Robust Monetary Policy under Time-Varying Model Uncertainty

Bachelor's Thesis

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Abstract

This paper introduces time-varying uncertainty into a simple New Keynesian model where the central bank seeks a decision rule that is robust to model misspecification. The paper finds that variation in the central bank's concern for robustness leads to time-varying, nonnormally distributed impulse responses of output gap, inflation, and the interest rate. These predictions are confirmed by the impulse responses estimated from US quarterly data from 1954 to 2015. Quantitatively, the estimates confirm previous findings that a robust decision maker responds more aggressively than the central bank does empirically.

Table of Contents

List of Tables and Figures

1 Introduction

At a Jackson Hole Symposium, Carl E. Walsh (2003), an economics professor at the University of California, Santa Cruz, emphasized that "policy choices are made in the face of tremendous uncertainty about the true structure of the economy, the impact policy actions have on the economy, and even about the current state of the economy. […] uncertainty is pervasive" (pp. 297-298). Moreover, uncertainty is not constant but fluctuates over time. Changes in the structure of the economy and in the political environment do not occur in fixed intervals but are clustered. Periods of relative constancy are followed by periods of rapid change and uncertainty is inherent to them. Arguably, the past one-and-a-half years were such a period of rapid change, with substantial geopolitical and economic shifts occurring in Eastern Europe and the Middle East, for example. To central bankers the Ukrainian crises poses important questions about the future of Western relations to Russia, the economic contagion of sanctions, and about how persistent the shock will be (cf. Cœuré, 2014; Draghi, 2014). Yet, there are no definite answers to these questions. Another example of how fast uncertainty can build up and decrease is the recent turmoil in Greece that started with the electoral victory of Syriza, a left-wing party, and culminated in the referendum about the terms of another bailout on July 5, 2015. In face of the worsening economic conditions that resulted from the rising uncertainty about Greece's future, the ECB had to make important decisions about its emergency funding for Greek banks (cf. Cœuré, 2015). At the same time, it had to prepare for the worst case of a Greek exit from the Eurozone, an incident for which no past experience exists (cf. Cœuré, 2015).

Against this background, this paper examines the implications for the economy and monetary policy when a discretionary policy maker faces time-varying uncertainty. The planner's decision problem is framed in terms of the robust control framework developed by Hansen & Sargent (2008). To solve the decision problem, I use extensions proposed by Giordani & Söderlind (2004, pp. 2379–2380) for robust control under discretion. I modify their techniques to account for time-varying uncertainty. The paper does not assume that the reader is familiar with robust control as developed by Hansen, Sargent and coauthors. Hence, Section 2 outlines the problem of model uncertainty and summarizes some of the key concepts of robust control, such as entropy, the worst-case model, multiplier games, and detection error probabilities. These concepts will be used throughout the other sections. Similarly, Section 3 introduces the extensions of Giordani & Söderlind (2004, pp. 2379–2380) for forward-looking models under discretion.

The remainder of the paper is structured as follows. Section 4 develops a concept of time-varying uncertainty and shows how the solution algorithm of Giordani & Söderlind (2004, pp. 2379–2380) can be used to account for it. Section 5 applies the techniques presented in the previous two sections to a simple New Keynesian model and discusses the effects of time-varying uncertainty on monetary policy and the economy. The discussion focusses on the impulse responses of output gap, inflation, and the interest rate to supply and demand shocks. In Section 6 these predictions are compared with the impulse responses estimated from US quarterly data. Section 7 concludes.

2 An Introduction to Hansen-Sargent Robust Control

2.1 How to Deal with Model Uncertainty

Rational expectations models assume that all agents know the true model of the economy. In practice, however, economists treat their models as useful approximations of a true model that they do not – and often cannot – know (Greenspan, 2003, pp. 1–2; Onatski, 2008). Thus, the question arises how a decision maker such as the central bank can deal with the model misspecification and uncertainty that is inherent to its models. The most prominent approaches that have been proposed to this end are Bayesian ones and max-min approaches.

A Bayesian planner would deal with his uncertainty about the true model by forming a prior probability distribution over a set of alternative models. By combining the prior with data he attains a posterior probability distribution over all models. Hence, he reduces all uncertainty he faces to calculated risk by computing a weighted average across all models to arrive at a 'hypermodel', which he uses to minimize his expected loss (Giordani & Söderlind, 2004, p. 2369; Hansen & Sargent, 2008, Chapter 1.9). An early application of this approach to model uncertainty can be found in Brainard (1967) and a more recent one in Batini, Justiniano, Levine, & Pearlman (2006). As Hansen & Sargent (2008, Chapter 1.9) point out, the Bayesian approach requires that the decision maker is able to form a unique prior over all models, which – under certain assumptions that are beyond the scope of this paper (cf. Savage, 1954) – a rational person can always do.

However, there arguably arise issues when the Bayesian approach is applied to economics. First, the set of alternative models might be too large for the decision maker to be aware of all its members; let alone to have enough information to assign a unique probability to each member (Giordani & Söderlind, 2004, p. 2370; Hansen & Sargent, 2008, Chapter 1.9; Onatski, 2008). The latter is particularly difficult for rare outcomes or events for which no past experience exists (Walsh, 2003, p. 299), as in the example of Greece given in the introduction. In short, the decision maker might be unable to form a unique prior over the set of outcomes, that is, he might face uncertainty in the sense of Knight (1921, Chapter 8). Lastly, any rule that is based on a particular prior is geared towards this prior, whereas a decision maker might seek a rule that is robust to any prior (Hansen & Sargent, 2008, Chapter 1.9).

The unwillingness or inability of people to form a unique prior over possible outcomes in face of Knightian uncertainty is further supported by empirical evidence on an experiment conceived by Ellsberg (1961). Subsequently, Gilboa & Schmeidler (1989) provided a theoretical framework for non-unique priors by developing axioms under which it is fully rational for the decision maker to maintain multiple priors. The decision maker then solves for a decision rule using max-min expected utility: he maximizes his expected utility (minimizes his expected loss) with respect to the worst prior of his set of priors, that is, with respect to the prior that minimizes his expected utility (maximizes his expected loss).

2.2 Hansen-Sargent Robust Control

Another max-min approach was developed by Lars Hansen, Thomas Sargent, and coauthors, who apply and extend robust control – a branch of control theory, which has its origin in engineering and applied mathematics – to economics. At the center of their work, which is summarized in Hansen & Sargent (2008), is a planner who acknowledges the Knightian nature of his uncertainty and assumes that the true model is an unknown member of a set of unspecified models near his *approximating model*, a model that he thinks comes somewhat close to the true law of motion. He then seeks a rule that works well for all models of the set, i.e. all model that are not too far from his approximating model, instead of only working well if the approximating model correctly describes the data generating process. Here, ‘works well’ does not mean that the rule will be optimal for all members of the set of alternative models (Hansen & Sargent, 2008, p. 12). Instead, the decision maker wants a rule whose performance does not sharply drop if the true model turns out to be a model in the neighborhood of his approximating model rather than coinciding with it.

The planner finds such a rule by constructing a two-player game and using a max-min approach to solve the game. First, he installs a fictitious evil agent who chooses a model from the planner’s unspecified set of alternative models with the goal of maximizing the planner’s loss. Secondly, the planner chooses a policy function to minimize this loss. Note that the evil agent is only a computational aid of the planner. Therefore, the evil agent has the same approximating model and the same loss function as the planner, and the setting constitutes a zero-sum game (Hansen & Sargent, 2008, Chapter 2).

2.3 The Linear-Quadratic Case

For the remainder of the paper I focus on the important case where both the perturbed model, i.e. the model chosen by the evil agent, and the approximating model are linear, and where the planner's loss function is quadratic.[1] Stated similarly to Giordani & Söderlind (2004), the optimization problem for a linear-quadratic backward-looking model under discretionary policy making can be written as

$$\min_{\{u_j\}_{j=t}^{\infty}} \max_{\{v_{j+1}\}_{j=t}^{\infty}} E_t \sum_{j=t}^{\infty} \beta^{(j-t)} \left(x_j' Q x_j + u_j' R u_j + 2 x_j' U u_j \right), \tag{2.1}$$

$$\text{subject to } x_{t+1} = A x_t + B u_t + C(\epsilon_{t+1} + v_{t+1}), \tag{2.2}$$

$$E_t \sum_{\mathrm{j=t}}^{\infty} \beta^{(j-t)} v_{j+1}' v_{j+1} \leq \eta, \tag{2.3}$$

where x_t is a $(n \times 1)$ state vector with x_0 being given, where u_t is the planner's $(k \times 1)$ control vector, v_t is a $(n \times 1)$ vector of perturbations chosen by the evil agent, and where ϵ_{t+1} is a $(n \times 1)$ vector of zero-mean shocks that are normally i.i.d. with identity covariance matrix. Equation (2.1) is the planner's loss function where $0 < \beta < 1$ is his discount factor, and R and Q are symmetric matrices. Equation (2.2) is the law of motion, which corresponds to the approximating model for $v_{t+1} = 0$ and to a perturbed model when the evil agent's distortions are nonzero (Hansen & Sargent, 2008, p. 18). As Giordani & Söderlind (2004, p. 2371) point out, the evil agent's control vector v is indexed by $t+1$ although it is known in period t. Indexing the vector by $t+1$ emphasizes that the evil agent interferes by distorting the shocks ϵ. Note that the latter implies that uncertainty can only exist if the true model is at least partially concealed by random errors, that is, if $C \neq \mathbf{0}$. Moreover, equation (2.3) is the evil agent's intertemporal budget constraint, which limits the extent to which he can distort the planner's approximating model. Here, η is the budget of the evil agent with $0 \leq \eta < \bar{\eta}$, where $\bar{\eta}$ is a breakdown point to be defined later.

[1] Hansen and Sargent (2008, Chapter 3) show that the results obtained for more general, nonlinear models are similar to the ones obtained for linear models.

