3. The Logic of Quantified Statements

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3.1 Predicates and Quantified Statements I

Predicates and Quantified Statements I

3.1.1. Predicates and Quantified Statements I

In logic, predicates can be obtained by removing some or all of the nouns from a statement. For instance, let *P* stand for "is a student at Bedford College" and let *Q* stand for "is a student at." Then both *P* and *Q* are *predicate symbols.*

Predicate variables:

$P(x)$ = "*x* is a student at Bedford College"
$Q(x, y)$ = "*x* is a student at *y*"

When concrete values are substituted in place of predicate variables, a statement results.

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For simplicity, we define a predicate to be a predicate symbol together with suitable predicate variables.

In some treatments of logic, such objects are referred to as propositional functions or open sentences.

> ### Definition 3.1.1 (Predicate)
>
> A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
> The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Predicates and Quantified Statements I

When an element in the domain of the variable of a one-variable predicate is substituted for the variable, the resulting statement is either true or false. The set of all such elements that make the predicate true is called the truth set of the predicate.

Definition 3.1.2 (Truth set)

If $P(x)$ is a predicate and x has domain D, the **truth set** is the set of all elements of D that make $P(x)$ true when they are substituted for x.

The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$.

Predicates and Quantified Statements I: Finding the Truth Set of a Predicate

Let $Q(n)$ be the predicate "n is a factor of 8."

Find the truth set of $Q(n)$ if

a.   the domain of n is the set $\mathbf{Z^+}$ of all positive integers

{1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly.

b.   the domain of n is the set $\mathbf{Z}$ of all integers.

3.1.2. The Universal Quantifier: ∀

One sure way to change predicates into statements is to assign specific values to all their variables.

Example: If x represents the number 35, the sentence "x is divisible by 5" is a true statement.

Another way to obtain statements from predicates is to add quantifiers.

The Universal Quantifier: ∀

Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

The symbol ∀ denotes "for all" (or "for any", "for every", "for each") and is called the universal quantifier.

Definition 3.1.3 (Universal Statement)

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$".
- It is defined to be true iff $Q(x)$ is true for every x in D.
- It is defined to be false iff $Q(x)$ is false for at least one x in D.
A value for x for which $Q(x)$ is false is called a **counterexample.**

The Universal Quantifier: Truth and Falsity of Universal Statements

Truth and Falsity of Universal Statements

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is **true**.

Check that "$x^2 \geq x$" is true for each x in D.

$1^2 \geq 1,$ $2^2 \geq 2,$ $3^2 \geq 3,$ $4^2 \geq 4,$ $5^2 \geq 5.$

Hence "$\forall x \in D, x^2 \geq x$" is true.

This method is called the method of exhaustion.

The Universal Quantifier: Truth and Falsity of Universal Statements

Truth and Falsity of Universal Statements

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is **false**.

The Existential Quantifier: ∃

3.1.3. The Existential Quantifier: ∃

Example: "There is a student in CS1231" can be written as

> ∃ a person p such that p is a student in CS1231.

Or, more formally,

> $∃p \in P$ such that p is a student in CS1231.

where P is the set of all people.

The words *such that* are inserted just before the predicate. Some alternative expressions for "there exists" are "there is a", "we can find a", "there is at least one", "for some", and "for at least one".

The Existential Quantifier: ∃

Sentences that are quantified existentially are defined as statements by giving them the truth values specified in the following definition.

> ## Definition 3.1.4 (Existential Statement)
>
> Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".
> - It is defined to be true iff $Q(x)$ is true for at least one x in D.
> - It is defined to be false iff $Q(x)$ is false for all x in D.

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The Existential Quantifier: Truth and Falsity of Existential Statements

Truth and Falsity of Existential Statements

a.   Show that the following statement is true.

$$\exists m \in \mathbf{Z^+} \text{ such that } m^2 = m.$$

Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbf{Z^+}$ such that $m^2 = m$" is true.

b.   Let $E = \{5, 6, 7, 8\}$. Show that the following statement is false.

$$\exists m \in E \text{ such that } m^2 = m.$$

Note that $m^2 = m$ is not true for any integer m from 5 through 8: $5^2 = 25 \neq 5$, $6^2 = 36 \neq 6$, $7^2 = 49 \neq 7$, $8^2 = 64 \neq 8$.

Hence "$\exists m \in E$ such that $m^2 = m$" is false.

Formal Versus Informal Language

3.1.4. Formal Versus Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do <u>not</u> use the symbol $\forall$ or $\exists$.

a. $\forall x \in \mathbf{R}, x^2 \geq 0$.

All real numbers have non-negative squares.
Every/Any real number has a non-negative square.
The square of each real number is non-negative.

b. $\forall x \in \mathbf{R}, x^2 \neq -1$.

All real numbers have squares that are not -1.
No real numbers have squares equal to -1.

c. $\exists m \in \mathbf{Z^+}$ such that $m^2 = m$.

There is a positive integer whose square is itself.
We can find at least one positive integer equal to its own square.
Some positive integer equals its own square.

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Universal Conditional Statements

3.1.5. Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

Universal Conditional Statements Quick Quiz

Rewrite the following statement informally, without quantifiers or variables:

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

If a real number is greater than 2, then its square is greater than 4.

or

Whenever a real number is greater than 2, its square is greater than 4.

or

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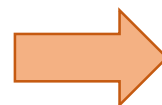Equivalent Forms of Universal and Existential Statements

3.1.6. Equivalent Forms of Universal and Existential Statements

Are these the same?

$\forall$ real numbers x, if x is an integer then x is rational.

$\forall$ integers x, x is rational.

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Equivalent Forms of Universal and Existential Statements

By narrowing *U* to be the domain *D* consisting of all values of the variable *x* that make *P*(*x*) true,

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x) \quad \Longrightarrow \quad \forall x \in D, Q(x)$$

Rewrite the statement "All squares are rectangles" in the two forms:

- $\forall x$, if _____ then _____.

- $\forall$ _____ *x*, _____.

Equivalent Forms of Universal and Existential Statements

Similarly,

$$\exists \, x \text{ such that } P(x) \text{ and } Q(x)$$

$$\exists \, x \in D \text{ such that } Q(x)$$

where D is the set of all x for which $P(x)$ is true.

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Equivalent Forms of Universal and Existential Statements

A **prime number** is an integer whose only positive integer factors are itself and 1. Consider the statement

"There is an integer that is both prime and even."

Let Prime(n) be "n is prime" and Even(n) be "n is even". Use the notation Prime(n) and Even(n) to rewrite this statement in the following two forms:

a. $\exists n$ such that _____ $\wedge$ _____.

b. $\exists$ _____ n such that _____.

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Implicit Quantification

3.1.7. Implicit Quantification

Mathematical writing contains many examples of implicitly quantified statements. Some occur, through the presence of the word *a* or *an*. Others occur in cases where the general context of a sentence supplies part of its meaning.

For example, in an algebra course in which the letter *x* is always used to indicate a real number, the predicate

If $x > 2$ then $x^2 > 4$

is interpreted to mean the same as the statement

$\forall$ real numbers *x*, if $x > 2$ then $x^2 > 4$.

Implicit Quantification: Using $\Rightarrow$ and $\Leftrightarrow$

Mathematicians often use a double arrow to indicate implicit quantification symbolically.

For instance, they might express the above statement as

$$x > 2 \Rightarrow x^2 > 4$$

Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation **$P(x) \Rightarrow Q(x)$** means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \rightarrow Q(x)$.
- The notation **$P(x) \Leftrightarrow Q(x)$** means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

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Implicit Quantification: Using $\Rightarrow$ and $\Leftrightarrow$

Let

      $Q(n)$ be "n is a factor of 8",

      $R(n)$ be "n is a factor of 4",

      $S(n)$ be "$n < 5$ and $n \neq 3$",

and suppose the domain of n is **Z$^+$**, the set of positive integers.

Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationship among $Q(n)$, $R(n)$, and $S(n)$.

Implicit Quantification: Using $\Rightarrow$ and $\Leftrightarrow$

> $Q(n)$ be "n is a factor of 8",
>
> $R(n)$ be "n is a factor of 4",
>
> $S(n)$ be "$n < 5$ and $n \neq 3$"

1. The truth set of $Q(n)$ is {1,2,4,8} and the truth set of $R(n)$ is {1,2,4}.

   Thus it is true that every element in the truth set of $R(n)$ is in the truth set of $Q(n)$, or,

   $$\forall n \text{ in } \mathbf{Z^+}, R(n) \rightarrow Q(n)$$

   so $R(n) \Rightarrow Q(n)$

   Or, equivalently,

   n is a factor of 4 $\Rightarrow$ n is a factor of 8

Implicit Quantification: Using $\Rightarrow$ and $\Leftrightarrow$

$Q(n)$ be "n is a factor of 8",

$R(n)$ be "n is a factor of 4",

$S(n)$ be "$n < 5$ and $n \neq 3$"

2. The truth set of $S(n)$ is {1,2,4} which is the same as the truth set of $R(n)$, or

$$\forall n \text{ in } \mathbf{Z^+}, R(n) \leftrightarrow S(n)$$

So

$$R(n) \Leftrightarrow S(n)$$

Or, equivalently,

$$n \text{ is a factor of } 4 \Leftrightarrow n < 5 \text{ and } n \neq 3.$$

Tarski's World

3.1.8. Tarski's World

Tarski's World is a computer program developed by information scientists Jon Barwise and John Etchemendy to help teach the principles of logic.

It is described in their book *The Language of First-Order Logic*, which is accompanied by a CD-ROM containing the program Tarski's World, named after the great logician Alfred Tarski.

Tarski's World

The program for Tarski's World provides pictures of blocks of various sizes, shapes, and colors, which are located on a grid.

- Shown in Figure 3.1.1 is a picture of an arrangement of objects in a two-dimensional Tarski world.

Figure 3.1.1

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Tarski's World
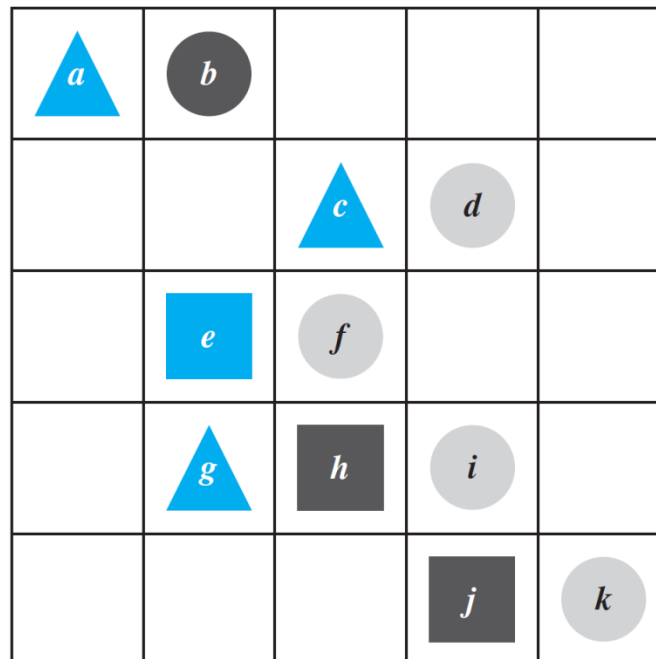
The configuration can be described using logical operators and — for the two-dimensional version — notation such as:

- Triangle(*x*), meaning "*x* is a triangle,"
- Blue(*y*), meaning "*y* is blue," and
- RightOf(*x, y*), meaning "*x* is to the right of *y* (but possibly in a different row)."

Individual objects can be given names such as *a*, *b*, or *c*.

Tarski's World

Determine the truth or falsity of the following statements. The domain for all variables is the set of objects in the Tarski's world shown on the right.

Figure 3.1.1

a. $\forall t$, Triangle(t) → Blue(t).

b. $\forall x$, Blue(x) → Triangle(x).

c. $\exists y$ such that Square(y) ∧ RightOf(d, y).

d. $\exists z$ such that Square(z) ∧ Gray(z).

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Predicates & Quantified Statement I / **II**   Statements with Multiple Quantifiers   Arguments with Quantified Statements

○ ○ ○ ○ ○ ○ ○ ○ ○   ● ○ ○ ○ ○ ○   ○ ○ ○ ○ ○ ○ ○ ○ ○   ○ ○ ○ ○ ○ ○ ○ ○ ○

3.2 Predicates and Quantified Statements II
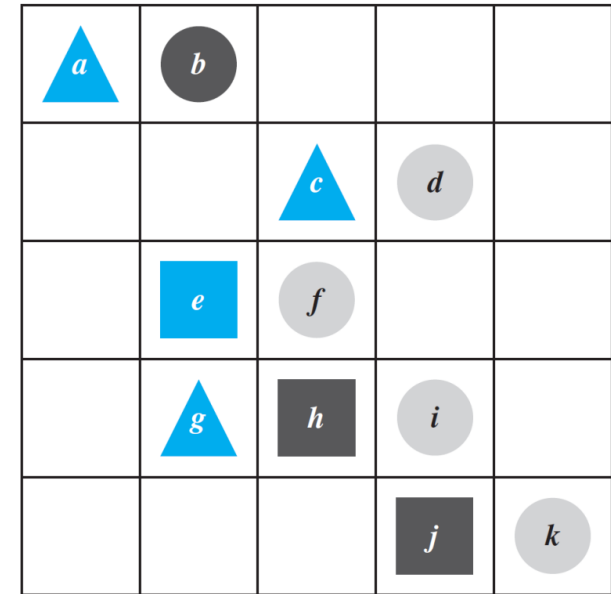
Negations of Quantified Statements: Negation of a Universal Statement

3.2.1. Negations of Quantified Statements

Theorem 3.2.1 Negation of a Universal Statement

The **negation** of a statement of the form

$$\forall x \text{ in } D, P(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim P(x)$$

Symbolically,

$$\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x)$$

That is, the negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Negations of Quantified Statements: Negation of an Existential Statement

Theorem 3.2.2 Negation of an Existential Statement

The **negation** of a statement of the form

$$\exists x \text{ in } D \text{ such that } P(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim P(x)$$

Symbolically,

$$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$$

That is, the negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Negations of Quantified Statements: Quick Quiz

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.

b. $\exists$ a triangle T such that the sum of the angles of T equals 200°.

Predicates & Quantified Statement I / **II**   Statements with Multiple Quantifiers   Arguments with Quantified Statements

○○ ○○○ ○○○   ○●○○○○   ○○ ○○○○○ ○○   ○○ ○○○ ○○○

Negations of Universal Conditional Statements

3.2.2. Negations of Universal Conditional Statements

Of special importance in mathematics.

$${\sim}(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } {\sim}(P(x) \rightarrow Q(x))$$ ... (A)

$${\sim}(P(x) \rightarrow Q(x)) \equiv P(x) \wedge {\sim}Q(x)$$ ... (B)

Substituting (B) into (A):

$${\sim}(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } P(x) \wedge {\sim}Q(x)$$

Negations of Universal Conditional Statements: Quick Quiz

Write a formal negation for statement (a) and an informal negation for statement (b):

a.   ∀ people *p*, if *p* is blond then *p* has blue eyes.

b.   If a computer program has more than 100,000 lines, then it contains a bug.

