REVIEW FOR FINAL EXAMINATIONS

1 Euclid's algorithm

Exercise 1.1. Using Euclide 's algorithm to calculate the following:

a) gcd(78, 42)

b) gcd(324, 240)

c) gcd(662,414)

d) gcd(252,34)

e) gcd(3122,4204)

f) gcd(1234,212)

Exercise 1.2. Using Euclide 's algorithm to calculate the following:

a) lcm(35,99)

b) lcm(42, 306)

c) lcm(78,270)

d) lcm (212, 36)

e) lcm(2341, 3122)

f) lcm(2010,2371)

Exercise 1.3. a) Fill a large trough in the field with exactly 1 litre of river water. Only two cans are available to scoop water from the river: one is exactly 13 litres when full, the other 11.

b) Fill a large trough in the field with exactly 1 litre of river water. Only two cans are available to scoop water from the river: one is exactly 17 litres when full, the other 11.

c) Fill a large trough in the field with exactly 1 litre of river water. Only two cans are available to scoop water from the river: one is exactly 17 litres when full, the other 13.

d) Fill a large trough in the field with exactly 1 litre of river water. Only two cans are available to scoop water from the river: one is exactly 19 litres when full, the other 13.

2 Modulo arithmetic

Exercise 2.1. Solve the following equation for integers x; y. List 3 pairs of integers x, y.

a) 3x + 7y = 69

b) 2x+5y=100

c) 3x+5y=90

d) 4x+3y=48

e) 3x-2y=21

f) 5x+7y=25

g) 2x-9y=20

3 Relations

Definition 3.1. For sets A, B, any subset of $A \times B$ is called a relation from A to B. Any subset of $A \times A$ is called a relation on A.

Example 3.1. *Let* $A = \{0, 1, 2\}$ *and* $B = \{a, b\}$*. Then*

$$\mathcal{R} = \{(0, a), (0, b), (1, a), (2, b)\}$$

is a relation from A *to* B*.*

Definition 3.2. Let $\mathcal{R}$ be a relation on A. We say that

i) $\mathcal{R}$ is *reflexive* $\Leftrightarrow \forall x \in A, x\mathcal{R}x$.

ii) $\mathcal{R}$ is *symmetric* $\Leftrightarrow \forall x, y \in A, x\mathcal{R}y \rightarrow y\mathcal{R}x$.

iii) $\mathcal{R}$ is *antisymmetric* $\Leftrightarrow \forall x, y \in A, x\mathcal{R}y \wedge y\mathcal{R}x \rightarrow x = y$.

iv) $\mathcal{R}$ is *transitive* $\Leftrightarrow \forall x, y, z \in A, x\mathcal{R}y \wedge y\mathcal{R}z \rightarrow x\mathcal{R}z$.

Definition 3.3. Let $\mathcal{R}$ be a relation on A. We say that $\mathcal{R}$ is an equivalence relation on A if $\mathcal{R}$ is reflexive, symmetric and transitive

Definition 3.4. Let $\mathcal{R}$ be an equivalence relation on A and $x \in A$. The set of all elements in A that is related to x is called the equivalence class of x, denoted by $\bar{x}$ or [x]. Therefore

$$\bar{x} = \{a \in A | a\mathcal{R}x\}$$

Definition 3.5. A relation $\mathcal{R}$ on A is called a partial ordering relation, or partial order if $\mathcal{R}$ is reflexive, antisymmetric, and transitive. $(A, \mathcal{R})$ is called a partially ordered set.

Let $\mathcal{R}$ is a partial order on A we denote $a \preceq b$ but $a \neq b$.

Example 3.2. *Let* R *be a binary relation defined on 2 integers as follow:*

$$\forall x, y \in \mathbb{Z}, x\mathcal{R}y \Leftrightarrow 2 \mid (x + y)$$

Is R *reflexive, symetric, anti-symetric, transitive? Prove your answer. Is it an equivalence relation? If it is, describe its equivalence class. Is it a partial order?*

Solution.

(i) $\forall x \in \mathbb{Z}$, since $x + x = 2x$ is even, we have $x\mathcal{R}x$. Hence $\mathcal{R}$ is reflexive.

(ii) $\forall x, y \in \mathbb{Z}$, if $x\mathcal{R}y$ then $x + y$ is even, hence $y + x$ is even. Therefore $y\mathcal{R}x$, this means $\mathcal{R}$ is symmetric.

(iii) We have $1\mathcal{R}3$ and $3\mathcal{R}1$, but $1 \neq 3$. Hence $\mathcal{R}$ is not antisymmetric.

(iv) $\forall x, y, z \in \mathbb{Z}$, if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x+y$ and $y+z$ are even. Since

$$x + z = (x + y) + (y + z) - 2y$$

we have $x+z$ is even, hence $x\mathcal{R}z$. Therefore $\mathcal{R}$ is transitive.

So $\mathcal{R}$ is reflexive, symmetric, transitive but not antisymmetric. Hence $\mathcal{R}$ is an equivalence relation but it is not a partial order.

