9. Counting and Probability 1

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9.1 Introduction

Tossing two coins

- Whether 0, 1 or 2 heads are obtained
- Does each of these events occur about 1/3 of the time?

No heads obtained

One head obtained

Two heads obtained

So if you repeatedly toss two balanced coins and record the number of heads, you should expect relative frequencies similar to those shown in Table 9.1.1.

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸 𝍸 𝍸 𝍸 ||	27	54%
0 heads obtained	𝍸 𝍸 ||	12	24%

To formalize this analysis and extend it to more complex situations, we introduce the notions of random process, sample space, event and probability.

To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability** of E, denoted $P(E)$, is

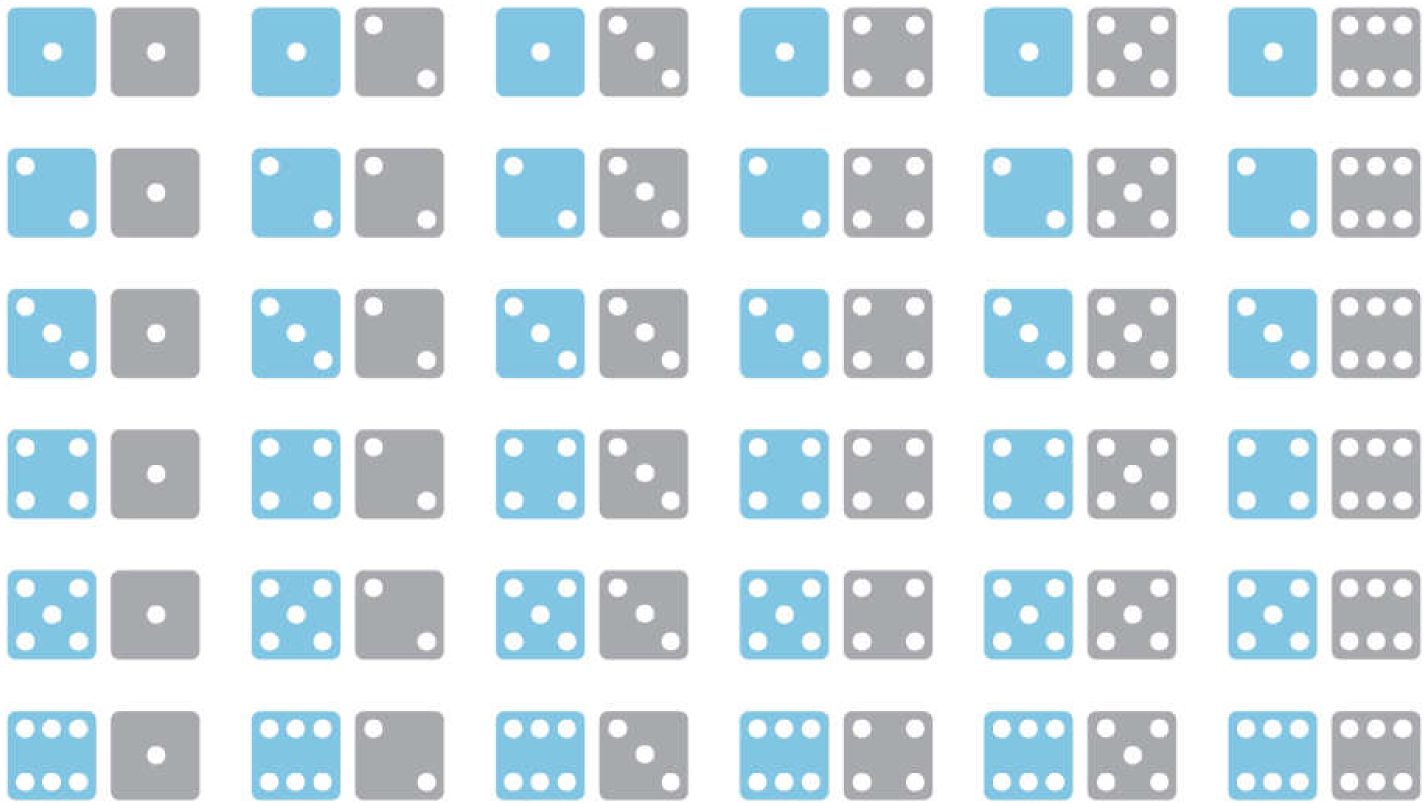
$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The total number of outcomes in } S}$$

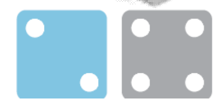
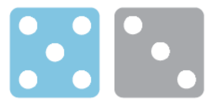
Notation

For a finite set A, $N(A)$ denotes the number of elements in A.

$$P(E) = \frac{N(E)}{N(S)}$$

Rolling a Pair of Dice



A more compact notation identifies, say,  with the notation 24,  with 53, and so forth.

a. Use the compact notation to write the sample space *S* of possible outcomes.

b. Use set notation to write the event *E* that the numbers showing face up have a sum of 6 and find the probability of this event.

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list.

For instance, how many integers are there from 5 through 12? To answer this question, imagine going along the list of integers from 5 to 12, counting each in turn.

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

More generally, if m and n are integers and $m \leq n$, how many integers are there from m through n?

Note that $n = m + (n - m)$, *where* $n - m \geq 0$ [*since* $n \geq m$].

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are

$$n - m + 1$$

integers from m to n inclusive.

Quiz

a. How many 3-digit integers (from 100 to 999 inclusive) are divisible by 5?

100 101 102 103 104 **105** 106 107 108 109 **110** ... 994 **995** 996 997 998 999

$5\times\mathbf{20}$ $5\times\mathbf{21}$ $5\times\mathbf{22}$ $5\times\mathbf{199}$

Number of multiples of 5 from 100 to 999 = number of integers from 20 to 199 inclusive.

Quiz

b. What is the probability that a randomly chosen 3-digit integer is divisible by 5?

By Theorem 9.1.1, total number of integers from 100 through 999 = 999 – 100 + 1 = 900.

9.2 Possibility Trees and the Multiplication Rule

Possibility Trees

A **tree structure** is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order.

Example 1: Possibilities for Tournament Play

Teams *A* and *B* are to play each other repeatedly until one wins two games in a row, or a total of three games.

One way in which this tournament can be played is for *A* to win the first game, *B* to win the second, and *A* to win the third and fourth games. Denote this by writing *A*–*B*–*A*–*A*.

Example 1: Possibilities for Tournament Play

a. How many ways can the tournament be played?

Possible ways are represented by the distinct paths from "root" (the start) to "leaf" (a terminal point) in the tree below.

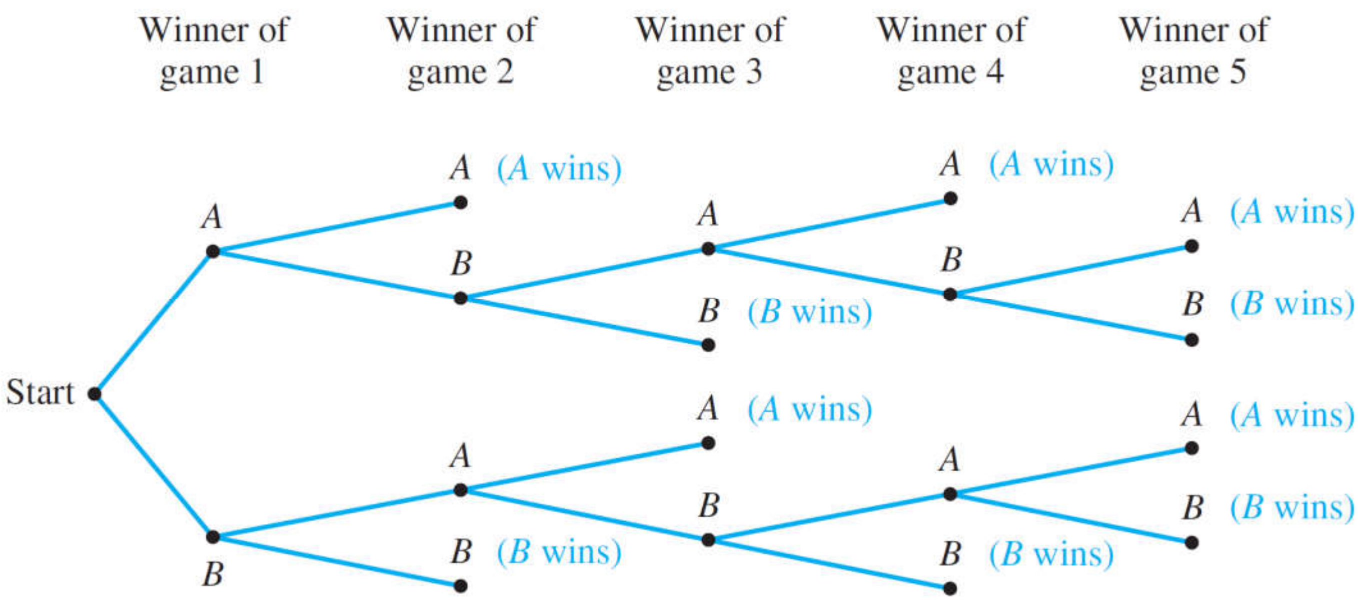


Figure 9.2.1 The Outcomes of a Tournament

Example 1: Possibilities for Tournament Play

a. How many ways can the tournament be played?

