Asymptotics of Localizing Entanglement

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1 Preliminaries

Let us first introduce common notations used throughout the document.

Table 1: Common notations in this document

Notation	Definition
$\mathcal{H}$	Finite-dimensional Hilbert space
$\mathcal{L}(\mathcal{H})$	Set of linear operators over $\mathcal{H}$
$\mathrm{U}(d)$	Unitary group, or $\{U \in \mathcal{L}(\mathbb{C}^d) \mid U^\dagger U = I\}$
d_A	Dimension of system A, or $\dim \mathcal{H}_A$
$\mathcal{C}(\mathcal{H})$	Collection of ordered orthonormal bases of $\mathcal{H}$
$\mathcal{P}(\bigotimes \mathcal{H}_i)$	$\bigotimes_i \mathcal{C}(\mathcal{H}_i) = \{\bigotimes_i \beta_i \mid \beta_i \in \mathcal{C}(\mathcal{H}_i)\}$
$\|i\rangle$	Orthonormal basis (a set of vectors)
ψ	Shorthand for $\|\psi\rangle\langle\psi\|$, the density matrix associated with $\|\psi\rangle$
$\|\varphi\rangle\|\psi\rangle$	Shorthand for $\|\varphi\rangle \otimes \|\psi\rangle$
$f(\psi)$	Shorthand for $f(\|\psi\rangle)$ for some function f
$\|\widetilde{\psi}\rangle$	Wootter's tilde, given by $\sigma_y^{\otimes n}\|\psi^*\rangle$
$\tau_n(\|\psi\rangle)$	n-tangle, given by $\|\langle\psi\|\widetilde{\psi}\rangle\|$
$L^\tau(\|\Psi\rangle)$	Localizable entanglement of $\|\Psi\rangle$ with respect to the n-tangle
$\mathrm{Haar}(d)$	Haar measure on the unitary group $\mathrm{U}(d)$

1.1 Linear Algebra

Definition 1 (Vector p-norm)**.** *For $|v\rangle \in \mathbb{C}^d$ and $p \in [1,\infty]$, the* ***p-norm*** *of $|v\rangle$ is given by $\||v\rangle\|_p := \big(\sum_{i=1}^d |v_i|^p\big)^{1/p}$. In particular, $\||v\rangle\|_2 = \sqrt{\langle v|v\rangle}$ and $\||v\rangle\|_\infty = \max_i |v_i|$.*

Definition 2 (Matrix p-norm)**.** *For an operator A and $p \in [1,\infty]$, the* ***p-norm*** *of A is given by $\|A\|_p := \mathrm{tr}\left[(\sqrt{A^\dagger A})^p\right]^{1/p}$, which corresponds to the p-norm of the vector of singular values of A. In particular, we call $\|\cdot\|_1$ the* ***trace norm*** *and $\|\cdot\|_2$ the* ***Hilbert-Schmidt norm****.*

Definition 3 (State)**.** *A* ***(quantum) state*** *is represented by a vector $|\psi\rangle \in \mathcal{H}$ with $\||\psi\rangle\|_2 = 1$.*

Lemma 4. *For all normalized $|\psi\rangle \in \mathbb{C}^d$ and for all operators $A \in \mathcal{L}(\mathbb{C}^d)$, we have*

$$\mathrm{tr}[A|\psi\rangle\langle\psi|] = \langle\psi|A|\psi\rangle. \tag{1}$$

Proof. Let $\{|v_i\rangle\}_{i=1}^d$ be an orthonormal basis with $|v_1\rangle = |\psi\rangle$. Since the trace of an operator is invariant under a similarity transformation, we have

$$\mathrm{tr}[A|\psi\rangle\langle\psi|] = \textstyle\sum_{i=1}^d \langle v_i|A|\psi\rangle\langle\psi|v_i\rangle = \langle\psi|A|\psi\rangle. \tag{2}$$

□

Lemma 5. *For all states $|u\rangle, |v\rangle \in \mathcal{H}$, we have*

$$\| |u\rangle\langle u| - |v\rangle\langle v| \|_1 = 2\sqrt{1 - |\langle u|v\rangle|^2}. \tag{3}$$

Proof. Let $A = |u\rangle\langle u| - |v\rangle\langle v|$ for simplicity. Obviously, $\operatorname{rank}(A) \leq 2$, which implies that there are at most 2 non-zero eigenvalues of A: let us denote them by λ_1 and λ_2. Then

$$\lambda_1 + \lambda_2 = \operatorname{tr}[A] = \operatorname{tr}[u - v] = \operatorname{tr}[u] - \operatorname{tr}[v] = 1 - 1 = 0 \implies \lambda_2 = -\lambda_1. \tag{4}$$

Observe that

$$\begin{aligned} 2\lambda_1^2 = \lambda_1^2 + \lambda_2^2 = \operatorname{tr}[A^2] &= \operatorname{tr}[(u - v)^2] && (5) \\ &= \operatorname{tr}[u^2 - uv - vu + v^2] && (6) \\ &= \operatorname{tr}[u^2] - \operatorname{tr}[uv] - \operatorname{tr}[vu] + \operatorname{tr}[v^2] && (7) \\ &= 2 - 2|\langle u|v\rangle|^2, \qquad \text{(by Lemma 4)} && (8) \end{aligned}$$

which gives $|\lambda_1| = \sqrt{1 - |\langle u|v\rangle|^2}$. Obviously, A is Hermitian, so $\lambda_1, \lambda_2 \in \mathbb{R}$. Finally,

$$\begin{aligned} \|u - v\|_1 = \|A\|_1 = \operatorname{tr}[\sqrt{A^\dagger A}] = \operatorname{tr}[\sqrt{A^2}] = \sqrt{\lambda_1^2} + \sqrt{\lambda_2^2} = |\lambda_1| + |\lambda_2| = 2|\lambda_1| \\ = 2\sqrt{1 - |\langle u|v\rangle|^2}. \qquad (9) \end{aligned}$$

□

Corollary 6. *For all states $|u\rangle, |v\rangle \in \mathcal{H}$, we have*

$$\| |u\rangle\langle u| - |v\rangle\langle v| \|_1 \leq 2\| |u\rangle - |v\rangle \|_2. \tag{10}$$

Proof. By Lemma 5,

$$\begin{aligned} \| |u\rangle\langle u| - |v\rangle\langle v| \|_1 = 2\sqrt{1 - |\langle u|v\rangle|^2} &\leq 2\sqrt{2 - 2|\langle u|v\rangle|} && (11) \\ &\leq 2\sqrt{2 - 2\operatorname{Re}(\langle u|v\rangle)} && (12) \\ &= 2\sqrt{\langle u|u\rangle - \langle u|v\rangle - \langle v|u\rangle + \langle v|v\rangle} && (13) \\ &= 2\sqrt{(\langle u| - \langle v|)(|u\rangle - |v\rangle)} && (14) \\ &= 2\| |u\rangle - |v\rangle \|_2. && (15) \end{aligned}$$

□

Finally, we state the following without proof.

