# ECS256 - Homework III

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## Problem 1.a

First, we'll derive $\pi_i$. The definition of the tree searching markov model leads to the following set of balance equations for the long-run state probabilities:

$$\pi_i = \pi_{i-1}q_{i-1} = \pi_0 \prod_{j=0}^{i-1} q_j \quad \text{for } i \geq 1, \text{ and}$$

$$\pi_0 = \sum_{i=1}^{\infty} \pi_i(1 - q_i) \quad \text{for } i = 0.$$

This definition for $\pi_0$ is a bit unweildy. We can also think of this quantity as one over the expected recurrance time, as in eq. (10.63) in the book:

$$\pi_0 = \frac{1}{E(T_{0,0})}$$

$$E(T_{0,0}) = 1 + \sum_{k \neq 0} p_{0,k}E(T_{k,0})$$

$$= 1 + p_{0,1}E(T_{1,0})$$

$$= 1 + p_{0,1}(1 + \sum_{k \neq 0} p_{1,k}E(T_{k,0}))$$

$$= 1 + p_{0,1}(1 + p_{1,2}E(T_{2,0}))$$

$$= 1 + p_{0,1}(1 + p_{1,2}(1 + \sum_{k \neq 0} p_{2,k}E(T_{k,0})))$$

$$= 1 + p_{0,1}(1 + p_{1,2}(1 + p_{2,3}E(T_{3,0})))$$

and so on. This unravels into a familiar closed form:

$$E(T_{0,0}) = 1 + q_0(1 + q_1(1 + q_2(1 + \ldots)\ldots)))$$

$$= 1 + q_0 + q_0q_1 + q_0q_1q_2 + \ldots$$

$$= 1 + \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j]$$

If the model is positive recurrent, then there exists some value $R$ such that

$$R = \sum_{i=1}^{\infty} [\prod_{j=0}^{i-1} q_j] < \infty.$$

Thus,
$$\pi_i = \frac{\prod_{j=0}^{i-1} q_j}{1+R} \quad \text{for } i \geq 0.$$

Next, $E(T_{i,0})$ follows a similar pattern.

$$E(T_{i,0}) = 1 + \sum_{k \neq 0} p_{i,k} E(T_{k,0})$$
$$= 1 + p_{i,i+1} E(T_{j+1,0})$$
$$= 1 + q_i + q_i q_{i+1} + q_i q_{i+1} q_{i+2} + \dots$$
$$= 1 + \sum_{j=i}^{\infty} \left[ \prod_{k=i}^{j} q_k \right].$$

## Problem 1.b

If $q_i = 0.5$ for all $i$, then $R$ is a geometric series that indeed converges.

$$\pi_2 = \frac{0.5 \cdot 0.5}{1 + \sum_{i=1}^{\infty} 0.5^{i-1}} = \frac{0.25}{1+2} \approx 0.083.$$

$$E(T_{2,0}) = 1 + \sum_{j=2}^{\infty} 0.5^{j-2} = 1 + \sum_{j=1}^{\infty} 0.5^{j-1} = 1 + 2 = 3.$$

## Problem 1.c

The rate of backtracking, in terms of the stationary probabilities $\pi_i$, is simply
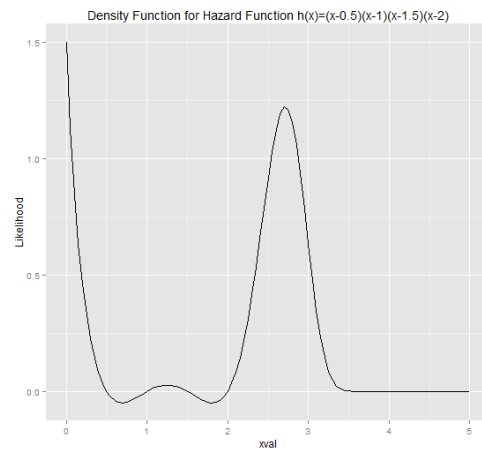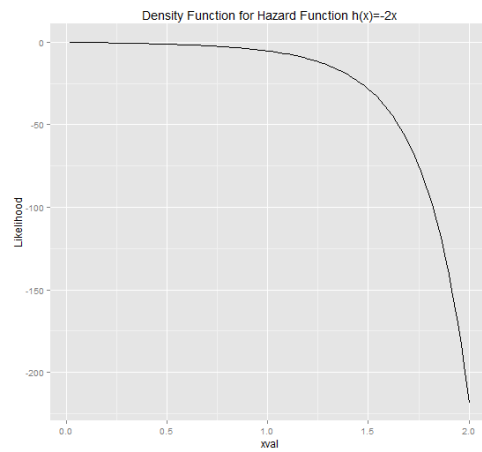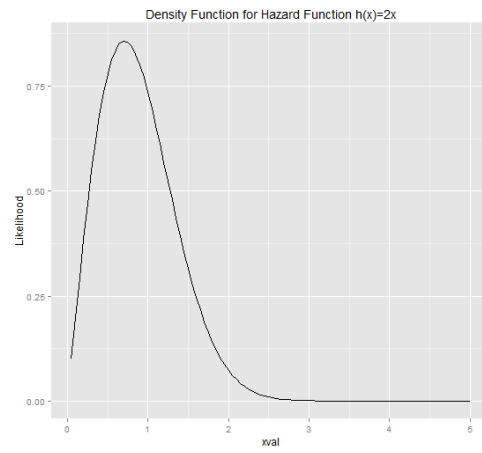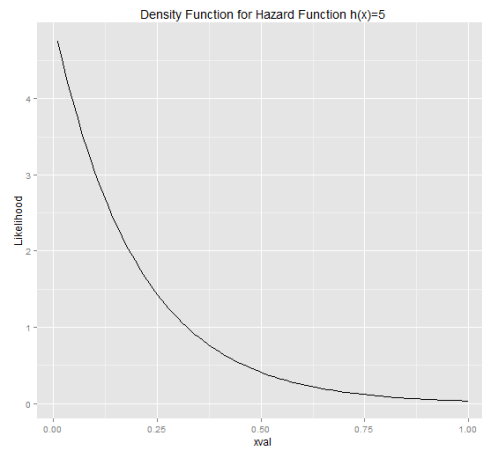
$$\sum_{i=1}^{\infty} \pi_i (1 - q_i).$$

