Modeling Taylor's Law with Exponential Growth and Migration

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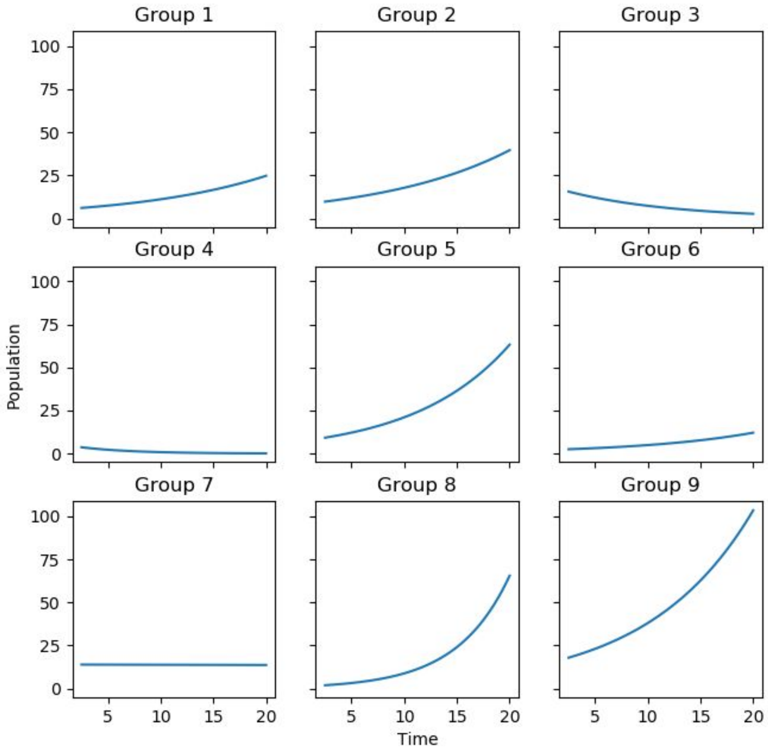
Presented by Clark Brown
BYU SRC February 29, 2020

What is Taylor’s Law?

Zooplankton Example



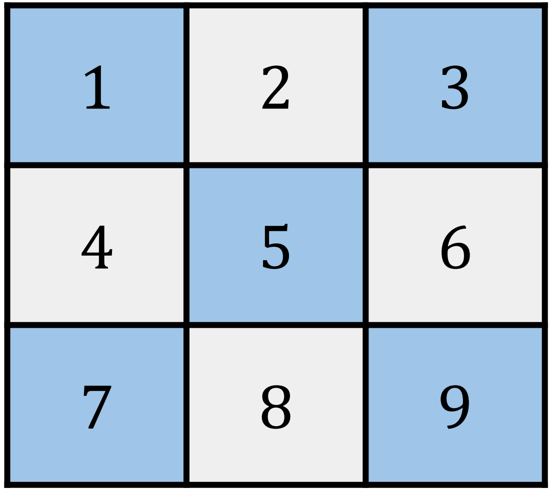
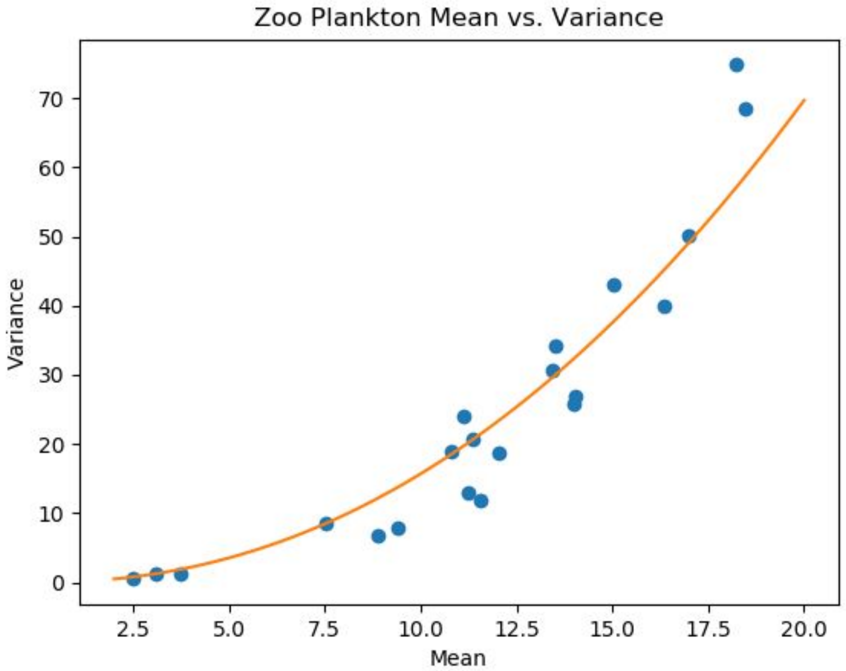


