ECS256 - Homework III

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Problem 1.a

First, we'll derive π_i. The definition of the tree searching markov model leads to the following set of balance equations for the long-run state probabilities:

$$\pi_i = \pi_{i-1} q_{i-1} = \pi_0 \prod_{j=0}^{i-1} q_j \quad \text{for } i \geq 1, \text{ and}$$

$$\pi_0 = \sum_{i=1}^{\infty} \pi_i (1 - q_i) \quad \text{for } i = 0.$$

This definition for π_0 is a bit unweildy. We can also think of this quantity as one over the expected recurrance time, as in eq. (10.63) in the book:

$$\begin{aligned}
\pi_0 &= \frac{1}{E(T_{0,0})} \\
E(T_{0,0}) &= 1 + \sum_{k \neq 0} p_{0,k} E(T_{k,0}) \\
&= 1 + p_{0,1} E(T_{1,0}) \\
&= 1 + p_{0,1}(1 + \sum_{k \neq 0} p_{1,k} E(T_{k,0})) \\
&= 1 + p_{0,1}(1 + p_{1,2} E(T_{2,0})) \\
&= 1 + p_{0,1}(1 + p_{1,2}(1 + \sum_{k \neq 0} p_{2,k} E(T_{k,0}))) \\
&= 1 + p_{0,1}(1 + p_{1,2}(1 + p_{2,3} E(T_{3,0})))
\end{aligned}$$

and so on. This unravels into a familiar closed form:

$$\begin{aligned}
E(T_{0,0}) &= 1 + q_0(1 + q_1(1 + q_2(1 + \dots) \dots))) \\
&= 1 + q_0 + q_0 q_1 + q_0 q_1 q_2 + \dots \\
&= 1 + \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j]
\end{aligned}$$

If the model is positive recurrent, then there exists some value R such that

$$R = \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j] < \infty.$$

Thus,

$$\pi_i = \frac{\prod_{j=0}^{i-1} q_j}{1+R} \quad \text{for } i \geq 0.$$

Next, $E(T_{i,0})$ follows a similar pattern.

$$\begin{aligned}
E(T_{i,0}) &= 1 + \sum_{k\neq 0} p_{i,k} E(T_{k,0}) \\
&= 1 + p_{i,i+1} E(T_{j+1,0}) \\
&= 1 + q_i + q_i q_{i+1} + q_i q_{i+1} q_{i+2} + \ldots \\
&= 1 + \sum_{j=i}^{\infty} \Big[\prod_{k=i}^{j} q_k \Big].
\end{aligned}$$

Problem 1.b

If $q_i = 0.5$ for all i, then R is a geometric series that indeed converges.

$$\pi_2 = \frac{0.5 \cdot 0.5}{1 + \sum_{i=1}^{\infty} 0.5^{i-1}} = \frac{0.25}{1+2} \approx 0.083.$$

$$E(T_{2,0}) = 1 + \sum_{j=2}^{\infty} 0.5^{j-2} = 1 + \sum_{j=1}^{\infty} 0.5^{j-1} = 1 + 2 = 3.$$

Problem 1.c

The rate of backtracking, in terms of the stationary probabilities π_i, is simply