Ten paths from the root of the tree to its leaves → ten possible ways for the tournament to be played.

1. A-A
2. A-B-A-A
3. A-B-A-B-A
4. A-B-A-B-B
5. A-B-B
6. B-A-A
7. B-A-B-A-A
8. B-A-B-A-B
9. B-A-B-B
10. B-B

Example 1: Possibilities for Tournament Play

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

1. A-A
2. A-B-A-A
3. A-B-A-B-A
4. A-B-A-B-B
5. A-B-B
6. B-A-A
7. B-A-B-A-A
8. B-A-B-A-B
9. B-A-B-B
10. B-B

Probability that 5 games are needed = 4/10 = **2/5**

The Multiplication Rule

Consider the following example. Suppose a computer installation has four input/output units (A, B, C, and D) and three central processing units (X, Y, and Z).

Any input/output unit can be paired with any central processing unit. How many ways are there to pair an input/output unit with a central processing unit?

Possibility tree:

The total number of ways to pair the two types of units...

is the same as the number of branches of the tree:

$3 + 3 + 3 + 3 = 4\times3 =$ **12**

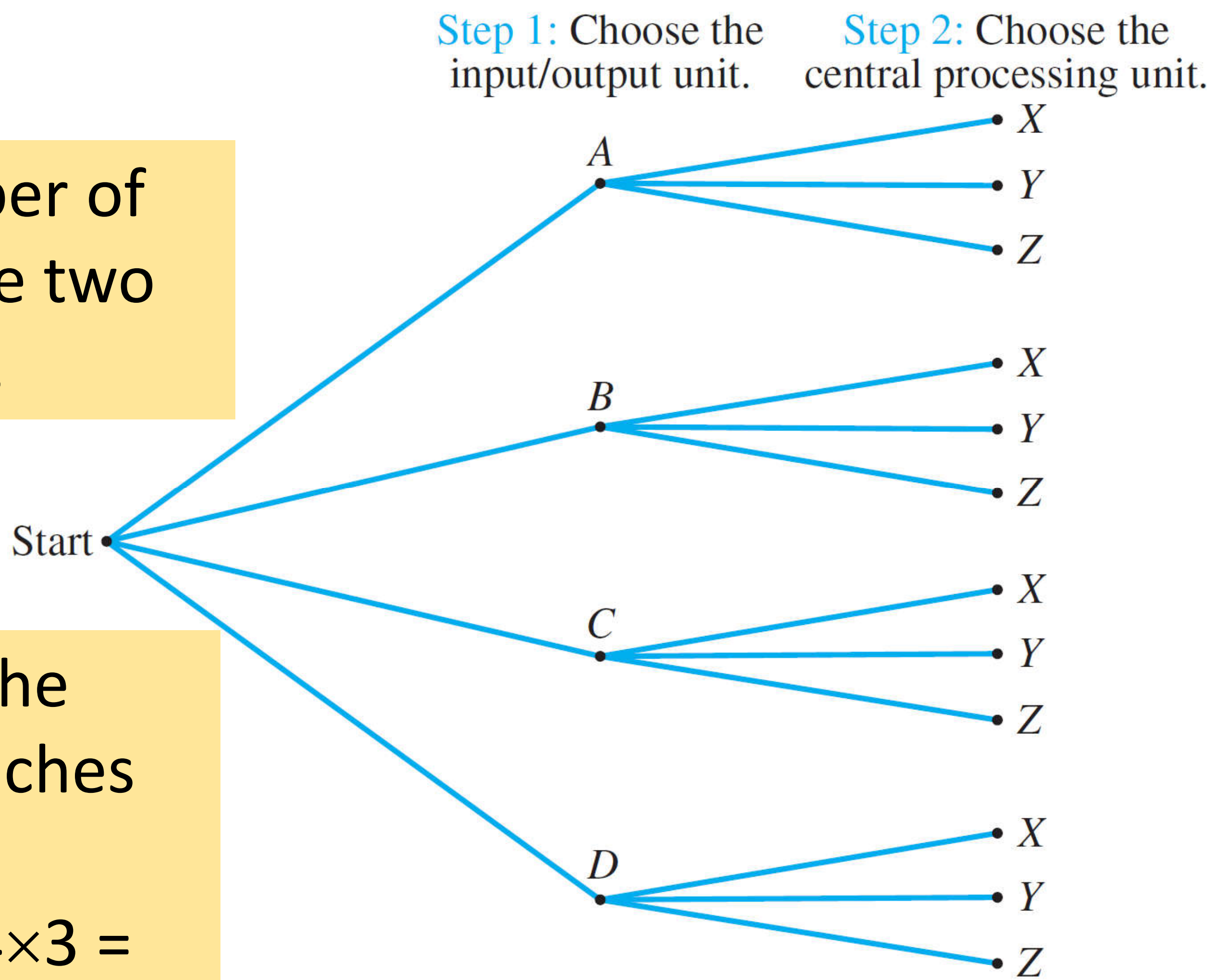


Figure 9.2.2 Pairing Objects Using a Possibility Tree

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways (regardless of how the first step was performed),

$\vdots$

the kth step can be performed in n_k ways (regardless of how the preceding steps were performed),

Then the entire operation can be performed in

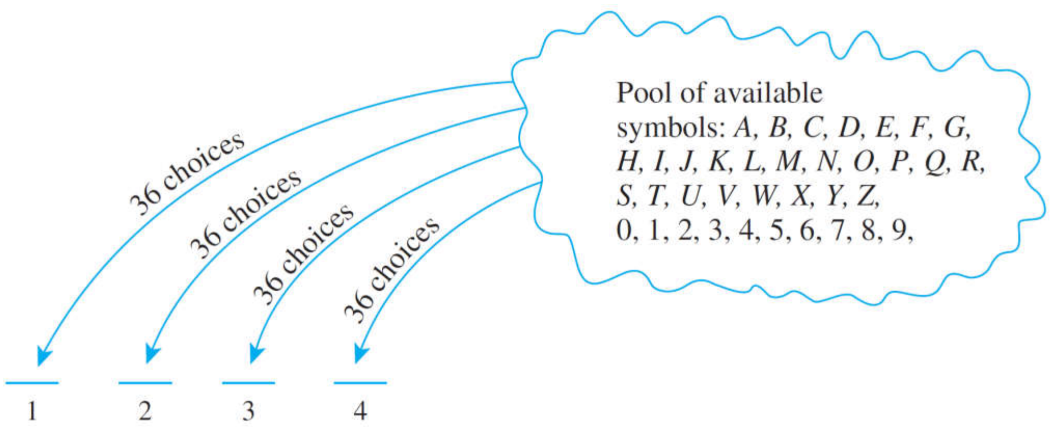
$$n_1 \times n_2 \times n_3 \times \ldots \times n_k \text{ ways.}$$

Example 2: No. of Personal Identification Numbers (PINs)

A typical PIN is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed. Examples: CARE, 3387, B32B, and so forth.

How many different PINs are possible?

You can think of forming a PIN as a four-step operation to fill in each of the four symbols in sequence.



Example 2: No. of Personal Identification Numbers (PINs)

Step 1: Choose the first symbol.
Step 2: Choose the second symbol.
Step 3: Choose the third symbol.
Step 4: Choose the fourth symbol.