3.2.3. The Relation among $\forall$, $\exists$, $\wedge$, and $\vee$

- The negation of a *for all* statement is a *there exists* statement, and the negation of a *there exists* statement is a *for all* statement.

- These facts are analogous to De Morgan's laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement.

- This similarity is not accidental. In a sense, universal statements are generalizations of *and* statements, and existential statements are generalizations of *or* statements.

The Relation among $\forall$, $\exists$, $\wedge$, and $\vee$

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, ..., x_n\}$, then

$$\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$$

Similarly, if $Q(x)$ is a predicate and $D = \{x_1, x_2, ..., x_n\}$, then

$$\exists x \in D \text{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

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Vacuous Truth of Universal Statements

3.2.4. Vacuous Truth of Universal Statements

- Suppose a bowl sits on a table and next to the bowl is a pile of five blue and five gray balls, any of which may be placed in the bowl.

- If three blue balls and one gray ball are placed in the bowl, as shown in Figure 3.2.1(a), the statement "All the balls in the bowl are blue" would be false (since one of the balls in the bowl is gray).

Figure 3.2.1(a)

- Now suppose that no balls at all are placed in the bowl, as shown in Figure 3.2.1(b).

- Consider the statement:

  All the balls in the bowl are blue.

Figure 3.2.1(b)

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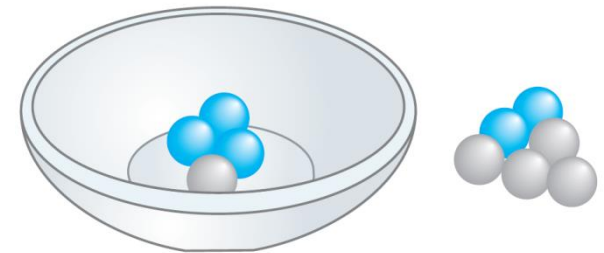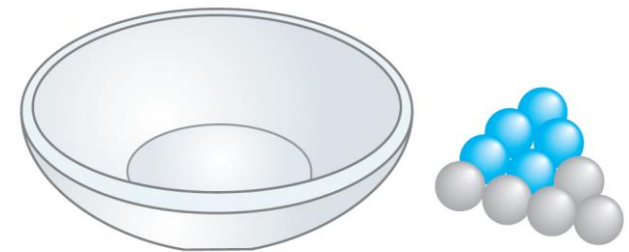
Vacuous Truth of Universal Statements

- Is the statement true or false? The statement is false if, and only if, its negation is true.

- And its negation is

   There exists a ball in the bowl that is not blue.

- But the only way this negation can be true is for there actually to be <u>a non-blue ball in the bowl</u>.

- And there is not! Hence the negation is false, and so the statement is true "by default".

> In general, a statement of the form
> $$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$
> is called vacuously true or true by default
> if, and only if, $P(x)$ is false for every x in D.

Variants of Universal Conditional Statements

3.2.5. Variants of Universal Conditional Statements

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.

2. Its **converse** is: $\forall x \in D$, if $Q(x)$ then $P(x)$.

3. Its **inverse** is: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Variants of Universal Conditional Statements

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

The formal version: $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.

Contrapositive: $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$.

If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse:

Inverse:

Variants of Universal Conditional Statements

Let $P(x)$ and $Q(x)$ be any predicates, let D be the domain of x, and consider the statement:

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

and its contrapositive

$$\forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$$

Any particular x in D that makes "if $P(x)$ then $Q(x)$" true also makes "if $\sim Q(x)$ then $\sim P(x)$" true (by the logical equivalence between $p \rightarrow q$ and $\sim q \rightarrow \sim p$).

It follows that "if $P(x)$ then $Q(x)$" is true for all x in D iff "if $\sim Q(x)$ then $\sim P(x)$" is true for all x in D.

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$$

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Predicates & Quantified Statement I / **II**   Statements with Multiple Quantifiers   Arguments with Quantified Statements

○○ ○○○ ○○○   ○○○○○●○   ○○ ○○○ ○○   ○○ ○○○ ○○○

Variants of Universal Conditional Statements

Consider the statement:

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4$$

True

and its converse

$$\forall x \in \mathbf{R}, \text{ if } x^2 > 4 \text{ then } x > 2$$

False

A universal conditional statement is <u>not</u> logically equivalent to its converse.

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } Q(x) \text{ then } P(x)$$

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Necessary and Sufficient Conditions, Only if

3.2.6. Necessary and Sufficient Conditions, Only if

The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$".

- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if ~$r(x)$ then ~$s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$".

- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if ~$s(x)$ then ~$r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$" .

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Necessary and Sufficient Conditions, Only if

Rewrite the following statements as quantified conditional statements. Do <u>not </u>use the word *necessary* or *sufficient*:

a. Squareness is a sufficient condition for rectangularity.

$\forall x$, if x is a square, then x is a rectangle.

Informal: If a figure is a square, then it is a rectangle.

b. Being at least 35 years old is a necessary condition for being President of the United States.

3.3 Statements with Multiple Quantifiers

Statements with Multiple Quantifiers

Consider the Tarski's world again.

Show that the following statement is true:

For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.

The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color.

There are only 3 triangles *d*, *f*, and *i*.
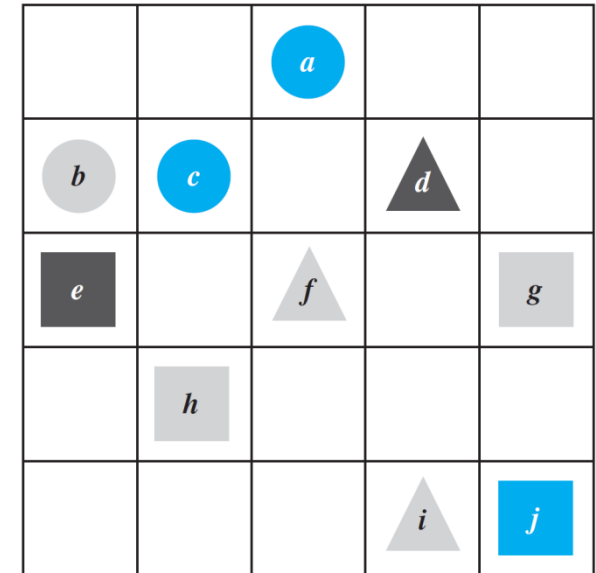
Figure 3.3.1

Given *x* =	choose *y* =	and check that *y* is the same color as *x*.
d	*e*	yes •
f or *i*	*h* or *g*	yes •

Statements with Multiple Quantifiers

3.3.1. Interpreting Multiply-Quantified Statements

If you want to establish the truth of a statement of the form:

$$\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)$$

your challenge is to allow someone else to pick whatever element x in D they wish and then you must find an element y in E that "works" for that particular x.

If you want to establish the truth of a statement of the form:

$$\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)$$

your job is to find one particular x in D that will "work" no matter what y in E anyone might choose to challenge you with.

Interpreting Multiply-Quantified Statements

A college cafeteria line has four stations: salads, main courses, desserts, and beverages.

