Number representation

Bit Pattern	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
Unsigned	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Sign & Magnitude	+0	+1	+2	+3	+4	+5	+6	+7	−0	−1	−2	−3	−4	−5	−6	−7
1s Complement	+0	+1	+2	+3	+4	+5	+6	+7	-7	-6	-5	-4	-3	-2	-1	-0
2s Complement	+0	+1	+2	+3	+4	+5	+6	+7	-8	-7	-6	-5	-4	-3	-2	-1
Excess-8	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
BCD	0	1	2	3	4	5	6	7	8	9	-	-	-	-	-	-

Number representation Excess-n

- **Excess-n TO Decimal number: convert to decimal, substract the n from the decimal**
- **Decimal number TO Excess-n: add the n to the decimal and convert result to binary**

-3 in Excess-8?
-3 + 8 = 5

5 in unsigned = 0101 = 5 in one-complement = 5 in two-complement = -3 in Excess 8

5 in Excess-8?

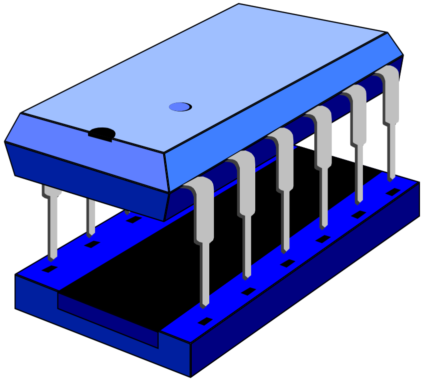
5 + 8 = 13
13 in unsigned: 1101 (beyond 2s complement range but positive (shift like a circular linked list in 2s complement!). No further processing necessary)

-7 in excess-6?
-7 + 6 = -1 = -1 in 1s complement: 0001 -> (negative number bit inversion rule) -> 1110 =
-1 in 2s complement: 1110 + 1 = 1111 = -7 in excess-6

-8 in excess-6?
-8+6 = -2 ->
2 in unsigned: 0010 -> 1s complement = 0010 -> (negative number bit inversion rule) -> 1101
In 2s complement = 1110 = -8 in excess-6

Number representation

Bit Pattern	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
Unsigned	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Sign & Magnitude	+0	+1	+2	+3	+4	+5	+6	+7	−0	−1	−2	−3	−4	−5	−6	−7
1s Complement	+0	+1	+2	+3	+4	+5	+6	+7	-7	-6	-5	-4	-3	-2	-1	-0
2s Complement	+0	+1	+2	+3	+4	+5	+6	+7	-8	-7	-6	-5	-4	-3	-2	-1
Excess-8	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
BCD	0	1	2	3	4	5	6	7	8	9	-	-	-	-	-	-
Excess-6	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	-8	-7



FLOATING POINT NUMBERS

Introduction

Bernhard Kainz (with thanks to **A. Gopalan, N. Dulay** and **E. Edwards**)

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Why do we need this: large, small and fractional numbers

World population	>7, 200, 000, 000 people
One light year	9, 130, 000, 000, 000 km
One solar mass	2, 000, 000, 000, 000, 000, 000, 000, 000, 000 kg
Electron diameter	0.000, 000, 000, 000, 000, 000, 01 m
Electron mass	0.000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 9 kg
Smallest measurable length of time	0.000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 1 sec
Pi (to 14 decimal places)	3.14159 26535 8979...
Standard rate of VAT	20%
Googol	1 followed by a 100 zeros ☺

Large integers

Example: How can we represent integers up to 30 decimal digits long?

- **Binary**: $2^X = 10^{30} \Rightarrow X = \log_2(10^{30}) \approx 100$ bits (1 decimal digit ≈ 3.32 bits)

- **BCD**: $30 \times 4 = 120$ bits
- **ASCII**: $30 \times 8 = 240$ bits

Floating point numbers

Recall scientific notation:

$M \times 10^E$ Decimal

$M \times 2^E$ Binary

This is the basis for most floating point representation schemes

- *M* is the **coefficient** (aka. **significand**, **fraction** or **mantissa**)
- *E* is the **exponent** (aka. **characteristic**)
- *10* (or for binary, *2*) is the **radix** (aka. **base**)

- No. of bits in **exponent** determines the **range** (**bigness/smallness**)
- No. of bits in **coefficient** determines the **precision** (**exactness**)

Real vs. floating point numbers

	Mathematical real	Floating point number
Range	$-\infty \ldots +\infty$	Finite
No. of values	(Uncountably) infinite	Finite
Spacing	?	Gap between numbers varies
Errors	?	Incorrect results are possible

