PRACTICE SESSION 2

EXERCISE 3

Given the system

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + 2u(t) \\ y(t) = 3x(t) \end{cases}$$

$F = \frac{1}{2}$ $\quad G = 2$ $\quad H = 3$ $\quad D = 0$

ⓐ Compute the first 5 samples of the impulse response

Method I: Directly from the system of difference equations

$$u(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad x(0) = x_0 = 0$$

If it is not specified consider the initial state equal to zero

t	x(t)	y(t)
0	$x(0) = 0$	$y(0) = 3x(0) = 0$
1	$x(1) = \frac{1}{2}\cancel{x(0)} + 2\underbrace{u(0)}^{1} = 2$	$y(1) = 3x(1) = 6$
2	$x(2) = \frac{1}{2}x(1) + 2\cancel{u(1)} = 1$	$y(2) = 3x(2) = 3$
3	$x(3) = \frac{1}{2}x(2) + 2\cancel{u(2)} = \frac{1}{2}$	$y(3) = 3x(3) = 3/2$
4	$x(4) = \frac{1}{2}x(3) + 2\cancel{u(3)} = \frac{1}{4}$	$y(4) = 3x(4) = 3/4$
5	$x(5) = \frac{1}{2}x(4) + \cancel{2u(4)} = \frac{1}{8}$	$y(5) = 3x(5) = 3/8$

$$\boxed{\omega(0) = 0 \quad \omega(1) = 6 \quad \omega(2) = 3 \quad \omega(3) = 3/2 \quad \omega(4) = 3/4 \quad \omega(5) = 3/8}$$

Method IV: Geometric series trick

First of all we need to compute the transfer function

$$W(z) = H(zI - F)^{-1}G + D$$

$$W(z) = 3\left(z - \frac{1}{2}\right)^{-1}2 = \frac{6}{z - \frac{1}{2}} = \frac{6z^{-1}}{1 - \frac{1}{2}z^{-1}} =$$

$$= 6z^{-1}\left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right) = 6z^{-1}\left(\sum_{k=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^k\right) =$$

$$= 6z^{-1}\left(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \frac{1}{16}z^{-4} + \dots\right) =$$

$$= 6z^{-1} + 3z^{-2} + \frac{3}{2}z^{-3} + \frac{3}{4}z^{-4} + \frac{3}{8}z^{-5} + \dots =$$

$$\boxed{\omega(0) = 0 \quad \omega(1) = 6 \quad \omega(2) = 3 \quad \omega(3) = \frac{3}{2} \quad \omega(4) = \frac{3}{4} \quad \omega(5) = \frac{3}{8}}$$

(b) Compute the 2nd order Henkel Matrix and check that it is not full rank, and justify it.

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

$H_2(2,:) = \frac{1}{2} H_2(1,:)$ $\rightarrow$ since the two rows are not linearly independent H_2 is not full rank.

$$\boxed{H_2 = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix} \qquad \text{rank}(H_2) < 2}$$

Justification: Consider the Henkel matrix of order i

if $i > n \Rightarrow \text{rank}(H_i) = n$

in our case $i = 2$ $n = 1$ $\Rightarrow \text{rank}(H_2) = 1$

(c) Identify the system matrices using the 4SID method

STEP 1: Identify the system order n

in this case we already know that $n = 1$

STEP 2: Build H_{n+1}

$$H_2 = \begin{bmatrix} 6 & 3 \\ 3 & 3/2 \end{bmatrix}$$

STEP 3: Find a factorization of $H_{n+1} = \mathcal{O}_{n+1} \mathcal{R}_{n+1}$

$\mathcal{O}_{n+1}$ $(n+1) \times (n)$ $\rightarrow$ in our case 2×1

$\mathcal{R}_{n+1}$ $(n) \times (n+1)$ $\rightarrow$ in our case 1×2

$$\underbrace{\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}_{\mathcal{O}_{n+1}} \underbrace{\begin{bmatrix} \boxed{6 \quad 3} \end{bmatrix}}_{\mathcal{R}_{n+1}} = \begin{bmatrix} \boxed{6 \quad 3} \\ 3 \quad 3/2 \end{bmatrix}$$

- Put n independent rows of H_{n+1} in $\mathcal{R}_{n+1}$
- Fill the rows of $\mathcal{O}_{n+1}$ such that $H_{n+1} = \mathcal{O}_{n+1} \mathcal{R}_{n+1}$

