$\overline{A} = \{x : B(x,\varepsilon)\cap A \neq \emptyset \ \forall \varepsilon > 0\}$

clearly $A \subset \overline{A}$

<u>Lemma 1</u> If $x \in A$ and $B(x, 2\varepsilon)\cap A \neq \emptyset$ then $B(x,\varepsilon)\cap \overline{A} = \emptyset$

<u>Corollary</u> - $\overset{\circ}{\overline{A^c}} = \overset{\circ}{\overline{A}^c}$

pf of <u>Lemma 1</u> - Assume $B(x,2\varepsilon)\cap A = \emptyset$ and $y \in B(x,\varepsilon)\cap\overline{A}$
Then $\exists\, z \in B(y,\varepsilon)\cap A$ (because $y\in\overline{A}$ so $B(z,\varepsilon)\cap A\neq\emptyset$). $\Rightarrow$
$d(x,z) \le d(x,y) + d(y,z) < \varepsilon + \varepsilon = 2\varepsilon$
But this implies $z \in A$ and $z \in B(x, 2\varepsilon)$

pf of <u>Corollary</u> — $\overline{A}\supset A \Rightarrow \overline{A}^c \subset A^c \Rightarrow \overset{\circ}{\overline{A}^c} \subset \overset{\circ}{A^c}$
Let $x \in \overset{\circ}{A^c} \Rightarrow B(x,2\varepsilon)\subset A^c \Rightarrow B(x,2\varepsilon)\cap A = \emptyset$
Lemma 1 $\Rightarrow B(x,\varepsilon)\cap\overline{A}\neq\emptyset \Rightarrow B(x,\varepsilon)\subset\overline{A}^c$
$\Rightarrow x \in \overset{\circ}{\overline{A}^c}$

<u>property</u> $\overline{A}$ is closed and $\overline{A}\supset A$

pf let $x \in \overline{A}^c \Rightarrow (x \notin \overline{A})$ so $B(x,2\varepsilon)\cap A = \emptyset$ for some $\varepsilon > 0$
$\Rightarrow B(x,\varepsilon)\cap\overline{A} = \emptyset \Rightarrow B(x,\varepsilon)\subset\overline{A}^c \Rightarrow \overline{A}^c$ is open $\Rightarrow$
$\overline{A}$ is closed

<u>property</u> — $\overline{A}$ is the smallest closed superset of A:

$$\overline{A} = \bigcap_{\substack{C\supset A \\ C \text{ closed}}} C$$

Since $\overset{\circ}{C^c}$ is open by corollary

pf Let $C \supset A$, C is closed $\Rightarrow C^c \subset A^c$, $C^c = \overset{\circ}{C^c} \subset \overset{\circ}{A^c} = \overline{A}^c \subset \overline{A}^c$

$\therefore C^c \subset \overline{A}^c \Rightarrow C \supset \overline{A}$

Ex $\overline{B}(z, R)$ is closed

wts $\overline{B}(z,R)^c$ is open

Let $x \in \overline{B}(z, R)^c \Rightarrow x \notin \overline{B}(z,R) \Rightarrow d(x,z) = s > R$.

Look at $B(x, s-R)$. Let $y \in B(x, s-R) \Rightarrow$

$d(z,y) + d(y,x) \geq d(z,x)$ $\boxed{\Delta \leq}$

$\therefore d(z,y) \geq d(z,x) - d(y,x) > s - (s-R)$

$d(z,y) > R \therefore B(x, s-R) \subset \overline{B}(z, R)^c \Rightarrow \overline{B}(x,R)^c$ is open

Remark - IS $\overline{B(z,R)} = \overline{B}(z,R)$?

No $M = \{0, 1\}$ or any discrete metric

$B(0,1) = \{0\} \Rightarrow \overline{B(0,1)} = \{0\}$

$\overline{B}(0,1) = \{0,1\}$

Accumulation point of A $B(x, \varepsilon) \cap (A \setminus \{x\}) \neq \emptyset$ $\forall \varepsilon > 0$