DESARGUES AND LEIBNIZ: IN THE BLACK BOX

A MATHEMATICAL MODEL OF THE LEIBNIZIAN MONAD

Here **Bernard Cache** provides a detailed analysis of a paper written in 1636 by the French mathematician, architect and engineer, Girard Desargues. Desargues is best known as the founder of projective geometry. Cache explains how he initally developed this significant concept in response to the very practical problems of producing a perspectival drawing. The introduction of projective geometry, though, had potentially more far-reaching implications on philosophical thought, informing the theory of monads developed by the German philosopher and mathematician Gottfried Leibniz in 1714 to explain the metaphysics of simple substances.

Strangely, the short essay on perspective published by Girard Desargues in 1636 makes no mention of any notion of projective geometry, explicitly at least, before reaching its rather contemplative conclusion. This remarkably curious text consists of a commentary of a single engraving presumed accessible enough for the knowing reader to apply its premises to any practical situation, a drawing expressing a 'universal way', in other words – as stated by the title of the essay itself: 'A Sample of the Universal Way of SGDL:[1] On the Practice of Perspective Without the Assistance of a Third Point, Distance Measurement Or Any Other Expedient External to the Task at Hand' ('*Exemple de l'une des manières universelles du SGDL. Touchant la pratique de la perspective sans employer aucun tiers point, de distance ni d'autre nature qui soit hors du champ de l'ouvrage*').[2]

The absence of any reference to projective geometry confirms the author's intent to address the material constraints of daily practice, primarily the fixed size of the board or sheet on which the drawing is laid out. The need to resort to 'third points' lying beyond the surface of the sheet, such as vanishing points, or those mapped from a transversal section onto the drawing plane (*rabattement*), was a frequent problem in practice – hence Leon Battista Alberti's famous recommendation that an extra sheet of paper be deployed next to the drawing itself.[3]

To address these two constraints, Desargues advocates a new method: 'The agent, in this instance, is a cage made simply of lines,'[4] he writes in a rather surprising comment, followed by the description of the plan and location of the cage, as well as a statement to the effect that 'the engraving itself is like a wood plank, a stone wall, or something like it' – seemingly turning on its head the accepted interplay between transparency and opacity advocated by most theoreticians of perspective before him.

In lieu of a solid and opaque body, such as the Baptistery of Florence depicted on Filippo Brunelleschi's experimental tablets (1415), Desargues presents the reader with a transparent cage.

Compare this to the very edifice that Flemish mathematician and military engineer Simon Stevin

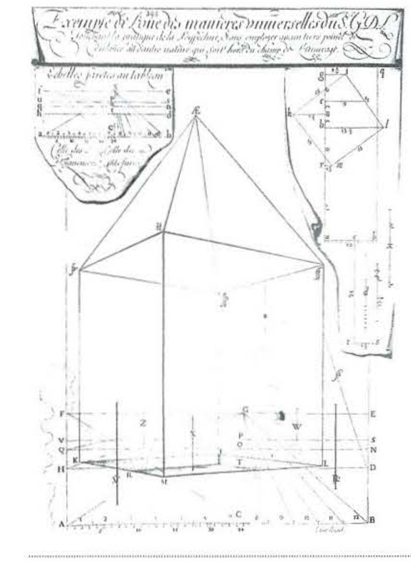


Figure 1. Girard Desargues, Exemple de l'une des manières universelles du SGDL (A Sample of the Universal Way of SGDL: On the Practice of Perspective Without the Assistance of a Third Point, Distance Measurement Or Any Other Expedient External to the Task at Hand'), 1636
The original 17th-century engraving.

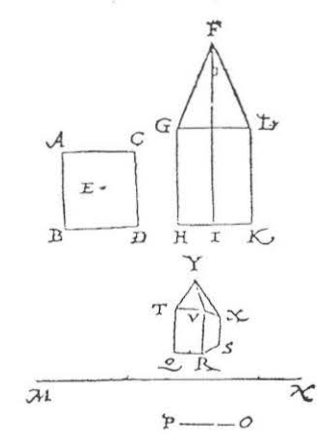


Figure 2. Simon Stevin, A Study in Perspective: top and front orthographic views and perspective projection of a solid modelled on the Baptistery of Florence
J Tuning's edition (1605), Vol III, Book I, prop Xi, problem V.