The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

- Uta: green salad, spaghetti, pie, milk
- Tim: fruit salad, fish, pie, milk, coffee
- Yuen: spaghetti, fish, pie, soda

Interpreting Multiply-Quantified Statements

These choices are illustrated in Figure 3.3.2.
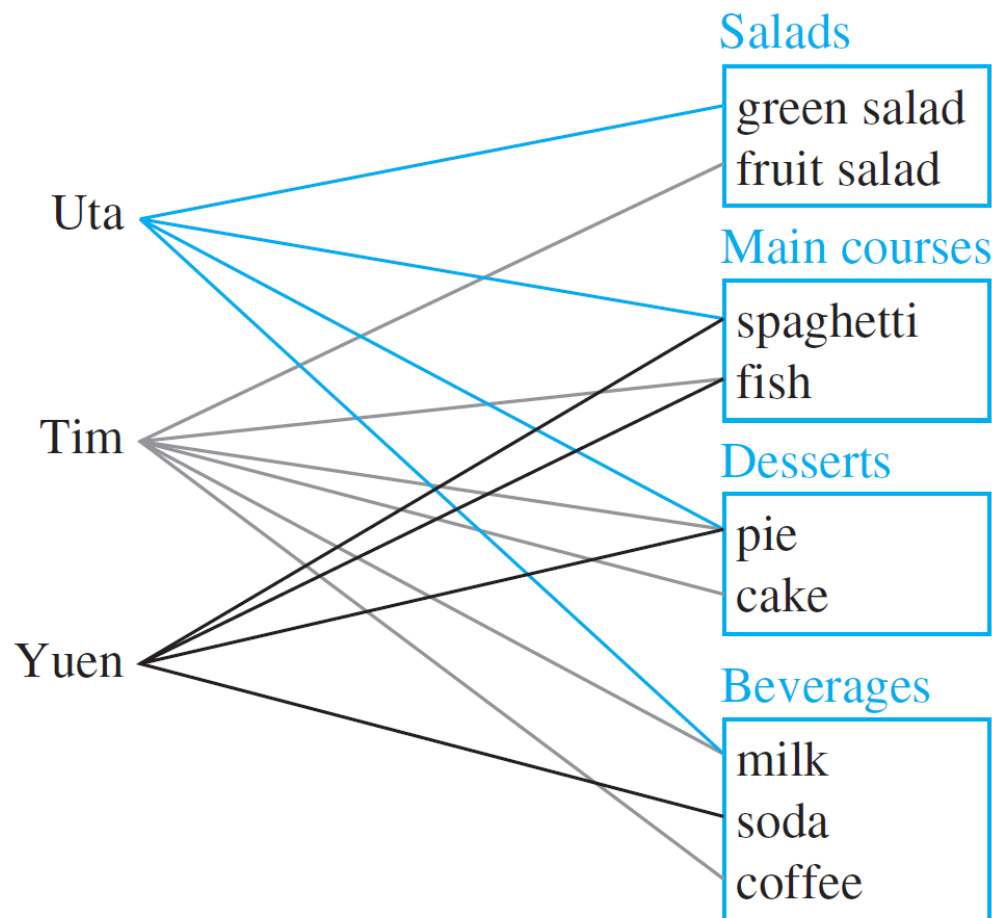
Figure 3.3.2

Interpreting Multiply-Quantified Statements

Write each of following statements informally and find its truth value.

a. $\exists$ an item I such that $\forall$ students S, S chose I.

<mark>There is an item that was chosen by every student. True of false?</mark>

b. $\exists$ a student S such that $\forall$ items I, S chose I.

c. $\exists$ a student S such that $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

d. $\forall$ students S and $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

Translating from Informal to Formal Language

3.3.2. Translating from Informal to Formal Language

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

Example: The **reciprocal** of a real number a is a real number b such that $ab = 1$. The following 2 statements are true. Rewrite them formally using quantifiers and variables:

a.   Every nonzero real number has a reciprocal.

$\forall$ nonzero real numbers u, $\exists$ a real number v such that $uv = 1$.

b.   There is a real number with no reciprocal.

$\exists$ a real numbers c such that $\forall$ real number d, $cd \neq 1$.

Ambiguous Language

3.3.3. Ambiguous Language

You are visiting a computer microchips factory. The factory guide tells you:

There is a person supervising every detail of the production process.

"there is" – existential quantifier; "every" – universal quantifier.

Which of the following best describes its meaning?

- There is one single person who supervises all the details of the production process.

- For any particular production detail, there is a person who supervises the detail, but there might be different supervisors for different details.

Ambiguous Language

Once you interpreted the sentence in one way, it may have been hard for you to see that it could be understood in the other way.

Perhaps you had difficulty even though the two possible meanings were explained.

Although statements written informally may be open to multiple interpretations, we cannot determine their truth or falsity without interpreting them one way or another.

Therefore, we have to use context to try to ascertain their meaning as best we can.

Negations of Multiply-Quantified Statements

3.3.4. Negations of Multiply-Quantified Statements

Recall in 3.2.1:

$\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x)$

$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$

(A) So, to find: $\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$

➔ $\exists x \text{ in } D \text{ such that } \sim(\exists y \text{ in } E \text{ such that } P(x, y))$

➔ $\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$.

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$

(B) Similarly, to find: $\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y))$

➔ $\forall x \text{ in } D, \sim(\forall y \text{ in } E, P(x, y))$

➔ $\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$.

$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$

Negations of Multiply-Quantified Statements

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

Figure 3.3.1

a.   For all squares *x*, there is a circle *y* such that *x* and *y* have the same color.
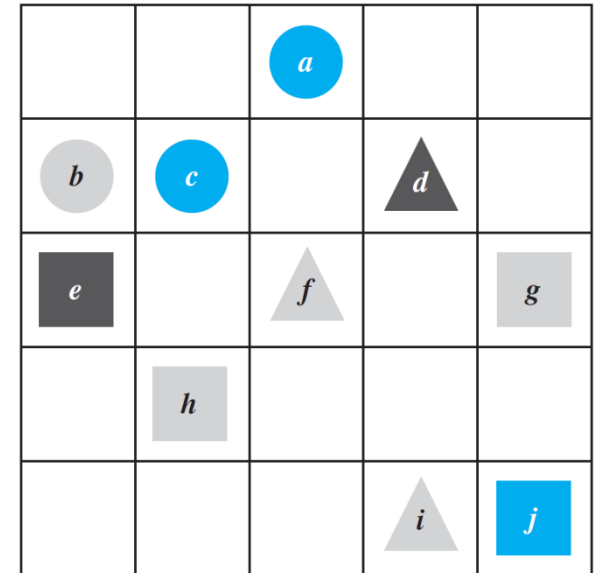
Negation:
∃ a square *x* such that ~(∃ a circle *y* such that *x* and *y* have the same color)

➔ ∃ a square *x* such that ∀ circles *y*, *x* and *y* do not have the same color.

TRUE (Square *e* is black and no circle is black).

Negations of Multiply-Quantified Statements

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

b.   There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.
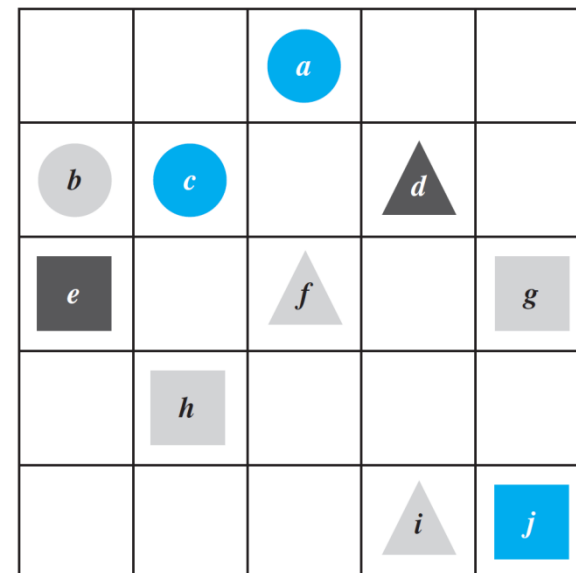
Figure 3.3.1

3.3.5. Order of Quantifiers

∀ people x, ∃ a person y such that x loves y.

