6. Sets

Terence Sim

6.1. Introduction

A set is a Many that allows itself to be thought of as a One.

Georg Cantor

Reading

Section 6.1 — 6.3 of Epp.
Section 3.1 — 3.4 of Campbell.

Familiar concepts

- Sets can be defined in extension, by explicitly listing its members: $\{1, 2, 3\}$, $\{apple, orange, red, unicorn\}$.
- Membership: $1 \in \{1, \{1, 2\}\}$.
- non-membership: $3 \notin \{1, 2\}$.
- No duplicates: $\{1, 1, 2, 2, 2\} = \{1, 2\}$
- Order does not matter: $\{1, 2\} = \{2, 1\}$.

A set can also be defined in intention, by specifying a property that characterizes its members:

$$\{X \mid X \in \mathbb{N} \wedge 1 < X \wedge X < 5\}.$$

Definition 6.1.1

S is a subset of T (or S is contained in T, or T contains S, or T is a superset of S) if all the elements of S are elements of T. We write $S \subseteq T$.[1]

Examples:

- $\{1,2\} \subseteq \{1,2,3\}$
- $\{1,2,3\} \subseteq \{1,2,3\}$. A set is a subset of itself.
- $\{3,4\} \not\subseteq \{1,2,3\}$

Warning

Do not confuse $S \in T$ with $S \subseteq T$!

Example: Let $S = \{1,2,\{3,4\}\}$, then

- $\{3,4\} \in S$, but $\{3,4\} \not\subseteq S$.
- $\{4\} \not\subseteq S$, and $4 \notin S$.

[1] Some authors use $X \subset Y$.

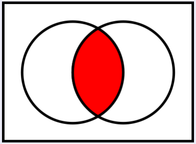
Note that the definition of subset allows a set to be a subset of itself. Sometimes we speak of a proper subset:

A set S is a proper subset, of T, denoted $S \subsetneq T$, iff $S \subseteq T$ and there is at least one element in T that is not in S.

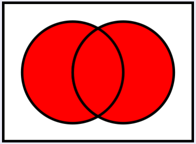
Example: $\{1, 2\} \subsetneq \{1, 2, 3\}$.

Therefore, a set cannot be a proper subset of itself.

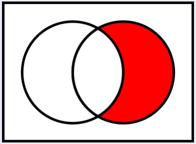
Venn diagrams and set operations



Intersection: $A \cap B$



Union: $A \cup B$



Difference: $B \backslash A$ or $B - A$

Most of the time, these intuitive concepts of sets serve us well.
But they can run into trouble.

Within 10 years after Cantor's naïve set theory, problems arose.

One of these is the famous Russell's Paradox.

Russell's Paradox

In a certain town, there is only one barber.
The barber is a man.
The barber shaves all and only those men who do not shave themselves.
Does the barber shave himself?

Russell's Paradox

Mathematically, let $\mathcal{U}$ be the set that contains all sets (the Universal set), and $M = \{A \in \mathcal{U} \mid A \notin A\}$.

Then, is $M \in M$?

Well, suppose $M \in M$. Then $M \notin M$.

On the other hand, if $M \notin M$, then we conclude that $M \in M$.

Set theory needed a firmer foundation. Intuition alone was not enough.

This foundation is called the *Zermelo-Fraenkel Set Theory with the Axiom of Choice*, or ZFC, in honor of its inventors.

Ernest Zermelo

Abraham Fraenkel

ZFC theory is outside the scope of CS1231.

6.3. Basic Set Theory

Instead of ZFC, we will study basic set theory.

Definition 6.3.1 (Empty set)

An empty set has no element, and is denoted as $\varnothing$ or $\{\}$.

Mathematically, $\varnothing$ is such that:

$$\forall Y \sim(Y \in \varnothing). \tag{1}$$

Theorem 6.2.4 (Epp) An Empty Set is a Subset of all Sets

$$\forall X\ \forall Z\ ((\forall Y \sim(Y \in X)) \rightarrow (X \subseteq Z))$$

Proof Sketch

This is a proof by contradiction. If there exists a set that does not contain the empty set, then the empty set is not empty.

Proof:

1. Let X be an empty set.

$$\forall Y \sim(Y \in X)$$

2. Suppose there exists a set in which X is not contained.

$$\exists Z \sim(X \subseteq Z)$$

...

cont'd

3. Then
   3.1 There exists an element that belongs to X and does not belong to Z by Definition 6.1.1[2]

   $$\exists Y \ (Y \in X \wedge \sim(Y \in Z))$$

   3.2 Therefore there exists an element that belongs to X.

   $$\exists Y \ (Y \in X)$$

   3.3 This a contradiction since no element belongs to X by Equation (1).

...

