# Passband Signals

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## Baseband and Passband Signals

*Definition:*

a time-continuous signal is said to be *baseband* when all its power (energy) spectral density is nonzero at $f = 0$ and it is zero at and at $|f| > f_{max}$, with $f_{max} < \infty$.

*Definition:*

a time-continuous signal is said to be *passband* when all its power (energy) spectral density lies in the range of frequencies $\{f\}$ such that

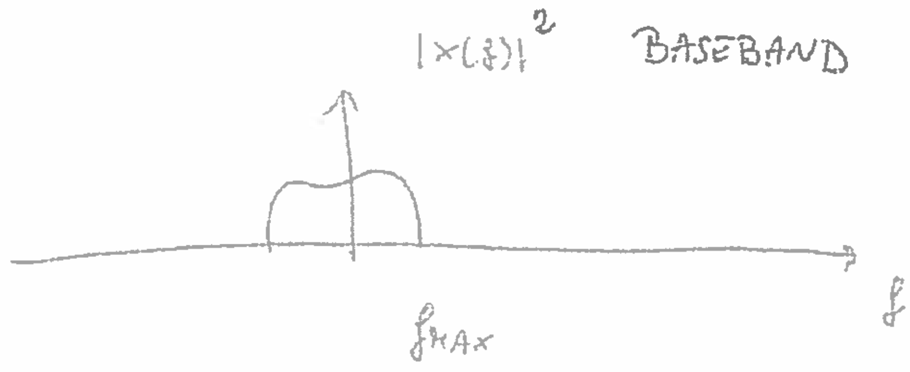
$$0 < f_{min} \leq |f| \leq f_{max} < \infty.$$

A passband signal is said to be passband around $f_0$ provided that

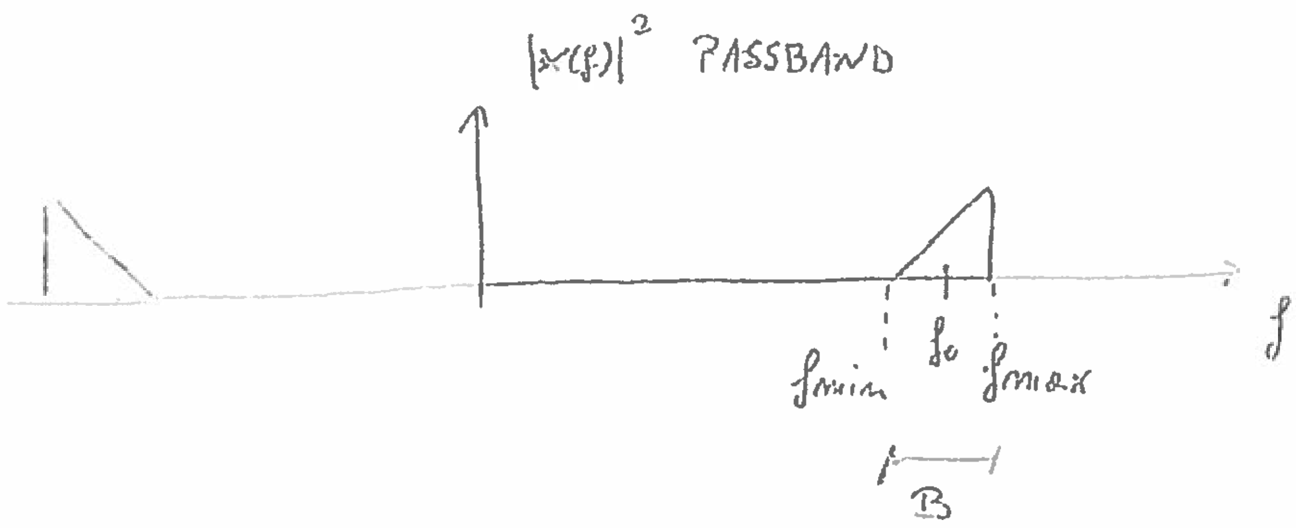
$$f_{min} \leq f_0 \leq f_{max}. \tag{1}$$

Although not strictly necessary, often one takes

$$f_0 = \frac{f_{max} + f_{min}}{2}.$$



|x(f)|²
BASEBAND
f
fmax

|x(f)|²
PASSBAND
f
fmin
f0
fmax
B

## Real Signals

Signals from the real world are real functions of time. The Fourier transform of a real signal $x(t)$ is Hermitian, that is

$$X^*(-f) = X(f).$$

Since the spectrum on the negative frequencies is obtained from the spectrum on the positive frequencies, all the properties of the signal can be expressed by looking to the positive frequencies only.