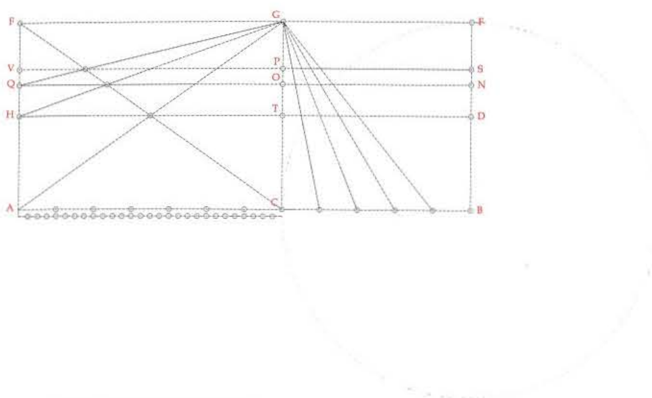


Figure 3. Reproduction of Desargues's original layout for the geometric demonstration
On this diagram (culled from the upper-left corner of the original engraving), three separate geometric constructions overlap onto a single space, and three sets of diagonal lines converge towards a single point.

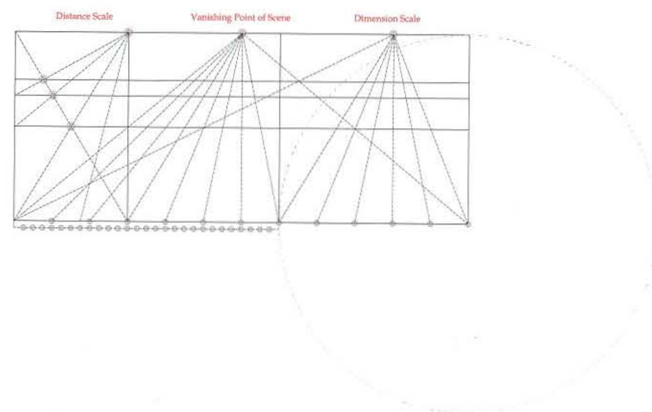
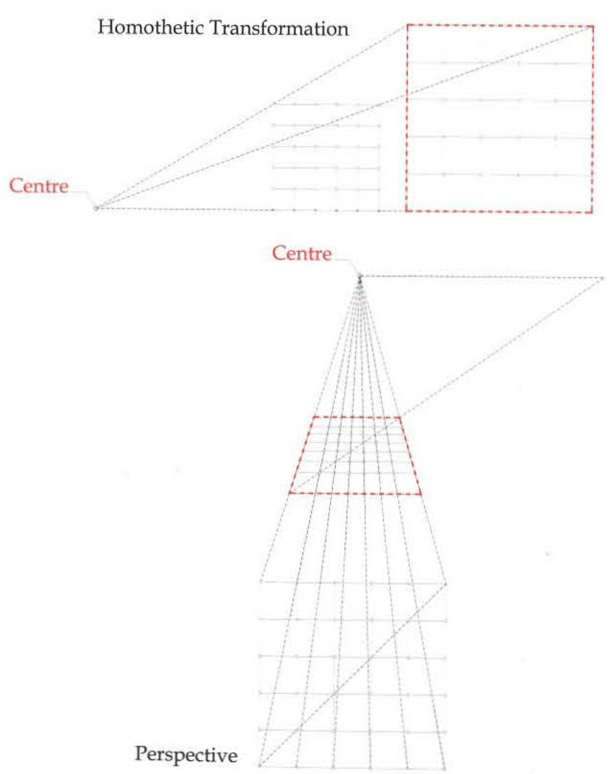


Figure 4. Graphical reorganisation of Desargues's original diagram
The reorganisation lays out the three separate geometric constructions side by side, revealing distinct points of convergence for each construction.



