*) or (x^*_1, x^*_2, x^*_3) as corresponding local Cartesian coordinates, $V_{x_{l_1}x_{l_2}\cdots x_{l_k}}$ and $V_{x^*_{l_1}x^*_{l_2}\cdots x^*_{l_k}}$ $(l_1, l_2, \cdots, l_k = 1, 2, 3)$ as the kth-order derivatives in the global and the local reference frames, respectively. The conversions between the vectors $\mathbf{r}$ and $\mathbf{r}^*$ are: $\mathbf{r}^* = \mathbf{R}(\mathbf{r} - \mathbf{r}_P)$, and $\mathbf{r} = \mathbf{R}^{\mathrm{T}}\mathbf{r}^* + \mathbf{r}_P$, where $\mathbf{R}$ is the rotation matrix from the the global to the local reference frames, and

$$\mathbf{R} = \begin{pmatrix} -\cos\vartheta\cos\lambda & -\cos\vartheta\sin\lambda & \sin\vartheta \\ \sin\lambda & -\cos\lambda & 0 \\ \sin\vartheta\cos\lambda & \sin\vartheta\sin\lambda & \cos\vartheta \end{pmatrix}. \tag{70}$$

In the Cartesian coordinate system, the covariant basis vector $\mathbf{e}_i$ (along the x_i-axis, $i = 1, 2, 3$) is equal to the contravariant basis vector $\mathbf{e}^i$, i.e. $\mathbf{e}_i = \mathbf{e}^i$, and then the corresponding covariant, contravariant and mixed components of a tensor are also equal. The rotation matrix is a second-order tensor and can be written as (Abraham et al. 2002, pp. 340-343; Huang et al. 2020, pp. 45): $\mathbf{R} = R_{ij}\mathbf{e}^i\mathbf{e}^j = R^{ij}\mathbf{e}_i\mathbf{e}_j = R_i^{\cdot j}\mathbf{e}^i\mathbf{e}_j = R^i_{\cdot j}\mathbf{e}_i\mathbf{e}^j$, where the juxtaposition of the vectors denotes the tensor product or the dyad, the covariant component R_{ij}, the contravariant component R^{ij} and the mixed components $R_i^{\cdot j}$, $R^i_{\cdot j}$ $(i, j = 1, 2, 3)$ are equal, and are the entry in the i-th row and j-th column element of the $\mathbf{R}$. The Einstein summation convention is used for the summations over the indexed terms in this section. Then, $\frac{\partial x_i^*}{\partial x_j} = R_{ij} = R^i_{\cdot j} = R_i^{\cdot j}$, $\frac{\partial x_i}{\partial x_j^*} = R_{ji} = R_j^{\cdot i} = R^j_{\cdot i}$.

The k-th order derivative can be expressed as the tensor form:

$$\mathbf{V}_k = T_{l_1 l_2 \cdots l_k}\mathbf{e}^{l_1}\mathbf{e}^{l_2}\cdots\mathbf{e}^{l_k} = T^*_{l_1 l_2 \cdots l_k}\mathbf{e}^{*l_1}\mathbf{e}^{*l_2}\cdots\mathbf{e}^{*l_k}, \tag{71}$$

where $\mathbf{V}_k$ is the tensor form of the k-th order derivative, the superscript $*$ denotes the quantity in the local reference frame, the covariant components $T_{l_1 l_2 \cdots l_k}$ and $T^*_{l_1 l_2 \cdots l_k}$ are actually the derivatives $V_{x_{l_1} x_{l_2} \cdots x_{l_k}}$ and $V_{x^*_{l_1} x^*_{l_2} \cdots x^*_{l_k}}$, i.e.,

$$V_{x_{l_1} x_{l_2} \cdots x_{l_k}} = T_{l_1 l_2 \cdots l_k},\ V_{x^*_{l_1} x^*_{l_2} \cdots x^*_{l_k}} = T^*_{l_1 l_2 \cdots l_k}, \tag{72}$$

From Borisenko and Tarapov (1968, pp. 88-91) and Huang et al. (2020, pp. 20-23), the relations for the two covariant components $T_{l_1 l_2 \cdots l_k}$ and $T^*_{l_1 l_2 \cdots l_k}$ are

$$T^*_{l_1 l_2 \cdots l_k} = T_{l'_1 l'_2 \cdots l'_k} R_{l_1}^{\cdot l'_1} R_{l_2}^{\cdot l'_2} \cdots R_{l_k}^{\cdot l'_k},\ T_{l_1 l_2 \cdots l_k} = T^*_{l'_1 l'_2 \cdots l'_k} R^{l'_1}_{\cdot l_1} R^{l'_2}_{\cdot l_2} \cdots R^{l'_k}_{\cdot l_k}. \tag{73}$$

where $l'_1, l'_2, \cdots, l'_k = 1, 2, 3$, and $R_{l_i}^{\cdot l'_i} = R_{l_i l'_i} \neq R^{l'_i}_{\cdot l_i} = R_{l'_i l_i}$ for $l_i \neq l'_i$ $(i = 1, 2, \cdots, k)$. Eq. (73) is the conversions between the derivatives of the potential in the global and local reference frames, and the corresponding expressions up to third-order can refer to Casotto and Fantino (2009) where the local reference frame in the literature is a right-handed East-North-Up system instead of the North-West-Up system in this paper, and the spherical latitude coordinate φ becomes the right angle minus the oblate spheroidal colatitude ϑ, i.e., $\varphi \to \pi/2 - \vartheta$.

5.2 Conversions of the derivatives of the potential from the oblate spheroidal to the Cartesian coordinates

The conversions of the natural components of the derivatives of the potential from the global to the local reference frames are given in Casotto and Fantino (2009), and the computations of the spherical harmonic expansions of the derivatives in the global and the local reference frames are discussed in Fantino and Casotto (2009). The scale factors of the oblate spheroidal coordinates h_u, h_ϑ and h_λ (also called as the Lamé coefficients) are: $h_u = L/v$, $h_\vartheta = L$, and $h_\lambda = v \sin\vartheta$. At the observation point, the following relations hold for the covariant basis vectors of the oblate spheroidal coordinates $(\mathbf{e}_u, \mathbf{e}_\vartheta, \mathbf{e}_\lambda)$ and the local Cartesian coordinates $(\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$: $\mathbf{e}_1^* = -\mathbf{e}_\vartheta/h_\vartheta$, $\mathbf{e}_2^* = -\mathbf{e}_\lambda/h_\lambda$, and $\mathbf{e}_3^* = \mathbf{e}_u/h_u$. The metric tensors of the oblate spheroidal coordinates in the covariant and contravariant forms can be given as

$$(g_{ij}) = \begin{pmatrix} \frac{L^2}{v^2} & 0 & 0 \\ 0 & L^2 & 0 \\ 0 & 0 & v^2\sin^2\vartheta \end{pmatrix}, \quad \left(g^{ij}\right) = \begin{pmatrix} \frac{v^2}{L^2} & 0 & 0 \\ 0 & \frac{1}{L^2} & 0 \\ 0 & 0 & \frac{1}{v^2\sin^2\vartheta} \end{pmatrix} \tag{74}$$

From Borisenko and Tarapov (1968, pp. 189) and Huang et al. (2020, pp. 123), the Christoffel symbols of the second kind for the oblate spheroidal coordinates can be obtained, as shown in Table 11. From the expressions of the Christoffel symbols of the second kind in Table 11 and Casotto and Fantino (2009), we get the conversions of the derivatives of the potential (up to third-order) from the oblate spheroidal coordinates in the global reference frame to the Cartesian coordinates in the local north-oriented reference frame, see Table 12. The results of the first- and second-order derivatives have also been given in Koop (1993, pp. 31) and Vershkov and Petrovskaya (2016), and the conversions in the local South-East-Up frame are given in Sebera et al. (2016). When $E = 0$, the conversions in Table 12 degenerate into the spherical coordinate case. We can first solve the expressions of the derivatives of the potential with respect to the oblate spheroidal coordinates (e.g. the discussions in the Appendix of this work), and then get the derivatives in the local Cartesian coordinates from Table 12. According to Eqs. (72) and (73), we finally obtain the derivatives in the global Cartesian coordinates. If the non-singular expressions of derivatives with respect to the oblate spheroidal and the local Cartesian coordinates in Table 12 are given, the computations of the derivatives in the global Cartesian coordinates do not produce singular values.

Table 11 Christoffel symbols of the second kind of the oblate spheroidal coordinates

Christoffel symbols	Expressions
$\left(\Gamma^1_{ij}\right)$	$\begin{pmatrix} \frac{uE^2\sin^2\vartheta}{L^2v^2} & -\frac{E^2\sin\vartheta\cos\vartheta}{L^2} & 0 \\ -\frac{E^2\sin\vartheta\cos\vartheta}{L^2} & -\frac{uv^2}{L^2} & 0 \\ 0 & 0 & -\frac{uv^2\sin^2\vartheta}{L^2} \end{pmatrix}$
$\left(\Gamma^2_{ij}\right)$	$\begin{pmatrix} \frac{E^2\sin\vartheta\cos\vartheta}{v^2L^2} & \frac{u}{L^2} & 0 \\ \frac{u}{L^2} & -\frac{E^2\sin\vartheta\cos\vartheta}{L^2} & 0 \\ 0 & 0 & -\frac{v^2\sin\vartheta\cos\vartheta}{L^2} \end{pmatrix}$
$\left(\Gamma^3_{ij}\right)$	$\begin{pmatrix} 0 & 0 & \frac{u}{v^2} \\ 0 & 0 & \cot\vartheta \\ \frac{u}{v^2} & \cot\vartheta & 0 \end{pmatrix}$

Table 12 Conversions of the derivatives of the gravitational potential with respect to the oblate spheroidal coordinates into the local Cartesian coordinates

$$
\begin{aligned}
V_{x^*} &= -\tfrac{1}{L}V_\vartheta,\ V_{y^*} = -\tfrac{1}{v\sin\vartheta}V_\lambda,\ V_{z^*} = \tfrac{v}{L}V_u \\
V_{x^*x^*} &= \tfrac{uv^2}{L^4}V_u + \tfrac{E^2\sin\vartheta\cos\vartheta}{L^4}V_\vartheta + \tfrac{1}{L^2}V_{\vartheta\vartheta},\ V_{x^*y^*} = -\tfrac{\cos\vartheta}{Lv\sin^2\vartheta}V_\lambda + \tfrac{1}{Lv\sin\vartheta}V_{\vartheta\lambda} \\
V_{x^*z^*} &= -\tfrac{vE^2\sin\vartheta\cos\vartheta}{L^4}V_u + \tfrac{uv}{L^4}V_\vartheta - \tfrac{v}{L^2}V_{u\vartheta},\ V_{y^*y^*} = \tfrac{u}{L^2}V_u + \tfrac{\cos\vartheta}{L^2\sin\vartheta}V_\vartheta + \tfrac{1}{v^2\sin^2\vartheta}V_{\lambda\lambda} \\
V_{y^*z^*} &= \tfrac{u}{Lv^2\sin\vartheta}V_\lambda - \tfrac{1}{L\sin\vartheta}V_{u\lambda},\ V_{z^*z^*} = -\tfrac{uE^2\sin^2\vartheta}{L^4}V_u - \tfrac{E^2\sin\vartheta\cos\vartheta}{L^4}V_\vartheta + \tfrac{v^2}{L^2}V_{uu} \\
V_{x^*x^*x^*} &= -\tfrac{6uv^2E^2\sin\vartheta\cos\vartheta}{L^7}V_u + \tfrac{2L^4-3L^2(u^2+v^2)+6u^2v^2}{L^7}V_\vartheta - \tfrac{3uv^2}{L^5}V_{u\vartheta} \\
&\quad - \tfrac{3E^2\sin\vartheta\cos\vartheta}{L^5}V_{\vartheta\vartheta} - \tfrac{1}{L^3}V_{\vartheta\vartheta\vartheta} \\
V_{x^*x^*y^*} &= -\tfrac{2\cos^2\vartheta}{L^2v\sin^3\vartheta}V_\lambda - \tfrac{uv}{L^4\sin\vartheta}V_{u\lambda} + \tfrac{(3L^2-v^2)\cos\vartheta}{L^4v\sin^2\vartheta}V_{\vartheta\lambda} - \tfrac{1}{L^2v\sin\vartheta}V_{\vartheta\vartheta\lambda} \\
V_{x^*x^*z^*} &= \tfrac{v(L^2-2u^2)(3v^2-2L^2)}{L^7}V_u - \tfrac{6uvE^2\sin\vartheta\cos\vartheta}{L^7}V_\vartheta + \tfrac{uv^3}{L^5}V_{uu} + \tfrac{3vE^2\sin\vartheta\cos\vartheta}{L^5}V_{u\vartheta} \\
&\quad - \tfrac{2uv}{L^5}V_{\vartheta\vartheta} + \tfrac{v}{L^3}V_{u\vartheta\vartheta} \\
V_{x^*y^*y^*} &= -\tfrac{2uE^2\sin\vartheta\cos\vartheta}{L^5}V_u + \tfrac{1}{L^5}\left(2u^2 - 2L^2 + \tfrac{L^2}{\sin^2\vartheta}\right)V_\vartheta - \tfrac{u}{L^3}V_{u\vartheta} - \tfrac{\cos\vartheta}{L^3\sin\vartheta}V_{\vartheta\vartheta} \\
&\quad + \tfrac{2\cos\vartheta}{Lv^2\sin^3\vartheta}V_{\lambda\lambda} - \tfrac{1}{Lv^2\sin^2\vartheta}V_{\vartheta\lambda\lambda} \\
V_{x^*y^*z^*} &= \tfrac{2u\cos\vartheta}{L^2v^2\sin^2\vartheta}V_\lambda + \tfrac{(v^2-2L^2)\cos\vartheta}{L^4\sin^2\vartheta}V_{u\lambda} - \tfrac{u(L^2+v^2)}{L^4v^2\sin\vartheta}V_{\vartheta\lambda} + \tfrac{1}{L^2\sin\vartheta}V_{u\vartheta\lambda} \\
V_{x^*z^*z^*} &= \tfrac{2uE^2(3v^2-L^2)\sin\vartheta\cos\vartheta}{L^7}V_u - \tfrac{2L^4-3L^2(u^2+v^2)+6u^2v^2}{L^7}V_\vartheta - \tfrac{2v^2E^2\sin\vartheta\cos\vartheta}{L^5}V_{uu} \\
&\quad + \tfrac{u(3v^2-L^2)}{L^5}V_{u\vartheta} + \tfrac{E^2\sin\vartheta\cos\vartheta}{L^5}V_{\vartheta\vartheta} - \tfrac{v^2}{L^3}V_{uu\vartheta} \\
V_{y^*y^*y^*} &= \tfrac{2}{v^3\sin^3\vartheta}V_\lambda - \tfrac{3u}{L^2v\sin\vartheta}V_{u\lambda} - \tfrac{3\cos\vartheta}{L^2v\sin^2\vartheta}V_{\vartheta\lambda} - \tfrac{1}{v^3\sin^3\vartheta}V_{\lambda\lambda\lambda} \\
V_{y^*y^*z^*} &= \tfrac{v(L^2-2u^2)}{L^5}V_u - \tfrac{2uv\cos\vartheta}{L^5\sin\vartheta}V_\vartheta + \tfrac{uv}{L^3}V_{uu} + \tfrac{v\cos\vartheta}{L^3\sin\vartheta}V_{u\vartheta} - \tfrac{2u}{Lv^3\sin^2\vartheta}V_{\lambda\lambda} + \tfrac{1}{Lv\sin^2\vartheta}V_{u\lambda\lambda} \\
V_{y^*z^*z^*} &= -\tfrac{2u^2}{L^2v^3\sin\vartheta}V_\lambda + \tfrac{u(L^2+v^2)}{L^4v\sin\vartheta}V_{u\lambda} + \tfrac{E^2\cos\vartheta}{L^4v}V_{\vartheta\lambda} - \tfrac{v}{L^2\sin\vartheta}V_{uu\lambda} \\
V_{z^*z^*z^*} &= -\tfrac{3v(L^2-2u^2)E^2\sin^2\vartheta}{L^7}V_u + \tfrac{6uvE^2\sin\vartheta\cos\vartheta}{L^7}V_\vartheta - \tfrac{3uvE^2\sin^2\vartheta}{L^5}V_{uu} - \tfrac{3vE^2\sin\vartheta\cos\vartheta}{L^5}V_{u\vartheta} \\
&\quad + \tfrac{v^3}{L^3}V_{uuu}
\end{aligned}
$$

6 Numerical experiments

In this section, we choose the uniform rectangular parallelepiped (prism) to test the algorithms of this work. The closed-form solutions (in space-domain) of the gravitational field up to third-order derivatives of the potential are given by Nagy et al. (2000), in which there are clerical errors in equations (11), (12) and (33) that need to be modified, see Table 13. The lengths of three sides of the prism body are 2 km, 2 km and 1 km, and the cross section of the prism parallel to the xy-plane is square, as shown in Fig. 1. The origin of the reference frame is located in the geometric center of the prism, and the density of the uniform prism is assumed as $\rho = 2.67$ g/cm^3. The semi-major and semi-minor axes of the reference oblate spheroid can be: $a = 1.6$ km, $b = 1.07$ km. Analogous to Sebera et al. (2016), the oblate spheroidal harmonic coefficients can be computed from the gravitational potential along a confocal oblate spheroid $u = u_0$:

$$\begin{pmatrix} \overline{C}_{n,m} \\ \overline{S}_{n,m} \end{pmatrix} = \frac{a}{4\pi\mu} \frac{Q^b_{n,m}}{Q^{u_0}_{n,m}} \int_0^{2\pi} \int_0^{\pi} V(u_0, \vartheta, \lambda) \overline{P}_{n,m}(\cos\vartheta) \begin{pmatrix} \cos(m\lambda) \\ \sin(m\lambda) \end{pmatrix} \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\lambda. \quad (75)$$

The known potential $V(u_0, \vartheta, \lambda)$ can be solved using the analytical solutions Nagy et al. (2000). We take $u_0 = b$ for computing the harmonic coefficients, i.e. applying the values of the potential on the reference oblate spheroid, and then the ratio $Q^b_{n,m}/Q^{u_0}_{n,m}$ is eliminated in Eq. (75). From the surface integral forms (75) or the volume integral forms of the oblate spheroidal harmonic coefficients (Jekeli 1981, pp. 100), the integrand contains the Legendre polynomial $\overline{P}_{n,m}(\cos\vartheta)$ and a factor $\cos(m\lambda)$ or $\sin(m\lambda)$. Due to the symmetry of the uniform prism and the identical relations: $\cos m(\pi \pm \lambda) = (-1)^m \cos(m\lambda)$, $\cos m(\frac{\pi}{2} - \lambda) = (-1)^{m/2} \cos(m\lambda)$ (for even m), $\sin m(\pi + \lambda) = (-1)^m \sin(m\lambda)$ and $\sin m(\pi - \lambda) = (-1)^{m+1} \sin(m\lambda)$, for the oblate spheroidal coefficients of the potential of the uniform prism we have: $\overline{C}_{n,m} = 0$ for the order $m \neq 4\ell_2$ ($\ell_2 = 0, 1, 2, \cdots$) and $\overline{S}_{n,m} = 0$ for all the order m. Since $\overline{P}_{n,m}(\cos(\pi - \vartheta)) = (-1)^{n-m} \overline{P}_{n,m}(\cos\vartheta)$, the harmonic coefficients $\overline{C}_{n,m} \neq 0$ when $n = 2\ell_1$ ($\ell_1 = 0, 1, 2, \cdots$) and $m = 4\ell_2$. Hence, we need only compute the harmonic coefficients $\overline{C}_{2\ell_1, 4\ell_2}$ ($4\ell_2 \leqslant 2\ell_1$). The surface integrals in Eq. (75) with fixed intervals can be computed by the Gauss-Legendre quadrature, and the numbers of the Gaussian nodes used in this work are 8/8. In order to obtain the harmonic coefficients up to d/o

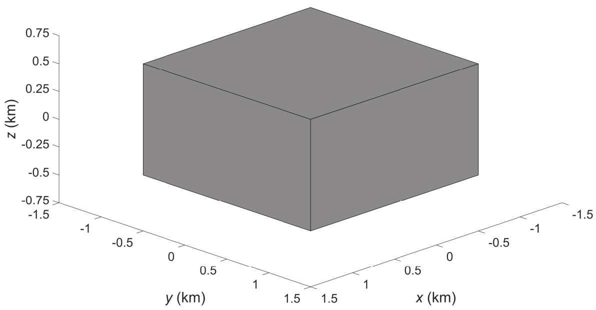

Fig. 1 The shape model of the prism

Table 13 Corrections for Eqs. (11), (12) and (33) in Nagy et al. (2000)

Equation numbers	Corrections
Eq. (11)	$u_x = \left\|\left\|\left\| y\ln(z+r) + z\ln(y+r) - x\tan^{-1}\frac{yz}{xr} \right\|_{x_1}^{x_2}\right\|_{y_1}^{y_2}\right\|_{z_1}^{z_2}$
Eq. (12)	$u_y = \left\|\left\|\left\| z\ln(x+r) + x\ln(z+r) - y\tan^{-1}\frac{zx}{yr} \right\|_{x_1}^{x_2}\right\|_{y_1}^{y_2}\right\|_{z_1}^{z_2}$
Eq. (33)	$u_{zzy} = \left\|\left\|\left\| -\frac{xz}{r}\left(\frac{1}{y^2+z^2}\right) \right\|_{x_1}^{x_2}\right\|_{y_1}^{y_2}\right\|_{z_1}^{z_2}$

Table 14 Some of the oblate spheroidal harmonic coefficients of the gravitational potential of the uniform prism

n	m	$C_{n,m}$
0	0	1.127483985998813E+00
2	0	6.165520401788543E-02
4	0	-6.069746076073939E-03
4	4	-1.366037767234713E-02
6	0	1.973318252751883E-04
6	4	-3.414404709283945E-03
60	0	2.223217284365850E-08
60	32	6.399577418009466E-07
60	60	-1.649410561998574E-08
120	0	-1.514123488639912E-08
120	60	-4.180710391554825E-08
120	120	1.538229284684741E-12
180	0	3.530816987068349E-09
180	92	-5.625285954137012E-09
180	180	1.536323590223830E-16

180, the latitude and longitude grids in Eq. (75) are discretized with $\Delta\vartheta = \Delta\lambda = 0.5°$ (Šprlák et al. 2018). The program codes for computing the harmonic coefficients using double-precision were written in Fortran 90, and compiled by the Intel Fortran Compiler 19.0 executed at a PC (Workstation) with an Intel Xeon W-2295 CPU and a 64 GB main memory under the 64 bit Windows 10. The processor base and max turbo frequencies of the W-2295 CPU are 3.00 GHz and 4.60 GHz. The Fortran programs were implemented in parallel programming using OpenMP with 36 (the number of the threads of the W-2295 CPU) threads for solving the harmonic coefficients (up to d/o 180), and the executed time is 6149.797 s, i.e., 1 hours 42 minutes 29.797 seconds. Some of the harmonic coefficients are listed in Table 14, and complete coefficients are provided at https://github.com/chengchengit/ohphderi_prep_code_file.