2.4 The Evil Agent's Budget Constraint

The evil agent's intertemporal budget η defines which models are 'not too far' from the planner's approximating model, i.e., for which models the planner seeks a rule that works well. Figure 1 illustrates that η can be interpreted as the radius of a ball around the approximating model, where the ball is the set of alternative models that the planner considers (Giordani & Söderlind, 2004, p. 2371). Put differently, any model on or within the ball is considered to be 'in the neighborhood' of the approximating model. This perspective highlights that there is an infinite number of ways how the approximating model can be disturbed (Hansen & Sargent, 2008, p. 3), and that η quantifies the planner's degree of uncertainty; the higher η, the less he trusts his approximating model and the larger are the distortions against which he seeks robustness. The limiting case $\eta = 0$ means the planner fully trusts his approximating model. Thus, $\eta = 0$ corresponds to the standard rational expectations case.

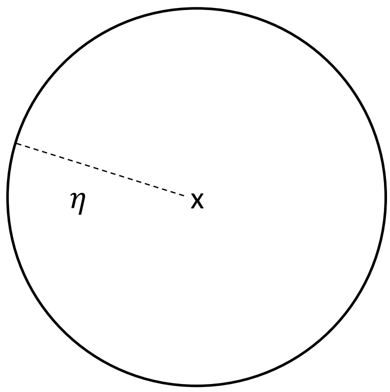

Figure 1. The planner's set of alternative models around his approximating model (x). Adapted from Hansen & Sargent (2008, p. 12).

Hansen & Sargent (2008) use conditional relative entropy, a measure of the statistical discrepancy between two probability distributions (Kullback & Leibler, 1951), to quantify the evil agent's budget η in a statistical way. To do so, note that the approximating model can be written as the one-period transition density

$$f_o(x_{t+1}|x_t) \sim \mathcal{N}(Ax_t + Bu_t, CC'), \tag{2.4}$$

while a perturbed model corresponds to the one-period transition density

$$f(x_{t+1}|x_t) \sim \mathcal{N}(Ax_t + Bu_t + Cv_{t+1}, CC'). \tag{2.5}$$

This formulation highlights that we can think of a model as a joint probability distribution, i.e. a particular prior, over sequences of outcomes (Hansen & Sargent, 2008, p. 8). From densities (2.4) and (2.5) we can construct the conditional relative entropy as

$$I(f_o, f)(x_t) = \int \ln\left(\frac{f(x_{t+1}|x_t)}{f_o(x_{t+1}|x_t)}\right) f(x_{t+1}|x_t) dx_{t+1}, \tag{2.6}$$

which is the amount of information lost when f_o is used to approximate f (Burnham & Anderson, 2002, p. 51). Moreover, an intertemporal version of conditional relative entropy can be written as

$$\mathcal{I}_t(f_o, f) = \sum_{j=\mathrm{t}}^{\infty} \beta^{(j-t)} \int \ln\left(\frac{f(x_{j+1}|x_t)}{f_o(x_{j+1}|x_t)}\right) f(x_{j+1}|x_t) dx_{j+1}. \tag{2.7}$$

For linear-quadratic settings with normally distributed shocks as in equations (2.1)–(2.3), Hansen & Sargent (2008, pp. 30, 146) show that equation (2.6) can be related to $v_{t+1}'v_{t+1}$. It follows that the evil agent's budget constraint (2.3) can be viewed as a limit on discounted conditional relative entropy (2.7) and hence on the statistical discrepancy between the approximating model and the perturbed models.[2] Another important implication of this relation is that the set of alternative models, represented by the ball around the approximating model in Figure 1, corresponds to a nondenumerable set of priors. In this way, the planner's set of alternative models in a robust control problem can be related to the multiple priors in Gilboa & Schmeidler (1989), as Hansen, Sargent, Turmuhambetova, & Williams (2006) show in detail.

2.5 The Worst-Case Model

Among the members of the set of alternative models, the worst-case model is of particular interest. Given a specific concern for robustness η,[3] the worst case for the planner is that the evil agent uses all his budget, i.e., his budget constraint holds with equality. Equivalently, worst-case models can be described as those models that lie on the boundary of the ball in Figure 1. Since the evil agent maximizes the planner's loss, the budget constraint is always

[2] Hansen & Sargent (2008) use this equivalence to derive a way to calibrate reasonable values for η. The technique will be introduced in Section 2.9.

[3] To avoid confusion, note that the terms 'planner's concern for robustness (to model misspecification)', 'planner's trust in his approximating model', and 'planner's uncertainty' all have the same meaning in the Hansen-Sargent framework and will be used interchangeably throughout this paper.

binding in a linear-quadratic setting, that is, the evil agent always chooses a worst-case model (Giordani & Söderlind, 2004, p. 2377). Thus, the worst-case model is also called ex-post model (Hansen & Sargent, 2012, p. 424).

2.6 Constraint and Multiplier games

The robust control problem in equations (2.1)–(2.3) can be stated in two equivalent ways, which Hansen & Sargent (2008, Chapter 7) call a constraint game and a multiplier game respectively. Equations (2.1)–(2.3) constitute a constraint game as the constraint on intertemporal entropy is stated explicitly via equation (2.3). The multiplier game can be obtained by applying a Lagrange multiplier theorem, which leads to

$$\min_{\{u_j\}_{j=t}^{\infty}} \max_{\{v_{j+1}\}_{j=t}^{\infty}} E_t \sum_{j=t}^{\infty} \beta^{(j-t)} \left(x_j' Q x_j + u_j' R u_j + 2x_j' U u_j - \theta v_{j+1}' v_{j+1} \right), \tag{2.8}$$

$$\text{subject to } x_{t+1} = Ax_t + Bu_t + C(\epsilon_{t+1} + v_{t+1}) \tag{2.9}$$

with $0 < \underline{\theta} < \theta < \infty$.[4] Here, intertemporal entropy is restricted implicitly by penalizing the time t contribution to intertemporal entropy (Hansen & Sargent, 2008, p. 141). The Lagrange multiplier θ represents the sensitivity of the planner's loss in the optimum to a marginal relaxation of the constraint (2.3) (Luenberger, 1969, p. 223) and can be interpreted as the shadow price of the evil agent's actions (cf. Hansen & Sargent, 2008, p. 39). As does the evil agent's budget η, the multiplier θ quantifies the planner's trust in his approximating model. In contrast to η, the multiplier θ is inversely related to the planner's trust: a lower θ corresponds to a higher concern for robustness while a lower η corresponds to a lower concern for robustness (Giordani & Söderlind, 2004, p. 2371). Note that $\eta = 0$ is equivalent to $\theta = \infty$, and both parameter values describe a decision maker who fully trusts his approximating model, which corresponds to a standard rational expectations model.

2.7 A Breakdown Point

Regardless of the form of the robust control problem, the planner's concern for robustness cannot be arbitrarily high. Instead, there exists a breakdown point, that is, a lower bound $\underline{\theta}$

[4] Note that η can be dropped from the term $\theta\left(v_{j+1}' v_{j+1} - \eta\right)$ because θ is fixed as it is chosen by the planner to express his concern for robustness to model misspecification (Hansen & Sargent, 2008, p. 128).

(corresponding to an upper bound $\bar{\eta}$) beyond which no additional robustness can be achieved because the evil agent is sufficiently unconstrained such that he can distort the law of motion in a way that pushes the planner's loss function to $-\infty$ regardless of the planner's control vector (Hansen & Sargent, 2008, p. 32). As Hansen & Sargent (2008, p. 40) point out, the breakdown point corresponds to the largest set of alternative models for which a robust rule can be obtained, and this rule will be robust to the largest feasible set of misspecifications. Moreover, the breakdown point is context-specific: it depends on the objective function, the discount factor, and the transition law.

2.8 Commitment and Discretion

While this paper focusses on discretionary policy making, a robust control problem for the optimal robust commitment policy can easily be constructed. To do so, Hansen & Sargent (2008, Chapter 7) condition the mathematical expectation in equation (2.1) on time zero information instead of time t information. This yields a Stackelberg game where both the planner and the evil agent commit to history-dependent sequences of controls in period zero. In contrast, in the discretionary setting the planner and evil agent choose their controls in each period. The optimal robust rule under discretion is obtained by solving for a Markov-perfect equilibrium, which will be discussed in more detail when solving the forward-looking model of Section 3.

2.9 Detection Error Probabilities

Hansen & Sargent (2008, Chapter 9) use the fact that the entropy penalty parameter θ limits the statistical discrepancy between the approximating model and the alternative models to calibrate reasonable values for θ given a data set of length T. The idea is to choose θ such that it is difficult to distinguish the approximating model and the worst-case model, which is the most different alternative model, statistically based on the available data. To quantify the meaning of 'difficult to distinguish' they map θ to the probability of making a model detection error, i.e., the probability of attributing observed data to the worst-case model although it was generated by the approximating model or vice versa.

Hansen & Sargent (2008, Chapter 9.3) calculate this detection error probability via a likelihood-ratio test. They assume that the planner has used maximum likelihood to estimate a model for a given sample of size T, where the sample consists of observations on the $(n \times 1)$

state vector x_t for $t = 0,1,\dots,T$. Let A denote the approximating model and let B denote the worst-case model that is implied by θ. Moreover, L_i denotes the likelihood of the sample for model i. A likelihood-ratio test selects model A if the log-likelihood ratio $\ln\frac{L_A}{L_B}$ is above zero and model B if $\ln\frac{L_A}{L_B}$ is below zero. Thus, the probability of attributing the observed data to the worst-case model although it was generated by the approximating model is

$$p_A = Prob\left(\ln\frac{L_A}{L_B} < 0 \middle| A\right). \tag{2.10}$$

Similarly, the detection error probability is

$$p_B = Prob\left(\ln\frac{L_A}{L_B} > 0 \middle| B\right) \tag{2.11}$$

when the data is generated by the worst-case model. To obtain the overall detection error probability $p(\theta)$, Hansen & Sargent (2008, Chapter 9.3) average p_A and p_B under the assumption that models A and B are equally likely to be the data generating process, that is,

$$p(\theta) = 0.5(p_A + p_B). \tag{2.12}$$

Ceteris paribus, the probability of a model detection error is lower the more different the approximating and the worst-case model are, i.e., the higher the planner's concern for robustness is. Equivalently, one finds that $p(\theta)$ is increasing in θ. Under rational expectations ($\theta = \infty$) the detection error probability is 0.5 because approximating and worst-case model coincide. Besides, the detection error probability is context specific: the detection error probability that corresponds to a specific θ depends on the objective function, the discount factor, the approximating model, and the sample size. Naturally, the detection error probability is decreasing in the sample size. For a given context, a reasonable value for θ is found by first setting $p(\theta)$ to a plausible probability – Hansen & Sargent (2008, pp. 219, 320) suggest 0.1 to 0.2 as a lower bound of this probability – and then inverting $p(\theta)$.