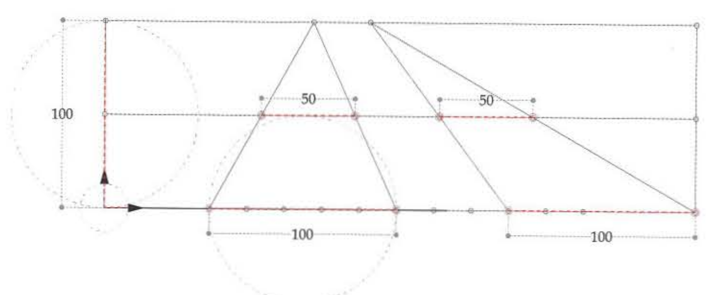


Figure 5. The Dimension Scale
(*above*) Diagram demonstrating the use of the Dimension Scale to figure out a length on the projection on the picture plane of a line parallel to the picture plane, assuming we know its elevation (in this instance, halfway up the Y axis of the Dimension Scale).

Figure 6. Homothetical (scaling) transformation and conic projection (perspective)
(*left*) Top: The small square maps onto the large one relative to a fixed point on the plane. The transformation alters the measurements of the original figure, but preserves its angles. Bottom: The transformation alters the measurements as well as the angles of the original figure, but preserves some given dimensional ratios.

(1548–1620) chose to illustrate his own take on the problem: the two volumes are nearly identical, with the difference that Desargues proposes a wireframe, whereas Stevin does not even bother to dot in the hidden edges of the solid. Rather than opening a window onto the world, Desargues walls us into a windowless black box, reminiscent of Gottfried Leibniz's monad.[5] Let us examine what this means in practice.

The engraving commented on by Desargues features several parts. In the upper left corner we find a greatly simplified plan view, consisting of a few simple strokes and annotated with measurements (such drawings used to be known as *géometral*, or orthographic, projections). In the middle of the engraving we find the perspectival view of the cage, drawn over some sort of diagram reproduced, at a smaller scale, in the upper left corner of the sheet.

Desargues's overlapping of three separate geometric constructions onto a single diagram is most certainly confusing, hence the importance of pulling this diagram apart in order to analyse it. The interpretation of it lays out the three constructions side by side, rather than on top of one another. Desargues's method seems to work just as well, if not better, with this alternative layout, where, unlike what is shown on the original engraving of Sieur Girard Desargues de Lyon, the three sets of diagonal lines do not converge towards a single point – each construction has a distinct point of convergence.

Let us begin with the central part of the diagram, where Desargues hardly innovates at all. Here the lines normal to the picture plane converge towards a single vanishing point, illustrating the orthogonal projection of the gaze on the picture. Desargues says no more than Alberti did on the same subject – nothing, that is.

Desargues's method is based on the commonly known (at the time) expedient of transferring points from one grid to another, traditionally deployed for the purpose of scaling figures (now known as a homothetic transformation).[6] By extension, applying the same method to map a point from an orthogonal grid to another kind of grid resulted in a projective transformation. To configure

