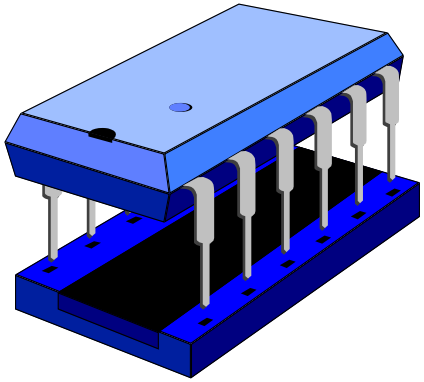

BOOLEAN ALGEBRA AND LOGIC

Bernhard Kainz (with thanks to **A. Gopalan, N. Dulay** and **E. Edwards**)

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Learning Objectives

- At the end of this lecture you should:
  - Understand how logic relates to computing problems
  - Be able to represent Boolean logic problems as:
    - Truth tables
    - Logic circuits
    - Boolean algebra

What is Logic?

- Dictionary definitions (dictionary.com - reduced!)
  - reason or sound judgement
  - a system of principles of reasoning
  - the science that investigates the principles governing correct or reliable inference
- Branch of philosophy
  - Principles of inference
- You use logic all the time in your everyday life

Propositional Logic

- The Ancient Greek philosophers created a system to formalise arguments called propositional logic

- A proposition is a statement that can be TRUE or FALSE

- Propositions can be compounded by means of the operators AND, OR and NOT

Propositional Logic Example

- Propositions may be TRUE or FALSE, for example:
  - It is raining
  - The weather forecast is bad
- A combined proposition example is:
  - It is raining OR the weather forecast is bad

Propositional Logic Example

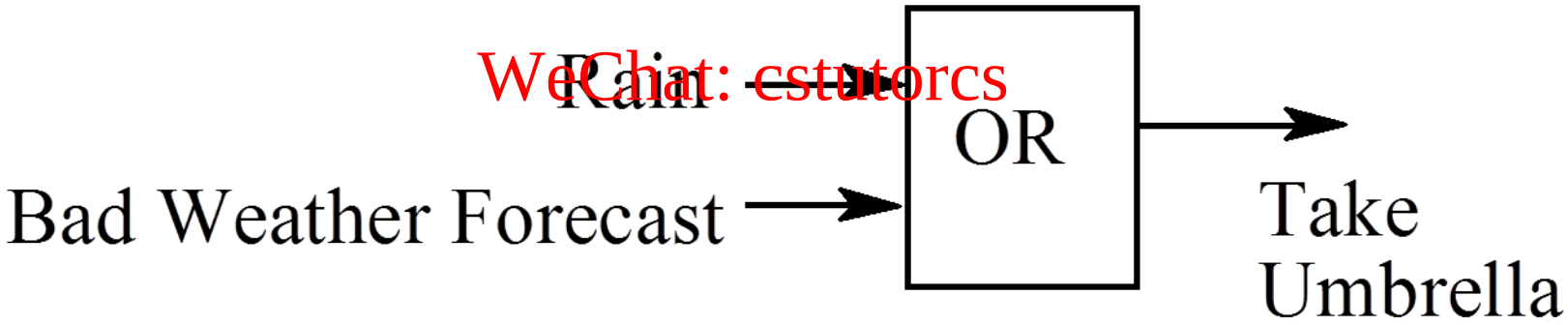
- Can assign values to propositions, for example: I will take an umbrella if it is raining OR the weather forecast is bad

  - Means that the proposition “I will take an umbrella” is the result of the Boolean combination (OR) between raining and weather forecast being bad. In fact we could write:

    - I will take an umbrella = it is raining OR the weather forecast is bad

Diagrammatic Representation

- Can think of the umbrella proposition as a result that we calculate from the weather forecast and the fact that it is raining by means of a logical OR



