ECS289F Progress report
— Opinion Dynamics with reluctant agents —

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This document reports on the recent progress we have made for the course project in the two weeks after the project proposal's submission. The goal of this project is to consider a new aspect for the DeGroot's model by considering an opinion dynamic model where a subset of agents are *reluctant* to update their opinion.

Let us first recap on the system model. We consider an undirected graph $G = (V, E)$ with $|V| = n$. Each agent $i \in V$ holds an initial opinion $\boldsymbol{w}_i^{(0)} \in \mathbb{R}^L$. At time k, the agents exchange their beliefs with the others to compute:

$$\hat{\boldsymbol{w}}_i^{(k)} = \sum_{j \in \mathcal{N}_i} A_{ij}^{(k)} \boldsymbol{w}_j^{(k-1)}, \tag{1}$$

where $0 \leq A_{ij}^{(k)} \leq 1$ models the trust agent i has on agent j at time k. Importantly, we assume $\sum_{j=1}^{|V|} A_{ij}^{(k)} = 1$ and $\sum_{i=1}^{|V|} A_{ij}^{(k)} = 1$. Notice that the matrix $\boldsymbol{A}^{(k)}$ is time-variant and it is randomly generated at each k.

In the subsequent analysis, we shall focus on the so-called *randomized gossip model* []. The mixing matrix $\boldsymbol{A}^{(k)}$ is given as:

$$\boldsymbol{A}^{(k)} = \mathbf{I} - \frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)(\mathbf{e}_i + \mathbf{e}_j)^T,$$

where $(i, j) \in E$ is an edge of E selected uniformly at time k. The model represents the scenario when agents communicate by 'gossiping', i.e., at each time there can only be two agents exchanging idea with each other.

The vector $\hat{\boldsymbol{w}}_i^{(k)}$ is the opinion that agent i is supposed to hold at time k. In DeGroot's model, the agents are updating instantly such that $\boldsymbol{w}_i^{(k)} = \hat{\boldsymbol{w}}_i^{(k)}$. In this case, it is known that $\boldsymbol{w}_i^{(k)}$ converges to the average of $\{\boldsymbol{w}_i^{(0)}\}$ asymptotically, i.e., achieving the 'wisdom of the crowd', under some mild assumptions. However, in our model, some agents are reluctant such that they don't update *immediately*. Instead, $\boldsymbol{w}_i^{(k)}$ is updated by:

$$\boldsymbol{w}_i^{(k)} = \frac{\min\{c_i^{(k)}, \tau_i\}}{\tau_i} \hat{\boldsymbol{w}}_i^{(k - c_i^{(k)} + 1)} + \frac{\tau_i - \min\{c_i^{(k)}, \tau_i\}}{\tau_i} \boldsymbol{w}_i^{(k - c_i^{(k)})}, \; i \in V_r, \tag{2}$$

where $V_r \subseteq V$ is the set of reluctant agents and

$$c_i^{(k)} = \begin{cases} 1 & , \text{ if } A_{ij}^{(k)} \neq 0, \text{ for some } j \in V \text{ (agent } i \text{ talked at time } k). \\ c_i^{(k-1)} + 1 & , \text{ otherwise.} \end{cases}$$

is a counter variable and $\tau_i \in \mathbb{Z}$ is the adaptation rate of i. In other words, the reluctant agent will slowly adapt to the new opinion in τ_i time steps. Notice that the 'normal' agents are special case of this with $\tau_i = 1$.

1 (Preliminary) Convergence Analysis

In this section, we perform a convergence analysis for the proposed opinion dynamics model using randomized gossip exchange. The main result (so far) is that we can establish that the opinions will converge to a (biased) consensus almost surely. As a preliminary observation, we found that the converged opinion will be biased towards the initial opinions of the reluctant agent in expectation.

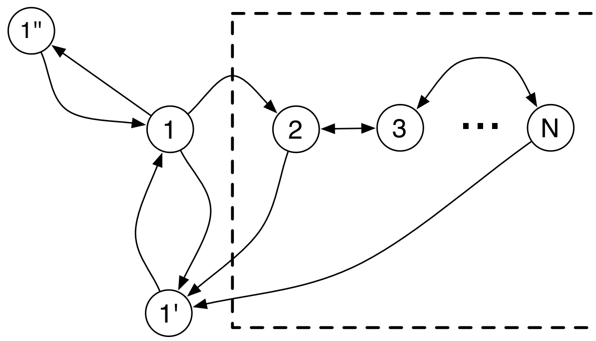

Figure 1: Example of the augmented system for $V_r = \{1\}$.