[2] We should prove it.

cont'd

4. Therefore there is no set in which X is not contained.

$$\sim(\exists Z \sim(X \subseteq Z))$$

5. Therefore X is a subset of all sets.

$$\forall Z \ (X \subseteq Z) \qquad \blacksquare$$

Definition 6.3.2 (Set Equality)

Two sets are equal if and only if they have the same elements.

$$\forall X \ \forall Y \ ((\forall Z \ (Z \in X \leftrightarrow Z \in Y)) \leftrightarrow X = Y)$$

Examples:

- $\{1, 2, 3\} = \{2, 1, 3, 2\}$.
- $\{\} \neq \{\{\}\}$.

Proposition 6.3.3

For any two sets X and Y, X is a subset of Y and Y is a subset of X if, and only if, $X = Y$.

$$\forall X \ \forall Y \ ((X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y)$$

Proof omitted. You try!

Note that this gives us a way to check if two sets are equal: by checking if one is a subset of the other, and vice versa.

Corollary 6.2.5 (Epp) The Empty Set is Unique

$\forall X_1 \; \forall X_2 \; ((\forall Y \; (\sim(Y \in X_1)) \wedge (\forall Y \; \sim(Y \in X_2))) \rightarrow X_1 = X_2)$

Proof:

1. Let X_1 and X_2 be two empty sets.
2. Therefore $X_1 \subseteq X_2$ by Theorem 6.2.4 (Epp).
3. Therefore $X_2 \subseteq X_1$ by Theorem 6.2.4 (Epp).
4. Therefore $X_1 = X_2$ by Proposition 6.3.3. ■

Definition 6.3.4 (Power Set)

Given any set S, the power set of S, denoted by $\mathcal{P}(S)$, or 2^S, is the set whose elements are all the subsets of S, nothing less and nothing more.

That is, given set S, if $T = \mathcal{P}(S)$, then:

$$\forall X \; ((X \in T) \leftrightarrow (X \subseteq S))$$

Examples:

- $\mathcal{P}(\{x, y\}) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$.
- $\mathcal{P}(\varnothing) = \{\varnothing\}$.

If S has n elements, then 2^S has 2^n elements.

6.4. Operation on Sets

Definition 6.4.1 (Union)

Let S be a set of sets, then we say that T is the union of the sets in S, and write:

$$T = \bigcup S = \bigcup_{X \in S} X$$

iff each element of T belongs to *some* set in S, nothing less and nothing more. That is, given S, the set T is such that:

$$\forall Y \ ((Y \in T) \leftrightarrow \exists Z((Z \in S) \wedge (Y \in Z)))$$

For two sets A, B, we may simply write $T = A \cup B$.

Examples:

- Let $\{1,2\} \cup \{3,1\} = \{1,2,3\}$.
- Let $S = \{\{1,2\},\{3\},\{1,\{2\}\}\}$. Then $T = \{1,2,3,\{2\}\}$.

Proposition 6.4.2 (Some easy propositions)

Let A, B, C be sets. Then,

- $\bigcup \varnothing = \bigcup_{A \in \varnothing} A = \varnothing$
- $\bigcup \{A\} = A$
- $A \cup \varnothing = A$
- $A \cup B = B \cup A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cup A = A$
- $A \subseteq B \leftrightarrow A \cup B = B$

All are easy to prove. You try!

Definition 6.4.3 (Intersection)

Let S be a non-empty set of sets. The intersection of the sets in S is the set T whose elements belong to *all* the sets in S, nothing less and nothing more.

That is, given S, the set T is such that:

$$\forall Y\ ((Y \in T) \leftrightarrow \forall Z\ ((Z \in S) \rightarrow (Y \in Z)))$$

We write:

$$T = \bigcap S = \bigcap_{X \in S} X$$

For two sets A, B, we may simply write $T = A \cap B$.

Examples:

- $\{1, 2, 3\} \cap \{1, 4, 2, 5\} = \{1, 2\}$.
- Let $S = \{\{1, 2\}, \{3\}, \{\{1\}, \{2\}\}\}$. Then $T = \varnothing$.

Proposition 6.4.4 (Some easy propositions)

Let A, B, C be sets. Then,

- $A \cap \varnothing = \varnothing$
- $A \cap B = B \cap A$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \subseteq B \leftrightarrow A \cap B = A$

Distributivity laws:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proofs omitted. You try!

Definition 6.4.5 (Disjoint)

Let S and T be two sets. S and T are disjoint iff $S \cap T = \varnothing$.

Definition 6.4.6 (Mutually disjoint)

Let V be a set of sets. The sets $T \in V$ are mutually disjoint iff every two distinct sets are disjoint.

$$\forall X, Y \in V \; (X \neq Y \to X \cap Y = \varnothing)$$

.

Example:
The sets in $V = \{\{1,2\},\{3\},\{\{1\},\{2\}\}\}$ are mutually disjoint.