By restraining the possible values of the free parameter θ such that it corresponds to a detection error probability above 0.1 for the worst-case model, Hansen & Sargent (2008, p. 16) ensure that any alternative model and the approximating model fit the data almost equally well. Consequently, although the planner bases his decision on the worst-case model, he does not think that it is the true model and he will not be surprised that the observed values of the state x_t will generally differ from the ones implied by the worst-case model. Instead of believing in the worst-case model, the planner views it as a computational aid: a cautious but not overcautious inference about the true law of motion in order to attain a robust decision rule (cf. Hansen & Sargent, 2012, p. 432). Another implication of the statistical similarity of the alternative models is that the decision maker cannot simply learn the true model in the course of time. To do so, he would need a large sample to distinguish the models empirically. However, waiting for this data to arrive is not a feasible solution for the decision maker because his discount factor makes him sufficiently impatient (Hansen & Sargent, 2008, pp. 17, 213). In many cases, any learning is further impeded by the fact that the true model is not constant but changes over time (Macklem, 2001, p. 231).

3 Robust Control in Forward-Looking Models under Discretion

The previous section introduced some of the key concepts of Hansen-Sargent robust control such as entropy, the worst-case model, constraint and multiplier games, the breakdown point, and detection error probabilities. It did so in a setting that consisted of a single decision maker and a backward-looking law of motion. In this section, I turn to a multi-agent setting with two forward-looking agents: a planner and a private sector. After discussing the key assumptions, I outline the solution algorithm that Giordani & Söderlind (2004, p. 2392) use to obtain the planner's optimal robust discretionary rule. The algorithm serves as a basis for modifications and simulations in later sections.

3.1 The Setting

In multi-agent settings, each agent's set of models and his beliefs about the other agents' sets of models need to be specified (Hansen & Sargent, 2012, p. 422). As for the two-agent forward-looking model to be discussed in this section, I follow Hansen & Sargent (2012) and assume that the private sector fully trusts its model, i.e., it views its model as the true law of motion.[5] Moreover, the private sector thinks that the planner's decision rule is a function of the current state. In contrast, the planner does not know the true model. Instead, he entertains an approximating model which he does not trust. Furthermore, he does not know the private sector's model either, but he presumes that it coincides with the true model.[6] As a result, the

[5] Hansen & Sargent (2012) make this assumption and the following ones for a Stackelberg game, but they can easily be adapted to the discretionary case considered in this paper.

[6] The setting described here mirrors Hansen & Sargent's (2012) type I ambiguity.

planner evaluates his loss function and the private sector's expectations using the same worst-case probability distribution (Hansen & Sargent, 2012, p. 423).[7] However, this does not mean that the planner thinks the worst-case model correctly describes the true or the private sector's model (Hansen & Sargent, 2012, p. 432). Instead, the worst-case model is only a device to obtain an optimally robust decision rule.

However, while Hansen & Sargent (2008, Chapter 16, 2012) model a Stackelberg game, I assume that the planner cannot commit. As Giordani & Söderlind (2004, p. 2379) point out, this is a common assumption in models for monetary policy, such as the one considered in Section 5. It is supported by the fact that currently no central bank has adopted a commitment policy except for a fixed exchange rate (J. Tetlow & von zur Muehlen, 2001, p. 913).

Stated similarly to Giordani & Söderlind (2004), the planner's decision problem in time t for a two-agent forward-looking model can be written as

$$\min_{\{u_j\}_{j=t}^{\infty}} \max_{\{v_{j+1}\}_{j=t}^{\infty}} E_t \sum_{j=t}^{\infty} \beta^{(j-t)} \left(x_j' Q x_j + u_j' R u_j + 2 x_j' U u_j - \theta v_{j+1}' v_{j+1} \right), \tag{3.1}$$

$$\text{subject to } \begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = A \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + B u_t + C(v_{t+1} + \epsilon_{t+1}), \tag{3.2}$$

$$\text{with } x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}, C = \begin{bmatrix} C_1 \\ \mathbf{0}_{n_2 \times n_1} \end{bmatrix}, \tag{3.3}$$

where x_{1t} is a $(n_1 \times 1)$ vector of backward-looking state variables with x_{10} being given, and where x_{2t} is a $(n_2 \times 1)$ vector of forward-looking variables. Shocks only affect the backward-looking variables such that ϵ_{t+1} is a $(n_1 \times 1)$ vector of zero-mean shocks that are normally i.i.d. with identity covariance matrix. Since the evil agent interferes by distorting the shocks, his control v_{t+1} is a $(n_1 \times 1)$ vector, too. As before, u_t is the planner's $(k \times 1)$ control vector, $0 < \beta < 1$ is the planner's discount factor, and R and Q are symmetric.

[7] Giordani & Söderlind (2004, pp. 2374, 2379) use different assumptions to justify this result. They assume that the planner and the private sector have the same approximating model, loss function, and concern for robustness. However, Hansen & Sargent (2012, p. 432) emphasize that even if this is the case, the planner and the private sector will very likely face different constraints such that they will still obtain different worst-case models. Thus, I follow Hansen and Sargent and use the assumptions stated above. Since both sets of assumptions motivate the same result, Giordani & Söderlind's (2004, pp. 2379–2380) solution algorithm, which I describe below, can be used regardless of the differently framed assumptions.

3.2 A Solution Algorithm

To solve for the optimal robust discretionary rule, Giordani & Söderlind (2004) transform the model such that the rational expectations techniques used by Backus & Driffill (1986, pp. 14–16) can be applied. First, write equations (3.1)–(3.3) as

$$\min_{\{u_j\}_{j=t}^{\infty}} \max_{\{v_{j+1}\}_{j=t}^{\infty}} E_t \sum_{j=t}^{\infty} \beta^{(j-t)} \left(x_j' Q x_j + u_j^{*\prime} R^* u_j^* + 2 x_j' U^* u_j^* \right), \tag{3.4}$$

$$\text{subject to} \begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = A \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + B^* u_t^* + C\epsilon_{t+1}, \tag{3.5}$$

$$\text{with } x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}, R^* = \begin{bmatrix} R & \mathbf{0}_{k\times n_1} \\ \mathbf{0}_{n_1\times k} & -\theta \boldsymbol{I}_{n_1} \end{bmatrix}, u_t^* = \begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix},$$

$$B^* = [B \quad C], C = \begin{bmatrix} C_1 \\ \mathbf{0}_{n_2\times n_1} \end{bmatrix}, U^* = [U \quad \mathbf{0}_{(n_1+n_2)\times n_1}], \tag{3.6}$$

where x_{10} is given. Under discretion one solves for a Markov-perfect equilibrium, which is a subgame-perfect equilibrium where the agents' decision rules only depend on the current state (cf. Ljungqvist & Sargent, 2004, p. 198; Maskin & Tirole, 2001). Thus, expectations of x_{2t+1} must depend only on x_{1t+1}, i.e.,

$$E_t x_{2t+1} = K_{t+1} E_t x_{1t+1}, \tag{3.7}$$

where K_{t+1} is a $(n_2 \times n_1)$ matrix to be defined below. Now, the Bellman equation for the optimization problem (3.4)–(3.6) can be written as

$$x_{1t}' V_t x_{1t} + w_t = \min_{u_t^*} \max \left[x_{1t}' \tilde{Q} x_{1t} + u_t^{*\prime} \tilde{R} u_t^* + 2 x_{1t}' \widetilde{U} u_t^* + \beta E_t (x_{1t+1}' V_{t+1} x_{1t+1} + w_{t+1}) \right]$$

$$\text{s. t. } x_{1t+1} = \tilde{A}_t x_{1t} + \tilde{B}_t u_t^* + C_1 \epsilon_{t+1}, \tag{3.8}$$

where the time t value function is $x_{1t}' V_t x_{1t} + w_t$ with w_t being a constant and x_{1t} given, and where

$$\begin{aligned}
D_t &= (A_{22} - K_{t+1} A_{12})^{-1} (K_{t+1} A_{11} - A_{21}), \\
G_t &= (A_{22} - K_{t+1} A_{12})^{-1} (K_{t+1} B_1^* - B_2^*), \\
\tilde{A}_t &= A_{11} + A_{12} D_t, \\
\tilde{B}_t &= B_1^* + A_{12} G_t, \\
\tilde{Q}_t &= Q_{11} + Q_{12} D_t + D_t' Q_{21} + D_t' Q_{22} D_t, \\
\widetilde{U}_t &= Q_{12} G_t + D_t' Q_{22} G_t + U_1^* + D_t' U_2^*, \\
\tilde{R}_t &= R^* + G_t' Q_{22} G_t + G_t' U_2^* + U_2^{*\prime} G_t.
\end{aligned} \tag{3.9}$$

The Bellman equation and the constraint in (3.8) yield the first order condition

$$u_t^* = -F_{1t}x_{1t}, \text{ with } F_{1t} = \left(\tilde{R}_t + \beta \tilde{B}_t'{}' V_{t+1}\tilde{B}_t\right)^{-1}\left(\tilde{U}_t' + \beta \tilde{B}_t'{}' V_{t+1}\tilde{A}_t\right). \tag{3.10}$$

It follows that

$$x_{2t} = K_t x_{1t}, \text{ with } K_t = D_t - G_t F_{1t}, \text{ and} \tag{3.11}$$

$$V_t = \tilde{Q} + F_{1t}'\tilde{R}F_{1t} - \tilde{U}F_{1t} - F_{1t}'\tilde{U}' + \beta\left(\tilde{A}_t - \tilde{B}_t F_{1t}\right)' V_{t+1}\left(\tilde{A}_t - \tilde{B}_t F_{1t}\right). \tag{3.12}$$

In an equilibrium K_t and F_{1t} must be constant, which allows Giordani & Söderlind (2004, p. 2394) to solve for K and F_1 by iterating backwards to convergence on equations (3.9)–(3.12). If an equilibrium exists, i.e. if K_{t+1} and F_{1t} do converge, the agent's control vectors are given by

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} = -F_1 x_{1t}, \text{ with } F_1 = \begin{bmatrix} F_u \\ F_v \end{bmatrix}, \tag{3.13}$$

where F_u is a $(k \times n_1)$ matrix and F_v is a $(n_1 \times n_1)$ matrix. Note that both rules are not history dependent but depend only on the current state. Moreover, the forward-looking variables, as a function of the backward-looking variables, can be stated as

$$x_{2t} = Kx_{1t}. \tag{3.14}$$

Lastly, the transition law of the state variable is

$$x_{1t+1} = Mx_{1t} + C_1\epsilon_{t+1}, \tag{3.15}$$

where $M = A_{11} + A_{12}K - B_1^*F_1$ for the worst-case model and $M = A_{11} + A_{12}K - B_1F_u$ for the approximating model. The latter follows from the fact that $v_{t+1} = 0$ under the approximating model.