The first-, second- and third-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $u = b = 1.07$ km are illustrated in Figs. 2, 3 and 4, and the figures on the southern half are very similar which are not illustrated. The angular resolution of the latitude and longitude grids is $0.1° \times 1°$. The root-mean-square (RMS) errors $\delta V^{deri} = |V^{deri} -$

$V^{deri*}|/\max(|V^{deri*}|)$ for the derivatives of the prism on the confocal spheroid $u = 1.5$ km varying with the colatitude $\vartheta = 0.00001^\circ \to 90^\circ$ or the latitude $\phi = \frac{\pi}{2} - \vartheta = 0^\circ \to 89.99999^\circ$ are plotted in Figs. 5, 6 and 7, where V^{deri} denotes the numerical value of a k-order derivative (k=1,2,3) computed by the oblate spheroidal harmonic method including the non-singular and the regular expressions, V^{deri*} with the asterisk superscript denotes for the closed-form solutions Nagy et al. (2000) and is regarded as a truth-value of the derivative, $\max(|V^{deri*}|)$ is the maximum value of all the absolute values of the k-order derivatives (i.e. $\max(|V^{deri*}|) = \max(|V_x^*|, |V_y^*|, |V_z^*|)$, $\max(|V_{xx}^*|, |V_{xy}^*|, |V_{xz}^*|, |V_{yy}^*|, |V_{yz}^*|, |V_{zz}^*|)$, or $\max(|V_{xxx}^*|, |V_{xxy}^*|, |V_{xxz}^*|, |V_{xyy}^*|, |V_{yyy}^*|, |V_{yyz}^*|, |V_{xzz}^*|, |V_{yzz}^*|, |V_{zzz}^*|, |V_{xyz}^*|)$), the RMS errors are taken for a certain colatitude (latitude) along the longitude $\lambda = 0^\circ - 359^\circ$ with the difference $\Delta\lambda = 1^\circ$, and the value of the latitude is $\phi = 0^\circ \to 89^\circ$ with the difference $\Delta\phi = 1^\circ$ or $\phi = \phi_{\ell-1} \to \phi_\ell$ ($\ell = 1, 2, 3, 4, 5$) with the difference $\Delta\phi = 1^\circ \times 10^{-\ell} = 0.1^\circ, 0.01^\circ, 0.001^\circ, 0.0001^\circ, 0.00001^\circ$ and the values $\phi_0 = 89^\circ$, $\phi_1 = 89.9^\circ$, $\phi_2 = 89.99^\circ$, $\phi_3 = 89.999^\circ$, $\phi_4 = 89.9999^\circ$ and $\phi_5 = 89.99999^\circ$. The time costs for computing the first-, second- and third-order derivatives of the potential using the closed-form solutions and the oblate spheroidal harmonic method are listed in Table 15, where the number of the computational points is 2520 with the values $\vartheta = 5^\circ \to 175^\circ$, $\lambda = 0^\circ \to 355^\circ$ on the confocal spheroid $u = 1.5$ km and the differences $\Delta\vartheta = \Delta\lambda = 5^\circ$, the time cost is taken from the mean value of the executed costs of 2520 points with serial program (not parallel program), and the time cost of the closed-form solutions are also the mean cost of the 1000 times repeated executions to obtain accurate value since it is very small. We also give the values of the derivatives of the potential at the poles (0 km,0 km,1.6 km) computed by the closed-form solutions and the oblate spheroidal harmonic method as well as the spherical harmonic method in Table 16, where the spherical harmonic expansions for the derivatives up to third-order adopt the way of Cunningham (1970), Petrovskaya and Vershkov (2010), Chen et al. (2019) and Jamet and Tsoulis (2020). The bold denotes for the numbers in the (non-zero) results of the oblate spheroidal harmonic expansions coincident with the closed-form solutions. The reference radius of the spherical harmonic expansions of the gravitational field is $R = \sqrt{1^2 + 1^2 + 0.5^2} = 1.5$ km. The algorithms to compute the spherical harmonic coefficients of the potential of the homogeneous polyhedral bodies can refer to Jamet and Tsoulis (2020), and the spherical harmonic coefficients $\overline{C}^{sh}_{2\ell_1, 4\ell_2}$ ($4\ell_2 \leqslant 2\ell_1$) are needed only considered due to $\overline{C}^{sh}_{n,m} = 0$ for other indices n, m and $\overline{S}^{sh}_{n,m} = 0$ for all indices n, m. The time cost of the Fortran programs implemented in parallel programming using OpenMP with 36 threads for solving the spherical harmonic coefficients up to d/o 180 with double-precision is 3.995 s, and the results of the spherical harmonic coefficients are high-precision. The spherical harmonic coefficients up to d/o 180 are also presented at https://github.com/chengchengit/ohphderi_prep_code_file.

Figures 2, 3 and 4 demonstrate the derivatives of the potential up to third-order are computable and their values are continuous along the whole reference oblate spheroid when using the non-singular oblate spheroidal harmonic expressions. The derivatives of the potential of the uniform prism are bilaterally or transversely symmetrical. When the observation point tends to the poles, the derivatives V_z, V_{xx}, V_{yy}, V_{zz}, V_{xxz}, V_{yyz}

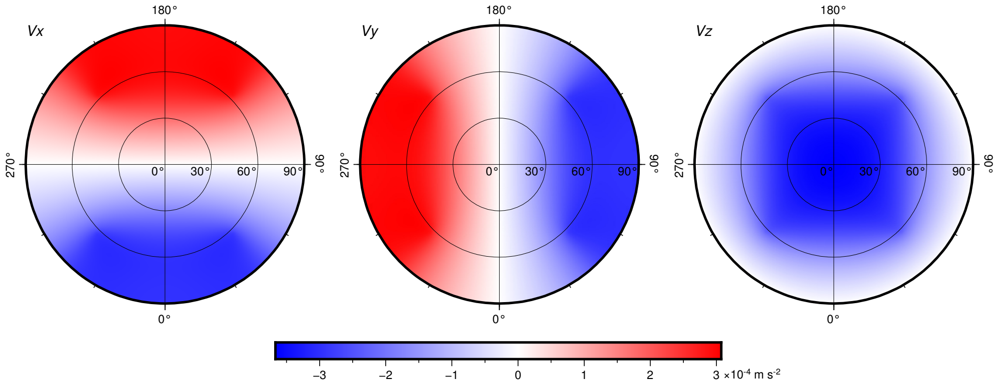

Fig. 2 First-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $u = b$. The value of the colatitude ϑ is labeled every 30°, and the value of the longitude λ every 90°. The oblate spheroidal harmonic expansions have been computed up to d/o 180

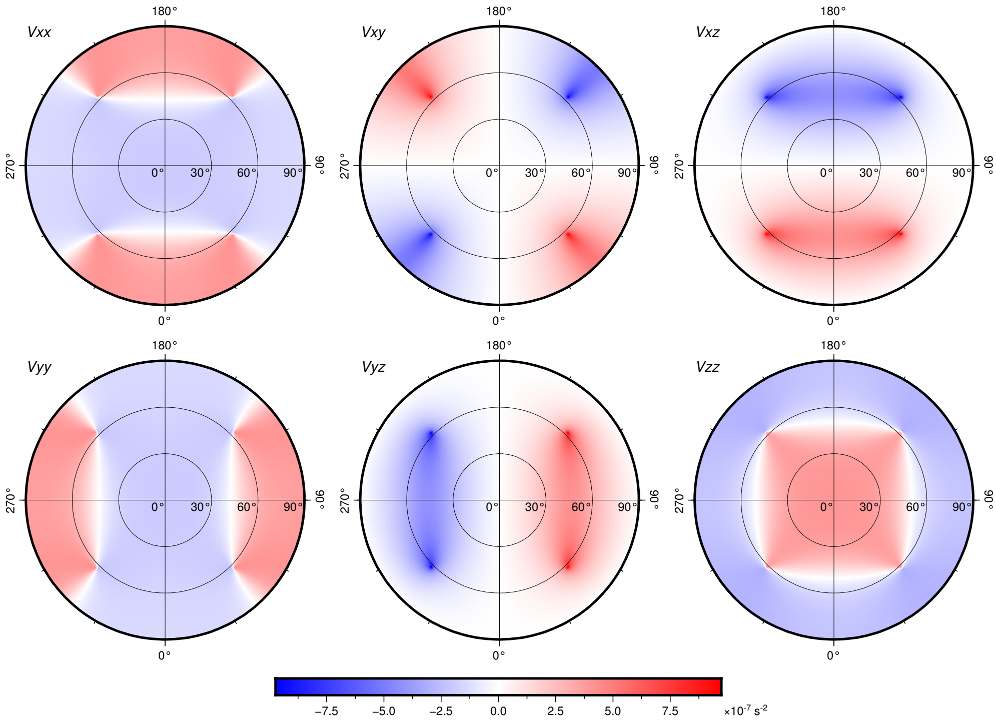

Fig. 3 Second-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $u = b$. The oblate spheroidal harmonic expansions have been computed up to d/o 180

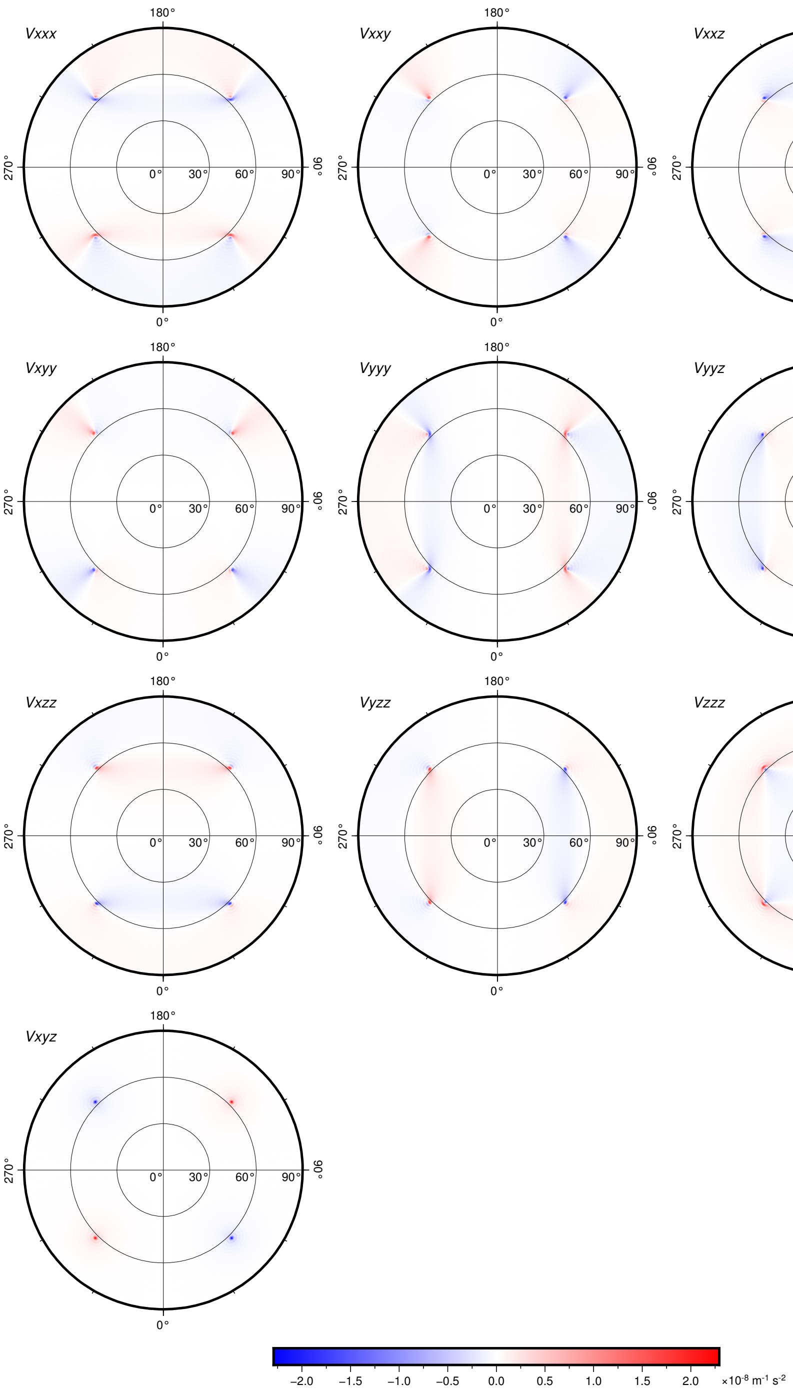

Fig. 4 Third-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $u = b$. The oblate spheroidal harmonic expansions have been computed up to d/o 180

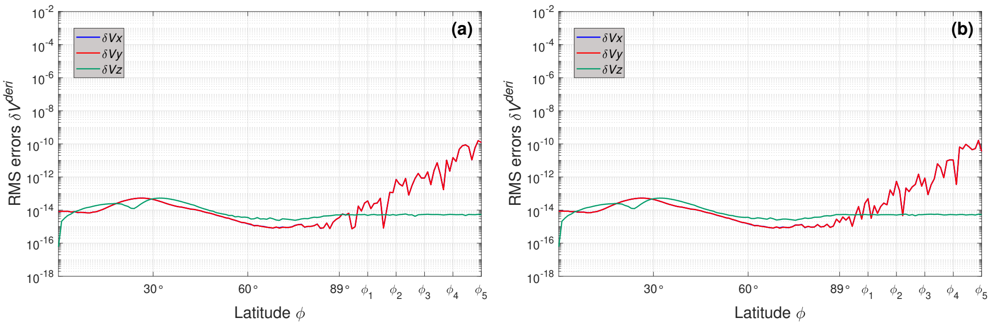

Fig. 5 The RMS errors δV^{deri} for the first-order derivatives of the gravitational potential of the uniform prism on the confocal spheroid $u = 1.5$ km computed by the non-singular (**a**) and regular (**b**) oblate spheroidal harmonic expressions. The oblate spheroidal harmonic expansions have been computed up to d/o 180

and V_{zzz} tend to non-zero values, and other derivatives become zero, which are also shown in Table 16. In Fig. 5 the changing curves of the RMS errors δV_x and δV_y are almost the same, and in Fig. 6c the RMS errors δV_{xx} and δV_{xy} are coincident when the point tends to the poles. For the third-order derivatives, the RMS errors δV_{xzz} and δV_{yzz} are coincident (in Figs. 7c and 7f), and the RMS errors δV_{xxx} and δV_{xxy} are also coincident when the point tends to the poles (in Figs. 7d and 7e). From Figs. 5, 6 and 7, the non-singular and the regular oblate spheroidal harmonic expressions are both highly accurate for the observation point not near the poles. When the point near the poles, the non-singular expressions are still highly accurate, but the regular expressions produce larger errors, including mainly the derivatives V_{xx}, V_{yy}, V_{zz}, V_{xxx}, V_{xxy}, V_{xxz}, V_{xyy}, V_{yyy}, V_{yyz} and V_{xyz}. Although the first-order derivatives computed by the regular expressions are accurate in Fig. 5, when the latitude $\phi \geqslant 90° - 1° \times 10^{-7}$ or $90° - 1° \times 10^{-8}$ (the colatitude $\vartheta \leqslant 1° \times 10^{-7}$ or $1° \times 10^{-8}$) the computed values of all the derivatives are NaN using the regular expressions, and become zero using the non-singular expressions. The increasing errors of the derivatives V_x, V_y, V_{xz}, V_{yz}, V_{xxx}, V_{xxy}, V_{xyy}, V_{yyy}, V_{xzz} and V_{yzz} computed by the non-singular expressions for the point near the poles may be caused by the small values of these derivatives and the double-precision floating point arithmetic, and do not indicate the non-singular expressions are inaccurate. These errors are still in about 10^{-10} at the latitude $\phi = 89.99999°$. Table 15 demonstrates the closed-form solutions are very fast, and the computation speed of the regular oblate spheroidal harmonic expressions is faster than the non-singular expressions, about 16-17 times. Note for more complicated shape bodies, the computation speed of the oblate spheroidal harmonic expressions is constant, and of the closed-form solutions in space-domain becomes slow, e.g., for the uniform polyhedral body (D'Urso 2014). When the observation point near or on the poles, no singular values are produced for the derivatives. From Table 16, the derivatives of the gravitational potential at the poles computed by the oblate spheroidal harmonic method are also high-precision, i.e. 13-14 digits achievable precision, and the precisions are better than the spherical harmonic method when computed up to d/o 180, i.e. the oblate spheroidal harmonic expansions converging faster.