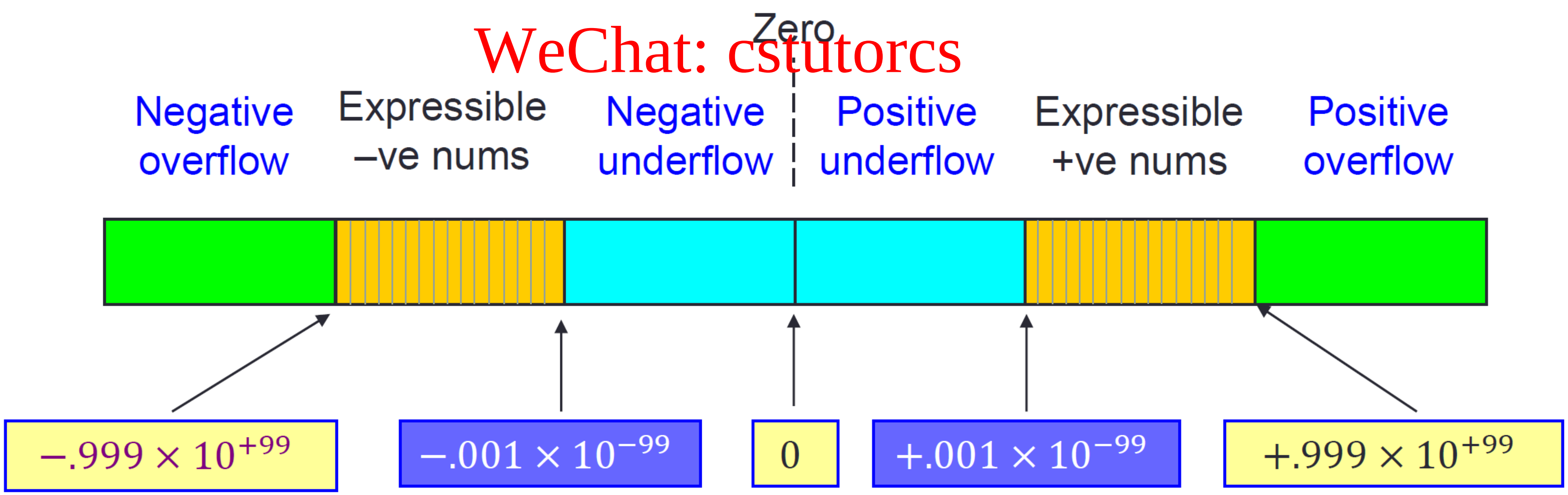
Some questions **(assume signed 3-digit coefficient** and a **signed 2-digit exponent as before):**

- What are the **closest** floating point numbers to $\mathbf{.001 \times 10^{-99}}$? What is the **gap** between this number and them?
- What about $\mathbf{.001 \times 10^{-50}}$?

Zones of expressibility

- **Example**: assume numbers are formed with a **signed 3-digit coefficient** and a **signed 2-digit exponent**

- **Zones of expressibility:**



Normalised floating point numbers

- Depending on how you interpret the coefficient, floating point numbers **can have multiple forms**, e.g.:

$$
\begin{aligned}
0.023 \times 10^4 &= 0.230 \times 10^3 \\
&= 2.3 \times 10^2 \\
&= 0.0023 \times 10^5
\end{aligned}
$$

- For hardware implementations it is desirable for each number to have a unique floating point representation, a **normalised form**
- We'll normalise coefficients in the **range** $[\mathbf{1}, \ldots \boldsymbol{R})$ where $\boldsymbol{R}$ is the base, e.g.:

$[1, \ldots, 10)$ for decimal

$[1, \ldots, 2)$ for binary

Normalised forms (base 10)

Number	Normalised form
23.2×10^4	

Normalised forms (base 10)

Number	Normalised form
23.2×10^4	2.32×10^5

Normalised forms (base 10)

Number	Normalised form
23.2×10^{4}	2.32×10^{5}
-4.01×10^{-3}	

Normalised forms (base 10)

Number	Normalised form
23.2×10^4	2.32×10^5
-4.01×10^{-3}	-4.01×10^{-3}

Normalised forms (base 10)

Number	Normalised form
23.2×10^{4}	2.32×10^{5}
-4.01×10^{-3}	-4.01×10^{-3}
$343\ 000 \times 10^{0}$	3.43×10^{5}
$0.000\ 000\ 098\ 9 \times 10^{0}$	

Normalised forms (base 10)

Number	Normalised form
23.2×10^{4}	2.32×10^{5}
-4.01×10^{-3}	-4.01×10^{-3}
$343\,000 \times 10^{0}$	3.43×10^{5}
$0.000\,000\,098\,9 \times 10^{0}$	9.89×10^{-8}

Normalised forms (base 2)

Number	Normalised form
100.01×2^1	1.0001×2^3
1010.11×2^2	1.01011×2^5
0.00101×2^{-2}	1.01×2^{-5}
1100101×2^{-2}	1.100101×2^4

Binary fractions

Binary	Decimal
0.1	

Binary fractions

Binary	Decimal
0.1	0.5

Binary fractions

Binary	Decimal
0.1	0.5
0.01	

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75
0.111	

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75
0.111	0.875

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75
0.111	0.875
0.011	

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75
0.111	0.875
0.011	0.375

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75
0.111	0.875
0.011	0.375
0.101	

Binary fractions

Binary	Decimal
0.1	0.5
0.01	0.25
0.001	0.125
0.11	0.75
0.111	0.875
0.011	0.375
0.101	0.625

Binary fraction to decimal fraction

- What is the binary value **0.01101** in decimal?