When solving for the optimal robust rule it is important to ensure that θ is above the breakdown point $\underline{\theta}$. This can be done by checking whether V_t is positive definite, i.e. all its eigenvalues are greater than zero, after each iteration (cf. Giordani & Söderlind, 2004, p. 2394).

The algorithm presented in this section will be used in Section 5 to simulate a simple New Keynesian model. First, however, I modify it to account for a planner whose trust in his approximating model varies over time.

4 Modelling Time-Varying Uncertainty

The examples of Greece and Ukraine in the introduction are but two that illustrate how rapidly the economic and political environment can fundamentally change and thereby cause the planner's uncertainty to change. Against this background, this section shows how time-varying uncertainty can be introduced in the two-agent forward-looking model of the previous section. To do so, it first discusses the interpretation of time-varying uncertainty in the Hansen-Sargent framework. It then states key assumptions that characterize the type of uncertainty variation that I consider in this paper. The section then moves to discussing the impact of time-varying uncertainty on the solution algorithm presented in the previous section. Lastly, detection error probabilities are used to discipline how uncertainty can evolve over time, and a stochastic process for its evolution is proposed. The methods developed in this section will be applied to a simple New Keynesian model in Section 5.

4.1 Time-Varying Uncertainty in Robust Control

In the Hansen-Sargent robust control framework, time-varying model uncertainty can be framed as a scenario where the parameter η, or equivalently θ, varies over time. Instead of being constant, the planner's trust in his approximating model is given by a stochastic processes $\{\theta_t\}$ or $\{\eta_t\}$ respectively. In terms of the constraint game, the setting can be described as one where the evil agent's budget η changes over time. As illustrated in Figure 2, this corresponds to the radius of the ball being time-varying such that the allowed statistical discrepancy fluctuates. This implies that the set of alternative models, i.e. which models are considered to be in the neighborhood of the approximating model, changes over time. Consequently, the worst-case model varies as well.[8] Note that variation in the worst-case model is also possible under a constant concern for robustness. In this case, however, any variation requires changes in the approximating model, the loss function, or the planner's discount factor, whereas I assume that all of them are constant.

In terms of the multiplier game, time-varying uncertainty corresponds to variation in the shadow price θ of the evil agent's actions. Due to the equivalence of the constraint and multiplier game discussed in Section 2, a time-varying shadow price has the same impact as variation in the evil agent's budget. Besides, the breakdown point $\bar{\eta}$, or equivalently $\underline{\theta}$, does

[8] An alternative approach to induce time-varying worst-case models in the Hansen-Sargent framework can be found in Bidder & Smith (2012).

not change since it only depends on the approximating model, the loss function, and the discount factor, all of which I assume to be constant.

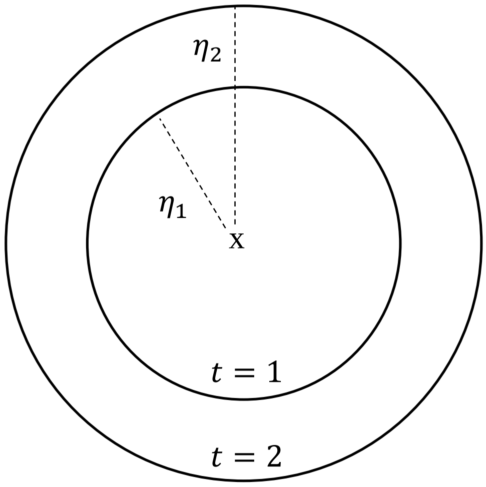

Figure 2. The set of alternative models around the approximating model (x) under time-varying uncertainty.

4.2 Key Assumptions

Several key assumptions help to specify the types of uncertainty variation that I consider. First, I assume that the planner's concern for robustness may vary exogenously, i.e., unrelated to economic fundamentals. This feature is shared by the confidence shocks in Ilut & Schneider (2014), the sentiment shocks in Angeletos & La'O (2013), sunspots (see e.g. Shell, 2008), stochastic bubbles (Martin & Ventura, 2012), and exogenous shocks in learning (see e.g. Bullard, Evans, & Honkapohja, 2008; Milani, 2011). Secondly, I presume that a loss in trust in the approximating model is not necessarily followed by worse economic outcomes. I do so because the planner's concern for robustness and the worst-case model implied by it are only computational devices, not predictors of future economic fundamentals. The latter also holds for the confidence shocks in Ilut & Schneider (2014). As they point out, this distinguishes confidence shocks from signals (or 'news') about future economic outcomes, which are on average followed by outcomes that validate the signal. Finally, as it is the case with a constant concern for robustness, I assume that the decision maker knows the degree to which he currently trusts his approximating model. This means that η_t and θ_t are known conditional on time t information.

4.3 A Modified Solution Algorithm

When the planner's concern for robustness varies over time, the Bellman equation for the two-agent forward-looking model from Section 3 can be stated as

$$x_{1t}'V_t x_{1t} + w_t = \min_{u_t^*}\max\left[x_{1t}'\tilde{Q}x_{1t} + u_t^{*\prime}\tilde{R}u_t^* + 2x_{1t}'\tilde{U}u_t^* + \beta E_t(x_{1t+1}'V_{t+1}x_{1t+1} + w_{t+1})\right]$$
$$\text{s. t } x_{1t+1} = \tilde{A}_t x_{1t} + \tilde{B}_t u_t^* + C_1\epsilon_{t+1}, \tag{4.1}$$

where the matrices with a tilde are given by (3.9), except that R^* in the equation $\tilde{R}_t = R^* + G_t'Q_{22}G_t + G_t'U_2^* + U_2^{*\prime}G_t$ is given by

$$R_t^* = \begin{bmatrix} R & \mathbf{0}_{k\times n_1} \\ \mathbf{0}_{n_1\times k} & -\theta_t \boldsymbol{I}_{n_1}\end{bmatrix}, \tag{4.2}$$

which varies exogenously over time, driven by the process we assume for $\{\theta_t\}$. Thus, the matrix

$$F_{1t} = \left(\tilde{R}_t + \beta\tilde{B}_t'{}'V_{t+1}\tilde{B}_t\right)^{-1}\left(\tilde{U}_t' + \beta\tilde{B}_t'{}'V_{t+1}\tilde{A}_t\right), \tag{4.3}$$

in the decision rule $u_t^* = -F_{1t}x_{1t}$ will also vary exogenously such that F_{1t} will not converge to a constant when iterating backwards. To obtain a decision rule nonetheless, note that in period t the decision maker only needs a rule that is optimally robust for period t. This is so because discretionary policy making allows the planner to update his decision rule in the next period such that it reflects his contemporaneous trust in the approximating model. He can find such a rule by treating θ as a constant being equal to his current concern for robustness, θ_t, and iterating backwards to convergence. Any rule obtained in this way will necessarily be optimal for the current period t. In period $t+1$, the planner repeats this process with θ equalling his contemporaneous concern for robustness, θ_{t+1}, in order to obtain a rule that is optimally robust for period $t+1$. By updating his decision rule each period, the planner will use the optimal robust rule for his contemporaneous concern for robustness in each period. Thus, his behavior can be described by a sequence of decision rules where the only variation in the rules is due variations in θ. Nonetheless, one cannot express the planner's decision rule explicitly as a function of θ because the convergence dynamics of K_t and F_{1t} change with θ as well, which is in line with the Lucas critique[9] (Lucas, 1976).

[9] In his famous paper, Robert E. Lucas, Jr. (1976) criticized contemporaneous macroeconomic policy evaluation because it did not take into account that the optimal decision rules of agents vary when the policy regime changes.

4.4 Modelling the Process $\{\boldsymbol{\theta_t}\}$

I do not allow θ to vary arbitrarily. As in the case of a constant concern for robustness, detection error probabilities can be used to discipline the values that θ can assume. I propose to specify the stochastic process $\{\theta_t\}$ such that it has a lower bound $\tilde{\theta}$ which fulfils two conditions. First, the lower bound has to ensure that all realizations of $\{\theta_t\}$ are above the breakdown point $\underline{\theta}$, i.e., $\tilde{\theta} > \underline{\theta}$. This condition is required because for θ below the breakdown point, the evil agent is sufficiently unconstrained such that he can distort the law of motion in a way that pushes the planner's loss function to $-\infty$ regardless of the planner's control vector (cf. Section 2.7; Hansen & Sargent, 2008, p. 32). Secondly, the lower bound $\tilde{\theta}$ has to correspond to a detection error probability above 0.1. Otherwise, the approximating model and the worst-case model are statistically too different such that the planner would very likely be able to distinguish between them based on the data at hand (cf. Section 2.9; Hansen & Sargent, 2008, p. 219).

Definition 4.1. A scalar $\tilde{\theta}$ is said to be a feasible lower bound if (i) $\tilde{\theta} > \underline{\theta}$, and (ii) $p(\tilde{\theta}) > 0.1$ where $p(\theta)$ is the detection error probability (2.12).

In contrast, it is not necessary to introduce an upper bound for $\{\theta_t\}$. For example, Ellison & Sargent (2015) work with detection error probabilities between 0.4 and 0.5 (recall that 0.5 is the largest possible detection error probability and corresponds to standard rational expectations). Note that using detection error probabilities to specify $\{\theta_t\}$ means that the appropriate values for the process's parameters will depend on the sample size for which the detection error probabilities are calculated (cf. Section 2.9).

