Proof of Taylor's Law for Exponential Growth Models with Migration

Dr. Ben Webb, Clark Brown, Sam Carpenter, Scout Callens

Presented by Clark Brown

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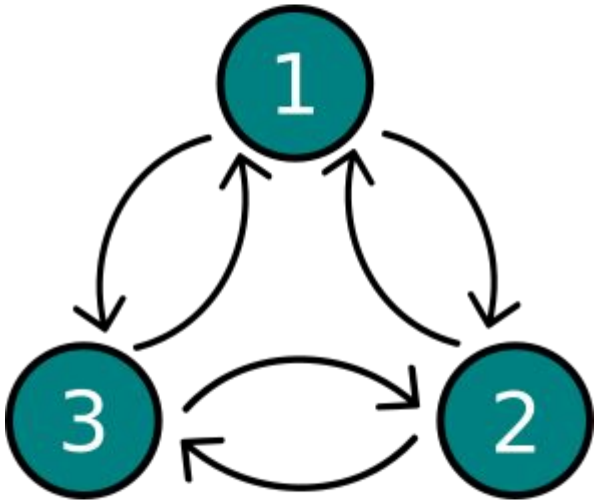
An Application of Perron-Frobenius

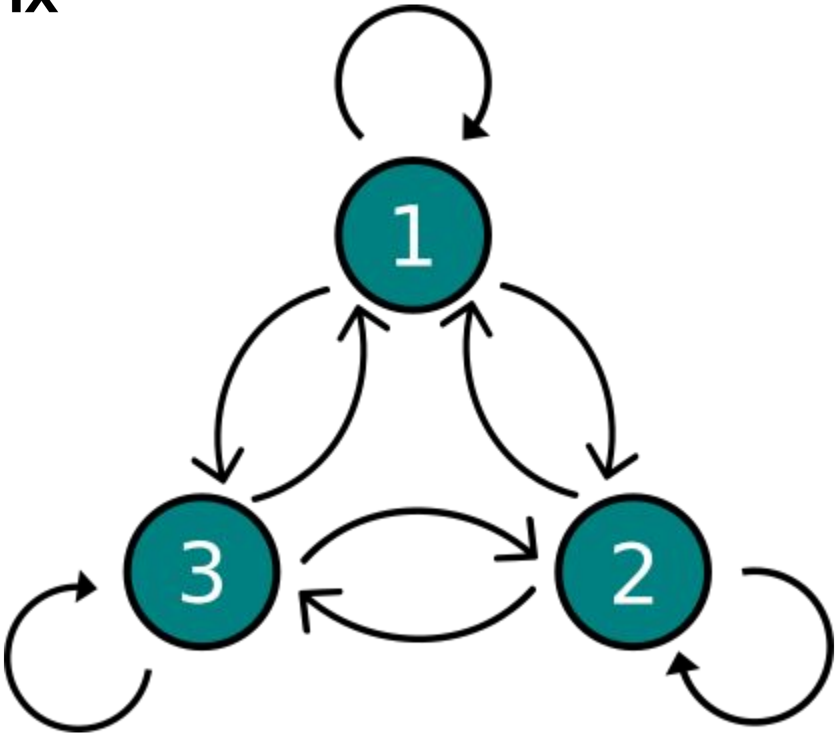
Irreducibility of Transition Matrix

$$M = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$



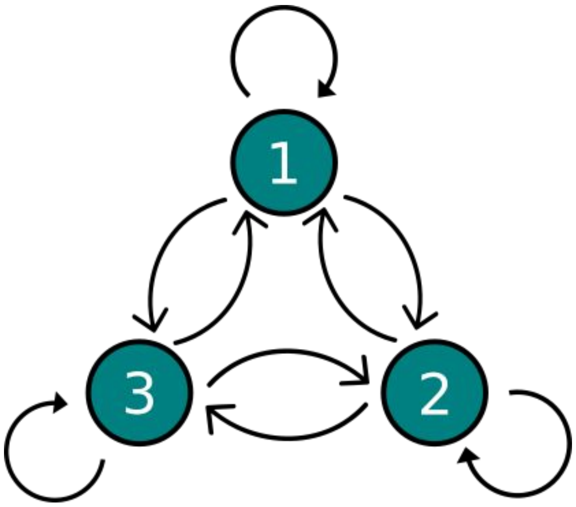
Irreducibility of Transition Matrix

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



Irreducibility of Transition Matrix

$$A = M + B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$



$$\mathbf{N}'(t) = A\mathbf{N}(t)$$

Perron-Frobenius for Nonnegative Irreducible Matrices

Theorem. ***(Perron-Frobenius Theorem)*** *Let $A \in \mathbb{R}^{n\times n}$ be nonnegative irreducible matrix. Then (a) $r = \rho(A)$ is a simple eigenvalue of A; and*
(b) any eigenvector $\boldsymbol{x}$ corresponding to the eigenvalue r has strictly positive entries, i.e. $\mathbf{x} > 0$.

Although the Perron-Frobenius theorem is typically stated with more than parts (a) and (b) these are the only results of the theorem we will use to prove the following proposition.