## Bandwidth of Real Signals

The bandwidth is a notion referred to the positive frequencies only. The bandwidth of a real baseband signal is the maximum frequency contained in the spectrum:

$$B = f_{max}.$$

The bandwidth of a real passband signal is

$$B = f_{max} - f_{min}.$$

## Upconversion in Time Domain

A passband signal is a sinusoid modulated in amplitude and phase

$$s_{pb}(t) = \sqrt{2}a(t)cos(2\pi f_0 t + \phi(t)).$$

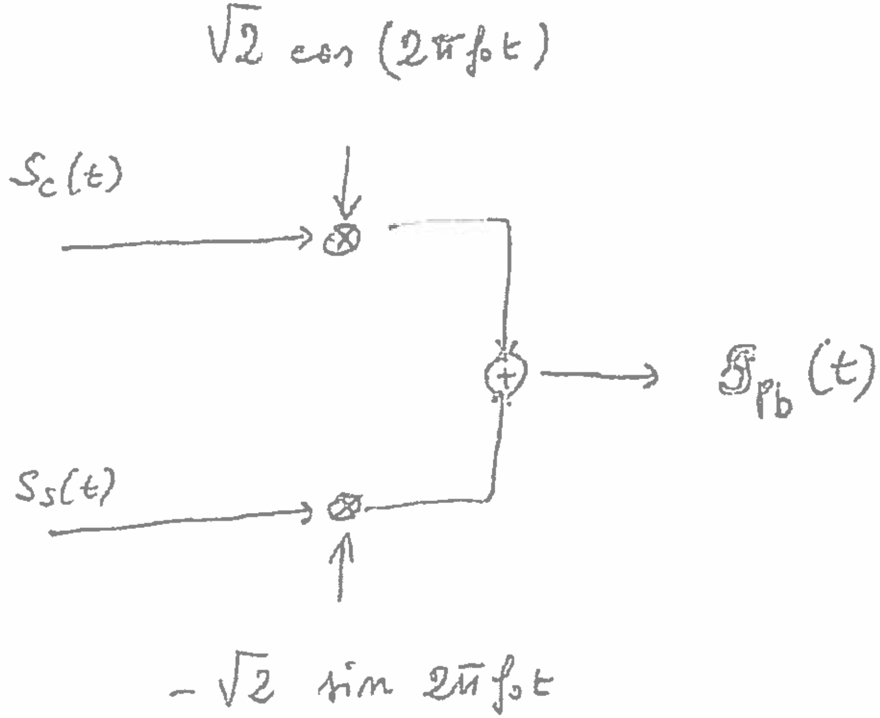
By defining

$$s_c(t) = a(t)cos(\phi(t)), \;\; s_s(t) = a(t)sin(\phi(t)),$$

one writes

$$s_{pb}(t) = \sqrt{2}s_c(t)cos(2\pi f_0 t) - \sqrt{2}s_s(t)sin(2\pi f_0 t), \qquad (2)$$

where $s_c(t)$ and $s_s(t)$ are real signals. If the spectrum of the passband signal contains energy (power) at $f = f_0$, then at least one among $s_c(t)$ and $s_s(t)$ is a baseband signal. In the technical language, upconversion (translation from two baseband signals to one passband signal) of $s_c(t)$ and $s_s(t)$ according to (2) is called *I/Q modulation*.


$\sqrt{2}\cos(2\pi f_0 t)$
$S_c(t)$
$S_s(t)$
$S_{pb}(t)$
$-\sqrt{2}\sin 2\pi f_0 t$

## Complex Envelope

The baseband equivalent model of a passband signal, also called *complex envelope*, is a mathematical model of a passband signal. It is a complex function of time defined as

$$s(t) = s_c(t) + js_s(t) = a(t)e^{j\phi(t)}.$$

The Fourier transform of the complex envelope is

$$S(f) = S_c(f) + jS_s(f).$$

Note that since, in the general case, $s(t)$ is not a real signal, $S(f)$ is not Hermitian, therefore the spectrum on the negative frequencies cannot be obtained from the spectrum on the positive frequencies. In other words, both the negative and the positive frequencies contribute to define the complex envelope in a nontrivial manner.