Against the background of the lower bound requirement, I propose to model $\{\theta_t\}$ as a shifted lognormal process

$$\theta_t - \tilde{\theta} = \exp(a_t), \tag{4.4}$$

where $\tilde{\theta}$ is a feasible lower bound and $\{a_t\}$ is a stationary Gaussian process with mean $\bar{a}$ and variance σ_a^2. It is straightforward to verify that the lower bound requirement is fulfilled. Because the right-hand side of equation (4.4) is always positive, θ_t is always larger than $\tilde{\theta}$.

Moreover, stationarity of $\{a_t\}$ implies that $\{\theta_t\}$ is also stationary and thus has constant mean and variance, which I define as $\bar{\theta}$ and σ_θ^2 respectively. Hence,

$$E(\theta_t) = \tilde{\theta} + \exp(\bar{a} + 0.5\sigma_a^2) \equiv \bar{\theta}, \text{ and} \tag{4.5}$$

$$Var(\theta_t) = \exp(2\bar{a} + \sigma_a^2)(\exp(\sigma_a^2) - 1) \equiv \sigma_\theta^2. \tag{4.6}$$

Since detection error probabilities are expressed as a function of θ, it is convenient to specify $\bar{a}$ and σ_a^2 in terms of the mean and variance of $\{\theta_t\}$. To do so, one can use equations (4.5) and (4.6) to obtain

$$\bar{a} = \ln \frac{\bar{\theta}-\tilde{\theta}}{\sqrt{1+\sigma_\theta^2/(\bar{\theta}-\tilde{\theta})^2}}, \text{ and} \tag{4.7}$$

$$\sigma_a^2 = \ln\left(1 + \sigma_\theta^2/(\bar{\theta} - \tilde{\theta})^2\right). \tag{4.8}$$

4.5 Introducing Persistence

A benefit of modelling $\{\theta_t\}$ as in (4.4) is that it enforces the lower bound $\tilde{\theta}$ regardless of the properties imposed on $\{\theta_t\}$ via $\{a_t\}$. This allows me to easily include persistence in the planner's uncertainty, such as in Ilut & Schneider (2014, p. 2375). I follow them and assume that uncertainty variation is driven by an AR(1) process of the form

$$a_t = \rho\bar{a} + (1-\rho)a_{t-1} + \sigma_Z Z_t, \tag{4.9}$$

where Z_t is a standard normally i.i.d. shock and where the persistence parameter ρ is chosen such that $0 < \rho < 1$. The latter ensures that $\{a_t\}$ is stationary.

Since it is the distribution of θ_t that is ultimately of interest, I specify the parameters of $\{a_t\}$ in terms of the mean and variance of $\{\theta_t\}$. To do so, note that stationarity implies that $\{a_t\}$ has a MA(∞) representation. It follows that $\{a_t\}$ is a Gaussian process with mean $\bar{a}$ and variance $\sigma_a^2 = \sigma_Z^2/(1-\rho^2)$. While $\bar{a}$ is given by equation (4.8) as before, we can obtain an expression for σ_Z^2 by equating $\sigma_a^2 = \sigma_Z^2/(1-\rho^2)$ and equation (4.7). This yields

$$\sigma_Z^2 = (1-\rho^2)\ln\left(1 + \sigma_\theta^2/(\bar{\theta} - \tilde{\theta})^2\right). \tag{4.10}$$

Taken together, $\{\theta_t\}$ can be modelled as

$$\theta_t - \tilde{\theta} = \exp(\rho\bar{a} + (1-\rho)a_{t-1} + \sigma_Z Z_t), \tag{4.11}$$

where $\bar{a}$ and σ_Z^2 are described by equations (4.7) and (4.10) with $\tilde{\theta}$, $\bar{\theta}$, σ_θ^2, and ρ being given. The latter highlights the costs of modelling uncertainty variation as proposed: Instead of a single free parameter, θ, there are now four parameters. However, by restricting the parameters based on the detection error probabilities, the parsimony of the Hansen-Sargent approach is at least partially maintained.

In this section, I have shown how it is possible to account for time-varying uncertainty in the forward-looking model of Section 3. I have proposed a shifted lognormal process to model variation in the planner's uncertainty and have shown how persistence can be included in this process. Ultimately, the process drives variation in the planner's decision rule such that his behavior is given by a sequence of decision rules. The planner attains these rules by solving each period for a rule that is optimally robust for period t given his contemporaneous concern for robustness. He does so by viewing θ as a constant equal to his time t degree of uncertainty and using the algorithm described in Section 3. The next section discusses the simulation results of introducing time-varying uncertainty into a simple New Keynesian model.

5 Time-Varying Uncertainty in a Simple New Keynesian Model

In this section I use a standard New Keynesian model as in Clarida, Gali, & Gertler (1999, p. 1665) to illustrate the implications for monetary policy and the economy when the central bank's trust in its approximating model varies over time. There are two key findings. First, time-varying uncertainty causes the impulse response functions to vary over time. Secondly, the impulse responses are nonnormally distributed. To establish a benchmark, the section first summarizes the results of robust decision making in the simple case that the central bank's trust is constant. It then moves to discussing the results obtained for time-varying uncertainty.

The New Keynesian model consists of an IS curve and a Calvo-style Phillips curve, and can be stated as

$$y_t = E_t y_{t+1} - \gamma(i_t - E_t \pi_{t+1}) + e_{1t}, \tag{5.1}$$

$$\pi_t = \beta E_t \pi_{t+1} + \alpha y_t + e_{2t}, \tag{5.2}$$

$$e_{1t} = \rho_1 e_{1t-1} + \xi_{1t}, \tag{5.3}$$

$$e_{2t} = \rho_2 e_{2t-1} + \xi_{2t}, \tag{5.4}$$

where y_t is the output gap, i_t is the nominal interest rate, and π_t is inflation. Moreover, it is $0 < \rho_1, \rho_2 < 1$, and the demand shock ξ_{1t} and the supply shock ξ_{2t} are normally i.i.d. random variables with zero mean and variances σ_1^2 and σ_2^2 respectively. The central bank sets the interest rate i_t with the goal of minimizing the loss function

$$E_t \sum_{j=t}^{\infty} \beta^{(j-t)} \left(\pi_j^2 + \lambda_y y_j^2 + \lambda_i i_j^2\right). \tag{5.5}$$

As for the parameter values used in the simulation, I follow Giordani & Söderlind (2004) and set $\alpha = 0.645$, $\beta = 0.99$, $\gamma = 0.5$, $\rho_1 = \rho_2 = 0.5$, $\sigma_1 = \sigma_2 = 1$, $\lambda_y = 0.5$, and $\lambda_i = 0.2$. Moreover, the central bank's concern for robustness evolves according to

$$\theta_t - \tilde{\theta} = \exp(\rho \bar{a} + (1-\rho) a_{t-1} + \sigma_Z Z_t), \tag{5.6}$$

where $\rho = 0.5$ and Z_t is a standard normally i.i.d. random variable. For the given parameter values, a feasible lower bound is $\tilde{\theta} = 53$, as this corresponds to a detection error probability of about 0.1 when the detection error probabilities are estimated by averaging over 10,000 samples of 244 observations.[10] Moreover, $\tilde{\theta} = 53$ is above the breakdown point $\underline{\theta}$, which can be calculated numerically as approximately 46.0365.[11] Furthermore, it seems reasonable to set the expected value of θ_t, $\bar{\theta}$, to 77 as this corresponds to a detection error probability of about 0.25. Using the detection error probabilities, I then set the standard deviation of θ_t, σ_θ, to 70 such that about 96% of all realizations are expected to represent detection error probabilities of 40% or less. Table 1 summarizes the parameter values. Note that $\bar{a}$ and σ_Z^2 can be calculated via equations (4.7) and (4.10) respectively. The model represented by equations (5.1) to (5.6) is simulated using the techniques outlined in Section 3 and Section 4.

[10] This number of observations is chosen for better comparability with the empirical results presented in Section 6.

[11] The MATLAB program Breakdown.m does this.

Parameter	Description	Value
α	Coefficient of output gap in Phillips curve	0.645
β	Discount factor	0.99
γ	Intertemporal elasticity of substitution in IS curve	0.5
ρ_1, ρ_2	Persistence of supply and demand shocks	0.5
σ_1, σ_2	Std. dev. of supply and demand shocks	1
λ_y	Weight of output gap in planner's loss function	0.5
λ_i	Weight of interest rate in planner's loss function	0.2
ρ	Persistence of AR(1) process that drives uncertainty	0.5
$\tilde{\theta}$	Lower bound of $\{\theta_t\}$	53
$\bar{\theta}$	Unconditional expectation of $\{\theta_t\}$	77
σ_θ	Std. dev. of $\{\theta_t\}$	70

Table 1. Summary of parameters used for simulations.

5.1 Simulation with Constant Uncertainty ($\boldsymbol{\sigma_\theta = 0}$)

In this subsection I briefly discuss the impact on monetary policy and the economy when the central bank has a constant concern for robustness to model misspecification. Similarly to Giordani & Söderlind (2004, pp. 2380–2381), I simulate the impulse responses of output gap, inflation, and the interest rate to a cost-push shock of one standard deviation. The shock occurs in period zero and the responses are calculated up to period four with one period representing one year. The following results were computed by averaging across 125 simulations.[12] Each quadrant of Figure 3 displays the impulse response of a variable for the rational expectations case ($\theta = \infty$), the approximating model, and the worst-case model.

When the central bank is concerned about model misspecification, the responses of all variables are higher in absolute value across all periods. The reason is that the evil agent chooses to distort the approximating model by increasing the persistence of the shock, as Figure 3 shows. This means that central bank bases its policy on a scenario where shocks are more persistent and consequently responds more aggressively to the shock than in the rational expectations case (cf. Giordani & Söderlind, 2004, p. 2381; Leitemo & Söderström, 2008). Similar results can be obtained for a demand shock, as Figure A.1 in Appendix A shows.

[12] As before, this number is chosen for better comparability with the empirics in Section 6.

