Counter examples — $\mathbb{Q}$ and not $\mathbb{Q}$

M, d

$A \subset M$ is said to be closed if A^c is open

x is said to be a closure point of A if
$B(x, \varepsilon) \cap A \neq \emptyset \quad \forall \varepsilon > 0$

$A \subset \bar{A}$

The set of closure points of A is denoted $\bar{A}$ $(cl(A))$

~~properties~~ — of closed sets

$\emptyset$ and M are closed

finite union of closed sets is closed

Any intersection of closed sets is closed

$\bar{A}$ is closed ie. $\bar{A}^c$ is open

pf Let $x \in \bar{A}^c$ and find $\varepsilon > 0 \ni B(x, \varepsilon) \subset \bar{A}^c$

But $x \notin \bar{A}$ so $\exists$ some $\varepsilon > 0 \ni B(x, \varepsilon) \cap A = \emptyset \Rightarrow B(x, \varepsilon) \subset A^c$

WRONG

$\bar{A} \supset A$ and $\bar{A}$ is closed
$\mathring{A} \subset A$ and $\mathring{A}$ is open

$\bar{A}$ is the smallest closed superset of A
ie. $\bar{A} = \bigcap\limits_{\substack{B \supset A \\ B \text{ is closed}}} B$

pf $B \supset A \Rightarrow \bigcap B \supset A$ $\quad$ B closed $\Rightarrow \bigcap B$ is closed
$\therefore \bigcap B \supset A$ and is closed
wts. $\bigcap B \supset \bar{A}$ and $\bar{A} \subset B$ whenever $B \supset A$, B is closed

Let $x \in B^c$ then $\exists\, \varepsilon > 0 \ni B(x, \varepsilon) \subset B^c \Rightarrow B(x, \varepsilon) \cap B = \emptyset$
But $B \supset A$ so $B(x, \varepsilon) \cap A = \emptyset \Rightarrow x \notin \bar{A} \Rightarrow x \in \bar{A}^c \Rightarrow$
$B^c \subset \bar{A}^c \Rightarrow \bar{A} \subset B$

Accumulation Point - x is an accumulation point of A is
$B(x, \varepsilon) \cap A$ contains a point in A other than x $\forall \varepsilon > 0$
$B(x, \varepsilon) \cap A \setminus \{x\} \neq \emptyset$ $\forall \varepsilon > 0$

$A = \{\frac{1}{n} | n \geq 1\}$ $\quad$ $\bar{A} = A \cup \{0\}$
Accumulation points, $\text{Accum}(A) = \{0\}$

Section on Accumulation points