Perron-Frobenius for Transition Matrices

Proposition. ***(Perron-Frobenius for Transition Matrices)*** *Let $A \in \mathbb{R}^{n \times n}$ be an irreducible transition matrix so that $a_{ij} \geq 0$ for all $i \neq j$. Then there is an eigenvalue $\lambda_1 \in \sigma(A)$ such that*
(a) $\lambda_1 \in \mathbb{R}$ is simple;
(b) $\lambda_1 > Re(\lambda_i)$ for all $\lambda_i \in \sigma(A)$ where $\lambda_i \neq \lambda_1$; and
(c) the eigenvector $\mathbf{x}_1$ corresponding to λ_1 has strictly positive entries, i.e. $\mathbf{x}_1 > 0$.

Perron-Frobenius for Transition Matrices

$$A = Q + \mathrm{diag}[m, \ldots, m]$$

$$m = \min_{1 \leq i \leq n} a_{ii}$$

Perron-Frobenius for Transition Matrices

Q is nonnegative and irreducible, so Perron-Frobenius gives us

$$\lambda \in \sigma(Q) \qquad \exists\ \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq \mathbf{0}$$

$$Q\mathbf{v} = \lambda\mathbf{v}$$

Perron-Frobenius for Transition Matrices

$$A\mathbf{v} = (Q + mI)\mathbf{v} = \lambda\mathbf{v} + m\mathbf{v} = (\lambda + m)\mathbf{v}$$

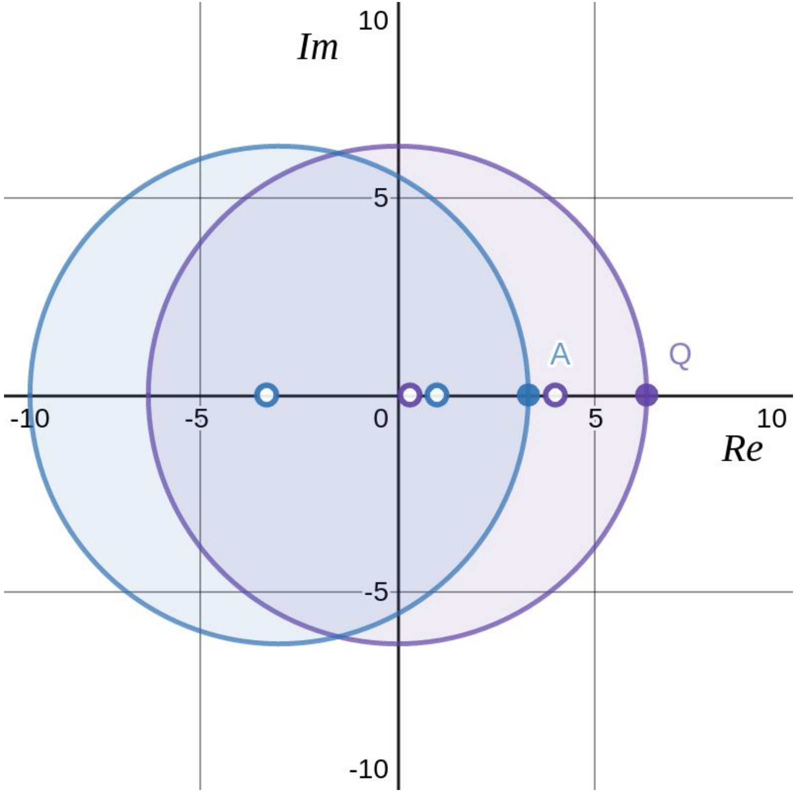
$$\lambda + m \in \sigma(A)$$

Perron-Frobenius for Transition Matrices (EX)