## Upconversion in Frequency Domain

Taking the Fourier transform of both sides of (2) one has

$$\begin{aligned} S_{pb}(f) &= \frac{1}{\sqrt{2}}(S_c(f-f_0) + S_c(f+f_0) + jS_s(f-f_0) - jS_s(f+f_0)) \\ &= \frac{1}{\sqrt{2}}(S(f-f_0) + S^*(f+f_0)). \end{aligned}$$

By straightforward use of the Hermitian property of $S_c(f)$ and $S_s(f)$, it can be shown that $S^*(f+f_0) = S^*(-f-f_0)$, therefore

$$S_{pb}(f) = \frac{1}{\sqrt{2}}(S(f-f_0) + S^*(-f-f_0)).$$

Note that we have to take

$$f_0 > B/2, \tag{3}$$

where $B$ is the bandwidth of the passband signal, to guarantee that the spectrum of the passband signal is zero at $f = 0$.

## Energy Spectral Density

The energy of the complex envelope is equal to the energy of the passband signal:

$$E_{pb} = E = E_c + E_s.$$

For the energy spectral density we have

$$|S_{pb}(f)|^2 = \frac{1}{2}|S(f - f_0)|^2 + \frac{1}{2}|S(f + f_0)|^2,$$

$$|S(f)|^2 = 2|S_{pb}(f + f_0)|^2, \quad f_{min} - f_0 \leq f \leq f_{max} - f_0.$$

We can say that half of the energy spectral density of the complex envelope is translated to the positive frequencies, half is frequency-reversed and translated to the negative frequencies. Note that, since $s_{pb}(t)$ is a real signal, its energy spectral density is real and even, while the energy spectral density of the complex envelope is real but not necessarily even. Power spectral densities are treated in a similar way.

## Downconversion

Note that $S(f)$ is obtained by translating around $f = 0$ the portion of $S_{pb}(f)$ that insists on the positive frequencies:

$$S(f) = \sqrt{2} S_{pb}(f + f_0), \quad f_{min} - f_0 \leq f \leq f_{max} - f_0. \tag{4}$$

In other words, $S(f)$ is the Fourier transform of the lowpass portion of $\sqrt{2} s_{pb}(t) e^{-j2\pi f_0 t}$:

$$s(t) = \text{lowpass portion of } \{\sqrt{2} s_{pb}(t) e^{-j2\pi f_0 t}\},$$

$$s_c(t) = \text{lowpass portion of } \{\sqrt{2} s_{pb}(t) cos(2\pi f_0 t)\},$$

$$s_s(t) = \text{lowpass portion of } \{-\sqrt{2} s_{pb}(t) sin(2\pi f_0 t)\}.$$

In the technical language, downconversion (translation of a passband signal to two baseband signals) of $s_{pb}(t)$ according to the last two equations is called *I/Q demodulation*.

## Downconversion

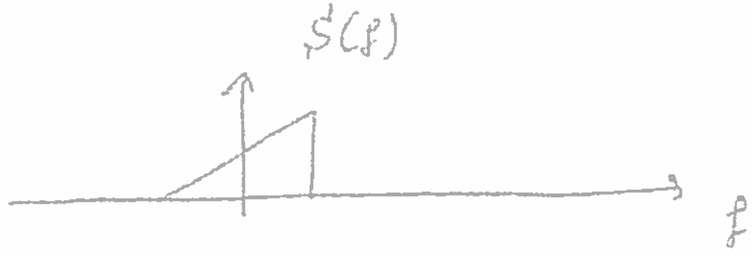
Multiplying the passband signal around $f_0$ by a sinusoid at frequency $f_0$, one obtains a signal whose spectrum can be seen as the sum of two portions: one portion of the spectrum is around $f = 0$, another portion of the spectrum is around $f = 2f_0$. To guarantee that the baseband signal can be reconstructed by the low-pass filter, one has to guarantee that the portion around $f = 0$ does not overlap with the portion around $f = 2f_0$. This is guaranteed when both (1) and (3) are fulfilled, that is when