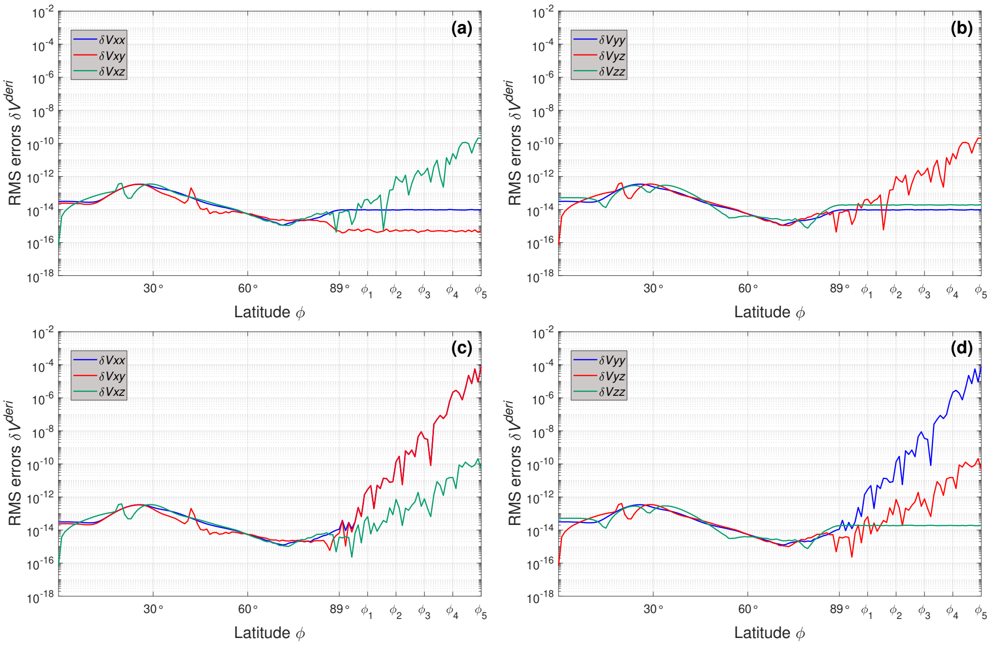

Fig. 6 The RMS errors δV^{deri} for the second-order derivatives of the gravitational potential of the uniform prism on the confocal spheroid $u = 1.5$ km computed by the non-singular (**a**, **b**) and regular (**c**, **d**) oblate spheroidal harmonic expressions. The oblate spheroidal harmonic expansions have been computed up to d/o 180

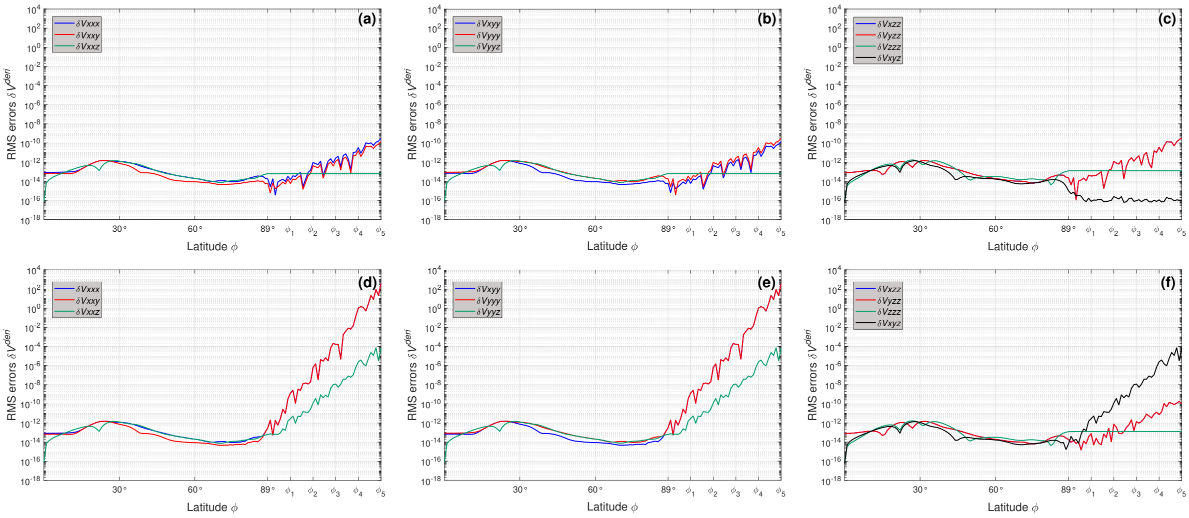

Fig. 7 The RMS errors δV^{deri} for the third-order derivatives of the gravitational potential of the uniform prism on the confocal spheroid $u = 1.5$ km computed by the non-singular (**a**, **b**, **c**) and regular (**d**, **e**, **f**) oblate spheroidal harmonic expressions. The oblate spheroidal harmonic expansions have been computed up to d/o 180

Table 15 The time costs for computing the derivatives up to third-order of the uniform prism at an observation point using the closed-form solutions and the oblate spheroidal harmonic method. The unit is seconds (s), and OH denotes for the oblate spheroidal harmonic expansion. The oblate spheroidal harmonic expansions have been computed up to d/o 180

d/o N	Closed-form solutions	Regular OH	Non-singular OH
60	0.000005026	0.006661	0.1082
120	0.000005026	0.02541	0.4203
180	0.000005026	0.05620	0.9609

Table 16 Derivatives of the gravitational potential of the uniform prism at the pole (0 km,0 km,1.6 km). SH denotes for the spherical harmonic expansion, and OH the (non-singular) oblate spheroidal harmonic expansion. The spherical and the oblate spheroidal harmonic expansions have been computed up to d/o 180

Derivatives	Closed-form solutions	SH	OH
V_x	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_y	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_z	-2.129976655912064E-04	-2.129976655881591E-04	**-2.12997665591207**4E-04
V_{xx}	-1.010922089468464E-07	-1.010922087691659E-07	**-1.010922089468**473E-07
V_{xy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{xz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yy}	-1.010922089468464E-07	-1.010922087691659E-07	**-1.010922089468**473E-07
V_{yz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{zz}	2.021844178936928E-07	2.021844175383318E-07	**2.021844178936**947E-07
V_{xxx}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{xxy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{xxz}	1.279831308230781E-10	1.279831099996278E-10	**1.279831308230**849E-10
V_{xyy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yyy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yyz}	1.279831308230781E-10	1.279831099996278E-10	**1.279831308230**849E-10
V_{xzz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yzz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{zzz}	-2.559662616461562E-10	-2.559662199992555E-10	**-2.559662616461**697E-10
V_{xyz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00

7 Conclusions

In this work we discuss the expressions of the oblate spheroidal harmonic expansions of the first-, second- and third-order derivatives of the gravitational potential in the global reference frame, and the conversions to the derivatives in the local north-oriented reference frame. In order to improve the applicability of the computations of the associated Legendre functions of the second kind in the derivatives of the potential for arbitrary oblate celestial bodies, we also convert the ratios of the Legendre functions into the ratios of the Gaussian hypergeometric series and give the recursive algorithm of the hypergeometric series. The relations for the product of the associated Legendre functions of the first and second kinds and the trigonometric function in the oblate spheroidal harmonic expansions of the gravitational field and its derivatives with respect to the Cartesian coordinates are proposed. Since the derivatives of the product can also be written in the forms of the linear combinations of the product with same/different integer indices, the relations can be applied to find the oblate spheroidal harmonic expressions of the gravitational vector and higher-order derivatives. The expressions are also non-singular for the observation point near or on the poles due to the non-singularity of the relations of the product and its derivatives. The expressions for higher-order derivatives are on the basis of the explicit form of the third-order derivatives, i.e., the second formula of Eq. (48), which contains more product terms. Compared the non-singular and the regular expressions of the derivatives in computational precision and time cost, the regular expressions are recommended for computing the gravitational field at the observation point which are not near the poles (e.g. the latitude $\phi \in [-89°, 89°]$ or the colatitude $\vartheta \in [1°, 179°]$), and the non-singular expressions are suitable for the point near the poles (e.g. the latitude $\phi \in (89°, 90°] \cup [-90°, -89°)$ or the colatitude $\vartheta \in [0°, 1°) \cup (179°, 180°]$). The formulas in Sects. 3.2, 3.3, 5.2 and Appendix of this work can be verified by the Mathematica codes.

Data Availability. The datasets for the oblate spheroidal harmonic and the spherical harmonic coefficients of the gravitational potential of the uniform prism up to d/o 180 are available at https://github.com/chengchengit/ohphderi_prep_code_file.

Appendix: Regular expressions of the derivatives of the gravitational potential

The first-order derivative of the gravitational potential with respect to the Cartesian coordinates can be written in the form of the derivatives with respect to the oblate spheroidal coordinates:

$$V_{x_i} = \sum_{l=1}^{3} \frac{\partial u_l}{\partial x_i} V_{u_l}, \tag{76}$$

where the indices $i, l = 1, 2, 3$, and the symbols $(u_1, u_2, u_3) = (u, \vartheta, \lambda)$. The partial derivatives $\frac{\partial u_l}{\partial x_i}$ including u_x, u_y, u_z, ϑ_x, ϑ_y, ϑ_z, λ_x, λ_y and λ_z $(= 0)$ are given in Eq.

(8), in which the subscript denotes for the derivative with respect to the Cartesian coordinate, e.g., $u_x = \frac{uv\sin\vartheta\cos\lambda}{L^2}$. Eq. (76) is the regular formula to compute the first-order derivatives in the literature (Hu and Jekeli 2015). Differentiating Eq. (76) with respect to the Cartesian coordinate, the second-order derivative of the potential can be obtained as follows

$$V_{x_ix_j} = \sum_{l=1}^{3} \frac{\partial^2 u_l}{\partial x_i \partial x_j} V_{u_l} + \sum_{1\leqslant l_1,l_2\leqslant 3} \frac{\partial u_{l_1}}{\partial x_i}\frac{\partial u_{l_2}}{\partial x_j} V_{u_{l_1}u_{l_2}}, \tag{77}$$

where the indices $i,j,l,l_1,l_2 = 1,2,3$. The expressions of the derivatives $\frac{\partial^2 u_l}{\partial x_i \partial x_j}$ are given in Table 17 using the partial derivative relation (8), and the coefficients f_2 are

$$\begin{aligned}
f_2^{u0} =& \frac{u}{L^2},\ f_2^{u} = \frac{uv^2\sin^2\vartheta}{L^6}\Big(v^2 - 4E^2 + 3E^2\sin^2\vartheta\Big),\\
f_2^{uz} =& \frac{v\sin\vartheta\cos\vartheta}{L^6}\Big(u^2v^2 - E^2v^2 + (3v^2-2E^2)E^2\sin^2\vartheta\Big),\\
f_2^{\vartheta 0} =& \frac{\cos\vartheta}{L^2\sin\vartheta},\ f_2^{\vartheta} = \frac{v^2\cos\vartheta}{L^6\sin\vartheta}\Big(v^2 + (2v^2-5E^2)\sin^2\vartheta + 2E^2\sin^4\vartheta\Big),\\
f_2^{\vartheta z} =& \frac{uv}{L^6}\Big(v^2 - (2v^2-3E^2)\sin^2\vartheta - 2E^2\sin^4\vartheta\Big),\ f_2^{\vartheta zz} = \frac{\sin\vartheta\cos\vartheta}{L^6}\Big((2v^2-3E^2)v^2 + (2v^2-E^2)E^2\sin^2\vartheta\Big),\\
f_2^{\lambda 0} =& \frac{1}{v^2\sin^2\vartheta},\ f_2^{\lambda} = \frac{2}{v^2\sin^2\vartheta},
\end{aligned} \tag{78}$$

Differentiating Eq. (77) with respect to the Cartesian coordinate, the third-order derivative of the potential can be further written as

$$\begin{aligned}
V_{x_ix_jx_k} =& \sum_{l=1}^{3} \frac{\partial^3 u_l}{\partial x_i\partial x_j\partial x_k} V_{u_l} + \sum_{1\leqslant l_1,l_2\leqslant 3}\Bigg(\frac{\partial^2 u_{l_1}}{\partial x_i\partial x_k}\frac{\partial u_{l_2}}{\partial x_j} + \frac{\partial u_{l_1}}{\partial x_i}\frac{\partial^2 u_{l_2}}{\partial x_j\partial x_k}\\
&+ \frac{\partial^2 u_{l_1}}{\partial x_i\partial x_j}\frac{\partial u_{l_2}}{\partial x_k}\Bigg)V_{u_{l_1}u_{l_2}} + \sum_{1\leqslant l_1,l_2,l_3\leqslant 3}\frac{\partial u_{l_1}}{\partial x_i}\frac{\partial u_{l_2}}{\partial x_j}\frac{\partial u_{l_3}}{\partial x_k}V_{u_{l_1}u_{l_2}u_{l_3}},
\end{aligned} \tag{79}$$

where the indices $i,j,k,l,l_1,l_2,l_3 = 1,2,3$. For the derivative $V_{x_ix_jx_k}$, we need also know the third-order derivative $\frac{\partial^3 u_l}{\partial x_i\partial x_j\partial x_k}$, whose expressions are completely given in Tables 18 and 19 by applying Eq. (8). The following relations can be observed: $f_3^u + f_3^{uxy} = f_3^{uzz}$, $f_3^{\vartheta} - \frac{3}{2}f_3^{\vartheta xy} = f_3^{\vartheta zz}$, and $4f_3^{\lambda xy} = f_3^{\lambda}$. The expressions of $\frac{\partial^2 u_l}{\partial x_i\partial x_j}$ and $\frac{\partial^3 u_l}{\partial x_i\partial x_j\partial x_k}$ have analogous forms to the coefficient terms of the non-singular expressions, see Tables 1 and 4.

For the partial derivatives V_{u_l}, $V_{u_{l_1}u_{l_2}}$ and $V_{u_{l_1}u_{l_2}u_{l_3}}$ with respect to the oblate spheroidal coordinates in Eqs. (76), (77) and (79), we define the general form of these derivatives as

$$V_{u^{k_1}\vartheta^{k_2}\lambda^{k_3}} = \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\widehat{Q}_{n,n,m}^{(k_1)}\overline{P}_{n,m}^{(k_2)}\overline{T}_{n,m,m}^{(k_3)}, \tag{80}$$

where the integer indices $k_1, k_2, k_3 = 0, 1, 2, \cdots$, and the symbols $\widehat{Q}^{(k_1)}_{n,n,m}$, $\overline{P}^{(k_2)}_{n,m}$ and $\overline{T}^{(k_3)}_{n,m,m}$ represent the k_1-, k_2- and k_3-order derivatives of the functions $\widehat{Q}_{n,n,m}(u/E)$, $\overline{P}_{n,m}(\cos\vartheta)$ and $\overline{T}_{n,m,m}(\lambda)$ with respect to the u-, ϑ- and λ-coordinates, respectively, i.e., $\widehat{Q}^{(k_1)}_{n,n,m} = \frac{\mathrm{d}^{k_1}\widehat{Q}_{n,n,m}}{\mathrm{d}u^{k_1}}$, $\overline{P}^{(k_2)}_{n,m} = \frac{\mathrm{d}^{k_2}\overline{P}_{n,m}}{\mathrm{d}\vartheta^{k_2}}$, and $\overline{T}^{(k_3)}_{n,m,m} = \frac{\mathrm{d}^{k_3}\overline{T}_{n,m,m}}{\mathrm{d}\lambda^{k_3}}$. The symbol $u^{k_1}\vartheta^{k_2}\lambda^{k_3}$ in Eq. (80) denotes the subscript including k_1 u-coordinate, k_2 ϑ-coordinate and k_3 λ-coordinate, e.g. being $\vartheta\lambda\lambda$, when the integers $(k_1, k_2, k_3) = (0, 1, 2)$. From Eq. (12) and the second-order derivative for the Legendre function $Q_{n,m}$ (Gradshteyn and Ryzhik 2007, pp. 958), we can obtain the relations for the derivatives $\widehat{Q}^{(k_1)}_{n,n,m}$

Table 17 Expressions of the derivatives $\frac{\partial^2 u_l}{\partial x_i \partial x_j}$ $(1 \leqslant l \leqslant 3,\ 1 \leqslant i \leqslant j \leqslant 3)$

$u_{xx} = f_2^{u0} - f_2^{u}\cos^2\lambda$, $\vartheta_{xx} = f_2^{\vartheta 0} - f_2^{\vartheta}\cos^2\lambda$, $\lambda_{xx} = f_2^{\lambda}\sin\lambda\cos\lambda$
$u_{xy} = -f_2^{u}\sin\lambda\cos\lambda$, $\vartheta_{xy} = -f_2^{\vartheta}\sin\lambda\cos\lambda$, $\lambda_{xy} = f_2^{\lambda 0} - f_2^{\lambda}\cos^2\lambda$
$u_{xz} = -f_2^{uz}\cos\lambda$, $\vartheta_{xz} = -f_2^{\vartheta z}\cos\lambda$, $\lambda_{xz} = 0$
$u_{yy} = f_2^{u0} - f_2^{u}\sin^2\lambda$, $\vartheta_{yy} = f_2^{\vartheta 0} - f_2^{\vartheta}\sin^2\lambda$, $\lambda_{yy} = -f_2^{\lambda}\sin\lambda\cos\lambda$
$u_{yz} = -f_2^{uz}\sin\lambda$, $\vartheta_{yz} = -f_2^{\vartheta z}\sin\lambda$, $\lambda_{yz} = 0$
$u_{zz} = f_2^{u}$, $\vartheta_{zz} = f_2^{\vartheta zz}$, $\lambda_{zz} = 0$

Table 18 Expressions of the derivatives $\frac{\partial^3 u_l}{\partial x_i \partial x_j \partial x_k}$ $(1 \leqslant l \leqslant 3,\ 1 \leqslant i \leqslant j \leqslant k \leqslant 3)$

$u_{xxx} = f_3^{u0}\cos\lambda + (f_3^{uxy} + f_3^{u}\cos^2\lambda)\cos\lambda$, $\vartheta_{xxx} = -f_3^{\vartheta 0}\cos\lambda - (f_3^{\vartheta xy} - f_3^{\vartheta}\cos^2\lambda)\cos\lambda$, $\lambda_{xxx} = (f_3^{\lambda xy} - f_3^{\lambda}\cos^2\lambda)\sin\lambda$
$u_{xxy} = f_3^{u0}\sin\lambda + f_3^{u}\cos^2\lambda\sin\lambda$, $\vartheta_{xxy} = -f_3^{\vartheta 0}\sin\lambda + f_3^{\vartheta}\cos^2\lambda\sin\lambda$, $\lambda_{xxy} = (f_3^{\lambda xy} - f_3^{\lambda}\sin^2\lambda)\cos\lambda$
$u_{xxz} = f_3^{uz0} + f_3^{uz}\cos^2\lambda$, $\vartheta_{xxz} = -f_3^{\vartheta z0} + f_3^{\vartheta z}\cos^2\lambda$, $\lambda_{xxz} = 0$
$u_{xyy} = f_3^{u0}\cos\lambda + f_3^{u}\sin^2\lambda\cos\lambda$, $\vartheta_{xyy} = -f_3^{\vartheta 0}\cos\lambda + f_3^{\vartheta}\sin^2\lambda\cos\lambda$, $\lambda_{xyy} = -(f_3^{\lambda xy} - f_3^{\lambda}\cos^2\lambda)\sin\lambda$
$u_{yyy} = f_3^{u0}\sin\lambda + (f_3^{uxy} + f_3^{u}\sin^2\lambda)\sin\lambda$, $\vartheta_{yyy} = -f_3^{\vartheta 0}\sin\lambda - (f_3^{\vartheta xy} - f_3^{\vartheta}\sin^2\lambda)\sin\lambda$, $\lambda_{yyy} = -(f_3^{\lambda xy} - f_3^{\lambda}\sin^2\lambda)\cos\lambda$
$u_{yyz} = f_3^{uz0} + f_3^{uz}\sin^2\lambda$, $\vartheta_{yyz} = -f_3^{\vartheta z0} + f_3^{\vartheta z}\sin^2\lambda$, $\lambda_{yyz} = 0$
$u_{xzz} = -f_3^{uzz}\cos\lambda$, $\vartheta_{xzz} = -f_3^{\vartheta zz}\cos\lambda$, $\lambda_{xzz} = 0$
$u_{yzz} = -f_3^{uzz}\sin\lambda$, $\vartheta_{yzz} = -f_3^{\vartheta zz}\sin\lambda$, $\lambda_{yzz} = 0$
$u_{zzz} = -f_3^{uz}$, $\vartheta_{zzz} = f_3^{\vartheta zzz}$, $\lambda_{zzz} = 0$
$u_{xyz} = f_3^{uz}\sin\lambda\cos\lambda$, $\vartheta_{xyz} = f_3^{\vartheta z}\sin\lambda\cos\lambda$, $\lambda_{xyz} = 0$

Table 19 Expressions of the notations f_3 in Table 18

$f_3^{u0} = \frac{uv\sin\vartheta}{L^6}\left(4E^2 - v^2 - 3E^2\sin^2\vartheta\right)$
$f_3^{u} = \frac{3uv^3\sin^3\vartheta}{L^{10}}\left(16E^4 - 12E^2v^2 + v^4 - 10E^2(2E^2 - v^2)\sin^2\vartheta + 5E^4\sin^4\vartheta\right)$
$f_3^{uxy} = \frac{2uv\sin\vartheta}{L^{10}}\left(v^4(4E^2 - v^2) - E^2v^2(8E^2 + v^2)\sin^2\vartheta + E^4(4E^2 + 5v^2)\sin^4\vartheta - 3E^6\sin^6\vartheta\right)$
$f_3^{uz0} = \frac{\cos\vartheta}{L^6}\left(E^2v^2 - u^2v^2 + E^2(2E^2 - 3v^2)\sin^2\vartheta\right)$
$f_3^{uz} = \frac{3v^2\sin^2\vartheta\cos\vartheta}{L^{10}}\left(v^2(8E^4 - 8E^2v^2 + v^4) + 2E^2(4E^4 - 10E^2v^2 + 5v^4)\sin^2\vartheta - E^4(4E^2 - 5v^2)\sin^4\vartheta\right)$
$f_3^{uzz} = \frac{uv\sin\vartheta}{L^{10}}\left(2v^4(4E^2 - v^2) + v^2(32E^4 - 38E^2v^2 + 3v^4)\sin^2\vartheta + 2E^2(4E^4 - 25E^2v^2 + 15v^4)\sin^4\vartheta - 3E^4(2E^2 - 5v^2)\sin^6\vartheta\right)$
$f_3^{\vartheta 0} = \frac{v\cos\vartheta}{L^6\sin^2\vartheta}\left(v^2 - (5E^2 - 2v^2)\sin^2\vartheta + 2E^2\sin^4\vartheta\right)$
$f_3^{\vartheta} = \frac{v^3\cos\vartheta}{L^{10}\sin^2\vartheta}\left(3v^4 - 2v^2(9E^2 - 2v^2)\sin^2\vartheta + (63E^4 - 44E^2v^2 + 8v^4)\sin^4\vartheta - 8E^2(7E^2 - 4v^2)\sin^6\vartheta + 8E^4\sin^8\vartheta\right)$
$f_3^{\vartheta xy} = \frac{2v\cos\vartheta}{L^{10}\sin^2\vartheta}\left(v^6 - v^4(7E^2 - 2v^2)\sin^2\vartheta + E^2v^2(11E^2 - 2v^2)\sin^4\vartheta - E^4(5E^2 + 2v^2)\sin^6\vartheta + 2E^6\sin^8\vartheta\right)$
$f_3^{\vartheta z0} = \frac{u}{L^6\sin\vartheta}\left(v^2 + (3E^2 - 2v^2)\sin^2\vartheta - 2E^2\sin^4\vartheta\right)$
$f_3^{\vartheta z} = \frac{uv^2}{L^{10}\sin\vartheta}\left(v^4 - 2v^2(7E^2 - 2v^2)\sin^2\vartheta - (35E^4 - 52E^2v^2 + 8v^4)\sin^4\vartheta + 8E^2(5E^2 - 4v^2)\sin^6\vartheta - 8E^4\sin^8\vartheta\right)$
$f_3^{\vartheta zz} = \frac{v\cos\vartheta}{L^{10}}\left(v^4(3E^2 - 2v^2) + 2v^2(15E^4 - 19E^2v^2 + 4v^4)\sin^2\vartheta + E^2(15E^4 - 50E^2v^2 + 32v^4)\sin^4\vartheta - 2E^4(3E^2 - 4v^2)\sin^6\vartheta\right)$
$f_3^{\vartheta zzz} = \frac{u\sin\vartheta}{L^{10}}\left(3v^4(5E^2 - 2v^2) + 2v^2(15E^4 - 25E^2v^2 + 4v^4)\sin^2\vartheta + E^2(3E^4 - 38E^2v^2 + 32v^4)\sin^4\vartheta - 2E^4(E^2 - 4v^2)\sin^6\vartheta\right)$
$f_3^{\lambda xy} = \frac{2}{v^3\sin^3\vartheta}$
$f_3^{\lambda} = \frac{8}{v^3\sin^3\vartheta}$

