ECS289F Progress report
— Opinion Dynamics with reluctant agents —

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This document reports on the recent progress we have made for the course project in the two weeks after the project proposal's submission. The goal of this project is to consider a new aspect for the DeGroot's model by considering an opinion dynamic model where a subset of agents are *reluctant* to update their opinion.

Let us first recap on the system model. We consider an undirected graph $G = (V, E)$ with $|V| = n$. Each agent $i \in V$ holds an initial opinion $\boldsymbol{w}_i^{(0)} \in \mathbb{R}^L$. At time k, the agents exchange their beliefs with the others to compute:

$$\hat{\boldsymbol{w}}_i^{(k)} = \sum_{j \in \mathcal{N}_i} A_{ij}^{(k)} \boldsymbol{w}_j^{(k-1)}, \tag{1}$$

where $0 \leq A_{ij}^{(k)} \leq 1$ models the trust agent i has on agent j at time k. Importantly, we assume $\sum_{j=1}^{|V|} A_{ij}^{(k)} = 1$ and $\sum_{i=1}^{|V|} A_{ij}^{(k)} = 1$. Notice that the matrix $\boldsymbol{A}^{(k)}$ is time-variant.

The vector $\hat{\boldsymbol{w}}_i^{(k)}$ is the opinion that agent i is supposed to hold at time k. In DeGroot's model, the agents are updating instantly such that $\boldsymbol{w}_i^{(k)} = \hat{\boldsymbol{w}}_i^{(k)}$. In this case, it is known that $\boldsymbol{w}_i^{(k)}$ converges to the average of $\{\boldsymbol{w}_i^{(0)}\}$ asymptotically, i.e., achieving the 'wisdom of the crowd', under some mild assumptions. However, in our model, some agents are reluctant such that they don't update *immediately.* Instead, $\boldsymbol{w}_i^{(k)}$ is updated by:

$$\boldsymbol{w}_i^{(k)} = \frac{\min\{c_i^{(k)}, \tau_i\}}{\tau_i} \hat{\boldsymbol{w}}_i^{(k-c_i^{(k)}+1)} + \frac{\tau_i - \min\{c_i^{(k)}, \tau_i\}}{\tau_i} \boldsymbol{w}_i^{(k-c_i^{(k)})},\ i \in V_r, \tag{2}$$

where $V_r \subseteq V$ is the set of reluctant agents and

$$c_i^{(k)} = \begin{cases} 1 & \text{, if } A_{ij}^{(k)} \neq 0, \text{ for some } j \in V \text{ (agent } i \text{ talked at time } k). \\ c_i^{(k-1)} + 1 & \text{, otherwise.} \end{cases}$$

is a counter variable and $\tau_i \in \mathbb{Z}$ is the adaptation rate of i. In other words, the reluctant agent will slowly adapt to the new opinion in τ_i time steps. Notice that the 'normal' agents are special case of this with $\tau_i = 1$.

1 (Preliminary) Convergence Analysis

In this section, we perform a convergence analysis for the proposed opinion dynamics model based on []. The main result (so far) is that we have proved that the opinions will converge to a (biased) consensus with high probability. As a preliminary observation, we found that the converged opinion will be biased towards the initial opinions of the reluctant agent in expectation.

We found that our proposed model can be analyzed under the framework of [], which studied a delayed consensus model. Specifically, [] considered an augmented system with a few extra nodes proportional to the maximum delay allowed in the system. The augmented system is delay-free and it can subsequently be analyzed using the available tools.

Let us apply the above approach to our model. We consider a directed graph $G' = (V', E')$ where V' contains all the nodes from V together with a few augmented node, defined as follows. For each $i \in V_r$, we define $1 + 2(\tau_i - 1)$ new nodes denoted by $\{i'\} \cup \{i + N, ..., i + (\tau_i - 1)N\} \cup \{i' + N, ..., i' + (\tau_i - 1)N\}$. Here, the

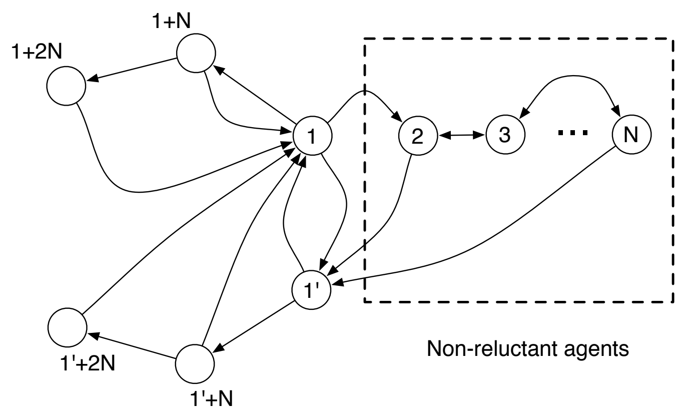

Figure 1: Example of the augmented system. Here, $V_r = \{1\}$ and $\tau_1 = 3$.

i'th node stores the value of $\hat{\boldsymbol{w}}_i^{(k)}$. The nodes $\{i+N, ..., i+(\tau_i-1)N\}$ and $\{i'+N, ..., i'+(\tau_i-1)N\}$ stores the delayed version of $\boldsymbol{w}_i$ and $\hat{\boldsymbol{w}}_i$, respectively. The inter-connectivity of these nodes are best illustrated by the example in Fig. 1.