Theorem 7 (Triangle inequality for trace norms)**.** *For complex matrices M and N with same dimensions, we have the following inequality:*

$$\|M + N\|_1 \leq \|M\|_1 + \|N\|_1. \tag{16}$$

1.2 Basis Epsilon-Net

Definition 8 (ε-net)**.** *For $\varepsilon > 0$, if a set $\mathcal{N} \subseteq \mathcal{H}$ satisfies that for all states $|\varphi\rangle \in \mathcal{H}$ there exists a state $|\eta\rangle \in \mathcal{N}$ such that $\| |\varphi\rangle\langle\varphi| - |\eta\rangle\langle\eta| \|_1 \leq \varepsilon$, then we call $\mathcal{N}$ an **ε-net** on $\mathcal{H}$.*

Lemma 9. *For $\varepsilon \in (0, 1)$ and $\dim \mathcal{H} = d$ there exists an ε-net $\mathcal{M}$ on $\mathcal{H}$ with $|\mathcal{M}| \leq (5/\varepsilon)^{2d}$ [1, Lemma II.4].*

Definition 10 (Collection of bases)**.** *The collection of orthonormal bases in $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$.*

Definition 11 (Basis-norm)**.** *For $\dim \mathcal{H} = d$ and $\beta = \{|\varphi_i\rangle\}_{i=1}^d$, $\gamma = \{|\eta_i\rangle\}_{i=1}^d \in \mathcal{C}(\mathcal{H})$, we define*

$$\|\beta - \gamma\|_B := \max_{1\le i\le d} \||\varphi_i\rangle\langle\varphi_i| - |\eta_i\rangle\langle\eta_i|\|_1. \tag{17}$$

Definition 12 (Product basis)**.** *For $\beta = \{|\varphi_i\rangle\}_{i=1}^{d_1} \in \mathcal{C}(\mathcal{H}_1)$ and $\gamma = \{|\eta_j\rangle\}_{j=1}^{d_2} \in \mathcal{C}(\mathcal{H}_2)$, we define $\beta \otimes \gamma \in \mathcal{C}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to be the ordered basis*

$$\beta \otimes \gamma := \{|\varphi_1\rangle|\eta_1\rangle, \dots, |\varphi_{d_1}\rangle|\eta_1\rangle, |\varphi_1\rangle|\eta_2\rangle, \dots, |\varphi_{d_1}\rangle|\eta_2\rangle, \dots, |\varphi_1\rangle|\eta_{d_2}\rangle, \dots, |\varphi_{d_1}\rangle|\eta_{d_2}\rangle\}. \tag{18}$$

Definition 13 (Collection of product bases)**.** *For Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\dots$, $\mathcal{H}_n$ and $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$, we define*

$$\mathcal{P}(\mathcal{H}) := \textstyle\bigotimes_{i=1}^n \mathcal{C}(\mathcal{H}_i) = \{\bigotimes_{i=1}^n \beta_i \mid \forall i \in [\![n]\!],\, \beta_i \in \mathcal{C}(\mathcal{H}_i)\}. \tag{19}$$

Obviously, $\mathcal{P}(\mathcal{H}) \subset \mathcal{C}(\mathcal{H})$.

Proposition 14 (Basis-norm additivity)**.** *Given $\beta_1, \beta_2 \in \mathcal{C}(\mathcal{H}_1)$ and $\gamma_1, \gamma_2 \in \mathcal{C}(\mathcal{H}_2)$, we have*

$$\|\beta_1 \otimes \gamma_1 - \beta_2 \otimes \gamma_2\|_B \le \|\beta_1 - \beta_2\|_B + \|\gamma_1 - \gamma_2\|_B. \tag{20}$$

Proof. Let $n = \dim \mathcal{H}_1$ and $m = \dim \mathcal{H}_2$. We may write

$$\beta_1 = \{|\varphi_i\rangle\}_{i=1}^n, \quad \beta_2 = \{|\varphi_i'\rangle\}_{i=1}^n, \quad \gamma_1 = \{|\eta_j\rangle\}_{j=1}^m, \quad \gamma_2 = \{|\eta_j'\rangle\}_{j=1}^m.$$

Let $i \in [\![n]\!]$ and $j \in [\![m]\!]$ be integers such that

$$\|\beta_1 \otimes \gamma_1 - \beta_2 \otimes \gamma_2\|_B = \|\varphi_i \otimes \eta_j - \varphi_i' \otimes \eta_j'\|_1.$$

Then

$$\begin{aligned}
\|\beta_1 \otimes \gamma_1 - \beta_2 \otimes \gamma_2\|_B &= \|\varphi_i \otimes \eta_j - \varphi_i' \otimes \eta_j'\|_1 && (21)\\
&= \|\varphi_i \otimes \eta_j - \varphi_i' \otimes \eta_j + \varphi_i' \otimes \eta_j - \varphi_i' \otimes \eta_j'\|_1 && (22)\\
&\le \|\varphi_i \otimes \eta_j - \varphi_i' \otimes \eta_j\|_1 + \|\varphi_i' \otimes \eta_j - \varphi_i' \otimes \eta_j'\|_1 \qquad \text{(by Thm. 7)} && (23)\\
&= \|(\varphi_i - \varphi_i') \otimes \eta_j\|_1 + \|\varphi_i' \otimes (\eta_j - \eta_j')\|_1 && (24)\\
&= \|\varphi_i - \varphi_i'\|_1 + \|\eta_j - \eta_j'\|_1 && (25)\\
&\le \|\beta_1 - \beta_2\|_B + \|\gamma_1 - \gamma_2\|_B. \qquad \text{(by Def. 11)} && (26)
\end{aligned}$$