The set of equivalence classes is $\{[0], [1]\}$ where $[0] = \{n \in \mathbb{Z} : 2|n\}$ and $[1] = \{n \in \mathbb{Z} : \text{n is not divisible by } 2\}$ ∎

Exercise 3.1. Let R be a binary relation between 2 real numbers such that "its product is not negative" defined as follow:

$$\forall x, y \in \mathbb{R} \leftrightarrow x * y \geq 0$$

Is R reflexive, symetric, anti-symetric, transitive? Prove your answer. Is it an equivalence relation? If it is, describe its equivalence class. Is it a partial order?

Exercise 3.2. Let $\mathcal{R}$ be a relation on the set $A = \{-19, -17, -8, -6, -5, -3, 1, 2, 3, 5\}$ defined by:

$$\forall x, y \in A : x\mathcal{R}y \Leftrightarrow x - 19y \text{ is even.}$$

a) Prove that $\mathcal{R}$ is an equivalence relation on A.
b) Find the equivalence class [1].

SOLUTION.

a) (i) $\forall x \in A$, since $x - 19x = -18x$ is even we have $x\mathcal{R}x$. Hence $\mathcal{R}$ is reflexive.

(ii) $\forall x, y \in A$, if $x\mathcal{R}y$ then $x - 19y$ is even, hence $y - 19x = 20(y - x) + (x - 19y)$ is even. Therefore $y\mathcal{R}x$, this means $\mathcal{R}$ is symmetric.

(iii) $\forall x, y, z \in A$, if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x - 19y$ and $y - 19z$ are even. Since

$$x - 19z = (x - 19y) + (y - 19z) + 18y$$

we have $x - 19z$ is even, hence $x\mathcal{R}z$. Therefore $\mathcal{R}$ is transitive.

So $\mathcal{R}$ is reflexive, symmetric, transitive. Hence $\mathcal{R}$ is an equivalence relation.

b) $[1] = \{x \in A : x\mathcal{R}1\} = \{x \in A : x - 19 \text{ is even}\} = \{-19, -17, -5, -3, 1, 3, 5\}$

∎

Exercise 3.3. Let $\mathcal{R}$ be a relation on the set $A = \{-13, -11, -7, -3, 0, 1, 2, 3, 5, 6, 8\}$ defined by:

$$\forall x, y \in A : x\mathcal{R}y \Leftrightarrow x - 23y \text{ is even.}$$

a) Prove that $\mathcal{R}$ is an equivalence relation on A.
b) Find the equivalence class [3].

Exercise 3.4. Let $\mathcal{R}$ be a relation on the set $A = \{-9, -8, -7, -5, -4, -3, 0, 1, 2, 4, 5, 11\}$ defined by:

$$\forall x, y \in A : x\mathcal{R}y \Leftrightarrow x - 17y \text{ is even.}$$

a) Prove that $\mathcal{R}$ is an equivalence relation on A.
b) Find the equivalence class [5].

4 Set

Exercise 4.1. Given $A = \{1,5,8,9\}; B = \{2,3,5,7,8,9\}$. Find the union, intersect, non-symmetric difference, and symmetric difference of A and B.

Solution.

$$A \bigcup B = \{1,2,3,5,7,8,9\}$$

$$A \bigcap B = \{5,8,9\}$$

$$A \backslash B = \{1\}$$

$$B \backslash A = \{2,3,7\}$$

$$A \Delta B = \{1,2,3,7\}$$

■

Exercise 4.2. Given $A = \{1,2,3,4,5,6,78,9\}; B = \{1,3,5,7,9,11\}$. Find the union, intersect, non-symmetric difference, and symmetric difference of A and B.

Exercise 4.3. Given $A = \{2,4,6,...,30\}; B = \{3,6,9,...,30\}$. Find the union, intersect, non-symmetric difference, and symmetric difference of A and B

5 Recurrence Relation

Exercise 5.1. Find the explicit formula of the following recurrence relation:

$$a_k = 3a_{k-1} - 2a_{k-2}$$

Given initial conditions $a_0 = 0, a_1 = 1$

Solution. Characteristic equation: $X^2 = 3X - 2 \Leftrightarrow X^2 - 3X + 2 = 0$. Hence $X = 1$ or $X = 2$. Therefore, $a_n = a.1^n + b.2^n$. We have $0 = a_0 = a + b$ and $1 = a_1 = a + 2b$. Hence $a = -1$ and $b = 1$. It follows that $a_n = 2^n - 1$ ■

Exercise 5.2. Find the explicit formula of the following recurrence relation:

$$a_k = 5a_{k-1} - 6a_{k-2}$$

Given initial conditions $a_0 = 4, a_1 = 2$

Exercise 5.3. Find the explicit formula of the following recurrence relation:

$$a_k = 7a_{k-1} - 12a_{k-2}$$

Given initial conditions $a_0 = 5, a_1 = 1$

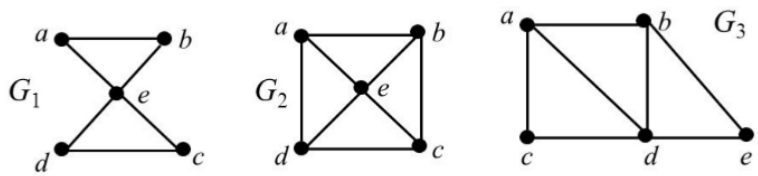
Exercise 5.4. Find the explicit formula of the following recurrence relation:

$$a_k = 11a_{k-1} - 30a_{k-2}$$

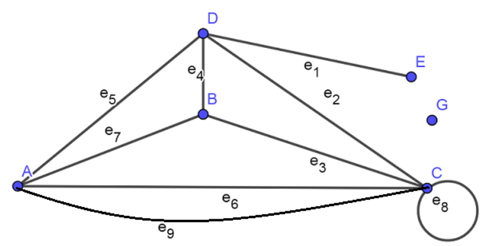
Given initial conditions $a_0 = 7, a_1 = 5$