	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
.	0	1	1	0	1

- $\frac{1}{4} + \frac{1}{8} + \frac{1}{32} = \frac{13}{32} = 0.40625$

32	16	8	4	2	1
.	0	1	1	0	1

- $\frac{8+4+1}{2^5} = \frac{13}{32}$
- What about 0.000 110 011?
- Answer: $\frac{32+16+2+1}{2^9} = \frac{51}{512} = 0.099\ 609\ 375$

Decimal fraction to binary fraction

- What is the decimal value **0.6875** in binary?

$$0.6875 = \frac{1.375}{2} = \frac{1}{2} + \frac{0.375}{2} = \frac{1}{2} + \frac{0.75}{4} = \frac{1}{2} + \frac{1.5}{8}$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{0.5}{8} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$$

So the answer is **0.1011**

- What is the decimal value **0.1** in binary?

$$0.1 = \frac{1.6}{16} = \frac{1}{16} + \frac{0.6}{16} = \frac{1}{16} + \frac{1.2}{32} = \frac{1}{16} + \frac{1}{32} + \frac{0.2}{32} = \frac{1}{16} + \frac{1}{32} + \frac{1.6}{256}$$

...

Floating point multiplication

$$N_1 \times N_2 = \left(M_1 \times 10^{E_1}\right) \times \left(M_2 \times 10^{E_2}\right)$$
$$= (M_1 \times M_2) \times \left(10^{E_1} \times 10^{E_2}\right)$$
$$= (M_1 \times M_2) \times \left(10^{E_1+E_2}\right)$$

- That is, we **multiply the coefficients** and **add the exponents**
- Example:

$$\left(2.6 \times 10^6\right) \times \left(5.4 \times 10^{-3}\right) = (2.6 \times 5.4) \times \left(10^3\right)$$
$$= 14.04 \times 10^3$$

- We must also **normalise the result**, so final answer is 1.404×10^4

Truncation and rounding

- For many computations, the result of a floating point operation is **too large to store in the coefficient**
- Example (with a **2-digit coefficient**):

$$(2.3 \times 10^{1}) \times (2.3 \times 10^{1}) = 5.29 \times 10^{2}$$

- **Truncation** ➔ 5.2×10^{2} (biased error)
- **Rounding** ➔ 5.3×10^{2} (unbiased error)

Floating point addition

- A floating point addition such as $4.5 \times 10^3 + 6.7 \times 10^2$ is not a simple coefficient addition, unless the exponents are the same. Otherwise, we need to align them first

$$
\begin{aligned}
N_1 + N_2 &= \left(M_1 \times 10^{E_1}\right) + \left(M_2 \times 10^{E_2}\right) \\
&= \left(M_1 + M_2 \times 10^{E_2 - E_1}\right) \times 10^{E_1}
\end{aligned}
$$

- To align, **choose the number with the smaller exponent** and **shift its coefficient** the corresponding number of digits to the right

$$
\begin{aligned}
4.5 \times 10^3 + 6.7 \times 10^2 &= 4.5 \times 10^3 + 0.67 \times 10^3 \\
&= 5.17 \times 10^3 = 5.2 \times 10^3
\end{aligned}
$$

(rounded)

Exponent overflow and underflow

- **Exponent overflow** occurs when the result is **too large** i.e. when the **result's exponent > maximum exponent**

- **Example:** if max exponent is 99 then $10^{99} \times 10^{99} = 10^{198}$ **(overflow)**

To handle **overflow**, set value as infinity or raise an exception

- **Exponent underflow** occurs when the result is **too small** i.e. when the **result's exponent < smallest exponent**

- **Example:** if min exponent is -99 then $10^{-99} \times 10^{-99} = 10^{-198}$ **(underflow)**

To handle **underflow**, set value as zero or raise an exception

Comparing floating point values

- Because of the potential for producing **inexact results**, comparing floating point values should **account for close results**
- If we know the **desired magnitude and precision** of results, we can adjust for closeness (**epsilon**). For example:

$a = b$ $\quad\quad$ $(b - \epsilon) < a < (b + \epsilon)$

$a = 1$ $\quad\quad$ $1 - 0.000005 < a < 1 + 0.000005$

$0.999995 < a < 1.000005$

- A more general approach is to calculate closeness of two numbers based on the **relative size** of the two numbers being compared