∃ a person y such that ∀ people x, x loves y.

Except for the order of the quantifiers,
these statements are identical.

Given any person, it is possible to find
someone whom that person loves.

They are not
logically
equivalent!

There is one amazing individual
who is loved by all people.

Order of Quantifiers

In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.

However, if one quantifier immediately follows another quantifier _of the same type_, then the order of the quantifiers does not affect the meaning.

Order of Quantifiers

Refer to the Tarski's world of Figure 3.3.1. What are the truth values of the following two statements?

a. For every square x, there is a triangle y such that x and y have different colors.

b. There exists a triangle y such that for every square x, x and y have different colors.

Figure 3.3.1
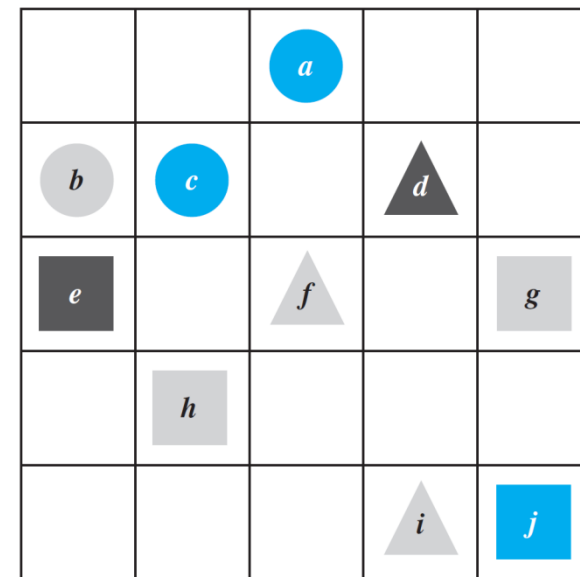
Formal Logical Notation

3.3.6. Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words *such as* in existential statements.

"$\forall x$ in D, $P(x)$" written as $\forall x\, (x \text{ in } D \rightarrow P(x))$

"$\exists x$ in D such that $P(x)$" written as $\exists x\, (x \text{ in } D \wedge P(x))$

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Formal Logical Notation: Formalizing Statements in a Tarski's World

Example:

- Tarski's world.
- Let the common domain *D* of all variables be the set of all the objects in the Tarski's world.

Triangle(*x*): "*x* is a triangle"
Circle(*x*): "*x* is a circle"
Square(*x*): "*x* is a square"

Blue(*x*): "*x* is blue"
Gray(*x*): "*x* is gray"
Black(*x*): "*x* is black"

RightOf(*x,y*): "*x* is to the right of *y*"
Above(*x,y*): "*x* is above *y*"
SameColorAs(*x,y*): "*x* has the same color as *y*"
x = *y*: "*x* is equal to *y*"

Formal Logical Notation: Formalizing Statements in a Tarski's World

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

a.   For all circles x, x is above f.

Statement: $\forall x\ (\text{Circle}(x) \rightarrow \text{Above}(x, f))$

Negation: $\sim(\forall x\ (\text{Circle}(x) \rightarrow \text{Above}(x, f)))$

$\equiv$ $\exists x\ \sim(\text{Circle}(x) \rightarrow \text{Above}(x, f))$ $\equiv$ $\exists x\ (\text{Circle}(x) \wedge \sim\text{Above}(x, f))$

b.   There is a square x such that x is black.

Statement: $\exists x\ (\text{Square}(x) \wedge \text{Black}(x))$

Negation: $\sim(\exists x\ (\text{Square}(x) \wedge \text{Black}(x)))$

$\equiv$ $\forall x\ \sim(\text{Square}(x) \wedge \text{Black}(x))$ $\equiv$ $\forall x\ (\sim\text{Square}(x) \vee \sim\text{Black}(x))$

Formal Logical Notation: Formalizing Statements in a Tarski's World

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

c. For all circles *x*, there is a square *y* such that *x* and *y* have the same color.

Statement: $\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

Negation:

Formal Logical Notation: Formalizing Statements in a Tarski's World

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

d.   There is a square *x* such that for all triangles *y*, *x* is to right of *y*.

Statement:

Negation:

Formal Logical Notation

Formal logical notation is used in branches of computer science such as *artificial intelligence*, *program verification*, and *automata theory and formal languages*.

Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the **language of first-order logic**.

Even though this language is simpler in many respects than the language we use every day, learning it requires the same kind of practice needed to acquire any foreign language.

Prolog

3.3.7. Prolog

The programming language **Prolog** (short for *pro*gramming in *log*ic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence.

A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements.

This frees the programmer from the necessity of having to write separate programs to answer each type of question.
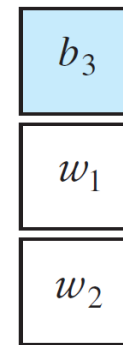
Prolog: A Prolog Program

Consider the following picture, which shows colored blocks stacked on a table.
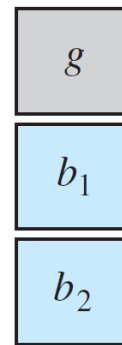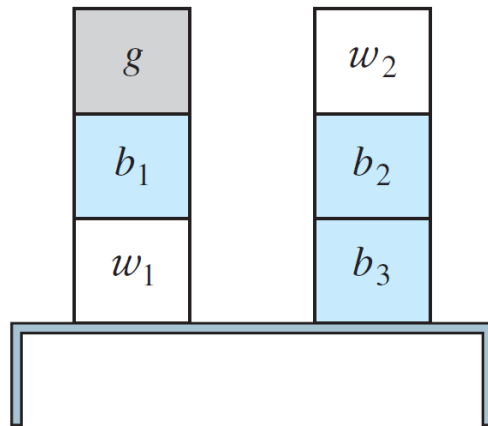
g = gray block b_3 = blue block 3

b_1 = blue block 1 w_1 = white block 1

b_2 = blue block 2 w_2 = white block 2

The following are statements in Prolog that describe this picture and ask two questions about it.

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Prolog: A Prolog Program

isabove(g, b_1)   color(g, gray)   color(b_3, blue)

isabove(b_1, w_1)   color(b_1, blue)   color(w_1, white)

isabove(w_2, b_2)   color(b_2, blue)   color(w_2, white)

isabove(b_2, b_3)   isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

?color(b_1, blue)   ?isabove(X, w_1)
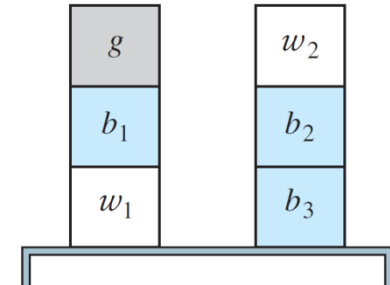
The statements "isabove(g, b_1)" and "color(g, gray)" are to be interpreted as "g is above b_1" and "g is colored gray".

The statement "isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)" is to be interpreted as "For all X, Y, and Z, if X is above Y and Y is above Z, then X is above Z."

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Prolog: A Prolog Program

The program statement

$$?\text{color}(b_1, \text{blue})$$

is a question asking whether block b_1 is colored blue.

Prolog answers this by writing

Yes

The program statement

$$?\text{isabove}(X, w_1)$$

is a question asking for which blocks X the predicate "X is above w_1" is true..