$(k_1 = 1, 2, 3)$:

$$\begin{aligned}
\widehat{Q}^{(1)}_{n,n,m} &= -(n-m+1)\frac{E}{v^2}\widehat{Q}_{n+1,n,m} - (n+1)\frac{u}{v^2}\widehat{Q}_{n,n,m},\\
\widehat{Q}^{(2)}_{n,n,m} &= -\frac{2u}{v^2}\widehat{Q}^{(1)}_{n,n,m} + \left(n(n+1) - m^2\frac{E^2}{v^2}\right)\frac{\widehat{Q}_{n,n,m}}{v^2},\\
\widehat{Q}^{(3)}_{n,n,m} &= -\frac{2u}{v^2}\widehat{Q}^{(2)}_{n,n,m} + \left(n(n+1) - \frac{(m^2+2)E^2 - 2u^2}{v^2}\right)\frac{\widehat{Q}^{(1)}_{n,n,m}}{v^2}\\
&\quad - \left(n(n+1) - 2m^2\frac{E^2}{v^2}\right)\frac{2u}{v^4}\widehat{Q}_{n,n,m},
\end{aligned} \tag{81}$$

The expression of the second-order derivative $\widehat{Q}^{(2)}_{n,n,m}$ is derived by differentiating the first-order derivative $\widehat{Q}^{(1)}_{n,n,m}$. Analogously, the formulas to compute the derivatives $\overline{P}^{(k_2)}_{n,m}$ $(k_2 = 1, 2, 3)$ are (Gradshteyn and Ryzhik 2007, pp. 958-965; Fantino and

Casotto 2009; Fukushima 2012b):

$$
\begin{aligned}
\overline{P}^{(1)}_{n,m} =& \frac{n\cos\vartheta}{\sin\vartheta}\overline{P}_{n,m} - \sqrt{\frac{(2n+1)(n+m)(n-m)}{2n-1}}\frac{\overline{P}_{n-1,m}}{\sin\vartheta},\\
\overline{P}^{(2)}_{n,m} =& -\frac{\cos\vartheta}{\sin\vartheta}\overline{P}^{(1)}_{n,m} - \left(n(n+1) - \frac{m^2}{\sin^2\vartheta}\right)\overline{P}_{n,m},\\
\overline{P}^{(3)}_{n,m} =& -\frac{\cos\vartheta}{\sin\vartheta}\overline{P}^{(2)}_{n,m} - \left(n(n+1) - \frac{m^2+1}{\sin^2\vartheta}\right)\overline{P}^{(1)}_{n,m} - \frac{2m^2\cos\vartheta}{\sin^3\vartheta}\overline{P}_{n,m},
\end{aligned}
\tag{82}
$$

According to Eq. (13), the derivatives $\overline{T}^{(k_3)}_{n,m,m}$ ($k_3 = 1, 2, 3$) can be expressed as

$$
\begin{aligned}
\overline{T}^{(1)}_{n,m,m} =& m\overline{T}'_{n,m,m},\\
\overline{T}^{(2)}_{n,m,m} =& -m^2\overline{T}_{n,m,m},\\
\overline{T}^{(3)}_{n,m,m} =& -m^3\overline{T}'_{n,m,m} = -m^2\overline{T}^{(1)}_{n,m,m},
\end{aligned}
\tag{83}
$$

The singular factors $\frac{1}{\sin\vartheta}$, $\frac{1}{\sin^2\vartheta}$ and $\frac{1}{\sin^3\vartheta}$ is included in Eq. (82), and the non-singular fixed-degree algorithms can refer to Bosch (2000) and Fukushima (2012) which can slow the increasing tendency of the error of the derivatives of the potential. In numerical experiments of this work, we test only Eq. (82) with worse digital precision. The higher precision (e.g. quad-precision) floating-point arithmetic can be applied for computing the values of the regular expressions to obtain better precision results. Another singularity of the regular expressions is caused by the coefficient terms ϑ_x, ϑ_y, $f_2^{\vartheta 0}$, f_2^{ϑ}, $f_2^{\lambda 0}$, f_2^{λ}, $f_3^{\vartheta 0}$, f_3^{ϑ}, $f_3^{\vartheta xy}$, $f_3^{\vartheta z0}$, $f_3^{\vartheta z}$, $f_3^{\lambda xy}$ and f_3^{λ} which also contain the singular factor $\frac{1}{\sin\vartheta}$ and its powers. Considering the expressions of these singular coefficient terms and the non-singular forms of the derivatives of the Legendre function $\overline{P}_{n,m}(\cos\vartheta)$, analogous to the spherical harmonic case in Eshagh (2008) and Hamáčková (2016) the non-singular expressions of the derivatives of the potential based on the regular relations (76), (77) and (79) can also be derived, in which the cases $m = 0, 1, 2$ and $m > 2$ may need to be discussed respectively.

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II: prolate spheroidal harmonic expressions

Abstract

In space explorations, the gravimetric observation is important on determining the physical properties of the celestial bodies. The prolate spheroidal harmonic function is a suited tool for handling the gravitational field of the elongated small bodies in the solar system. In this work we present the prolate spheroidal harmonic expressions for the derivatives of the gravitational potential of celestial bodies up to third-order, including the non-singular and the regular forms. The non-singular expressions produce are valid for arbitrary external observation points outside the Brillouin prolate spheroid, and produce correct values near or at the poles. The derivatives of the potential can be written in linear combinations of the product of the associated Legendre functions and the trigonometric function in the prolate spheroidal harmonics with same or different integer subscripts. The regular expressions are singular at the poles, and may be suitable for computing the derivatives of the potential with less time cost for the observation points outside the polar regions. The prolate spheroidal harmonic expressions of the derivatives in the local north-oriented reference frame is also preliminarily discussed. The main algorithms of this work are verified by the numerical experiments of the gravitational field of the uniform prolate prism.

Keywords: Prolate spheroidal harmonics, Derivatives of the gravitational potential, Legendre function, Non-singular expressions, Elongated small bodies

1 Introduction

With the increase of the number of space exploration missions, there is paid more and more attention to the small celestial bodies in the solar system (Souchay and Dvorak 2010; Scheeres 2016). These bodies produce irregular gravitational fields due to their irregular shape. For an elongated small body, its gravitational field can be modelled using the prolate spheroidal harmonics. Compared with the reference spherical surface the spherical harmonic series of the external gravitational field, the reference prolate spheroidal surface of the prolate spheroidal harmonic series is more fitted to the shape of elongated small bodies. In addition, the prolate spheroidal harmonic expansions converge faster than the spherical harmonic expansions (Hobson 1931; Fukushima 2014; Reimond and Baur 2016). The external gravitational potential of 3D bodies can be described by a standard prolate spheroidal harmonic expansion, and the harmonic expansions of the derivatives of the potential are also expected to be solved.

Analogous to the oblate spheroidal harmonic expressions, the prolate spheroidal harmonic expansions of the derivatives contain the reciprocal factor of the sine function of the colatitude coordinate, and then have the singularity problems for the observation point near the poles. In the literature the derivatives in the global reference frame centered at a fixed point no more than first order, i.e. the gravitational accelerations, are considered using the direct relations of the derivatives of the potential with respect to the prolate spheroidal coordinates and the derivatives of the prolate spheroidal coordinates with respect to three Cartesian coordinates (Fukushima 2014), and the derivatives up to second-order in the local South-East-Up frame are computed using the conversions of the derivatives of the potential from the prolate spheroidal coordinates in the global reference frame to the local Cartesian coordinates centered at the observation point (Sebera et al. 2016), which may be incalculable near the poles, and high-order derivatives are not taken into account. The topic of the prolate spherical harmonic expressions of the derivatives of the gravitational potential up to second- or higher-order in the global and local reference frames is useful, and the expressions applicable to arbitrary external observation point may also be required.

The aim of this work is to find the prolate spherical harmonic expressions of the derivatives of the gravitational potential. The relations of the prolate spheroidal harmonic expansion and its derivatives without singularity problems in the global reference frame are established in Sect. 2. We carefully derive the expressions of the derivatives of the potential up to third-order in Sect. 3, and give the corresponding computations of derivatives at the poles in Sect. 4. The prolate spheroidal harmonic expressions of the derivatives in the local north-oriented reference frame is discussed in Sect. 5. The numerical tests of the algorithms of this work are considered in Sect. 6. Finally, we draw the conclusions in Sect. 7. The regular expressions of the derivatives are discussed in the Appendix.

2 Relations of the prolate spheroidal harmonic expansion and its derivatives

2.1 Expansion of the gravitational potential

The prolate spheroidal coordinates consist of the semi-major axis v of the confocal oblate spheroid $x^2/u^2+y^2/u^2+z^2/v^2=1$, the colatitude ϑ and the longitude λ, where u is the semi-minor axis ($u=\sqrt{v^2-E^2}$ with the linear eccentricity E). The relations between the global Cartesian coordinates (x,y,z) and prolate spheroidal coordinates (v,ϑ,λ) are: $x=u\sin\vartheta\cos\lambda$, $y=u\sin\vartheta\sin\lambda$, and $z=v\cos\vartheta$ (Hobson 1931, pp. 412; Fukushima 2014). From Hobson (1931, pp. 417-421), the external gravitational potential V in the body-fixed frame can be expressed as the prolate spheroidal harmonic expansion:

$$V=\frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\frac{Q_{n,m}(v/E)}{Q_{n,m}(a/E)}\overline{P}_{n,m}(\cos\vartheta)\Big(\overline{C}_{n,m}\cos(m\lambda)+\overline{S}_{n,m}\sin(m\lambda)\Big),\qquad(1)$$

where $\mu = GM$ is the gravitational constant of the body, a is the semi-major axis of the reference prolate spheroid (b being the semi-minor axis and $E = \sqrt{a^2 - b^2}$), $\overline{P}_{n,m}(\cos\vartheta)$ is the fully normalized associated Legendre function of the first kind, $Q_{n,m}(v/E)$ is the associated Legendre function of the second kind, $\overline{C}_{n,m}$ and $\overline{S}_{n,m}$ are the normalized prolate spheroidal harmonic coefficients. The relations of the harmonic coefficients $C_{n,m}$, $S_{n,m}$ and the Legendre function $P_{n,m}(\cos\vartheta)$ and their normalized forms are: $\overline{C}_{n,m} = C_{n,m}/N_{n,m}$, $\overline{S}_{n,m} = S_{n,m}/N_{n,m}$ and $\overline{P}_{n,m}(\cos\vartheta) = N_{n,m}P_{n,m}(\cos\vartheta)$ with $N_{n,m} = \sqrt{(2-\delta_{0,m})(2n+1)(n-m)!/(n+m)!}$, where $\delta_{0,m}$ is the Kronecker symbol. The Legendre function $Q_{n,m}(v/E)$ can be expressed as (Fukushima 2014)

$$Q_{n,m}(v/E) = (-1)^m \frac{(n+m)!}{(2n+1)!!} \left(\frac{E}{v}\right)^{n+1} \left(\frac{v}{u}\right)^m F_{n,m}(v/E), \tag{2}$$

where

$$F_{n,m}(v/E) = {}_2F_1\left(\frac{n-m+1}{2}, \frac{n-m+2}{2}; n+\frac{3}{2}; \frac{E^2}{v^2}\right), \tag{3}$$

and then the ratio $Q_{n,m}(v/E)/Q_{n,m}(a/E)$ in Eq. (1) is

$$\frac{Q_{n,m}(v/E)}{Q_{n,m}(a/E)} = \left(\frac{a}{v}\right)^{n-m+1} \left(\frac{b}{u}\right)^m \frac{F_{n,m}(v/E)}{F_{n,m}(a/E)} \tag{4}$$

The hypergeometric series $F_{n,m}(v/E)$ converges due to the variable $E^2/v^2 < 1$ for arbitrary $u > 0$, and its recursive algorithm has been given in Fukushima (2014). When N is large, the X-number method (Fukushima 2012) or the logarithm method (Reimond and Baur 2016) can be used to solve the possible floating-point underflow or overflow problems in computing the hypergeometric series $F_{n,m}(v/E)$ and the associated Legendre functions of the first and second kinds in Eq. (1).

2.2 Relations of the harmonic expansion and its derivatives

Solving the inverse solutions of the conversions from the derivative operators in the Cartesian coordinates $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ to the prolate spheroidal coordinates $(\frac{\partial}{\partial v}, \frac{\partial}{\partial \vartheta}, \frac{\partial}{\partial \lambda})$, we have

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{uv\sin\vartheta\cos\lambda}{L^2}\frac{\partial}{\partial v} + \frac{u\cos\vartheta\cos\lambda}{L^2}\frac{\partial}{\partial\vartheta} - \frac{\sin\lambda}{u\sin\vartheta}\frac{\partial}{\partial\lambda},\\
\frac{\partial}{\partial y} &= \frac{uv\sin\vartheta\sin\lambda}{L^2}\frac{\partial}{\partial v} + \frac{u\cos\vartheta\sin\lambda}{L^2}\frac{\partial}{\partial\vartheta} + \frac{\cos\lambda}{u\sin\vartheta}\frac{\partial}{\partial\lambda},\\
\frac{\partial}{\partial z} &= \frac{u^2\cos\vartheta}{L^2}\frac{\partial}{\partial v} - \frac{v\sin\vartheta}{L^2}\frac{\partial}{\partial\vartheta},
\end{aligned} \tag{5}$$

with $L = \sqrt{u^2 + E^2 \sin^2 \vartheta}$. When $E = 0$, the prolate spheroidal coordinates (v, ϑ, λ) degenerate into the spherical coordinates (r, θ, λ), and Eq. (2) also becomes the spherical coordinate case. Analogous to the oblate spheroidal coordinates, the factor $1/\sin\vartheta$ in the derivatives of the prolate spheroidal harmonics with respect to the Cartesian coordinates produces the singular value at the poles ($\sin\vartheta = 0$).

We assume

$$\begin{aligned} V_{n,n_1,n_2,m,m_2} &= Q_{n_1,m} P_{n_2,m_2} T_{n,m,m_2}, \\ V'_{n,n_1,n_2,m,m_2} &= Q_{n_1,m} P_{n_2,m_2} T'_{n,m,m_2}, \end{aligned} \tag{6}$$

where n, n_1, n_2, m and m_2 are non-negative integers, the coordinate variables in the functions $V_{n,n_1,n_2,m,m_2}(v, \vartheta, \lambda)$, $V'_{n,n_1,n_2,m,m_2}(v, \vartheta, \lambda)$, $Q_{n_1,m}(v/E)$, $P_{n_2,m_2}(\cos\vartheta)$, $T_{n,m,m_2}(\lambda)$ and $T'_{n,m,m_2}(\lambda)$ are omitted for simplified representation By considering the normalizations of the Legendre functions and the harmonic coefficients, the normalized forms of V_{n,n_1,n_2,m,m_2} and V'_{n,n_1,n_2,m,m_2} can be defined as

$$\begin{aligned} \overline{V}_{n,n_1,n_2,m,m_2} &= \frac{N_{n_2,m_2} V_{n,n_1,n_2,m,m_2}}{N_{n,m} Q^a_{n,m}} = \widehat{Q}_{n_1,n,m} \overline{P}_{n_2,m_2} \overline{T}_{n,m,m_2} \\ \overline{V}'_{n,n_1,n_2,m,m_2} &= \frac{N_{n_2,m_2} V'_{n,n_1,n_2,m,m_2}}{N_{n,m} Q^a_{n,m}} = \widehat{Q}_{n_1,n,m} \overline{P}_{n_2,m_2} \overline{T}'_{n,m,m_2} \end{aligned} \tag{7}$$

where $\widehat{Q}_{n_1,n,m}(v/E) = Q_{n_1,m}/Q^a_{n,m}$, and $Q^a_{n,m}$ denotes the Legendre function $Q_{n,m}(a/E)$. Then, the gravitational potential can be written in the form of $V_{n,n,n,m,m}$ or $\overline{V}_{n,n,n,m,m}$ as

$$V = \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{V_{n,n,n,m,m}}{Q^a_{n,m}} = \frac{\mu}{a} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \overline{V}_{n,n,n,m,m}. \tag{8}$$

For $n_1 = n+1, n+2, n+3$ (extending to the third-order derivatives of the potential), the term $\widehat{Q}_{n_1,m}$ in Eq. (7) can be computed by the same relation as the oblate case, in which the zero-order term $\widehat{Q}_{n,m}$ has been computed in Eq. (4), and the hypergeometric series $F_{n,m}$ refers to Eq. (3), i.e. being $F_{n,m}(v/E)$. The derivative of the Legendre function $Q_{n,m}(v/E)$ with respect to the v-coordinate can be expressed as (Gradshteyn and Ryzhik 2007, pp. 965):

$$u^2 \frac{\mathrm{d}Q_{n,m}(v/E)}{\mathrm{d}v} = (n - m + 1) E Q_{n+1,m}(v/E) - (n+1) v Q_{n,m}(v/E) \tag{9}$$

From Eqs. (5), (6) and (9), and the formulas for the derivatives of $P_{n,m}$, the ratio of $P_{n,m}/\sin\vartheta$ and the functions $T_{n,m,m_2}(\lambda)$ and $T'_{n,m,m_2}(\lambda)$ used in the oblate spheroidal harmonic expressions, the derivative of the function V_{n,n_1,n_2,m,m_2} with respect to the

Cartesian coordinate x can be written as

$$\begin{aligned}\partial_x V_{n,n_1,n_2,m,m_2} =& \frac{1}{L^2 u \sin\vartheta}\Bigg(\frac{\mathrm{d}Q_{n_1,m}}{\mathrm{d}v} P_{n_2,m_2} T_{n,m,m_2} u^2 v \sin^2\vartheta \cos\lambda + Q_{n_1,m}\frac{\mathrm{d}P_{n_2,m_2}}{\mathrm{d}\vartheta} T_{n,m,m_2} \\ & \times u^2 \sin\vartheta\cos\vartheta\cos\lambda - Q_{n_1,m} P_{n_2,m_2}\frac{\mathrm{d}T_{n,m,m_2}}{\mathrm{d}\lambda}(u^2 + E^2\sin^2\vartheta)\sin\lambda\Bigg) \\ =& \frac{\sin\vartheta}{L^2 u}\Bigg([(n_1 - m + 1)E v Q_{n_1+1,m} - ((n_1+1)E^2 + (n_1 - n_2)u^2)Q_{n_1,m}] \\ & \times P_{n_2,m_2} T_{n,m,m_2}\cos\lambda - m_2 Q_{n_1,m} P_{n_2,m_2} T'_{n,m,m_2} E^2 \sin\lambda\Bigg) + \frac{u}{L^2}\Bigg(Q_{n_1,m}\frac{m_2 P_{n_2,m_2}}{\sin\vartheta} \\ & \times (T_{n,m,m_2}\cos\lambda - T'_{n,m,m_2}\sin\lambda) - Q_{n_1,m} P_{n_2+1,m_2+1} T_{n,m,m_2}\cos\lambda\Bigg)\end{aligned} \tag{10}$$

Then, when $m_2 \geqslant 1$,

$$\begin{aligned}\partial_x V_{n,n_1,n_2,m,m_2} =& a^x_{n_1,m} V_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2} V_{n,n_1,n_2,m,m_2} + c^y_{m_2} V'_{n,n_1,n_2,m,m_2} \\ & + d_{n_2,m_2} V_{n,n_1,n_2+1,m,m_2-1} + e_{m_2} V_{n,n_1,n_2+1,m,m_2+1},\end{aligned} \tag{11}$$

where the coefficients on the right-hand side of the expression are

$$\begin{aligned} a^x_{n_1,m} &= (n_1 - m + 1)\frac{E v \sin\vartheta\cos\lambda}{L^2 u}, \\ b^x_{n_1,n_2} &= -((n_1+1)E^2 + (n_1 - n_2)u^2)\frac{\sin\vartheta\cos\lambda}{L^2 u}, \\ c^y_{m_2} &= -m_2\frac{E^2\sin\vartheta\sin\lambda}{L^2 u}, \\ d_{n_2,m_2} &= (n_2 - m_2 + 1)(n_2 - m_2 + 2)\frac{u}{2L^2}, \\ e_{m_2} &= -(1 + \delta_{0,m_2})\frac{u}{2L^2}. \end{aligned} \tag{12}$$