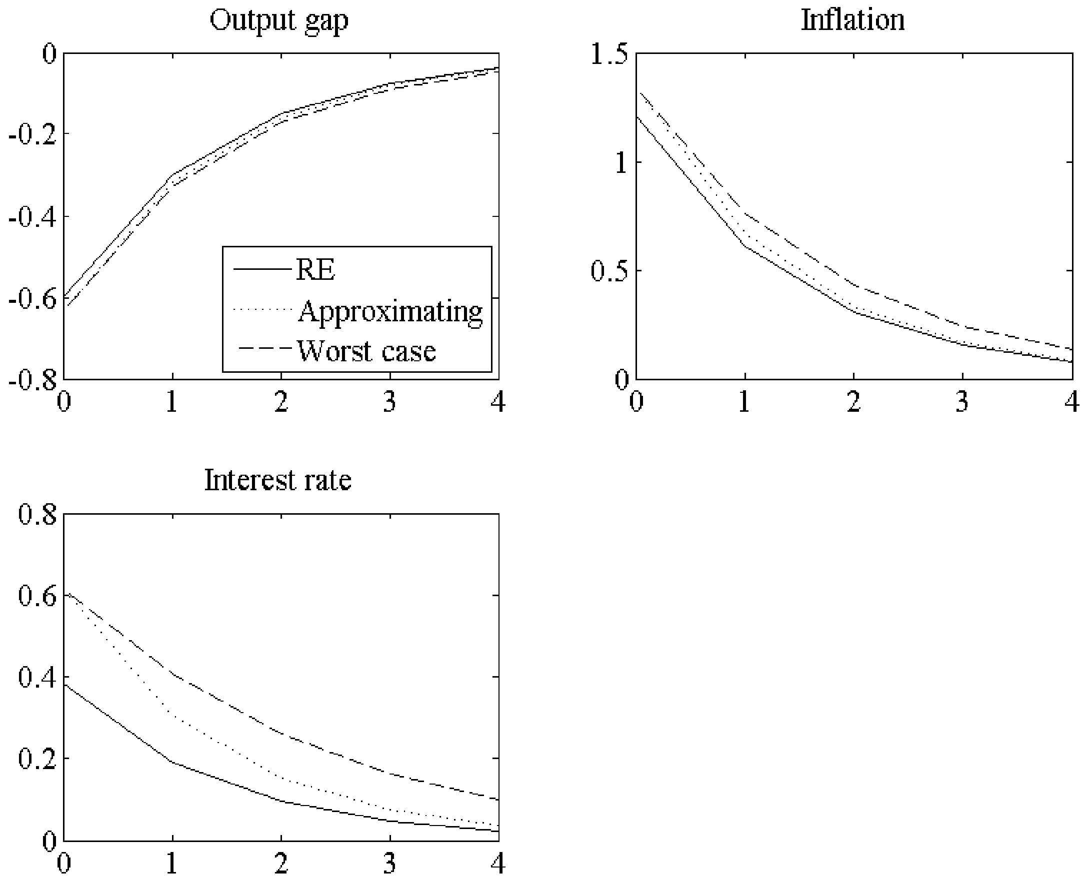

Figure 3. Impulse responses of output, inflation, and the interest rate with constant uncertainty.

5.2 Implications of Time-Varying Uncertainty ($\boldsymbol{\sigma_\theta > 0}$)

The setting is the same as in the previous subsection except that the central bank's concern for robustness fluctuates. As discussed in Section 4.3, this causes the planner's decision rule to vary over time. The same holds for the impulse responses of output gap, inflation, and the interest rate, as I show in this subsection.

Figure 4 shows the mean responses (solid lines) to a cost-push shock and the responses ± 2 standard deviations away from the mean (dashed lines). From Figure 4 several conclusions can be drawn. First, the cost-push shock has the expected macroeconomic effects: In the short term, the output gap decreases while inflation and the interest rate increase. Secondly, the observed effects disappear in the long run, here after four periods (approximately four years). Lastly, there is variation in the short-term effects of a cost-push shock, i.e., mean and ± 2 standard deviation responses do not coincide. This is so because the more uncertainty the central bank faces in a given period, the more persistent it fears the shock will be in the worst case and the more aggressive it will thus respond. A decrease in the central bank's concern for robustness has the opposite effect: it will reduce the aggressiveness of its response. Moreover, note that although the approximating model does not change over time, there is still variation in the impulse responses under the approximating model. This is due to the fact that the central bank constructs its policy based on the worst-case model, not the approximating model, and the former does change over time because it is constructed based on the uncertainty parameter θ_t, which varies each period.

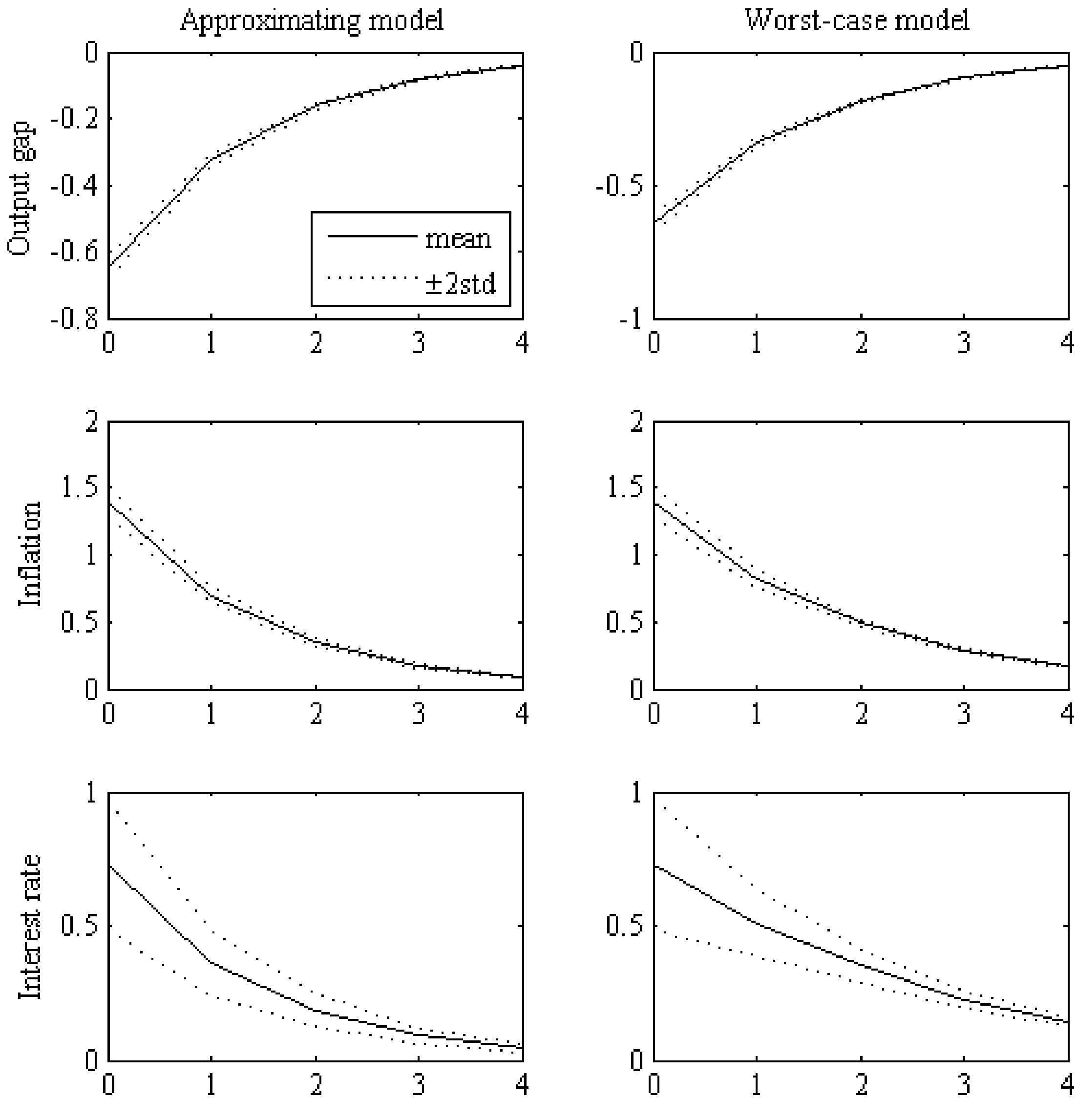

Figure 4. Impulse responses of output gap, inflation, and the interest rate under time-varying uncertainty. 125 observations.

Therefore, a central bank that updates its concern for robustness each period anticipates time-varying impulse responses. Put differently, impulse responses depend on how the central bank's trust in its approximating model develops in the periods after a shock. If its doubts increase, the central bank will respond more aggressively than it would have if its trust had stayed constant. This is so because the more the central bank distrusts its approximating model, the more persistent it fears the shocks will be, and the more aggressive it will respond. Conversely, the more the central bank trusts its approximating model, the more it will attenuate its policy response.

Another way to display the time-varying nature of impulse responses is the histogram of the impulse response one period after the cost-push shock.

Figure 5 does this for the response of the interest rate to a cost-push shock. The histogram reveals that the variation under the worst-case model is substantially larger than under the approximating model: the standard deviation and range are approximately twice as large. Moreover, the impulse responses are not normally distributed, the Jarque-Bera test[13] rejects normality at about 1% significance level.[14] The same results are obtained for the responses of output gap and inflation, as can be seen in the more detailed Figure A.2 in Appendix A. The latter also contains the responses to a demand shock, for which similar results can be obtained. Moreover, the results hold for a wide range of parameter values provided that the variance of the planner's uncertainty, σ_Z^2, is sufficiently high.

[13] The test, proposed by Jarque & Bera (1980), is a two-sided goodness-of-fit test to check whether the skewness and kurtosis of sample data match those of a normal distribution. The test statistic is $JB = (n/6)[s^2 + (k-3)^2/4]$, where n is the sample size, s is the sample skewness, and k is the sample kurtosis. If the data comes from a normal distribution, this statistic is expected to be chi-square distributed with two degrees of freedom for large samples. For small samples, MATLAB uses critical values computed via Monte Carlo simulations (see http://de.mathworks.com/help/stats/jbtest.html#btv1ru_).

[14] The larger the sample of impulse responses, the more decisive is the rejection.

