COMS 633:
Advanced Topics in Computational Randomness

Lecture Notes - Fall 2017

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1 Introduction

These notes were prepared by Alex Scheel.

2 Finite-State Gambling

Let Σ be an alphabet with $2 \leq |\Sigma| < \infty$. That is, Σ is finite. A probability measure on Σ is a function:

$$\pi : \Sigma \to [0, \infty)$$

satisfying:

$$\sum_{a \in \Sigma} \pi(a) = 1$$

A rational probability measure on Σ is a function:

$$\pi : \Sigma \to \mathbb{Q} \cap [0, \infty)$$

that is a probability measure on Σ. We write:

$$\begin{aligned}
\Delta(\Sigma) &= \{\text{probability measures on } \Sigma\} \\
\Delta_{\mathbb{Q}} &= \{\text{rational probability measures on } \Sigma\} \\
\Delta^{+}(\Sigma) &= \{\pi \in \Delta(\Sigma) | (\forall a \in \Sigma) \pi(a) > 0\} \\
\Delta^{+}_{\mathbb{Q}}(\Sigma) &= \{\pi \in \Delta_{\mathbb{Q}}(\Sigma) | (\forall a \in \Sigma) \pi(a) > 0\}
\end{aligned}$$

Note that $\Delta(\Sigma)$ is a $(|\Sigma| - 1)$-dimensional simplex in $\mathbb{R}^{|\Sigma|}$.

Definition 2.1. A finite-state automaton (FSA) on Σ is a triple:

$$A = (Q, \delta, s)$$

where:

$$\begin{aligned}
&Q \text{ is a finite set of states,} \\
&\delta : Q \times \Sigma \to Q \text{ is a transition function,} \\
&s \in Q \text{ is a start state.}
\end{aligned}$$

Example 2.2. DIAGRAM

Given an FSA, $A = (Q, \delta, s)$ on Σ, we define the extended-transition function:

$$\hat{\delta} : Q \times \Sigma^* \to Q$$

by the recursion:

$$\hat{\delta}(q, \lambda) = q$$
$$\hat{\delta}(q, wa) = \delta(\hat{\delta}(q, w), a)$$

$\forall q \in Q, w \in \Sigma^*$, and $a \in \Sigma$.

Notational Conventions:

1. We write $\delta(q, w)$ for $\hat{\delta}(q, w)$.
2. We write $\delta(w)$ for $\delta(s, w)$.

Our next objective is to endow FSAs with the ability to gamble.

Definition 2.3. A bet on Σ is a rational probability measure $\beta \in \Delta_{\mathbb{Q}}(\Sigma)$.

Intuition: Assume that:

- You have $d \in \mathbb{Q} \cap [0, \infty)$ dollars
- You are confronting an experiment whose outcome is some element of Σ
- You place the bet $\beta \in \Delta_{\mathbb{Q}}(\Sigma)$

This means that $\forall a \in \Sigma$, you are betting $d \cdot \beta(a)$ dollars that the outcome is a. Since $\sum_{a\in\Sigma} d\beta(a) = d$, you have to bet all your money in this scenario. After the bet, if the martingale's outcome was a, you will have $d(a)$, an amount that we now specify.

Suppose that the outcomes of this experiment occur according to a probability measure $\pi \in \Delta(\Sigma)$. What is your expected amount of money after the bet, given that you had d dollars?

Answer:

$$\sum_{a\in\Sigma} \pi(a)d(a) = \mathbb{E}_\pi[d(a)|d]$$

Therefore "the payoffs are fair" if:

$$d = \sum_{a\in\Sigma} \pi(a)d(a)$$

Recall: A bet on Σ is a probability measure $\beta \in \Delta_{\mathbb{Q}}(\Sigma)$. For now, a payoff rule on Σ is a probability measure $\rho \in \Delta(\Sigma)$.

Intuition: Assume that a gambler has $d \in [0, \infty)$ dollars and places a bet, β on an experiment that will have an outcome that is an element of Σ. Placing this bet means that, $(\forall a \in \Sigma)$, the gambler is betting $d\beta(a)$ dollars that the outcome is a. Note that $d = \sum_{a\in\Sigma} d\beta(a)$, so the gambler is required to bet all its money.

If the payoff rule is $\rho \in \Delta(\Sigma)$, and the actual outcome is $a \in \Sigma$, then the gambler will have:

$$d(a) := \frac{\beta(a)}{\rho(a)} \tag{1}$$

dollars after the bet.

Now assum that the outcome of the experiment occurs according to a probabilty measure, $\pi \in \Delta(\Sigma)$, which we call the actual probability measure of the experiment. The expected value of the gambler's amount of money after the bet (given that it has d dollars before the bet) is:

$$\mathbb{E}_{a\sim\pi}[d(a)] := \sum_{a\in\Sigma} d(a)\pi(a), \tag{2}$$

where $a \sim \pi$ means "a is drawn according to π". By (2.1) and (2.2),

$$\mathbb{E}_{a \sim \pi}[d(a)] = d \sum_{a \in \Sigma} \frac{\beta(a)\pi(a)}{\rho(a)} \tag{3}$$

Observation: If $\rho = \pi$, i.e., the payoff rule is the actual probability measure on Σ, then the payoffs are fair in the sense that $\mathbf{E}_{a \sim \pi}[d(a) = d$.

Definition 2.4. A finite state gambler (FSA) on Σ is a 5-tuple, $G = (Q, \delta, s, \beta, c)$, where:

- (Q, δ, s) is a FSA,
- $\beta : Q \to \Delta_{\mathbb{Q}}(\Sigma)$ is the betting function,
- $c \in \mathbb{Q} \cap [0, \infty)$ is the initial capital.

Example 2.5. DIAGRAM

Intuition:

Before defining the semantics of FSGs formally, let us use example 1.3 to gain some insight. Assume that G is given an input string $w \in \{0,1\}^*$, whose bits are chosen by independent tosses of a fair coin, and asusme that the payoff rule coincides with this. Let $d_G(w)$ denote the amount of money that G has after betting on w. Then:

$$d_G(\lambda) = c = 1 \qquad d_G(1) = \frac{4}{3} = 2\frac{2}{3} d_G(\lambda) d_G(11) = 2\frac{1}{3} d_G(1) = \frac{8}{9} \qquad d_G(110) = 2\frac{2}{3} d_G(11) = frac3227...$$

Definition 2.6. If $G = (Q, \delta, s, \beta, c)$ is an FSG and $\pi \in \Delta^+(\Sigma)$, then the π-martingale of G is the function:

$$d_G^\pi : \Sigma^* \to [0, \infty)$$

defined by the recursion:

$$d_G^\pi(\lambda) = c d_G^\pi(wa) = d_G^\pi(w) \frac{\beta(\delta(w))(a)}{\pi(a)}$$

Definition 2.7. (Ville - 1939). Let $\pi \in \Delta(\Sigma)$. A π-martingale on Σ is a function:

$$d : \Sigma^* \to [0, \infty)$$

that satisfies, $(\forall w \in \Sigma^*)$:

$$d(w) = \sum_{a \in \Sigma} d(wa)\pi(a) \tag{4}$$

Obervation: For every FSG G on Σ and every probability measure $\pi \in \Delta^+(\Sigma)$, d_G^π is a π-martingale.