Hence, by the **multiplication rule**, there are:
$36 \times 36 \times 36 \times 36 = 36^4 =$ **1,679,616**
PINs in all.

There is a fixed number of ways to perform each step, namely 36, regardless how preceding steps were performed.

Example 3: No. of PINs without Repetition

Now, suppose that repetition is not allowed.

a. How many different PINs are there?

Step 1: Choose the first symbol.
Step 2: Choose the second symbol.
Step 3: Choose the third symbol.
Step 4: Choose the fourth symbol.

Example 3: No. of PINs without Repetition

b. If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbols?

When the Multiplication Rule is Difficult/Impossible to Apply

Example 4: Consider the following problem:

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

Is this correct?

It is natural to try to solve this problem using the multiplication rule. A person might answer as follows:

There are three choices for president (all except Ann), three choices for treasurer (all except the one chosen as president), and two choices for secretary (Cyd or Dan).

Therefore, by the **multiplication rule**, $3\times3\times2$ = **18**

It is incorrect. The number of ways to choose the secretary varies depending on who is chosen for president and treasurer.

For instance, if Bob is chosen for president and Ann for treasurer, then there are two choices for secretary: Cyd and Dan.

But if Bob is chosen for president and Cyd for treasurer, then there is just one choice for secretary: Dan.

The clearest way to see all the possible choices is to construct the possibility tree, as shown below.

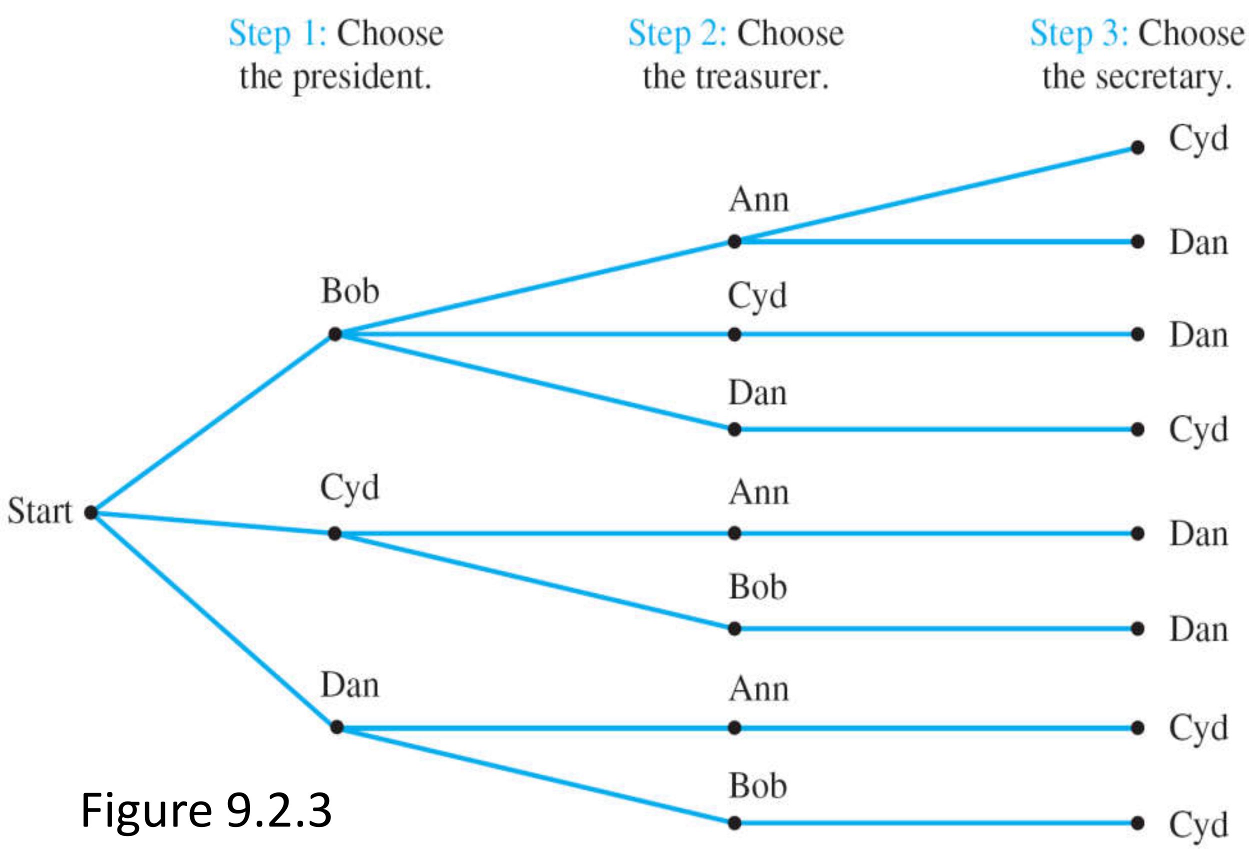


Figure 9.2.3

How many ways?

8 ways.

Example 5: A More Subtle Use of the Multiplication Rule

Reorder the steps for choosing the officers in the previous example so that the total number of ways to choose officers can be computed using the multiplication rule.

Step 1: Choose the secretary.
Step 2: Choose the president.
Step 3: Choose the treasurer.

2 ways: Either Cyd or Dan.

2 ways: Neither Ann nor the person chosen in step 1 may be chosen, but either of the other two may.

2 ways: Either of the 2 persons not chosen in steps 1 and 2 may be chosen.

Hence, total number of ways = $2 \times 2 \times 2 = \mathbf{8}$

Example 5: A More Subtle Use of the Multiplication Rule

A possibility tree illustrating this sequence of choices:

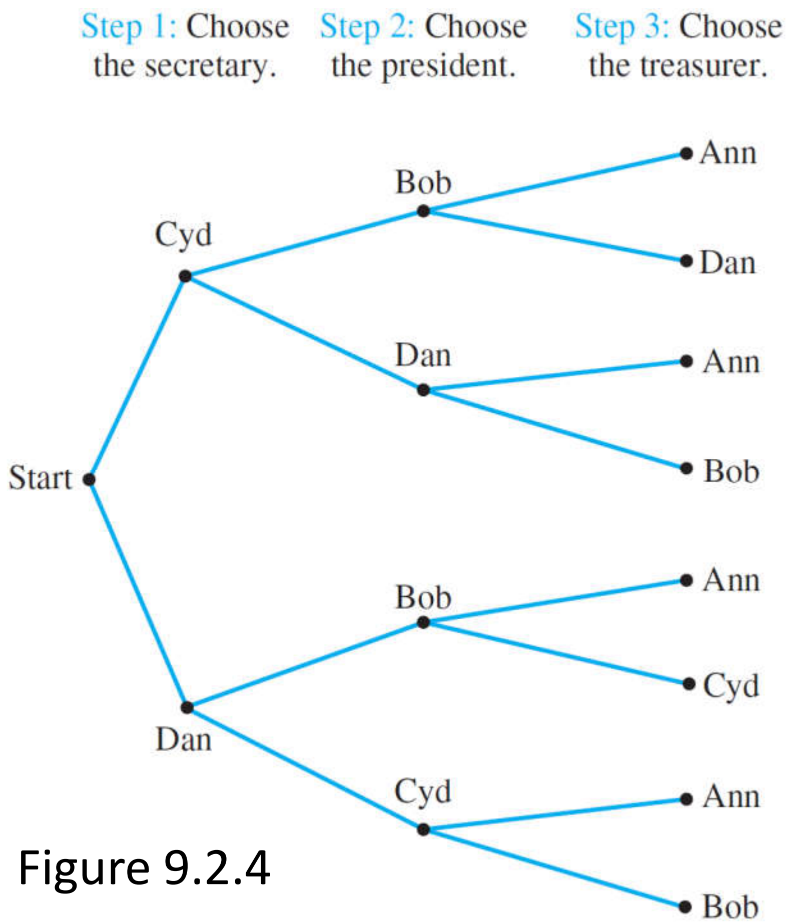


Figure 9.2.4

Quiz

Given 26 scrabble tiles with letters ‘A’ to ‘Z’, what is the probability of drawing “ICANDOIT” if…

a. you are not allowed to return the tile after it is drawn.

b. you are allowed to return the tile after it is drawn.