Likewise, when $m_2 \geqslant 1$ the derivative of the function V_{n,n_1,n_2,m,m_2} with respect to the coordinate y can be given as

$$\begin{aligned}\partial_y V_{n,n_1,n_2,m,m_2} =& a^y_{n_1,m} V_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2} V_{n,n_1,n_2,m,m_2} - c^x_{m_2} V'_{n,n_1,n_2,m,m_2} \\ & + d_{n_2,m_2} V'_{n,n_1,n_2+1,m,m_2-1} - e_{m_2} V'_{n,n_1,n_2+1,m,m_2+1},\end{aligned} \tag{13}$$

where

$$\begin{aligned}
a^y_{n_1,m} &= (n_1 - m + 1)\frac{Ev\sin\vartheta\sin\lambda}{L^2u} \\
b^y_{n_1,n_2} &= -((n_1+1)E^2 + (n_1-n_2)u^2)\frac{\sin\vartheta\sin\lambda}{L^2u} \\
c^x_{m_2} &= -m_2\frac{E^2\sin\vartheta\cos\lambda}{L^2u}.
\end{aligned} \tag{14}$$

Analogous to the oblate case, the derivative of the function V_{n,n_1,n_2,m,m_2} with respect to the coordinate z is

$$\partial_z V_{n,n_1,n_2,m,m_2} = p_{n_1,m}V_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2}V_{n,n_1,n_2,m,m_2} + t_{n_2,m_2}V_{n,n_1,n_2+1,m,m_2}, \tag{15}$$

where $m_2 = 0, 1, 2, \cdots$, and the coefficients $p_{n_1,m}$, q_{n_1,n_2}, and t_{n_2,m_2} are

$$\begin{aligned}
p_{n_1,m} &= (n_1 - m + 1)\frac{E\cos\vartheta}{L^2}, \\
q_{n_1,n_2} &= -(n_1 - n_2)\frac{v\cos\vartheta}{L^2}, \\
t_{n_2,m_2} &= -(n_2 - m_2 + 1)\frac{v}{L^2}.
\end{aligned} \tag{16}$$

According to the derivations of $\partial_x V_{n,n_1,n_2,m,m_2}$, $\partial_y V_{n,n_1,n_2,m,m_2}$ and $\partial_z V_{n,n_1,n_2,m,m_2}$, it's not hard to get the derivatives of the function V'_{n,n_1,n_2,m,m_2} with respect to three Cartesian coordinates:

$$\begin{aligned}
\partial_x V'_{n,n_1,n_2,m,m_2} &= a^x_{n_1,m}V'_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2}V'_{n,n_1,n_2,m,m_2} - c^y_{m_2}V_{n,n_1,n_2,m,m_2} \\
&\quad + d_{n_2,m_2}V'_{n,n_1,n_2+1,m,m_2-1} + e_{m_2}V'_{n,n_1,n_2+1,m,m_2+1}, \\
\partial_y V'_{n,n_1,n_2,m,m_2} &= a^y_{n_1,m}V'_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2}V'_{n,n_1,n_2,m,m_2} + c^x_{m_2}V_{n,n_1,n_2,m,m_2} \\
&\quad - d_{n_2,m_2}V_{n,n_1,n_2+1,m,m_2-1} + e_{m_2}V_{n,n_1,n_2+1,m,m_2+1}, \\
\partial_z V'_{n,n_1,n_2,m,m_2} &= p_{n_1,m}V'_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2}V'_{n,n_1,n_2,m,m_2} + t_{n_2,m_2}V'_{n,n_1,n_2+1,m,m_2},
\end{aligned} \tag{17}$$

where the integer $m_2 \geqslant 1$ for the expressions of $\partial_x V'_{n,n_1,n_2,m,m_2}$ and $\partial_y V'_{n,n_1,n_2,m,m_2}$.

The normalized forms of Eqs. (11), (13), (15) and (17) for the prolate spheroidal harmonic expressions are the same as the oblate case, and also given as following for the convenience to the readers:

$$
\begin{aligned}
\partial_x \overline{V}_{n,n_1,n_2,m,m_2} =& a^x_{n_1,m} \overline{V}_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2} \overline{V}_{n,n_1,n_2,m,m_2} + c^y_{m_2} \overline{V}'_{n,n_1,n_2,m,m_2} \\
& + \overline{d}_{n_2,m_2} \overline{V}_{n,n_1,n_2+1,m,m_2-1} + \overline{e}_{n_2,m_2} \overline{V}_{n,n_1,n_2+1,m,m_2+1} \\
\partial_y \overline{V}_{n,n_1,n_2,m,m_2} =& a^y_{n_1,m} \overline{V}_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2} \overline{V}_{n,n_1,n_2,m,m_2} - c^x_{m_2} \overline{V}'_{n,n_1,n_2,m,m_2} \\
& + \overline{d}_{n_2,m_2} \overline{V}'_{n,n_1,n_2+1,m,m_2-1} - \overline{e}_{n_2,m_2} \overline{V}'_{n,n_1,n_2+1,m,m_2+1} \\
\partial_z \overline{V}_{n,n_1,n_2,m,m_2} =& p_{n_1,m} \overline{V}_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2} \overline{V}_{n,n_1,n_2,m,m_2} + \overline{t}_{n_2,m_2} \overline{V}_{n,n_1,n_2+1,m,m_2}
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\partial_x \overline{V}'_{n,n_1,n_2,m,m_2} =& a^x_{n_1,m} \overline{V}'_{n,n_1+1,n_2,m,m_2} + b^x_{n_1,n_2} \overline{V}'_{n,n_1,n_2,m,m_2} - c^y_{m_2} \overline{V}_{n,n_1,n_2,m,m_2} \\
& + \overline{d}_{n_2,m_2} \overline{V}'_{n,n_1,n_2+1,m,m_2-1} + \overline{e}_{n_2,m_2} \overline{V}'_{n,n_1,n_2+1,m,m_2+1} \\
\partial_y \overline{V}'_{n,n_1,n_2,m,m_2} =& a^y_{n_1,m} \overline{V}'_{n,n_1+1,n_2,m,m_2} + b^y_{n_1,n_2} \overline{V}'_{n,n_1,n_2,m,m_2} + c^x_{m_2} \overline{V}_{n,n_1,n_2,m,m_2} \\
& - \overline{d}_{n_2,m_2} \overline{V}_{n,n_1,n_2+1,m,m_2-1} + \overline{e}_{n_2,m_2} \overline{V}_{n,n_1,n_2+1,m,m_2+1} \\
\partial_z \overline{V}'_{n,n_1,n_2,m,m_2} =& p_{n_1,m} \overline{V}'_{n,n_1+1,n_2,m,m_2} + q_{n_1,n_2} \overline{V}'_{n,n_1,n_2,m,m_2} + \overline{t}_{n_2,m_2} \overline{V}'_{n,n_1,n_2+1,m,m_2}
\end{aligned}
\tag{19}
$$

where the coefficients $\overline{d}_{n_2,m_2}$, $\overline{e}_{n_2,m_2}$, $\overline{t}_{n_2,m_2}$ with same normalizations as the oblate case are

$$
\begin{aligned}
\overline{d}_{n_2,m_2} &= w^d_{n_2,m_2} \frac{u}{L^2} \\
\overline{e}_{n_2,m_2} &= -w^e_{n_2,m_2} \frac{u}{L^2} \\
\overline{t}_{n_2,m_2} &= -w^t_{n_2,m_2} \frac{v}{L^2}
\end{aligned}
\tag{20}
$$

The expressions of $w^d_{n_2,m_2}$, $w^e_{n_2,m_2}$ and $w^t_{n_2,m_2}$ are in accord with the oblate case.

The relations for the derivatives $\partial_x \overline{V}_{n,n_1,n_2,m,m_2}$, $\partial_y \overline{V}_{n,n_1,n_2,m,m_2}$, $\partial_x \overline{V}'_{n,n_1,n_2,m,m_2}$ and $\partial_y \overline{V}'_{n,n_1,n_2,m,m_2}$ in Eqs. (18) and (19) are valid for $m_2 \geqslant 1$. When $m_2 = 0$, the forms of the expressions are also the same as the oblate case, including the derivatives $\partial_x \overline{V}_{n,n_1,n_2,0,0}$, $\partial_y \overline{V}_{n,n_1,n_2,0,0}$, $\partial_x \overline{V}'_{n,n_1,n_2,0,0}$, $\partial_y \overline{V}'_{n,n_1,n_2,0,0}$, $\partial_x \overline{V}_{n,n_1,n_2,m,0}$, $\partial_y \overline{V}_{n,n_1,n_2,m,0}$, $\partial_x \overline{V}'_{n,n_1,n_2,m,0}$ and $\partial_y \overline{V}'_{n,n_1,n_2,m,0}$ $(m \geqslant 1)$.

Assuming the coefficients $a^x_{n_1,m}$ and $a^y_{n_1,m}$ are the derivatives of a function $a_{n_1,m}$ with respect to x- and y-coordinates, the coefficients $b^x_{n_1,n_2}$ and $b^y_{n_1,n_2}$ are the derivatives of a function b_{n_1,n_2} with respect to x- and y-coordinates, and the coefficients $c^x_{m_2}$ and $c^y_{m_2}$ are the derivatives of a function c_{m_2} with respect to x- and y-coordinates, from Eq. (5) we can also have

$$
\begin{aligned}
a_{n_1,m} &= \int (n_1 - m + 1)\frac{E}{u^2}\mathrm{d}v = (n_1 - m + 1)\ln\Big(\frac{u}{v+E}\Big),\\
b_{n_1,n_2} &= -\int \Big((n_1+1)\frac{E^2}{u^2 v} + (n_1 - n_2)\frac{1}{v}\Big)\mathrm{d}v = (n_2+1)\ln\Big(\frac{v}{E}\Big) - (n_1+1)\ln\Big(\frac{u}{E}\Big),\\
c_{m_2} &= -\int m_2\frac{E^2}{u^2 v}\mathrm{d}v = -m_2\ln\Big(\frac{u}{v}\Big).
\end{aligned}
\tag{21}
$$

The relation $\partial_z a_{n_1,m} = p_{n_1,m}$ in the oblate case also holds for the prolate case. Hence, the superscripts x, y and z on $a_{n_1,m}$, b_{n_1,n_2} and c_{m_2} can be indicated for the derivatives of $a_{n_1,m}$, b_{n_1,n_2} and c_{m_2}, where the vertical coordinate z does not appear in the first coordinate symbols of the superscripts of b_{n_1,n_2} and c_{m_2}.

3 Prolate spheroidal harmonic expressions of the derivatives

It is obvious that the forms of the expressions of the first-order derivatives are the same as the oblate spheroidal harmonic expansions, and then the high-order derivatives have also the same forms, i.e.,

$$
\begin{aligned}
V_{x_i} =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_1}\mathcal{C}_{n,m,l}^{x_i}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}},\\
V_{x_ix_j} =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_1}\Big(\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}\partial_{x_j}\mathcal{C}_{n,m,l}^{x_i} + \mathcal{C}_{n,m,l}^{x_i}\partial_{x_j}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}\Big)\\
=& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_2}\mathcal{C}_{n,m,l}^{x_ix_j}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}},\\
V_{x_ix_jx_k} =& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_2}\Big(\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}\partial_{x_k}\mathcal{C}_{n,m,l}^{x_ix_j} + \mathcal{C}_{n,m,l}^{x_ix_j}\partial_{x_k}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}\Big)\\
=& \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{l=1}^{\ell_3}\mathcal{C}_{n,m,l}^{x_ix_jx_k}\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}},
\end{aligned}
\tag{22}
$$

where the subscripts $i, j, k = 1, 2, 3$ ($i \leqslant j \leqslant k$), and other symbols are in accord with the oblate case. The relations for the coefficients of the high-order derivatives ($\mathcal{C}_{n,m,l}^{x_ix_j}$, $\partial_{x_k}\mathcal{C}_{n,m,l}^{x_ix_j}$) and the lower-order derivatives ($\mathcal{C}_{n,m,l}^{x_i}$, $\mathcal{C}_{n,m,l}^{x_ix_j}$) are in accord in form with the oblate case. Hence, we just need to find the expressions of $\mathcal{C}_{n,m,l}^{x_i}$, $\partial_{x_j}\mathcal{C}_{n,m,l}^{x_i}$ and a part of $\partial_{x_k}\mathcal{C}_{n,m,l}^{x_ix_j}$ for the prolate spheroidal harmonic expressions.

Table 1 Expressions of the derivatives $\partial_{x_j}\mathcal{C}^{x_i}_{n,m,l}$ $(1 \leqslant i \leqslant j \leqslant 3)$

$a^{xx}_{n,m} = (n-m+1)f^0_2 v - (n-m+1)f^a_2\cos^2\lambda$	$a^{yz}_{n,m} = -(n-m+1)f^{az}_2\sin\lambda$
$b^{xx}_{n,n} = -(n+1)f^0_2 E + (n+1)f^{bc}_2\cos^2\lambda$	$b^{yz}_{n,n} = (n+1)f^{bcz}_2\sin\lambda$
$c^{xy}_m = mf^{bc}_2\sin\lambda\cos\lambda$	$c^{xz}_m = mf^{bcz}_2\cos\lambda$
$a^{xy}_{n,m} = -(n-m+1)f^a_2\sin\lambda\cos\lambda$	$p^z_{n,m} = -(n-m+1)f^{pz}_2$
$b^{xy}_{n,n} = (n+1)f^{bc}_2\sin\lambda\cos\lambda$	$\bar{t}^z_{n,m} = w^t_{n,m}f^{tz}_2$
$c^{yy}_m = -mf^0_2 E + mf^{bc}_2\sin^2\lambda$	$\bar{d}^x_{n,m} = -w^d_{n,m}f^{de}_2\cos\lambda$
$a^{xz}_{n,m} = -(n-m+1)f^{az}_2\cos\lambda$	$\bar{e}^x_{n,m} = w^e_{n,m}f^{de}_2\cos\lambda$
$b^{xz}_{n,n} = (n+1)f^{bcz}_2\cos\lambda$	$\bar{d}^y_{n,m} = -w^d_{n,m}f^{de}_2\sin\lambda$
$c^{yz}_m = mf^{bcz}_2\sin\lambda$	$\bar{e}^y_{n,m} = w^e_{n,m}f^{de}_2\sin\lambda$
$a^{yy}_{n,m} = (n-m+1)f^0_2 v - (n-m+1)f^a_2\sin^2\lambda$	$\bar{d}^z_{n,m} = -w^d_{n,m}f^{dez}_2$
$b^{yy}_{n,n} = -(n+1)f^0_2 E + (n+1)f^{bc}_2\sin^2\lambda$	$\bar{e}^z_{n,m} = w^e_{n,m}f^{dez}_2$

The coefficients $\mathcal{C}^{x_i}_{n,m,l}$ in Eq. (22) for the first-order derivatives have been given in Eqs. (12), (14), (16) and (20). The coefficients $\partial_{x_j}\mathcal{C}^{x_i}_{n,m,l}$ are listed in Table 1, where the notations f_2 represents

$$\begin{aligned}
&f^0_2 = \frac{E}{L^2u^2},\ f^a_2 = \frac{Ev\sin^2\vartheta}{L^6u^2}\Big(3u^2v^2+3E^2u^2-(u^2-2E^2)E^2\sin^2\vartheta\Big),\\
&f^{bc}_2 = \frac{E^2\sin^2\vartheta}{L^6u^2}\Big(4u^2v^2+2E^2u^2+2E^4\sin^2\vartheta\Big),\ f^{de}_2 = \frac{\sin\vartheta}{L^6}\Big(u^2v^2+2E^2u^2-(3u^2+E^2)E^2\sin^2\vartheta\Big),\\
&f^{az}_2 = \frac{Eu\sin\vartheta\cos\vartheta}{L^6}\Big(3u^2+4E^2-E^2\sin^2\vartheta\Big),\ f^{bcz}_2 = \frac{4E^2uv\sin\vartheta\cos\vartheta}{L^6},\ f^{dez}_2 = \frac{uv\cos\vartheta}{L^6}\Big(u^2-3E^2\sin^2\vartheta\Big),\\
&f^{pz}_2 = \frac{Ev}{L^6}\Big(2u^2-(3u^2+2E^2)\sin^2\vartheta+E^2\sin^4\vartheta\Big),\ f^{tz}_2 = \frac{\cos\vartheta}{L^6}\Big(u^2v^2+E^2u^2-(3u^2+2E^2)E^2\sin^2\vartheta\Big).
\end{aligned} \tag{23}$$

Expressions of the derivatives of the quantities in Table 1 with respect to the Cartesian coordinates are also given in Tables 2 and 3. For the third-order derivatives of the gravitational potential, we also need find the expressions of the quantities $b^{xx}_{n+1,n}$, $b^{xy}_{n+1,n}$, $b^{xz}_{n+1,n}$, $b^{yy}_{n+1,n}$, $b^{yz}_{n+1,n}$, $b^{xx}_{n,n+1}$, $b^{xy}_{n,n+1}$, $b^{xz}_{n,n+1}$, $b^{yy}_{n,n+1}$, $b^{yz}_{n,n+1}$, c^{xx}_m, $p^x_{n,m}$, $p^y_{n,m}$, $\bar{t}^x_{n,m}$, $\bar{t}^y_{n,m}$, $q^x_{n+1,n}$, $q^y_{n+1,n}$, $q^z_{n+1,n}$, $q^x_{n,n+1}$, $q^y_{n,n+1}$ and $q^z_{n,n+1}$. The coefficient c^{xx}_m can be obtained by differentiating c^x_m with respect to the x-coordinate:

$$c^{xx}_m = -mf^0_2 E + mf^{bc}_2\cos^2\lambda. \tag{24}$$

Table 2 Expressions of the derivatives of the quantities in Table 1 with respect to the Cartesian coordinates

$a^{xxx}_{n,m} = -3(n-m+1)f^{a0}_3 \cos\lambda + (n-m+1)f^a_3 \cos^3\lambda$
$b^{xxx}_{n,n} = 3(n+1)f^{bc0}_3 \cos\lambda - (n+1)f^{bc}_3 \cos^3\lambda$
$c^{xxy}_m = m f^{bc0}_3 \sin\lambda - m f^{bc}_3 \cos^2\lambda \sin\lambda$
$a^{xxy}_{n,m} = -(n-m+1)f^{a0}_3 \sin\lambda + (n-m+1)f^a_3 \cos^2\lambda \sin\lambda$
$b^{xxy}_{n,n} = (n+1)f^{bc0}_3 \sin\lambda - (n+1)f^{bc}_3 \cos^2\lambda \sin\lambda$
$c^{xyy}_m = m f^{bc0}_3 \cos\lambda - m f^{bc}_3 \sin^2\lambda \cos\lambda$
$a^{xxz}_{n,m} = -(n-m+1)f^{az0}_3 + (n-m+1)f^{az}_3 \cos^2\lambda$
$b^{xxz}_{n,n} = (n+1)f^{bcz0}_3 - (n+1)f^{bcz}_3 \cos^2\lambda$
$c^{xyz}_m = -m f^{bcz}_3 \sin\lambda \cos\lambda$
$a^{xyy}_{n,m} = -(n-m+1)f^{a0}_3 \cos\lambda + (n-m+1)f^a_3 \sin^2\lambda \cos\lambda$
$b^{xyy}_{n,n} = (n+1)f^{bc0}_3 \cos\lambda - (n+1)f^{bc}_3 \sin^2\lambda \cos\lambda$
$a^{xyz}_{n,m} = (n-m+1)f^{az}_3 \sin\lambda \cos\lambda$
$b^{xyz}_{n,n} = -(n+1)f^{bcz}_3 \sin\lambda \cos\lambda$
$c^{yyz}_m = m f^{bcz0}_3 - m f^{bcz}_3 \sin^2\lambda$
$a^{yyy}_{n,m} = -3(n-m+1)f^{a0}_3 \sin\lambda + (n-m+1)f^a_3 \sin^3\lambda$
$b^{yyy}_{n,n} = 3(n+1)f^{bc0}_3 \sin\lambda - (n+1)f^{bc}_3 \sin^3\lambda$
$a^{yyz}_{n,m} = -(n-m+1)f^{az0}_3 + (n-m+1)f^{az}_3 \sin^2\lambda$
$b^{yyz}_{n,n} = (n+1)f^{bcz0}_3 - (n+1)f^{bcz}_3 \sin^2\lambda$
$\bar{d}^{xx}_{n,m} = -w^d_{n,m}(f^{de0}_3 - f^{de}_3 \cos^2\lambda)$
$\bar{e}^{xx}_{n,m} = w^e_{n,m}(f^{de0}_3 - f^{de}_3 \cos^2\lambda)$
$\bar{d}^{xy}_{n,m} = w^d_{n,m} f^{de}_3 \sin\lambda \cos\lambda$
$\bar{e}^{xy}_{n,m} = -w^e_{n,m} f^{de}_3 \sin\lambda \cos\lambda$
$\bar{d}^{xz}_{n,m} = w^d_{n,m} f^{dez}_3 \cos\lambda$
$\bar{e}^{xz}_{n,m} = -w^e_{n,m} f^{dez}_3 \cos\lambda$
$\bar{d}^{yy}_{n,m} = -w^d_{n,m}(f^{de0}_3 - f^{de}_3 \sin^2\lambda)$
$\bar{e}^{yy}_{n,m} = w^e_{n,m}(f^{de0}_3 - f^{de}_3 \sin^2\lambda)$
$\bar{d}^{yz}_{n,m} = w^d_{n,m} f^{dez}_3 \sin\lambda$
$\bar{e}^{yz}_{n,m} = -w^e_{n,m} f^{dez}_3 \sin\lambda$
$p^{xz}_{n,m} = (n-m+1)f^p_3 \cos\lambda$
$p^{yz}_{n,m} = (n-m+1)f^p_3 \sin\lambda$
$p^{zz}_{n,m} = (n-m+1)f^{pz}_3$
$\bar{t}^{xz}_{n,m} = -w^t_{n,m} f^t_3 \cos\lambda$
$\bar{t}^{yz}_{n,m} = -w^t_{n,m} f^t_3 \sin\lambda$
$\bar{t}^{zz}_{n,m} = -w^t_{n,m} f^{tz}_3$

