ECS256 - Homework III

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Problem 1.a

First, we'll derive π_i. The definition of the tree searching markov model leads to the following set of balance equations for the long-run state probabilities:

$$\pi_i = \pi_{i-1}q_{i-1} = \pi_0 \prod_{j=0}^{i-1} q_j \quad \text{for } i \geq 1\text{, and}$$

$$\pi_0 = \sum_{i=1}^{\infty} \pi_i(1 - q_i) \quad \text{for } i = 0.$$

This definition for π_0 is a bit unweildy. Since the chain is positive recurrent, we can also think of this quantity as one over the expected recurrance time, as in eq. (10.63) in the book:

$$\begin{aligned}
\pi_0 &= \frac{1}{E(T_{0,0})} \\
E(T_{0,0}) &= 1 + \sum_{k \neq 0} p_{0,k} E(T_{k,0}) \quad \text{By eq. (10.65)} \\
&= 1 + p_{0,1}E(T_{1,0}) \\
&= 1 + p_{0,1}(1 + \sum_{k \neq 0} p_{1,k} E(T_{k,0})) \\
&= 1 + p_{0,1}(1 + p_{1,2}E(T_{2,0})) \\
&= 1 + p_{0,1}(1 + p_{1,2}(1 + \sum_{k \neq 0} p_{2,k} E(T_{k,0}))) \\
&= 1 + p_{0,1}(1 + p_{1,2}(1 + p_{2,3}E(T_{3,0})))
\end{aligned}$$

and so on. This unravels into a familiar closed form:

$$\begin{aligned}
E(T_{0,0}) &= 1 + q_0(1 + q_1(1 + q_2(1 + \dots) \dots))) \\
&= 1 + q_0 + q_0q_1 + q_0q_1q_2 + \dots \\
&= 1 + \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j]
\end{aligned}$$

If the model is positive recurrent, then there exists some value R such that

$$R = \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j] < \infty.$$

Thus,

$$\pi_i = \frac{\prod_{j=0}^{i-1} q_j}{1+R} \quad \text{for } i \geq 0.$$

Next, $E(T_{i,0})$ follows a similar pattern.

$$\begin{aligned}
E(T_{i,0}) &= 1 + \sum_{k \neq 0} p_{i,k} E(T_{k,0}) \\
&= 1 + p_{i,i+1} E(T_{j+1,0}) \\
&= 1 + q_i + q_i q_{i+1} + q_i q_{i+1} q_{i+2} + \dots \\
&= 1 + \sum_{j=i}^{\infty} \Big[\prod_{k=i}^{j} q_k \Big].
\end{aligned}$$

Problem 1.b

If $q_i = 0.5$ for all i, then R is a geometric series that indeed converges.

$$\pi_2 = \frac{0.5 \cdot 0.5}{1 + \sum_{i=1}^{\infty} 0.5^{i-1}} = \frac{0.25}{1+2} \approx \boxed{0.083.}$$

$$E(T_{2,0}) = 1 + \sum_{j=2}^{\infty} 0.5^{j-2} = 1 + \sum_{j=1}^{\infty} 0.5^{j-1} = 1 + 2 = \boxed{3.}$$

Problem 1.c

The rate of backtracking, in terms of the stationary probabilities π_i, is simply

$$\sum_{i=1}^{\infty} \pi_i (1 - q_i).$$

Problem 2.a

In this problem, we are given the task of generating a method-of-stages approximation of a distribution, given its quantile function. To accomplish the approximation, we seek to combine a set of erlang distributions and receive the approximation as the sum of the erlang distributions.

