Advanced Digital Modulation

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Differential Encoding of PSK

Differential encoding is a technique that can be used to remove the effect of the M-fold phase ambiguity of M-PSK Constellations. Consider 4-PSK and let $e^{j\psi_k}$ be the k-element of the PSK sequence. The encoded sequence is

$$e^{j\phi_k} = e^{j(\psi_k + \phi_{k-1})},$$

which means that the information is encoded in the phase jumps:

$$e^{j\psi_k} = e^{j(\phi_k - \phi_{k-1})}.$$

At the receive side, after ambiguous coherent detection, the ambiguous decisions $\hat{\phi}_k = \phi_k + n\pi/2$ are taken, then $\hat{\psi}_k$ is unambiguously recovered as

$$\hat{\psi}_k = \hat{\phi}_k - \hat{\phi}_{k-1}.$$

The penalty of differential encoding with ambiguous coherent detection is error propagation: an error on $\hat{\phi}_k$ causes errors on $\hat{\psi}_k$ and on $\hat{\psi}_{k+1}$.

Noncoherent Differential Decoding of PSK

Suppose that, at the receive side, besides being impossible to remove the M-fold phase ambiguity, also the phase in the range $(0, 2\pi/M]$ of the incoming carrier cannot be recovered. In this case detection of D-PSK can be noncoherent, that is, without knowledge of the phase of the incoming carrier. Specifically, two adjacent noisy signals are multiplicated as

$$(e^{-j(\phi_{k-1}+\theta)} + w_{k-1})(e^{j(\phi_k+\theta)} + w_k) \approx e^{j\psi_k} + w'_{k-1} + w'_k,$$

where θ is the unknown phase of the incoming carrier, w' is statistically equivalent to w, and the product $w_k w_{k-1}$ has been neglected. The penalty of noncoherent differential decoding is about 3 dB, since the noise power is approximately two times the power of the noise in coherent detection.

Differential Encoding of QAM

A similar encoding method can be used with M^2-QAM to cope with the four-fold phase ambiguity due to the four-fold symmetry of the constellation. Specifically, the M^2-QAM constellation with points in the grid of odd integers is decomposed as

$$a = b + \frac{M}{\sqrt{2}} e^{j\phi},$$

where $e^{j\phi}$ is taken from the 4-PSK constellation (assume the set $\{\pi/4 + n\pi/2\}$ for ϕ), and b is taken from the $(M^2/4)$-QAM constellation. The term $Me^{j\phi}/\sqrt{2}$ encodes the quadrant of the complex plane. The phasor $e^{j\phi}$ is produced by differential encoding two bits as in the 4-PSK case, while b is produced by encoding the remaining $log_2(M^2/4)$ bits in a rotationally invariant fashion.

Differential Encoding of QAM

Another mapping with differential encoding for M^2-QAM is

$$a = 2b + \sqrt{2}e^{j\phi},$$

where $e^{j\phi}$ is taken from the 4-PSK constellation, and b is taken from the $(M^2/4)$-QAM constellation. The term $\sqrt{2}e^{j\phi}$ encodes the the two LSBs of the natural mapping of the two rails of QAM. The phasor $e^{j\phi}$ is produced by differential encoding two bits as in the 4-PSK case, while b is produced by encoding the remaining $log_2(M^2/4)$ bits in a rotationally invariant fashion.

Transmission Based on Orthogonal Signals

Let the M-ary alphabet used for transmission of $\log_2 M$ bits per symbol be

$$\mathcal{A} = \{A\phi_1(t), A\phi_2(t), \cdots, A\phi_M(t)\},$$

where A is a scalar, and the waveforms $\{\phi_k(t)\}$ have the property

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = \delta(i,j),$$

hence they form an orthonormal basis. Suppose of transmitting an isolated impulse, or, equivalently, that there is no ISI, and that at the receive side the baseband equivalent of the received signal is

$$x(t) = A\phi_i(t) + w(t),$$

where $w(t)$ is complex AWGN with power spectral density N_0. The SNR is

$$\text{SNR} = \frac{E_s}{N_0} = \frac{|A|^2}{N_0}.$$

Correlation Receiver

Reception can be based on a bank of M correlations that produce the M numbers

$$c_i = \int_{-\infty}^{\infty} \phi_i(t)x(t)dt, \quad i = 1, 2, \cdots, M.$$

Suppose that $A\phi_1(t)$ is transmitted. Then vector $\mathbf{c} = (c_1, c_2, \cdots, c_M)$ looks as $(A + n_1, n_2, \cdots, n_M)$ where n_i is the the correlation between $\phi_i(t)$ and the noise $w(t)$. In other words, $x(t)$ can be represented as vector with M entries in a M-dimensional space, where the constellation is the set of points

$$\{(A, 0, \cdots, 0), (0, A, \cdots, 0), \cdots, (0, 0, \cdots, A)\}.$$

Note that correlation is equivalent to filtering the signal through the bank of filters with impulse responses $h_i(t) = \phi_i(-t)$ and then sampling the M outputs at $t = 0$.

