ECS256 - Homework III

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Problem 1.a

First, we'll derive π_i. The definition of the tree searching markov model leads to the following set of balance equations for the long-run state probabilities:

$$\pi_i = \pi_{i-1} q_{i-1} = \pi_0 \prod_{j=0}^{i-1} q_j \quad \text{for } i \geq 1, \text{ and}$$

$$\pi_0 = \sum_{i=1}^{\infty} \pi_i (1 - q_i) \quad \text{for } i = 0.$$

This definition for π_0 is a bit unweildy. Since the chain is positive recurrent, we can also think of this quantity as one over the expected recurrance time, as in eq. (10.63) in the book:

$$\pi_0 = \frac{1}{E(T_{0,0})}$$

$$E(T_{0,0}) = 1 + \sum_{k \neq 0} p_{0,k} E(T_{k,0}) \quad \text{By eq. (10.65)}$$

$$= 1 + p_{0,1} E(T_{1,0})$$

$$= 1 + p_{0,1}(1 + \sum_{k \neq 0} p_{1,k} E(T_{k,0}))$$

$$= 1 + p_{0,1}(1 + p_{1,2} E(T_{2,0}))$$

$$= 1 + p_{0,1}(1 + p_{1,2}(1 + \sum_{k \neq 0} p_{2,k} E(T_{k,0})))$$

$$= 1 + p_{0,1}(1 + p_{1,2}(1 + p_{2,3} E(T_{3,0})))$$

and so on. This unravels into a familiar closed form:

$$E(T_{0,0}) = 1 + q_0(1 + q_1(1 + q_2(1 + \ldots)\ldots)))$$

$$= 1 + q_0 + q_0 q_1 + q_0 q_1 q_2 + \ldots$$

$$= 1 + \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j]$$

If the model is positive recurrent, then there exists some value R such that

$$R = \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j] < \infty.$$

Thus,

$$\pi_i = \frac{\prod_{j=0}^{i-1} q_j}{1 + R} \quad \text{for } i \geq 0.$$

Next, $E(T_{i,0})$ follows a similar pattern.

$$\begin{aligned}
E(T_{i,0}) &= 1 + \sum_{k \neq 0} p_{i,k} E(T_{k,0}) \\
&= 1 + p_{i,i+1} E(T_{j+1,0}) \\
&= 1 + q_i + q_i q_{i+1} + q_i q_{i+1} q_{i+2} + \dots \\
&= 1 + \sum_{j=i}^{\infty} \left[\prod_{k=i}^{j} q_k \right].
\end{aligned}$$

Problem 1.b

If $q_i = 0.5$ for all i, then R is a geometric series that indeed converges.

$$\pi_2 = \frac{0.5 \cdot 0.5}{1 + \sum_{i=1}^{\infty} 0.5^{i-1}} = \frac{0.25}{1+2} \approx 0.083.$$

$$E(T_{2,0}) = 1 + \sum_{j=2}^{\infty} 0.5^{j-2} = 1 + \sum_{j=1}^{\infty} 0.5^{j-1} = 1 + 2 = 3.$$

Problem 1.c

The rate of backtracking, in terms of the stationary probabilities π_i, is simply