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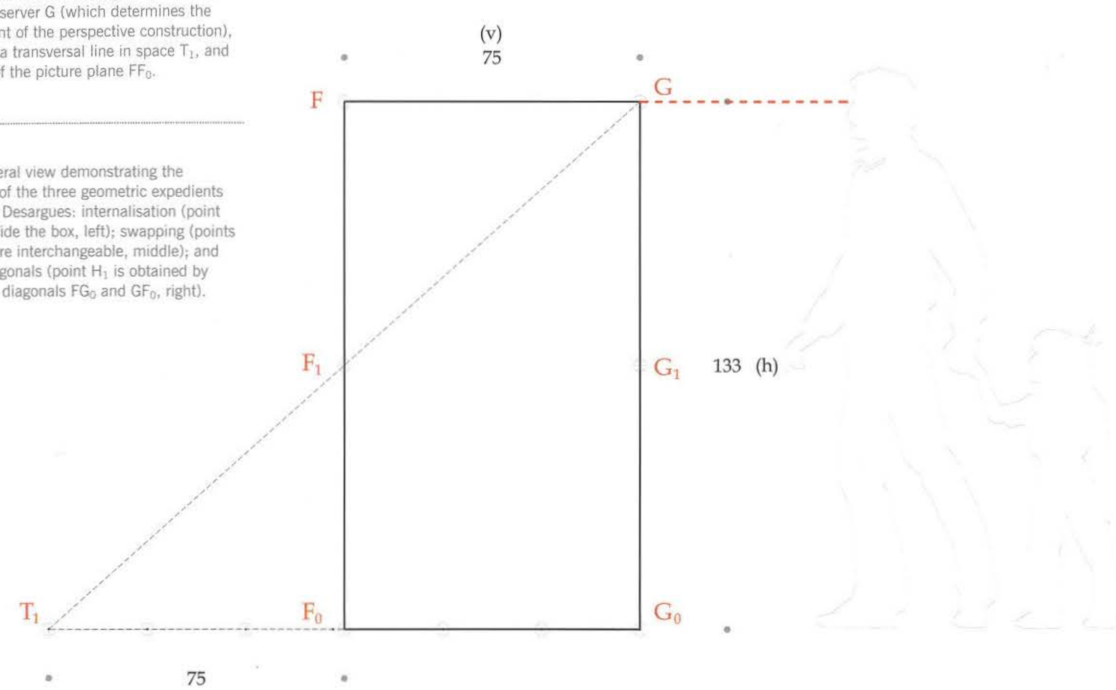
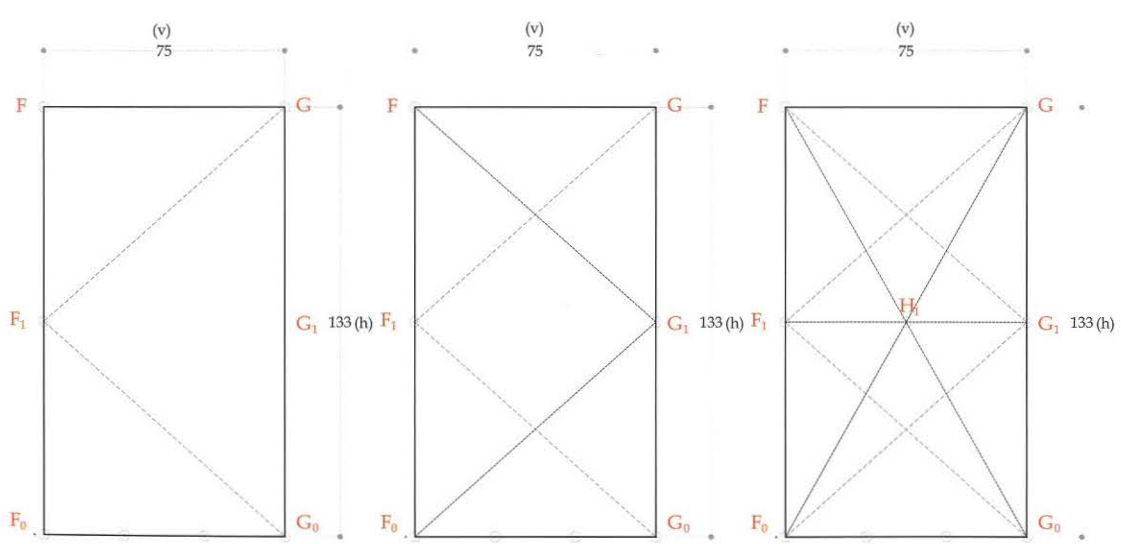
Figure 7. Desargues's Black Box
(*right*) Lateral view indicating the position of the ideal observer G (which determines the vantage point of the perspective construction), the edge of a transversal line in space T_1, and the profile of the picture plane FF_0.

Figure 8
(*below*) Lateral view demonstrating the application of the three geometric expedients available to Desargues: internalisation (point G_0 stays inside the box, left); swapping (points F_1 and G_1 are interchangeable, middle); and drawing diagonals (point H_1 is obtained by intersecting diagonals FG_0 and GF_0, right).





this new grid, Desargues needs to locate the vanishing point of all lines normal to the picture plane (on a separate note, the method does not guarantee that the said lines will converge within the extents of the page – especially where oblique projection is concerned).

Once the 'converging lines' (those lines normal to the picture plane) have been drawn, we need to establish dimensions along the 'transversal lines', or 'transversals' (those lines parallel to the picture plane), which Desargues calls the 'Dimension Scale'.