Permutations

A **permutation** of a set of objects is an ordering of the objects in a row. For example, the set of elements *a*, *b*, and *c* has six permutations.

abc *acb* *cba* *bac* *bca* *cab*

In general, given a set of *n* objects, how many permutations does the set have?

Permutations

Imagine forming a permutation as an *n*-step operation:

Step 1: Choose an element to write first.	← n ways
Step 2: Choose an element to write second.	← $n - 1$ ways
Step 3: Choose an element to write third.	← $n - 2$ ways
⋮	
Step *n*: Choose an element to write *n*th.	← 1 way

By the **multiplication rule**, there are

$$n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = \boldsymbol{n!}$$

ways to perform the entire operation.

In other words, there are $n!$ permutations of a set of n elements.

Theorem 9.2.2 Permutations

The number of permutations of a set with n ($n \geq 1$) elements is $\boldsymbol{n!}$

Example 6 – Permutations of the Letters in a Word

a. How many ways can the letters in the word *COMPUTER* be arranged in a row?

All the eight letters in the word *COMPUTER* are distinct. Hence, **8!** = **40320**.

b. How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

There are effectively only seven objects "CO", "M", "P", "U", "T", "E" and "R". Hence, **7!** = **5040**.

Example 6 – Permutations of the Letters in a Word

c. If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

Permutations of Selected Elements

Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of two elements of $\{a, b, c\}$ is called a ***2-permutation*** of $\{a, b, c\}$?.

Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set.
The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

Theorem 9.2.3 *r*-permutations from a set of *n* elements

If n and r are integers and $1 \le r \le n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$ first version

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!}$$ second version

Quiz

a. Evaluate $P(5, 2)$.

b. How many 4-permutations are there of a set of seven objects?

c. How many 5-permutations are there of a set of five objects?

Example 7 – Proving a Property of $P(n, r)$

Prove that for all integers $n \geq 2$,

$$\boxed{P(n, 2) + P(n, 1) = n^2}$$

Suppose $n \geq 2$. By Theorem 9.2.3,

$$P(n, 2) = n!/(n - 2)! = n(n - 1)$$

and

$$P(n, 1) = n!/(n - 1)! = n$$

Hence,

$$P(n, 2) + P(n, 1) = n(n - 1) + n = n^2 - n + n = n^2$$

■

9.3 Counting Elements of Disjoint Sets

Counting Elements of Disjoint Sets: The Addition Rule

The basic rule underlying the calculation of the number of elements in a union or difference or intersection is the **addition rule**.

This rule states that the number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_1, \ldots, A_k$. Then

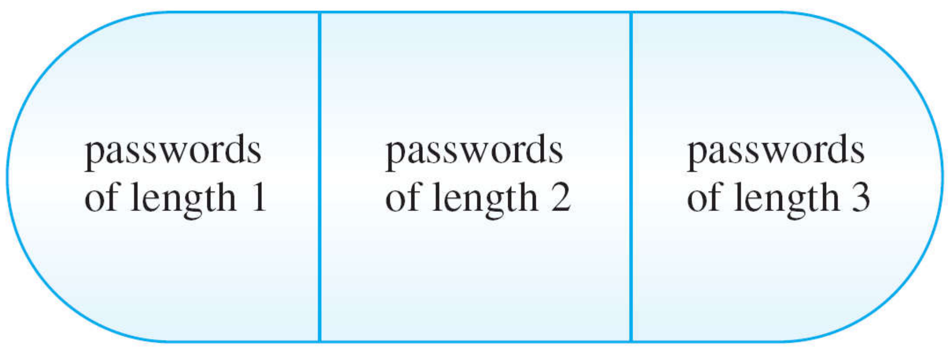
$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k).$$

Example 8 – Counting Passwords with 3 or fewer Letters

A computer access password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different passwords are possible?

The set of all passwords can be partitioned into subsets consisting of those of length 1, length 2, and length 3:

Figure 9.3.1
Set of all passwords of length ≤ 3



Example 8 – Counting Passwords with 3 or fewer Letters

By the **addition rule**, the total number of passwords equals the sum of the number of passwords of length 1, length 2, and length 3. Now,

Number of passwords of length 1 = 26

Number of passwords of length 2 = 26^2

Number of passwords of length 3 = 26^3

Hence, total number of passwords = $26 + 26^2 + 26^3$
$= \mathbf{18{,}278}$.

The Difference Rule

An important consequence of the addition rule is the fact that if the number of elements in a set A and the number in a subset B of A are both known, then the number of elements that are in A and not in B can be computed.

Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$

The Difference Rule

The difference rule is illustrated in Figure 9.3.3.

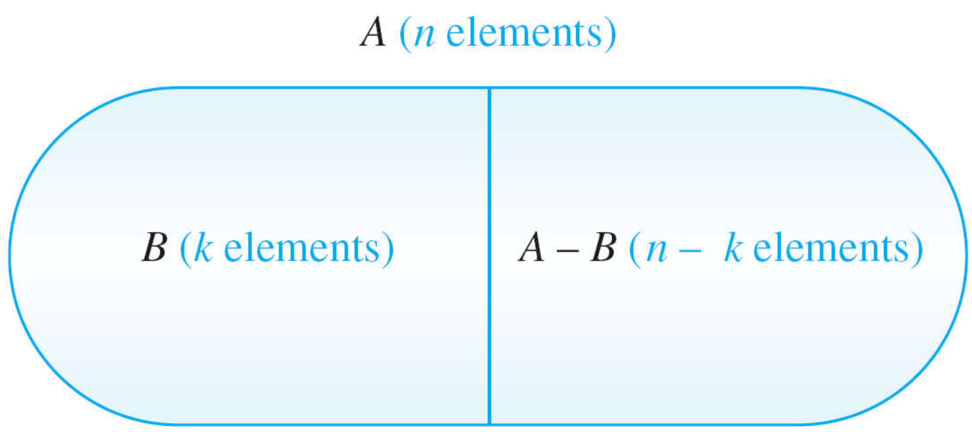


Figure 9.3.3 The Difference Rule

The difference rule holds for the following reason: If B is a subset of A, then the two sets B and $A - B$ have no elements in common and $B \cup (A - B) = A$. Hence, by the addition rule,

$$N(B) + N(A - B) = N(A).$$

Subtracting $N(B)$ from both sides gives the equation

$$N(A - B) = N(A) - N(B)$$

Example 9 – Counting PINs with Repeated Symbols

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed.

a. How many PINs contain repeated symbols?

There are $36^4 = 1{,}679{,}616$ PINs when repetition is allowed, and there are $36 \times 35 \times 34 \times 33 = 1{,}413{,}720$ PINs when repetition is not allowed.

By the **difference rule**, there are

$$1{,}679{,}616 - 1{,}413{,}720 = \mathbf{265{,}896}$$

PINS that contain at least one repeated symbol.

Example 9 – Counting PINs with Repeated Symbols

b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

There are 1,679,616 PINs in all, and by part (a) 265,896 of these contain at least one repeated symbol.

Thus, the probability that a randomly chosen PIN contains a repeated symbol is

$$\frac{265{,}896}{1{,}679{,}616} \cong \mathbf{0.158}$$

Example 9 – Counting PINs with Repeated Symbols

An alternative solution to part (b) is based on the observation that if S is the set of all PINs and A is the set of all PINs with no repeated symbol, then $S - A$ is the set of all PINs with at least one repeated symbol.

It follows that:

$$P(S-A) = \frac{N(S-A)}{N(S)} = \frac{N(S)-N(A)}{N(S)}$$

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$

$$= 1 - P(A)$$

$$P(A) = \frac{1{,}413{,}720}{1{,}679{,}616}$$

$$\cong 0.842$$

$$P(S-A) \cong 1 - 0.842$$

$$\cong \mathbf{0.158}$$

Probability of the Complement of an Event

This solution illustrates a more general property of probabilities: that the probability of the complement of an event is obtained by subtracting the probability of the event from the number 1.

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

The Inclusion/Exclusion Rule

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a union of sets when **some of the sets overlap**.

For simplicity, begin by looking at a union of two sets *A* and *B*, as shown in Figure 9.3.5.