Truth Tables

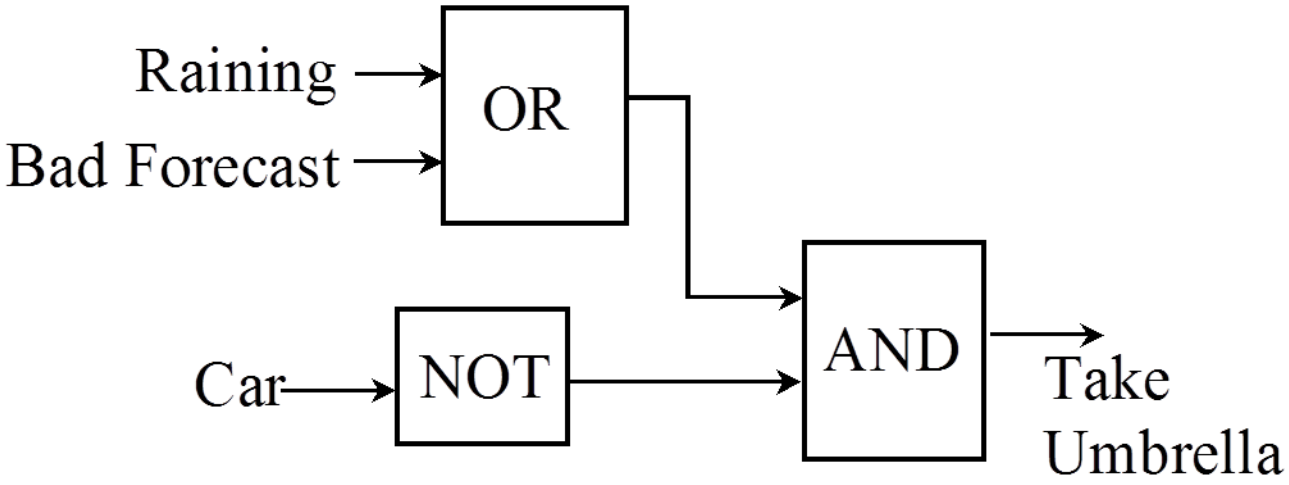
- Since propositions can only take two values, we can express all possible outcomes of the umbrella proposition by a table

Raining	Bad Forecast	Umbrella
FALSE	FALSE	FALSE
FALSE	TRUE	TRUE
TRUE	FALSE	TRUE
TRUE	TRUE	TRUE

Complex Propositions

- Can make our propositions more complex, for example:
  - (Take Umbrella) = (NOT (Take Car)) AND ((Bad Forecast) OR (Raining))