6 Euler's circuit/trail

Exercise 6.1. Does each of the following graphs have an Euler 's circuit/trail?

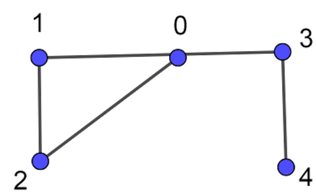


Exercise 6.2. Let G be the graph



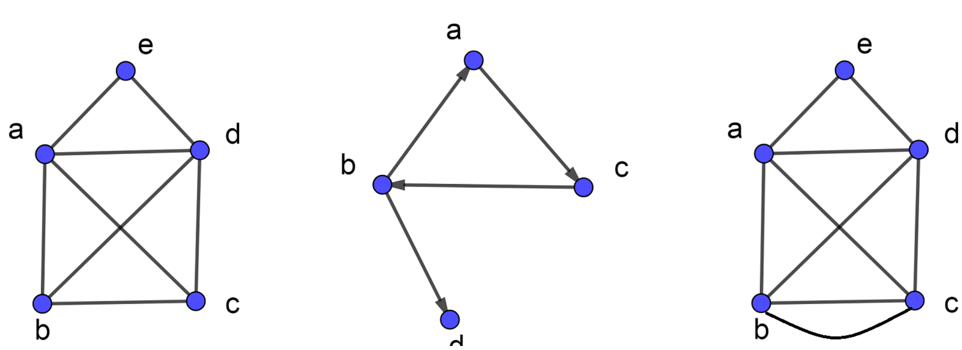
a) Determine the degree of each vertex of G.
b) Does G have an Euler 's trail? Why? Find such a trail if one exists.

Exercise 6.3. Does the following graph have an Euler 's circuit?

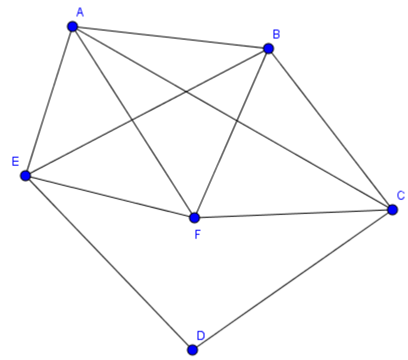


Solution. We have $deg(0) = 3, deg(1) = deg(2) = deg(3)$ and $deg(4) = 1$. Hence the graph G has exactly two vertices with odd degrees. It follows that G has an Eulerian trail.
An Eulerian trail is 4 3 0 1 2 0. ∎

Exercise 6.4. Does each of the following graphs have an Euler 's circuit/trail?

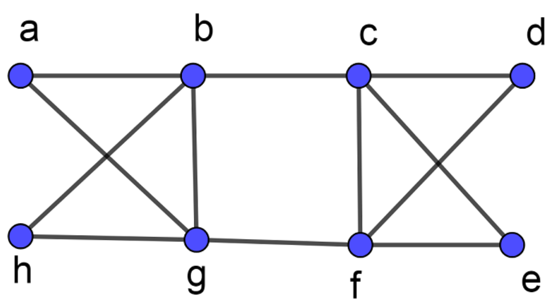


Exercise 6.5. Is the following graph an Eulerian graph? If so, determine an Euler 's circuit.



SOLUTION. We have $deg(A) = deg(B) = deg(C) = deg(E) = deg(F) = 4$ and $deg(D) = 2$. Hence every vertex of G has even degree. Therefore G is an Eulerian graph.
An Eulerian circuit is A B C D E F A E B F C A. ■

Exercise 6.6. Is the following graph an Euler 's graph? If so, determine an Euler 's circuit.

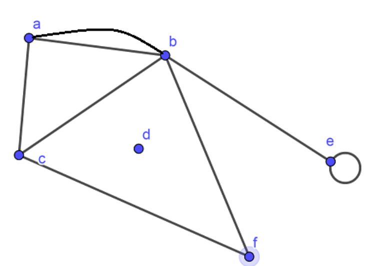


7 Adjacency matrix

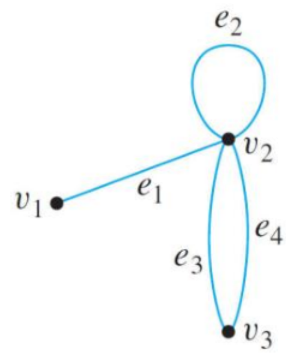
Exercise 7.1. Find directed unweighted graph that have the following adjacency matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