Image from biologydicitonary.net

Zooplankton Example



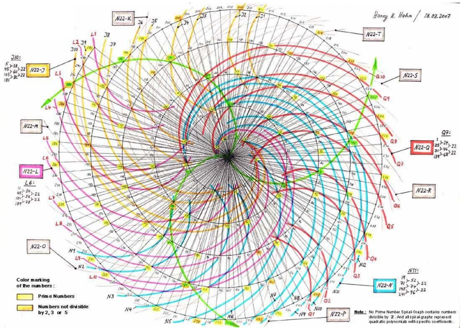
$$\mathrm{Var}(X) = a \cdot \mathrm{E}(X)^{\,b}$$

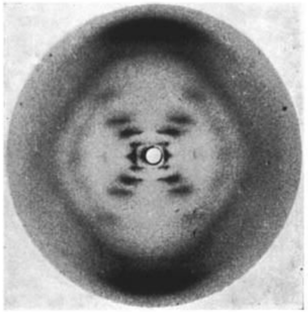
Time	Mean	Variance
0	9.4	13.7
5	11.1	19.6
10	13.5	29.9
15	12.0	23.4
20	18.5	58.8
25	13.9	32.2
30	10.8	18.5

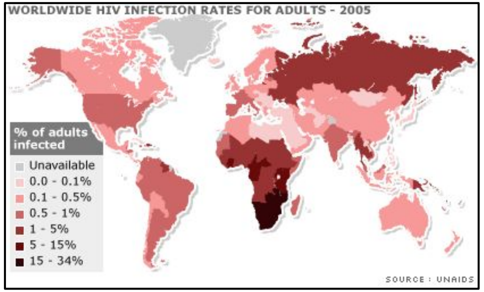
Data from Taylor, L. R. (1961). "Aggregation, variance and the mean". *Nature*. **189** (4766): 732–735.

Empirical Observations

- Thousands of Species Populations
- Cancer Metastasis
- HIV Epidemiology
- Homes Built over Tonami Plain in Japan
- Distribution of Stores in Beijing
- Gene Structure
- Distribution of Prime Numbers







Images from zekkeijapan.com, proofmathisbeautiful.tumblr.com, bbc.co.uk, jstor.org

Taylor’s Law Definition

Let $N_i(t)$ be the i^{th} subpopulation and $\mathbf{N}(t)$ be a vector of our subpopulations.

$$\mathrm{Var}(\mathbf{N}(t)) = a \cdot E(\mathbf{N}(t))^{\,b}$$

Taylor's Law Definition

Taylor's Law holds *in the limit of large time* if there exist a > 0 and b such that

$$\lim_{t \to \infty} \left[\log(\mathrm{Var}(\mathbf{N}(t))) - b \cdot \log(E(\mathbf{N}(t))) \right] = \log(a)$$

Taylor's Law Definition

Taylor's Law holds i*n the limit of large time* if there exist $a > 0$ and b such that

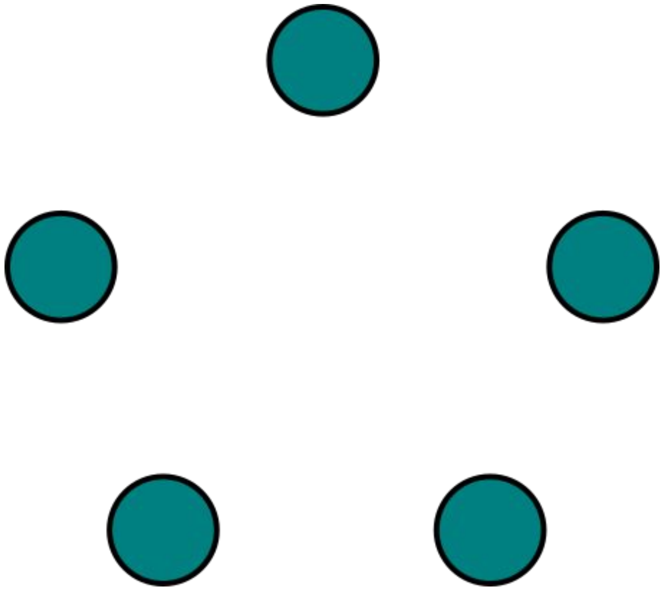
$$\lim_{t \to \infty} b(t) = \frac{d \log(\mathrm{Var}(\mathbf{N}(t)))}{d \log(E(\mathbf{N}(t)))} = 2$$