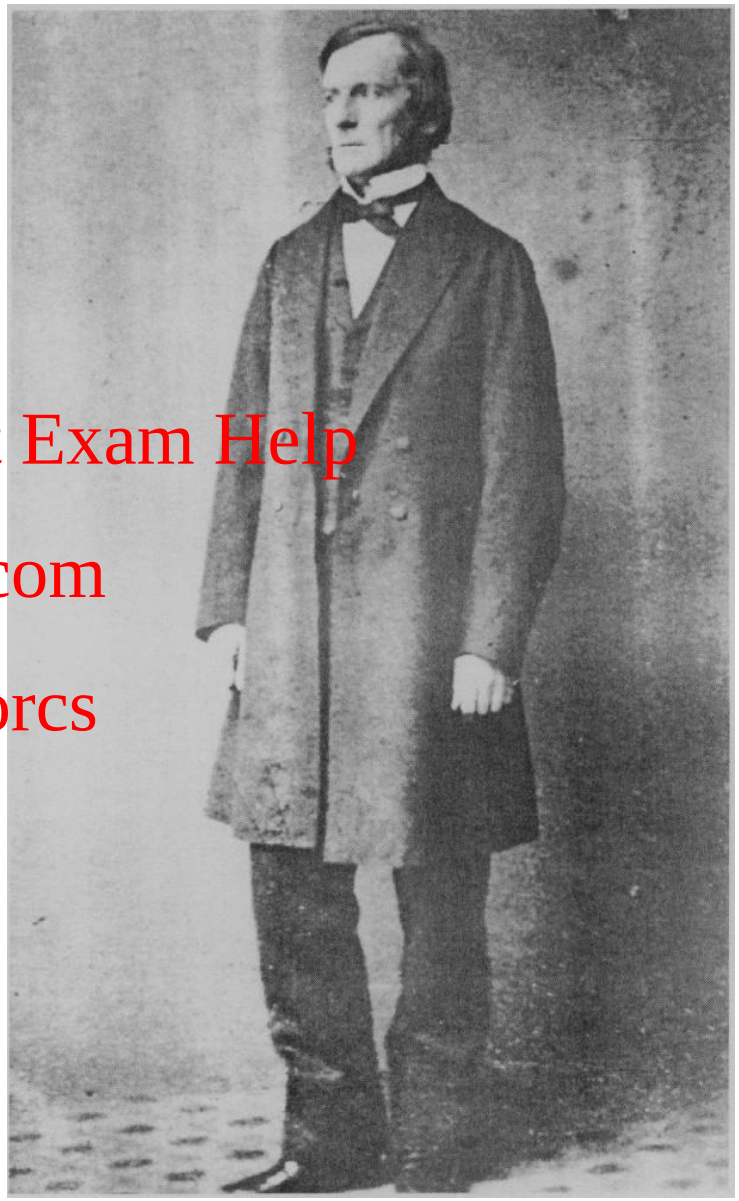
- Diagrammatical representation



Boolean Logic

- To perform calculations quickly and efficiently we need a more succinct notation than propositional logic
- Need to have well-defined semantics for all the “operators”, or connectives that we intend to use

- Boolean Algebra satisfies the criterion above
- Named after George Boole
- Provides a system of logical operations
- Rules for combining operations
- Describes their application to binary numbers



George Boole: 1815-1864

Boolean Algebra – Fundamentals

- The truth values are replaced by 1 and 0
  - 1 = TRUE 0 = FALSE

- Propositions are replaced by variables
  - R = it is raining W = The weather forecast is bad

- Operators are replaced by symbols
  - ' = NOT + = OR • = AND

Boolean Algebra – Simplify Propositions

- Recall:
  - (Take Umbrella) = (NOT (Take Cat) AND (Heavy Forecast) OR (Raining))
- Using notations notations, we get:
  - U = (C') • (W + R)

Boolean Algebra – Precedence

- Operator Precedence
  - Highest precedence operator is evaluated first

OPERATOR	SYMBOL	PRECEDENCE
NOT	' (¬)	Highest
AND	• (∧)	Middle
OR	+ (∨)	Lowest

math (logic) symbol

- Note that: (C') • (W + R) is not the same as C' • W + R
- Logic operators in, e.g., C:
  - Logical: AND: && OR: || NOT: !
  - (Binary: AND: & OR: | NOT: ~)

Boolean Algebra – Truth Tables

- All possible outcomes of the operators can be written as truth tables

AND

•

A	B	R
0	0	0
0	1	0
1	0	0
1	1	1

OR

+

A	B	R
0	0	0
0	1	1
1	0	1
1	1	1

NOT

'

A	R
0	1
1	0

Boolean Algebra – Truth Tables

- Given any Boolean expression e.g.: U = C' • (W + R)
- We can calculate a truth table for every possible value of the variables on the right hand side
- **For n variables there are 2^n possibilities**

Boolean Algebra – Truth Tables

- Truth table for “Umbrella”
  - U = C' • (W + R)

R	W	C	X1=R+W	X2=C'	U=X1•X2
0	0	0	0	1	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	1	0	0
Inputs			Partial Results		Outputs

Boolean Algebra – Rules

- **Note: A and B can be any Boolean Expression**

Negation:

(A')' = A

A • A' = 0

A + A' = 1

Associative:

(A • B) • C = A • (B • C)

(A + B) + C = A + (B + C)

Commutative:

A • B = B • A

A + B = B + A

Distributive:

A • (B + C) = A • B + A • C

A + (B • C) = (A + B) • (A + C)

Note the precedence

Boolean Algebra – Rules

Single variables (Idempotent law):

$A \cdot A = A$

$A + A = A$

Simplification rules with 1 and 0:

$A \cdot 0 = 0$

$A \cdot 1 = A$

$A + 0 = A$

$A + 1 = 1$

Boolean Algebra – de Morgan's Rule

(A + B)' = A' • B'