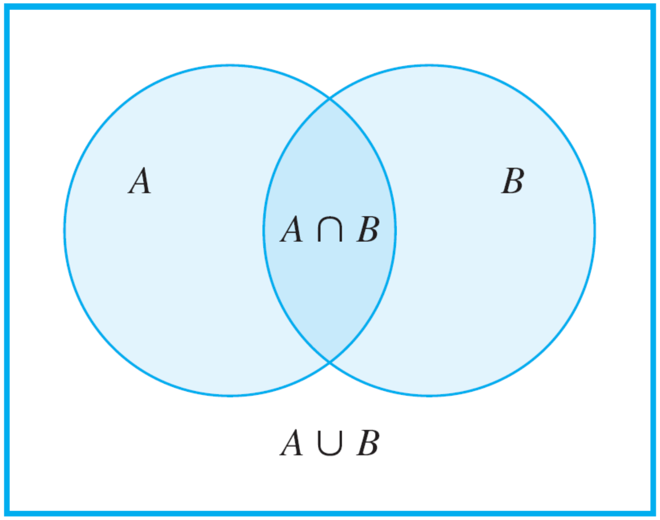


Figure 9.3.5

To get an accurate count of the elements in $A \cup B$, it is necessary to subtract the number of elements that are in both A and B. Because these are the elements in $A \cap B$,

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

Theorem 9.3.3 The Inclusion/Exclusion Rule for 2 or 3 Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

Example 10 – Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

Let A = the set of all integers in [1..1000] that are multiples of 3.

Let B = the set of all integers in [1..1000] that are multiples of 5.

Then $A \cup B$ = the set of all integers in [1..1000] that are multiples of 3 or multiples of 5.

Then $A \cap B$ = the set of all integers in [1..1000] that are multiples of both 3 and 5

= the set of all integers in [1..1000] that are multiples of 15.

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

As every third integer from 3 through 999 is a multiple of 3, each can be represented in the form $3k$, for some integer k in [1..333].

Hence there are 333 multiples of 3 from 1 through 1000, and so

$$N(A) = 333$$

Similarly, every multiple of 5 from 1 through 1000 has the form $5k$, for some integer k in [1..200].

Hence there are 200 multiples of 5 from 1 through 1000, and so

$$N(B) = 200$$

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

Finally, every multiple of 15 from 1 through 1000 has the form $15k$, for some integer k in [1..66] (since $990 = 66 \times 15$).

Hence there are 66 multiples of 15 from 1 through 1000, and so

$$N(A \cap B) = 66$$

It follows by the **inclusion/exclusion rule** that

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

There are 1000 integers from 1 through 1000.

By part (a), 467 of these are multiples of 3 or multiples of 5.

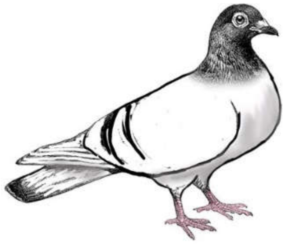
Note that the solution to part (b) of Example 10 hid a use of De Morgan's law.

The number of elements that are neither in A nor in B is $N(A^c \cap B^c)$.
By De Morgan's law, $A^c \cap B^c = (A \cup B)^c$.

So $N((A \cup B)^c)$ was then calculated using the set difference rule: $N((A \cup B)^c) = N(U) - N(A \cup B)$, where the universe U was the set of all integers from 1 through 1,000.

9.4 The Pigeonhole Principle

The Pigeonhole Principle: Introduction



If n pigeons fly into m pigeonholes and $n > m$, then at least one hole must contain two or more pigeons.

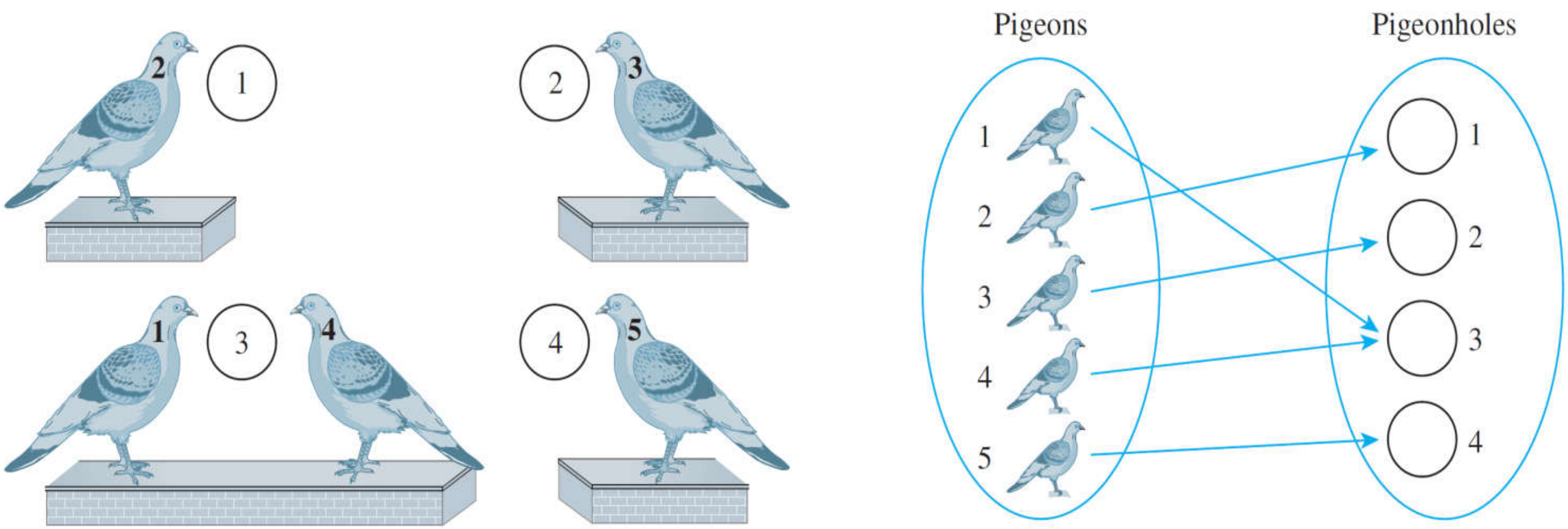
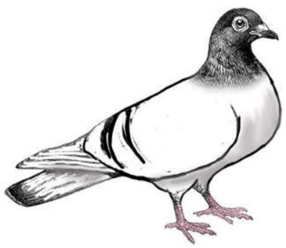


Figure 9.4.1 $n = 5$ and $m = 4$

The Pigeonhole Principle: Introduction

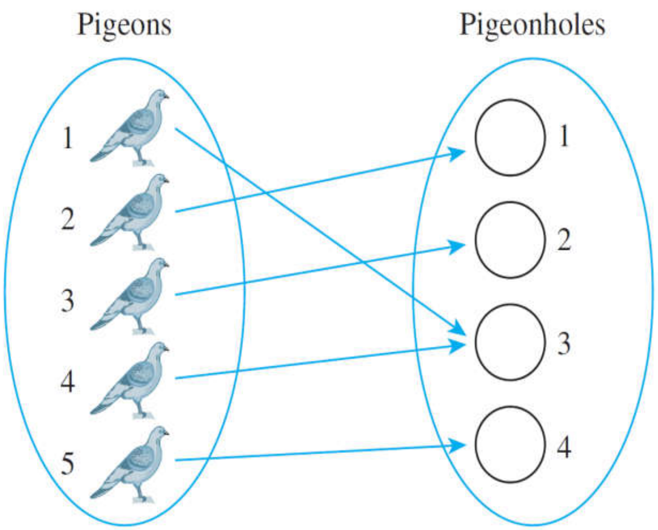
The pigeonhole principle is sometimes called the *Dirichlet box principle* because it was first stated formally by J. P. G. L. Dirichlet (1805–1859).