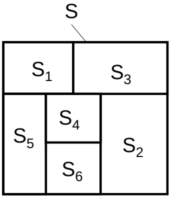
Definition 6.4.7 (Partition)

Let S be a set, and let V be a set of non-empty subsets of S. Then V is called a partition of S iff

(i) The sets in V are mutually disjoint.

(ii) The union of the sets in V equals S.

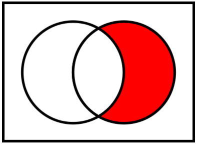
The Venn diagram shows $\{S_1, \ldots, S_6\}$ is a partition of S.



Definition 6.4.8 (Non-symmetric Difference)

Let S and T be two sets. The (non-symmetric) difference (or relative complement) of S and T, denoted $S - T$[3] is the set whose elements belong to S and do not belong to T, nothing less and nothing more.

$$\forall X\ (X \in S - T \leftrightarrow (X \in S \wedge \sim(X \in T)))$$



Examples:

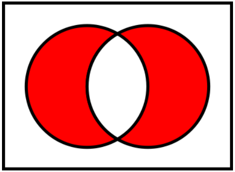
- $\{1, 2, 3, 5, 8\} - \{1, 2, 4, 8, 16, 32\} = \{3, 5\}$.
- $\{1, 2, 4, 8, 16, 32\} - \{1, 2, 3, 5, 8\} = \{4, 16, 32\}$.

[3] Some authors use $S \setminus T$.

Definition 6.4.9 (Symmetric Difference)

Let S and T be two sets. The symmetric difference of S and T, denoted $S \ominus T$[4] is the set whose elements belong to S or T but not both, nothing less and nothing more.

$$\forall X\ (X \in S \ominus T \leftrightarrow (X \in S \oplus X \in T))$$

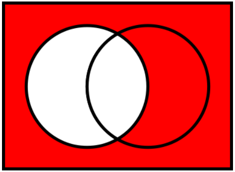


Example:
$\{1, 2, 3, 5, 8\} \ominus \{1, 2, 4, 8, 16, 32\} = \{3, 4, 5, 16, 32\}$.

[4]Some authors use $S \Delta T$.

Definition 6.4.10 (Set Complement)

Let $\mathcal{U}$ be the Universal set (or the Universe of Discourse).[5] And let A be a subset of $\mathcal{U}$. Then, the complement (or absolute complement) of A, denoted A^c, is $\mathcal{U} - A$.



Example:
$\mathcal{U} = \mathbb{N}$, and $A = \{\text{positive even numbers}\}$. Then $A^c = \{\text{positive odd numbers}\} \cup \{0\}$.

[5] This set contains all objects under discussion, eg. the set of integers if we are doing Number Theory.

There are many useful identities and theorems in set theory, many of which are already familiar to you.

Please read Theorems 6.2.1 to 6.2.3 (Epp), and their proofs. You may use and cite them as needed. Example:

$$\text{For all sets } A, B : (A \cup B)^c = A^c \cap B^c.$$

This is De Morgan's law on sets. Let's prove this.

Proof:

1. Take any two sets: A, B.
2. (Need to show that $(A \cup B)^c \subseteq A^c \cap B^c$)
3. For any $x \in (A \cup B)^c$:
4. $\quad x \notin (A \cup B)$, by definition of complement.
5. $\quad$ So $\sim(x \in A \vee x \in B)$, by definition of union.
6. $\quad$ Thus $x \notin A \wedge x \notin B$, by De Morgan's laws.
7. $\quad$ Thus $x \in A^c \wedge x \in B^c$, by definition of complement.
8. $\quad$ Thus $x \in A^c \cap B^c$, by definition of intersection.
9. $\quad$ Thus $(A \cup B)^c \subseteq A^c \cap B^c$, by definition of subset.

$\ldots$

Proof cont'd:

10. (Now, need to show that $A^c \cap B^c \subseteq (A \cup B)^c$)
11. For any $x \in A^c \cap B^c$:
12. $x \in A^c \wedge x \in B^c$, by definition of intersection.
13. Thus $x \notin A \wedge x \notin B$, by definition of complement.
14. Thus $\sim(x \in A \vee x \in B)$, by De Morgan's laws.
15. Thus $x \notin A \cup B$, by definition of union.
16. Thus $x \in (A \cup B)^c$, by definition of complement.
17. Thus $A^c \cap B^c \subseteq (A \cup B)^c$, by definition of subset.
18. Hence $(A \cup B)^c = A^c \cap B^c$, by Proposition 6.3.3. ■

Summary

- Sets may be defined in extension or in intention.
- Set membership, subset, and set equality are basic properties.
- ZFC puts set theory on a firm axiomatic foundation.
- Operations on sets include: union, intersection, difference, complement.
- Set identities mirror those of logic equivalences.