5. Sequences & Recursion

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A mathematician, like a painter or poet, is a maker of patterns.

Godfrey Harold Hardy, 1877— 1947

Reading

Sections 5.1 — 5.4, 5.6 — 5.8 of Epp.
Section 2.10 of Campbell.

5.1. Sequences

You may already be familiar with the following sequences:

$\mathbb{N}$:	$0, 1, 2, 3, 4, 5, \dots$
Even numbers:	$0, 2, 4, 6, 8, 10, \dots$
Primes:	$2, 3, 5, 7, 11, 13, \dots$
Geometric:	$1, 2, 4, 8, 16, 32, \dots$
Squares:	$0, 1, 4, 9, 16, 25, \dots$
Fibonacci:	$0, 1, 1, 2, 3, 5, \dots$

5.1.1. Explicit formula

In general, denote a sequence of numbers by:

$$a_0, a_1, a_2, a_3, a_4, a_5, \ldots$$

That is, $a_n = f(n)$, for some function $f()$ and $n \in \mathbb{N}$. The indexing variable is n.

Thus, one way to express a sequence is to specify the function $f()$, eg.

- Even numbers: $f(n) = 2n$.
- Squares: $f(n) = n^2$.
- Geometric: $f(n) = 2^n$.
- Fibonacci: $f(n) = \frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right]$.

The function $f()$ permits direct computation of the n^{th} term of the sequence. However, this may not be "efficient".

Quiz 1: What is the next number in the sequence?

1, 3, 5, 7, ?

Answer: **217341**

Because:

$$f(n) = \frac{18111}{2}n^4 - 90555n^3 + \frac{633885}{2}n^2 - 452773n + 217331$$

So, $f(1) = 1,$

$f(2) = 3,$

$f(3) = 5,$

$f(4) = 7,$

$f(5) = 217341$

Moral: For an infinite sequence, you cannot guess $f()$ just by examining a finite number of terms.

Quiz 2: What is the next word in this sequence?

I, CAN, DO, ?

5.1.2. Recurrence relation

Another way to express a sequence is to specify how a_n is related to its predecessors $a_{n-1}, a_{n-2}, \ldots$, called the recurrence relation, together with some initial conditions.

Examples:

Sequence	Recurrence relation	Initial conditions
Even numbers:	$a_n = a_{n-1} + 2$	$a_0 = 0$
Geometric:	$a_n = 2a_{n-1}$	$a_0 = 1$
Fibonacci:	$a_n = a_{n-1} + a_{n-2}$	$a_1 = 1, a_0 = 0$

Starting from the initial conditions, we can simply apply the recurrence relation repeatedly to "work forwards" to get any subsequent term we wish.

Example: Given the following recurrence relation and initial conditions, work out the next few terms.

$$c_k = c_{k-1} + kc_{k-2} + 1, \text{ for all integers } k \geq 2$$
$$c_0 = 1, \text{ and } c_1 = 2$$

Thus,

$$c_2 = c_1 + 2c_0 + 1 = 2 + 2(1) + 1 = 5$$
$$c_3 = c_2 + 3c_1 + 1 = 5 + 3(2) + 1 = 12$$
$$c_4 = c_3 + 4c_2 + 1 = 12 + 4(5) + 1 = 33$$

A recurrence relation is easily implemented in code. For example, the previous sequence written as a recursive function in Python:

```
def cTermR(n):
    if n < 2:
        return n+1
    else:
        return cTermR(n-1) + n * cTermR(n-2) + 1
```

However, this runs slowly for large n, eg. $n = 40$.

A better way is to re-write it as a `while` loop to "work forwards" from the initial conditions:

```
def cTermI(n):
    if n == 0:
        return 1

    I = 2   #) Initial conditions
    CAN = 1 #)
    DOIT = 2
    while n > 1:
        (I,CAN) = (I+CAN*DOIT+1,I) #Recurrence relation
        DOIT = DOIT + 1
        n = n - 1

    return I
```

This runs significantly faster for large n.

What sequence does this recurrence generate?

$$a_n = a_{n-1} + 2$$

Answer: It depends on the initial conditions.

- If $a_0 = 0$, we get $0, 2, 4, 6, 8, \ldots$
- If $a_0 = 1$, we get $1, 3, 5, 7, 9, \ldots$

Thus, a recurrence relation alone, without initial conditions, does not define a unique sequence.