Prolog answers this by giving a list of all such blocks. In this case, the answer is

$$X = b_1, X = g.$$

Infer the solution $X = g$ from the following statements:
- isabove(g, b_1)
- isabove(b_1, w_1)
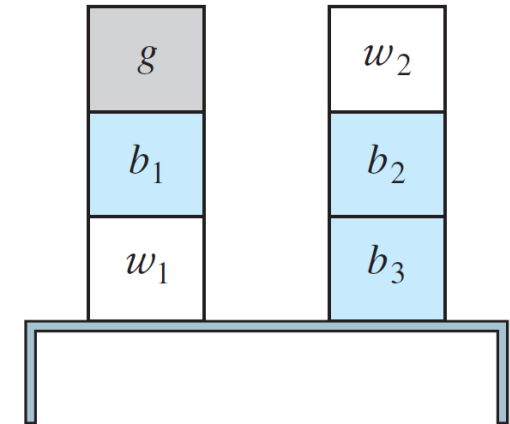- isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

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Prolog: Quick Quiz

Write the answers Prolog would give if the following questions were added to the program above.

a. ?isabove(b_2, w_1)   "No"

b. ?color(w_1, X)

c. ?color(X, blue)

---

isabove(g, b_1)   color(g, gray)   color(b_3, blue)

isabove(b_1, w_1)   color(b_1, blue)   color(w_1, white)

isabove(w_2, b_2)   color(b_2, blue)   color(w_2, white)

isabove(b_2, b_3)   isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

3.4 Arguments with Quantified Statements

3.4.1. Universal Instantiation

The rule of universal instantiation:

> If some property is true of *everything* in the set, then it is true of *any particular* thing in the set.

Universal instantiation is the fundamental tool of deductive reasoning.

Universal Modus Ponens

3.4.2. Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called universal modus ponens.

<div style="border:1px solid blue">

Universal Modus Ponens

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

</div>

Recognizing Universal Modus Ponens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

> If an integer is even, then its square is even.
> k is a particular integer that is even.
> • k^2 is even.

Solution:

Premise: $\forall x$, if x is an even integer then x^2 is even.

Let $E(x)$ be "x is an even integer", let $S(x)$ be "x^2 is even", and let k stand for a particular integer that is even.

$\forall x$, if $E(x)$ then $S(x)$.
$E(x)$, for a particular k.
• $S(k)$.

This argument has the form of universal modus ponens and is therefore valid.

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Predicates & Quantified Statement I / II   Statements with Multiple Quantifiers   **Arguments with Quantified Statements**
∘∘ ∘∘∘ ∘ ∘∘∘ ∘   ∘∘ ∘∘∘ ∘ ∘   ∘∘ ∘∘∘∘ ∘∘   ∘∘ ● ∘∘ ∘ ∘∘∘

Use of Universal Modus Ponens in a Proof

3.4.3. Use of Universal Modus Ponens in a Proof

Proof: The sum of any two even integers is even.

$\forall$ integers x, x is even iff $\exists$ an integer k such that $x = 2k$.

Suppose m and n are particular but arbitrarily chosen even integers, then $m = 2r$ for some integer r[1], and $n = 2s$ for some integer s[2].

Hence

$$m + n = 2r + 2s = 2(r + s) \text{ [3]}$$

Now $(r + s)$ is an integer[4], and so $2(r + s)$ is even[5].

Thus $m + n$ is even.

Use of Universal Modus Ponens in a Proof

How universal modus ponens is used in the proof.

Suppose m and n are particular but arbitrarily chosen even integers, then $m = 2r$ for some integer $r^{(1)}$, and $n = 2s$ for some integer $s^{(2)}$.

(1) If an integer is even, then it equals twice some integer.

 m is a particular even integer.

 • m equals twice some integer r.

(2) Similar to (1).

Use of Universal Modus Ponens in a Proof

How universal modus ponens is used in the proof.

Hence

$$m + n = 2r + 2s = 2(r + s) \;^{(3)}$$

(3) If a quantity is an integer, then it is a real number.

r and s are particular integers.

• r and s are real numbers.

For all a, b, and c, if a, b, and c are real numbers, then $ab + ac = a(b + c)$.

2, r, and s are particular real numbers.

• $2r + 2s = 2(r + s)$.

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Use of Universal Modus Ponens in a Proof

How universal modus ponens is used in the proof.

Now $(r + s)$ is an integer[(4)], and so $2(r + s)$ is even[(5)].

Thus $m + n$ is even.

(4) For all u and v, if u and v are integers, then $(u + v)$ is an integer.

   r and s are two particular integers.

   • $(r + s)$ is an integer.

(5) If a number equals twice some integer, then that number is even.

   $2(r + s)$ equals twice the integer $(r + s)$.

   • $2(r + s)$ is even.

Universal Modus Tollens

3.4.4. Universal Modus Tollens

Another crucially important rule of inference is universal modus tollens. Its validity results from combining universal instantiation with modus tollens.

Universal modus tollens is the heart of proof of contradiction.

Universal Modus Tollens

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
~$Q(a)$ for a particular a.	a does not make $Q(x)$ true.
• ~$P(a)$.	• a does not makes $P(x)$ true.

Recognizing Universal Modus Tollens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

> All human beings are mortal.
> Zeus is not mortal.
> ● Zeus is not human.

Solution:

Premise: $\forall x$, if x is human then x is mortal.

Let $H(x)$ be "x is human", let $M(x)$ be "x is mortal", and let Z stand for Zeus.

$\forall x$, if $H(x)$ then $M(x)$.
~$M(Z)$.
● ~$H(Z)$.

This argument has the form of universal modus tollens and is therefore valid.

Proving Validity of Arguments with Quantified Statements

3.4.5. Proving Validity of Arguments with Quantified Statements

The intuitive definition of validity for arguments with quantified statements is the same as for arguments with compound statements.

An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises.

Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

Using Diagrams to Test for Validity

3.4.6. Using Diagrams to Test for Validity

Consider the statement: All integers are rational numbers.

$\forall$integers *n, n* is a rational number.
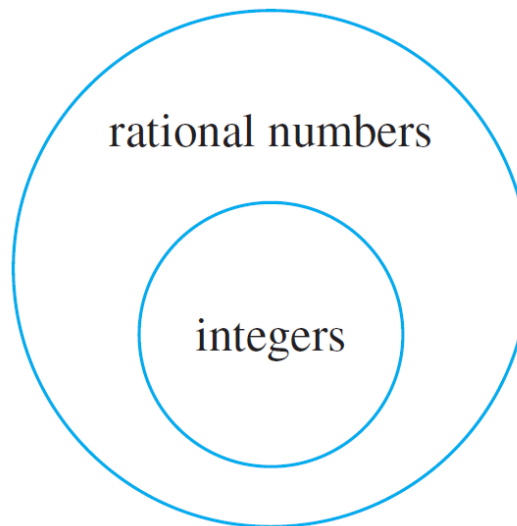
Figure 3.4.1

Using Diagrams to Show Invalidity

Use a diagram to show the invalidity of the following argument:

All human beings are mortal.
Felix is mortal.