To figure out a length on the image of a given transversal, assuming of course we know its elevation on the picture plane, we will report in true length the segment of reference on the base of the Dimension Scale, then draw two convergent projectors from its endpoints. The intersections of these projectors with the image of the transversal at the given elevation will give us two points – and the distance between these two points, the desired length. It is critical to note that the point where these projectors come to a focus is not the same as the central point where converging lines meet. In other words, the focal point of the Dimension Scale does not have to coincide with the vanishing point of the scene. The two points must be located at the same elevation, but the former will move laterally by any amount without altering the reported measurements from one transversal line to the next.

Why insist on this particular aspect of Desargues's strategy? Because this is the crux of the originality of Desargues's perspectival method, which implies that the Distance Scale[7] cannot be understood unless it is radically distinguished from those lines converging towards the vanishing point of the scene. How does Desargues determine the elevation of the image of a transversal line then? What exactly is going on here?

Let us recall Alberti's *Construzione Legittima* and his section of a tapering cone of vision. For the sake of the demonstration, let us imagine that the eye, notated G, lies at a distance d from the picture, the section of which determines the vertical line FF_0. The elevation of observer G relative to the ground line is equal to h.

Desargues cunningly prioritises the determination of the image of the transversal line T_1, a line located in the ground plane at a distance d from the section FF_0, but on the side opposite to observer G. Since line T_1 and eye G lie on two vertical planes symmetrically disposed about section FF_0, the ray of vision GT_1 will interest section FF_0 at its midpoint, notated F_1. This point will remain the midpoint of section FF_0 regardless of any fluctuations of distance d taken between eye G and the picture F, provided of course that the distance between the transversal line T_1 and the picture is adjusted accordingly. Prefiguring in some way the projective method of homogeneous coordinates, the transversal lines of Desargues are located at a distance equal to a multiple of the distance separating the eye G from the picture F. Following, the position of these transversal lines can vary arbitrarily, provided the elevation of observer G relative to the ground line remains equal to h.

This is the first benefit of Desargues's method. Notwithstanding the risk of proposing something most historians of descriptive geometry would regard as an anachronism, we will refer to Desargues's system as a system of homogeneous coordinates. Assuming that height h of rectangle GG_0F_0F is correct, the width of this rectangle may vary arbitrarily, given that T_1 will always remain symmetrical to G about the axis FF_0. The maintenance of the symmetry between the eye G and any transversal line T_1 opens up three more potential configurations, which, taken in tandem, make it possible to determine the height F_0F_1 of the image of line T_1 relative to G without using a point located outside the rectangle itself. These three provisions may be summarised as follows:

(a) Internalising Moves

All geometric moves will take place within rectangle GG_0F_0F. So long as $T_1F_0=F_0G_0=d$, the width of the rectangle (and the correlative position of any transversal line in space) will vary as needed, but from now on everything takes place indoors. Like an inward reflection, the lines meeting the boundaries of the back box will bounce back

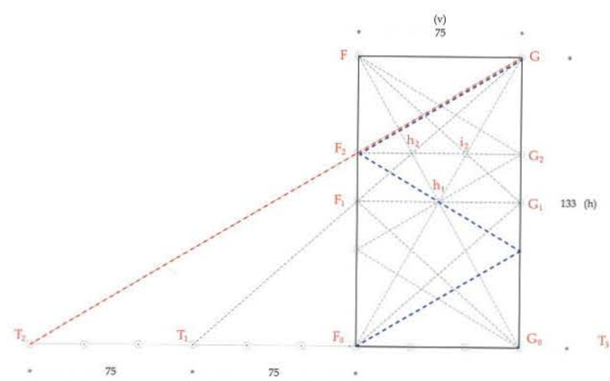


Figure 9. Desargues's use of the Black Box
Desargues's use of the Black Box (and the three geometric expedients shown in Figure 8) to determine the image (projection) of a line T_2 located two times further than the observer from the picture plane (but on the opposite side). The length of the red dotted line is equal to that of the blue one, providing a vivid and accurate visual account of the folding of outer space within the confines of the Black Box.