5.2. Summation & Product

Summing a sequence will yield another sequence

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n = S_n, \ \forall n \in \mathbb{N}.$$

Example:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} = \triangle_n.$$

This is the sum of the first $n+1$ integers, starting from 0. It generates the sequence called the Triangle numbers.

n	0	1	2	3	4	5	6
$\triangle_n$	0	1	3	6	10	15	21

Multiplying a sequence will also yield another sequence

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \ldots \times a_{n-1} \times a_n = P_n, \forall n \in \mathbb{N}.$$

Example:

$$\prod_{i=1}^{n} i = n!$$

This is the product of the first n integers, starting from 1, which is known as factorials.

n	0	1	2	3	4	5	6
$n!$	1	1	2	6	24	120	720

Note that the formula above gives $0! = \prod_{i=1}^{0} i = 1$ because an empty product defaults to 1.

Sums and Products may be written recursively too:

$$\sum_{i=m}^{n} a_i = \begin{cases} 0, & \text{if } n < m, \\ \left(\sum_{i=m}^{n-1} a_i\right) + a_n, & \text{otherwise.} \end{cases}$$

$$\prod_{i=m}^{n} a_i = \begin{cases} 1, & \text{if } n < m, \\ \left(\prod_{i=m}^{n-1} a_i\right) \cdot a_n, & \text{otherwise.} \end{cases}$$

[1] Note: the cases when $n < m$ are called the empty sum and empty product, respectively. The empty sum is 0 by definition, while the empty product is 1 by definition.

[1]Things highlighted in green are the corrections of typos.

Theorem 5.1.1 (Epp)

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$

2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n}(a_k \cdot b_k).$

Sometimes, it is more convenient to change the indexing variable in a sum. To do this, we (i) change the term; (ii) change both lower and upper limits.

Example

Replace k with $j = k - 1$ in $\sum_{k=1}^{n+1} \frac{k}{n+k}$.

Answer: since $j = k - 1$, so $k = j + 1$.

1. Replace k in the term: $\frac{j+1}{n+j+1}$.
2. Change the lower limit: $k = 1$ becomes $j + 1 = 1$, so $j = 0$.
   Change the upper limit: $k = n + 1$ becomes $j + 1 = n + 1$, so $j = n$

Thus:

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{j=0}^{n} \frac{j+1}{n+j+1} = \sum_{k=0}^{n} \frac{k+1}{n+k+1}.$$

The last step is obtained by changing "dummy" variable j back to k.

5.3. Common sequences

We will study a few common sequences:

- Arithmetic sequence
- Geometric sequence
- Square numbers
- Triangle numbers
- Fibonacci numbers
- Binomial numbers

5.3.1. Arithmetic sequence

An arithmetic sequence is given by the recurrence:

$$\forall n \in \mathbb{N},\ a_n = \begin{cases} a, & \text{if } n = 0, \\ a_{n-1} + d, & \text{otherwise.} \end{cases}$$

where a, d are real constants.
The explicit formula is: $a_n = a + nd$, $\forall n \in \mathbb{N}$.

Example: The positive odd numbers $1, 3, 5, 7, 9, \ldots$ is an arithmetic sequence generated by $a = 1$, and $d = 2$.

The sum of the first n terms is defined as $S_n = \sum_{i=0}^{n-1} a_i$. This sequence has the explicit closed form[2] formula:

(1) $$S_n = \frac{n}{2}\left[2a + (n-1)d\right], \forall n \in \mathbb{N}, \text{ and } a, d \in \mathbb{R}.$$

[2] "Closed form" means "not using recursion, summation, product, or ellipsis"

5.3.2. Geometric sequence

A geometric sequence is given by the recurrence:

$$\forall n \in \mathbb{N},\ a_n = \begin{cases} a, & \text{if } n = 0, \\ ra_{n-1}, & \text{otherwise.} \end{cases}$$

where a, r are real constants.
The explicit formula is: $a_n = ar^n, \forall n \in \mathbb{N}$.

Example: The power sequence $1, 2, 4, 8, 16, \ldots$ is a geometric sequence generated by $a = 1, r = 2$.

The sum of the first n terms is defined as $S_n = \sum_{i=0}^{n-1} a_i$. This sequence has the explicit closed form formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}, \forall n \in \mathbb{N},\ \text{and } a, r \in \mathbb{R}, r \neq 1. \tag{2}$$

For the special case $|r| < 1$, the sum to infinity is $S_\infty = \frac{a}{1-r}$.

5.3.3. Square numbers

The square numbers is the sequence $0, 1, 4, 9, 16, 25, \ldots$.

The explicit formula is $\forall n \in \mathbb{N}, \ \square_n = n^2$.

Interestingly, $\square_n =$ sum of the first n odd numbers.