Note that Eq. (25) is obtained using Lemma 5:

$$\begin{aligned}
\|(\varphi_i - \varphi_i') \otimes \eta_j\|_1 &= \|\varphi_i \otimes \eta_j - \varphi_i' \otimes \eta_j\|_1 && (27)\\
&= 2\sqrt{1 - |(\langle\varphi_i|\langle\eta_j|)(|\varphi_i'\rangle|\eta_j\rangle)|^2} && (28)\\
&= 2\sqrt{1 - |\langle\varphi_i|\varphi_i'\rangle\langle\eta_j|\eta_j\rangle|^2} && (29)\\
&= 2\sqrt{1 - |\langle\varphi_i|\varphi_i'\rangle|^2} && (30)\\
&= \|\varphi_i - \varphi_i'\|_1 && (31)
\end{aligned}$$

and similarly $\|\varphi_i' \otimes (\eta_j - \eta_j')\|_1 = \|\eta_j - \eta_j'\|_1$. □

Definition 15 (Basis ε-net)**.** *Let $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$ be an n-partite Hilbert space. For $\varepsilon > 0$, if a set $\mathcal{N} \subseteq \mathcal{P}(\mathcal{H})$ satisfies that for all $\beta \in \mathcal{P}(\mathcal{H})$ there exists $\gamma \in \mathcal{N}$ such that $\|\beta - \gamma\|_B \le \varepsilon$, then we call $\mathcal{N}$ a **basis ε-net** on $\mathcal{P}(\mathcal{H})$.*

Proposition 16. *For $\varepsilon \in (0,1)$, there exists a basis $[(1+2\sqrt{2})\sqrt{\varepsilon}]$-net $\mathcal{N}$ on $\mathcal{C}(\mathbb{C}^2)$ with $|\mathcal{N}| \le (5/\varepsilon)^8$.*

Proof. Lemma 9 states that there exists an ε-net $\mathcal{N}_s$ on $\mathbb{C}^2$ with $|\mathcal{N}_s| \le (5/\varepsilon)^4$. Let

$$\mathcal{N} = \left\{ \left\{ |\eta_1\rangle, \frac{(I-\eta_1)|\eta_2\rangle}{\sqrt{\langle\eta_2|(I-\eta_1)|\eta_2\rangle}} \right\} : |\eta_1\rangle, |\eta_2\rangle \in \mathcal{N}_s \text{ and } \langle\eta_2|(I-\eta_1)|\eta_2\rangle > 0 \right\}. \tag{32}$$

Notice that

$$\frac{(I-\eta_1)|\eta_2\rangle}{\sqrt{\langle\eta_2|(I-\eta_1)|\eta_2\rangle}}$$

is the normalized projection of $|\eta_2\rangle$ onto the orthogonal complement of $|\eta_1\rangle\langle\eta_1|$. Hence, we verified that every element in $\mathcal{N}$ is an orthonormal basis, i.e., $\mathcal{N} \subseteq \mathcal{C}(\mathbb{C}^2)$.

Let $\beta = \{|\varphi_1\rangle, |\varphi_2\rangle\} \in \mathcal{C}(\mathbb{C}^2)$ be arbitrary. By Def. 8 there exist $|\eta_1\rangle, |\eta_2\rangle \in \mathcal{N}_s$ such that $\|\eta_1 - \varphi_1\|_1 \le \varepsilon$ and $\|\eta_2 - \varphi_2\|_1 \le \varepsilon$. Let

$$|\eta_2'\rangle = \frac{(I-\eta_1)|\eta_2\rangle}{\sqrt{\langle\eta_2|(I-\eta_1)|\eta_2\rangle}} \tag{33}$$

and $\gamma = \{|\eta_1\rangle, |\eta_2'\rangle\} \in \mathcal{N}$. Def. 11 gives

$$\|\beta - \gamma\|_B = \max\{\|\varphi_1 - \eta_1\|_1,\, \|\varphi_2 - \eta_2'\|_1\}. \tag{34}$$

We already have $\|\varphi_1 - \eta_1\|_1 \le \varepsilon$. Observe that

$$\begin{aligned}
\|\eta_1 - \eta_2\|_1 &= \|(\eta_1 - \varphi_1) - (\eta_2 - \varphi_2) + (\varphi_1 - \varphi_2)\|_1 && (35)\\
&\ge \|\varphi_1 - \varphi_2\|_1 - \|(\eta_1 - \varphi_1) - (\eta_2 - \varphi_2)\|_1 && (36)\\
&\ge \|\varphi_1 - \varphi_2\|_1 - (\|(\eta_1 - \varphi_1)\|_1 + \|(\eta_2 - \varphi_2)\|_1) && (37)\\
&\ge 2\sqrt{1 - |\langle\varphi_1|\varphi_2\rangle|^2} - 2\varepsilon \qquad \text{(by Lemma 5)} && (38)\\
&= 2 - 2\varepsilon. && (39)
\end{aligned}$$

The following relations give an upper bound for $\|\varphi_2 - \eta_2'\|_1$:

$$\begin{aligned}
\|\varphi_2 - \eta_2'\|_1 &= \|\varphi_2 - \eta_2 + \eta_2 - \eta_2'\|_1 && (40)\\
&\le \|\varphi_2 - \eta_2\| + \|\eta_2 - \eta_2'\|_1 && (41)\\
&\le \varepsilon + 2\sqrt{1 - |\langle\eta_2|\eta_2'\rangle|^2} \qquad \text{(by Lemma 5)} && (42)\\
&= \varepsilon + 2\sqrt{1 - \langle\eta_2|(I-\eta_1)|\eta_2\rangle} \qquad \text{(by Eq. (33))} && (43)\\
&= \varepsilon + 2\sqrt{1 - \langle\eta_2|\eta_2\rangle + |\langle\eta_1|\eta_2\rangle|^2} && (44)\\
&= \varepsilon + 2|\langle\eta_1|\eta_2\rangle| \qquad (\because \langle\eta_2|\eta_2\rangle = 1) && (45)\\
&= \varepsilon + 2\sqrt{1 - (\|\eta_1 - \eta_2\|_1/2)^2} \qquad \text{(by Lemma 5)} && (46)\\
&\le \varepsilon + 2\sqrt{1 - (1-\varepsilon)^2} \qquad \text{(by Eq. (35)–(39))} && (47)\\
&= \varepsilon + 2\sqrt{2\varepsilon - \varepsilon^2} && (48)\\
&\le \varepsilon + 2\sqrt{2}\sqrt{\varepsilon} && (49)\\
&\le (1+2\sqrt{2})\sqrt{\varepsilon} \qquad (\because \varepsilon < 1) && (50)
\end{aligned}$$