## Problem 3

Given a hazard function, $h(t)$, the density function, $f(t)$, can be found as follows:
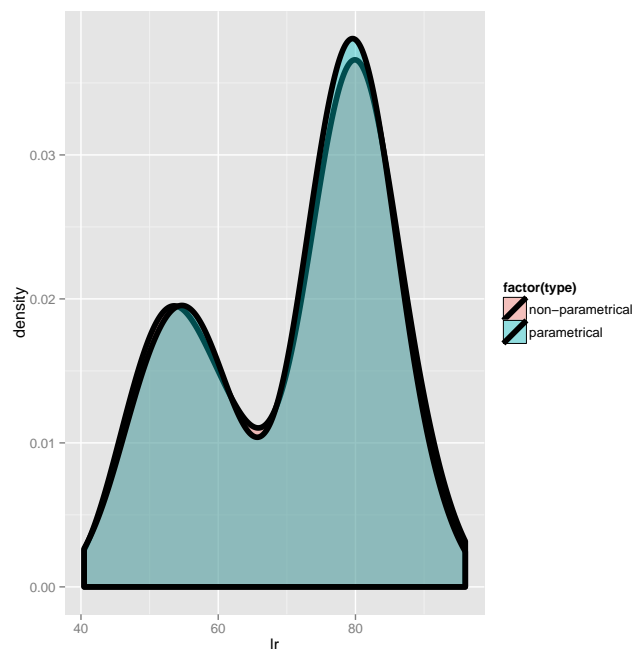$$f(t) = h(t) \cdot e^{-\int_0^t h(s) \, ds}$$

We looked at the following hazard functions to explore what their density would look like:

$$h(t) = 5$$
$$h(t) = 2t$$
$$h(t) = -2t$$
$$h(t) = (t - 0.5)(t - 1)(t - 1.5)(t - 2)$$

Density Function for Hazard Function h(x)=5

Density Function for Hazard Function h(x)=2x

Density Function for Hazard Function h(x)=-2x

Density Function for Hazard Function h(x)=(x-0.5)(x-1)(x-1.5)(x-2)

# Problem 4

**4.a-b**

# Appendix

Problem 4

```
1  #install.packages("ggplot2")
2  #install.packages("mixtools")
3  library(mixtools)
4  library(ggplot2)
5
6  #4.a ##############################
7  p<-ggplot(data.frame(faithful))
8  p+geom_density(aes(x=faithful$waiting))
9
10 #4.b ##############################
11 simulateFromDist <- function(n,p1,m1,s1,m2,s2){
12         k1 <- p1*n #proportion of type 1
13         k2 <- n-k1+1 #proportion of type 2
14         x1 <- rnorm(k1, mean=m1, sd=s1)
15         x2 <- rnorm(k2, mean=m2, sd=s2)
16         c(x1,x2) #order of events doesn't matter for histogram, so simply concatenate
17 }
18
19 ####from mixtools simulation
20 mixout<-normalmixEM(faithful$waiting,lambda=0.5,mu=c(55,80),sigma=10,k=2)
21 str(mixout)
22 # $ lambda    : num [1:2]  0.361  0.639
23 # $ mu        : num [1:2]  54.6  80.1
24 # $ sigma     : num [1:2]  5.87  5.87
25
26
27 # Is it necessary to simulate this? Can we plot the function directly?
28 sim_waiting<-simulateFromDist(length(faithful$waiting),0.361,54.6,5.87,80.1,5.87)
29
30 data <- rbind( data.frame(type="non-parametrical", lr=faithful$waiting), data.frame(type="parametr
```

4

```
31 m <- ggplot(data, aes(x=lr))
32 m <- m + geom_density(aes(fill=factor(type)), size=2, alpha=.4)
33
34 #save m
35 pdf('plot4b.pdf')
36 m
37 dev.off()
38
39 #4.f  ####################
40
41 EL2 <- ( mixout$sigma[1]^2 * mixout$lambda[1] + mixout$sigma[2]^2 * mixout$lambda[2] ) + (mixout$m
42 EL <- mixout$mu[1] * mixout$lambda[1] + mixout$mu[2] * mixout$lambda[2]
```