Our analysis is inspired by [] and applies the result in []. Specifically, we introduce the augmented nodes such that the reluctant opinion dynamics model can be recast into the simple form of

$$\tilde{\boldsymbol{w}}_i^{(k)} = \sum_{j=1}^{|V'|} \tilde{A}_{ij}^{(k)} \tilde{\boldsymbol{w}}_j^{(k-1)}$$

To this end, let us consider a directed graph $G' = (V', E')$ where V' contains all the nodes from V together with a few augmented node, defined as follows. For each $i \in V_r$, we define 2 new nodes denoted by $\{i', i''\}$. Here, the i'th node stores the value of $\hat{\boldsymbol{w}}_i^{(k)}$ and the i''th node stores the value of old $\boldsymbol{w}_i^{(k)}$. The interconnectivity of these nodes are best illustrated by the example in Fig. 1.

Using the augmented system G', we can describe the equivalent opinion dynamics that executes on the graph. In particular, we split the update dynamics at time k to two steps k and k'. Step k corresponds to the update stage (1) and step k' corresponds to the adaptation stage (2). Additionally, the counter variable $c_i^{(k)}$ is defined as before and they are updated before step k'.

Let $\tilde{\boldsymbol{w}}_i$ be the opinion held by agent i in the augmented system, the system is initialized as follows.

$$c_i^{(0)} = M, \ \forall \ i \in V' \text{ and } \tilde{\boldsymbol{w}}_i^{(0)} = \begin{cases} \boldsymbol{w}_i^{(0)} & , \ i \in V \\ \boldsymbol{0} & , \ i \in V' \setminus V, \end{cases}$$

where $M > \tau_i$ for all i enforces a 'reset' state for the adaptation process.

Updating stage: At the *updating* stage, we have that

$$\tilde{\boldsymbol{w}}_i^{(k)} = \sum_{j \in \mathcal{N}_i} A_{ij}^{(k)} \tilde{\boldsymbol{w}}_j^{(k-1)'}, \ i \in V \setminus V_r,$$

i.e., the non-reluctant agents are updated immediately. As for node $i \in V_r$, we have

$$\tilde{\boldsymbol{w}}_i^{(k)} = \tilde{\boldsymbol{w}}_i^{(k-1)'}$$

such that it is left intact. For node i' where $i \in V_r$, we have:

$$\tilde{\boldsymbol{w}}_{i'}^{(k)} = \begin{cases} \sum_{j \in \mathcal{N}_i} A_{ij}^{(k)} \tilde{\boldsymbol{w}}_j^{(k-1)'} & , \text{ if } c_i^{(k)} = 1. \\ \tilde{\boldsymbol{w}}_{i'}^{(k-1)'} & , \text{ otherwise}, \end{cases}$$

and for node i'' where $i \in V_r$,

$$\tilde{\boldsymbol{w}}_{i''}^{(k)} = \begin{cases} \tilde{\boldsymbol{w}}_{i}^{(k-1)'} & \text{, if } c_i^{(k)} = 1. \\ \tilde{\boldsymbol{w}}_{i''}^{(k-1)'} & \text{, otherwise.} \end{cases}$$

The two equations above describes the opinion dynamics that when the counter variable $c_i^{(k)}$ is reset, then the stored variables will be updated.

Let us define $\tilde{\boldsymbol{A}}^{(k)}$ as the overall updating matrix in this stage, i.e., we have:

$$\tilde{\boldsymbol{w}}_{i}^{(k)} = \sum_{j=1}^{|V'|} \tilde{A}_{ij}^{(k)} \tilde{\boldsymbol{w}}_{j}^{(k-1)'}.$$

It can be verified that $\tilde{\boldsymbol{A}}^{(k)}$ is stochastic.