(A • B)' = A' + B'

as before, A and B can be any Boolean expression

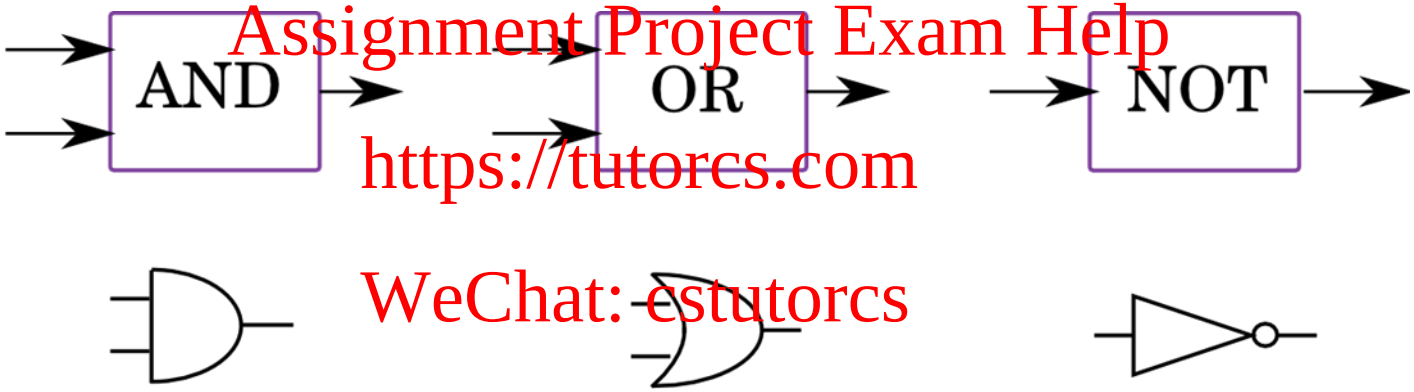
Can generalise to n Boolean variables:

(A + B + C + D + ...)' = A' • B' • C' • D' • ...

(A • B • C • D • ... • X)' = A ' + B' + C' + D' + ... + X'

Boolean Functions – Schematic Representation

- A standard set of easy-to-recognise symbols is used to represent Boolean functions



- A circle is all that is required to indicate NOT. The triangle is just to indicate Input/Output direction

Inverting Functions

- A circle can be added to the AND and OR symbol outputs to create their inverted functions – NotAnd (NAND) and NotOr (NOR) gates

Building Blocks for Circuits

- NAND/NOR are the commonly used building blocks for most circuits

  - NAND / NOR can easily be constructed from transistors

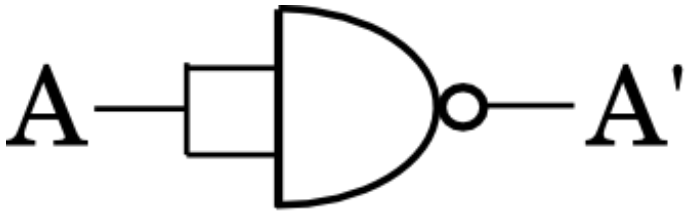
  - NAND is complete

    - A set of Boolean functions f1,f2,… is “complete” if and only if any Boolean function can be generated by a combination these functions

    - Also called “universal gate”