Problem 2.b

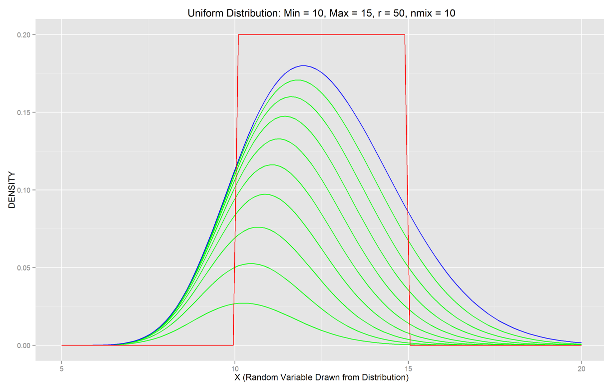
Using the **ermixobj** object generated in Problem 2.a, we are able to generate a set of **nmix** erlang distributions with parameters given as:

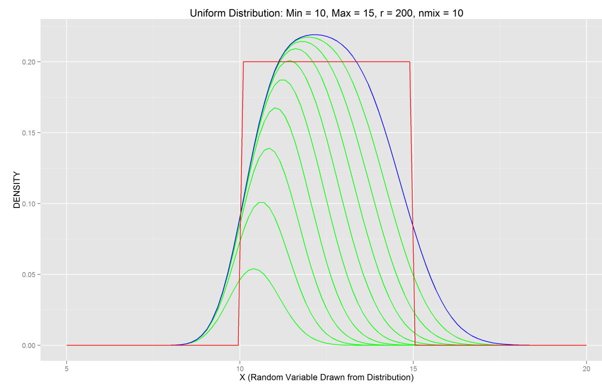
$$\text{Shape} = \mathbf{r}$$
$$\text{Rate} = \mathbf{lamb}$$

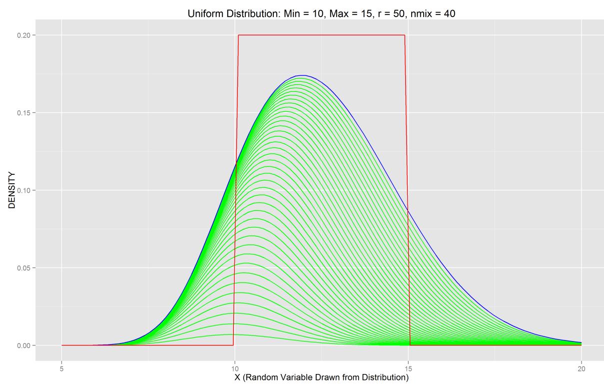
The combination of all nmix erlang distributions yields our method-of-stages approximation of the quantile function fed into erlangmix() in Problem 2.a.

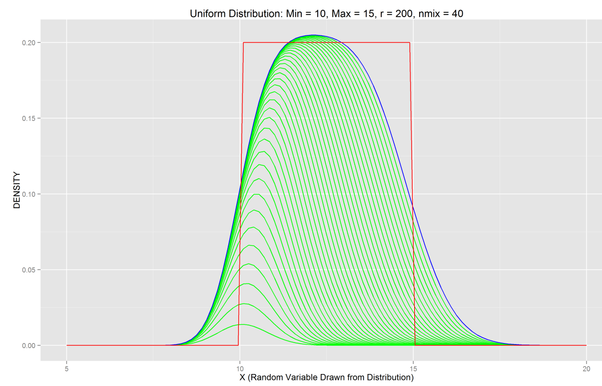
Here, we explored the effect of different values of r and nmix on the approximation.