Thus, $\|\beta - \gamma\|_B$ is bounded above by $\max\{\varepsilon, (1+2\sqrt{2})\sqrt{\varepsilon}\} = (1+2\sqrt{2})\sqrt{\varepsilon}$. By Def. 15, $\mathcal{N}$ is indeed a basis $[(1+2\sqrt{2})\sqrt{\varepsilon}]$-net over $\mathcal{C}(\mathbb{C}^2)$.

As for the cardinality bound, Eq. (32) implies $\mathcal{N} \subseteq N_s \times N_s$. Thus, $|\mathcal{N}| \le |\mathcal{N}_s|^2 = (5/\varepsilon)^8$. □

Theorem 17. *For $n \in \mathbb{N}$ and $\varepsilon \in (0,(1+2\sqrt{2})n)$, there exists a basis ε-net $\mathcal{N}$ for $\mathcal{P}((\mathbb{C}^2)^{\otimes n})$ with*

$$|\mathcal{N}| \leq \left(\frac{5(1+2\sqrt{2})^2 n^2}{\varepsilon^2}\right)^{8n}.$$

Proof. Let $\delta \in (0,1)$ such that

$$\delta = \frac{(\varepsilon/n)^2}{(1+2\sqrt{2})^2} \iff (1+2\sqrt{2})\sqrt{\delta} = \frac{\varepsilon}{n}. \tag{51}$$

Prop. 16 states that there exists a basis (ε/n)-net $\mathcal{M}$ on $\mathcal{C}(\mathbb{C}^2)$ with

$$|\mathcal{M}| \leq \left(\frac{5}{\delta}\right)^8 = \left(\frac{5(1+2\sqrt{2})^2 n^2}{\varepsilon^2}\right)^8. \tag{52}$$

Set

$$\mathcal{N} = \left\{ \bigotimes_{i=1}^n \gamma_i \;\middle|\; \forall i \in [\![n]\!],\, \gamma_i \in \mathcal{M} \right\} \subseteq \mathcal{P}((\mathbb{C}^2)^{\otimes n}). \tag{53}$$

Let $\beta = \bigotimes_{i=1}^n \beta_i$ be an arbitrary basis in $\mathcal{P}((\mathbb{C}^2)^{\otimes n})$ and $\gamma_1, \ldots, \gamma_n$ be in $\mathcal{M}$. We apply Prop. 14 repetitively and obtain the following relations:

$$\begin{aligned} \left\| \bigotimes_{i=1}^n \beta_i - \bigotimes_{i=1}^n \gamma_i \right\|_B \leq \left\|\beta_1 - \gamma_1\right\|_B + \left\| \bigotimes_{i=2}^n \beta_i - \bigotimes_{i=2}^n \gamma_i \right\|_B \leq \cdots \\ \leq \textstyle\sum_{i=1}^n \|\beta_i - \gamma_i\|_B. \end{aligned} \tag{54}$$

Since every γ_i is in the basis (ε/n)-net $\mathcal{M}$, we can choose γ_i such that $\|\beta_i - \gamma_i\|_B \leq \varepsilon/n$ for all i. Therefore, continuing from Eq. (54), we have

$$\left\| \bigotimes_{i=1}^n \beta_i - \bigotimes_{i=1}^n \gamma_i \right\|_B \leq \textstyle\sum_{i=1}^n \|\beta_i - \gamma_i\|_B \leq n \cdot \frac{\varepsilon}{n} = \varepsilon, \tag{55}$$

which by Def. 15 shows that $\mathcal{N}$ is indeed a basis ε-net.

As for the cardinality bound, Eq. (53) implies that $|\mathcal{N}| = |\mathcal{M}|^n \leq [5(1+2\sqrt{2})^2 n^2/\varepsilon^2]^{8n}$. $\square$

1.3 Entanglement in Multipartite Systems

Let A and B be quantum systems containing qubits $A_1, \ldots, A_{N_A}$ and $B_1, \ldots, B_{N_B}$, respectively, i.e., $\mathcal{H}_A = \bigotimes_{i=1}^{N_A} \mathcal{H}_{A_i}$ and $\mathcal{H}_B = \bigotimes_{j=1}^{N_B} \mathcal{H}_{B_j}$ with $\dim \mathcal{H}_{A_i} = \dim \mathcal{H}_{B_j} = 2$ for all $i \in [\![N_A]\!]$ and $j \in [\![N_B]\!]$. Let $d_A = \dim \mathcal{H}_A = 2^{N_A}$ and $d_B = \dim \mathcal{H}_B = 2^{N_B}$. Suppose that we are given states $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, $|v\rangle \in \mathcal{H}_A$, and we perform the projective measurement on system A, associated with the measurement operator $|v\rangle\langle v|_A \otimes I_B$.