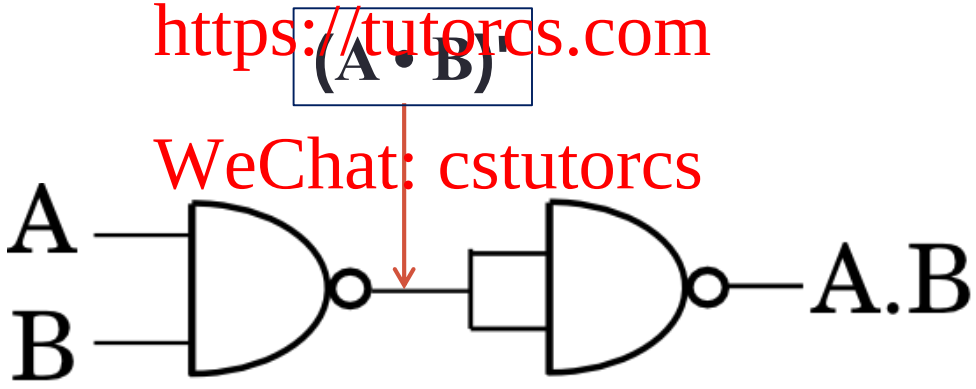
NAND Gate – NOT

- It is possible to build all other gates out of NAND gates
- Create a NOT gate using the idempotent law:
  - A • A = A therefore (A • A)' = A'



NAND Gate – AND

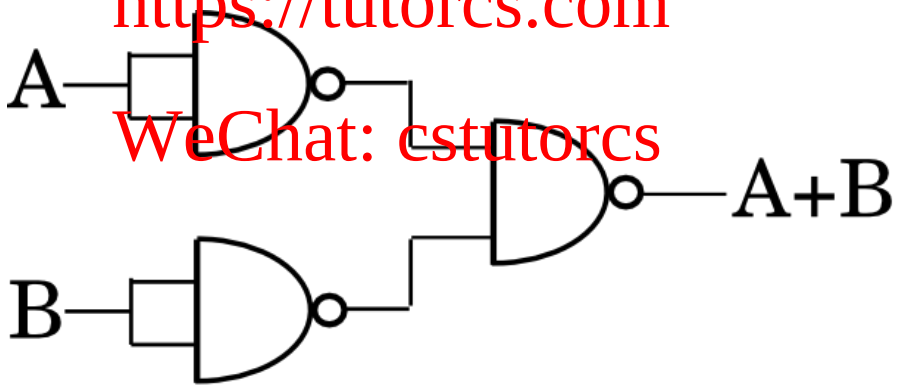
- Create an AND gate using the Involution law:
  - (A')' = A



NAND Gate – OR / NOR

- To make an OR gate we need to apply de Morgan's theorem:

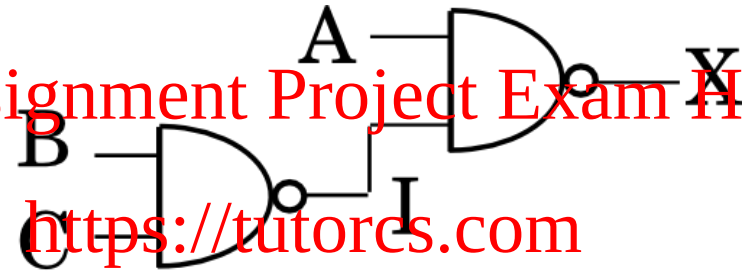
  - A + B = (A' • B')'



- Just invert output to get a **NOR** gate ☺

NAND – Complex Circuits

- Consider two cascading NAND Gates



- What circuit have we created?
  - Use Boolean Algebra to find out
  - I = (B • C)'
  - X = (A • I)' = (A • (B • C)')'
  - Apply de Morgan's law, we get
  - X = A' + ((B • C)')' = A' + (B • C)

NAND – Complex Circuits

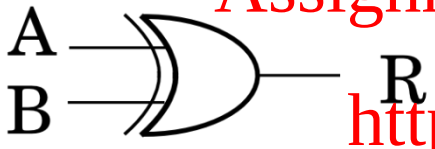
- Truth table for X = A' + (B • C)

A	B	C	B • C	X = A' + (B • C)
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

XOR and XNOR Gates

- Very useful gates

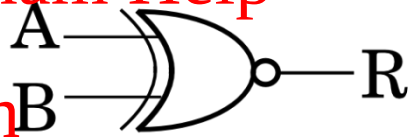
Exclusive Or (XOR)



R = A.B' + A'.B

A	B	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Exclusive Nor (XNOR)