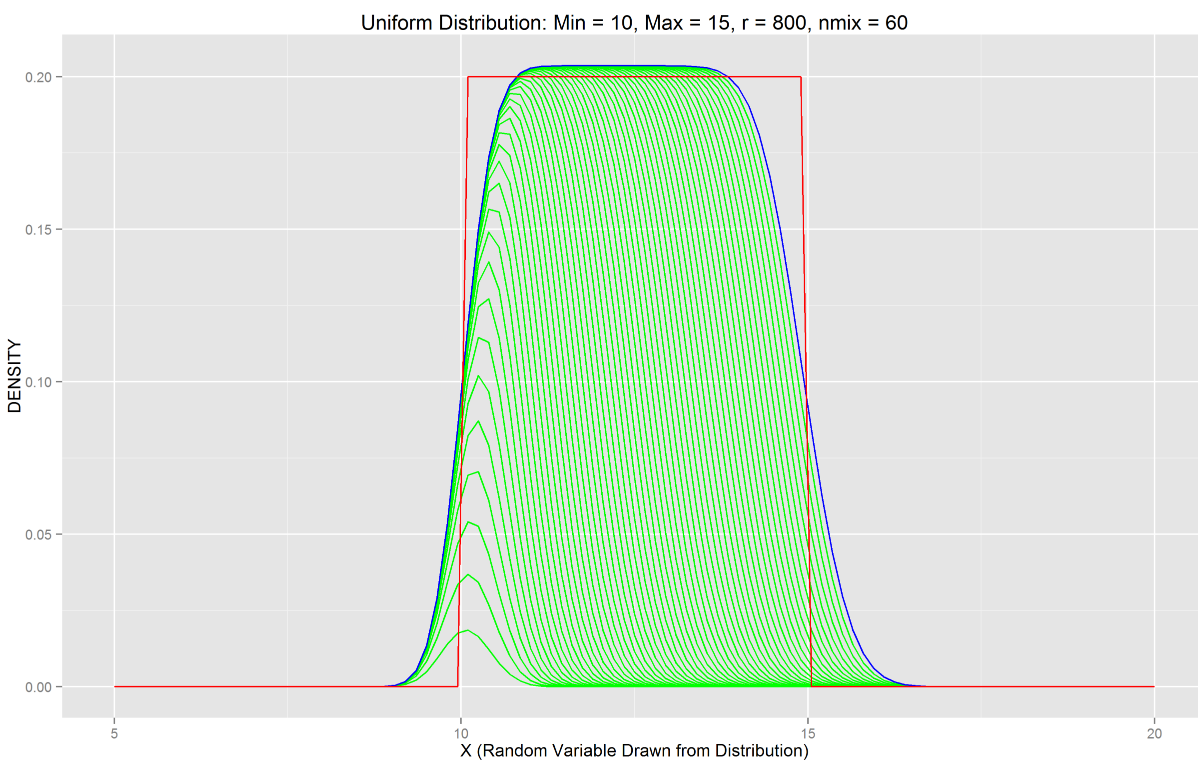
For a uniform distribution with minimum = 10, maximum = 15:



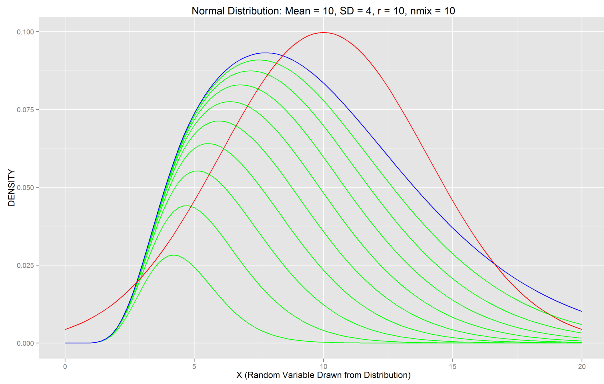


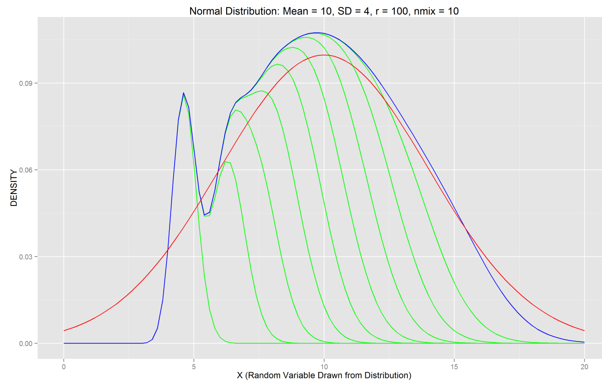


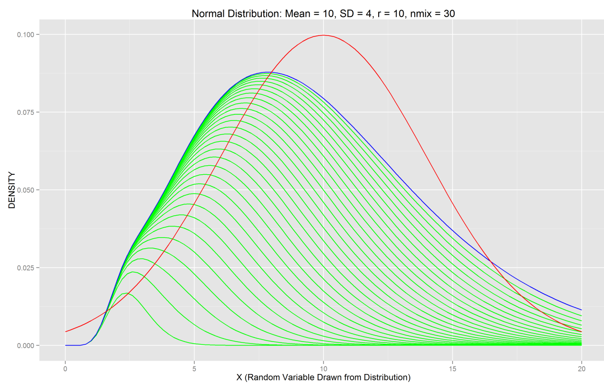


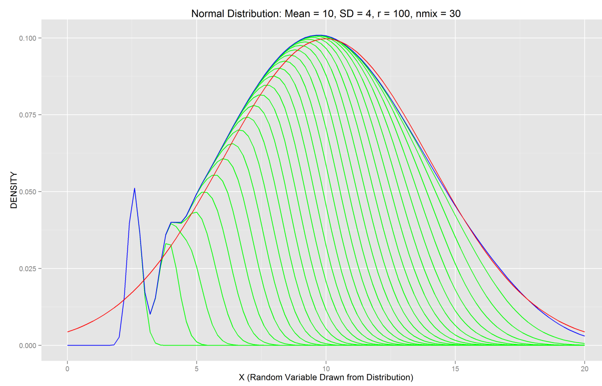


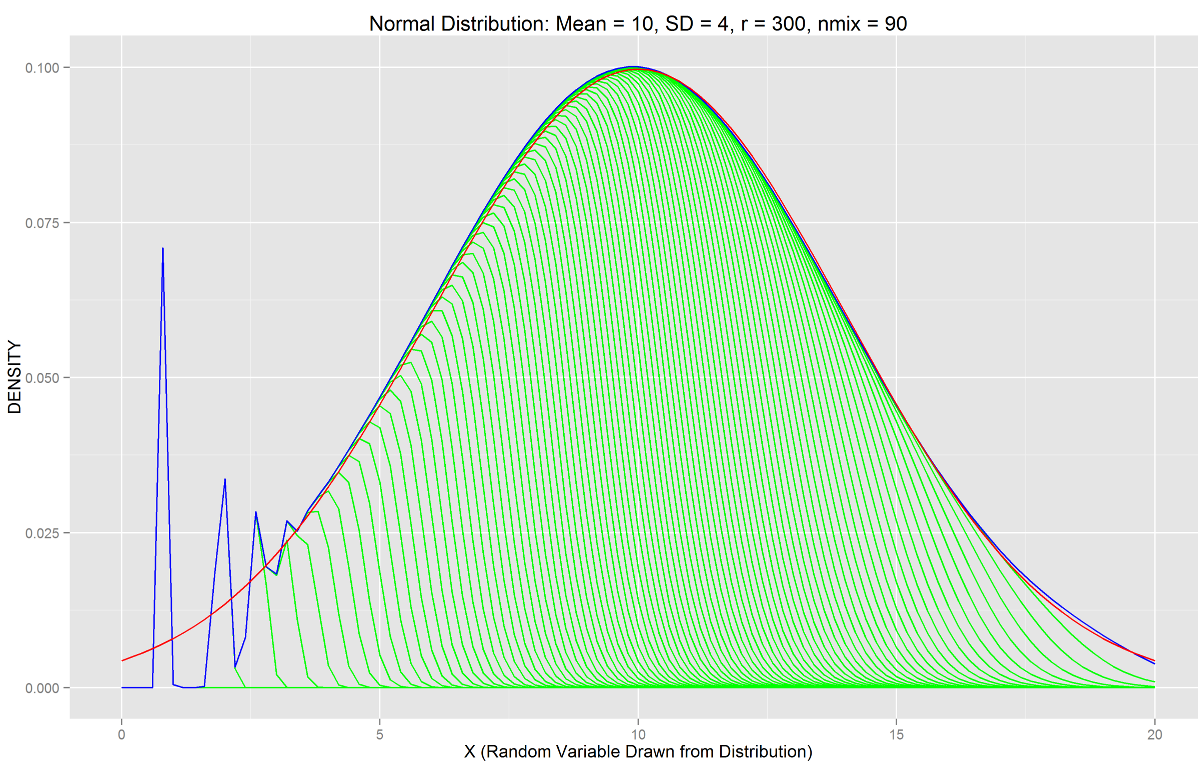
For a normal distribution with mean = 10, standard deviation = 4:











As can be seen, as textbfr modifies the magnitude of each component within the approximation while textbfnmix controls the resolution of the approximation. By increasing both, we can get an increasingly accurate approximation of the given distribution.