$$\sum_{i=1}^{\infty} \pi_i (1 - q_i).$$

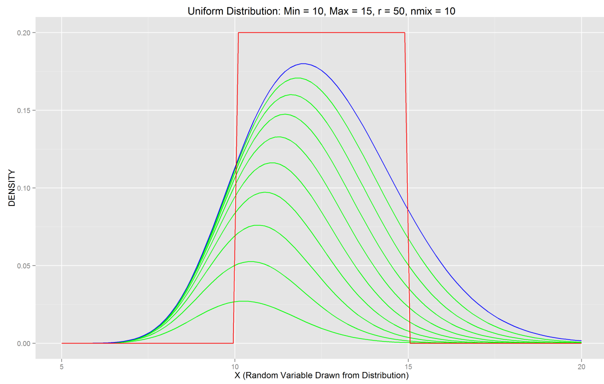
Problem 2.b

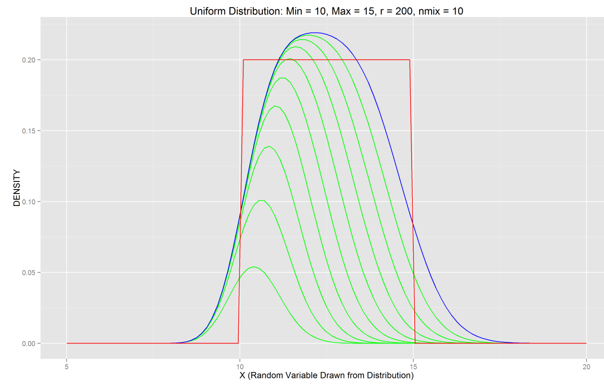
Using the lambdas generated by erlangmix(), we are able to generate a set of nmix erlang distributions with parameters given as: Shape = R Rate = lambda

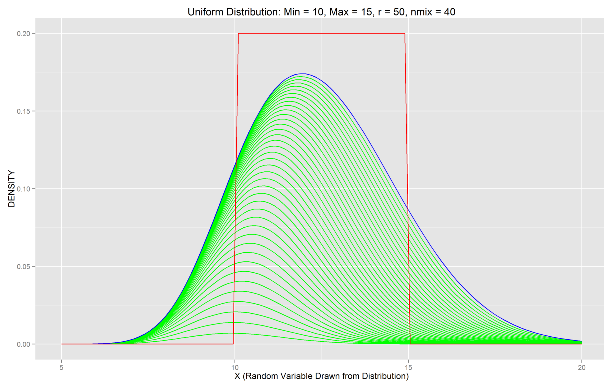
The combination of all nmix erlang distributions yields our method-of-stages approximation of the quantile function fed into erlangmix().

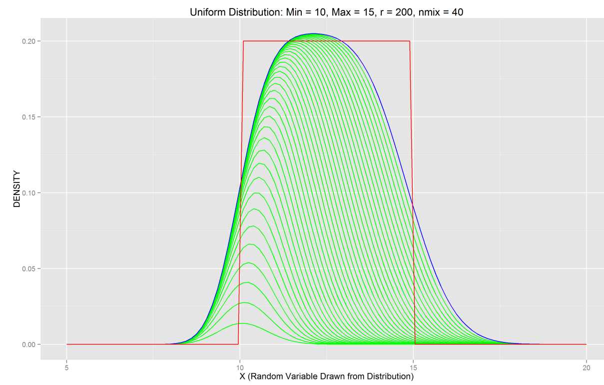
Here, we explored the effect of different values of r and nmix on the approximation.