Definition 18. *The post-measurement state for the measurement operator $|v\rangle\langle v|_A \otimes I_B$ is given by*

$$\frac{(|v\rangle\langle v|_A \otimes I_B)|\Psi\rangle}{\sqrt{\langle\Psi|(|v\rangle\langle v|_A \otimes I_B)|\Psi\rangle}}, \tag{56}$$

and the probability of obtaining such state is given by

$$p_v(\Psi) \coloneqq \langle\Psi|(|v\rangle\langle v|_A \otimes I_B)|\Psi\rangle. \tag{57}$$

We can show that indeed $0 \leq p_v(\Psi) \leq 1$: Suppose that $|i\rangle \in \mathcal{C}(\mathcal{H}_A)$ contains $|v\rangle$ and $|j\rangle \in \mathcal{C}(\mathcal{H}_B)$. Then

$$\begin{aligned} 0 \leq p_v(\Psi) &= \langle\Psi|(|v\rangle\langle v|_A \otimes I_B)|\Psi\rangle && (58)\\ &= \textstyle\sum_j \langle\Psi|(|v\rangle\langle v|_A \otimes |j\rangle\langle j|_B)|\Psi\rangle && (59)\\ &= \textstyle\sum_j |(\langle v|_A\langle j|_B)|\Psi\rangle|^2 && (60)\\ &\leq \textstyle\sum_i \sum_j |\langle ij|\Psi\rangle|^2 && (61)\\ &= 1. && (62)\end{aligned}$$

Proposition 19 (Post-measurement disentanglement)**.** *The following relation indicates that the post-measurement state will not be entangled:*

$$(|v\rangle\langle v|_A \otimes I_B)|\Psi\rangle = |v\rangle_A \otimes (\langle v|_A \otimes I_B)|\Psi\rangle. \tag{63}$$

Proof. The following relations will prove Eq. (63):

$$LHS = (|v\rangle_A \otimes I_B)(\langle v|_A \otimes I_B)|\Psi\rangle = (|v\rangle_A \otimes I_B)\big(1 \otimes ((\langle v|_A \otimes I_B)|\Psi\rangle)\big) = RHS \tag{64}$$

□

Definition 20. *The unnormalized post-measurement state with system A discarded is given by*

$$|P_v(\Psi)\rangle \coloneqq (\langle v|_A \otimes I_B)|\Psi\rangle \in \mathcal{H}_B \tag{65}$$

and the normalized one is given by

$$|M_v(\Psi)\rangle \coloneqq \frac{|P_v(\Psi)\rangle}{\sqrt{p_v(\Psi)}} = \frac{(\langle v|_A \otimes I_B)|\Psi\rangle}{\sqrt{\langle\Psi|(|v\rangle\langle v|_A \otimes I_B)|\Psi\rangle}} \in \mathcal{H}_B. \tag{66}$$

Note that $0 \leq \||P_v(\Psi)\rangle\|_2 \leq 1$ *because* $\langle P_v(\Psi)|P_v(\Psi)\rangle = p_v(\Psi)$.

Definition 21 (Wootter's tilde)**.** *For an n-qubit state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, we define*

$$|\widetilde{\psi}\rangle \coloneqq \sigma_y^{\otimes n}|\psi^*\rangle. \tag{67}$$

Proposition 22. *Wootter's tilde preserves length, i.e.,* $\||\widetilde{\psi}\rangle\|_2 = \||\psi\rangle\|_2$.

Proof. Since $(\sigma_y^{\otimes n})^\dagger \sigma_y^{\otimes n} = I$, we have $\langle\widetilde{\psi}|\widetilde{\psi}\rangle = \langle\psi^*|\psi^*\rangle = \langle\psi|\psi\rangle$. □

Definition 23 (n-tangle)**.** *The entanglement measurement* ***n-tangle*** *of an n-qubit state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is given by*

$$\tau_n(\psi) \coloneqq |\langle\psi|\widetilde{\psi}\rangle|. \tag{68}$$

Proposition 24. *The n-tangle for any state is bounded above by* 1.

Proof. Trivial by Cauchy-Schwartz and Prop. 22. □

Definition 25 (Average n-tangle)**.** *The* ***average post-measurement n-tangle*** *given $|v\rangle \in \mathcal{H}_A$ is defined as*

$$F_v(\Psi) \coloneqq p_v(\Psi)\tau(|M_v(\Psi)\rangle) = |\langle P_v(\Psi)|\widetilde{P}_v(\Psi)\rangle| \tag{69}$$

Definition 26 (Average n-tangle over a basis)**.** *For an orthonormal basis $\beta = \{|\varphi_i\rangle\}_{i=1}^{d_A} \in \mathcal{C}(\mathcal{H}_A)$, we define*

$$\overline{\tau}_\beta(\Psi) \coloneqq \textstyle\sum_{i=1}^{d_A} F_{\varphi_i}(\Psi) = \sum_{i=1}^{d_A} p_{\varphi_i}(\Psi)\tau(|M_{\varphi_i}(\Psi)\rangle). \tag{70}$$

Lemma 27. *For all states $|v\rangle, |w\rangle \in \mathcal{H}_A$, we have*

$$|F_v(\Psi) - F_w(\Psi)| \leq \sqrt{2} d_B \, \||v\rangle\langle v| - |w\rangle\langle w|\|_1 \, . \tag{71}$$

Proof. For all $|u\rangle \in \mathcal{H}_A$ and for all $\theta \in \mathbb{R}$, we have

$$F_u(e^{i\theta}|\Psi\rangle) = \left|\langle P_u(e^{i\theta}|\Psi\rangle)|\widetilde{P}_u(e^{i\theta}|\Psi\rangle)\rangle\right| = \left|e^{-2i\theta}\langle P_u(\Psi)|\widetilde{P}_u(\Psi)\rangle\right| = F_u(\Psi). \tag{72}$$

Thus, we may assume

$$\langle v|w\rangle = |\langle v|w\rangle| \in [0,1] \tag{73}$$

WLOG. Observe the following relations (for simplicity we omit Ψ):