Mathematical formulation:

Pigeonhole Principle

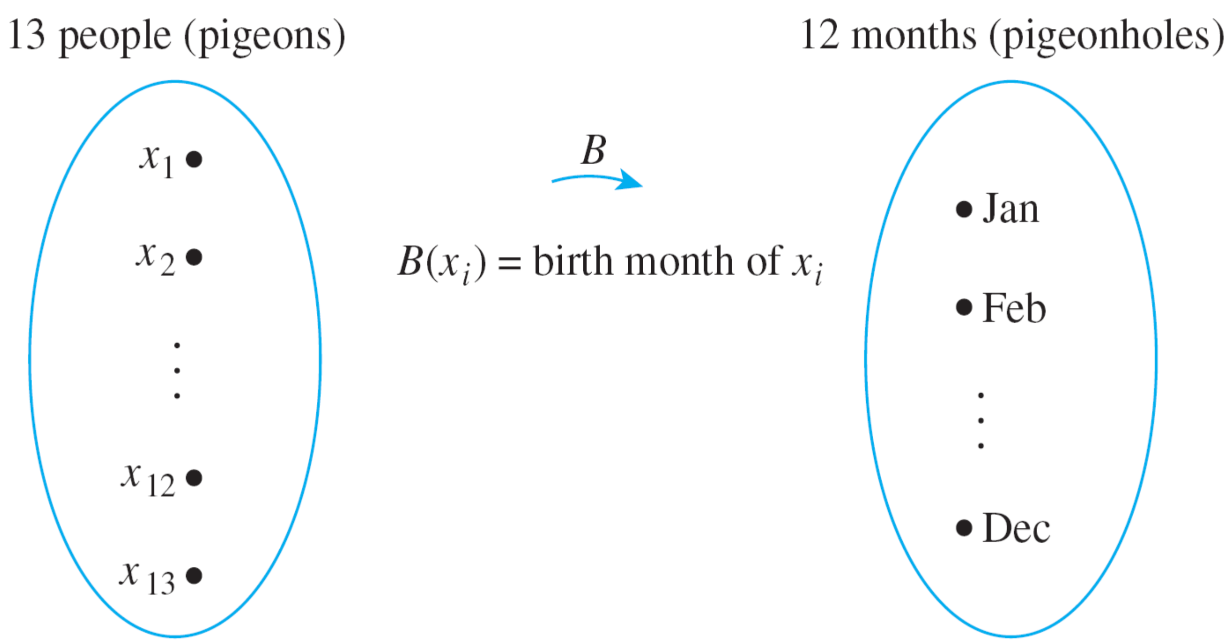
A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.



Example 11 – Applying the Pigeonhole Principle

a. In a group of six people, must there be at least two who were born in the same month? No.

In a group of 13 people, must there be at least two who were born in the same month? Why?



Yes. At least 2 people must have been born in the same month.

Example 11 – Applying the Pigeonhole Principle

b. Among the population of Singapore, are there at least two people with the same number of hairs on their heads? Why?

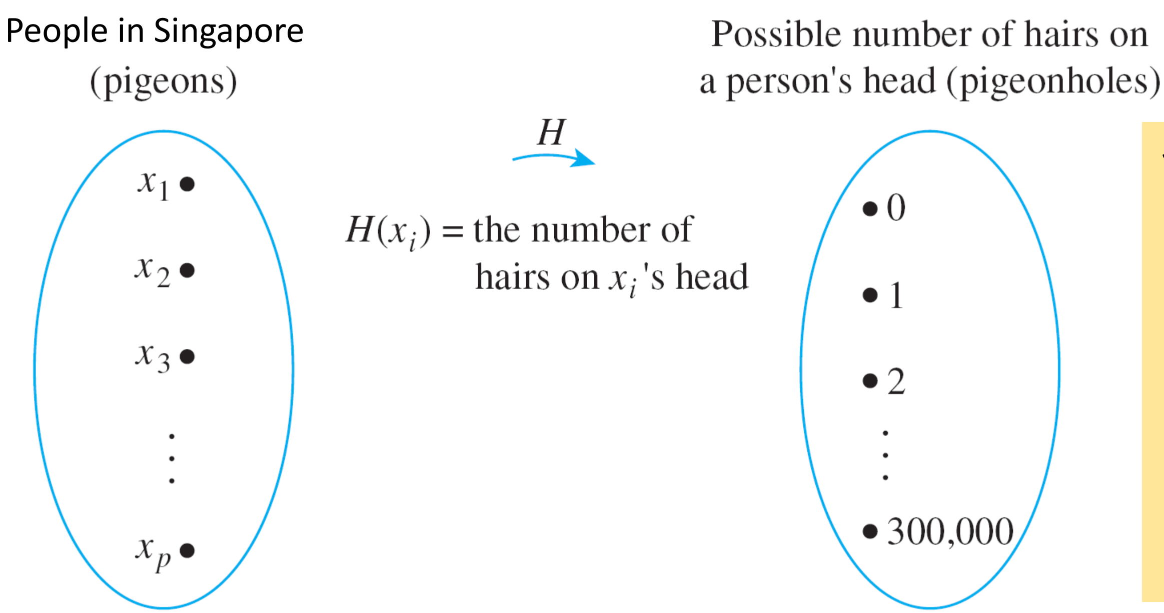
Population of Singapore: 5.47m (June 2014).

Hairs on head: average up to 150,000; no more than 300,000.

Define a function *H* from the set of people in Singapore $\{x_1, x_2, \ldots x_p\}$ to the set $\{0, 1, 2, \ldots 300000\}$ as shown in the next slide.

Example 11 – Applying the Pigeonhole Principle

b. Among the population of Singapore, are there at least two people with the same number of hairs on their heads? Why?



Yes. At least 2 people must have the same number of hairs on their heads.

Application to Decimal Expansions of Fractions

One important consequence of the piegonhole principle is the fact that *the decimal expansion of any rational number either terminates or repeats*.

A terminating decimal: 3.625

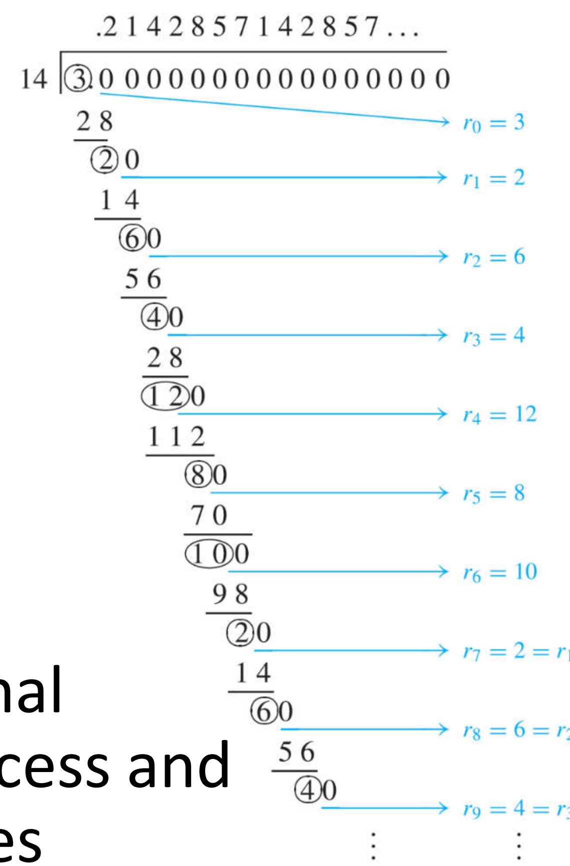
A repeating decimal: $2.38\overline{246}$ (2.38246246246246…)

A rational number can be written as a fraction *a*/*b*.

When one integer is divided by another, it is the pigeonhole principle (together with the quotient-remainder theorem) that guarantees that such a repetition of remainders (and hence decimal digits) must always occur.

Consider a/b, where for simplicity assume that a and b are positive. The decimal expansion of a/b is obtained by dividing a by b as illustrated here for a = 3 and b = 14.

Let $r_0 = a$ and let $r_1, r_2, r_3, \ldots$ be the successive remainders obtained in the long division of a by b.