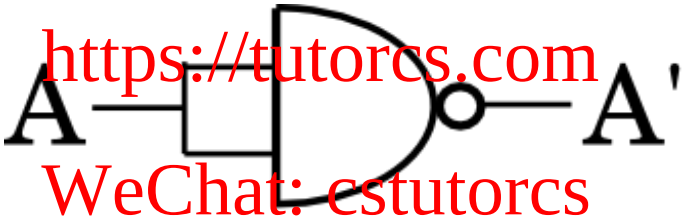


R = A'.B' + A.B

A	B	XNOR
0	0	1
0	1	0
1	0	0
1	1	1

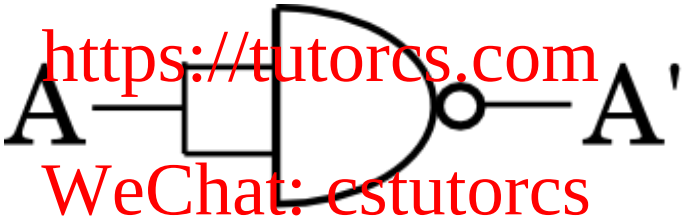
Behind the scenes

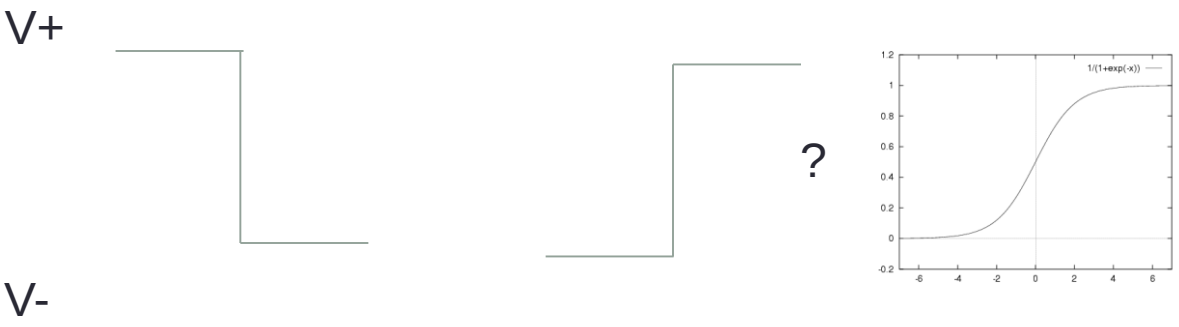
- Instantaneous on IC?



Behind the scenes

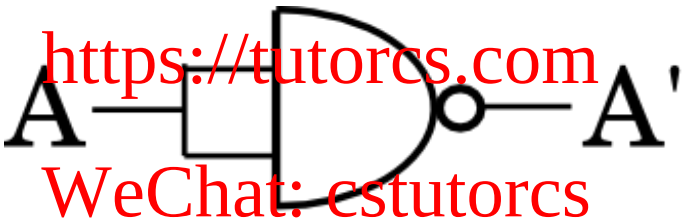
- Instantaneous on IC?