STEP 4: matrix extraction $\hat{F}, \hat{G}, \hat{H}, \hat{D}$

$$\hat{F} = \mathcal{O}_{n+1}(1:n,:)^{-1} \mathcal{O}_{n+1}(2:n+1,:) = (1)^{-1} \tfrac{1}{2} = 1/2$$

$$\hat{G} = \mathcal{R}_{n+1}(:,1) = 6$$

$$\hat{H} = \mathcal{O}_{n+1}(1,:) = 1$$

$\hat{D} = 0$ (since the system is strictly proper)

$$\boxed{\begin{array}{ll} \hat{F} = \frac{1}{2} & \hat{G} = 6 \\ \hat{H} = 1 & \hat{D} = 0 \end{array}}$$

Remark: we have found a different but equivalent state space representation.

d) Compute the transfer function starting from the identified matrices

$$\hat{W}(z) = \hat{H}(zI - \hat{F})^{-1}\hat{G} + \hat{D} =$$

$$= 1\left(z - \frac{1}{2}\right)^{-1} 6 = \frac{6}{z - \frac{1}{2}} = \frac{6z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$\boxed{\hat{W}(z) = \frac{6z^{-1}}{1 - \frac{1}{2}z^{-1}}}$$

The transfer function is the same, the I/O relationship is maintained

EXERCISE 4

Given the transfer function

$$W(z) = \frac{(z+2)}{(z+\frac{1}{2})(z+2)}$$

REMARK ZERO-POLE CANCELLATION

$$W_1(z) = \frac{(z+2)}{(z+\frac{1}{2})(z+2)} \qquad W_2(z) = \frac{1}{z+\frac{1}{2}}$$

W_1 and W_2 represents the same I/O relationship.

$$W_1(z) = \frac{z+2}{(z+\frac{1}{2})(z+2)} = \frac{z+2}{z^2+\frac{5}{2}z+1} \quad \leftarrow \text{2}^{\text{nd}} \text{ order unstable system}$$

$$W_2(z) = \frac{1}{z+\frac{1}{2}} \quad \leftarrow \text{1}^{\text{ST}} \text{ order stable system}$$

WRONG CONCLUSION

DO NOT CANCEL NUMERATOR-DENOMINATOR COMMON-TERMS!

ⓐ Compute the state space system in control form

$$W(z) = \frac{z+2}{z^2+\frac{5}{2}z+1} \qquad b_0 = 1 \quad b_1 = 2 \qquad a_1 = \frac{5}{2} \quad a_2 = 1$$

$$\boxed{F = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = [\,2 \quad 1\,] \quad D = [\,0\,]}$$

ⓑ Check observability and Reachability

$$n = 2 \rightarrow \mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} \qquad \mathcal{R} = [\,G \quad FG\,]$$

$$\mathcal{O}(1,:) = [\,2 \quad 1\,]$$

$$\mathcal{O}(2,:) = [\,2 \quad 1\,]\begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} = [\,-1 \quad -\frac{1}{2}\,]$$

$$\mathcal{O} = \begin{bmatrix} 2 & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \longrightarrow \text{the two rows are not linearly independent}$$

$$\mathcal{O}(2,:) = -\frac{1}{2}\,\mathcal{O}(1,:)$$

$rank(\mathcal{O}) < 2 \longrightarrow$ The system S is not observable

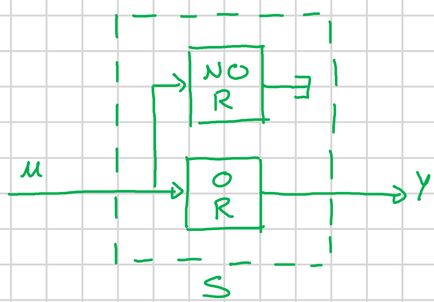
$$\mathcal{R}(:,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathcal{R}(:,2) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$$

$$\mathcal{R} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{5}{2} \end{bmatrix}$$ All the rows are linearly independent

$rank(\mathcal{R}) = 2 \longrightarrow$ The system S is reachable

A zero-pole cancellation corresponds to a hidden part of the systems: something that the I/O representation cannot catch.

In this case there is a nonobservable part



$w(z)$ represents just the reachable and observable part of the system

ⓒ Compute the first 4 samples of the Impulse Response and plot the state and output evolution.

We have to use method I:

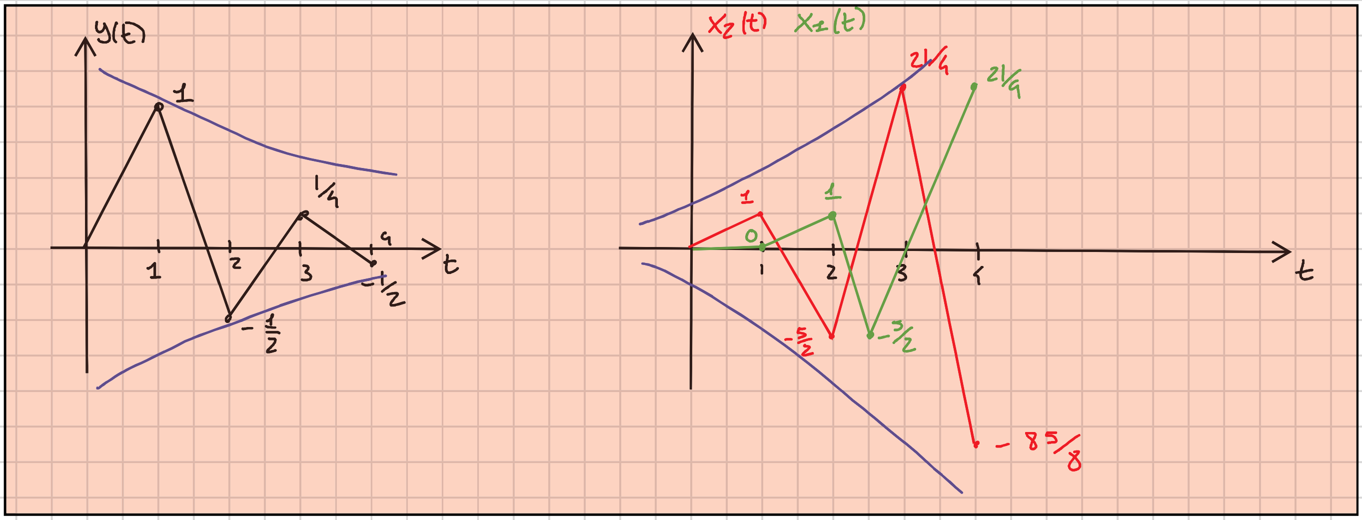
$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = -x_1(t) - \frac{5}{2} x_2(t) + u(t) \\ y(t) = 2x_1(t) + x_2(t) \end{cases}$$

$$u(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

t	x(t)	y(t)
0	$x_1(0) = 0$ $x_2(0) = 0$	$y(0) = 2\cancel{x_1(0)} + \cancel{x_2(0)} = 0$
1	$x_1(1) = \cancel{x_2(0)} = 0$ $x_2(1) = -\cancel{x_1(0)} - \frac{5}{2}\cancel{x_2(0)} + \underbrace{u(0)}_{1} = 1$	$y(1) = 2\cancel{x_1(1)} + x_2(1) = 1$
2	$x_1(2) = x_2(1) = 1$ $x_2(2) = -\cancel{x_1(1)} - \frac{5}{2}x_2(1) + \cancel{u(1)} = -\frac{5}{2}$	$y(2) = 2x_1(2) + x_2(2) = -\frac{1}{2}$

3	$x_1(3) = x_2(2) = -5/2$ $x_2(3) = -x_1(2) - \frac{5}{2}x_2(2) + \cancel{u(2)} = 21/4$	$y(3) = 2x_1(3) + x_2(3) = \frac{1}{4}$
4	$x_1(4) = x_2(3) = 21/4$ $x_2(4) = -x_1(3) - \frac{5}{2}x_2(3) + \cancel{u(3)} = -\frac{85}{8}$	$y(4) = 2x_1(4) + x_2(4) = -\frac{1}{8}$

$\boxed{w(0) = 0 \quad w(1) = 1 \quad w(2) = -1/2 \quad w(3) = 1/4 \quad w(4) = -1/8}$



Remark: Looking at the I/O relationship the system seems to be stable, but the states diverge then there is an internal instability.

d) Identify the matrices using the 4SID method

STEP 1: Identify the system order

$\text{rank}(H_i) = n \quad \forall i \geq n \qquad n$ is the system order

$\left.\begin{array}{l}\text{rank}(H_n) = n \\ \text{rank}(H_{n+1}) = n\end{array}\right\}$ the rank stops increasing

$H_1 = w(1) = 1 \qquad \text{rank}(H_1) = 1$

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix} \qquad H_2(2,:) = -\frac{1}{2} H_2(1,:)$$

$\text{rank}(H_2) = 1$

$\left.\begin{array}{l}\text{rank}(H_1) = 1 \\ \text{rank}(H_2) = 1\end{array}\right. \longrightarrow n = 1$

STEP 2 Build H_{n+1}

$$H_2 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$$

STEP 3: Find a factorization of $H_{n+1} = \mathcal{O}_{n+1} \mathcal{R}_{n+1}$

$$\underbrace{\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}}_{\mathcal{O}_2} \underbrace{\begin{bmatrix} \boxed{1 \quad -\frac{1}{2}} \end{bmatrix}}_{\mathcal{R}_2} = \underbrace{\begin{bmatrix} \boxed{1 \quad -\frac{1}{2}} \\ -\frac{1}{2} \quad \frac{1}{4} \end{bmatrix}}_{H_2}$$