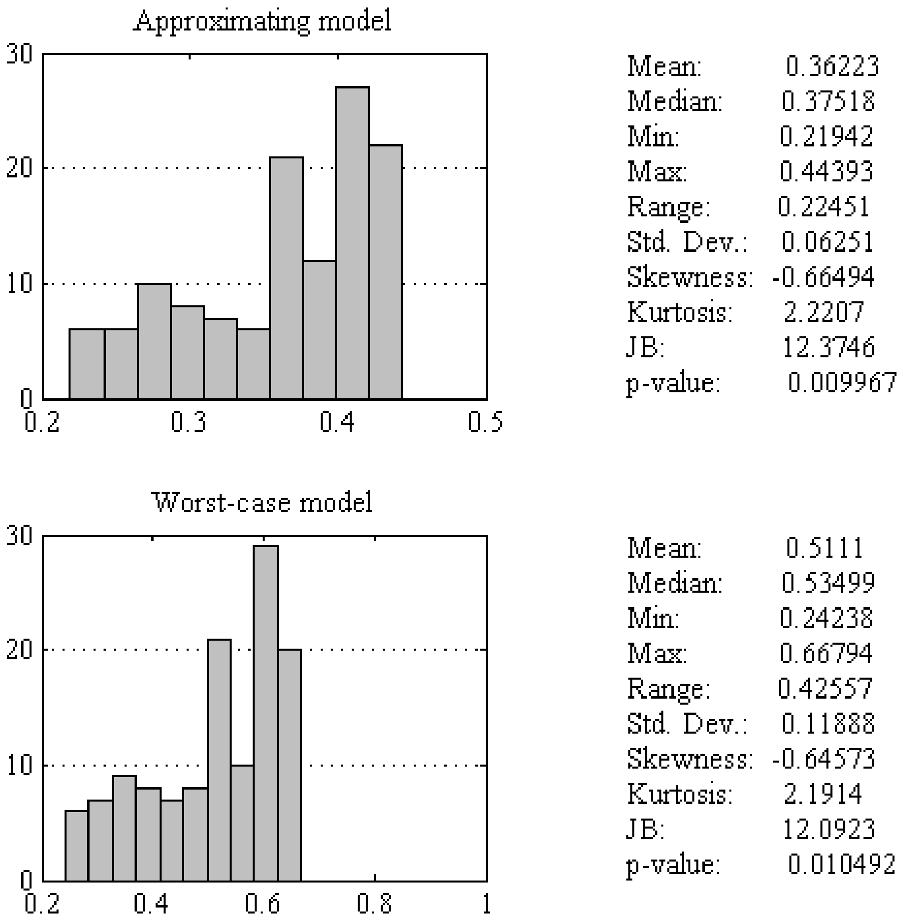

Figure 5. Histogram of the impulse response of the interest rate to a cost-push shock. 125 observations.

In sum, the main qualitative prediction of time-varying uncertainty is that the impulse responses depend on the timing of the shock and that they are nonnormally distributed. This prediction will be confronted with data in the next section.

6 Empirics

To test the model of time-varying uncertainty empirically, I use an approach of indirect inference as in De Grauwe (2012, Chapter 8), i.e., I confront the model's predictions with data instead of estimating an econometric model. I do so because the New Keynesian model used in Section 5 is rather simple and thus suffers most likely from missing variables and incomplete dynamics. To confront the prediction of time-varying, nonnormally distributed impulse responses with data, I first estimate a VAR of inflation, output gap, and the interest rate for the United States, using a Choleski decomposition and quarterly data from Q3 1954 until Q2 2015.

Appendix B describes the data sources. Similar to De Grauwe (2012, Chapter 8), I then compute the impulse responses for different subsamples. I use rolling sample periods of 30 years where the first period begins in Q3 1954 and the last sample period ends in Q2 2015. This leads to 125 rolling sample periods in total. For each sample period the short-term effects of a supply and a demand shock are calculated for each variable. Here short-term effect is defined as the effect one year after the shock occurred.

Figure 6 shows the histogram of the short-term effect that a cost-push shock has on the interest rate. It confirms the prediction that impulse responses vary substantially over time and are nonnormally distributed. The Jarque-Bera test rejects normality at a 1% significance level.[15] Quantitatively, the empirical moments do not coincide with the predicted ones shown in Figure 5, but partly they are not too far off. Overall, a robust planner tends to respond more aggressively than the central bank does empirically[16], which is in line with the findings of other authors (cf. J. Tetlow & von zur Muehlen, 2001; Macklem, 2001, pp. 232–233).

These results, most importantly nonnormality, also hold for the empirical impulse responses of output gap and inflation, and for the empirical impulse responses to a demand shock. Figure A.2 and Figure A.3 in Appendix A highlight this fact.

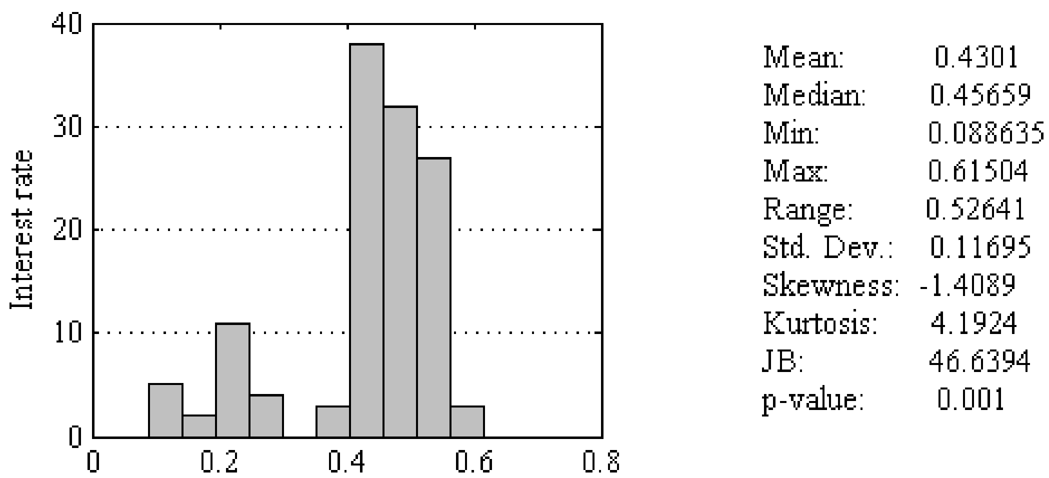

Figure 6. Histogram of the empirical impulse response of the interest rate to a cost-push shock. 125 observations.

[15] This result also holds when the length of the subsamples is altered or when the overall sample size is reduced. In those cases, however, the results partly hold only at 5% significance.

[16] While the tendency is found to be relatively small for a cost-push shock, it is rather large for a demand shock, as can be seen in Figure A.3 of Appendix A.

On the whole, the findings indicate that modelling the central bank as a robust planner who updates his concern for robustness each period can partly explain the data in the sense that it leads to distributions of the impulse responses that share important characteristics with the distributions observed in the data.

7 Conclusion

This paper has introduced time-varying uncertainty into the solution algorithm of Giordani & Söderlind (2004, pp. 2379–2380) for forward-looking robust control problems under discretion. A robust planner whose trust in his approximating model varies over time is found to use a sequence of decision rules, each of which is optimally robust for his contemporaneous concern for robustness. Any variation in his decision rule is due to variations in the uncertainty he faces. The paper has applied this method to a simple New Keynesian model as in Clarida, Gali, & Gertler (1999, p. 1665) and has tested the predictions via an approach of indirect inference (cf. De Grauwe, 2012, Chapter 8). The simulations reveal that variations in the planner's concern for robustness can generate time-varying and nonnormally distributed impulse responses of output gap, inflation, and the interest rate. Qualitatively, these predictions are confirmed by the impulse responses estimated from US quarterly data covering about 60 years. In this way, I find that time-varying model uncertainty can partly explain characteristics observed in the data. Quantitatively, the comparison of predicted and empirical impulse responses confirms previous findings that the robust policy responds more aggressively than the central bank does empirically (cf. J. Tetlow & von zur Muehlen, 2001; Macklem, 2001, pp. 232–233).

While this paper has focused on discretionary policy making, future research could examine time-varying uncertainty for the commitment case. On a normative level, the implications for the costs of adhering to the commitment policy seem a promising topic. As for further positive analysis, a next step is to explore the impact of time-varying uncertainty in the context of more sophisticated New Keynesian models. In addition to being of interest in itself, this allows for more direct testing approaches. Lastly, the beliefs of the planner and the private sector chosen in this paper correspond to what Hansen & Sargent (2012) call type I ambiguity. Importantly, this is but one out of many scenarios that are possible in a multi-agent setting. Therefore, one could extent the analysis to other types of ambiguity, such as type III where the central bank trusts its approximating model but thinks the private sector does not.

Appendix A Additional Figures

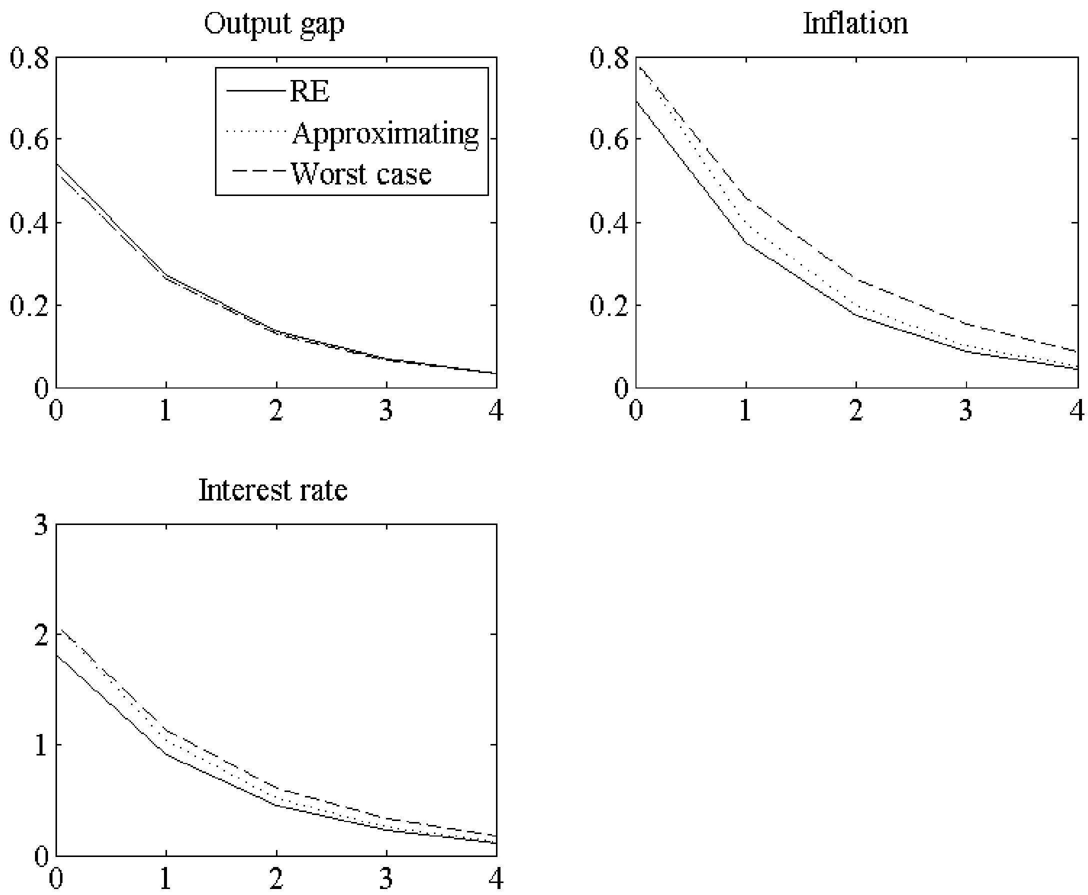

Figure A.1. Impulse responses of output gap, inflation, and the interest rate to a demand shock under constant uncertainty.