A Simple Model

No Migration

Simple Model

$$R = \begin{bmatrix} r_{11} & 0 & \dots & 0 \\ 0 & r_{22} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$$



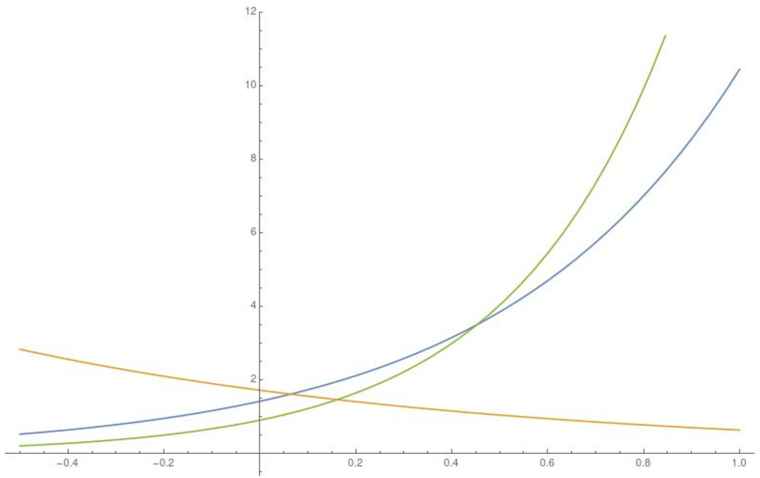
$$\frac{d\mathbf{N}(t)}{dt} = R\mathbf{N}(t)$$

$$\mathbf{N}(t) = e^{Rt}\mathbf{N}(0)$$

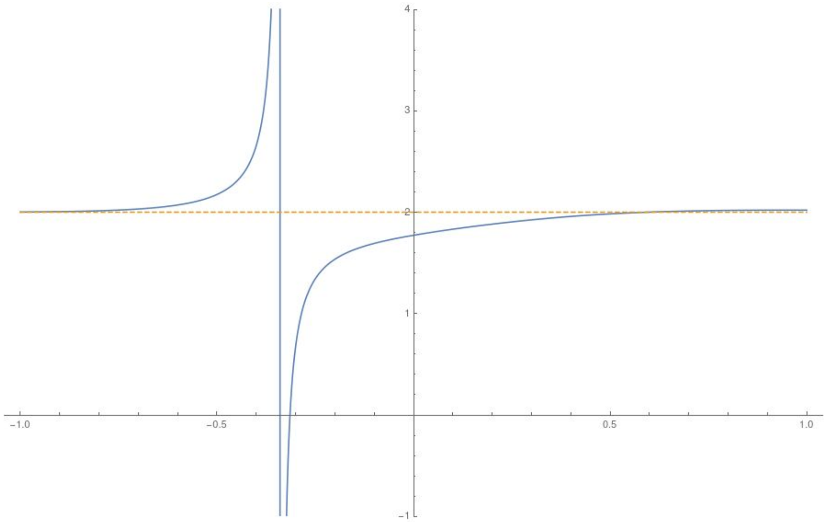
From Joel E. Cohen, “Taylor’s power law of fluctuation scaling and the growth-rate theorem”, Theoretical Population Biology, Volume 88, 2013, Pages 94-100.