$$\begin{aligned}
\left|F_v - F_w\right| &= \left||\langle P_v|\widetilde{P}_v\rangle| - |\langle P_w|\widetilde{P}_w\rangle|\right| && \text{(by Def. 25)} \quad (74)\\
&\leq \left|\langle P_v|\widetilde{P}_v\rangle - \langle P_w|\widetilde{P}_w\rangle\right| && (75)\\
&= \left|\langle P_v|\widetilde{P}_v\rangle - \langle P_w|\widetilde{P}_v\rangle + \langle P_w|\widetilde{P}_v\rangle - \langle P_w|\widetilde{P}_w\rangle\right| && (76)\\
&= \left|(\langle P_v| - \langle P_w|)|\widetilde{P}_v\rangle + \langle P_w|(|\widetilde{P}_v\rangle - |\widetilde{P}_w\rangle)\right| && (77)\\
&\leq \left|(\langle P_v| - \langle P_w|)|\widetilde{P}_v\rangle\right| + \left|\langle P_w|(|\widetilde{P}_v\rangle - |\widetilde{P}_w\rangle)\right| && (78)\\
&\leq \||P_v\rangle - |P_w\rangle\|_2 \||\widetilde{P}_v\rangle\|_2 + \||P_w\rangle\|_2 \||\widetilde{P}_v\rangle - |\widetilde{P}_w\rangle\|_2 && \text{(by Cauchy-Schwartz)} \quad (79)\\
&= \||P_v\rangle - |P_w\rangle\|_2 \||P_v\rangle\|_2 + \||P_w\rangle\|_2 \||P_v\rangle - |P_w\rangle\|_2. && \text{(by Prop. 22)} \quad (80)\\
&\leq 2\||P_v\rangle - |P_w\rangle\|_2. && \text{(by Def. 20)} \quad (81)
\end{aligned}$$

Let $|\alpha\rangle = |v\rangle - |w\rangle$ and $|i\rangle \in \mathcal{C}(\mathcal{H}_B)$. We continue from Eq. (81):

$$\begin{aligned}
\left|F_v - F_w\right| \leq 2\||P_v\rangle - |P_w\rangle\| &= 2\|(\langle\alpha| \otimes I_B)|\Psi\rangle\|_2 && \text{(by Def. 20)} \quad (82)\\
&= 2\sqrt{\langle\Psi|(|\alpha\rangle\langle\alpha|_A \otimes I_B)|\Psi\rangle} && (83)\\
&= 2\sqrt{\textstyle\sum_{i=1}^{d_B}\langle\Psi|(|\alpha\rangle\langle\alpha|_A \otimes |i\rangle\langle i|_B)|\Psi\rangle} && (84)\\
&\leq 2\textstyle\sum_{i=1}^{d_B}\sqrt{\langle\Psi|(|\alpha\rangle\langle\alpha|_A \otimes |i\rangle\langle i|_B)|\Psi\rangle} && (85)\\
&= 2\textstyle\sum_{i=1}^{d_B}\left|(\langle\alpha|_A\langle i|_B)|\Psi\rangle\right| && (86)\\
&\leq 2\textstyle\sum_{i=1}^{d_B}\||\alpha\rangle_A|i\rangle_B\|_2 \cdot \||\Psi\rangle\|_2 && \text{(by Cauchy-Schwartz)} \quad (87)\\
&= 2\textstyle\sum_{i=1}^{d_B}\||\alpha\rangle\|_2 \cdot \||i\rangle\|_2 \cdot \||\Psi\rangle\|_2 && (88)\\
&= 2d_B\||\alpha\rangle\|_2. && (89)\\
&= 2d_B\||v\rangle - |w\rangle\|_2 && (90)\\
&= 2d_B\sqrt{(\langle v| - \langle w|)(|v\rangle - |w\rangle)} && (91)\\
&= 2d_B\sqrt{2 - 2\,\mathrm{Re}(\langle v|w\rangle)} && (92)\\
&= 2d_B\sqrt{2 - 2|\langle v|w\rangle|} && \text{(by Eq. (73))} \quad (93)\\
&\leq 2d_B\sqrt{2 - 2|\langle v|w\rangle|^2} && (\because |\langle v|w\rangle| \leq 1) \quad (94)\\
&= \sqrt{2}d_B\||v\rangle\langle v| - |w\rangle\langle w|\|_1. && \text{(by Lemma 5)} \quad (95)
\end{aligned}$$

□

Lemma 28. *Given $\beta = \{|\varphi_i\rangle\}_{i=1}^{d_A}$, $\gamma = \{|\eta_i\rangle\}_{i=1}^{d_A} \in \mathcal{C}(\mathcal{H}_A)$, we have*

$$|\overline{\tau}_\beta(\Psi) - \overline{\tau}_\gamma(\Psi)| \leq \sqrt{2} d_A d_B \|\beta - \gamma\|_B \tag{96}$$

Proof.

$$
\begin{aligned}
\left|\overline{\tau}_\beta(\Psi) - \overline{\tau}_\gamma(\Psi)\right| &= \left| \textstyle\sum_{i=1}^{d_A} F_{\varphi_i}(\Psi) - \sum_{i=1}^{d_A} F_{\eta_i}(\Psi)\right| && \text{(by Def. 26)} && (97)\\
&= \left| \textstyle\sum_{i=1}^{d_A} \left(F_{\varphi_i}(\Psi) - F_{\eta_i}(\Psi)\right)\right| && && (98)\\
&\le \textstyle\sum_{i=1}^{d_A} \left|F_{\varphi_i}(\Psi) - F_{\eta_i}(\Psi)\right| && && (99)\\
&\le \sqrt{2} d_B \textstyle\sum_{i=1}^{d_A} \|\varphi_i - \eta_i\|_1 && \text{(by Lemma 27)} && (100)\\
&\le \sqrt{2} d_A d_B \|\beta - \gamma\|_B. && \text{(by Def. 11)} && (101)
\end{aligned}
$$

□

2 Results

Definition 29 (LME). *Given a state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the* ***localizable multipartite entanglement (LME)*** *of $|\Psi\rangle$ with respect to the entanglement measurement τ (as in Def. 23) is defined as the maximum average post-measurement n-tangle, given by*

$$L^\tau(\Psi) := \max_{\beta \in \mathcal{P}(\mathcal{H}_A)} \overline{\tau}_\beta(\Psi), \tag{102}$$

as stated in [2]. Also, we denote by $\beta_{\max} \in \mathcal{P}(\mathcal{H}_A)$ a basis such that $\overline{\tau}_{\beta_{\max}}(\Psi) = L^\tau(\Psi)$.

According to [2], its counterpart ***multipartite entanglement assistance*** is defined as

$$L^\tau_{\text{global}}(\Psi) := \max_{\beta \in \mathcal{C}(\mathcal{H}_A)} \overline{\tau}_\beta(\Psi). \tag{103}$$

Notice the difference in the range of β between Eq. (102) and Eq. (103).