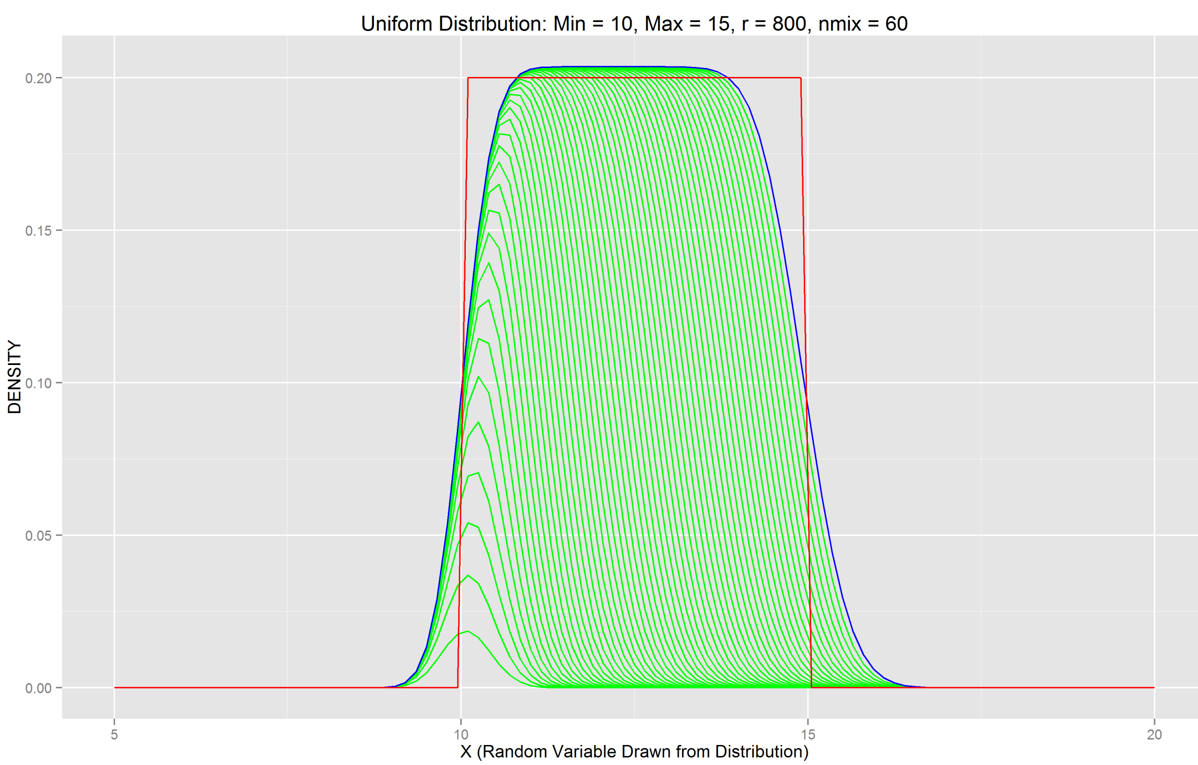
For a uniform distribution with minimum = 10, maximum = 15:



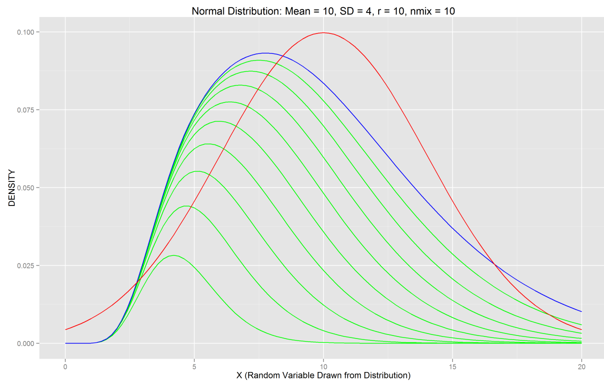


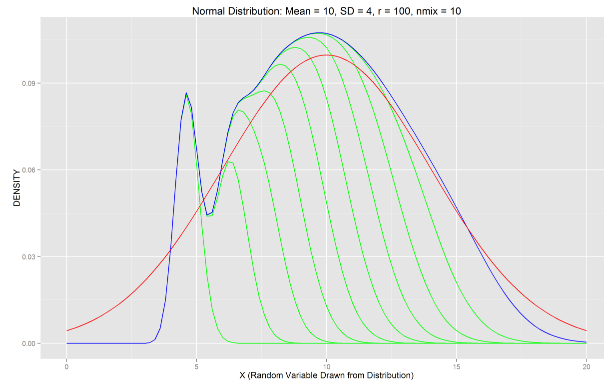


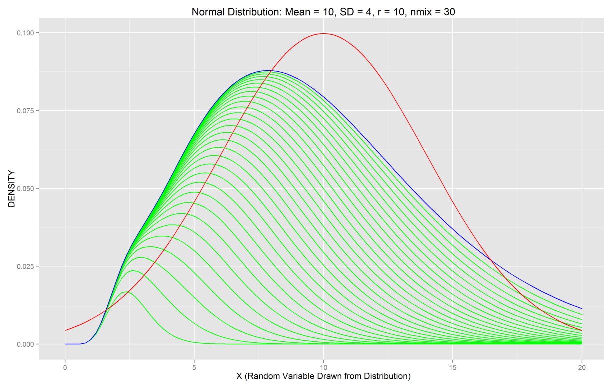


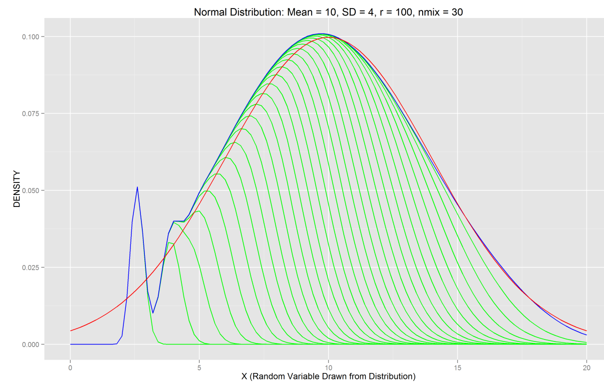


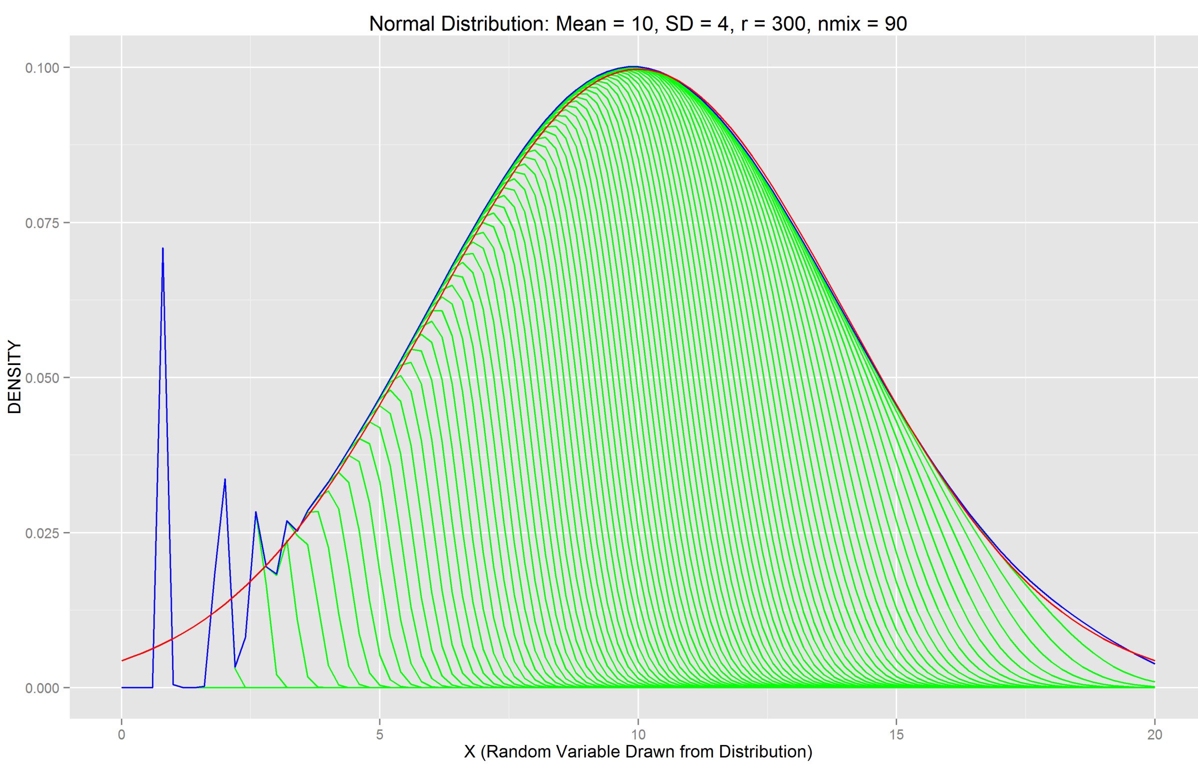
For a normal distribution with mean = 10, standard deviation = 4:











Problem 3.a

See `HtoF.R`.

```r
htof = function(hftn,t,lower){
  density_val = c()
  for(val in t)
  {
    density_val = c(density_val, hftn(val) *
          exp(-1*integrate(hftn,lower,val)$value))
  }
  return (density_val)
}
```

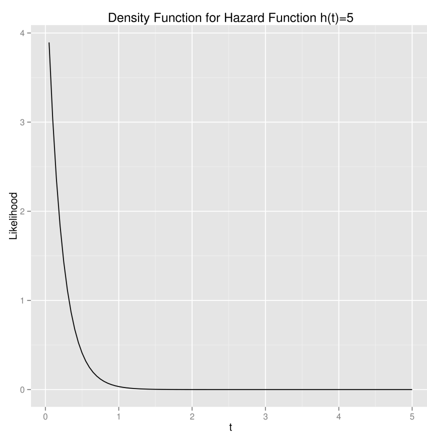
Problem 3.b

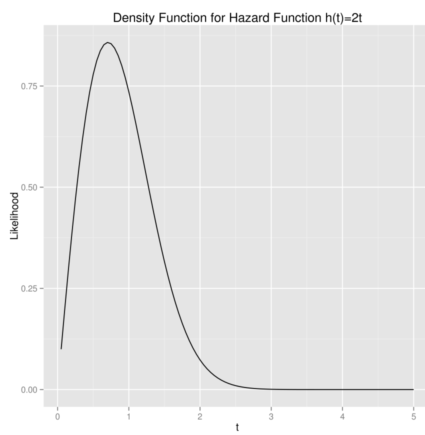
Given a hazard function, $h(t)$, the density function, $f(t)$, can be found as follows:

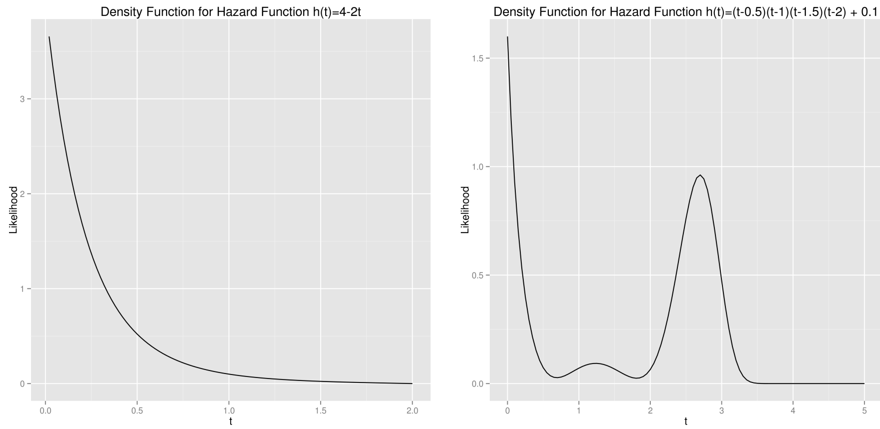
$$f(t) = h(t) \cdot e^{-\int_0^t \mathrm{h(s)}\,\mathrm{d}s}$$

We looked at the following hazard functions to explore what their density would look like:

$$\begin{aligned}
&h(t) = 5 \\
&h(t) = 2t \\
&h(t) = 4 - 2t \\
&h(t) = (t-0.5)(t-1)(t-1.5)(t-2) + 0.1
\end{aligned}$$







(Plots generated with `3.R`).

Problem 4.a-b

See `4.R` for generating these plots.