Simple Model Example

$$R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$



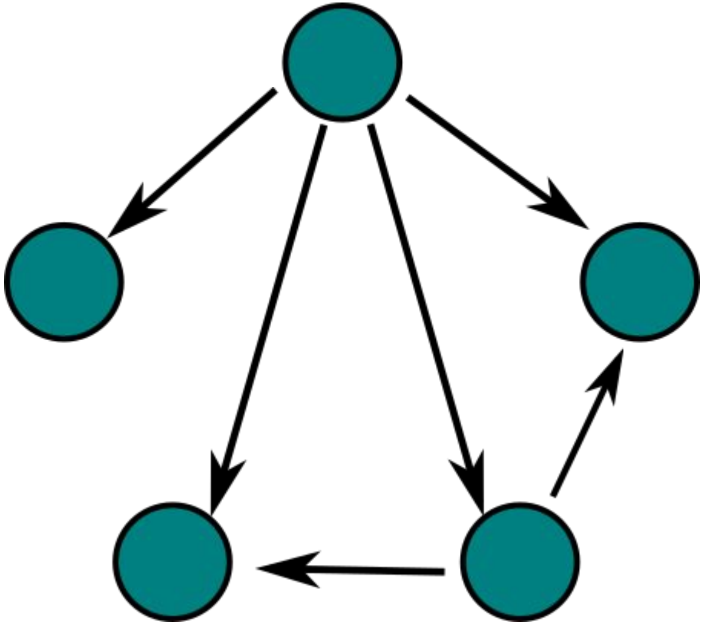
$\mathbf{N}(t) = e^{Rt}\mathbf{N}(0)$



$b(t)$

What happens if we introduce migration into the system?

Our Model: Migration Model



$$R = \begin{bmatrix} r_{11} & 0 & \dots & 0 \\ 0 & r_{22} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$$

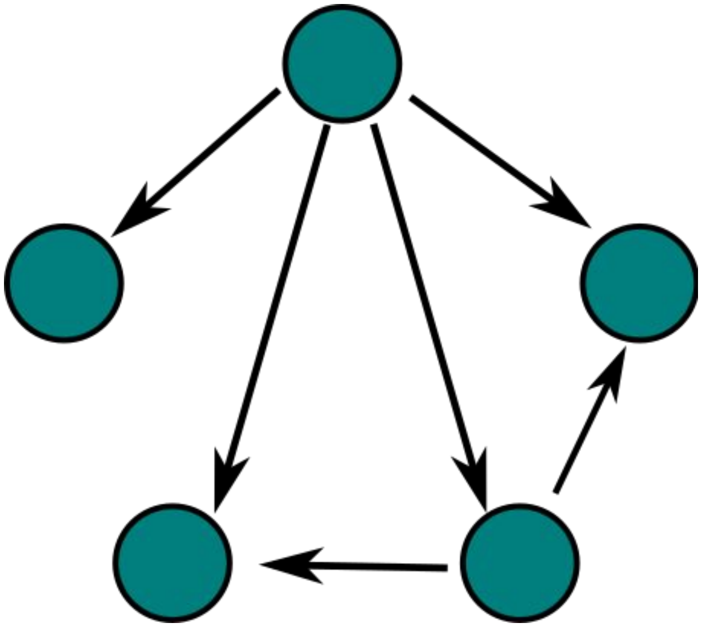
$$M = \begin{bmatrix} -\sum_{j\neq 1} m_{j1} & m_{12} & \dots & m_{1n} \\ m_{21} & -\sum_{j\neq 2} m_{j2} & \dots & m_{2n} \\ \vdots & & & \vdots \\ m_{n1} & m_{n2} & \dots & -\sum_{j\neq n} m_{jn} \end{bmatrix} \text{ where } m_{ij} \geq 0,\ i \neq j.$$

Our Model: Migration Model

$$A = R + M$$

$$\mathbf{N}(t) = e^{At}\mathbf{N}(0)$$

$$b(t) = \frac{d \log(\mathrm{Var}(e^{At}\mathbf{N}(0)))}{d \log(E(e^{At}\mathbf{N}(0)))}$$



Migration Model Example

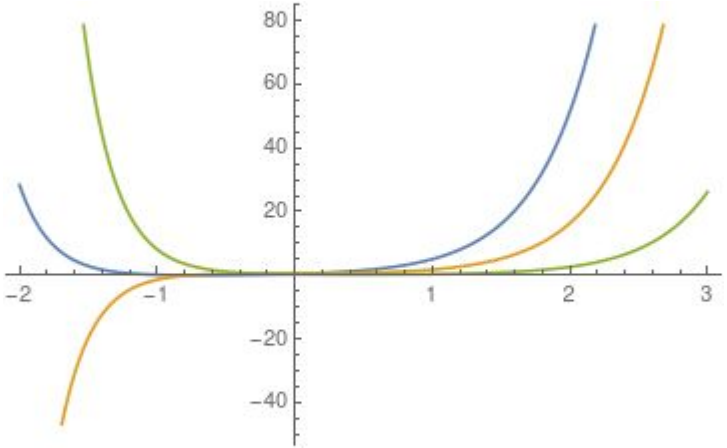
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} = R + M$$