Lemma 30. *For all states $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_B$, by [2, Lem. 6] we have*

$$|\tau_{N_B}(|\psi\rangle) - \tau_{N_B}(|\psi'\rangle)| \le \sqrt{2}\|\psi - \psi'\|_1. \tag{104}$$

Suppose that a concave, increasing $f : [0,\infty) \to [0,\infty)$ satisfies

$$\left|\tau(|\psi\rangle) - \tau(|\psi'\rangle)\right| \le f(\|\psi - \psi'\|_1) \tag{105}$$

and

$$\tau(|\psi\rangle) \le f(\|\psi\|_1). \tag{106}$$

for all states $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_B$.

Lemma 31. *Given $\beta \in \mathcal{C}(\mathcal{H}_A)$, for all states $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, by [2, Lem. 16] we have*

$$|\overline{\tau}_\beta(|\Psi\rangle) - \overline{\tau}_\beta(|\Psi'\rangle)| \le f(2\|\Psi - \Psi'\|_1) + \|\Psi - \Psi'\|_1. \tag{107}$$

Corollary 32. *For $\beta \in \mathcal{C}(\mathcal{H}_A)$ and states $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we have*

$$|\overline{\tau}_\beta(|\Psi\rangle) - \overline{\tau}_\beta(|\Psi'\rangle)| \le (1 + 2\sqrt{2})\|\Psi - \Psi'\|_1 \tag{108}$$

Proof. Choose $f : [0,\infty) \to [0,\infty)$ defined by $f(x) = \sqrt{2}x$. Obviously, it is concave and monotone. By Lemma 30, the condition as in Eq. (105) is satisfied. Also, Prop. 24 implies

$$\tau(|\psi\rangle) \le 1 < f(\|\psi\|_1) = \sqrt{2}\|\psi\|_1 \le \sqrt{2}, \tag{109}$$

so Eq. (106) holds. Finally, Lemma 31 finishes the proof. □

Lemma 33. *For all $\beta \in \mathcal{C}(\mathcal{H}_A)$, by [2, Lem. 22] we have*

$$\mathop{\mathbb{E}}_{|\Psi\rangle\sim\mathrm{Haar}(d_A d_B)}[\overline{\tau}_\beta(\Psi)] \leq \sqrt{\frac{2}{d_B+1}}. \tag{110}$$

Definition 34. *For convenience let us define*

$$K(d_B) \coloneqq \sqrt{\frac{2}{d_B+1}}. \tag{111}$$

Lemma 35 (Levy's Lemma [3])**.** *Consider $\mathbb{S}^{2d-1} \coloneqq \{|\Phi\rangle \in \mathbb{C}^d : \||\Phi\rangle\|_2 = 1\}$. Let $f : \mathbb{S}^{2d-1} \to \mathbb{R}$ be a function satisfying the Lipschitz condition*

$$\exists L \geq 0 \ s.t. \ \forall |\Phi\rangle, |\Phi'\rangle \in \mathbb{S}^{2d-1},\ |f(\Phi) - f(\Phi')| \leq L\||\Phi\rangle - |\Phi'\rangle\|_2$$

— we call such f Lipschitz continuous and such L the Lipschitz constant. Then for all $\varepsilon > 0$, we have the probability bound

$$\Pr_{|\Phi\rangle\sim\mathrm{Haar}(d)}\left(\left|f(\Phi) - \mathop{\mathbb{E}}_{|\Phi'\rangle\sim\mathrm{Haar}(d)}[f(\Phi')]\right| \geq \varepsilon\right) \leq 2\exp\left(-\frac{2d\varepsilon^2}{9\pi^3 L^2}\right). \tag{112}$$

Lemma 36. *Given a fixed $\gamma \in \mathcal{P}(\mathcal{H}_A)$, for all $\varepsilon, \delta > 0$ such that $\varepsilon - \sqrt{2}d_A d_B\delta > 0$, we have*

$$\begin{aligned}\Pr_{|\Psi\rangle\sim\mathrm{Haar}(d_A d_B)}&\left(\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon \ \text{ and } \ \|\beta_{\max} - \gamma\|_B \leq \delta\right) \\ &\leq 2\exp\left(-\frac{2d_A d_B(\varepsilon - \sqrt{2}d_A d_B\delta)^2}{9\pi^3(4\sqrt{2}+2)^2}\right)\end{aligned} \tag{113}$$

Proof. Let us claim that

$$\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon \ \text{ and } \ \|\beta_{\max} - \gamma\|_B \leq \delta \implies \overline{\tau}_\gamma(\Psi) \geq K(d_B) + \varepsilon - \sqrt{2}d_A d_B\delta.$$

Proof:

$$\begin{aligned}
\overline{\tau}_\gamma(\Psi) &= \overline{\tau}_{\beta_{\max}}(\Psi) - (\overline{\tau}_{\beta_{\max}}(\Psi) - \overline{\tau}_\gamma(\Psi)) && (114)\\
&\geq \overline{\tau}_{\beta_{\max}}(\Psi) - |\overline{\tau}_{\beta_{\max}}(\Psi) - \overline{\tau}_\gamma(\Psi)| && (115)\\
&\geq \overline{\tau}_{\beta_{\max}}(\Psi) - \sqrt{2}d_A d_B\|\beta_{\max} - \gamma\|_B \qquad \text{(by Lemma 28)} && (116)\\
&\geq K(d_B) + \varepsilon - \sqrt{2}d_A d_B\delta. && (117)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Pr_{|\Psi\rangle\sim\mathrm{Haar}(d_A d_B)}\left(\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon \ \text{ and } \ \|\beta_{\max} - \gamma\|_B \leq \delta\right) && (118)\\
&\quad\leq \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\overline{\tau}_\gamma(\Psi) \geq K(d_B) + \varepsilon - \sqrt{2}d_A d_B\delta\right) \qquad \text{(by Eq. (114)–(117))} && (119)\\
&\quad\leq \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\overline{\tau}_\gamma(\Psi) \geq \mathop{\mathbb{E}}_{|\Phi\rangle\sim\mathrm{Haar}}[\overline{\tau}_\gamma(\Phi)] + \varepsilon - \sqrt{2}d_A d_B\delta\right) \qquad \text{(by Lemma 33)} && (120)\\
&\quad= \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\overline{\tau}_\gamma(\Psi) - \mathop{\mathbb{E}}_{|\Phi\rangle\sim\mathrm{Haar}}[\overline{\tau}_\gamma(\Phi)] \geq \varepsilon - \sqrt{2}d_A d_B\delta\right) && (121)
\end{aligned}$$

$$\leq \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\left|\overline{\tau}_\gamma(\Psi) - \mathop{\mathbb{E}}_{|\Phi\rangle\sim\mathrm{Haar}}[\overline{\tau}_\gamma(\Phi)]\right| \geq \varepsilon - \sqrt{2}d_A d_B \delta\right). \tag{122}$$