To see this, note that the positive odd numbers is an arithmetic sequence with $a = 1, d = 2$. Using Equation (1) for the sum of an arithmetic sequence:

$$\begin{aligned}\forall n \in \mathbb{N}, \ S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \\ &= \frac{n}{2}\left[2(1) + (n-1)(2)\right] \\ &= n^2 \\ &= \square_n.\end{aligned}$$

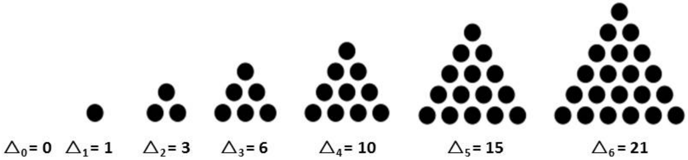
5.3.4. Triangle numbers

The triangle numbers is the sequence $0, 1, 3, 6, 10, 15, 21, \ldots$.

The explicit formula is $\forall n \in \mathbb{N},\ \triangle_n = \frac{n(n+1)}{2}$.

As noted previously, $\triangle_n =$ sum of the first $n+1$ integers. Interestingly, the sum of two consecutive terms is a square number:

$$\begin{aligned}\forall n \in \mathbb{Z}^+,\ \triangle_n + \triangle_{n-1} &= \left[\frac{n(n+1)}{2} + \frac{(n-1)n}{2}\right] \\ &= \frac{1}{2}\left[n^2 + n + n^2 - n\right] \\ &= n^2 = \square_n = (\triangle_n - \triangle_{n-1})^2.\end{aligned}$$



5.3.5. Fibonacci numbers

The Fibonacci sequence is usually defined recursively:

$$\forall n \in \mathbb{N},\ F_0 = 0, \quad F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2}$$

This gives the famous sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$

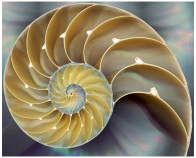
The explicit formula is:

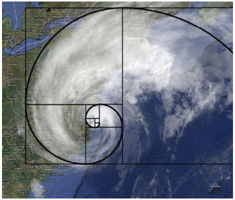
$$\forall n \in \mathbb{N},\ F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

where $\phi = (1+\sqrt{5})/2 \approx 1.6180339887\dots$ is called the golden ratio.

Strangely, even though each term involves $\sqrt{5}$, the resulting number is an integer. The above formula is never used in a computer program, since manipulating $\sqrt{5}$ introduces rounding errors.

Fibonacci numbers and the golden ratio occur frequently in nature.


(a) Nautilus shell


(b) Hurricane


(c) Galaxy


(d) Ratios in the human face

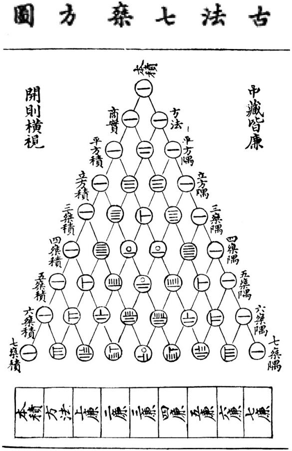
Video: `https://youtu.be/SjSHVDfXHQ4`

5.3.6. Binomial numbers

The binomial numbers is the triangular sequence called Pascal's triangle or Yang Hui's triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's triangle



Yang Hui's triangle

Note that there are two indices in the triangle: the rows are indexed by n from top to bottom, while the columns are indexed by r from left to right.

The explicit formula is:

$$\forall n, r \in \mathbb{N} \text{ such that } r \leq n, \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

This formula crops up when choosing r things from n things. We will futher explore combinatorics in a future lecture.

The recurrence relation and initial conditions are:

$$\forall n, r \in \mathbb{N}, \binom{n}{r} = \begin{cases} 1, & \text{if } r = 0 \text{ and } n \geq 0, \\ \binom{n-1}{r} + \binom{n-1}{r-1}, & \text{if } 0 < r \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

The second line above also says that adding two consecutive terms in one row gives one term in the next row.

Other interesting identities,

$$\binom{n}{r} = \binom{n}{n-r}$$

This says that choosing r things from n is the same as omitting $n-r$ things from n.

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

which says the sum of all possible ways to choose r things from n is 2^n.

$$\text{Now, } 2^n = 2 \times 2^{n-1}$$

$$\text{Thus, } \sum_{r=0}^{n} \binom{n}{r} = 2 \times \sum_{r=0}^{n-1} \binom{n-1}{r}$$

That is, the sum of numbers in one row of the triangle is twice that of the previous row.