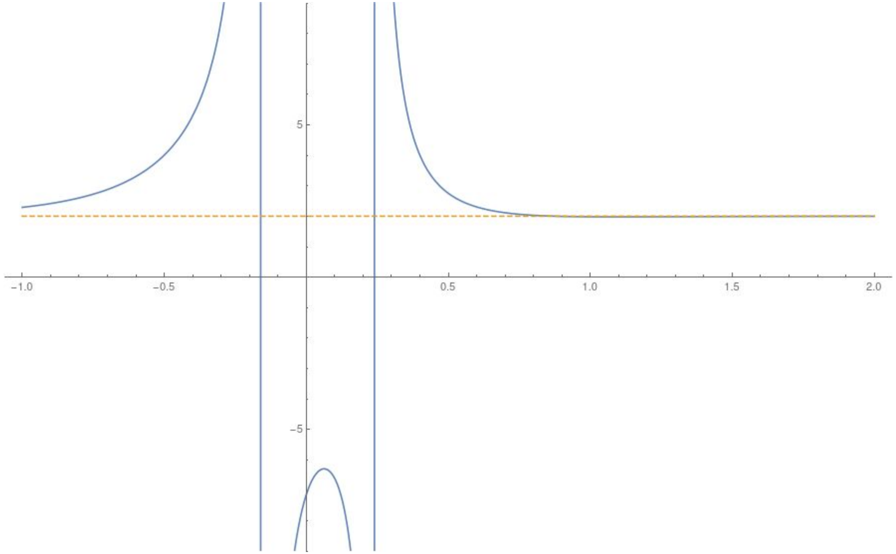
$$\sigma(A) = \{-1 - \sqrt{11}, -1 + \sqrt{11}, -1\}$$

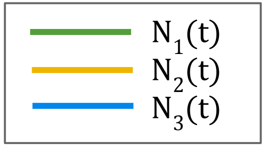
$$\lambda_\ell = -1 + \sqrt{11} > 0$$

$$\mathbf{v}_\ell = (10 + 3\sqrt{11}, 3 + \sqrt{11}, 1) \succeq \mathbf{0}$$

Migration Model Example







$$\mathbf{N}(t) = e^{At}\mathbf{N}(0)$$

$b(t)$

Properties of the Model with Migration

Proposition 1. *In the exponential model with migration, we have:*

(1) If $\mathbf{N}(0) \succeq 0$, then $\mathbf{N}(t) \succeq 0$ for all $t > 0$.

(2) The matrix A has strictly non-negative entries off the diagonals.

(3) Any matrix A that satisfies (2) can be decomposed into unique matrices M and D that represent exponential growth and migration rates.

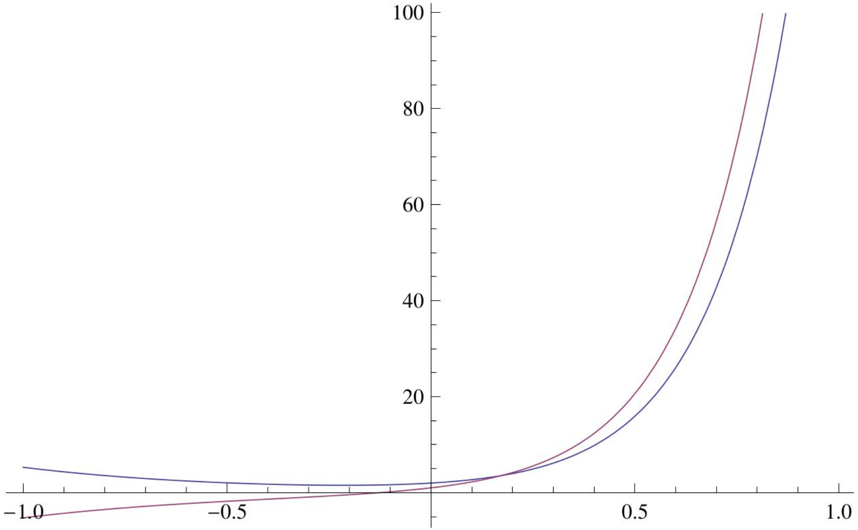
(4) There is a $\lambda_\ell \in \mathbb{R}$ with $\lambda_\ell \geq Re(\lambda)$ for all $\lambda \in \sigma(A)$. We call λ_ℓ the "leading eigenvalue."