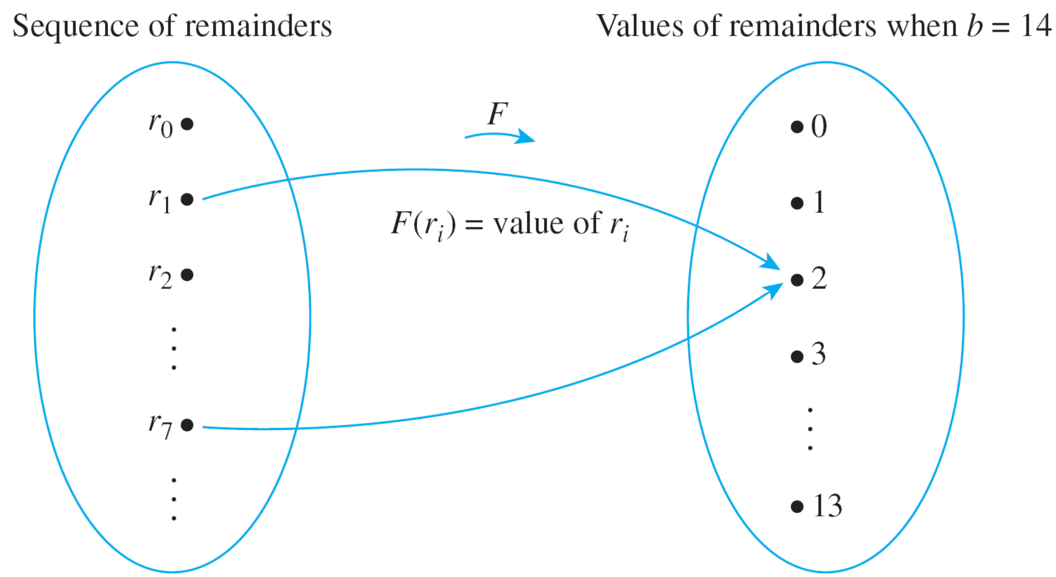


By the quotient-remainder theorem, each remainder must be between 0 and $b - 1$. (Here, a is 3 and b is 14, and so the remainders are from 0 to 13.)

If some remainder $r_i = 0$, then the division terminates and a/b has a terminating decimal expansion. If no $r_i = 0$, then the division process and hence the sequence of remainders continues forever.

By the pigeonhole principle, since there are more remainders than values that the remainders can take, some remainder value must repeat: $r_j = r_k$, for some indices j and k with $j < k$.

It follows that the decimal digits obtained from the divisions between r_j and r_{k-1} repeat forever.



In the case of 3/14, the repetition begins with $r_7 = 2 = r_1$ and the decimal expansion repeats the quotients obtained from the divisions from r_1 through r_6 forever:

$$3/14 = 0.2\overline{142857}$$

Generalized Pigeonhole Principle

If n pigeons fly into m pigeonholes and, for some positive integer k, $k < n/m$, then at least one pigeonhole contains $k + 1$ or more pigeons.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

Example 12 – Applying the General Pigeonhole Principle

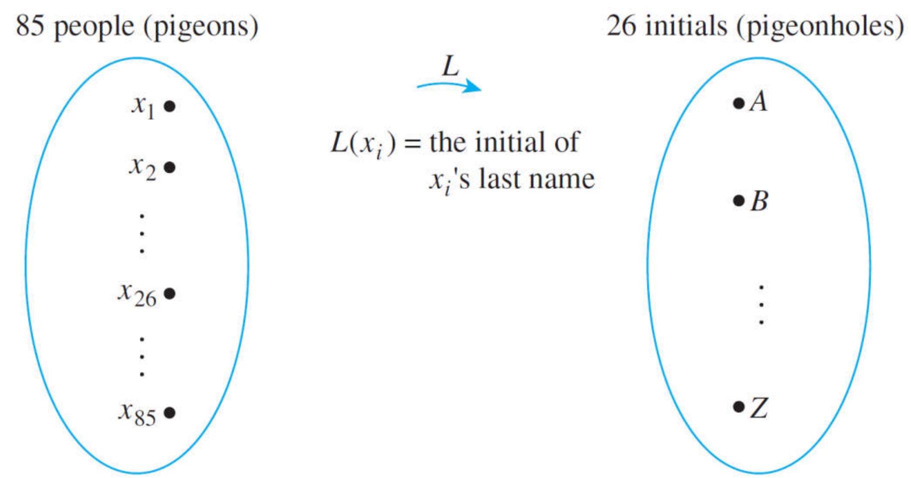
Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial (‘A’, ‘B’, …, ‘Z’).

In this example the pigeons are the 85 people and the pigeonholes are the 26 possible last initials of their names. Note that

$$3 < 85/26 \cong 3.27$$

Example 12 – Applying the General Pigeonhole Principle

Consider the function *L* from people to initials defined by the following arrow diagram.



Since 3 < 85/26, the generalized pigeonhole principle states that some initial must be the image of at least four (3 + 1) people.

Thus, at least 4 people have the same last initial. ■

Consider the following contrapositive form of the generalized pigeonhole principle. You may find it natural to use the contrapositive form of the generalized pigeonhole principle in certain situations

Generalized Pigeonhole Principle (Contrapositive Form)

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most km elements; in other words, $n \leq km$.

For instance, the result of Example 12 can be explained as follows:

Suppose no 4 people out of the 85 had the same last initial. Then at most 3 would share any particular one.

By the generalized pigeonhole principle (contrapositive form), this would imply that the total number of people is at most $3 \times 26 = 78$. But this contradicts the fact that there are 85 people in all.

Hence at least 4 people share a last initial.

Example 13 – Using the General Pigeonhole Principle (Contrapositive)

There are 42 students who are to share 12 computers. Each student uses exactly 1 computer, and no computer is used by more than 6 students. Show that at least 5 computers are used by 3 or more students.

Using an Argument by Contradiction: Suppose not. Suppose that 4 or fewer computers are used by 3 or more students. *[A contradiction will be derived.]* Then 8 or more computers are used by 2 or fewer students.

Divide the set of computers into two subsets: C_1 and C_2.

Into C_1 place 8 of the computers used by 2 or fewer students; into C_2 place the computers used by 3 or more students plus any remaining computers (to make a total of 4 computers in C_2).

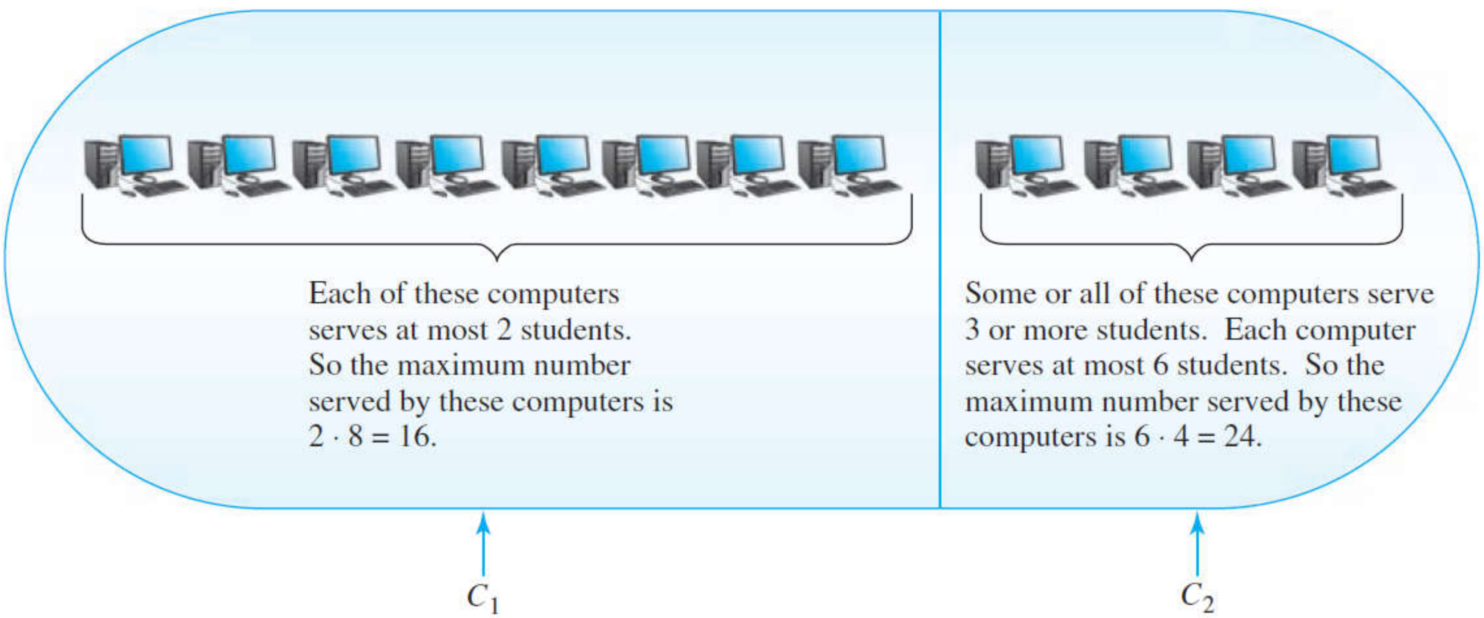


Figure 9.4.3 The set of 12 computers

Since at most 2 students are served by any one computer in C_1, by the generalized pigeonhole principle (contrapositive form), the computers in set C_1 serve at most $2 \times 8 = 16$ students.