Using the augmented system G', we can describe the equivalent opinion dynamics that executes on the graph. In particular, we split the update dynamics at time k to two steps k and k'. Step k corresponds to the update stage (1) and step k' corresponds to the adaptation stage (2). Additionally, the counter variable $c_i^{(k)}$ is defined as before and they are updated before step k'.

Let $\tilde{\boldsymbol{w}}_i$ be the opinion held by agent i in the augmented system, the system is initialized as follows.

$$c_i^{(0)} = M, \; \forall \; i \in V' \text{ and } \tilde{\boldsymbol{w}}_i^{(0)} = \begin{cases} \boldsymbol{w}_i^{(0)} & , \; i \in V \\ \boldsymbol{0} & , \; i \in V' \setminus V, \end{cases}$$

where $M > \tau_i$ for all i denotes that all the reluctant agents are not in the process of adapting to a new opinion.

Updating stage: At the *updating* stage, we have that

$$\tilde{\boldsymbol{w}}_i^{(k)} = \sum_{j \in \mathcal{N}_i} A_{ij}^{(k)} \tilde{\boldsymbol{w}}_j^{(k-1)'}, \; i \in V \setminus V_r,$$

i.e., the non-reluctant agents are updated instantaneously. Moreover, the updated opinion will be stored in the i'th node if i is a reluctant agent:

$$\tilde{\boldsymbol{w}}_{i'}^{(k)} = \sum_{j \in \mathcal{N}_i} A_{ij}^{(k)} \tilde{\boldsymbol{w}}_j^{(k-1)'}, \; i \in V_r,$$

The opinions at the other augmented nodes are updated as follows:

$$\tilde{\boldsymbol{w}}_{i+jN}^{(k)} = \tilde{\boldsymbol{w}}_{i+(j-1)N}^{(k-1)'}, \; j = 1, ..., \tau_i - 1, \; i \in V_r,$$

$$\tilde{\boldsymbol{w}}_{i'+jN}^{(k)} = \tilde{\boldsymbol{w}}_{i'+(j-1)N}^{(k-1)'}, \; j = 1, ..., \tau_i - 1, \; i \in V_r.$$

To summarize, let us define $\tilde{\boldsymbol{A}}^{(k)}$ as the updating matrix in this stage, we have:

$$\tilde{\boldsymbol{w}}_i^{(k)} = \sum_{j=1}^{|V'|} \tilde{A}_{ij}^{(k)} \tilde{\boldsymbol{w}}_j^{(k-1)'}$$

where $\tilde{\boldsymbol{A}}^{(k)}$ is:

$$\tilde{A}_{ij}^{(k)} = \begin{cases} A_{ij}^{(k)} & , \; i \in V \setminus V_r, \\ A_{ij}^{(k)} & , \; i = i', \\ 1 & , \; i = j \text{ and } i \in V_r, \\ 1 & , \; i = m + \ell N, \; j = m + (\ell - 1)N \text{ for some } m \in V_r, \\ 1 & , \; i = m' + \ell N, \; j = m' + (\ell - 1)N \text{ for some } m \in V_r, \\ 0 & , \; \text{otherwise.} \end{cases} \tag{3}$$

Adaptation stage: At the *adaptation* stage, the non-reluctant agents and the augmented nodes will remain stationary, i.e.,

$$\tilde{\boldsymbol{w}}_i^{(k')} = \tilde{\boldsymbol{w}}_i^{(k)}, \; i \in V' \setminus V_r.$$

Instead, the opinion of the reluctant agent will be adapted using information from the counter variable as follows:

$$\tilde{\boldsymbol{w}}_i^{(k')} = \begin{cases} \frac{c_i^{(k)}}{\tau_i} \tilde{\boldsymbol{w}}_{i' + (c_i^{(k)} - 1)N}^{(k)} + \frac{\tau_i - c_i^{(k)}}{\tau_i} \tilde{\boldsymbol{w}}_{i + c_i^{(k)} N}^{(k)} & , \text{ if } c_i^{(k)} \le \tau_i, \\ \tilde{\boldsymbol{w}}_i^{(k)} & , \text{ otherwise.} \end{cases}$$

With some efforts, the dynamics described above can also be written using an update matrix $\tilde{\boldsymbol{A}}^{(k')}$ as in the above.

To obtain our main result about the convergence of the proposed model, we observe that both $\tilde{\boldsymbol{A}}^{(k)}$ and $\tilde{\boldsymbol{A}}^{(k')}$ are stochastic as long as $\boldsymbol{A}^{(k)}$ is. To summarize, we have:

Proposition 1. *Assuming that $\{A_{ij}^{(k)}\}$ constitutes a sequence of opinion updates that lead to consensus in the DeGroot's model as $k \to \infty$, then the opinions in reluctant updates model will also reach a consensus as $k \to \infty$. In other words, we have*

$$\lim_{k \to \infty} \prod_{\ell=1}^{k} \tilde{\boldsymbol{A}}^{(\ell')} \tilde{\boldsymbol{A}}^{(\ell)} = \mathbf{1} (\boldsymbol{\pi}^\star)^T,$$

for some $\boldsymbol{\pi}^\star \in \mathbb{R}^{|V'|}$

2 (Preliminary) Simulation Studies