(5) There exists an eigenvector associated with λ_ℓ with strictly non-negative entries. We call $\mathbf{v}_\ell$ the "leading eigenvector."

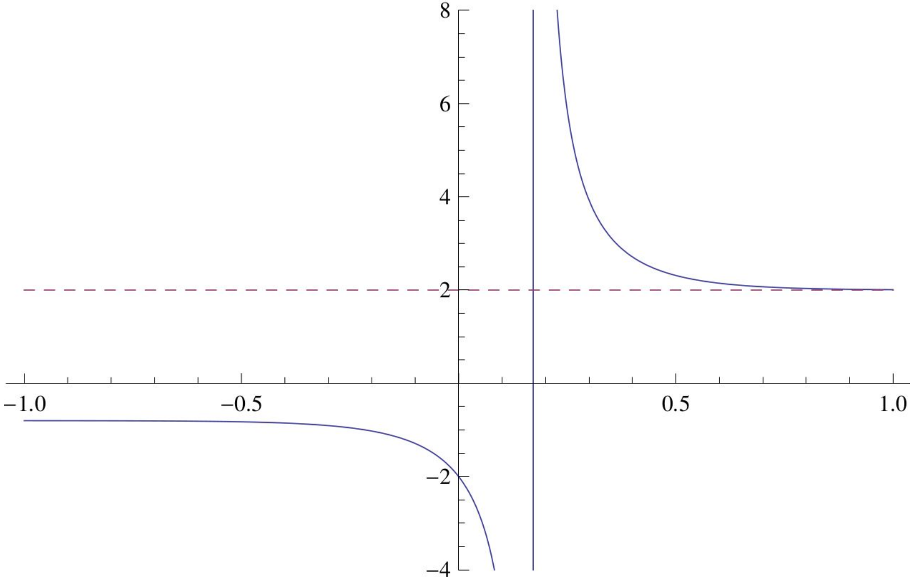
Under which conditions does Taylor's Law hold?

Example 1

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = diag[-5,-5] + \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}$$