Since at most 6 students are served by any one computer in C_2, by the generalized pigeonhole principle (contrapositive form), the computers in set C_2 serve at most $6 \times 4 = 24$ students.

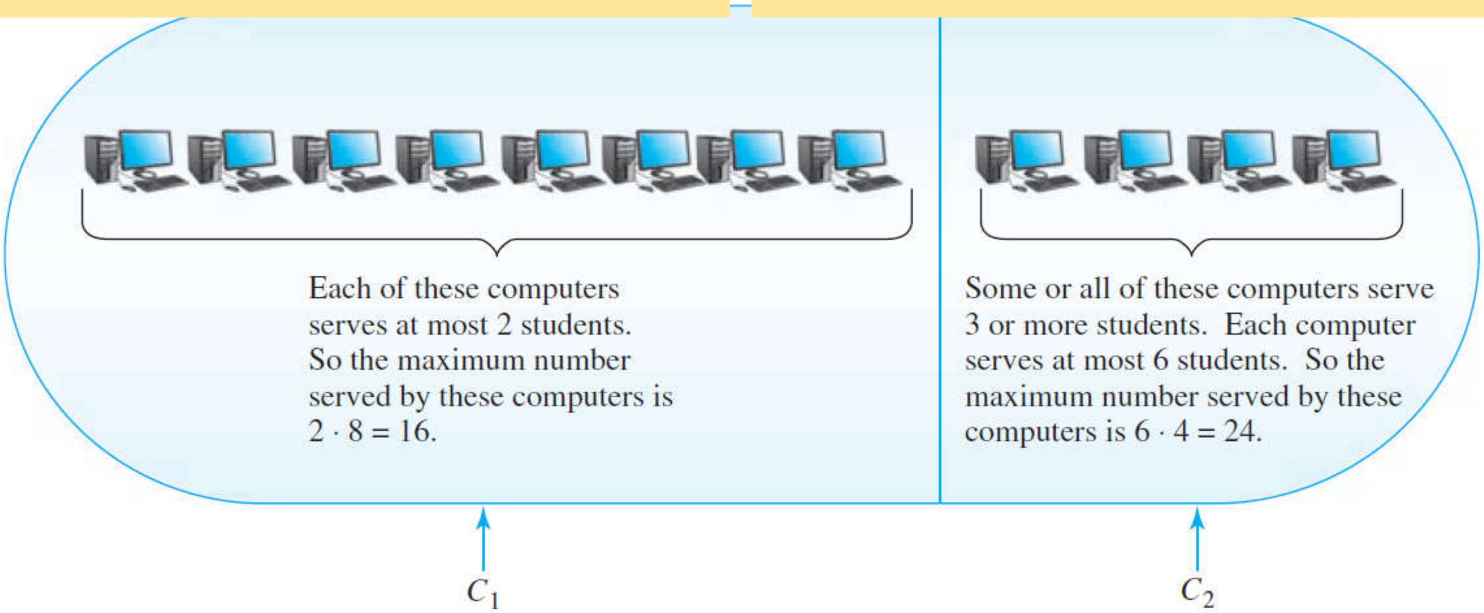


Figure 9.4.3 The set of 12 computers

Hence the total number of students served by the computers is 24 + 16 = 40.

But this contradicts the fact that each of the 42 students is served by a computer.

Therefore, the supposition is false. ■

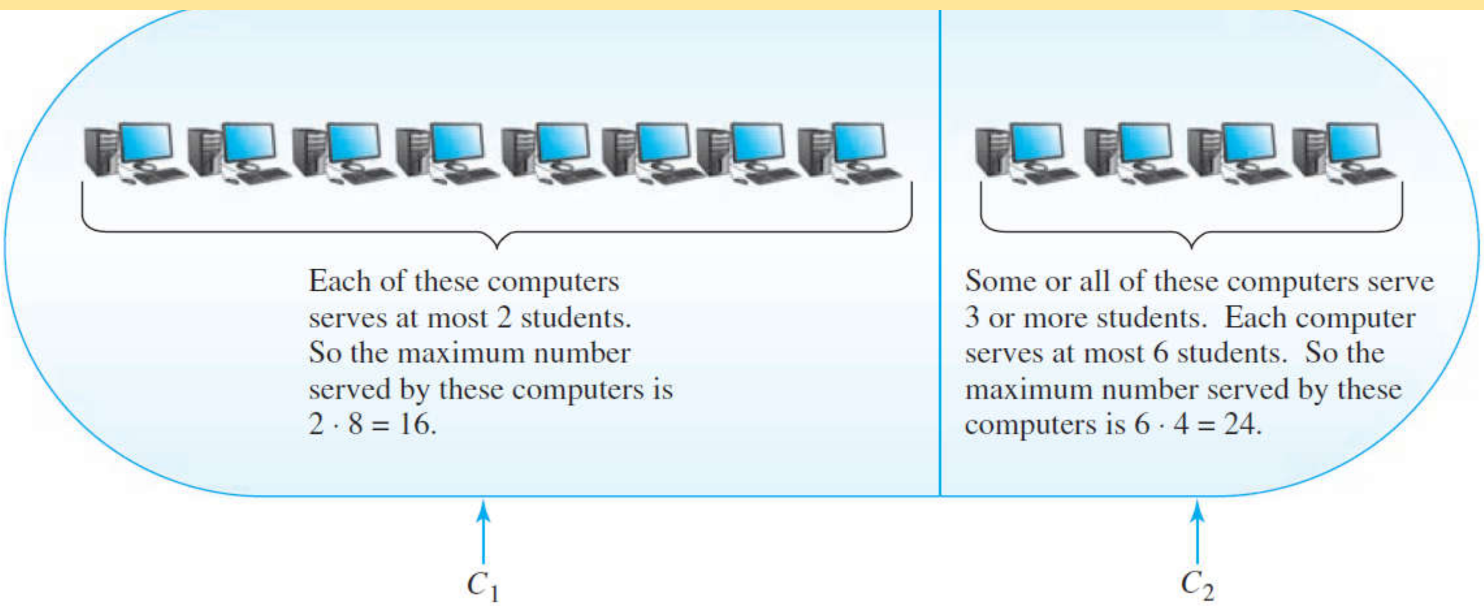


Figure 9.4.3 The set of 12 computers

Using a Direct Argument: Let k be the number of computers used by 3 or more students. *[We must show that* $k \geq 5$*.]*

Because each computer is used by at most 6 students, these computers are used by at most $6k$ students (by the generalized pigeonhole principle (contrapositive form)).

Each of the remaining $12 - k$ computers is used by at most 2 students.

Taken together, they are used by at most $2(12 - k) = 24 - 2k$ students (again, by the generalized pigeonhole principle).

Thus the maximum number of students served by the computers is $6k + (24 - 2k) = 4k + 24$.

Thus the maximum number of students served by the computers is $6k + (24 - 2k) = 4k + 24$.

Because 42 students are served by the computers,
$4k + 24 \geq 42$.

Solving for k gives that $k \geq 4.5$, and since k is an integer, this implies that $k \geq 5$. ■

Using Pigeonhole Principle on Properties of Functions

Recall the following definitions in Chapter 7:

Definition: One-to-One Function

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

if $F(x_1) = F(x_2)$, then $x_1 = x_2$,

or, equivalently, if $x_1 \neq x_2$, then $F(x_1) \neq F(x_2)$.

Definition: Onto Function

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X such that $y = F(x)$.

Theorem 9.4.1 The Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not one-to-one.

Theorem 9.4.2 One-to-One and Onto for Finite Sets

Let X and Y be finite sets with the same number of elements and suppose f is a function from X to Y. Then f is one-to-one if, and only if, f is onto.

Next week's lectures

More on Counting and Probability!

- Combinations
- *r*-Combinations
- Pascal's Formula and the Binomial Theorem
- Probability Axioms and Expected Value
- Conditional Probability, Bayes' Formula, and Independent Events

END OF FILE