ECS256 - Homework III

Olga Prilepova, Christopher Patton, Alexander Rumbaugh,
John Chen, Thomas Provan

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Problem 1.a

First, we'll derive π_i. The definition of the tree searching markov model leads to the following set of balance equations for the long-run state probabilities:

$$\pi_i = \pi_{i-1} q_{i-1} = \pi_0 \prod_{j=0}^{i-1} q_j \quad \text{for } i \geq 1 \text{, and}$$

$$\pi_0 = \sum_{i=1}^{\infty} \pi_i (1 - q_i) \quad \text{for } i = 0.$$

This definition for π_0 is a bit unweildy. We can also think of this quantity as one over the expected recurrance time, as in eq. (10.63) in the book:

$$\begin{aligned}
\pi_0 &= \frac{1}{E(T_{0,0})} \\
E(T_{0,0}) &= 1 + \sum_{k \neq 0} p_{0,k} E(T_{k,0}) \\
&= 1 + p_{0,1} E(T_{1,0}) \\
&= 1 + p_{0,1}(1 + \sum_{k \neq 0} p_{1,k} E(T_{k,0})) \\
&= 1 + p_{0,1}(1 + p_{1,2} E(T_{2,0})) \\
&= 1 + p_{0,1}(1 + p_{1,2}(1 + \sum_{k \neq 0} p_{2,k} E(T_{k,0}))) \\
&= 1 + p_{0,1}(1 + p_{1,2}(1 + p_{2,3} E(T_{3,0})))
\end{aligned}$$

and so on. This unravels into a familiar closed form:

$$\begin{aligned}
E(T_{0,0}) &= 1 + q_0(1 + q_1(1 + q_2(1 + \dots) \dots))) \\
&= 1 + q_0 + q_0 q_1 + q_0 q_1 q_2 + \dots \\
&= 1 + \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j]
\end{aligned}$$

If the model is positive recurrent, then there exists some value R such that

$$R = \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j] < \infty.$$

Thus,

$$\pi_i = \frac{\prod_{j=0}^{i-1} q_j}{1+R} \quad \text{for } i \geq 0.$$

Next, $E(T_{i,0})$ follows a similar pattern.

$$\begin{aligned}
E(T_{i,0}) &= 1 + \sum_{k\neq 0} p_{i,k} E(T_{k,0}) \\
&= 1 + p_{i,i+1} E(T_{j+1,0}) \\
&= 1 + q_i + q_i q_{i+1} + q_i q_{i+1} q_{i+2} + \dots \\
&= 1 + \sum_{j=i}^{\infty} \Big[\prod_{k=i}^{j} q_k \Big].
\end{aligned}$$

Problem 1.b

If $q_i = 0.5$ for all i, then R is a geometric series that indeed converges.

$$\pi_2 = \frac{0.5 \cdot 0.5}{1 + \sum_{i=1}^{\infty} 0.5^{i-1}} = \frac{0.25}{1+2} \approx 0.083.$$

$$E(T_{2,0}) = 1 + \sum_{j=2}^{\infty} 0.5^{j-2} = 1 + \sum_{j=1}^{\infty} 0.5^{j-1} = 1 + 2 = 3.$$

Problem 1.c

The rate of backtracking, in terms of the stationary probabilities π_i, is simply

$$\sum_{i=1}^{\infty} \pi_i (1 - q_i).$$

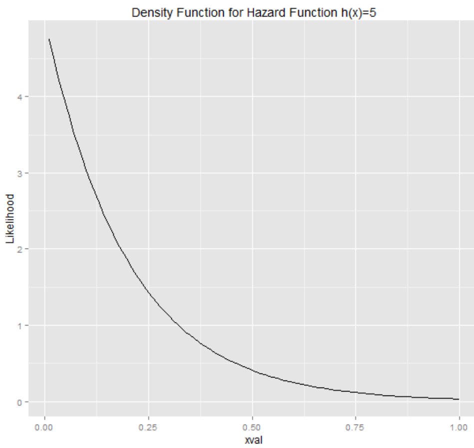
Problem 3

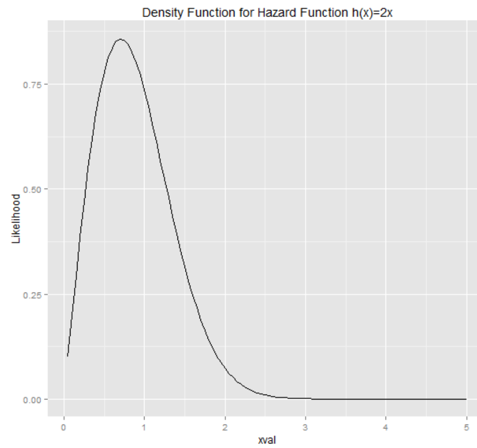
Given a hazard function, $h(t)$, the density function, $f(t)$, can be found as follows:

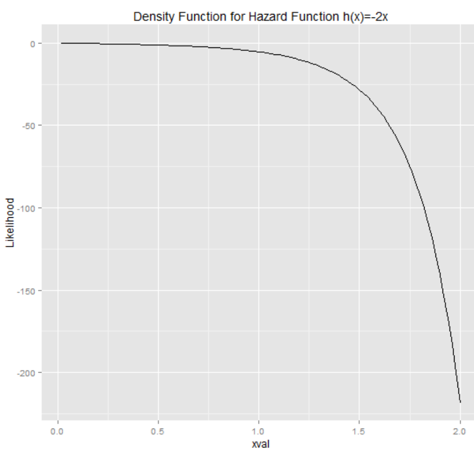
$$f(t) = h(t) \cdot e^{-\int_0^t \mathrm{h(s)\,d}s}$$

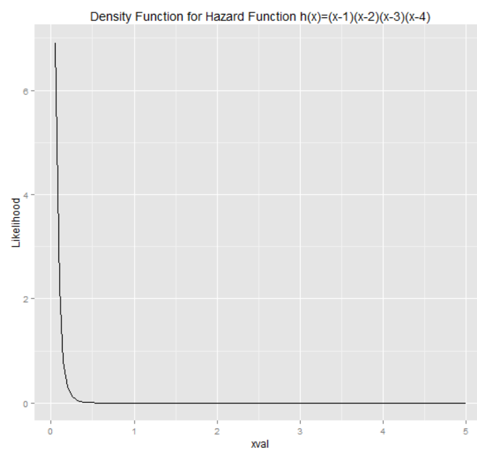
We looked at the following hazard functions to explore what their density would look like:

$$\begin{aligned}
h(t) &= 5 \\
h(t) &= 2t \\
h(t) &= -2t \\
h(t) &= (t-1)(t-2)(t-3)(t-4)
\end{aligned}$$



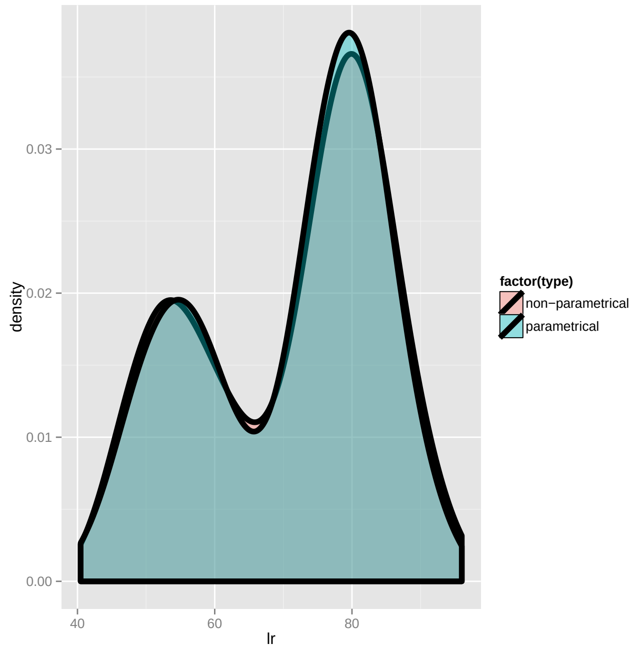






Problem 4

4.a-b



Appendix

Problem 4

```
#install.packages("ggplot2")
#install.packages("mixtools")
library(mixtools)
library(ggplot2)

#4.a ##############################
p<-ggplot(data.frame(faithful))
p+geom_density(aes(x=faithful$waiting))

#4.b ##############################
simulateFromDist <- function(n,p1,m1,s1,m2,s2){
        k1 <- p1*n #proportion of type 1
        k2 <- n-k1+1 #proportion of type 2
        x1 <- rnorm(k1, mean=m1, sd=s1)
        x2 <- rnorm(k2, mean=m2, sd=s2)
        c(x1,x2) #order of events doesn't matter for histogram, so simply concatenate
}

####from mixtools simulation
mixout<-normalmixEM(faithful$waiting,lambda=0.5,mu=c(55,80),sigma=10,k=2)
str(mixout)
# $ lambda     : num [1:2] 0.361 0.639
# $ mu         : num [1:2] 54.6 80.1
# $ sigma      : num [1:2] 5.87 5.87


# Is it necessary to simulate this? Can we plot the function directly?
sim_waiting<-simulateFromDist(length(faithful$waiting),0.361,54.6,5.87,80.1,5.87)

data <- rbind( data.frame(type="non-parametrical", lr=faithful$waiting), data.frame(type="parametr
```

```r
m <- ggplot(data, aes(x=lr))
m <- m + geom_density(aes(fill=factor(type)), size=2, alpha=.4)

#save m
pdf('plot4b.pdf')
m
dev.off()

#4.f   ####################

EL2 <- ( mixout$sigma[1]^2 * mixout$lambda[1] + mixout$sigma[2]^2 * mixout$lambda[2] ) + (mixout$m
EL <- mixout$mu[1] * mixout$lambda[1] + mixout$mu[2] * mixout$lambda[2]
```