Table 3 Expressions of the notations f_3 in Table 2

$f_3^{a0} = \frac{Ev\sin\vartheta}{L^6u^3}\big(3u^2(2E^2+u^2)+E^2(2E^2-u^2)\sin^2\vartheta\big)$
$f_3^{bc0} = \frac{2E^2\sin\vartheta}{L^6u^3}\big(u^2(3E^2+2u^2)+E^4\sin^2\vartheta\big)$
$f_3^{de0} = \frac{1}{L^6u}\big(u^2(3E^2+u^2)-E^2(E^2+3u^2)\sin^2\vartheta\big)$
$f_3^{a} = \frac{Ev\sin^3\vartheta}{L^{10}u^3}\big(5u^4(16E^4+16E^2u^2+3u^4)+10E^2u^2(4E^4-2E^2u^2-3u^4)\sin^2\vartheta$
$\quad +E^4(8E^4-4E^2u^2+3u^4)\sin^4\vartheta\big)$
$f_3^{bc} = \frac{E^2\sin^3\vartheta}{L^{10}u^3}\big(8u^4(10E^4+12E^2u^2+3u^4)+8E^2u^2(5E^4-3u^4)\sin^2\vartheta+8E^8\sin^4\vartheta\big)$
$f_3^{de} = \frac{\sin^2\vartheta}{L^{10}u}\big(u^4(35E^4+30E^2u^2+3u^4)-2E^2u^2(7E^4+30E^2u^2+15u^4)\sin^2\vartheta$
$\quad -E^4(E^4-6E^2u^2-15u^4)\sin^4\vartheta\big)$
$f_3^{az0} = \frac{E\cos\vartheta}{L^6}\big(4E^2+3u^2-E^2\sin^2\vartheta\big)$
$f_3^{bcz0} = \frac{4E^2v\cos\vartheta}{L^6}$
$f_3^{az} = \frac{3Eu^2\sin^2\vartheta\cos\vartheta}{L^{10}}\big((16E^4+20E^2u^2+5u^4)-2E^2(6E^2+5u^2)\sin^2\vartheta+E^4\sin^4\vartheta\big)$
$f_3^{bcz} = \frac{24E^2u^2v\sin^2\vartheta\cos\vartheta}{L^{10}}\big(2E^2+u^2-E^2\sin^2\vartheta\big)$
$f_3^{dez} = \frac{3v\sin\vartheta\cos\vartheta}{L^{10}}\big(u^4(5E^2+u^2)-10E^2u^2(E^2+u^2)\sin^2\vartheta+E^4(E^2+5u^2)\sin^4\vartheta\big)$
$f_3^{p} = \frac{3Euv\sin\vartheta}{L^{10}}\big(4u^2(2E^2+u^2)-(8E^4+20E^2u^2+5u^4)\sin^2\vartheta+2E^2(4E^2+5u^2)\sin^4\vartheta-E^4\sin^6\vartheta\big)$
$f_3^{t} = \frac{3u\sin\vartheta\cos\vartheta}{L^{10}}\big(u^2(8E^4+8E^2u^2+u^4)-2E^2(4E^4+10E^2u^2+5u^4)\sin^2\vartheta+E^4(4E^2+5u^2)\sin^4\vartheta\big)$
$f_3^{pz} = \frac{E\cos\vartheta}{L^{10}}\big(2u^4(4E^2+3u^2)-u^2(32E^4+50E^2u^2+15u^4)\sin^2\vartheta+2E^2(4E^4+19E^2u^2+15u^4)\sin^4\vartheta$
$\quad -E^4(2E^2+3u^2)\sin^6\vartheta\big)$
$f_3^{tz} = \frac{v}{L^{10}}\big(2u^4(4E^2+u^2)-u^2(32E^4+38E^2u^2+3u^4)\sin^2\vartheta+2E^2(4E^4+25E^2u^2+15u^4)\sin^4\vartheta$
$\quad -3E^4(2E^2+5u^2)\sin^6\vartheta\big)$

Analogous to the oblate case, the coefficient $b^x_{n_1,n_2}$ or $b^y_{n_1,n_2}$ can also be written in the form of the summation of $b^x_{n_1,n_1}$ and $(n_1-n_2)b^x_0$, or $b^y_{n_1,n_1}$ and $(n_1-n_2)b^y_0$, where $b^x_0 = -(u\sin\vartheta\cos\lambda)/L^2$, $b^y_0 = -(u\sin\vartheta\sin\lambda)/L^2$. The variable $b_0 = -\int(1/v)\mathrm{d}v = -\ln(v/E)$ meets the relations $\partial_x b_0 = b^x_0$ and $\partial_y b_0 = b^y_0$. The second-order derivatives b^{xx}_0, b^{xy}_0, b^{xz}_0, b^{yy}_0 and b^{yz}_0 have the same forms as the oblate case, with $f^0_1 = 1/L^2$ and

$$f^{b0}_2 = \frac{2u^2\sin^2\vartheta}{L^6}\Big(u^2+2E^2-E^2\sin^2\vartheta\Big),\ f^{bz0}_2 = \frac{2uv\sin\vartheta\cos\vartheta}{L^6}\Big(u^2-E^2\sin^2\vartheta\Big). \tag{25}$$

Then, the expressions of $b^{xx}_{n_1,n_2}$, $b^{xy}_{n_1,n_2}$, $b^{xz}_{n_1,n_2}$, $b^{yy}_{n_1,n_2}$ and $b^{yz}_{n_1,n_2}$ can be obtained, which also have the same forms as the oblate case. Differentiating the coefficient $p_{n,m}$ with respect to the horizontal coordinates x and y, we obtain the derivatives $p^x_{n,m}$ and $p^y_{n,m}$:

$$p^x_{n,m} = a^{xz}_{n,m} = -(n-m+1)f^p_2\cos\lambda,\ p^y_{n,m} = a^{yz}_{n,m} = -(n-m+1)f^p_2\sin\lambda, \tag{26}$$

with $f^p_2 = f^{az}_2$. For the coefficients $\bar{t}^x_{n,m}$ and $\bar{t}^y_{n,m}$, their expressions are in accord with the oblate case, and the variable f^t_2 is

$$f^t_2 = \frac{uv\sin\vartheta}{L^6}\Big(u^2+4E^2-3E^2\sin^2\vartheta\Big). \tag{27}$$

The coefficient q_{n_1,n_2} can be written as: $q_{n_1,n_2} = -(n_1-n_2)f^q_1$, with $f^q_1 = (v\cos\vartheta)/L^2$, and then the expressions of the derivatives $q^x_{n_1,n_2}$, $q^y_{n_1,n_2}$ and $q^z_{n_1,n_2}$ are also in accord with the oblate case, where the quantities f^q_2 and f^{qz}_2 become

$$\begin{aligned} f^q_2 &= \frac{2uv\sin\vartheta\cos\vartheta}{L^6}\Big(u^2+2E^2-E^2\sin^2\vartheta\Big),\\ f^{qz}_2 &= \frac{1}{L^6}\Big(u^2v^2+E^2u^2-2(u^2v^2+2E^2u^2+E^4)\sin^2\vartheta+E^2(E^2+2u^2)\sin^4\vartheta\Big). \end{aligned} \tag{28}$$

Now all the expressions of $\partial_{x_k}\mathcal{C}^{x_ix_j}_{n,m,l}$ are known, and the third-order derivatives are also be computable.

4 Computations of the gravitational field for the observation point at the poles

In this section, we find the computations of the derivatives of the gravitational potential for the poles, and take the same variable symbols as the oblate spheroidal harmonic expressions. When the observation point located on the poles, the factor $\sin\vartheta$ becomes zero, and then the coefficients $a^x_{n_1,m}$, $a^y_{n_1,m}$, $b^x_{n_1,n_2}$, $b^y_{n_1,n_2}$, $c^x_{m_2}$ and $c^y_{m_2}$ in Eqs. (18) and (19) with horizontal coordinate superscripts are eliminated, and the coefficients $\bar{d}_{n_2,m_2}$, $\bar{e}_{n_2,m_2}$, $p_{n_1,m}$, q_{n_1,n_2} and $\bar{t}_{n_2,m_2}$ without superscript are nonzero. The expressions of $\bar{d}_{n_2,m_2}$, $\bar{e}_{n_2,m_2}$, q_{n_1,n_2} and $\bar{t}_{n_2,m_2}$ are in accord with the oblate case in form, and the coefficient $p_{n_1,m}$ becomes

$$p_{n_1,m} = (n_1 - m + 1) f_{10}^{p}, \tag{29}$$

where the quantities f_{10} are

$$f_{10}^{0} = \frac{1}{u^2},\ f_{10}^{de} = \frac{1}{u},\ f_{10}^{p} = \pm\frac{E}{u^2},\ f_{10}^{q} = \pm\frac{v}{u^2},\ f_{10}^{t} = \frac{v}{u^2}. \tag{30}$$

For the poles, the coefficients $a_{n,m}^{xy}$, $a_{n,m}^{xz}$, $a_{n,m}^{yz}$, $b_{n,n}^{xy}$, $b_{n,n}^{xz}$, $b_{n,n}^{yz}$, c_{m}^{xy}, c_{m}^{xz}, c_{m}^{yz}, $\overline{d}_{n,m}^{x}$, $\overline{d}_{n,m}^{y}$, $\overline{e}_{n,m}^{x}$, $\overline{d}_{n,m}^{y}$ in Table 1 and the coefficients b_{n_1,n_2}^{xy}, b_{n_1,n_2}^{xz}, b_{n_1,n_2}^{yz}, $p_{n,m}^{x}$, $p_{n,m}^{y}$, q_{n_1,n_2}^{x}, q_{n_1,n_2}^{y}, $\overline{t}_{n,m}^{x}$ and $\overline{t}_{n,m}^{y}$ are eliminated. The nonzero coefficients have a pair of horizontal coordinate superscripts x or y, or a vertical coordinate superscript z. The expressions of the coefficients $\overline{d}_{n,m}^{z}$, $\overline{e}_{n,m}^{z}$, q_{n_1,n_2}^{z} and $\overline{t}_{n,m}^{z}$ are consistent in form with the oblate case, and the other nonzero coefficients are

$$\begin{aligned}
a_{n,m}^{xx} &= (n-m+1) f_{20}^{0} v,\ b_{n,n}^{xx} = -(n+1) f_{20}^{0} E,\ c_{m}^{xx} = -m f_{20}^{0} E,\\
a_{n,m}^{yy} &= (n-m+1) f_{20}^{0} v,\ b_{n,n}^{yy} = -(n+1) f_{20}^{0} E,\ c_{m}^{yy} = -m f_{20}^{0} E,\\
b_{n_1,n_2}^{xx} &= -(n_1+1) f_{20}^{0} E - (n_1 - n_2) f_{10}^{0},\ b_{n_1,n_2}^{yy} = -(n_1+1) f_{20}^{0} E - (n_1 - n_2) f_{10}^{0},\\
p_{n,m}^{z} &= -(n-m+1) f_{20}^{pz},
\end{aligned} \tag{31}$$

where the quantities f_{20} are

$$f_{20}^{0} = \frac{E}{u^4},\ f_{20}^{dez} = \pm\frac{v}{u^3},\ f_{20}^{pz} = \frac{2Ev}{u^4},\ f_{20}^{qz} = \frac{v^2+E^2}{u^4},\ f_{20}^{tz} = \pm\frac{v^2+E^2}{u^4}, \tag{32}$$

The coefficients in Table 2 are also eliminated except those with a pair of horizontal coordinate superscripts and a vertical coordinate superscript for the third-order derivatives of $a_{n,m}$, $b_{n,n}$ and c_m or a pair of horizontal/vertical coordinate superscripts for the second-order derivatives of $\overline{d}_{n,m}$, $\overline{e}_{n,m}$, $p_{n,m}$ and $\overline{t}_{n,m}$. The expression of $a_{n,m}^{xxz}$, $a_{n,m}^{yyz}$ and $p_{n,m}^{zz}$ are in accord with the oblate case in form, and the other nonzero coefficients can be written as

$$\begin{aligned}
&b_{n,n}^{xxz} = (n+1) f_{30}^{bcz0},\ b_{n,n}^{yyz} = (n+1) f_{30}^{bcz0},\ c_{m}^{yyz} = m f_{30}^{bcz0},\\
&\overline{d}_{n,m}^{xx} = -w_{n,m}^{d} f_{30}^{de0},\ \overline{e}_{n,m}^{xx} = w_{n,m}^{e} f_{30}^{de0},\ \overline{d}_{n,m}^{yy} = -w_{n,m}^{d} f_{30}^{dey},\ \overline{e}_{n,m}^{yy} = w_{n,m}^{e} f_{30}^{dey},\\
&\overline{t}_{n,m}^{zz} = -w_{n,m}^{t} f_{30}^{tz},
\end{aligned} \tag{33}$$

where the quantities f_{30} are

$$\begin{aligned}
f_{30}^{az0} &= \pm\frac{E(4E^2+3u^2)}{u^6},\ f_{30}^{bcz0} = \pm\frac{4E^2 v}{u^6},\ f_{30}^{de0} = \frac{3E^2+u^2}{u^5},\\
f_{30}^{pz} &= \pm\frac{2E(4E^2+3u^2)}{u^6},\ f_{30}^{tz} = \frac{2v(4E^2+u^2)}{u^6}.
\end{aligned} \tag{34}$$

For the observation point at the poles, when the integer $m_2 \geqslant 1$, $\overline{V}_{n,n_1,n_2,m,m_2} = \overline{V}'_{n,n_1,n_2,m,m_2} = 0$, and when $m_2 = 0$, the products $\overline{V}_{n,n_1,n_2,m,m_2}$ and $\overline{V}'_{n,n_1,n_2,m,m_2}$ can be expressed as

$$\overline{V}_{n,n_1,n_2,m,0} = \widehat{Q}_{n_1,n,m}\overline{P}_{n_2}\overline{C}_{n,m},\ \overline{V}'_{n,n_1,n_2,m,0} = \widehat{Q}_{n_1,n,m}\overline{P}_{n_2}\overline{S}_{n,m}. \tag{35}$$

Therefore, the prolate spheroidal harmonic expressions of the second- and third-order derivatives $V_{x_ix_j}$ and $V_{x_ix_jx_k}$ of the potential for the poles are the same in form as the oblate case, and the coefficients of the terms $\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}$ or $\partial_z\overline{V}^f_{n,n_{1,l},n_{2,l},m,m_{2,l}}$ are also the same in form. Specific computations of the coefficients have been given in the above discussions.

5 Prolate spheroidal harmonic expressions in the local north-oriented reference frame

The conversions between the derivatives of the potential with respect to the Cartesian coordinates in the global and local reference frames for the prolate spheroidal harmonic expressions are consistent with the oblate case, and the colatitude ϑ in the rotation matrix $\mathbf{R}$ is for the prolate spheroidal coordinates. We now consider the conversions of the derivatives of the potential from the prolate spheroidal coordinates in the global reference frame to the Cartesian coordinates in the local north-oriented reference frame, and then can derive the expressions of the derivatives in the local north-oriented reference frame. The scale factors of the prolate spheroidal coordinates h_v, h_ϑ and h_λ are: $h_v = L/u$, $h_\vartheta = L$, and $h_\lambda = u\sin\vartheta$. At the observation point, the covariant basis vectors of the local Cartesian coordinates are: $\mathbf{e}_1^* = -\mathbf{e}_\vartheta/h_\vartheta$, $\mathbf{e}_2^* = -\mathbf{e}_\lambda/h_\lambda$, and $\mathbf{e}_3^* = \mathbf{e}_v/h_v$, where ($\mathbf{e}_v$, $\mathbf{e}_\vartheta$, $\mathbf{e}_\lambda$) are the covariant basis vectors of the prolate spheroidal coordinates. Then, the metric tensors of the prolate spheroidal coordinates in the covariant and contravariant forms can be written as

$$(g_{ij}) = \begin{pmatrix} \frac{L^2}{u^2} & 0 & 0 \\ 0 & L^2 & 0 \\ 0 & 0 & u^2\sin^2\vartheta \end{pmatrix},\ \ (g^{ij}) = \begin{pmatrix} \frac{u^2}{L^2} & 0 & 0 \\ 0 & \frac{1}{L^2} & 0 \\ 0 & 0 & \frac{1}{u^2\sin^2\vartheta} \end{pmatrix} \tag{36}$$

We can derive the expressions of the Christoffel symbols of the second kind for the prolate spheroidal coordinates from Borisenko and Tarapov (1968, pp. 189) and Huang et al. (2020, pp. 123), and give the results in Table 4. Then, from Casotto and Fantino (2009), the conversions of the derivatives of the potential (up to third-order) from the prolate spheroidal coordinates in the global reference frame to the Cartesian coordinates in the local north-oriented reference frame can be solved, as shown in Table 5. The corresponding results up to second-order derivatives in the local South-East-Up frame can refer to Sebera et al. (2016). We can still first find the expressions of the derivatives of the potential with respect to the oblate spheroidal coordinates, and then from Table 5 obtain the derivatives in the local Cartesian coordinates.

Table 4 Christoffel symbols of the second kind of the prolate spheroidal coordinates

Christoffel symbols	Expressions
$\left(\Gamma_{ij}^{1}\right)$	$\begin{pmatrix} -\frac{vE^2\sin^2\vartheta}{L^2u^2} & \frac{E^2\sin\vartheta\cos\vartheta}{L^2} & 0 \\ \frac{E^2\sin\vartheta\cos\vartheta}{L^2} & -\frac{u^2v}{L^2} & 0 \\ 0 & 0 & -\frac{u^2v\sin^2\vartheta}{L^2} \end{pmatrix}$
$\left(\Gamma_{ij}^{2}\right)$	$\begin{pmatrix} -\frac{E^2\sin\vartheta\cos\vartheta}{L^2u^2} & \frac{v}{L^2} & 0 \\ \frac{v}{L^2} & \frac{E^2\sin\vartheta\cos\vartheta}{L^2} & 0 \\ 0 & 0 & -\frac{u^2\sin\vartheta\cos\vartheta}{L^2} \end{pmatrix}$
$\left(\Gamma_{ij}^{3}\right)$	$\begin{pmatrix} 0 & 0 & \frac{v}{u^2} \\ 0 & 0 & \cot\vartheta \\ \frac{v}{u^2} & \cot\vartheta & 0 \end{pmatrix}$