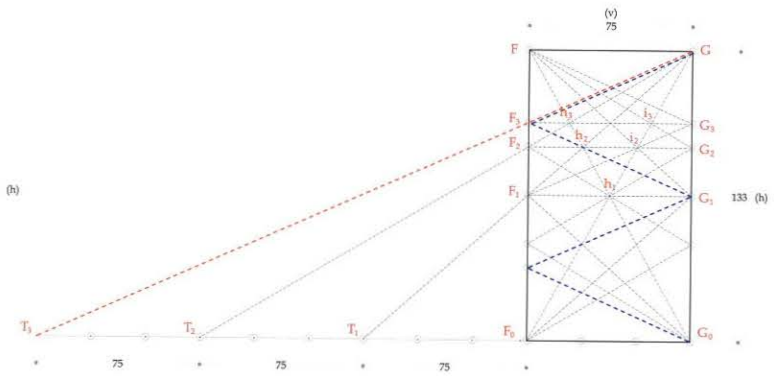


Figure 10
The same operation as in Figure 9, assuming that the transversal line is now located three times further than the observer from the picture plane. Here too the length of the red dotted line is equal to the length of the blue one.

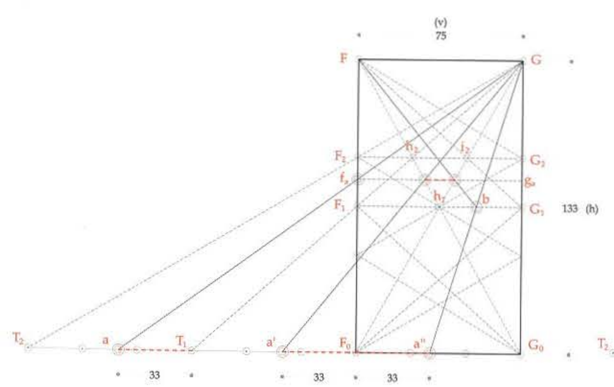


Figure 11
The same operation as in Figures 9 and 10, assuming now that the transversal line is located at a random distance from the picture plane (somewhere between lines T_2 and T_1). The same metrical equivalence between red and blue dotted lines applies, and space folds like an accordion.

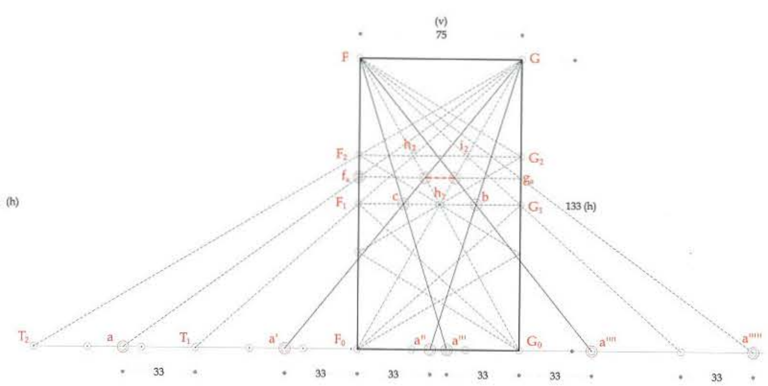


Figure 12
The full deployment of internalisation, swapping and extra diagonals helps determine the image of any transversal line in space thanks to a bevy of auxiliary points, two of which are strategically located within the Black Box (a" and a"').

in, and their image will be determined without reference to any external points.

(b) **Swapping Points**

Since all moves now take place within the box, points F and G are equivalent. The reflected image F_1 of G_0 relative to G is equivalent to the reflected image G_1 of F_0 relative to F. Points F and G may therefore be swapped at will, something Desargues himself does repeatedly in the course of his explanation on how to determine the image of a given point, invariably losing his reader who labours under the double misapprehension that G represents the vanishing point of the scene, and F some kind of distance point – or at least an accessory vanishing point of sorts. This is why it is critical to distinguish G from the vanishing point, as well as to move point F away from the left border of the picture. For the essential raison d'être of the Distance Scale is to fold and unfold itself at will, accordion-like, admittedly not the easiest of procedures when it comes to the wooden engravings of Desargues's time, but a mere trifle for today's parametric software routines.