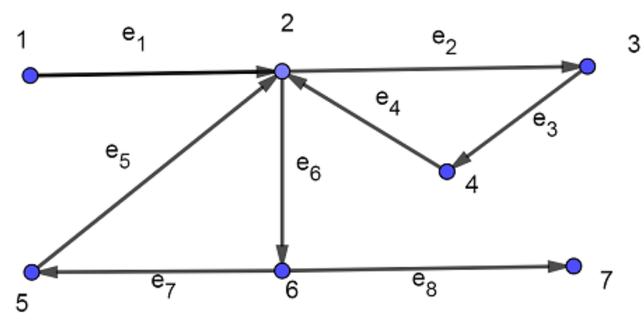
Exercise 7.2. Determine the Adjacency matrix of the following graph



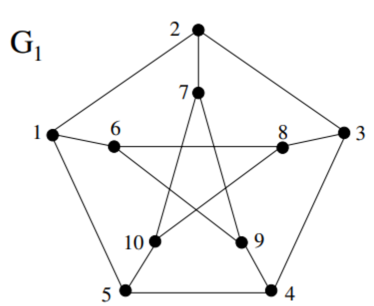
Exercise 7.3. Consider the following graph G. How many distinct walks of length 5 connect v_2 and v_3?

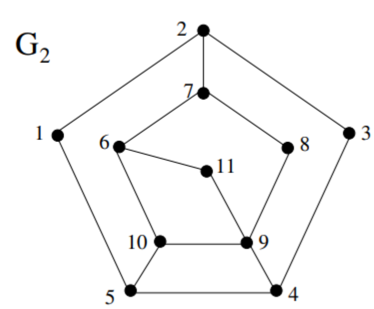


Exercise 7.4. Find a walk with length 4 from 1 to 2, and a walk of length 5 from 1 to 7.



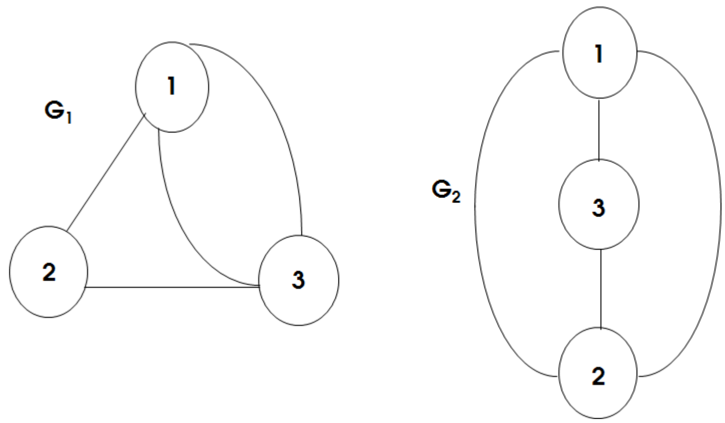
Exercise 7.5. In each of the following graphs, find paths of length 9 and 11, and cycles of length 5, 6, 8 and 9, if possible.





8 Isomorphisms of Graphs

Exercise 8.1. Which of these following pairs of graphs are isomophic? Prove your answer.

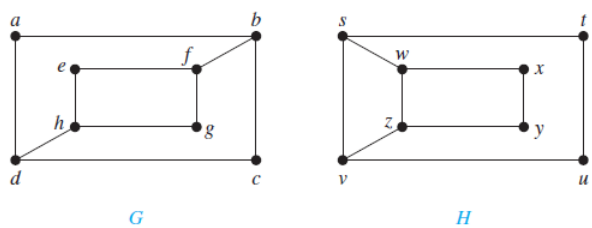


SOLUTION. G_1 and G_2 are isomorphic since there is the graph isomorphism: $\varphi : G_1 \to G_2$ as follows:

$$\varphi(1) = 1, \varphi(2) = 3, \varphi(3) = 2$$

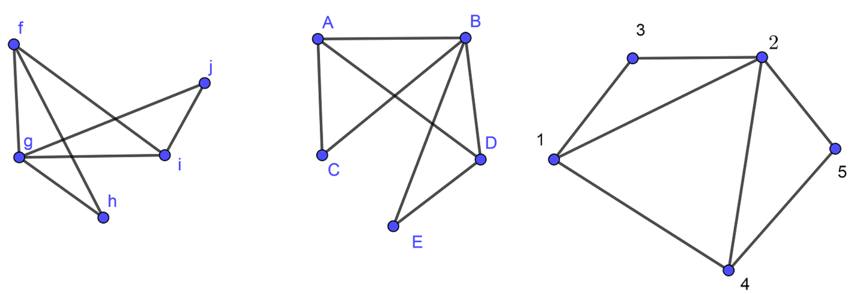
■

Exercise 8.2. Which of these following pairs of graphs are isomophic? Prove your answer.