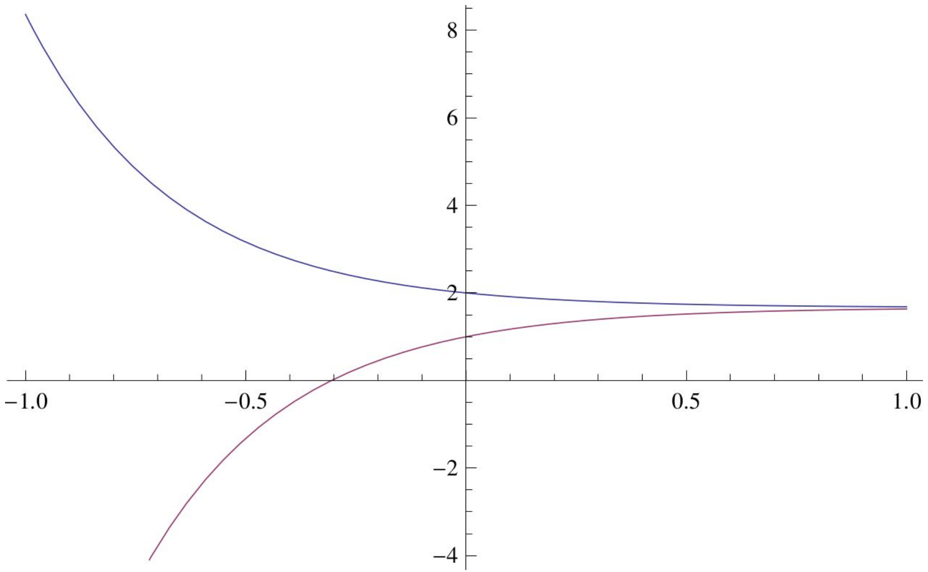
$\mathbf{N}(t) = e^{At}\mathbf{N}(0)$



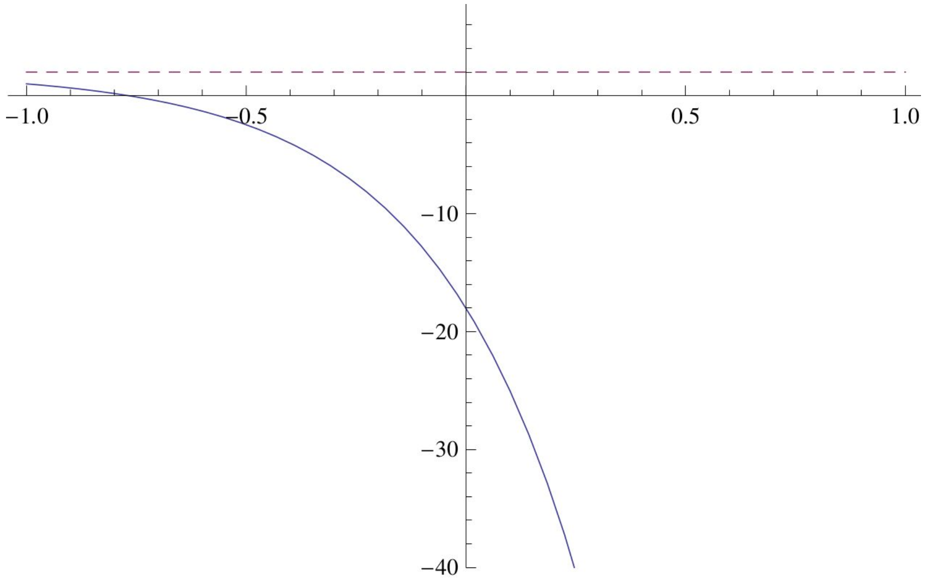
$b(t)$

Example 2

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} = diag[1, -1] + \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$$