(c) **Drawing Diagonals**

Since our goal is to determine from within the elevation of the image of transversal line T_1, which happens to coincide with the midpoint of the box's vertical edge, is it not simpler to look for the intersection of the box's two diagonals, leaving aside all reflections in F_1 or G_1? This final provision will prove essential when it comes to determining not only the image of a given transversal line T_1 located at distance d from the picture, but that of any transversal line in space.

To figure out the image of a line located two times further than transversal T_1 (n=2) from the plane of the picture, Desargues methodically generalises his application. First he takes into account the image of a transversal line T_n located n times further than transversal T_1 from the plane of the picture, at a distance $dn = n \cdot d$. The diagrams reproduced here should provide enough evidence to shore up the conclusion that the four provisions outlined in the case of the transversal line T_1 apply equally to lines T_2 and T_3, and by extension, to any line T_n. When it comes to multiple transversal lines, we note that, if G is the eye of the observer, any transversal line T_n located beyond the confines of the box will map internally to either F_0 or G_0, depending on the parity of integer n (if n is odd, it will be G_0; otherwise F_0). If F is posited as the eye of the observer, the reasoning is precisely reversed.

The same applies to determining the image of a line T_3 located three times further than transversal T_1 (n=3) from the plane of the picture.

It is important to note the key role diagonals play in the general case where a transversal line T_n is located at a random distance from the plane of the picture (this distance no longer being a multiple of d). Let us for instance determine the elevation of the image of transversal line T_2. The elevation is set by point h_2, located on a horizontal line going through point F_2, itself the intersection of lines GT_2 and F_0F. Point h_2 is also the intersection of line GT_1 and diagonal FG_0. The elevation of the image of point a, randomly located between T_1 and T_2, will be found within the diagonal segment h_1h_2, at the intersection with a line passing through eye G and a new point a', located between F_0 and T_1, at a distance equal to the distance between a and T_1.

To remain 'within the box', as it were, we will simply swap vantage points and look for the elevations of the images of any point a between T_1 and T_2 on the opposite diagonal segment GF_0, between h_1 and i_2. Starting with point a", located directly opposite point a' about F_0, we can trace a line a"G intersecting the horizontal F_1G_1 in point b, from which we can draw another line to point F. This line meets segment h_1i_2 at the desired elevation, notated f_a. Moreover, the line connecting F and b implies that b lies on a sight line originating in F and ending in a'''', a new point located directly opposite a' about a vertical axis through h_1, halfway across rectangle GG_0F_0F. In the end, the inclusion of this vertical axis of symmetry brings the number of points available to help us determine the image of a transversal line passing through T_1 and T_2 to a staggering six. Critically for Desargues, two of these points, a and a', are located inside the black box.

At this juncture of our demonstration, the reader will probably be subject to mixed emotions of shock and awe. Let us dispense with shock first. In essence, none of the elementary steps described here is truly complicated. The whole thing may perhaps be a little more complex than the 'enchanted description of a palace in a novel' which, upon reading the author's draft *Brouillon Project Sur les Coniques*,[8] Descartes had admonished Desargues to pursue. But let's face it; this is the stuff of elementary geometry: a few lines and their intersections here and there, some scaling applications, some symmetry. Yet this is where a 17th-century reader might have struggled a bit, lest we forget that the term 'symmetry' had other meanings before 1794, when Adrien-Marie Legendre recovered the moniker to designate an inversion of spatial orientation, whereby the right-handed becomes left-handed.[9]

Leibniz's monads are some kind of individuated atoms, combining in nature to produce a complete and optimal set of bodies, under God's coordination. Monads aggregate into bodies under the aegis of a higher monad, vested with the power of soulfulness.

We can think of at least one reader of Desargues who must have fully appreciated the awe and power of this perspective method – Leibniz himself, who indeed might have been surprised to find here something akin to his own preoccupations. Leibniz's monads are some kind of individuated atoms,[10] combining in nature to produce a complete and optimal set of bodies, under God's coordination.[11] Monads aggregate into bodies under the aegis of a higher monad, vested with the power of soulfulness. Given that 'it has no windows through which to come and go',[12] each monad is solely determined by itself. On the basis of this purely internal principle,[13] a given monad will be the locus of changing perceptions,[14] independent of any other source. When one looks at a moving body, suggests Leibniz, God has tuned one's internal principles of perception in harmony with the movement of the moving body, while blocking all communications between the two monads governing the body and the moving body.

