

# Stars and Bars



↳ Helps in finding number of ways a given input can be divided into subsets/partitions with constraints

## Mental model:

$$1 \leq x_1 + x_2 + x_3 + \dots + x_n \leq K$$

$x_i \rightarrow$  subset/partition/bucket

$K \rightarrow$  Items to be distributed.

## Some Constraints:

→ Sometimes it can be

asked like 10 items

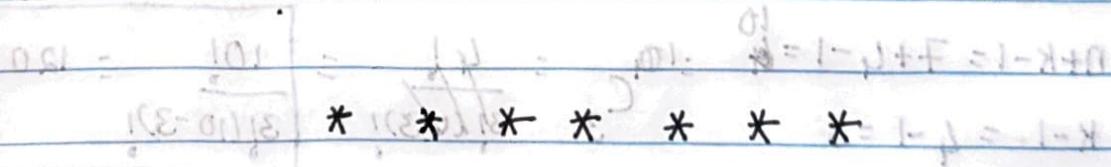
↳ All the items/ need to be "identical" with each unique id.

↳ No. of stars need to be constant. We no need to bother

↳ No. of bars need to be constant. about labelling. We need to bother only about

## Proof:

Divide "7" cookies among "4" children. How many way we can do that?



$K \rightarrow$  buckets/bars = 4

$n \quad * \rightarrow$  stars = 7

To divide into "4" buckets, we need " $K-1$ " =  $4-1=3$  bars.

\* \* \* \* \*

$$n=7 \quad k=4 \quad \text{Bars required} = k-1 = 4-1 = 3$$

We can deduce this problem into arranging "n" cookies and "k-1" kids.

So, total slots  $= n+k-1$

available for arranging

Now, we need to think about the problem in such a way that,

→ The cookies are already arranged (Since, they are identical, the order does not matter)

→ Find the number of ways in which bars can be placed so that it can be divided.

No. of combinations in which  $\binom{n+k-1}{k-1}$  =  $C$   
"k-1" bars can be placed among "n+k-1" positions

$$\begin{aligned} n+k-1 &= 7+4-1 = 10 \\ k-1 &= 4-1 = 3 \end{aligned} \quad \therefore C_3 = \frac{10!}{7!(10-7)!} = \boxed{\frac{10!}{3!7!}} = 120$$

### Constraint:

↳ Each kid gets at least "1" cookie each.

Pre allocate the cookies of "1" to each kid.

$$n = 7 \quad k = 4$$

After pre allocation = "4" cookies already selected

$$\text{Remaining cookies} = 7 - 4 = "3" \text{ cookies}$$

Now the problem can be deduced to "distributing 3 cookies to 4 children"

$$\text{That comes to } n = 3 \quad k = 4$$

$$\binom{n+k-1}{k-1} = \binom{3+4-1}{4-1} = \binom{6}{3}$$

= 20 combinations

Constraint:

$$n=7 \quad k=4$$

### Inclusion-Exclusion

↳ Each kid needs to have at least "1" cookie.

↳ 2<sup>nd</sup> kid can have at most "1" cookie.

Pre allocation = "1" for each kid = "4" cookies pre allocated.

$$\text{Remaining} = n - \text{pre allocated} = 7 - 4 = 3$$

Illegal State = 2<sup>nd</sup> Kid having  $\geq 2$  cookies.  $n=3, k=4$

$$\text{Total combinations} = \binom{n+k-1}{k-1} = \binom{4+3-1}{4-1} = \binom{6}{3} = 20 \text{ combinations}$$

↳ This will also contain illegal state where 2<sup>nd</sup> kid has  $\geq 2$  cookies. i.e.  $x_2 \geq 2$ .

Problem can be deduced to how "3" cookies can be distributed to "4" kids.

Calculating no. of illegal states:

How to induce illegal state? Few things to consider

↳ Already pre allocation is done. So, need to take 3 cookies only in consideration

↳ To induce illegal state, for 2<sup>nd</sup> kid,  $x_2 \geq 1$

↳ Pre allocate the extra cookie to the 2<sup>nd</sup> kid

Updated cookies =  $3 - 1 = 2$  (Done using minimum shift)

Now,  $n = 2$   $K = 4$  Total selections =  $n + K - 1 = 2 + 4 - 1 = 5$

$$\text{No. of combinations} = \binom{n+K-1}{K-1} = \binom{5}{3} = 5C_3 = 10 \text{ ways}$$

Originally: 7 cookies 4 kids  
7 (stars)  $K-1 = 3$  (bars)

- ↳ Out of "7"  $\rightarrow$  "3" pre allocated to accommodate at least 1 constraint
- ↳ Out of "3"  $\rightarrow$  "1" preallocated to find illegal states

$$\text{Original} = n + K - 1 = 7 + 4 - 1 = 10$$

$$\text{Already selected} = \underset{\text{(preallocate)}}{\cancel{3}} + \underset{\text{(induce illegal)}}{\cancel{1}} + \underset{= 5}{\cancel{4}} = 5$$

$$\text{Remaining cookies/stars} = 7 - 5 = 2$$

$$\text{Bars} = K - 1 = 4 - 1 = 3$$

Finally  $\rightarrow$  Total Combinations - Illegal Combinations  
 $= 20 - 10 = 10 \text{ ways}$