ECS 256 - Problem set 2

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Problem 1.a

A coin is flipped k times with p probability of heads. For each head, the coin is flipped one additional time (a bonus flip). The number of bonus flips is referred to as Y and the total number of heads X

Var(X) can be found using the Law of Total Variance, and properties of binomial distributions. We will also need to use part of the derivation of EX:

$$\begin{aligned} E(X|Y) &= E(X - Y + Y|Y) \\ &= E((X - Y)|Y) + E(Y|Y) && \text{(by 3.13)} \\ &= pY + Y && \text{(by 3.110)} \\ &= (1 + p)Y \end{aligned}$$

$$\begin{aligned} Var(X) &= E[Var(X|Y)] + Var[E(X|Y)] && \text{(by 9.8)} \\ &= E[Var(X|Y)] + Var[(1 + p)Y] && \text{(from above)} \\ &= E[Var(X|Y)] + (1 + p)^2 kp(1 - p) && \text{(by 3.34 and 3.109)} \\ &= E[Yp(l - p)] + (1 + p)^2 kp(1 - p) && \text{(by 3.111)} \\ &= kp^2(1 - p) + (1 + p)^2 kp(1 - p) && \text{(by 3.103)} \\ &= kp(1 - p)\left(p + (1 + p)^2\right) \end{aligned}$$

Using p=0.5

$$\begin{aligned} &= k(0.25)(0.5 + (1.5^2)) \\ &= 0.6875k \end{aligned}$$

Problem 1.b

In the trapped miner example, a miner chooses between three doors with only one leading to safety after 2 hours. The other two doors lead back to the door room after 3 and 5 hours respectively.

We are interesting the variance of Y, the time it takes to escape the mine. We will build upon Ahmed Ahmedin's solution to EY, where N refers to the total attempts needed to escape and U_i refers to the time spent traveling on the i^{th} attempt.

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\begin{aligned}
Var(Y) &= E[Var(Y|N)] + Var[E(Y|N)] && \text{(by 9.8)}\\
&= E[Var(Y|N)] + Var[4N-2] && \text{(by 9.16)}\\
&= E[Var(Y|N)] + 16Var[N] && \text{(by 3.34 and 3.41)}\\
&= E[Var(Y|N)] + 16 \cdot \frac{1-1/3}{(1/3)^2} && \text{(by 3.93)}\\
&= E[Var(U_1 + U_2 + ... + U_n|N)] + 96 &&\\
&= E[Var(U_1|N) + ... + Var(U_{N-1}|N + Var(U_N|N)] + 96 && \text{(by 3.51)}\\
&= E[1 + 1 + ...1 + 0] + 96 &&\\
&= E[N-1] + 96 &&\\
&= E[N] - 1 + 96 && \text{(by 3.17)}\\
&= 3 - 1 + 96 && \text{(by 3.92)}\\
&= 98 &&
\end{aligned}
$$

We know that Var(U_i—N) is independent because the miner's choice of door does not depend of a previous choice. Since we are conditioning this event on there being N attempts, the values of the first N-1 attempts will either be 3 or 5. So the variance of an individual attempt in this case is 1. The variance of the N^{th} attempt is 0 because that attempt always is the same tunnel.

Problem 2.a
For a vector Q of random variables $(Q_1, ..Q_n)$ we have:

$$Cov(Q) = E(QQ') - E(Q)E(Q') \quad \text{(by 13.53)}$$

Let $Q = Y|X$, where Y is vector valued. Then:

$$Cov(Y|X) = E\big((Y|X)(Y|X)'\big) - E(Y|X)E(Y|X)' \quad \text{(by 13.53)}$$

Taking expected value of both sides we have:

$$\begin{aligned} E\big(Cov(Y|X)\big) &= E\Big(E\big((Y|X)(Y|X)'\big) - E(Y|X)E(Y|X)'\Big) \\ &= E\Big(E\big((Y|X)(Y|X)'\big)\Big) - E\Big(E(Y|X)E(Y|X)'\Big) \\ &= E(YY') - E\Big(E(Y|X)E(Y|X)'\Big) && \text{(by Law of Tot. Expect.)} \end{aligned}$$

Now let $Q = E(Y|X)$, where Y is vector valued. Then:

$$\begin{aligned} Cov(E(Y|X)) &= E\big(E(Y|X)E(Y|X)'\big) - E\big(E(Y|X)\big)E\big(E(Y|X)\big)' && \text{(by 13.53)} \\ &= E\big(E(Y|X)E(Y|X)'\big) - E(Y)E(Y)' && \text{(by Law of Tot. Expect.)} \end{aligned}$$

Summing up the left sides and the right sides of these 2 equations we get:

$$\begin{aligned} E\big(Cov(Y|X)\big) + Cov(E(Y|X)) &= E(YY') - E\Big(E(Y|X)E(Y|X)'\Big) \\ &\quad + E\big(E(Y|X)E(Y|X)'\big) - E(Y)E(Y)' \\ E\big(Cov(Y|X)\big) + Cov(E(Y|X)) &= E(YY') - E(Y)E(Y)' \\ E\big(Cov(Y|X)\big) + Cov(E(Y|X)) &= Cov(Y) && \text{(by 13.53)} \end{aligned}$$

Problem 2.b

First, just an equation to remind us of what we're actually trying to find here, the correlation between X and Y.