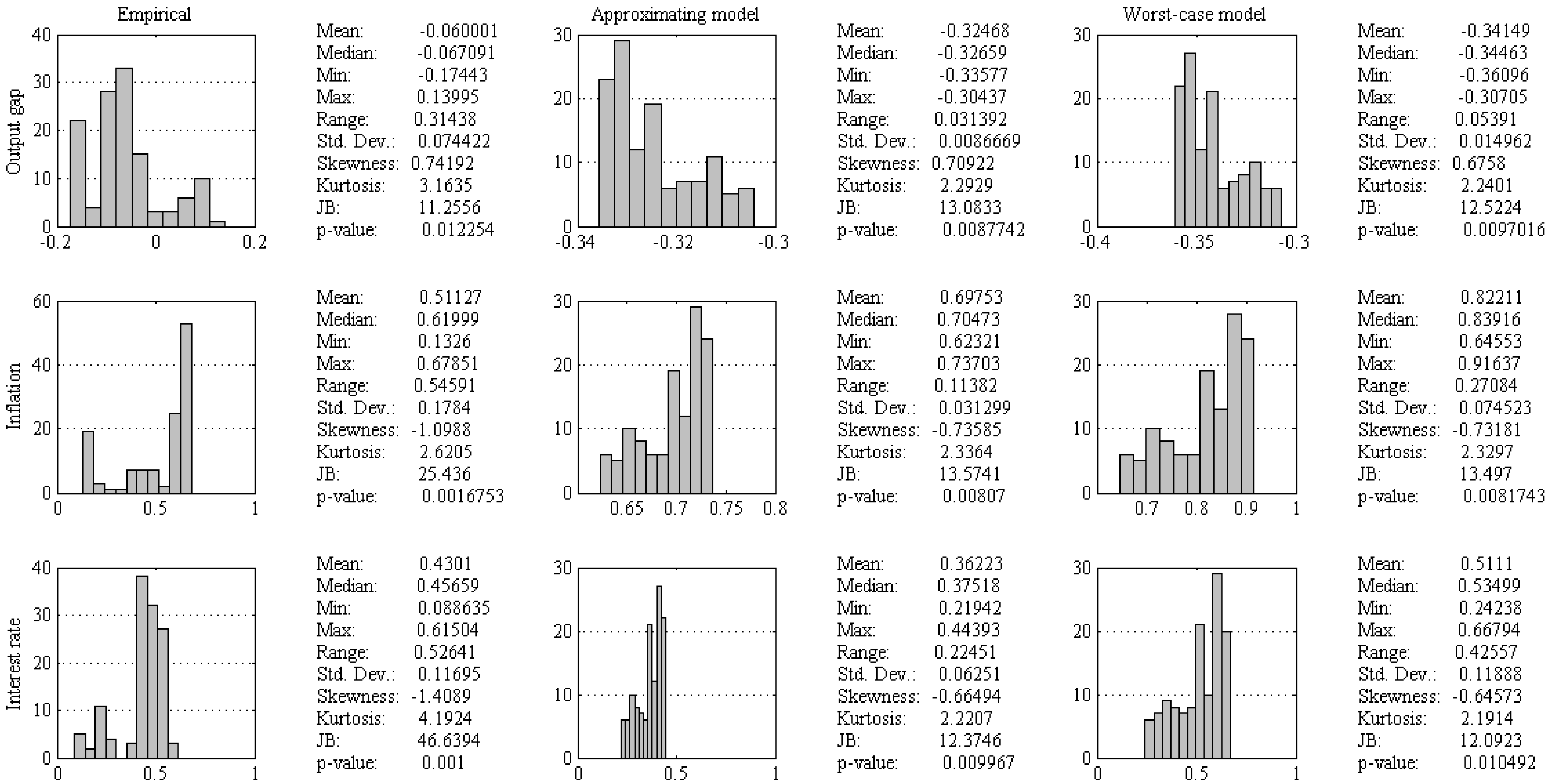

Figure A.2. Empirical and predicted impulse responses to a cost-push shock. 125 observations.

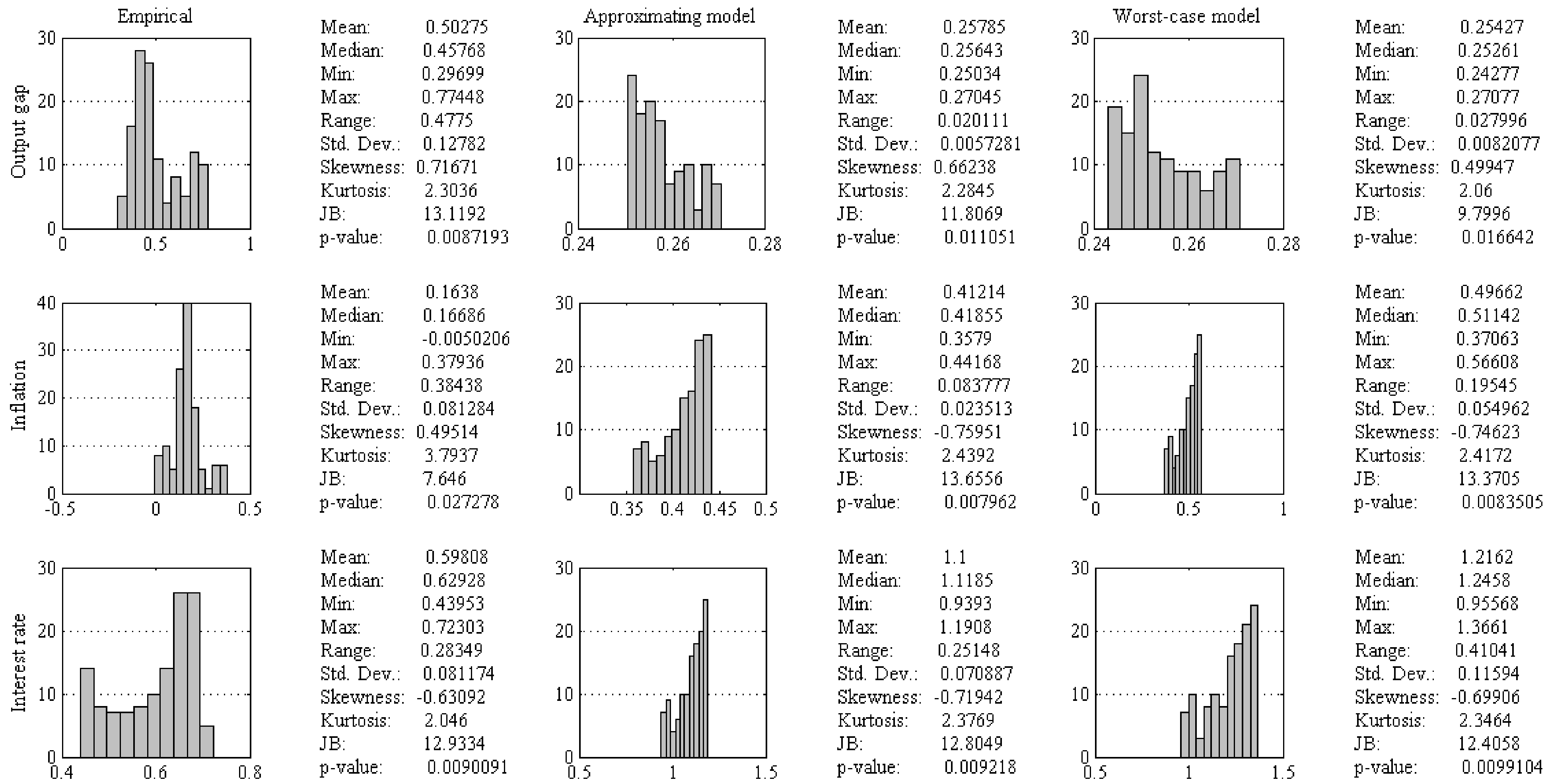

Figure A.3. Empirical and predicted impulse responses to a demand shock. 125 observations.

Appendix B Data Sources

The data used to estimate the impulse responses is:

(1) Price Indexes for Gross Domestic Product. Bureau of Economic Analysis. NIPA table 1.1.4, line 1, index numbers, 2009=100. Retrieved August 5, 2015, from http://www.bea.gov/iTable/index_nipa.cfm

(2) Effective Federal Funds Rate. Board of Governors of the Federal Reserve System. Retrieved August 5, 2015, http://www.federalreserve.gov/releases/h15/data.htm

(3) Real Potential Gross Domestic Product. Congressional Budget Office. Billions of chained 2009 Dollars. Retrieved August 5, 2015, from https://research.stlouisfed.org/fred2/series/GDPPOT

(4) Real Gross Domestic Product. Bureau of Economic Analysis. NIPA table 1.1.6, line 1, Billions of chained 2009 Dollars. Retrieved August 5, 2015, from http://www.bea.gov/iTable/index_nipa.cfm

Based on this data, I calculated

(5) Annualized inflation: $4 \cdot \left[(1)/(1)_{lag} - 1\right]$

(6) Output gap: $(4)/(3) - 1$

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