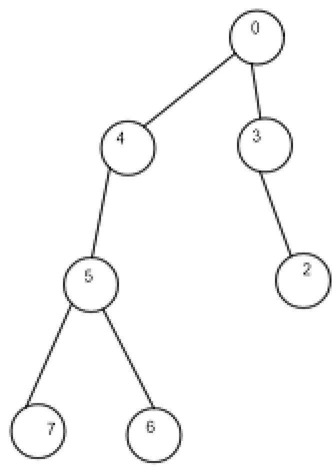
SOLUTION. On the one's hand, G has four vertices with degree 3 and each of them is adjacent to two vertices with degree 2 and another vertex with degree 3. On the other hand, every vertex with degree 3 of H is adjacent to two vertices with degree 3 and another vertex with degree 2. Hence G and H are not isomorphic. ■

Exercise 8.3. Which of these following pairs of graphs are isomophic? Prove your answer.



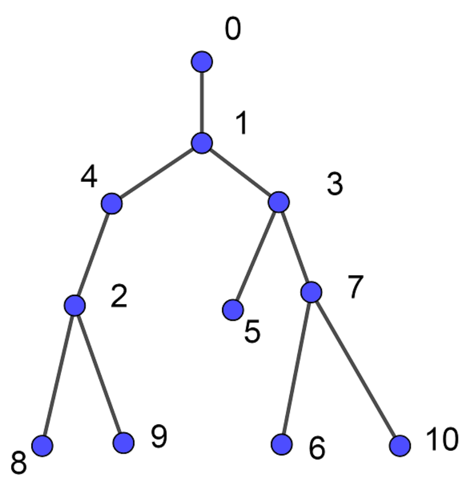
9 Trees

Example 9.1. *Tranverse the following trees in Pre-order, In-order and Post-order. What is the tree's height?*

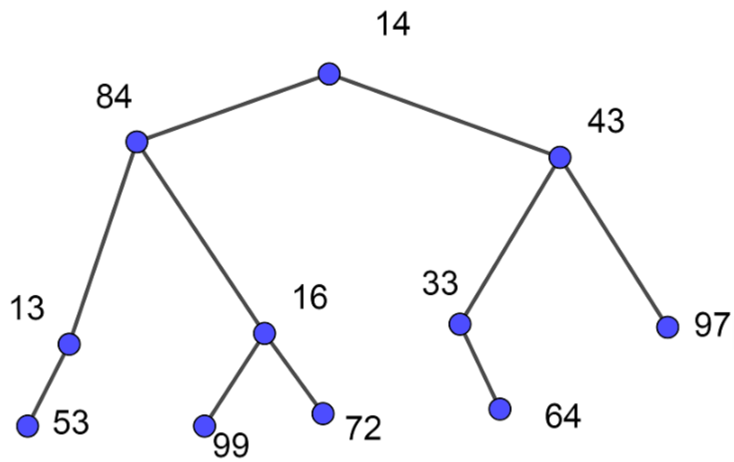


Solution. Preorder tranversal: 0 4 5 7 6 3 2
Inorder tranversal: 7 5 6 4 0 3 2
Postorder tranversal: 7 6 5 4 2 3 0
The tree's height is 3. ■

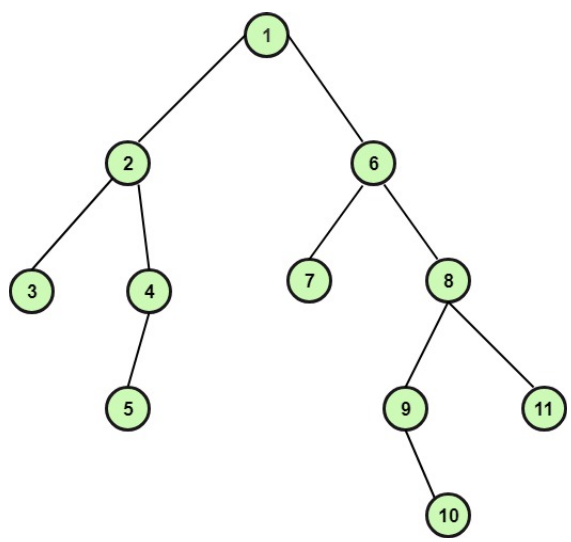
Exercise 9.1. Tranverse the following trees in Pre-order, In-order and Post-order. What is the tree's height?



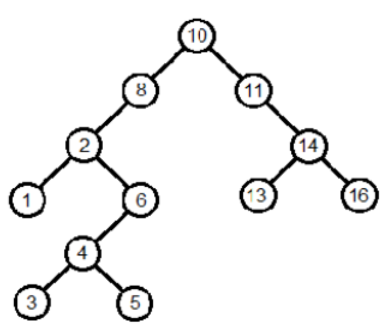
Exercise 9.2. Tranverse the following tree in Pre-order, In-order and Post-order. What is the tree's height?



Exercise 9.3. Tranverse the following tree in Pre-order, In-order and Post-order. What is the tree's height?



Exercise 9.4. Tranverse the following trees in Pre-order, In-order and Post-order. What is the tree's height?