Problem 3.a

See `HtoF.R`.

```
htof = function(hftn,t,lower){
  density_val = c()
  for(val in t)
  {
    density_val = c(density_val, hftn(val) *
          exp(-1*integrate(hftn,lower,val)$value))
  }
  return (density_val)
}
```

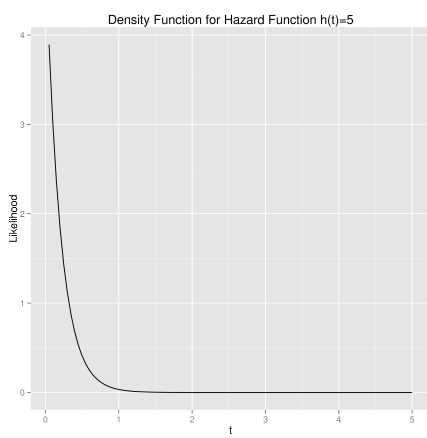
Problem 3.b

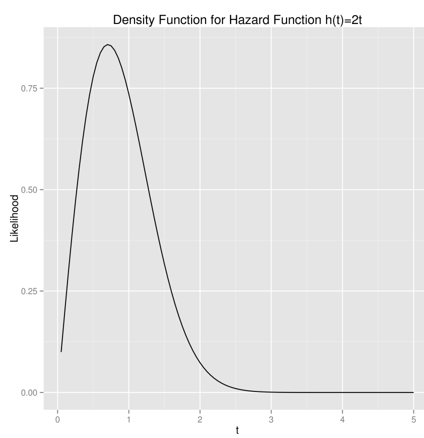
Given a hazard function, $h(t)$, the density function, $f(t)$, can be found as follows:

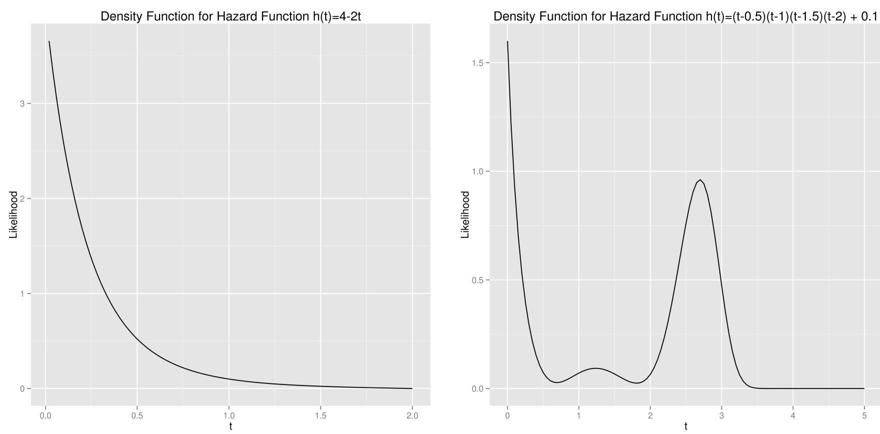
$$f(t) = h(t) \cdot e^{-\int_0^t \mathrm{h(s)}\,\mathrm{d}s}$$

We looked at the following hazard functions to explore what their density would look like:

$$h(t) = 5$$
$$h(t) = 2t$$
$$h(t) = 4 - 2t$$
$$h(t) = (t - 0.5)(t - 1)(t - 1.5)(t - 2) + 0.1$$







(Plots generated with `3.R`).

Problem 4.a-b

See `4.R` for generating these plots.