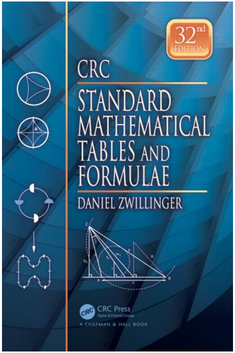
5.4. Solving recurrences

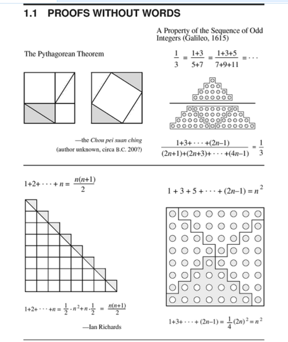
Given a recurrence relation with initial conditions, how do you solve it to get an explicit closed form formula?

1. Look it up.
2. Guess and check (aka iteration).
3. Use formula.

5.4.1. Consult math tables

The *CRC Standard Mathematical Tables and Formulae* lists many useful results and theorems.





https://www.crcpress.com/CRC-Standard-Mathematical-Tables-and-Formulae-32nd-Edition/Zwillinger/9781439835487

5.4.2. Guess and Check

Starting from the initial conditions, use the given recurrence relation to calculate the first few terms. Note the pattern that emerges. Guess the formula, and check it with Induction.

Example: solve the following

$$\forall k \in \mathbb{Z}^+,\ m_k = \begin{cases} 1, & \text{if } k = 1, \\ 2m_{k-1} + 1, & \text{if } k \geq 2. \end{cases}$$

Solution:

$$
\begin{aligned}
m_1 &= 1 \\
m_2 &= 2m_1 + 1 = 2(1) + 1 = 2 + 1 \\
m_3 &= 2m_2 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1 \\
m_4 &= 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1 \\
m_5 &= 2m_4 + 1 = 2(2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1
\end{aligned}
$$

Thus, we may guess the formula: $m_n = 2^{n-1} + 2^{n-2} + \ldots + 2^1 + 2^0$.

This is the sum of a geometric sequence. Using Equation (2), we get:

$$m_n = 2^n - 1, \quad \text{for all integers } n \geq 1.$$

We then check the formula using Induction.

Check using Induction

1. Let $P(n) = (\ m_n = 2^n - 1\ ),\ \forall n \in \mathbb{Z}^+$.
2. $m_1 = 1$, by the initial conditions.
3. Also, $2^1 - 1 = 1$. Thus $P(1)$ is true.
4. For any $k \in \mathbb{Z}^+$:
5. Assume $P(k)$ is true.
6. That is, $m_k = 2^k - 1$.
7. Now, $m_{k+1} = 2m_k + 1$, by the recurrence relation.
8. $= 2(2^k - 1) + 1$, using the Inductive hypothesis.
9. $= 2^{k+1} - 2 + 1 = 2^{k+1} - 1$, so $P(k+1)$ is true.
10. Thus by Mathematical Induction, the statement is true. ■

5.4.3. Using formula

Guess and check works for relatively simple recurrences. But for something like the Fibonacci sequence, things get tedious very quickly. Fortunately there is a better way.

Definition 5.4.1 (Second-order Linear Homogeneous Recurrence Relation with Constant Coefficients)

A second-order linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form:

$$a_k = Aa_{k-1} + Ba_{k-2}, \quad \forall k \in \mathbb{Z}_{\geq k_0}$$

where A, B are real constants, $B \neq 0$ and k_0 is an integer constant.

The Fibonacci sequence is an example of such a recurrence relation.

Theorem 5.8.3 (Epp) Distinct-Roots Theorem

Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for real constants A, B, with $B \neq 0$, and $k \in \mathbb{Z}_{\geq 2}$. If the characteristic equation

$$t^2 - At - B = 0$$

has two distinct roots r and s, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Ds^n, \quad \forall n \in \mathbb{N}$$

where C, D are real numbers determined by the initial conditions a_0, a_1.

Proof: see page 322 of Epp.

Example: solve the Fibonacci sequence

$$F_k = F_{k-1} + F_{k-2}, \quad \text{for all integers } k \geq 2$$

with initial conditions $F_0 = 0, F_1 = 1$.

