The Asymptotics of Localizing Entanglement

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Introduction

Localization is a protocol that involves measuring a state and discarding a subset of subsystems.

Definition 1. The ***n-tangle*** is the entanglement measurement $\tau\colon (\mathbb{C}^2)^{\otimes n} \to [0,1]$ given by $\tau_n(|\psi\rangle) := |\langle\psi|\tilde{\psi}\rangle|$ where $|\tilde{\psi}\rangle := \sigma_y^{\otimes n}|\psi^*\rangle$.

Definition 2. Suppose a von Neumann measurement Π_A on subsystem A of a state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ produces an ensemble $\{(p_i, |\psi\rangle_i)\}_i$ of states over subsystem B. The ***multipartite entanglement assistance (MEA)*** of $|\Psi\rangle$ with respect to τ is defined as the maximal average post-measurement entanglement:

$$L^\tau_{\text{global}}(|\Psi\rangle) := \max_{\Pi_A} \textstyle\sum_i p_i \tau_{N_B}(|\psi_i\rangle).$$

In particular, if Π_A is *local* — i.e., the operators assume a tensor product form with respect to the subsystems in A — then such L^τ_{global} is alternatively called the ***localizable multipartite entanglement (LME)*** and denoted by L^τ.

Theorem 3. For a known $C > 0$, we have

$$\Pr_{|\Psi\rangle\sim\text{Haar}}\left(L^\tau_{\text{global}}(|\Psi\rangle) \le 1 - \sqrt{2d_B/d_A} - \varepsilon\right) \le 2\exp\left(-Cd_Ad_B\varepsilon^2\right),$$

which shows that when $d_A \gg d_B \gg 1$, the quantity L^τ_{global} is near maximal [1, Thm. 8].

We aim to obtain similar concentration results for L^τ and study its asymptotic behavior. Preliminary numerical estimation (see Fig. 1) suggests our hypothesis as follows.

Conjecture 4. For some function K of d_A and d_B, we have

$$\Pr_{|\Psi\rangle\sim\text{Haar}}\left(L^\tau(|\Psi\rangle) \ge K(d_A, d_B) + \varepsilon\right) \le 2\exp\left(-Cd_Ad_B\varepsilon^2\right)$$

with $\lim_{d_B\to\infty} K(d_A, d_B) = 0$.

The above holds when $K = \mathop{\mathbb{E}}_{|\Psi\rangle\sim\text{Haar}}[L^\tau(|\Psi\rangle)]$ [1]. Yet, the explicit form of K is unknown.

Notation	Definition
$\mathcal{H}_A$	Hilbert space of subsystem A
N_A	Number of qubits in A
d_A	Dimension of $\mathcal{H}_A$. Equals 2^{N_A}
Haar	Haar measure
$\mathcal{C}(\mathcal{H}_{A_i})$	Collection of orthonormal bases in qubit A_i
$\mathcal{P}(\mathcal{H}_A)$	Collection of orthonormal bases that have a tensor product form with respect to the subsystems in A

Table 1: Notations used in this poster

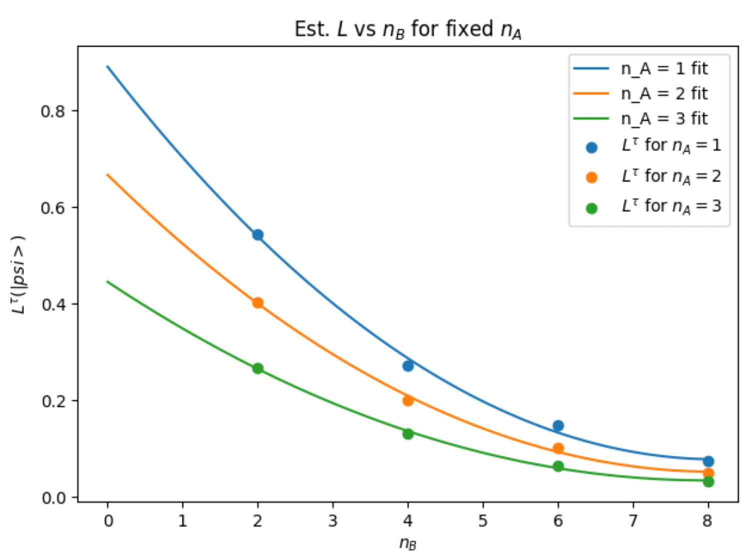

Figure 1: Exponential fitting of $\mathop{\mathbb{E}}_{|\Psi\rangle\sim\text{Haar}}[L^\tau(|\Psi\rangle)]$ for $N_A = 1, 2, 3$ as N_B increases

Methods

Given a basis $\beta = \{|\varphi_i\rangle\}_i \in \mathcal{P}(\mathcal{H}_A)$, the set $\{|\varphi_i\rangle\langle\varphi_i| \otimes I_B\}_i$ containing operators associated with a Π_A, and a state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we denote by $\overline{\tau}_\beta(|\Psi\rangle)$ the average N_B-tangle. We also denote $\beta_{\max} := \operatorname{argmax}_\beta \overline{\tau}_\beta(|\Psi\rangle)$. Note that by Def. 2, $\overline{\tau}_{\beta_{\max}}(|\Psi\rangle) = L^\tau(|\Psi\rangle)$

Recall that our goal is to bound $\Pr\left(L^\tau(|\Psi\rangle) \ge K(d_A, d_B) + \varepsilon\right)$. Lemma 22 of [1] suggests that

$$K(d_A, d_B) = \sqrt{\frac{2}{d_B + 1}}$$

is a decent choice as it is an upper bound of $\mathbb{E}\left[\overline{\tau}_\beta(|\Psi\rangle)\right]$. We can then prove the following using Levi's lemma [2, Lem. 53].