Adaptation stage: At the *adaptation* stage, the non-reluctant agents and the augmented nodes will remain unchanged, i.e.,

$$\tilde{\boldsymbol{w}}_{i}^{(k')} = \tilde{\boldsymbol{w}}_{i}^{(k)}, \; i \in V' \setminus V_r.$$

However, the opinion of the reluctant agent will be adapted using the stored information in node i' and i'':

$$\tilde{\boldsymbol{w}}_{i}^{(k')} = \begin{cases} \dfrac{c_i^{(k)}}{\tau_i} \tilde{\boldsymbol{w}}_{i'}^{(k)} + \dfrac{\tau_i - c_i^{(k)}}{\tau_i} \tilde{\boldsymbol{w}}_{i''}^{(k)} & \text{, if } c_i^{(k)} \le \tau_i, \\ \tilde{\boldsymbol{w}}_{i}^{(k)} & \text{, otherwise.} \end{cases}$$

With some efforts, the dynamics described above can also be written using an update matrix $\tilde{\boldsymbol{A}}^{(k')}$. Such $\tilde{\boldsymbol{A}}^{(k')}$ is also found to be stochastic.

To summarize, we can now write the opinion dynamics as:

$$\tilde{\boldsymbol{w}}_{i}^{(k')} = \sum_{j=1}^{|V'|} \tilde{A}_{ij}^{(k')} \sum_{\ell=1}^{|V'|} \tilde{A}_{j\ell}^{(k)} \tilde{\boldsymbol{w}}_{\ell}^{(k-1)'} = \sum_{j=1}^{|V'|} \bar{A}_{ij}^{(k)} \tilde{\boldsymbol{w}}_{\ell}^{(k-1)'},$$

where $\bar{\boldsymbol{A}}^{(k)} = \tilde{\boldsymbol{A}}^{(k')} \tilde{\boldsymbol{A}}^{(k)}$ is a stochastic matrix for all k.

Given the above model, we are interested in proving its convergence property. It turns out that the result is not trivial to obtain. In particular, the main challenge is that the update matrix $\bar{\boldsymbol{A}}^{(k)}$ is correlated with its previous realizations, e.g., $\bar{\boldsymbol{A}}^{(k-1)}, ..., \bar{\boldsymbol{A}}^{(1)}$.

To prove the main result of the convergence analysis, we apply a recent result from []. Let us define

$$\boldsymbol{\Phi}(s+t, s) = \bar{\boldsymbol{A}}^{(s+t)} \bar{\boldsymbol{A}}^{(s+t-1)} \ldots \bar{\boldsymbol{A}}^{(s)}$$

The result from [] requires the so-called *balancedness* property. We claim that there exists an $\eta, t > 0$ such that:

$$E\left([\Phi(k+t,k)]_{ij} | \mathcal{F}_k\right) \ge \eta E\left([\Phi(k+t,k)]_{ji} | \mathcal{F}_k\right), \; \forall \; i, j, k, \tag{3}$$

where $\mathcal{F}_k = \{\bar{\boldsymbol{A}}^{(k-1)}, \bar{\boldsymbol{A}}^{(k-2)}, ..., \bar{\boldsymbol{A}}^{(1)}\}$ denotes the events that happened in the past (prior to the kth time step).

A rigorous proof for this claim would require some clumsy notations and will not be pursued here. To provide some insight, we can consider the original graph G to fully connected. We see that (3) can be satisfied for $t = 2$. The reason is that in the augmented system, there is a non-zero probability for any two nodes to communicate with each other in 2 hops.

As a consequence of the claim and [], we have:

Proposition 1. *As the reluctant update model constitutes a balanced process, the opinions in reluctant updates model reach consensus almost surely, i.e., we have*

$$\lim_{k\to\infty} P\left(\prod_{\ell=1}^{k} \bar{\boldsymbol{A}}^{(\ell)} = \mathbf{1}(\boldsymbol{\pi}^\star)^T\right) = 1, \tag{4}$$

for some $\boldsymbol{\pi}^\star \in \mathbb{R}^{|V'|}$.

I haven't looked into the proof of the theorem in [] yet.

1.1 Insights from the analysis (empirical)

Using the augmented system model, we now demonstrate that some insights can be obtained. In particular, we are interested in the value of $\boldsymbol{\pi}^\star$ in (4).

2 (Preliminary) Simulation Studies