- Put n independent rows of H_{n+1} in $\mathcal{R}_{n+1}$
- Fill the rows of $\mathcal{O}_{n+1}$ such that $\mathcal{O}_{n+1} \mathcal{R}_{n+1} = H_{n+1}$

STEP 4: matrix Extraction

$$\hat{F} = \mathcal{O}_{n+1}(1:n,:)^{-1} \mathcal{O}_{n+1}(2:n+1,:) = (1)^{-1}\left(-\tfrac{1}{2}\right) = -\tfrac{1}{2}$$

$$\hat{G} = \mathcal{R}_{n+1}(:,1) = 1$$

$$\hat{H} = \mathcal{O}_{n+1}(1,:) = 1$$

$\hat{D} = 0$ strictly proper system $(\omega(0) = 0)$

$$\boxed{\hat{F} = -\tfrac{1}{2} \qquad \hat{G} = 1 \qquad \hat{H} = 1 \qquad \hat{D} = 0}$$

(e) Compute the transfer function starting from the identified matrices

$$\boxed{\hat{W}(z) = \hat{H}(zI - \hat{F})^{-1}\hat{G} + \hat{D} = \frac{1}{z + \frac{1}{2}}}$$

→ it represents just the reachable and observable part.

EXERCISE 5

Given the impulse response

$w(0)=0 \quad w(1)=0 \quad w(2)=2 \quad w(3)=0 \quad w(4)=1 \quad w(5)=0$

ⓐ Identify the system order

$H_1 = w(1) = 0 \qquad rank(H_1) = 0$

$H_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad rank(H_2) = 2$

$H_3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad H_3(3,:) = \frac{1}{2} H_3(1,:) \rightarrow rank(H_3) = 2$

$rank(H_2)=2$
$rank(H_3)=2$ $\rightarrow \boxed{n=2}$

ⓑ Compute the transfer function

$IR \rightarrow \hat{F}\ \hat{G}\ \hat{H}\ \hat{D} \rightarrow \hat{W}(z)$

STEP 1: Identify the system order

$n=2$ from question ⓐ

STEP 2: Build H_{n+1}

$H_3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

STEP 3: Find a factorization $H_{n+1} = \mathcal{O}_{n+1} R_{n+1}$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}}_{\mathcal{O}_3} \underbrace{\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{R_3} = \underbrace{\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{H_3}$$

• Put n independent rows of H_{n+1} in R_{n+1}

STEP 4: Matrix extraction

$$\hat{F} = \mathcal{O}_{n+1}(1:n,:)^{-1} \mathcal{O}_{n+1}(2:n+1,:) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\hat{G} = R_{n+1}(:,1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\hat{H} = \mathcal{O}_{n+1}(1,:) = [1 \quad 0]$

$\hat{D} = 0$ strictly proper system $(w(0)=0)$

$$\hat{W}(z) = \hat{H}(zI-\hat{F})^{-1}\hat{G} + \hat{D}$$

$$(zI-\hat{F}) = \begin{bmatrix} z & -1 \\ -\frac{1}{2} & z \end{bmatrix} \qquad (zI-\hat{F})^{-1} = \frac{1}{z^2-\frac{1}{2}} \begin{bmatrix} z & 1 \\ \frac{1}{2} & z \end{bmatrix}$$

$$\hat{W}(z) = \frac{1}{z^2-\frac{1}{2}} [1 \quad 0] \begin{bmatrix} z & 1 \\ \frac{1}{2} & z \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} =$$

$$= \frac{1}{z^2-\frac{1}{2}} [z \quad 1] \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{2}{z^2-\frac{1}{2}}$$

$$\boxed{\hat{W}(z) = \frac{2}{z^2-\frac{1}{2}}}$$

(c) Compute the first 4 samples of the Impulse response

Method II: matrix multiplication formula

$$w(t) = \begin{cases} 0 & t=0 \\ HF^{t-1}G & t \geq 1 \end{cases}$$

it is valid only for strictly proper systems

$w(0) = 0$

$$w(1) = HG = [1 \quad 0]\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$$

$$w(2) = HFG = [1 \quad 0]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \underbrace{\boxed{[0 \quad 1]}}_{HF}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2$$

$$w(3) = HF^2G = HF \cdot FG = [0 \quad 1]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \underbrace{\boxed{[\tfrac{1}{2} \quad 0]}}_{HF^2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$$

$$w(4) = HF^2 \cdot FG = [\tfrac{1}{2} \quad 0]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = [0 \quad \tfrac{1}{2}]\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 1$$

$$\boxed{w(0) = 0 \quad w(1) = 0 \quad w(2) = 2 \quad w(3) = 0 \quad w(4) = 1}$$