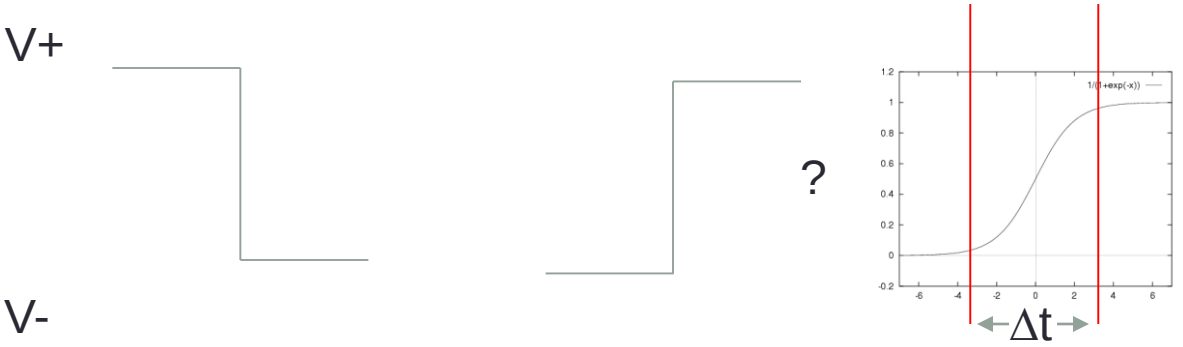




Behind the scenes

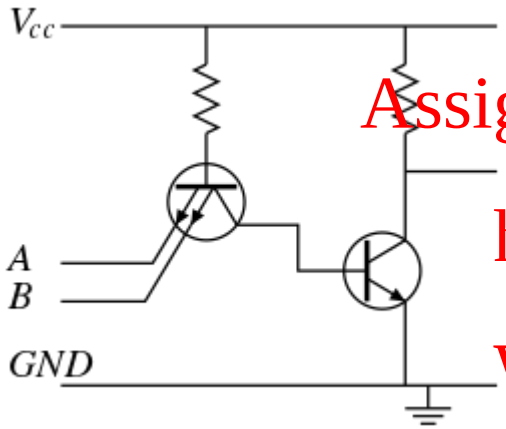
- Time delay and saturation limit state ‘switching’ speed of real in-silico circuits