Table 5 Conversions of the derivatives of the potential with respect to the prolate spheroidal coordinates into the local Cartesian coordinates

$$
\begin{aligned}
&V_{x^*} = -\tfrac{1}{L}V_\vartheta,\ V_{y^*} = -\tfrac{1}{u\sin\vartheta}V_\lambda,\ V_{z^*} = \tfrac{u}{L}V_v \\
&V_{x^*x^*} = \tfrac{u^2v}{L^4}V_v - \tfrac{E^2\sin\vartheta\cos\vartheta}{L^4}V_\vartheta + \tfrac{1}{L^2}V_{\vartheta\vartheta},\ V_{x^*y^*} = -\tfrac{\cos\vartheta}{Lu\sin^2\vartheta}V_\lambda + \tfrac{1}{Lu\sin\vartheta}V_{\vartheta\lambda} \\
&V_{x^*z^*} = \tfrac{uE^2\sin\vartheta\cos\vartheta}{L^4}V_v + \tfrac{uv}{L^4}V_\vartheta - \tfrac{u}{L^2}V_{v\vartheta},\ V_{y^*y^*} = \tfrac{v}{L^2}V_v + \tfrac{\cos\vartheta}{L^2\sin\vartheta}V_\vartheta + \tfrac{1}{u^2\sin^2\vartheta}V_{\lambda\lambda} \\
&V_{y^*z^*} = \tfrac{v}{Lu^2\sin\vartheta}V_\lambda - \tfrac{1}{L\sin\vartheta}V_{v\lambda},\ V_{z^*z^*} = \tfrac{vE^2\sin^2\vartheta}{L^4}V_v + \tfrac{E^2\sin\vartheta\cos\vartheta}{L^4}V_\vartheta + \tfrac{u^2}{L^2}V_{vv} \\
&V_{x^*x^*x^*} = \tfrac{6u^2vE^2\sin\vartheta\cos\vartheta}{L^7}V_v + \tfrac{2L^4-3L^2(u^2+v^2)+6u^2v^2}{L^7}V_\vartheta - \tfrac{3u^2v}{L^5}V_{v\vartheta} \\
&\qquad + \tfrac{3E^2\sin\vartheta\cos\vartheta}{L^5}V_{\vartheta\vartheta} - \tfrac{1}{L^3}V_{\vartheta\vartheta\vartheta} \\
&V_{x^*x^*y^*} = -\tfrac{2\cos^2\vartheta}{L^2u\sin^3\vartheta}V_\lambda - \tfrac{uv}{L^4\sin\vartheta}V_{v\lambda} + \tfrac{(3L^2-u^2)\cos\vartheta}{L^4u\sin^2\vartheta}V_{\vartheta\lambda} - \tfrac{1}{L^2u\sin\vartheta}V_{\vartheta\vartheta\lambda} \\
&V_{x^*x^*z^*} = \tfrac{u(L^2-2v^2)(3u^2-2L^2)}{L^7}V_v + \tfrac{6uvE^2\sin\vartheta\cos\vartheta}{L^7}V_\vartheta + \tfrac{u^3v}{L^5}V_{vv} - \tfrac{3uE^2\sin\vartheta\cos\vartheta}{L^5}V_{v\vartheta} \\
&\qquad - \tfrac{2uv}{L^5}V_{\vartheta\vartheta} + \tfrac{u}{L^3}V_{v\vartheta\vartheta} \\
&V_{x^*y^*y^*} = \tfrac{2vE^2\sin\vartheta\cos\vartheta}{L^5}V_v + \tfrac{1}{L^5}\left(2v^2 - 2L^2 + \tfrac{L^2}{\sin^2\vartheta}\right)V_\vartheta - \tfrac{v}{L^3}V_{v\vartheta} - \tfrac{\cos\vartheta}{L^3\sin\vartheta}V_{\vartheta\vartheta} \\
&\qquad + \tfrac{2\cos\vartheta}{Lu^2\sin^3\vartheta}V_{\lambda\lambda} - \tfrac{1}{Lu^2\sin^2\vartheta}V_{\vartheta\lambda\lambda} \\
&V_{x^*y^*z^*} = \tfrac{2v\cos\vartheta}{L^2u^2\sin^2\vartheta}V_\lambda + \tfrac{(u^2-2L^2)\cos\vartheta}{L^4\sin^2\vartheta}V_{v\lambda} - \tfrac{v(L^2+u^2)}{L^4u^2\sin\vartheta}V_{\vartheta\lambda} + \tfrac{1}{L^2\sin\vartheta}V_{v\vartheta\lambda} \\
&V_{x^*z^*z^*} = -\tfrac{2vE^2(3u^2-L^2)\sin\vartheta\cos\vartheta}{L^7}V_v - \tfrac{2L^4-3L^2(u^2+v^2)+6u^2v^2}{L^7}V_\vartheta + \tfrac{2u^2E^2\sin\vartheta\cos\vartheta}{L^5}V_{vv} \\
&\qquad + \tfrac{v(3u^2-L^2)}{L^5}V_{v\vartheta} - \tfrac{E^2\sin\vartheta\cos\vartheta}{L^5}V_{\vartheta\vartheta} - \tfrac{u^2}{L^3}V_{vv\vartheta} \\
&V_{y^*y^*y^*} = \tfrac{2}{u^3\sin^3\vartheta}V_\lambda - \tfrac{3v}{L^2u\sin\vartheta}V_{v\lambda} - \tfrac{3\cos\vartheta}{L^2u\sin^2\vartheta}V_{\vartheta\lambda} - \tfrac{1}{u^3\sin^3\vartheta}V_{\lambda\lambda\lambda} \\
&V_{y^*y^*z^*} = \tfrac{u(L^2-2v^2)}{L^5}V_v - \tfrac{2uv\cos\vartheta}{L^5\sin\vartheta}V_\vartheta + \tfrac{uv}{L^3}V_{vv} + \tfrac{u\cos\vartheta}{L^3\sin\vartheta}V_{v\vartheta} - \tfrac{2v}{Lu^3\sin^2\vartheta}V_{\lambda\lambda} + \tfrac{1}{Lu\sin^2\vartheta}V_{v\lambda\lambda} \\
&V_{y^*z^*z^*} = -\tfrac{2v^2}{L^2u^3\sin\vartheta}V_\lambda + \tfrac{v(L^2+u^2)}{L^4u\sin\vartheta}V_{v\lambda} - \tfrac{E^2\cos\vartheta}{L^4u}V_{\vartheta\lambda} - \tfrac{u}{L^2\sin\vartheta}V_{vv\lambda} \\
&V_{z^*z^*z^*} = \tfrac{3u(L^2-2v^2)E^2\sin^2\vartheta}{L^7}V_v - \tfrac{6uvE^2\sin\vartheta\cos\vartheta}{L^7}V_\vartheta + \tfrac{3uvE^2\sin^2\vartheta}{L^5}V_{vv} + \tfrac{3uE^2\sin\vartheta\cos\vartheta}{L^5}V_{v\vartheta} \\
&\qquad + \tfrac{u^3}{L^3}V_{vvv}
\end{aligned}
$$

6 Numerical experiments

The uniform prism body is tested to verify the algorithms of this work, and the lengths of three sides of the prism are 1 km, 1 km and 2 km, as shown in Fig. 1. The size of the prism along the z-axis is longer (than the equatorial size). For the uniform prism, the density is $\rho = 2.67$ g/cm^3 and its geometric center is the origin of the reference frame. The semi-major and semi-minor axes of the reference prolate spheroid enclosed the prism are choosed as: $a = 1.5$ km, $b = 0.949$ km. Analogous to the computations of the harmonic coefficients in the oblate spheroidal harmonic expression analysis, the prolate spheroidal harmonic coefficients can be obtained from the potential along a given confocal prolate spheroid $v = v_0$:

$$\begin{pmatrix} \overline{C}_{n,m} \\ \overline{S}_{n,m} \end{pmatrix} = \frac{a}{4\pi\mu} \frac{Q^{a}_{n,m}}{Q^{v_0}_{n,m}} \int_0^{2\pi} \int_0^{\pi} V(v_0, \vartheta, \lambda) \overline{P}_{n,m}(\cos\vartheta) \begin{pmatrix} \cos(m\lambda) \\ \sin(m\lambda) \end{pmatrix} \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\lambda \tag{37}$$

In this work, the semi-major axis $v_0 = a$, and then $V(v_0, \vartheta, \lambda)$ is the values of the potential on the reference prolate spheroid, which can be obtained by the closed-form solutions of the gravitational field of the prism in space-domain, e.g. Nagy et al. (2000). For the prolate spheroidal harmonic coefficients, $\overline{C}_{n,m}$ are non-zero only when the degree $n = 2\ell_1$ and and the order $m = 4\ell_2$ $(\ell_1, \ell_2 = 0, 1, 2, \cdots)$, and $\overline{S}_{n,m} = 0$ for all degrees and orders. We compute the surface integrals in Eq. (37) using the Gauss-Legendre quadrature with 8/8 Gaussian nodes, and discretize the latitude and longitude grids with $\Delta\vartheta = \Delta\lambda = 0.5°$ for computing the harmonic coefficients up to degree/order (d/o) 180 (Šprlák et al. 2018). The program codes for computing the harmonic coefficients using double-precision were written in Fortran 90, and compiled

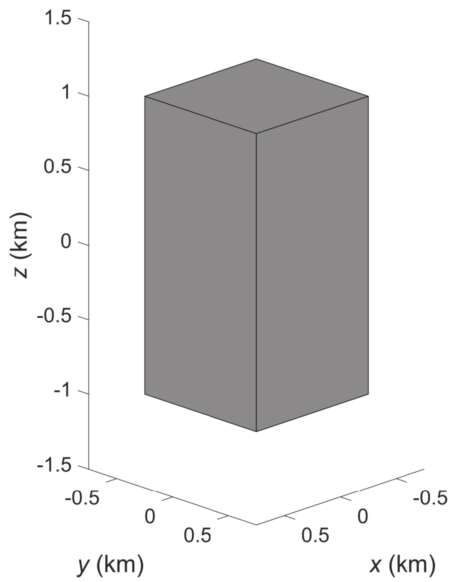

Fig. 1 The shape model of the prism

Table 6 Some of the prolate spheroidal harmonic coefficients of the gravitational potential of the uniform prism

n	m	$C_{n,m}$
0	0	1.331682998717521E+00
2	0	-8.131543511871320E-02
4	0	-1.734942451623555E-02
4	4	-5.390028954559478E-03
6	0	3.174451094475201E-03
6	4	-2.926746829242809E-03
60	0	1.243862461107105E-06
60	32	-7.171299248429795E-07
60	60	-3.950880341376749E-13
120	0	8.010263141083214E-08
120	60	5.433513799077741E-08
120	120	-6.954156083359933E-17
180	0	1.539124393164022E-08
180	92	-2.359011982770743E-09
180	180	6.177044504412598E-16

by the Intel Fortran Compiler 19.0 executed at a PC (Workstation) with an Intel Xeon W-2295 CPU and 64 GB main memory under the 64 bit Windows 10. We implemented the Fortran programs in parallel programming using OpenMP with 36 (the number of the threads of the W-2295 CPU) threads to save the time cost of the computations of the harmonic coefficients. Then, the executed time for solving the prolate harmonic coefficients up to d/o 180 is 7477.114 s, i.e., 2 hours 4 minutes 37.114 seconds. Some of the prolate harmonic coefficients are listed in Table 6. We also compute the spherical harmonic coefficients of the potential with the reference radius $R = \sqrt{0.5^2 + 0.5^2 + 1^2} \approx 1.225$ km using the algorithms of line integrals by Jamet and Tsoulis (2020), and the executed time for the Fortran programs of the solutions of the spherical harmonic coefficients implemented in parallel programming is 3.744 s. Complete prolate spheroidal and spherical harmonic coefficients up to d/o 180 are provided at https://github.com/chengchengit/ohphderi_prep_code_file.

The derivatives of the gravitational potential of the uniform prism up to third-order on the northern half of the reference prolate spheroid $v = a = 1.5$ km are plotted in Figs. 2, 3 and 4, and the angular resolution of the latitude and longitude grids is $0.1° \times 1°$. The root-mean-square (RMS) errors δV^{deri} for the derivatives of the prism on the confocal spheroid $v = 1.8$ km varying with the latitude $\phi = \frac{\pi}{2} - \vartheta = 0° \to 89.99999°$ (the colatitude $\vartheta = 0.00001° \to 90°$) are plotted in Figs. 5, 6 and 7, where δV^{deri} has the same meaning as the oblate spheroidal harmonic case, the taken values of the longitude and latitude are also the same and the truth-value of the derivative is taken from the closed-form solutions by Nagy et al. (2000). Table 7 shows the time costs for computing the derivatives of the potential up to third-order using the closed-form solutions and the regular and non-singular prolate spheroidal harmonic expressions, where the computational points is also taken on the confocal spheroid $v = 1.8$ km with the differences $\Delta\vartheta = \Delta\lambda = 5°$ and excluding the poles, the programs are executed as serial. The computed values of the derivatives of the potential at the poles (0 km,0

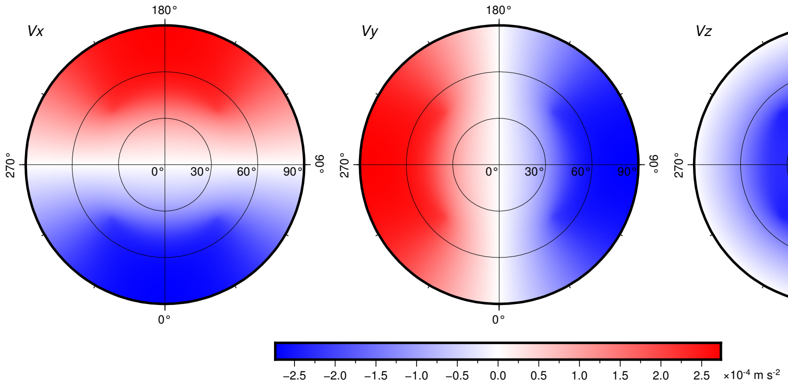

Fig. 2 First-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $v = a$. The value of the colatitude ϑ is labeled every 30°, and the value of the longitude λ every 90°. The prolate spheroidal harmonic expansions have been computed up to d/o 180

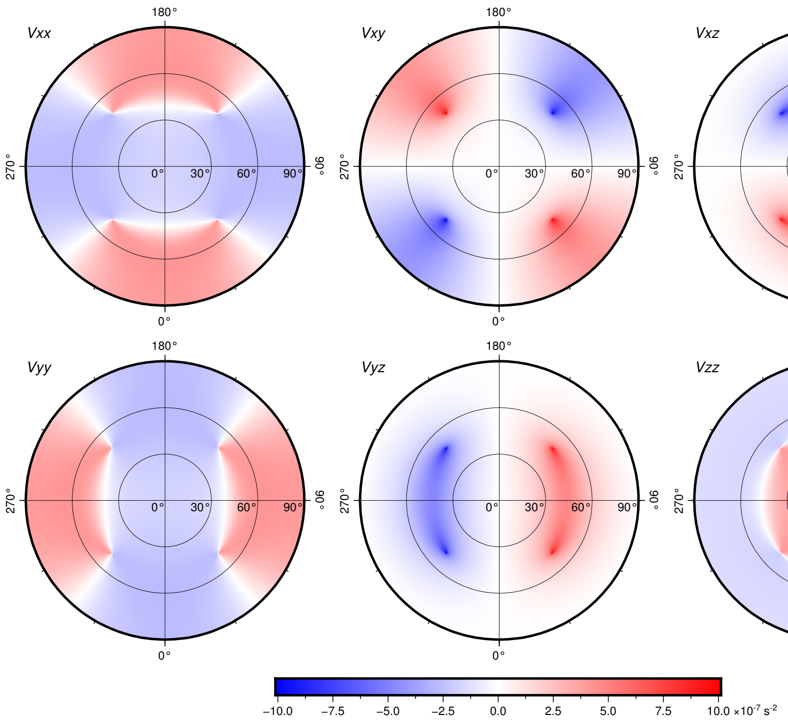

Fig. 3 Second-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $v = a$. The prolate spheroidal harmonic expansions have been computed up to d/o 180

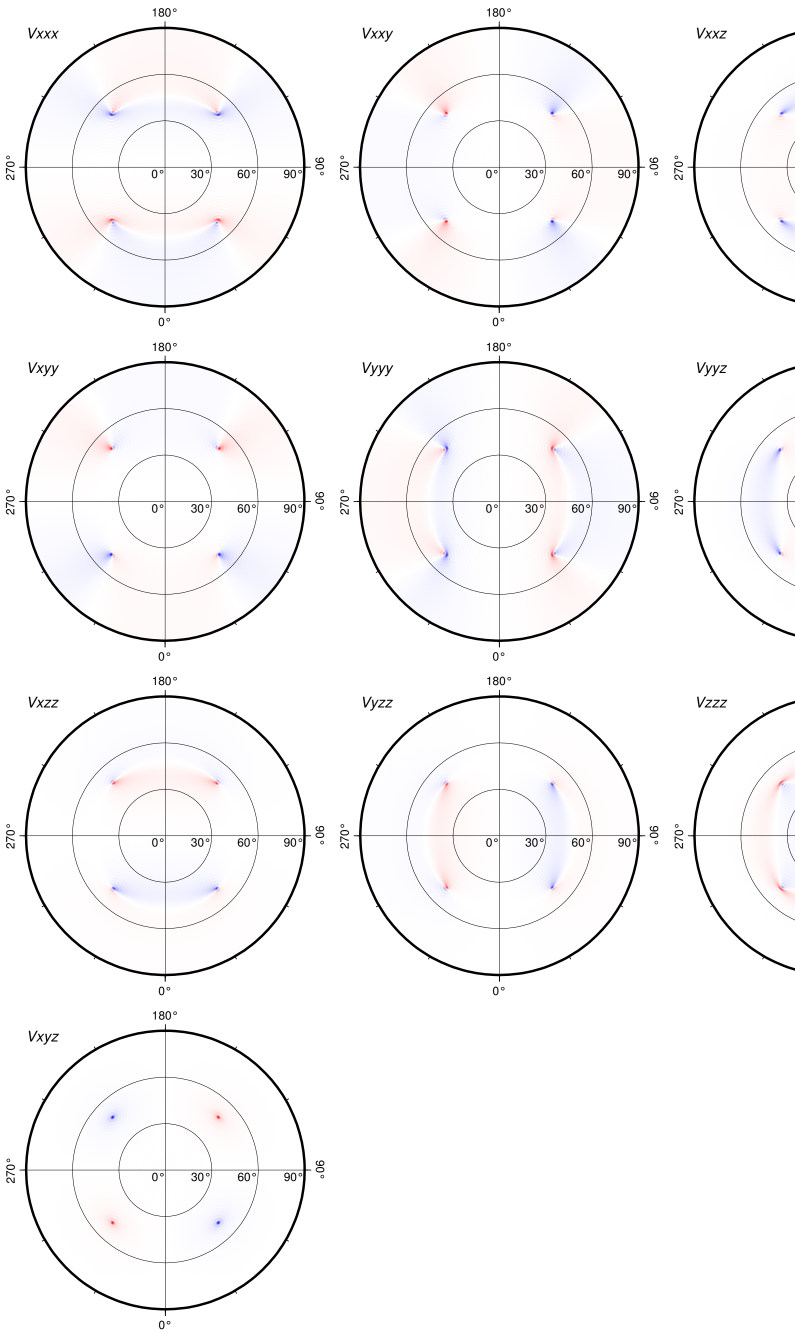

Fig. 4 Third-order derivatives of the gravitational potential of the uniform prism on the northern half of the reference spheroid $v = a$. The prolate spheroidal harmonic expansions have been computed up to d/o 180

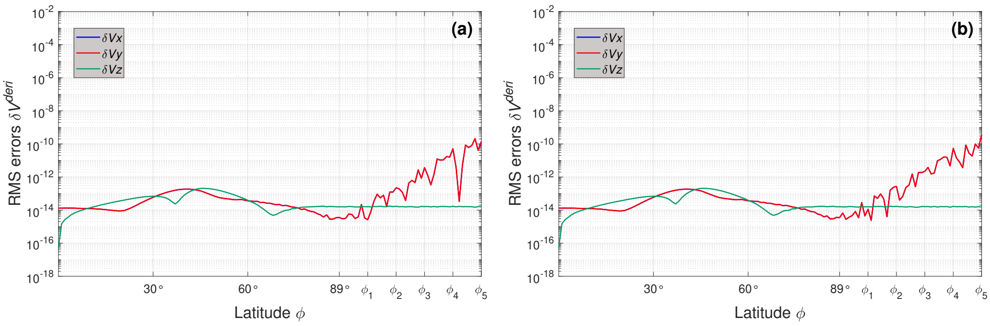

Fig. 5 The RMS errors δV^{deri} for the first-order derivatives of the gravitational potential of the uniform prism on the confocal spheroid $v = 1.8$ km computed by the non-singular (**a**) and regular (**b**) prolate spheroidal harmonic expressions. The prolate spheroidal harmonic expansions have been computed up to d/o 180

km,1.8 km) using the closed-form solutions and the harmonic method including the prolate spheroidal and the spherical harmonic method are given in Table 8, where the spherical harmonic expansions of the derivatives also refer to Cunningham (1970), Petrovskaya and Vershkov (2010), Chen et al. (2019) and Jamet and Tsoulis (2020).

Figures 2, 3, 4, 5, 6 and 7 and Table 8 demonstrate the computed values of the derivatives of the potential using the non-singular prolate spheroidal harmonic expressions are continuous along the whole reference prolate spheroid, and highly accurate for the point near or not near the poles. The numerical results of the regular prolate spheroidal harmonic expressions on the reference spheroid show almost the same variation tendencies as the oblate spheroidal harmonic case, and also produce the values of the derivatives of the potential (V_{xx}, V_{yy}, V_{zz}, V_{xxx}, V_{xxy}, V_{xxz}, V_{xyy}, V_{yyy}, V_{yyz} and V_{xyz}) with worse precision than the non-singular expressions for the points near the poles. When the points outside the polar regions, the regular expressions are also high precision. From Table 8, the times cost using the regular expressions is less than the non-singular expressions. In Table 8, the numerical results of the spherical harmonic expressions at the poles have higher precision than the (non-singular) prolate spheroidal harmonic expressions, and the reason is that the spherical harmonic coefficients are more accurate using line integral algorithms than the prolate spheroidal harmonic coefficients using boundary surface integrals and the spherical harmonic series of the gravitational field are also convergent well for the points near the polar axis as well as the radii $R < a$. For the points away from the polar axis, the prolate spherical harmonic series may have better convergence. The derivatives V_{xyy} and V_{xyz} at the poles computed by the closed-form solutions are non-zero, and their magnitudes are about 10^{-26} whose magnitudes divided by the other non-zero derivatives V_{xxz}, V_{yyz} and V_{zzz} are about 10^{-16} close to the machine epsilon of the double-precision floating point arithmetic, i.e. still being accurate.