- Felix is a human being.
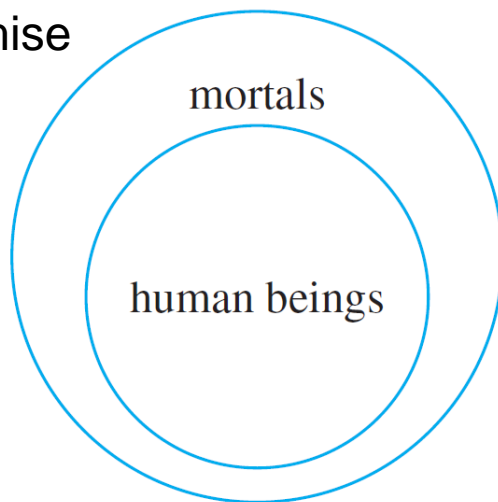
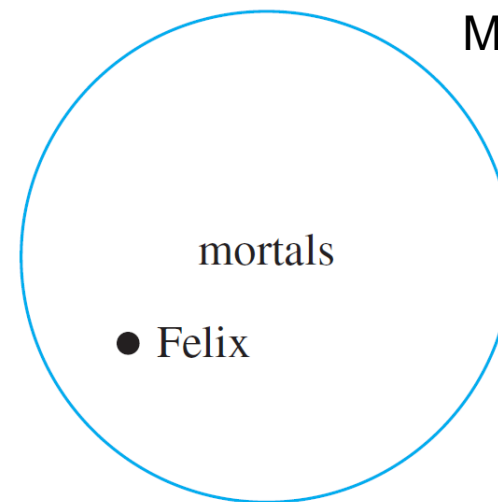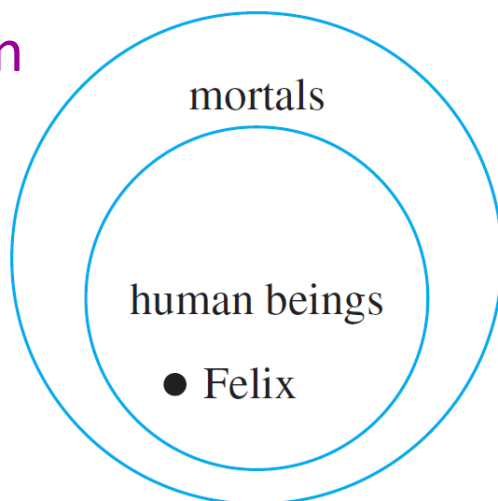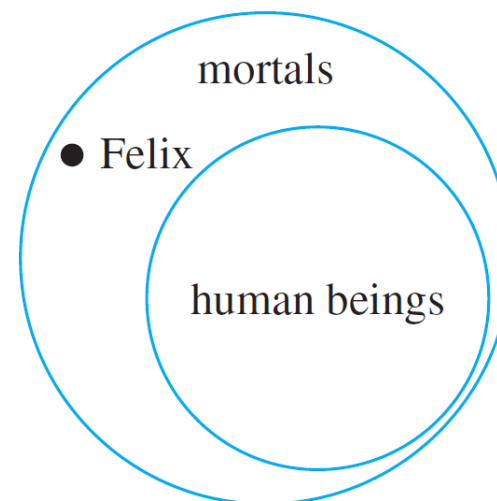Major premise   Minor premise

Figure 3.4.4

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Using Diagrams to Show Invalidity

Use a diagram to show the invalidity of the following argument:

All human beings are mortal.
Felix is mortal.
• Felix is a human being.

Hence, argument is invalid.

Conclusion is true.

Conclusion is false.

(a)   (b)

Figure 3.4.5

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Using Diagrams to Test for Validity

The argument of previous example would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made.

We say that this argument exhibit the converse error.

<div style="border:1px solid blue">

Converse Error (Quantified Form)

Formal version   *Informal version*

$\forall x$, if $P(x)$ then $Q(x)$.   If x makes $P(x)$ true, then x makes $Q(x)$ true.

$Q(a)$ for a particular a. a makes $Q(x)$ true.

● $P(a)$.   ● a makes $P(x)$ true.

</div>

Using Diagrams to Test for Validity

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid.

We say that this argument exhibit the inverse error.

<div style="border:1px solid blue">

Inverse Error (Quantified Form)

Formal version	*Informal version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
~$P(a)$ for a particular a.	a does not make $P(x)$ true.
• ~$Q(a)$.	• a does not make $Q(x)$ true.

</div>

An Argument with "No"

Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
• This function is not a polynomial function.

polynomial functions

functions with horizontal asymptotes
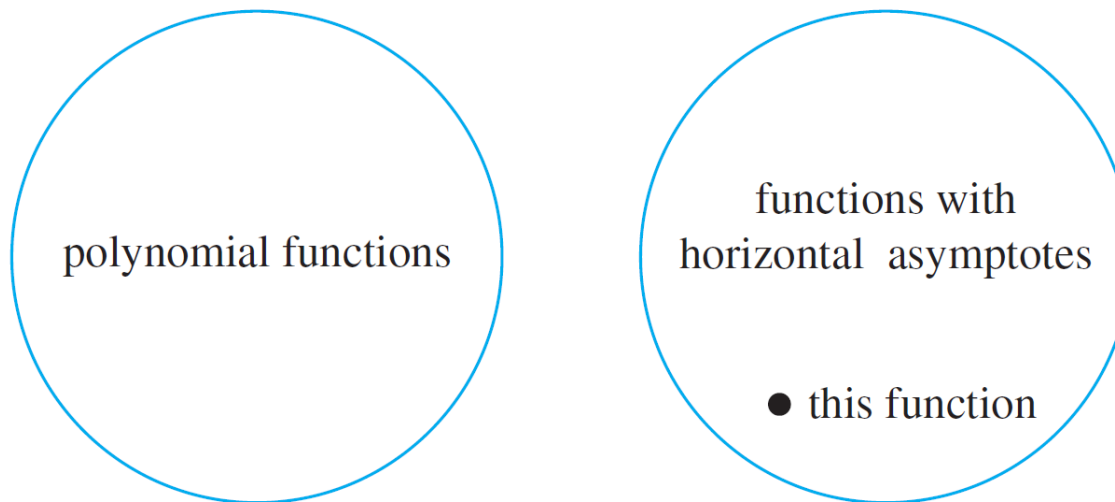
● this function

Hence argument is valid.

Figure 3.4.6

An Argument with "No"

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
- This function is not a polynomial function.

Alternatively, transform the first statement into:

$\forall x$, if x is a polynomial function, then x does not have a horizontal asymptote.

Then the argument has the form:
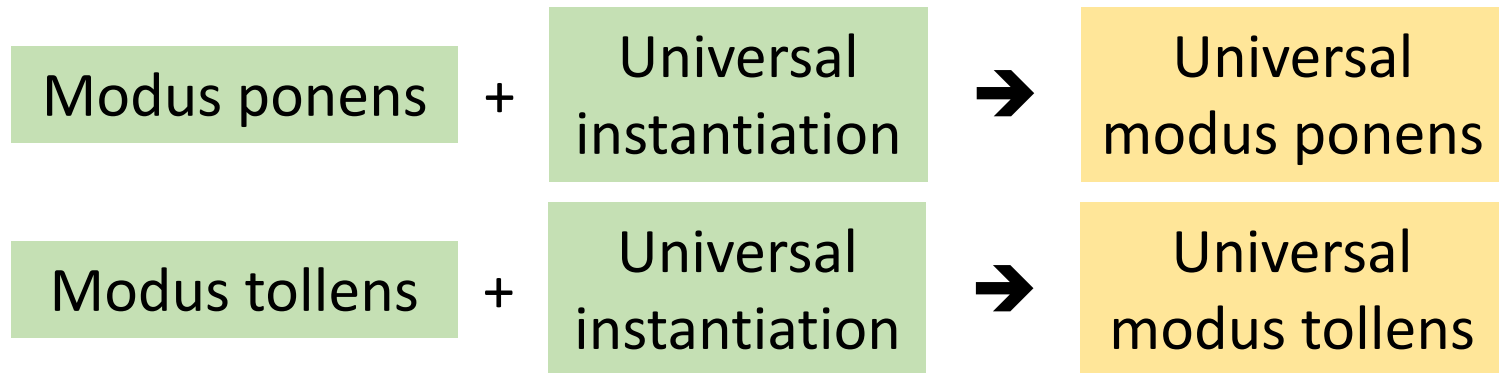
$\forall x$, if $P(x)$ then $Q(x)$.
$\sim Q(a)$, for a particular a.
- $\sim P(a)$.