$$f_{max} \geq f_0 \geq \min\left\{\frac{B}{2}, f_{min}\right\}, \tag{5}$$

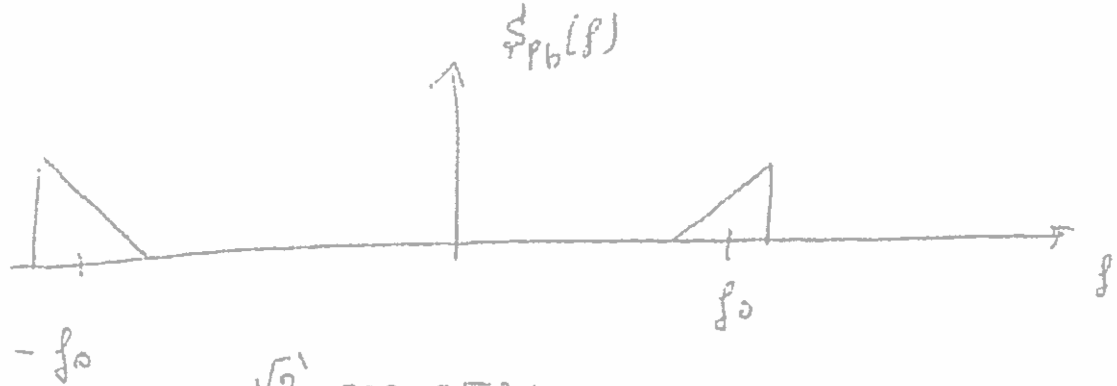
where $B$ is the bandwidth of the passband signal. When the above condition is satisfied, downconversion is the inverse of upconversion:

$$s(t) = D(U(s(t)),$$

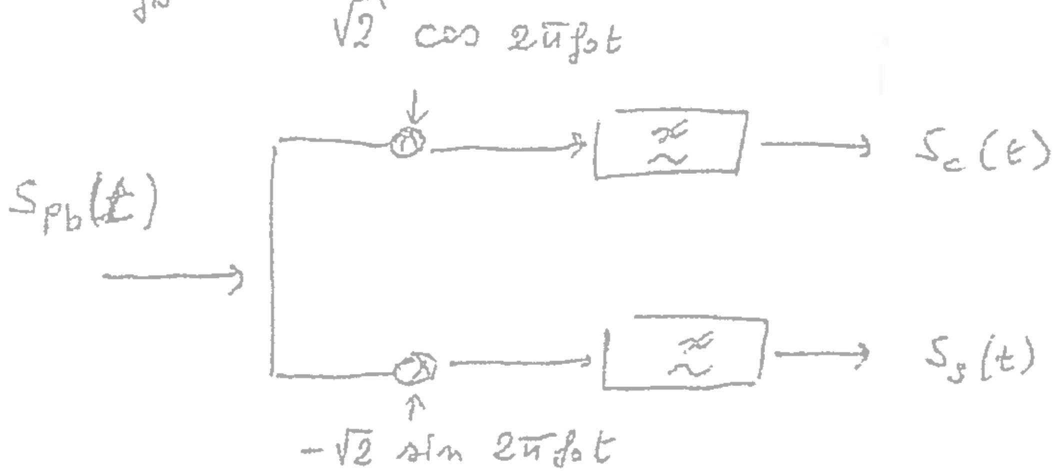
$$s_{pb}(t) = U(D(s_{pb}(t)).$$



$\underline{S}(f)$
$f$

$\underline{S}_{pb}(f)$
$-f_0$
$f_0$
$f$

$s_{pb}(t)$
$\sqrt{2}\cos 2\pi f_0 t$
$-\sqrt{2}\sin 2\pi f_0 t$
$s_c(t)$
$s_s(t)$

## Filtering

The Fourier transform of the complex envelope of a filtered signal can be found as follows.