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\sigma(A) = \left\{ \sqrt{11}, 1, -\sqrt{11} \right\}$$

Perron-Frobenius for Transition Matrices (EX)



Notation for Proof

General Solution of N(t)

The spectrum of the transition matrix A is given by:

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_m, \alpha_{m+1} + \beta_{m+1} i, \alpha_{m+2} - \beta_{m+2} i, \ldots, \alpha_{n-1} + \beta_{n-1} i, \alpha_n - \beta_n i\}$$

$$\overline{\alpha_j + \beta_j i} = \alpha_{j+1} - \beta_{j+1} i$$

With a generalized eigenbasis:

$$\mathbf{x}_1 \ldots, \mathbf{x}_n$$

General Solution of N(t)

$$\mathbf{N}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + \sum_{j=2}^{m} c_j t^{p_j} e^{\lambda_j t}\mathbf{x}_j + \sum_{j=m+1}^{n} c_j t^{p_j} e^{\alpha_j t} T_j(\beta_j t)\mathbf{x}_j$$

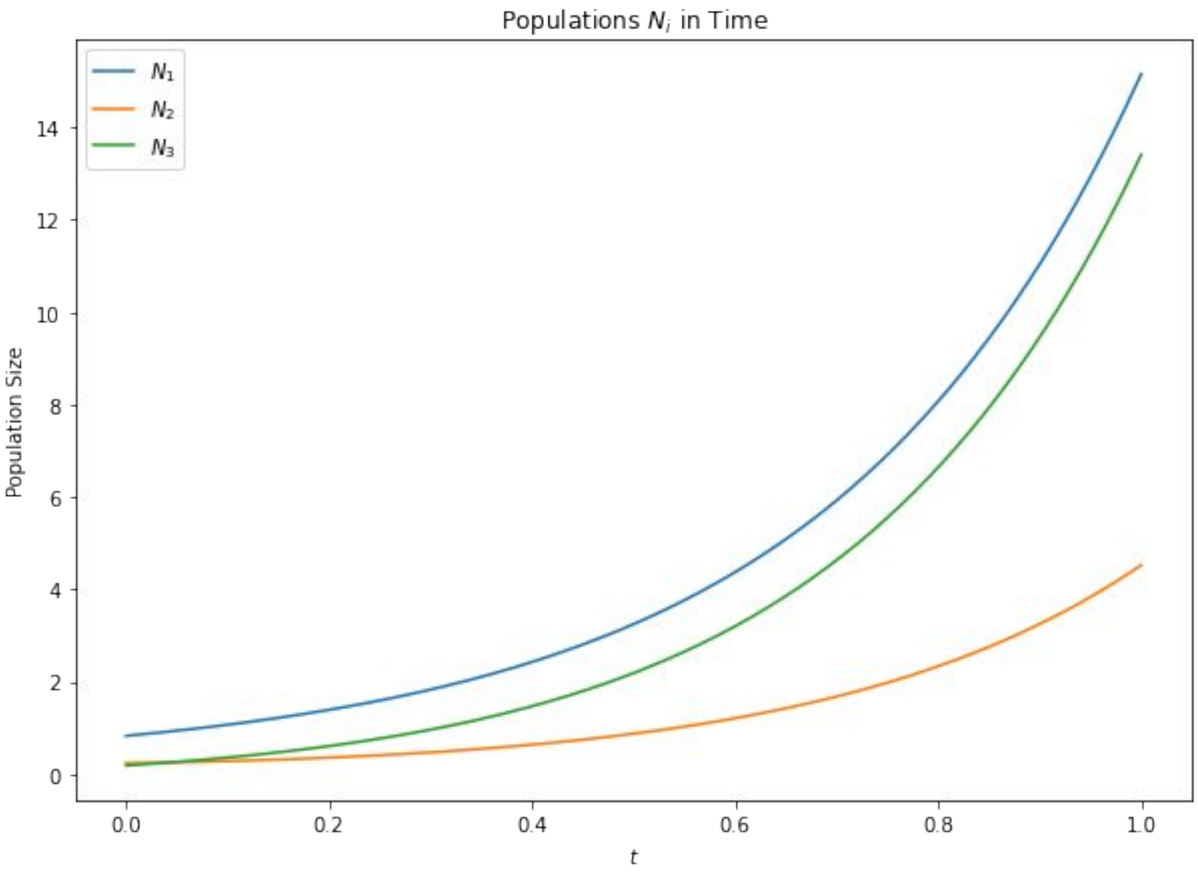
$$T_j(t) = \begin{cases} \cos(t) & \text{for} \quad j = m+1, m+3, \ldots, n-1 \\ \sin(t) & \text{for} \quad j = m+2, m+4, \ldots, n \end{cases}$$