This is valid by universal modus tollens.

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Creating Additional Forms of Argument

3.4.7. Creating Additional Forms of Argument

We have seen:

| Modus ponens | + | Universal instantiation | ➔ | Universal modus ponens |

| Modus tollens | + | Universal instantiation | ➔ | Universal modus tollens |

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

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Creating Additional Forms of Argument

Consider the following argument:

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\bullet \ p \rightarrow r$$

This can be combined with universal instantiation to obtain a valid argument form.

> ## Universal Transitivity
>
> | *Formal version* | *Informal version* |
> |---|---|
> | $\forall x, P(x) \rightarrow Q(x)$. | Any x that makes $P(x)$ true makes $Q(x)$ true. |
> | $\forall x, Q(x) \rightarrow R(x)$. | Any x that makes $Q(x)$ true makes $R(x)$ true. |
> | $\bullet \ \forall x, P(x) \rightarrow R(x)$. | $\bullet$ Any x that makes $P(x)$ true makes $R(x)$ true. |

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Figure 3.3.1

1. All the triangles are blue.

2. If an object is to the right of all the squares, then it is above all the circles.

3. If an object is not to the right of all the squares, then it is not blue.
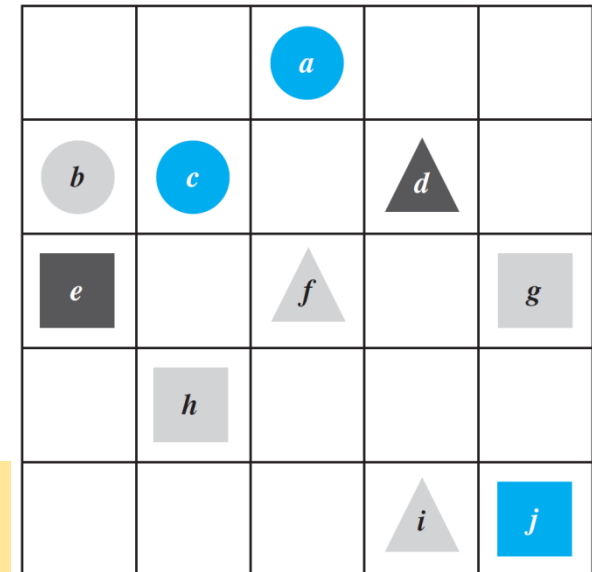
• All the triangles are above all the circles.

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 1:

1. $\forall x$, if x is a triangle, then x is blue.

2. $\forall x$, if x is to the right of all the squares, then x is above all the circles.

3. $\forall x$, if x is not to the right of all the squares, then x is not blue.

- $\forall x$, if x is a triangle, then x is above all the circles.

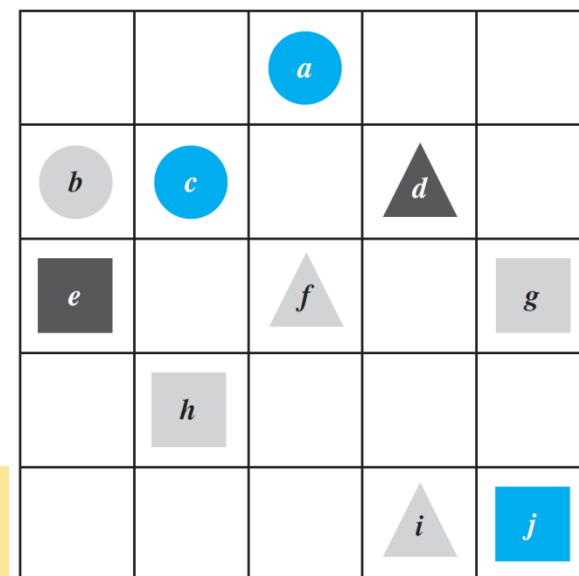**Figure 3.3.1**

Should be same as conclusion of the last premise.

Should be same as hypothesis of the first premise.

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Evaluating an Argument for Tarski's World

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 2:

1. $\forall x$, if x is a triangle, then x is blue.

2. $\forall x$, if x is not to the right of all the squares, then x is not blue.

3. $\forall x$, if x is to the right of all the squares, then x is above all the circles.

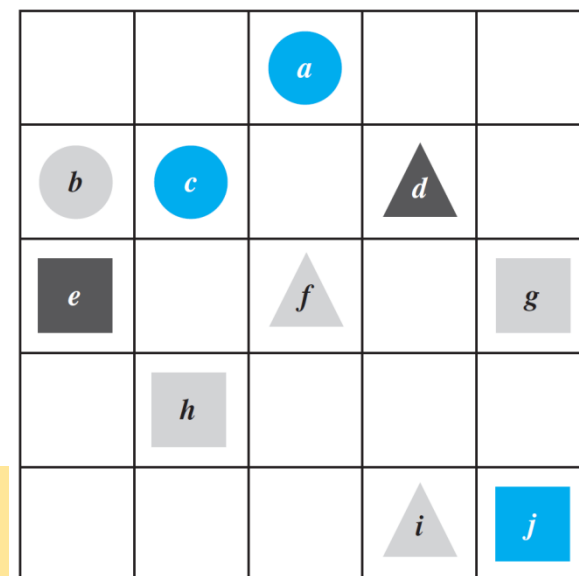• $\forall x$, if x is a triangle, then x is above all the circles.

Figure 3.3.1

Rewrite it in contrapositive form.

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Evaluating an Argument for Tarski's World

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 3:

1. $\forall x$, if x is a triangle, then x is blue.

2. $\forall x$, if x is blue, then x is to the right of all the squares.

3. $\forall x$, if x is to the right of all the squares, then x is above all the circles.

● $\forall x$, if x is a triangle, then x is above all the circles.

Figure 3.3.1

Remark on the Converser and Inverse Errors
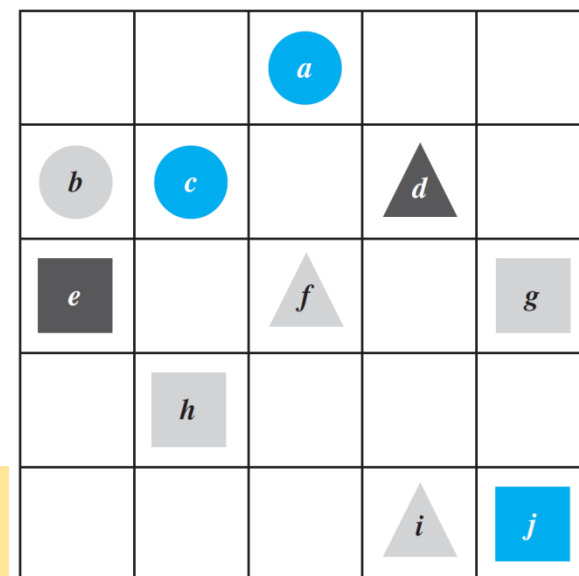
3.4.8. Remark on the Converse and Inverse Errors

Only for reading.

A variation of the converse error is a very useful reasoning tool, provided that it is used with caution.

It is the type of reasoning that is used by doctors to make medical diagnoses and by auto mechanics to repair cars.

It is the type of reasoning used to generate explanations for phenomena. It goes like this: If a statement of the form

For all x, if $P(x)$ then $Q(x)$

is true, and if

$Q(a)$ is true, for a particular a,

then check out the statement $P(a)$; it just might be true.

Remark on the Converser and Inverse Errors

Only for reading.

For instance, suppose a doctor knows that

> For all *x*, if *x* has pneumonia, then *x* has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

And suppose the doctor also knows that

> John has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty.

This form of reasoning has been named abduction by researchers working in artificial intelligence.

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