Lemma 5. Given a fixed basis $\gamma \in \mathcal{P}(\mathcal{H}_A)$, for all $\varepsilon, \delta > 0$ such that $\varepsilon - \sqrt{2}d_Ad_B\delta > 0$, we have

$$\Pr_{|\Psi\rangle\sim\text{Haar}}\left(L^\tau(|\Psi\rangle) \ge \sqrt{2/(d_B+1)} + \varepsilon \text{ and } \|\beta_{\max} - \gamma\|_B \le \delta\right) \le 2\exp\left(-2d_Ad_B(\varepsilon - \sqrt{2}d_Ad_B\delta)^2/9\pi^3(2+4\sqrt{2})^2\right).$$

Inspired by [3, Lem. II.4], we define Defs. 6 and 7.

Definition 6 (Basis-norm). For bases $\beta = \{|\varphi_i\rangle\}, \gamma = \{|\eta_i\rangle\}$ in $\mathcal{H}_A$,

$$\|\beta - \gamma\|_B := \max_i \||\varphi_i\rangle\langle\varphi_i| - |\eta_i\rangle\langle\eta_i|\|_1.$$

Definition 7 (Basis ε-net). For $\varepsilon > 0$, if a set $\mathcal{N} \subseteq \mathcal{P}(\mathcal{H}_A)$ satisfies that for all $\beta \in \mathcal{P}(\mathcal{H}_A)$ there exists $\gamma \in \mathcal{N}$ such that $\|\beta - \gamma\|_B \le \varepsilon$, then we call $\mathcal{N}$ a ***basis ε-net*** on $\mathcal{P}(\mathcal{H}_A)$.

Theorem 8. There exists a basis ε-net $\mathcal{N}$ on $\mathcal{P}(\mathcal{H}_A)$ with

$$|\mathcal{N}| \le [5(1+2\sqrt{2})^2N_A^2/\varepsilon^2]^{8N_A}.$$

The above theorem allows us to find a set of bases such that $\beta_{\max}$ is *close to* at least one of them. Combining the result with Lem. 5, we finally arrive at the Thm. 9.

Theorem 9. For $\varepsilon > 0$,

$$\Pr_{|\Psi\rangle\sim\text{Haar}}\left(L^\tau(|\Psi\rangle) \ge \sqrt{2/(d_B+1)} + \varepsilon\right) \le 2\left(\frac{40(1+2\sqrt{2})^2N_A^2d_A^2d_B^2}{\varepsilon^2}\right)^{8N_A}\exp\left(-\frac{d_Ad_B\varepsilon^2}{18\pi^3(2+4\sqrt{2})^2}\right).$$

Results

Observe the inequality displayed in Thm. 9: Suppose ε is fixed. We can notice that for a fixed d_A, the right-hand side can be written as

$$c_1 d_B^{16N_A}\exp\left(-c_2 d_B\right)$$

for some known positive reals c_1 and c_2, which obviously tends to 0 as d_B approaches infinity. Likewise, we can discover that for a fixed d_B, the right-hand side vanishes as d_A approaches infinity. In conclusion, we may state the following.

Corollary 10. Let $\varepsilon, \delta > 0$ be arbitrary. Then for any $d_B \ge 2$, there exists an $N_0 \in \mathbb{N}$ such that for all $d_A \ge 2^{N_0}$, we have

$$\Pr_{|\Psi\rangle\sim\text{Haar}}\left(L^\tau(|\Psi\rangle) \ge \sqrt{2/(d_B+1)} + \varepsilon\right) \le \delta.$$

Likewise, for any $d_A \ge 2$, there exists an $N_0' \in \mathbb{N}$ such that for all $d_B \ge 2^{N_0'}$, the above bound holds.

Discussion

Theorem 3 and Cor. 10 gives the following table.

	d_A fixed, $d_B \to \infty$	$d_A \to \infty$, d_B fixed
L^τ	$< \sqrt{2/(d_B+1)}$	$< \sqrt{2/(d_B+1)}$
L^τ_{global}	?	$> 1 - \sqrt{2d_B/d_A}$

Table 2: Bounds for typical values of L^τ and L^τ_{global} as d_A or d_B approaches infinity

Our next and perhaps last step is to find the value around which L^τ_{global} concentrates when d_A is fixed and d_B approaches infinity.

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