$$c_j \in \mathbb{R},\ p_j \in \mathbb{Z}^+$$

Example Solution of N(t)

$$\mathbf{N}(t) = c_1 e^{\sqrt{11}t}\mathbf{x}_1 + c_2 e^{t}\mathbf{x}_2 + c_3 e^{-\sqrt{11}t}\mathbf{x}_3$$

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3+\sqrt{11} \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 1 \\ -3+\sqrt{11} \\ 1 \end{pmatrix}$$



Populations N_i in Time
N_1
N_2
N_3
Population Size
t
0
2
4
6
8
10
12
14
0.0
0.2
0.4
0.6
0.8
1.0

Taylor's Law in Exponential Model with Migration

Definition. *We say that a vector* $\mathbf{v} \in \mathbb{R}^{n\times 1}$ *is diagonal if it is a scalar multiple of the vector* $\mathbf{1} = [1\ 1\ \ldots\ 1]^T$ *with components all equal to 1.*

Theorem. ***(Sufficient Conditions for Taylor's Law)*** *Let* $A \in \mathbb{R}^{n\times n}$ *be a transition matrix and* $\mathbf{N}$ *be the solution to the associated EM model as given previously. If*

(a) *the leading coefficient* $c_1 \neq 0$;

(b) *the leading eigenvalue* $\lambda_1 \neq 0$;

(c) *the leading eigenvector* $\mathbf{x}_1$ *is non-diagonal; and*

(d) *A is irreducible*

then this EM model satisfies Taylor's Law.

Proof of Taylor’s Law in Exponential Model with Migration

Proof that leading eigenvalue dominates

We write:

$$\mathbf{N}(t) \;=\; e^{\lambda_1 t}(c_1\mathbf{x}_1 \;+\; \mathbf{F}(t))$$

$$F_i(t) = \sum_{j=2}^{m} c_j t^{p_j} e^{(\lambda_j - \lambda_1)t} x_{ji} + \sum_{j=m+1}^{n} c_j t^{p_j} e^{(\alpha_j - \lambda_1)t} T_j(\beta_j t) x_{ji}$$

Proof that leading eigenvalue dominates

$$F_i(t) = \sum_{j=2}^{m} c_j t^{p_j} e^{(\lambda_j - \lambda_1)t} x_{ji} + \sum_{j=m+1}^{n} c_j t^{p_j} e^{(\alpha_j - \lambda_1)t} T_j(\beta_j t) x_{ji}$$

The Transition Matrix A is irreducible, so we have:

$$\lambda_j - \lambda_1 < 0, \ \forall j \neq 1$$

$$\alpha_j - \lambda_1 < 0, \ \forall j \geq m + 1$$

Proof that leading eigenvalue dominates

Hence:

$$\lim_{t \to \infty} F_i(t) \; = \; 0$$

Similarly:

$$\lim_{t \to \infty} F_i'(t) \; = \; 0$$

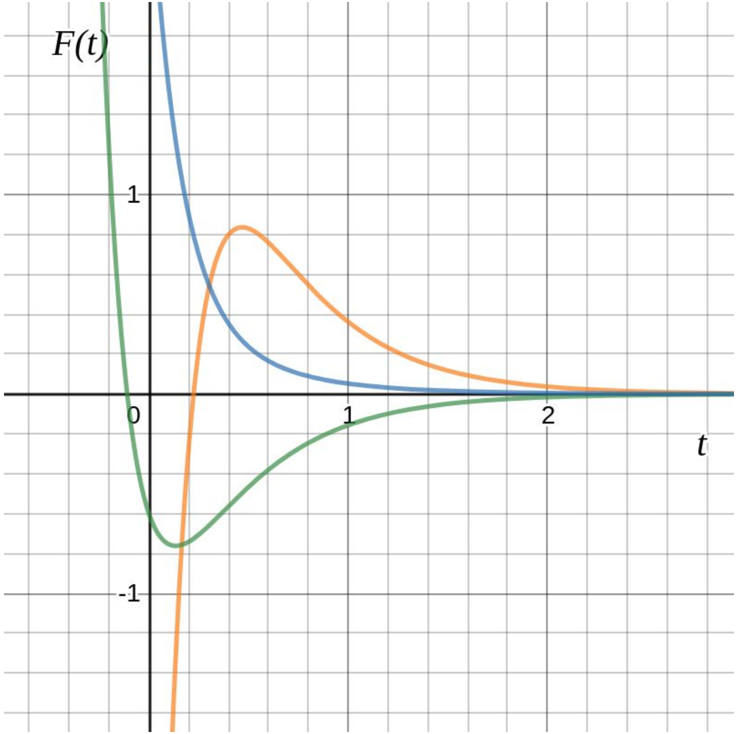
Proof that leading eigenvalue dominates (EX)

$$\mathbf{N}(t) = e^{\sqrt{11}t}\left(c_1\mathbf{x}_1 + c_2 e^{(1-\sqrt{11})t}\mathbf{x}_2 + c_3 e^{-2\sqrt{11}t}\mathbf{x}_3\right)$$

$$F_i(t) = c_2 e^{(1-\sqrt{11})t} x_{2i} + c_3 e^{-2\sqrt{11}t} x_{3i}$$

Proof that leading eigenvalue dominates (EX)

For various values of constants, we see that our F(t) approaches 0 in the limit of large t.



Limit of Taylor Law relationship

$$b(t) = \frac{d\log[\sum_{i<j}(N_i(t) - N_j(t))^2]}{d\log[\sum_{i=1}^{n} N_i(t)]}$$

We say Taylor's law holds if we observe the following power-law relationship:

$$\lim_{t\to\infty} b(t) = 2$$

Limit of Taylor Law relationship

$$b(t) = \frac{d\log[\big(\sum_{i<j}(N_i(t) - N_j(t))^2]}{d\log[\big(\sum_{i=1}^{n}(N_i)]}$$

$$= \frac{d\log[\big(e^{2\lambda_1 t}\sum_{i<j}((c_1x_{1i} + F_i(t)) - (c_1x_{1j} + F_j(t)))^2]}{d\log[e^{\lambda_1 t}\big(\sum_{i=1}^{n}(c_1x_{1i} + F_i(t))]}$$

$$= \frac{d\big(2\lambda_1 t + \log\big[\sum_{i<j}((c_1x_{1i} + F_i(t)) - (c_1x_{1j} + F_j(t)))^2\big]\big)}{d\big(\lambda_1 t + \log\big[e^{\lambda_1 t}\big(\sum_{i=1}^{n}(c_1x_{1i} + F_i(t))\big]\big)}$$

$$= \frac{2\lambda_1 + \frac{2\sum_{i<j}((c_1x_{1i}+F_i(t))-(c_1x_{1j}+F_j(t)))(F_i'(t)-F_j'(t))}{\sum_{i<j}((c_1x_{1i}+F_i(t))-(c_1x_{1j}+F_j(t)))^2}}{\lambda_1 + \frac{\sum_{i=1}^{n}F_i'(t)}{\sum_{i=1}^{n}(c_1x_{1i}+F_i(t))}}$$

Limit of Taylor Law relationship

$$\lim_{t\to\infty} b(t) = \frac{\boxed{2\lambda_1} + \overset{0}{\overline{\boxed{c_1^2}\sum_{i<j}\boxed{(x_{1i}-x_{1j})}^2}}}{\boxed{\lambda_1} + \overset{0}{\overline{\boxed{c_1}\sum_{i=1}^{n}\boxed{x_{1i}}}}}$$

And from the hypotheses, we have:

$$\boxed{\lambda_1 \neq 0,}\,\boxed{c_1 \neq 0,}\ \exists i, j \text{ such that } \boxed{x_{1i} \neq x_{1j}}$$