Start from the passband signal after the filter:

$$\begin{aligned} Y_{pb}(f) &= H_{pb}(f)X_{pb}(f) \\ &= \frac{1}{2}(H(f-f_0)+H^*(f+f_0))(X(f-f_0)+X^*(f+f_0)) \\ &= \frac{1}{2}(H(f-f_0)X(f-f_0)+H^*(f+f_0)X^*(f+f_0)) \\ &= \frac{1}{\sqrt{2}}(Y(f-f_0)+Y^*(f+f_0)), \end{aligned}$$

where

$$Y(f) = \frac{1}{\sqrt{2}}H(f)X(f).$$

## Filtering

Let

$$Y(f) = \frac{1}{\sqrt{2}} H(f)X(f) = Y_c(f) + jY_s(f),$$

then

$$\sqrt{2}Y_c(f) = H_c(f)X_c(f) - H_s(f)X_s(f),$$
$$\sqrt{2}Y_s(f) = H_c(f)X_s(f) + H_s(f)X_c(f),$$
$$\sqrt{2}y_c(t) = h_c(t) \otimes x_c(t) - h_s(t) \otimes x_s(t),$$
$$\sqrt{2}y_s(t) = h_c(t) \otimes x_s(t) + h_s(t) \otimes x_c(t),$$

where $\otimes$ denotes convolution. When $H(f)$ is Hermitian, the passband frequency response $H_{pb}(f)$ is Hermitian around $f_0$, hence frequency above $f_0$ and below $f_0$, that are mapped onto positive and negative frequencies, are treated in the same way. In this case $h(t)$ is real, that is $h(t) = h_c(t)$, and there is no crosstalk between the I/Q signals. Elsewhere, asymmetries of $H_{pb}(f)$ with respect to $f_0$ cause crosstalk through $h_s(t)$.

## Passband Signal and Complex Envelope: Example 1

Consider the sinusoid

$$s_{pb}(t) = \sqrt{2P} cos(2\pi f_0 t + \phi) = \sqrt{2P}\Re\{e^{j\phi}e^{j2\pi f_0 t}\}.$$

The complex envelope is the phasor

$$s(t) = \sqrt{P}e^{j\phi}.$$

For the power spectral density one has

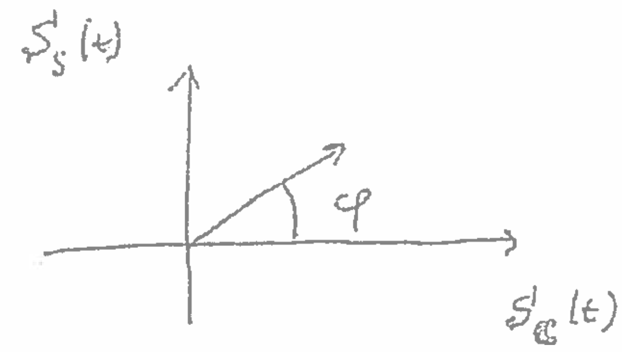
$$\Psi_{s_{pb}}(f) = \frac{P}{2}\delta(f - f_0) + \frac{P}{2}\delta(f + f_0),$$