With Cor. 32 and Cor. 6, for all states $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we have

$$|\overline{\tau}_\gamma(|\Psi\rangle) - \overline{\tau}_\gamma(|\Psi'\rangle)| \leq (1+2\sqrt{2})\|\Psi - \Psi'\|_1 \leq (2+4\sqrt{2})\|\Psi - \Psi'\|_2. \tag{123}$$

Therefore, we verified that the function $\overline{\tau}_\gamma(\,\cdot\,)$ is Lipschitz continuous with Lipschitz constant $L = 2+4\sqrt{2}$, and hence from Eq. (122), we may use Lemma 35 to finish the proof. □

Finally, we can turn to our main theorem.

Theorem 37. *For $0 < \varepsilon < (2\sqrt{2}+8)N_A d_A d_B$, we have*

$$\Pr_{|\Psi\rangle\sim\mathrm{Haar}(d_A d_B)}\left(L^\tau(\Psi) \geq K(d_B) + \varepsilon\right)$$
$$\leq 2\left(\frac{40(1+2\sqrt{2})^2 N_A^2 d_A^2 d_B^2}{\varepsilon^2}\right)^{8N_A} \exp\left(-\frac{d_A d_B \varepsilon^2}{18\pi^3(2+4\sqrt{2})^2}\right) \tag{124}$$

Proof. By Thm. 17, there exists a basis $(\varepsilon/2\sqrt{2}d_A d_B)$-net $\mathcal{N} = \{\gamma_i\}_{i=1}^N$ on $\mathcal{P}(\mathcal{H}_A)$ with

$$|\mathcal{N}| \leq \left(\frac{5(1+2\sqrt{2})^2 N_A^2}{(\varepsilon/2\sqrt{2}d_A d_B)^2}\right)^{8N_A} = \left(\frac{40(1+2\sqrt{2})^2 N_A^2 d_A^2 d_B^2}{\varepsilon^2}\right)^{8N_A}. \tag{125}$$

Then by Def. 15 there must be $i \in [\![|\mathcal{N}|]\!]$ such that $\|\beta_{\max} - \gamma_i\|_B \leq \varepsilon/2\sqrt{2}d_A d_B$, which suggests

$$\Pr_{|\Psi\rangle\sim\mathrm{Haar}(d_A d_B)}\left(\bigvee_{i=1}^{|\mathcal{N}|} \|\beta_{\max} - \gamma_i\|_B \leq \frac{\varepsilon}{2\sqrt{2}d_A d_B}\right) = 1. \tag{126}$$

Hence,

$$\Pr_{|\Psi\rangle\sim\mathrm{Haar}(d_A d_B)}\left(L^\tau(\Psi) \geq K(d_B) + \varepsilon\right)$$
$$= \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon\right) \qquad \text{(by Def. 29)} \tag{127}$$
$$= \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon \;\wedge\; \left(\bigvee_{i=1}^{|\mathcal{N}|} \|\beta_{\max} - \gamma_i\|_B \leq \frac{\varepsilon}{2\sqrt{2}d_A d_B}\right)\right) \qquad \text{(by Eq. (126))} \tag{128}$$
$$= \Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\bigcup_{i=1}^{|\mathcal{N}|}\left\{\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon \;\wedge\; \|\beta_{\max} - \gamma_i\|_B \leq \frac{\varepsilon}{2\sqrt{2}d_A d_B}\right\}\right) \tag{129}$$
$$\leq \sum_{i=1}^{|\mathcal{N}|}\Pr_{|\Psi\rangle\sim\mathrm{Haar}}\left(\overline{\tau}_{\beta_{\max}}(\Psi) \geq K(d_B) + \varepsilon \;\wedge\; \|\beta_{\max} - \gamma_i\|_B \leq \frac{\varepsilon}{2\sqrt{2}d_A d_B}\right) \tag{130}$$
$$\leq |\mathcal{N}| \cdot 2\exp\left(-\frac{d_A d_B \varepsilon^2}{18\pi^3(2+4\sqrt{2})^2}\right) \qquad \text{(by Lemma 36)} \tag{131}$$
$$\leq 2\left(\frac{40(1+2\sqrt{2})^2 N_A^2 d_A^2 d_B^2}{\varepsilon^2}\right)^{8N_A} \exp\left(-\frac{d_A d_B \varepsilon^2}{18\pi^3(2+4\sqrt{2})^2}\right). \qquad \text{(by Eq. (125))} \tag{132}$$

□

We also highlight the following corollary.

Corollary 38. *Let $\varepsilon, \delta > 0$ be arbitrary. Then for any $d_B \geq 2$, there exists an $N_0 \in \mathbb{N}$ such that for all $d_A \geq 2^{N_0}$, we have*

$$\Pr_{|\Psi\rangle \sim \mathrm{Haar}(d_A d_B)} \big(L^{\tau}(|\Psi\rangle) \geq K(d_B) + \varepsilon \big) \leq \delta. \tag{133}$$

Likewise, for any $d_A \geq 2$, there exists an $N_0' \in \mathbb{N}$ such that for all $d_B \geq 2^{N_0'}$, the above bound holds.

Explanation. In *RHS* of Eq. (124), the last term $\exp[-d_A d_B \varepsilon^2 / 18\pi^3 (2 + 4\sqrt{2})^2]$, which is decreasing with respect to both d_A and d_B, dominates asymptotically. Therefore, for a fixed d_B we can always choose d_A that is large enough for *RHS* of Eq. (124) to fall below δ, and vice versa.

3 References

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