Stars and Bars

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↳ Helps in finding number of ways a given input can be divided into subsets/partitions with constraints

Mental model:

$$x_1 + x_2 + x_3 + \dots + x_n \leq K$$

x_i → subset/partition/basket

K → Items to be distributed.

Some Constraints:

↳ All the items/stars need to be "identical"

↳ No. of stars need to be constant

↳ No. of bars need to be constant.

→ Sometimes it can be asked like 10 items with each unique id. We no need to bother about labelling. We need to bother only about identical items.

Proof:

Divide "7" cookies among "4" children. How many way we can do that?

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K → buckets/bars = 4

n * → stars = 7

To divide into "4" buckets, we need "K-1" = 4-1 = 3 bars.

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$n = 7 \quad K = 4 \quad$ Bars required $= K-1 = 4-1 = 3$

We can deduce this problem into arranging "n" cookies and "k-1" kids.

So, total slots available for arranging $= n+K-1$

Now, we need to think about the problem in such a way that,

→ The cookies are already arranged (Since, they are identical, the order does not matter)

→ Find the number of ways in which bars can be placed so that cookies can be divided.

No. of combinations in which "k-1" bars can be placed among "n+k-1" positions $= \binom{n+K-1}{K-1} = {}^{n+K-1}C_{K-1}$

$n+K-1 = 7+4-1 = 10 \quad = {}^{10}C_3 = \frac{\cancel{4!}}{\cancel{3!(4-3)!}} = \boxed{\frac{10!}{3!(10-3)!} = 120}$

$K-1 = 4-1 = 3$

Constraint:

↳ Each kid gets at least "1" cookie each.

$\boxed{\text{MINIMUM SHIFT}}$

Pre allocate the cookies of "1" to each kid.

$n = 7 \quad K = 4$

After pre allocation = "4" cookies already selected

Remaining cookies = $7 - 4$ = "3" cookies

Now the problem can be deduced to "distributing 3 cookies to 4 children"

That comes to $n \Rightarrow 3 \quad K \Rightarrow 4$ $\quad \binom{n+K-1}{K-1} = \binom{3+4-1}{4-1} = \binom{6}{3}$

$= \boxed{20 \text{ combinations}}$

Constraint: $n=7 \quad K=4$

$\boxed{\text{Inclusion-Exclusion}}$

↳ Each kid needs to have at least "1" cookie

↳ 2^{nd} kid can have atmost "1" cookie.

Pre allocation = "1" for each kid = "4" cookies preallocated.

Remaing = n - preallocated = $7-4=3$

Illegal State : 2^{nd} kid having ≥ 2 cookies. $n=3 \quad K=4$

Total combinations = $\binom{n+k-1}{k-1} = \binom{4+3-1}{4-1} = \binom{6}{3} = 20$ combinations

↳ This will also contain $\boxed{\text{illegal state}}$ where "2^{nd}" kid has ≥ 2 cookies. ie $x_2 \geq 2$.

→ Problem can be deduced to how "3" cookies can be distributed to "4" kids.

Calculating no. of illegal states:

How to induce illegal state? Few things to consider

↳ Already pre allocation is done. So, need to take 3 cookies only in consideration

↳ To induce illegal state, for 2^{nd} kid, $x_2 \geq 1$

↳ Pre allocate the extra cookie to the 2^{nd} kid

Updated cookies = 3 − 1 = 2 (Done using minimum shift)

Now, $n = 2$ $\quad k = 4$ $\quad$ Total selections = $n + k - 1 = 2 + 4 - 1 = 5$

No. of combinations = $\binom{n+k-1}{k-1} = \binom{5}{3} = {}^{5}C_{3} = 10$ ways

Originally : 7 cookies $\quad$ 4 kids
7 (Stars) $\quad$ $k - 1 = 3$ (bars)

↳ Out of "7" → "3" pre allocated to accomodate at least 1 constraint
↳ Out of "3" → "1" preallocated to find illegal states

Original = $n + k - 1 = 7 + 4 - 1 = 10$
Already selected = $\underset{\text{(preallocate)}}{\cancel{3}4} + \underset{\text{(induce illegal)}}{1}$ $\cancel{4} = \cancel{6}$ $= 5$
Remaining cookies/stars = $7 - 5 = 2$
Bars $= k - 1 = 4 - 1 = 3$

Finally → Total Combinations − Illegal Combinations
$= 20 - 10 = 10$ ways