Correlation Receiver

In the case of correlation reception, the probability of error can be upperbounded by the union bound. Specifically, one evaluates the argument of the Q-function by the Pythagorean theorem as

$$d^2_{min} = 2E_s,$$

$$\frac{d^2_{min}}{4\sigma^2_n} = \frac{2E_s}{4 \cdot (N_0/2)} = \frac{E_s}{N_0} = \text{SNR},$$

while the number of nearest neighbor is $M-1$, therefore, by the union bound, one has

$$P(e) \leq (M-1)Q(\sqrt{\text{SNR}}).$$

Examples of Orthogonal Signals: Continuous-Phase FSK

The alphabet of M-ary continuous-phase FSK is

$$\mathcal{A} = \left\{ A \cdot rect\left(\frac{t - T/2}{T}\right) \sqrt{\frac{2}{T}} cos\left(\frac{2\pi(n+i)t}{T} + \phi\right) \right\}, \;\; i = 1, 2, \cdots, M,$$

where n is an integer up to the designer that determines the frequency band at which the passband spectrum is located. The signal set is such that the symbol interval T contains an integer number of cycles of the sinusoid, and that there is phase continuity between two successive symbols. The frequencies of the sinusoids that form the signal set are $f_i = (n+i)T^{-1}$ hence all the frequencies are integer multiplies of the frequency T^{-1}. This is to say that the signal set is formed by orthogonal functions. Exercise: determine the spectrum of 2-FSK.

Examples of Orthogonal Signals: Continuous-Phase FSK

Although binary FSK require more bandwidth than BPSK and 3 dB more of SNR, it has some advantages. The most prominent are

- FSK admits noncoherent detection, for instance by
  - a limiter-differentiator, that is a circuit that counts the number of zero crossings in the symbol interval,
  - a bank of bandpass filters followed by envelope detectors and selection of the maximum; in its essence this is a special case of correlation detection that does not require knowledge of the phase ϕ of the carrier.
- The envelope of FSK is constant.

Examples of Orthogonal Signals: M-ary PPM

In Pulse Position Modulation the signal set is

$$\mathcal{S} = \left\{ A \cdot \sqrt{\frac{2M}{T}} rect\left(\frac{Mt}{T} - i\right) \right\}, \;\; i = 1, 2, \cdots, M.$$

More generally, one can build the signal set as

$$\mathcal{S} = \left\{ A \cdot \sqrt{\frac{1}{E_g}} \cdot g\left(t - \frac{iT}{M}\right) \right\}, \;\; i = 1, 2, \cdots, M.$$

Which condition should satisfy $g(t)$ for ISI free transmission and matched filter receiver with sampling frequency M/T?

Transmission Based on Orthogonal Signals

Consider the argument of the Q-function in the union bound

$$P(e) \leq (M-1)Q(\sqrt{\text{SNR}}).$$

For $M = 2$ there is a loss of 3 dB compared to BPSK, which is *antipodal*, not orthogonal. Moreover, here $M = N = 2$, while with BPSK, $M = 2$, $N = 1$, where N is the number of complex dimensions. For $M = 4$ the argument of the Q-function is the same as with 4-QAM, however here $N = M = 4$, while, with 4-QAM, $M = 4$, $N = 1$.

Transmission Based on Orthogonal Signals

We see that, with orthogonal signal, the SNR required to obtain a fixed argument of the Q-function is independent of the size of the symbol set, which seems to be an advantage. However, M independent dimensions are necessary for transmission of one symbol. Therefore, even if energy is allocated only in one of the M dimensions, the transmission of one symbol requires of using the channel M times, $M - 1$ of which with zero energy. The number of bits per complex channel use is therefore

$$\eta = \frac{log_2 M}{M} \; b/2D.$$

Since the number of dimensions M is proportional to two times the bandwidth of the passband signal, we conclude that orthogonal signalling is not bandwidth efficient compared to QAM.