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

From the problem **2.a**, we have the following.

Let $B = X - Y$ and use that random variable in the following computations.

$$Cov(W) = E\big(Cov(W|Z)\big) + Cov\big(E(W|Z)\big)$$

Now we'll allow $W = (B, Y)'$ and $Z = Y$ in order to calculate $Cov(B, Y)'$.

$$Cov(B, Y) = E\big[Cov(B, Y|Y)\big] + Cov\big[E(B, Y|Y)\big]$$

$$\begin{aligned} Var(B + Y) &= Var(B) + Var(Y) + 2Cov(B, Y) \\ Cov(B, Y) &= \big(Var(B + Y) - Var(B) - Var(Y)\big)/2 \end{aligned}$$

$$\begin{aligned} Cov\big(B, Y|Y\big) &= \big(Var(B + Y|Y) - Var(B|Y) - Var(Y|Y)\big)/2 \\ Var(B + Y|Y) &= Var(B|Y) \\ Var(Y|Y) &= 0 \\ Cov\big(B, Y|Y\big) &= 0 \end{aligned}$$

$$\begin{aligned} Cov\big(E(B, Y|Y)\big) &= \big(Var(EB + EY|Y) - Var(EB|Y) - Var(EY|Y)\big)/2 \\ &= \big(Var(1.5Y) - Var(0.5Y) - Var(Y)\big)/2 \\ &= \big(2.25Var(Y) - 0.25Var(Y) - Var(Y)\big)/2 \\ &= Var(Y) \end{aligned}$$

Hence:

$$\begin{aligned} Cov\big(B, Y\big) &= 0 + Var(Y) \\ &= Var(Y) \\ &= kp(1 - p) \\ &= 0.25k \end{aligned}$$

Now, let's see how we can get $Cov\big(X, Y\big)$ using that fact that we now know $Cov\big(B, Y\big)$. Remember, $X = B + Y$

$$\begin{aligned} Cov\big(X, Y\big) &= Cov\big(B + Y, Y\big) \\ Cov\big(B + Y, Y\big) &= Cov\big(B, Y\big) + Cov\big(Y, Y\big) \quad (13.2) \\ &= Cov\big(B, Y\big) + Var(Y) \end{aligned}$$

Now we have all the ingredients to find the correlation:

$$
\begin{aligned}
\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\
&= \frac{Cov\big(B,Y\big) + Var(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\
&= \frac{0.25k + 0.25k}{\sqrt{0.6875k}\sqrt{0.25k}} \\
&= \frac{0.5k}{k\sqrt{0.6875}\sqrt{0.25}} \\
&= \frac{0.5}{\sqrt{0.6875 * 0.25}} \\
&= 1.2
\end{aligned}
$$

Problem 3

$$\rho(X_{i+j}, X_i) = \frac{\text{Cov}(X_{i+j}, X_i)}{\sqrt{\text{Var}(X_{i+j})} \cdot \sqrt{\text{Var}(X_i)}}$$

A couple things we noticed:

$$E(X_i) = \sum_{l=1}^{n} l \cdot \pi_l$$

$$E(X_{i+j}|X_i) = \sum_{l=1}^{n} l \cdot m_{k,l}^{j}$$

where M^j is the transition matrix $M \cdot M \cdots M$ (j times), and $X_i = k$. First, let's derive $\text{Cov}(X_{i+j}, X_i)$.

$$\text{Cov}(X_{i+j}, X_i) = E(X_{i,j}X_i) - E(X_{i+j})E(X_i)$$

Suppose $Q = X_{i,j}X_i$. By the law of total expectations, $E(Q) = E(E(Q|X_i))$. Thus,

$$\begin{aligned} E(X_{i,j}X_i) &= \sum_{l=1}^{n} \pi_l l \cdot E(X_{i+j}|X_i) \\ &= \sum_{l=1}^{n} \left[\pi_l l \cdot \left[\sum_{k=1}^{n} k \cdot m_{l,k}^{j} \right] \right]. \end{aligned}$$

Since $X_{i,j}$ and X_i have the same distribution (Π), the covariance becomes

$$\text{Cov}(X_{i+j}, X_i) = \sum_{l=1}^{n} \left[\pi_l l \cdot \left[\sum_{k=1}^{n} k \cdot m_{l,k}^{j} \right] \right] - (EX_i)^2.$$

Next, we derive $\text{Var}(X_{i,j})$ and $\text{Var}(X_i)$. **TODO**