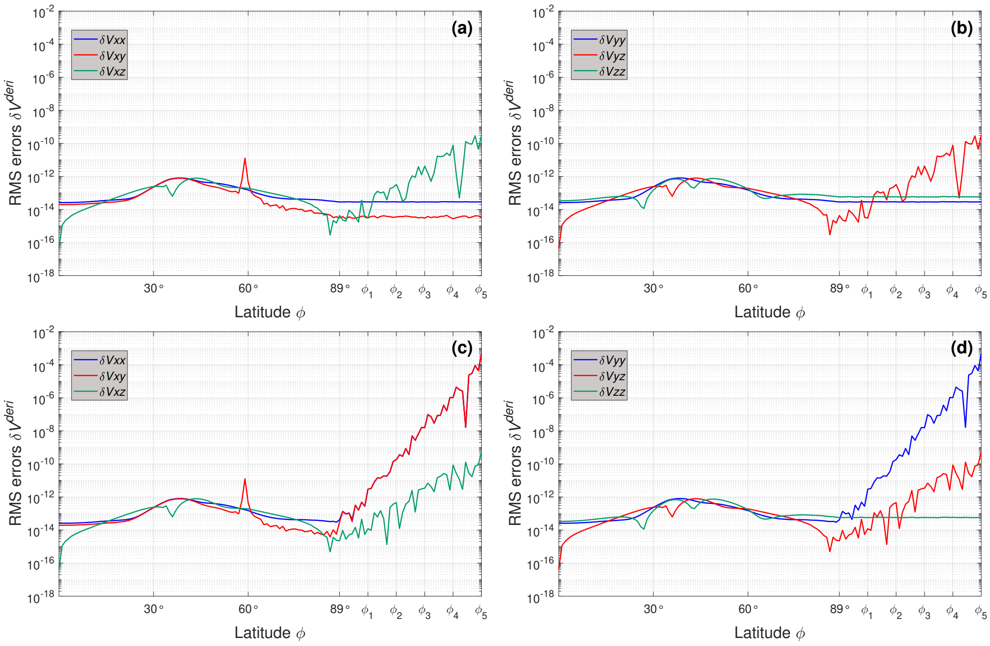

Fig. 6 The RMS errors δV^{deri} for the second-order derivatives of the gravitational potential of the uniform prism on the confocal spheroid $v = 1.8$ km computed by the non-singular (**a**, **b**) and regular (**c**, **d**) prolate spheroidal harmonic expressions. The prolate spheroidal harmonic expansions have been computed up to d/o 180

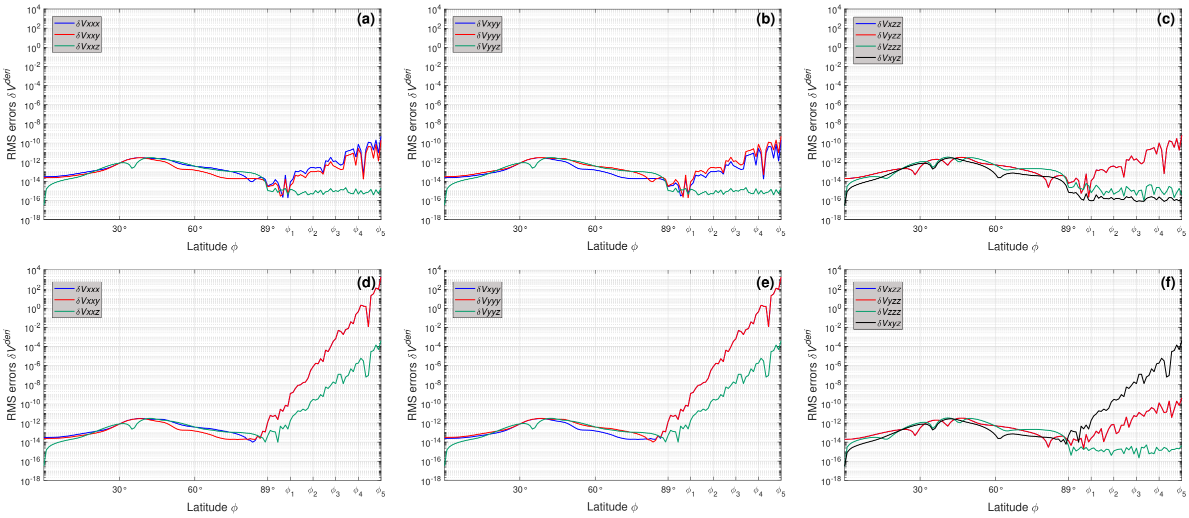

Fig. 7 The RMS errors δV^{deri} for the third-order derivatives of the gravitational potential of the uniform prism on the confocal spheroid $v = 1.8$ km computed by the non-singular (**a**, **b**, **c**) and regular (**d**, **e**, **f**) prolate spheroidal harmonic expressions. The prolate spheroidal harmonic expansions have been computed up to d/o 180

Table 7 The time costs for computing the derivatives up to third-order of the uniform prism at an observation point using the closed-form solutions and the prolate spheroidal harmonic method. The unit is seconds (s), and PH denotes for the prolate spheroidal harmonic expansion. The prolate spheroidal harmonic expansions have been computed up to d/o 180

d/o N	Closed-form solutions	Regular PH	Non-singular PH
60	0.000004977	0.006628	0.1104
120	0.000004977	0.02561	0.4282
180	0.000004977	0.05614	0.9752

Table 8 Derivatives of the gravitational potential of the uniform prism at the pole (0 km,0 km,1.8 km). SH denotes for the spherical harmonic expansion, and PH the (non-singular) prolate spheroidal harmonic expansion. The spherical and the prolate spheroidal harmonic expansions have been computed up to d/o 180

Derivatives	Closed-form solutions	SH	PH
V_x	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_y	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_z	-1.366750267767020E-04	-1.366750267767021E-04	**-1.36675026776**6998E-04
V_{xx}	-9.046455744666818E-08	-9.046455744666815E-08	**-9.046455744666**307E-08
V_{xy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{xz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yy}	-9.046455744666818E-08	-9.046455744666815E-08	**-9.046455744666**307E-08
V_{yz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{zz}	1.809291148933364E-07	1.809291148933363E-07	**1.809291148933**259E-07
V_{xxx}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{xxy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{xxz}	1.799040838830198E-10	1.799040838830198E-10	**1.799040838830**201E-10
V_{xyy}	-1.978459729379267E-26	0.000000000000000E+00	0.000000000000000E+00
V_{yyy}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yyz}	1.799040838830198E-10	1.799040838830198E-10	**1.799040838830**201E-10
V_{xzz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{yzz}	0.000000000000000E+00	0.000000000000000E+00	0.000000000000000E+00
V_{zzz}	-3.598081677660396E-10	-3.598081677660396E-10	**-3.59808167766039**3E-10
V_{xyz}	-1.978459729379267E-26	0.000000000000000E+00	0.000000000000000E+00

7 Conclusions

The prolate spheroidal harmonic expansions of the derivatives of the gravitational potential up to third-order in the global reference frame, and the conversions to the derivatives of the potential in the local north-oriented reference frame are proposed in this work. The non-singular and the regular expressions of the derivatives are similar to the oblate spheroidal harmonic case. The non-singular expressions of high-order derivatives are derived by lower-order derivatives (one order less) and the general linear relations (18) and (19). For the expressions of the fourth- and higher-order derivatives, we need to find the second formula of the third-order derivatives in Eq. (22). Since the singular factor $1/\sin\vartheta$ is eliminated in the linear relations, the expressions of the derivatives are non-singular for arbitrary external observation point outside the Brillouin prolate spheroid (more generally for $u \neq 0$, i.e. not on the line $\widetilde{\ell}_0$: $\{(x, y, z)|x = y = 0, -E \leqslant z \leqslant E\}$), which are also verified by the numerical implementation with the tested elongated uniform prism. For some complicated expressions of the coefficients f_1^{sup}, f_2^{sup}, f_3^{sup} and corresponding values f_{10}^{sup}, f_{20}^{sup}, f_{30}^{sup} on the poles, we take into account that the common factors of their denominators contain the semi-minor radius u, and then write their numerators in the form of the polynomial of u excluding the common factors of the numerators, where the symbol sup denotes the indeterminate superscript. Such phenomenon for some complicated expressions in the oblate case is actually the opposite, i.e. containing the semi-major axis v and the polynomial of v instead of u. We can observe some corresponding polynomials of u in the coefficients of the prolate case and of v in the oblate case have the same or opposite coefficients for the similar terms of polynomials. Considering the computational precision and cost, the non-singular expressions can be used for computations of the gravitational field at the observation points near the poles, and the regular expressions are more suitable for the observation points not near the poles. The formulas in Sects. 3 and 5 and Appendix of this work can also be verified by the Mathematica codes.

Data Availability. The datasets for the prolate spheroidal harmonic and spherical harmonic coefficients of the gravitational potential of the uniform prism up to d/o 180 are available at https://github.com/chengchengit/ohphderi_prep_code_file.

Appendix: Regular expressions of the derivatives of the gravitational potential

We can express the derivatives of the gravitational potential with respect to the Cartesian coordinate as the forms of the derivatives with respect to the prolate spheroidal coordinates, and the expressions are the same as the oblate spheroidal coordinates with replacing the coordinate symbol u by the symbol v. The expressions of the first-order derivatives can also be seen in Fukushima (2014). The expressions of the derivatives $\frac{\partial v_l}{\partial x_i}$ have been given in Eq. (5), where the integer indices $i, l = 1, 2, 3$, and $(v_1, v_2, v_3) = (v, \vartheta, \lambda)$. The expressions of the derivatives $\frac{\partial^2 v_l}{\partial x_i \partial x_j}$ $(i, j, l = 1, 2, 3)$ are also the same as oblate case with replacing u by v, and the coefficient terms f_2 are

$$
\begin{aligned}
f_2^{v0} =& \frac{v}{L^2},\ f_2^{v} = \frac{u^2 v \sin^2\vartheta}{L^6}\Big(u^2 + 4E^2 - 3E^2\sin^2\vartheta\Big),\\
f_2^{vz} =& \frac{u\sin\vartheta\cos\vartheta}{L^6}\Big(u^2v^2 + E^2u^2 - (3u^2+2E^2)E^2\sin^2\vartheta\Big),\\
f_2^{\vartheta 0} =& \frac{\cos\vartheta}{L^2\sin\vartheta},\ f_2^{\vartheta} = \frac{u^2\cos\vartheta}{L^6\sin\vartheta}\Big(u^2 + (2u^2+5E^2)\sin^2\vartheta - 2E^2\sin^4\vartheta\Big),\\
f_2^{\vartheta z} =& \frac{uv}{L^6}\Big(u^2 - (2u^2+3E^2)\sin^2\vartheta + 2E^2\sin^4\vartheta\Big),\ f_2^{\vartheta zz} = \frac{\sin\vartheta\cos\vartheta}{L^6}\Big((2u^2+3E^2)u^2 - (2u^2+E^2)E^2\sin^2\vartheta\Big),\\
f_2^{\lambda 0} =& \frac{1}{u^2\sin^2\vartheta},\ f_2^{\lambda} = \frac{2}{u^2\sin^2\vartheta},
\end{aligned}
\tag{38}
$$

The expressions of the derivatives $\frac{\partial^3 v_l}{\partial x_i\partial x_j\partial x_k}$ $(i,j,k,l = 1,2,3)$ are listed in Tables 9 and 10, in which we can observe: $-f_3^{v} + f_3^{vxy} = f_3^{vzz}$, $-f_3^{\vartheta} + \frac{3}{2}f_3^{\vartheta xy} = f_3^{\vartheta zz}$, and $4f_3^{\lambda xy} = f_3^{\lambda}$.

The derivatives V_{v_l}, $V_{v_{l_1}v_{l_2}}$ and $V_{v_{l_1}v_{l_2}v_{l_3}}$ $(l,l_1,l_2,l_3 = 1,2,3)$ with respect to the prolate spheroidal coordinates, their general form can also be expressed as

$$
V_{v^{k_1}\vartheta^{k_2}\lambda^{k_3}} = \frac{\mu}{a}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\widehat{Q}_{n,n,m}^{(k_1)}\overline{P}_{n,m}^{(k_2)}\overline{T}_{n,m,m}^{(k_3)}, \tag{39}
$$

where the integer indices $k_1,k_2,k_3 = 0,1,2,\cdots$, the notations $\widehat{Q}_{n,n,m}^{(k_1)}$ ($\widehat{Q}_{n,n,m}^{(k_1)} = \frac{\mathrm{d}^{k_1}\widehat{Q}_{n,n,m}}{\mathrm{d}v^{k_1}}$), $\overline{P}_{n,m}^{(k_2)}$ and $\overline{T}_{n,m,m}^{(k_3)}$ denote for the k_1-, k_2- and k_3-order derivatives of $\widehat{Q}_{n,n,m}(v/E)$, $\overline{P}_{n,m}(\cos\vartheta)$ and $\overline{T}_{n,m,m}(\lambda)$ with respect to the v-, ϑ- and λ-coordinates, respectively. From Eq. (9) and Gradshteyn and Ryzhik (2007, pp. 958), the derivatives

Table 9 Expressions of the derivatives $\frac{\partial^3 v_l}{\partial x_i\partial x_j\partial x_k}$ $(1 \leqslant l \leqslant 3,\ 1 \leqslant i \leqslant j \leqslant k \leqslant 3)$

$v_{xxx} = -f_3^{v0}\cos\lambda - (f_3^{vxy} - f_3^{v}\cos^2\lambda)\cos\lambda$, $\vartheta_{xxx} = -f_3^{\vartheta 0}\cos\lambda - (f_3^{\vartheta xy} - f_3^{\vartheta}\cos^2\lambda)\cos\lambda$, $\lambda_{xxx} = (f_3^{\lambda xy} - f_3^{\lambda}\cos^2\lambda)\sin\lambda$
$v_{xxy} = -f_3^{v0}\sin\lambda + f_3^{v}\cos^2\lambda\sin\lambda$, $\vartheta_{xxy} = -f_3^{\vartheta 0}\sin\lambda + f_3^{\vartheta}\cos^2\lambda\sin\lambda$, $\lambda_{xxy} = (f_3^{\lambda xy} - f_3^{\lambda}\sin^2\lambda)\cos\lambda$
$v_{xxz} = -f_3^{vz0} + f_3^{vz}\cos^2\lambda$, $\vartheta_{xxz} = -f_3^{\vartheta z0} + f_3^{\vartheta z}\cos^2\lambda$, $\lambda_{xxz} = 0$
$v_{xyy} = -f_3^{v0}\cos\lambda + f_3^{v}\sin^2\lambda\cos\lambda$, $\vartheta_{xyy} = -f_3^{\vartheta 0}\cos\lambda + f_3^{\vartheta}\sin^2\lambda\cos\lambda$, $\lambda_{xyy} = -(f_3^{\lambda xy} - f_3^{\lambda}\cos^2\lambda)\sin\lambda$
$v_{yyy} = -f_3^{v0}\sin\lambda - (f_3^{vxy} - f_3^{v}\sin^2\lambda)\sin\lambda$, $\vartheta_{yyy} = -f_3^{\vartheta 0}\sin\lambda - (f_3^{\vartheta xy} - f_3^{\vartheta}\sin^2\lambda)\sin\lambda$, $\lambda_{yyy} = -(f_3^{\lambda xy} - f_3^{\lambda}\sin^2\lambda)\cos\lambda$
$v_{yyz} = -f_3^{vz0} + f_3^{vz}\sin^2\lambda$, $\vartheta_{yyz} = -f_3^{\vartheta z0} + f_3^{\vartheta z}\sin^2\lambda$, $\lambda_{yyz} = 0$
$v_{xzz} = f_3^{vzz}\cos\lambda$, $\vartheta_{xzz} = f_3^{\vartheta zz}\cos\lambda$, $\lambda_{xzz} = 0$
$v_{yzz} = f_3^{vzz}\sin\lambda$, $\vartheta_{yzz} = f_3^{\vartheta zz}\sin\lambda$, $\lambda_{yzz} = 0$
$v_{zzz} = -f_3^{vz}$, $\vartheta_{zzz} = -f_3^{\vartheta zzz}$, $\lambda_{zzz} = 0$
$v_{xyz} = f_3^{vz}\sin\lambda\cos\lambda$, $\vartheta_{xyz} = f_3^{\vartheta z}\sin\lambda\cos\lambda$, $\lambda_{xyz} = 0$

Table 10 Expressions of the notations f_3 in Table 9

$f_3^{v0} = \frac{uv\sin\vartheta}{L^6}\left(4E^2+u^2-3E^2\sin^2\vartheta\right)$
$f_3^{v} = \frac{3u^3v\sin^3\vartheta}{L^{10}}\left(16E^4+12E^2u^2+u^4-10E^2(2E^2+u^2)\sin^2\vartheta+5E^4\sin^4\vartheta\right)$
$f_3^{vxy} = \frac{2uv\sin\vartheta}{L^{10}}\left(u^4(4E^2+u^2)+E^2u^2(8E^2-u^2)\sin^2\vartheta+E^4(4E^2-5u^2)\sin^4\vartheta-3E^6\sin^6\vartheta\right)$
$f_3^{vz0} = \frac{\cos\vartheta}{L^6}\left(E^2u^2+u^2v^2-E^2(2E^2+3u^2)\sin^2\vartheta\right)$
$f_3^{vz} = \frac{3u^2\sin^2\vartheta\cos\vartheta}{L^{10}}\left(u^2(8E^4+8E^2u^2+u^4)-2E^2(4E^4+10E^2u^2+5u^4)\sin^2\vartheta+E^4(4E^2+5u^2)\sin^4\vartheta\right)$
$f_3^{vzz} = \frac{uv\sin\vartheta}{L^{10}}\big(2u^4(4E^2+u^2)-u^2(32E^4+38E^2u^2+3u^4)\sin^2\vartheta+2E^2(4E^4+25E^2u^2+15u^4)\sin^4\vartheta$ $-3E^4(2E^2+5u^2)\sin^6\vartheta\big)$
$f_3^{\vartheta 0} = \frac{u\cos\vartheta}{L^6\sin^2\vartheta}\left(u^2+(5E^2+2u^2)\sin^2\vartheta-2E^2\sin^4\vartheta\right)$
$f_3^{\vartheta} = \frac{u^3\cos\vartheta}{L^{10}\sin^2\vartheta}\left(3u^4+2u^2(9E^2+2u^2)\sin^2\vartheta+(63E^4+44E^2u^2+8u^4)\sin^4\vartheta-8E^2(7E^2+4u^2)\sin^6\vartheta+8E^4\sin^8\vartheta\right)$
$f_3^{\vartheta xy} = \frac{2u\cos\vartheta}{L^{10}\sin^2\vartheta}\left(u^6+u^4(7E^2+2u^2)\sin^2\vartheta+E^2u^2(11E^2+2u^2)\sin^4\vartheta+E^4(5E^2-2u^2)\sin^6\vartheta-2E^6\sin^8\vartheta\right)$
$f_3^{\vartheta z0} = \frac{v}{L^6\sin\vartheta}\left(u^2-(3E^2+2u^2)\sin^2\vartheta+2E^2\sin^4\vartheta\right)$
$f_3^{\vartheta z} = \frac{u^2v}{L^{10}\sin\vartheta}\left(u^4+2u^2(7E^2+2u^2)\sin^2\vartheta-(35E^4+52E^2u^2+8u^4)\sin^4\vartheta+8E^2(5E^2+4u^2)\sin^6\vartheta-8E^4\sin^8\vartheta\right)$
$f_3^{\vartheta zz} = \frac{u\cos\vartheta}{L^{10}}\big(u^4(3E^2+2u^2)-2u^2(15E^4+19E^2u^2+4u^4)\sin^2\vartheta+E^2(15E^4+50E^2u^2+32u^4)\sin^4\vartheta$ $-2E^4(3E^2+4u^2)\sin^6\vartheta\big)$
$f_3^{\vartheta zzz} = \frac{v\sin\vartheta}{L^{10}}\big(3u^4(5E^2+2u^2)-2u^2(15E^4+25E^2u^2+4u^4)\sin^2\vartheta+E^2(3E^4+38E^2u^2+32u^4)\sin^4\vartheta$ $-2E^4(E^2+4u^2)\sin^6\vartheta\big)$
$f_3^{\lambda xy} = \frac{2}{u^3\sin^3\vartheta}$
$f_3^{\lambda} = \frac{8}{u^3\sin^3\vartheta}$

$\widehat{Q}^{(k_1)}_{n,n,m}$ $(k_1 = 1,2,3)$ can be written as

$$\begin{aligned}
\widehat{Q}^{(1)}_{n,n,m} =&(n-m+1)\frac{E}{u^2}\widehat{Q}_{n+1,n,m}-(n+1)\frac{v}{u^2}\widehat{Q}_{n,n,m},\\
\widehat{Q}^{(2)}_{n,n,m} =&-\frac{2v}{u^2}\widehat{Q}^{(1)}_{n,n,m}+\left(n(n+1)+m^2\frac{E^2}{u^2}\right)\frac{\widehat{Q}_{n,n,m}}{u^2},\\
\widehat{Q}^{(3)}_{n,n,m} =&-\frac{2v}{u^2}\widehat{Q}^{(2)}_{n,n,m}+\left(n(n+1)+\frac{(m^2+2)E^2+2v^2}{u^2}\right)\frac{\widehat{Q}^{(1)}_{n,n,m}}{u^2}\\
&-\left(n(n+1)+2m^2\frac{E^2}{u^2}\right)\frac{2v}{u^4}\widehat{Q}_{n,n,m},
\end{aligned} \tag{40}$$

The formulas for computing the derivatives $\overline{P}^{(k_2)}_{n,m}$ and $\overline{T}^{(k_3)}_{n,m,m}$ $(k_2, k_3 = 1,2,3)$ can refer to the oblate spheroidal harmonic case. We can also derive the non-singular prolate spheroidal harmonic expressions for the derivatives of the potential on the basis of the regular expressions, and may need consider the cases $m = 0,1,2$ and $m > 2$, respectively.

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