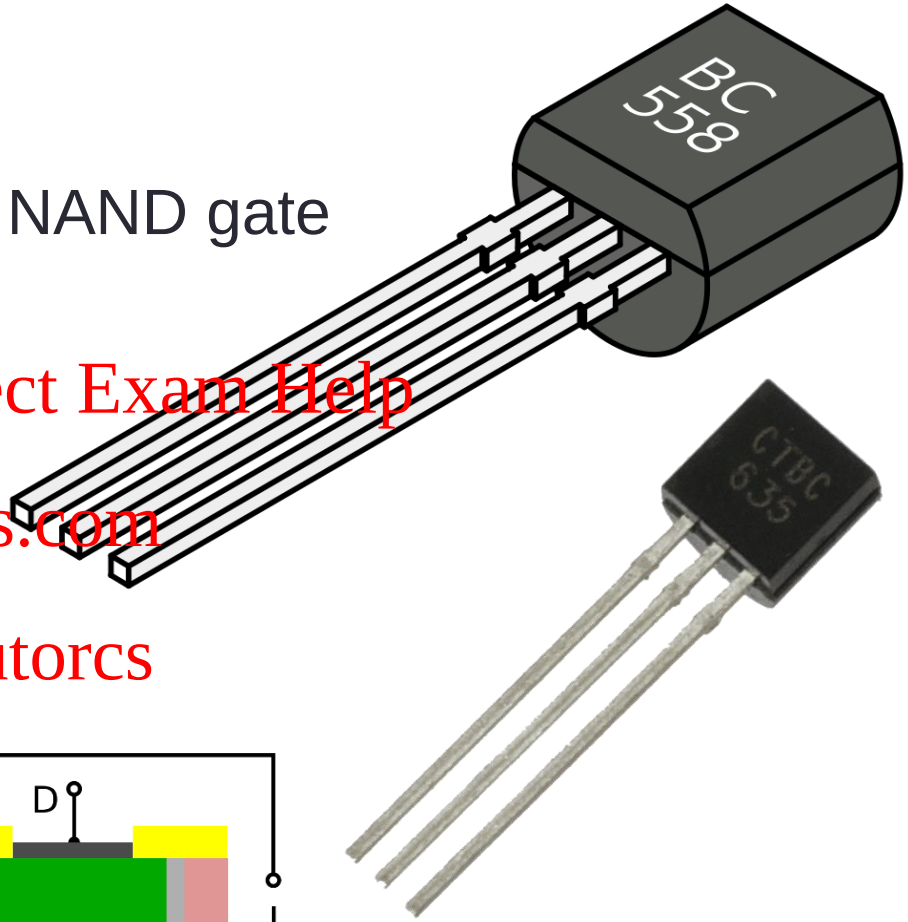


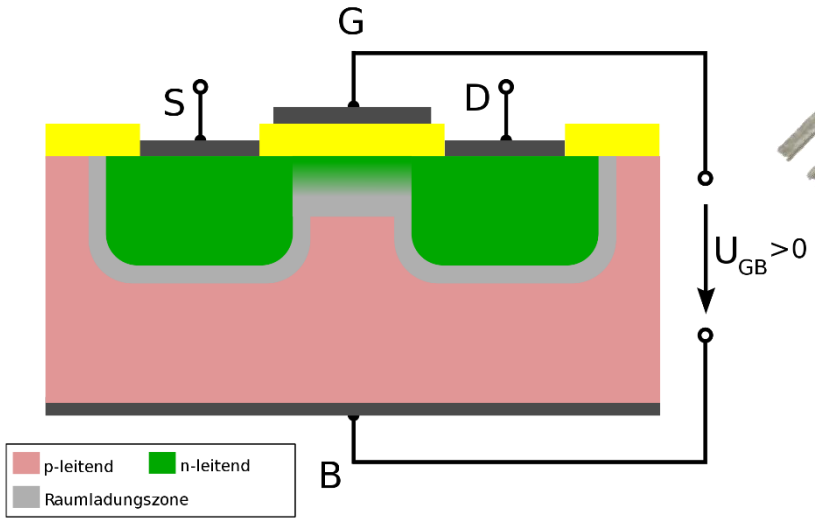


Behind the scenes

- Transistor-transistor logic TTL NAND gate

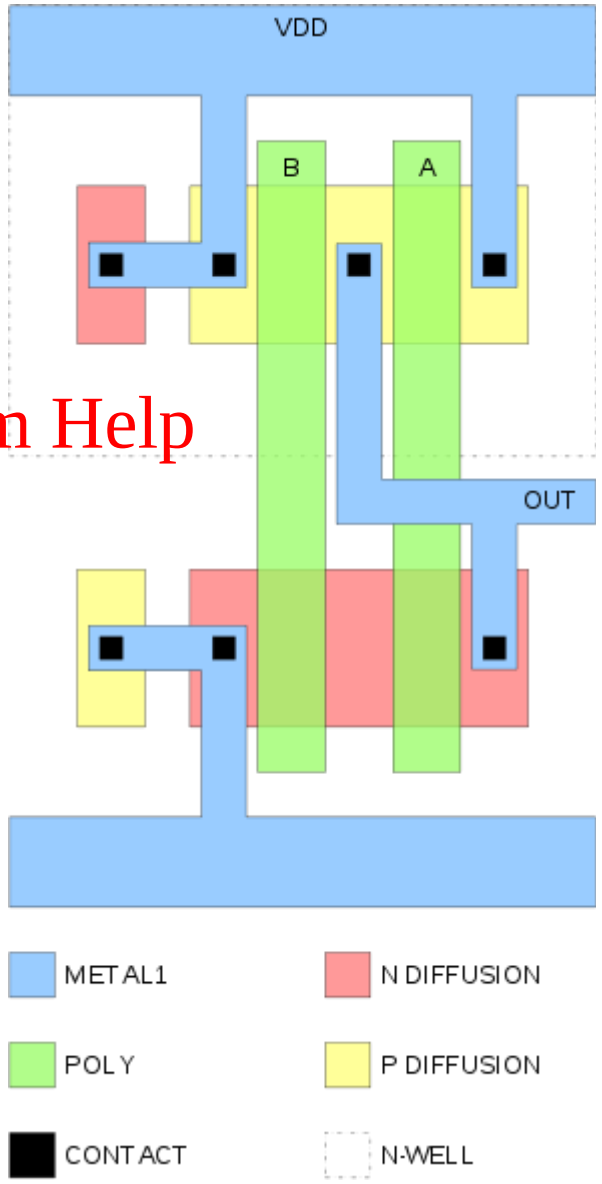






Behind the scenes

- CMOS NAND in silico



Behind the scenes

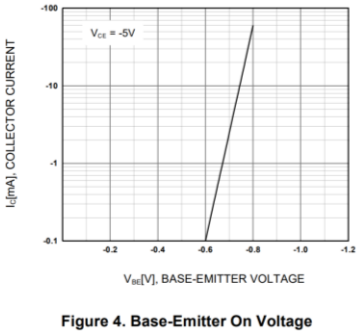


Figure 4. Base-Emitter On Voltage