Clearly, the Fibonacci recurrence is a second-order linear homogeneous recurrence relation, with $A = B = 1$. Its characteristic equation is:

$$t^2 - t - 1 = 0$$

Using the standard quadratic formula, we get:

$$t = \frac{1 \pm \sqrt{1 - 4(-1)}}{2}$$

which yields distinct roots:

$$\phi = (1 + \sqrt{5})/2, \quad \psi = (1 - \sqrt{5})/2$$

Thus, by Theorem 5.8.3 (Epp), the explicit formula is:

$$F_n = C\phi^n + D\psi^n, \quad \forall n \in \mathbb{N}$$

We now need to solve for C and D using the initial conditions.

$$F_0 = 0 = C\phi^0 + D\psi^0 = C + D$$
$$F_1 = 1 = C\phi^1 + D\psi^1 = C\phi + D\psi$$

These are two linear equations in two unknowns. Solving them gives:

$$C = 1/\sqrt{5}, \quad D = -1/\sqrt{5}$$

Note that $\psi = -1/\phi$. We can thus write:

$$\forall n \in \mathbb{N},\ F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

Theorem 5.8.5 (Epp) Single-Roots Theorem

Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for real constants A, B, with $B \neq 0$, and $k \in \mathbb{Z}_{\geq 2}$. If the characteristic equation

$$t^2 - At - B = 0$$

has a single real root r, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Dnr^n, \quad \forall n \in \mathbb{N}$$

where C, D are real numbers determined by the value a_0, and any other known value of the sequence.

Proof omitted.

5.5. Application

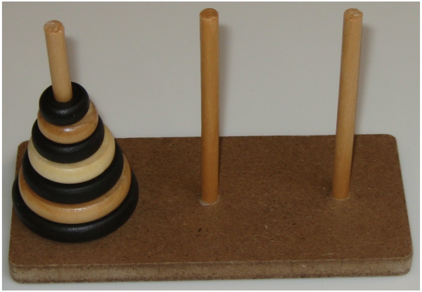
Tower of Hanoi

Éduoard Lucas invented this puzzle in 1883. The goal is to move a set of n disks of different sizes from one pole to another, subject to:

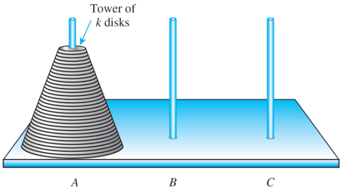
(i) Only one disk can be moved at a time.

(ii) At all times, no disk can rest on a smaller disk.

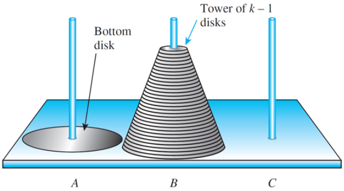
What is the minimum number of moves needed?



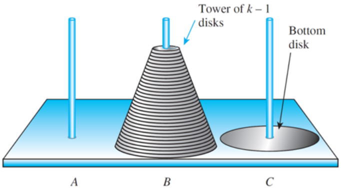
The key to solving this puzzle is to think recursively.



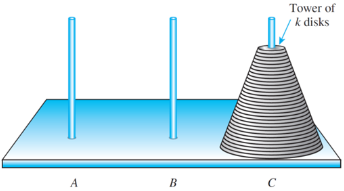
(a) Initial position.



(b) After moving $k-1$ disks from A to B.



(c) After moving bottom disk from A to C.



(d) After moving $k-1$ disks from B to C.

1. To move the whole stack of k disks from A to C, at some point you need to move the bottom (largest) disk from A to C.
2. Since the bottom disk cannot rest on something smaller, C must be empty when you move the bottom disk over. Also, there cannot be anything on top of the bottom disk when you want to move it.
3. Thus, the top $k-1$ disks must be on B.
4. Suppose an efficient genie helped you move the top $k-1$ disks from A to B, using m_{k-1} moves.

...

5. You can now move the bottom disk from A to C, using 1 move.
6. You then ask the genie to move the $k-1$ disks from B to C, using another m_{k-1} moves.
7. Thus, total number of moves is $m_k = 2m_{k-1} + 1$.
8. This is the most efficient way, because if the bottom disk did not move directly from A to C, but say, it went to B first, then this would incur even more moves.

Line 7 is thus the recurrence relation, and the initial condition is obviously $m_1 = 1$, because a tower of 1 disk requires just 1 move.

From Section 5.4.2, the solution to this recurrence relation is: $m_n = 2^n - 1$, for $n \in \mathbb{Z}^+$.

5.6. Summary

- Sequences are common in Computer Science. They can be expressed either by an explicit closed-form formula, or by recurrence relations plus some initial conditions.
- In either way, code can be written to generate the sequence.
- Common sequences include Arithmetic and Geometric sequences, Square and Triangle numbers, Fibonacci, and Binomial numbers.
- Sums and products of a sequence generate another sequence.
- Solving a recurrence relation may be done in several ways: looking it up, guessing and checking, using a formula.
- Recurrence relations are important for analyzing the behavior of algorithms and data structures.