10 Graph's Algorithms

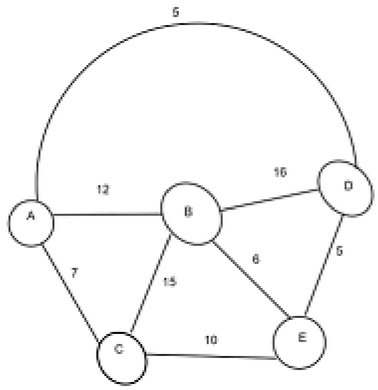
Kruskal's algorithm Given: A connected, weighted graph G.

i. Find an edge of minimum weight and mark it.

ii. Among all of the unmarked edges that do not form a cycle with any of the marked edges, choose an edge of minimum weight and mark it.

iii. If the set of marked edges forms a spanning tree of G, then stop. If not, repeat step ii.

Example 10.1. *Apply Kruskal's algorithm to the following graph to build a minimum spanning tree.*



Solution. A spanning tree is $T = \{AD, DE, BE, AC\}$
The total weight is $5 + 5 + 6 + 7 = 23$ ■

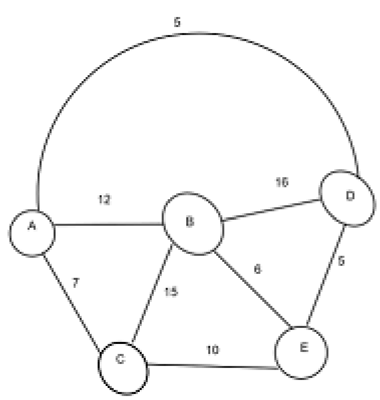
Prim's algorithm Given: A connected, weighted graph G.

i. Choose a vertex v, and mark it.

ii. From among all edges that have one marked end vertex and one unmarked end vertex, choose an edge e of minimum weight. Mark the edge e, and also mark its unmarked end vertex.

iii. If every vertex of G is marked, then the set of marked edges forms a minimum weight spanning tree. If not, repeat step ii.

Example 10.2. *Apply Prim's/Kruskal's algorithm to the following graph to build a minimum spanning tree.*



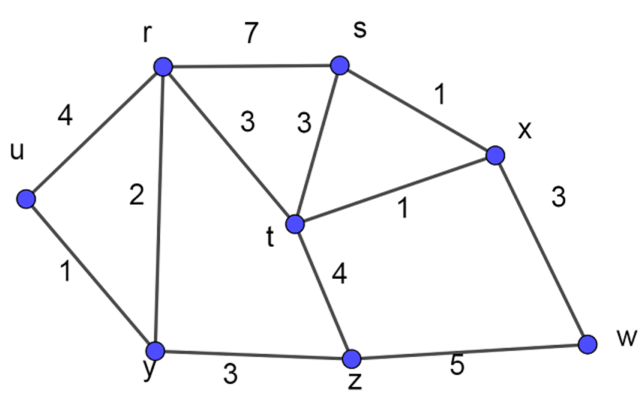
Solution. A spanning tree is $T = \{DE, AD, BE, AC\}$
The total weight is $5 + 5 + 6 + 7 = 23$ ■

Dijkstra Shortest-Path Algorithm
Let G = (V, E) be a weighted graph, with $|V| = n$. To find the shortest distance from a fixed vertex v_0 to all other vertices in G, as well as a shortest directed path for each of these vertices, we apply the following algorithm.

Dijkstra Shortest-Path Algorithm

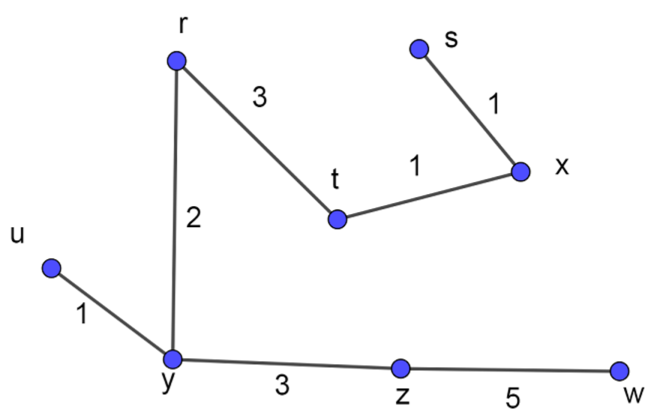
- **Step 1**: Set the counter i = 0 and $S_0 = \{v_0\}$. Label v_0 with $(0, -)$ and each $v \neq v_0$ with $(\infty, -)$.
  - If n = 1, then $V = \{v_0\}$ and the problem is solved.
  - If n $>$ 1, continue to step 2.
- **Step 2**: Let $\bar{S}_i := V \setminus S_i$. For each $v \in \bar{S}_i$ replace, when possible, the label on v by the new label $(L(v), y)$where
$$L(v) = min_{u \in S_i}\{L(v), L(u) + w(u, v)\},$$
and y is a vertex in S_i that produces the minimum $L(v)$. [When a replacement does take place, it is due to the fact that we can go from wo to v and travel a shorter distance by going along a path that includes the edge (y,v)].
- **Step 3**: If every vertex in $\bar{S}_i$ (for some $0 \leq i \leq n-2$) has the label $(\infty, -)$ then the labeled graph contains the information we are seeking.
If not, then there is at least one vertex $v \in \bar{S}_i$ that is not labeled by $(\infty, -)$ and we perform the following tasks:
  1. Select a vertex v_{i+1} where $L(v_{i+1})$ is a minimum (for all such v). There may be more than one such vertex, in which case we are free to choose among the possible candidates. The vertex v_{i+1} is an element of $\bar{S}_i$ that is closest to v_0.
  2. Assign $S_i \cup \{v_{i+1}\}$ to S_{i+1}.
  3. Increase the counter i by 1.
  If i = n -1, the labeled graph contains the information we want. If $i < n-1$, return to step 2.