$$\sum_{i=1}^{\infty} \pi_i (1 - q_i).$$

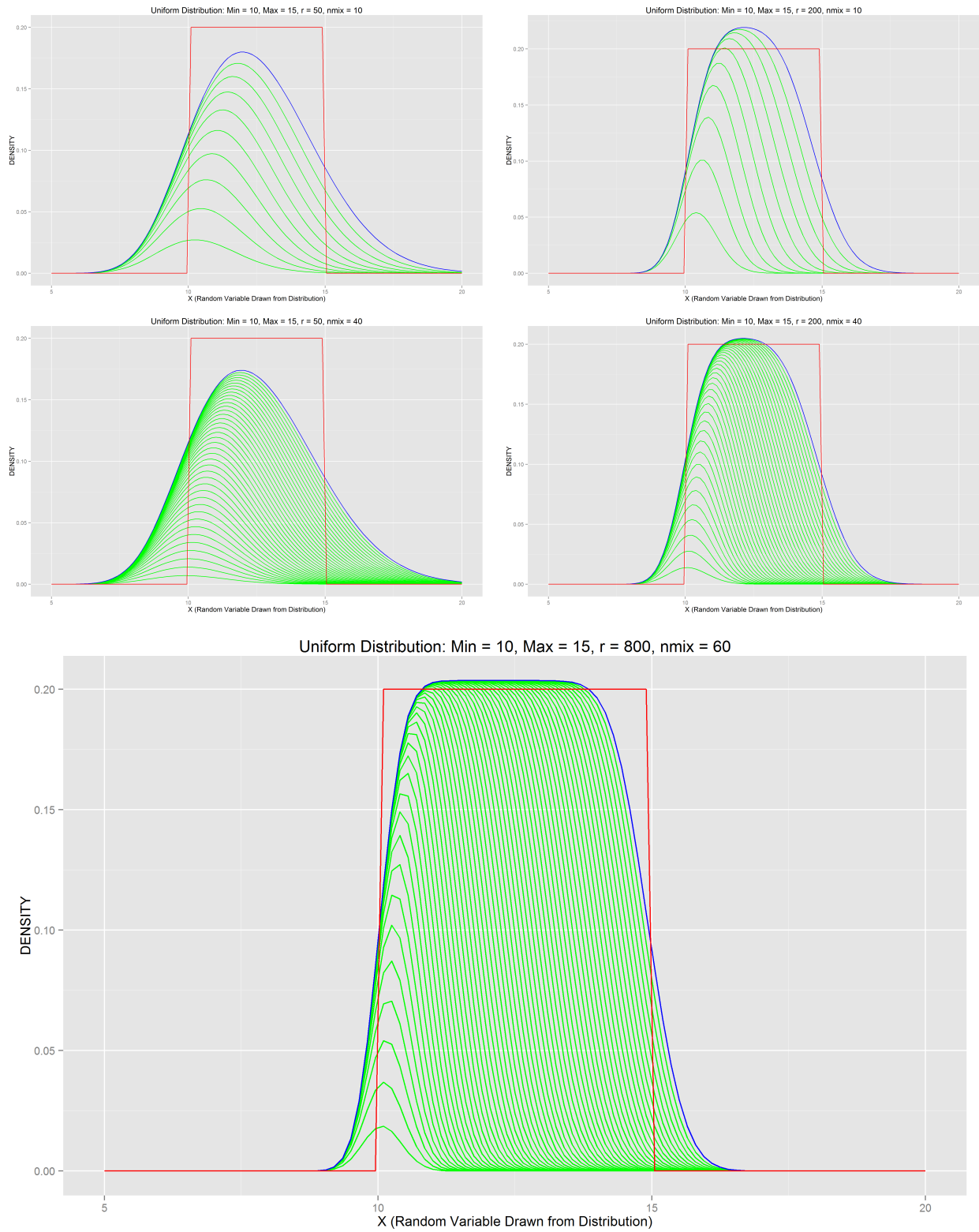## Problem 2.b

Using the lambdas generated by erlangmix(), we are able to generate a set of nmix erlang distributions with parameters given as: Shape = R Rate = lambda
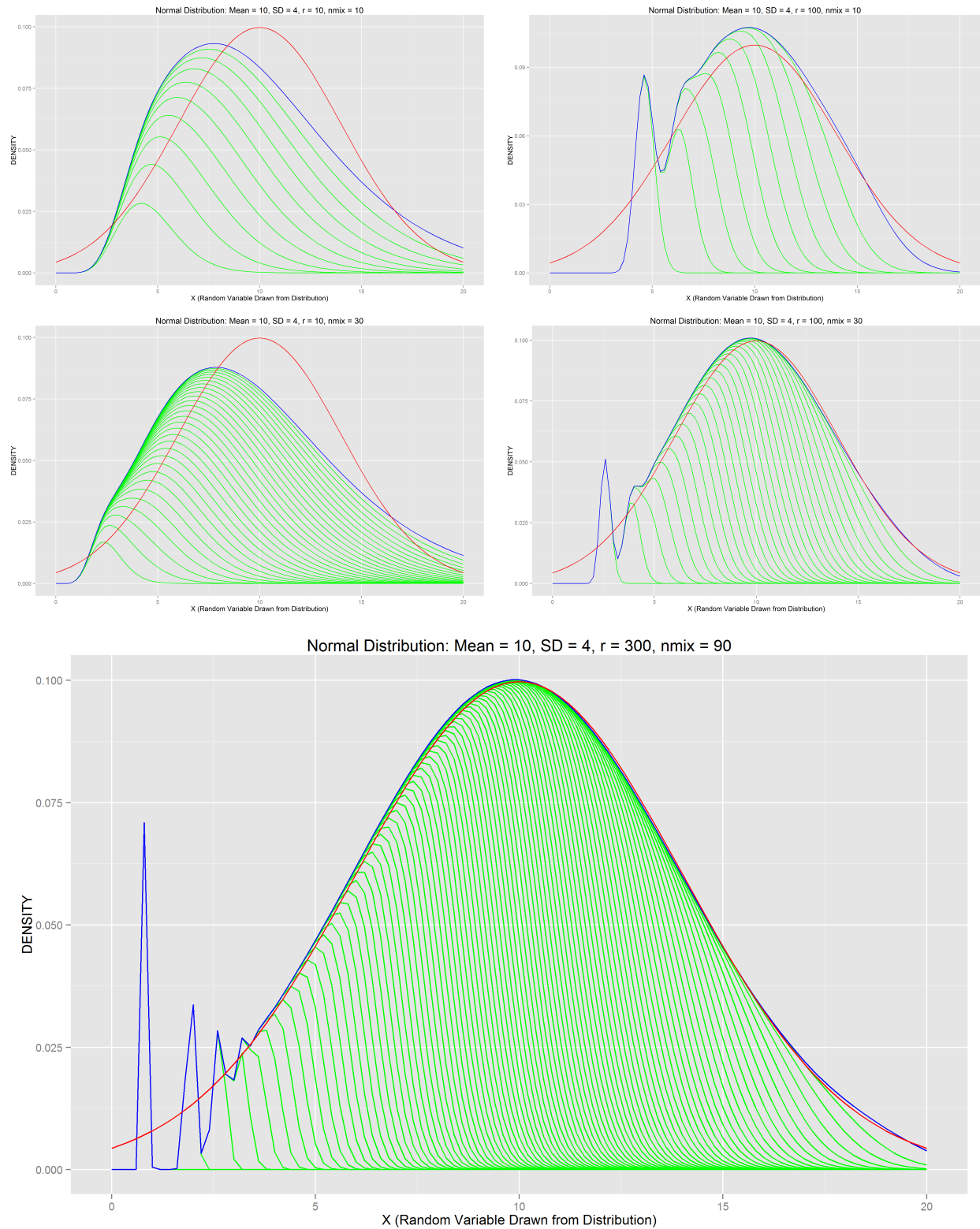
The combination of all nmix erlang distributions yields our method-of-stages approximation of the quantile function fed into erlangmix().

Here, we explored the effect of different values of r and nmix on the approximation.

For a uniform distribution with minimum = 10, maximum = 15:

Uniform Distribution: Min = 10, Max = 15, r = 50, nmix = 10

Uniform Distribution: Min = 10, Max = 15, r = 200, nmix = 10

Uniform Distribution: Min = 10, Max = 15, r = 50, nmix = 40

Uniform Distribution: Min = 10, Max = 15, r = 200, nmix = 40

Uniform Distribution: Min = 10, Max = 15, r = 800, nmix = 60

For a normal distribution with mean = 10, standard deviation = 4:

Normal Distribution: Mean = 10, SD = 4, r = 10, nmix = 10

Normal Distribution: Mean = 10, SD = 4, r = 100, nmix = 10

Normal Distribution: Mean = 10, SD = 4, r = 10, nmix = 30

Normal Distribution: Mean = 10, SD = 4, r = 100, nmix = 30

Normal Distribution: Mean = 10, SD = 4, r = 300, nmix = 90

Problem 3.a

See `HtoF.R`.

```
1  htof = function(hftn, t, lower){
2    density_val = c()
3    for(val in t)
4    {
5      density_val = c(density_val, hftn(val) *
6             exp(-1*integrate(hftn, lower, val)$value))
7    }
8    return (density_val)
9  }
```

Problem 3.b

Given a hazard function, $h(t)$, the density function, $f(t)$, can be found as follows:
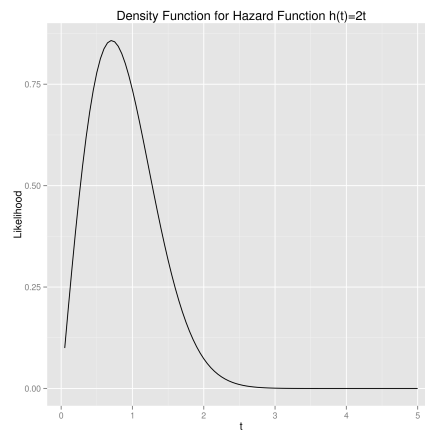
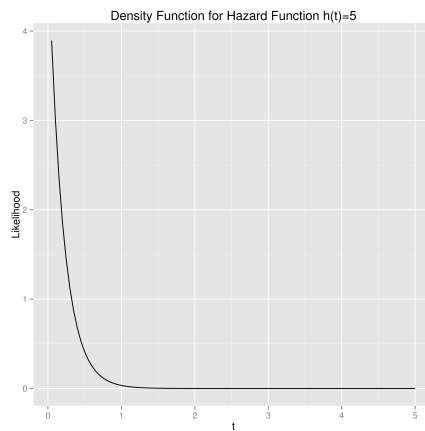$$f(t) = h(t) \cdot e^{-\int_0^t h(s)\,ds}$$

We looked at the following hazard functions to explore what their density would look like:
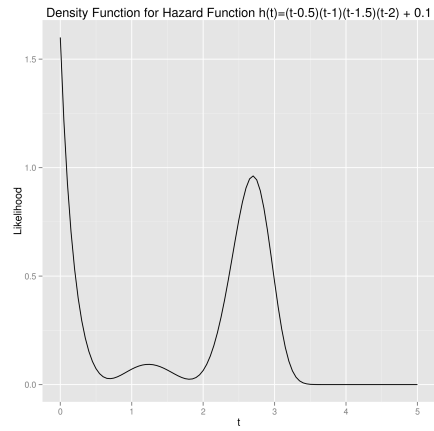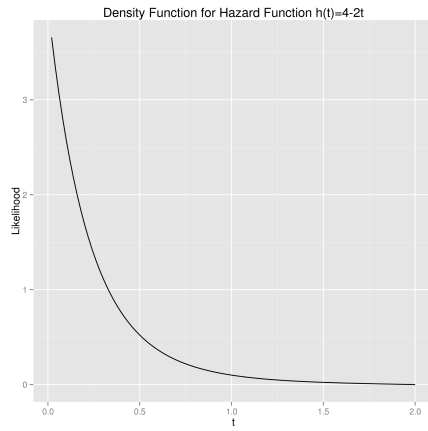
$$h(t) = 5$$
$$h(t) = 2t$$
$$h(t) = 4 - 2t$$
$$h(t) = (t - 0.5)(t - 1)(t - 1.5)(t - 2) + 0.1$$

Density Function for Hazard Function h(t)=4-2t

Density Function for Hazard Function h(t)=(t-0.5)(t-1)(t-1.5)(t-2) + 0.1

(Plots generated with `3.R`).

Problem 4.a-b

See `4.R` for generating these plots.