At this juncture, Leibniz calls upon the higher authority of perspective: 'Just like a city considered from different vantage points looks different every time, seemingly multiplied by perspective; likewise it so happens that an infinite multitude of simple substances will produce many distinct universes, which are nothing but alternative perspectives of a single universe, taken from the vantage point of each individual Monad.'[15]

How did Leibniz devise the notion of a perception ordinated internally by closed, windowless and individuated monads, moving independently from one another yet highly coherent as a whole? It is precisely on such a double regimen of interiority/exteriority that Desargues's Distance Scale is based. It is on a purely internal basis that the vertical boundary F_0G_0 determines the reflected image taken from G. Whether the perceived object is located outside in T_1, or inside in F_0, the internal and external procedures will yield the same image in F_1. Critically, the multiple reflections determine how the boundary F_0G_0 will allow us to calculate the image of any point, however remote, or even infinite, by folding space over and over. And as for the swapping of the viewing points F and G, this expedient cannot fail to remind us of Leibniz's fundamental distinction between perception and apperception.

No model is equivalent to the theory it is meant to subtend. Undoubtedly, Leibniz will have devised the monad from a multiplicity of models, later surveyed by the philosopher Michel Serres.[16] Naturally our own conjecture ought to account, upfront, for Leibniz's vague use of perspective in his philosophic writings at large. This is a tough question indeed, a question requiring us to precisely analyse the full extent of the many mathematical domains into which Leibniz wandered, specifically as well as relatively to one another. Yet moving on from Desargues's perspective to differential calculus, Leibniz may have simply laid to rest the precise workings of the practical application of the Distance Scale in the safe knowledge that his conception of the monad had been properly grounded. ᴅ

Notes

1. Sieur Girard Desargues de Lyon.
2. The essay first appeared in English in JV Field and JJ Gray, *The Geometrical Work of Girard Desargues*, Springer (New York, Berlin, Heidelberg, London, Paris, Tokyo), 1987, pp 147–60.
3. Leon Battista Alberti, *De Pictura* I, 20, 1485. 'I take a small surface …', specified in Italian as '*prendo un piccolo spazio*', or in Latin as '*habeo areolam*'.
4. See Field and Gray, op cit, p 147. Desargues reverses the accepted interplay between opacity and transparency that perspective is based on: for him the represented object is a wireframe, and his picture plane a screen. Alberti's picture plane, on the other hand, is like a window – or a light veil – and the object a solid and opaque mass.
5. The idea of the monad was first published in his *La Monadologie* (The Monadology) of 1714.
6. In this Desargues is among other things heir to Ptolemy, and his use of different coordinate systems in each of his three cartography mappings.
7. The Dimension Scale (*échelle des mesures*) unfolds *parallel* to the picture plane, and its base (the edge along which the Dimension Scale meets the ground) offers a ground line. The Distance Scale (*échelle des éloignements*) unfolds *perpendicularly* to the picture plane and records measurements extending in depth. Determining exact metric correspondences on the Distance Scale using only the Dimension Scale and a few elementary planar geometric operations, such as bisecting or mirroring a line, is the great innovation of Desargues, who manages to determine the exact foreshortening of distances in depth without resorting to perspective – as claimed in the subtitle of the *Universal Way* (translator's note).
8. Descartes, Letter to Desargues dated 19 June 1639. See Field and Gray, op cit, pp 176–7.
9. Giora Hon and Bernard R Goldstein, *From Summetria to Symmetry : The Making of a Revolutionary Scientific Concept*, Springer (New York), 2008.
10. Leibniz, *La Monadologie*, article 3.
11. Ibid, article 55.
12. Ibid, article 7.
13. Ibid, article 11.
14. Ibid, article 14.
15. Ibid, article 13.
16. Michel Serres, *Le système de Leibniz et ses modèles mathématiques*, PUF (Paris), 1968.

Article translated from the original French by George L Legendre