Limit of Taylor Law relationship

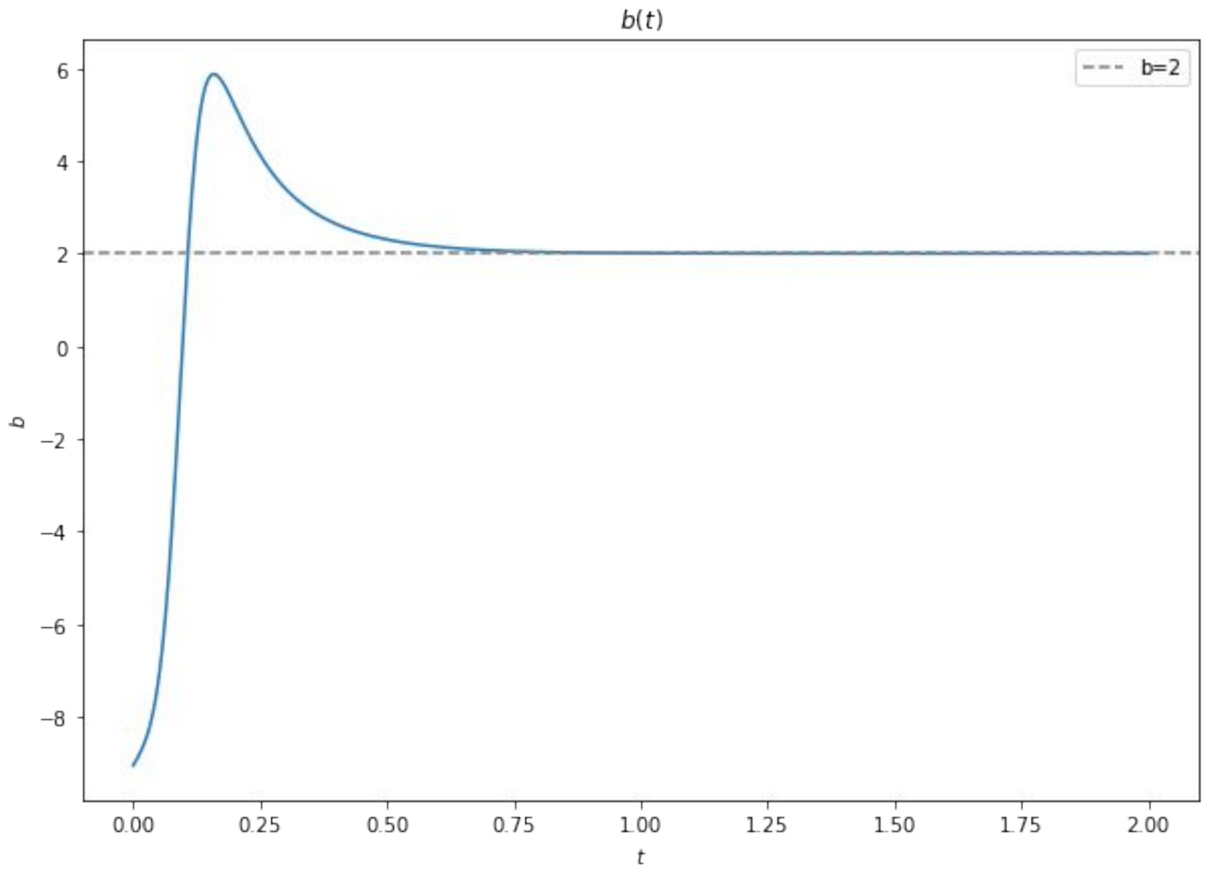
$$\lim_{t\to\infty} b(t) = \frac{2\lambda_1}{\lambda_1} = 2$$

So Taylor's Law holds!

Limit of Taylor Law relationship (EX)

$$\lim_{t\to\infty} b(t) = \frac{2\sqrt{11} + \frac{0}{c_1^2 \sum_{i<j}(x_{1i}-x_{1j})^2}}{\sqrt{11} + \frac{0}{c_1 \sum_{i=1}^{n} x_{1i}}}$$

$$\lim_{t\to\infty} b(t) = \frac{2\sqrt{11}}{\sqrt{11}} = 2$$



$b(t)$
b=2
6
4
2
0
−2
−4
−6
−8
b
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

Thank you

Q & A