Example 10.3. *Apply Dijstra's algorithm to the following graph to find shortest part from vertex u to vertex w.*



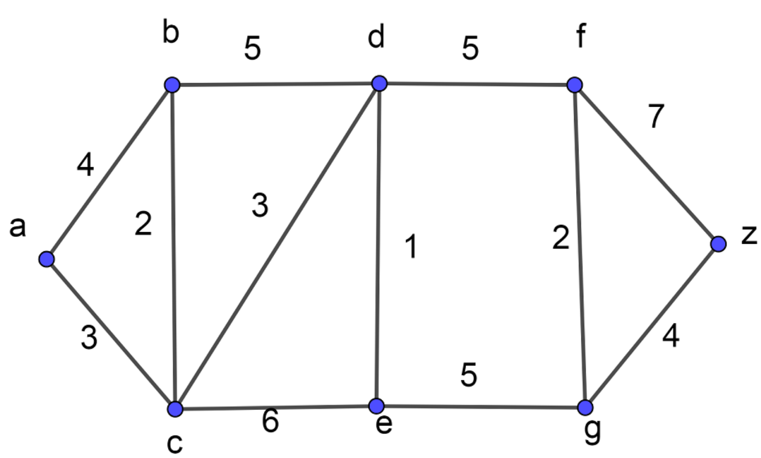
SOLUTION. ■

u	r	s	t	x	y	z	w
0*	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
-	$(4,u_0)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(1,u_0)$	$(\infty,-)$	$(\infty,-)$
-	(3,y)*	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	-	(4,y)	$(\infty,-)$
-	-	(10,r)	(6,r)	$(\infty,-)$	-	(4,y)*	$(\infty,-)$
-	-	(10,r)	(6,r)*	$(\infty,-)$	-	-	(9,z)
-	-	(9,t)	-	(7,t)*	-	-	(9,z)
-	-	(8,x) *	-	-	-	-	(9,z)
-	-	-	-	-	-	-	(9,z)*



Shortest path tree

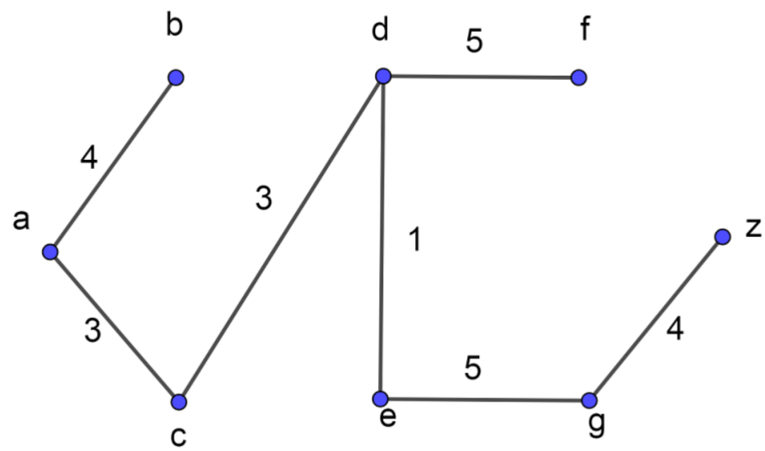
Example 10.4. *Apply Dijstra's algorithm to the following graph to find shortest part from vertex a to vertex z.*



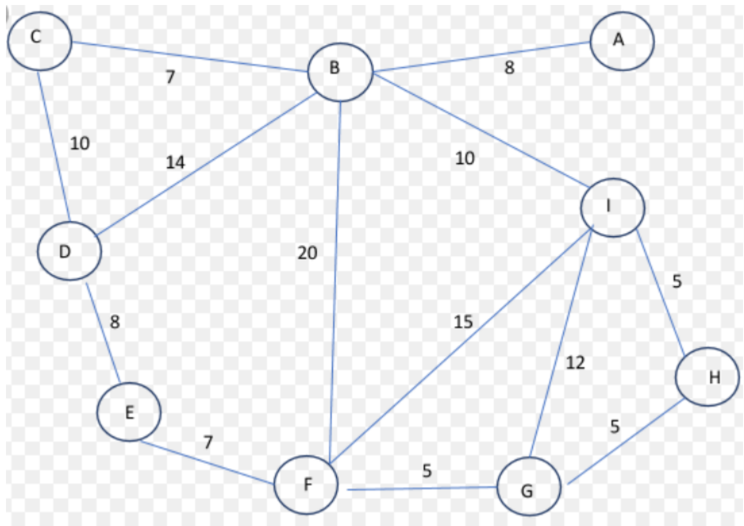
SOLUTION. ■

a	b	c	d	e	f	g	z
0*	(∞,-)	(∞,-)	(∞,-)	(∞,-)	(∞,-)	(∞,-)	(∞,-)
-	(4,a)	**(3,a)***	(∞,-)	(∞,-)	(∞,-)	(∞,-)	(∞,-)
-	**(4,a)***	-	(6,c)	(9,c)	(∞,-)	(∞,-)	(∞,-)
-	-	-	**(6,c)***	(9,c)	(∞,-)	(∞,-)	(∞,-)
-	-	-	-	**(7,d)***	(11,d)	(∞,-)	(∞,-)
-	-	-	-	-	**(11,d)***	(12,e)	(∞,-)
-	-	-	-	-	-	**(12,e)***	(18,f)
-	-	-	-	-	-	-	**(16,g)***