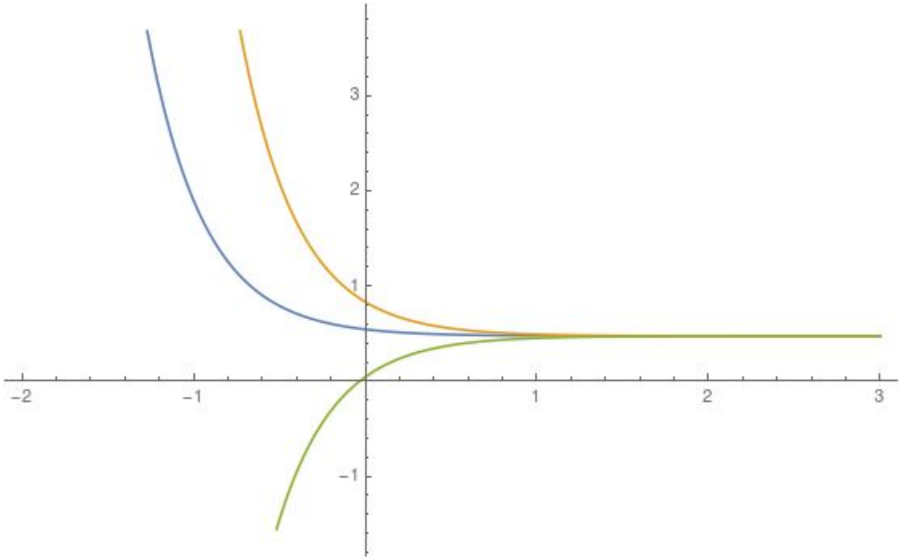
$\mathbf{N}(t) = e^{At}\mathbf{N}(0)$

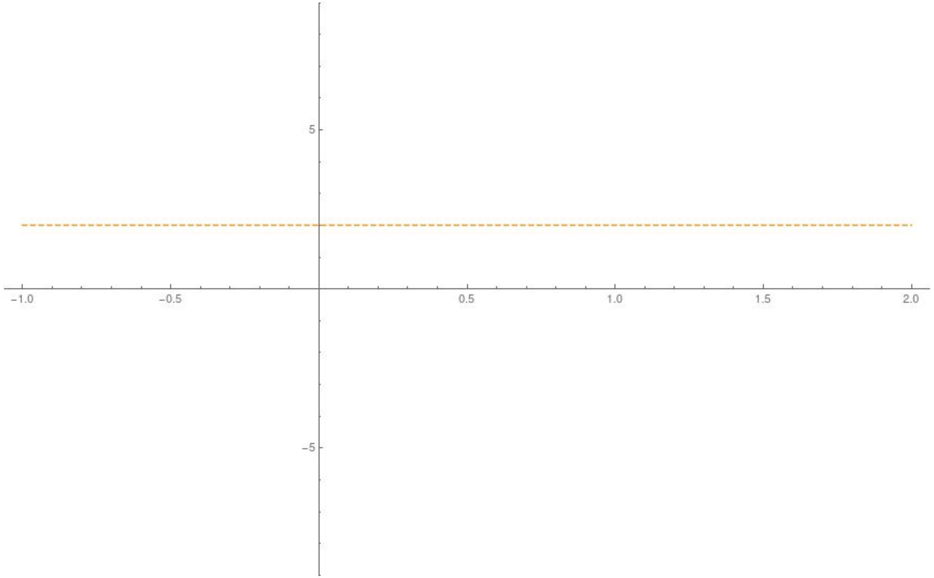


$b(t)$

Example 3

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} = diag[2,2,2] + \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$





$\mathbf{N}(t) = e^{At}\mathbf{N}(0)$

$b(t)$

Sufficient Conditions for Taylor's Law

Proposition 2 (Sufficient Conditions for Taylor's Law). *Let* $A \in \mathbb{R}^{n\times n}$ *where* $a_{ij} \geq 0$ *for all* $i \neq j$. *Let* $\mathbf{v}_\ell$ *be the "leading eigenvector" as defined earlier. If*

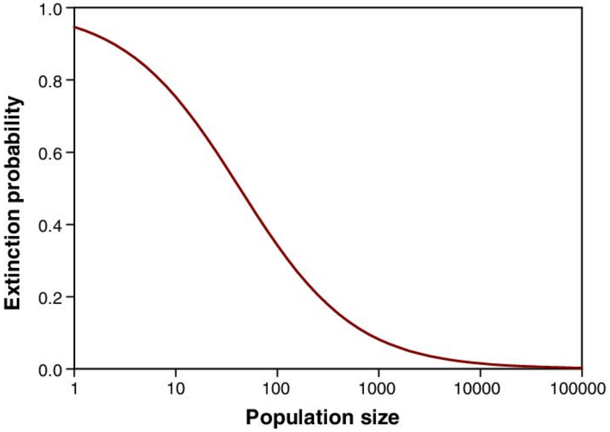
(1) $\mathbf{v}_\ell$ *is non-diagonal, i.e.* $\mathbf{v}_\ell \neq k * \mathbf{1}$

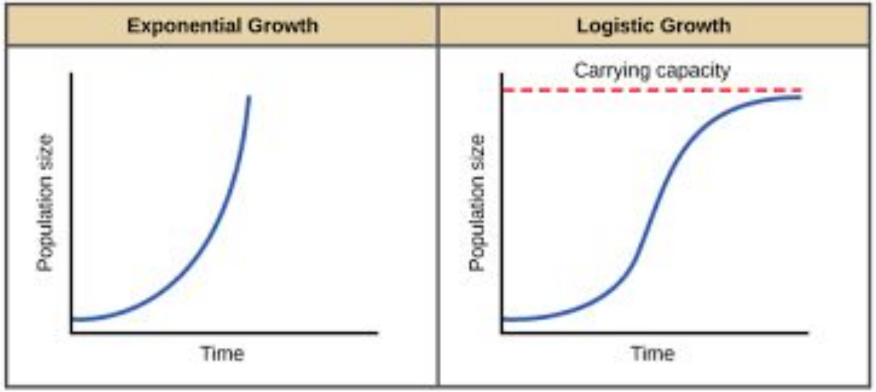
(2) c_n *is nonzero in the solution of* $\mathbf{N}(t) = e^{At}\mathbf{N}(0)$.

then Taylor's Law holds in the limit of large time.

Future Research Possibilities

- Modeling Logistic Growth with Migration
- Time to Extinction Prediction





Images from kahnacademy.org, researchgate.net

Thank you

Q & A