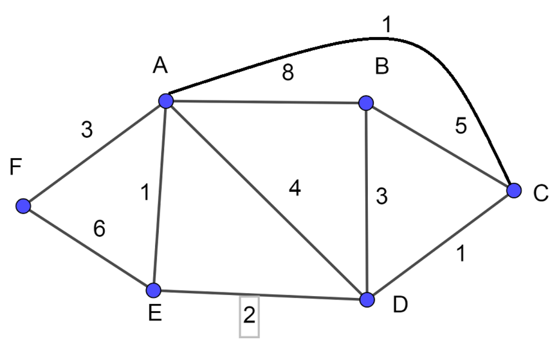
Shortest path tree



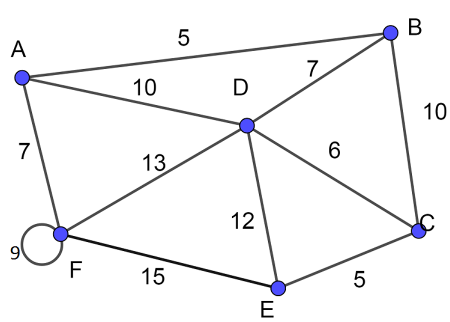
Exercise 10.1. Apply Prim's/Kruskal's algorithm to the following graph to build a minimum spanning tree.



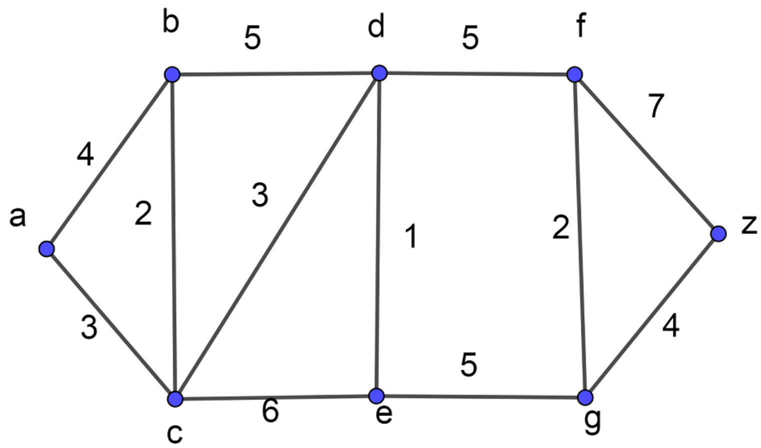
Exercise 10.2. Apply Prim's/Kruskal's algorithm to the following graph to build a minimum spanning tree



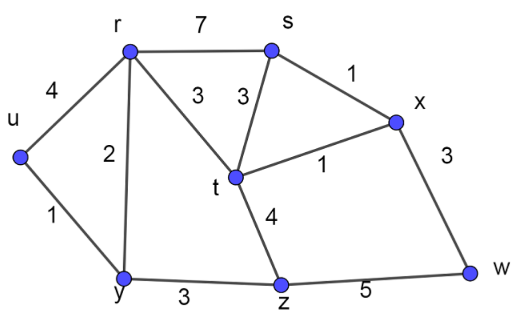
Exercise 10.3. Apply Prim's/Kruskal's algorithm to the following graph to build a minimum spanning tree



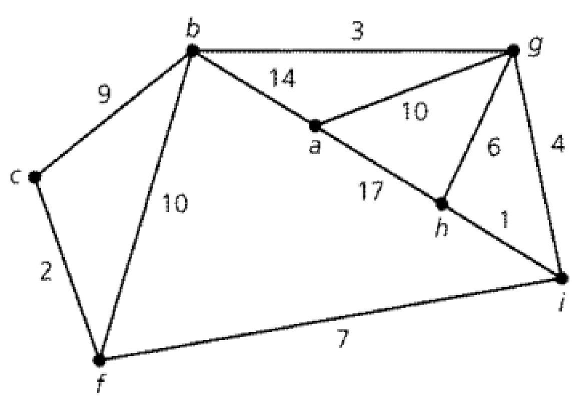
Exercise 10.4. Apply Dijstra's algorithm to the following graph to find shortest part from vertex a to vertex z.



Exercise 10.5. Apply Dijstra's algorithm to the following graph to find shortest part from u to the other vertices.



Exercise 10.6. Apply Dijstra's algorithm to the following graph to find shortest part from vertex c to vertex i.



Exercise 10.7. Apply Dijstra's algorithm to the following graph to find shortest part from vertex 0 to vertex 4.