$$\Psi_s(f) = P\delta(f).$$

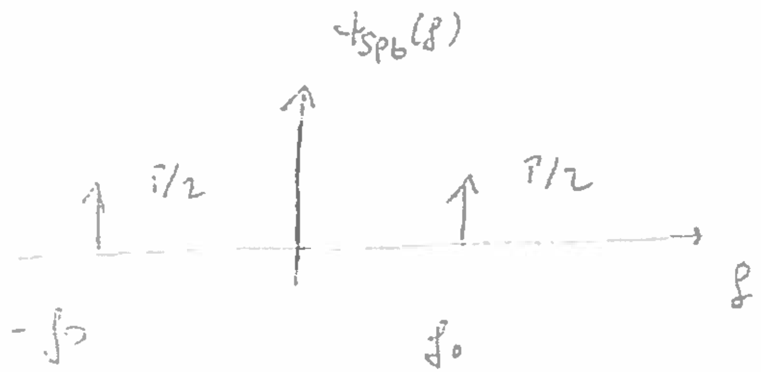
Note that

$$P_s = P\int_{-\infty}^{\infty} \delta(f)df = P.$$

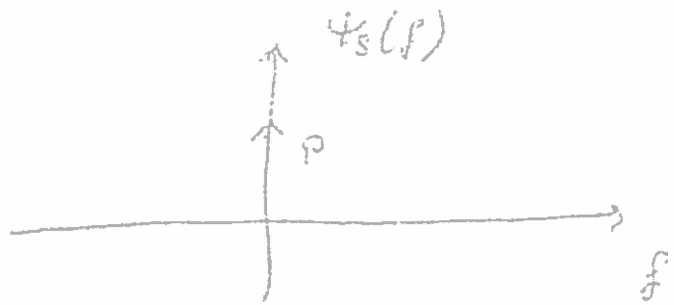
cos(2πf0t + φ )
t

s's(t)
s'c(t)
φ

Ψspb(f)
P/2
P/2
f
-f0
f0

Ψs(f)
P
f

## Passband Signal and Complex Envelope: Example 2

Consider the sum of two sinusoids

$$s_{pb}(t) = \sqrt{2P_1}cos(2\pi(f_0 + f_1)t + \phi_1) + \sqrt{2P_2}cos(2\pi(f_0 - f_2)t + \phi_2),$$

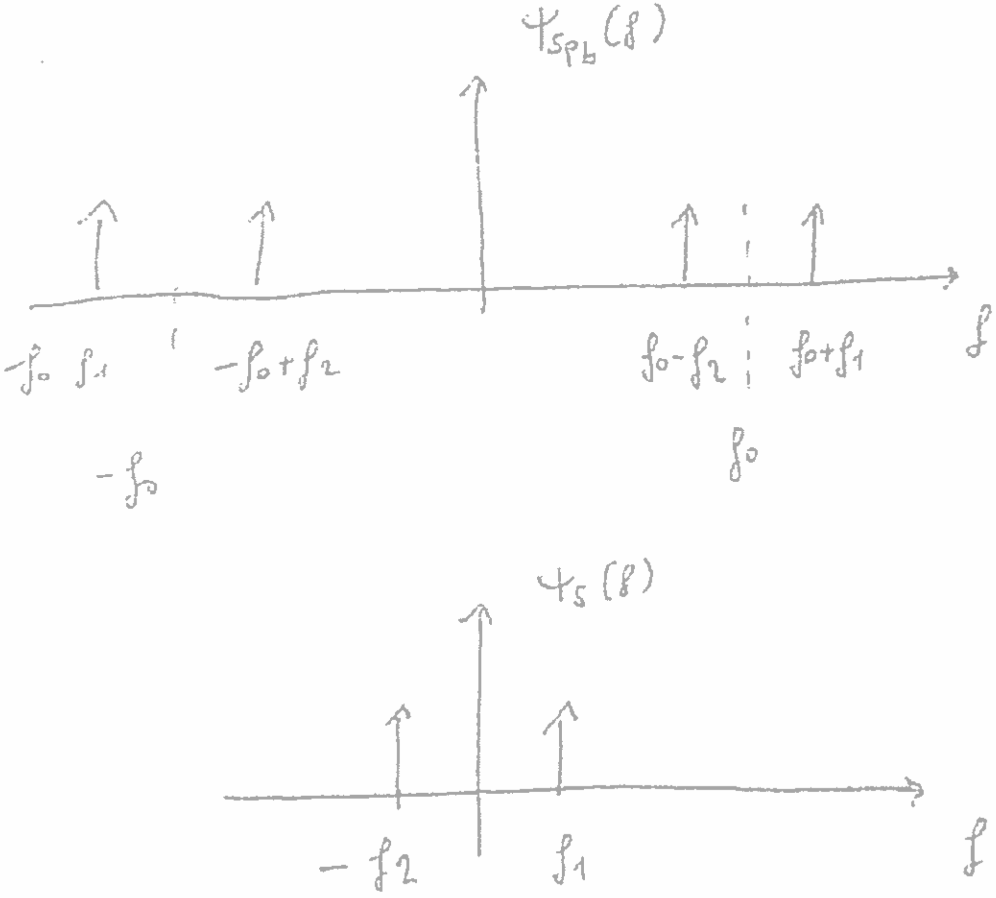
with

$$f_1 \geq 0, \ \ f_2 \geq 0.$$

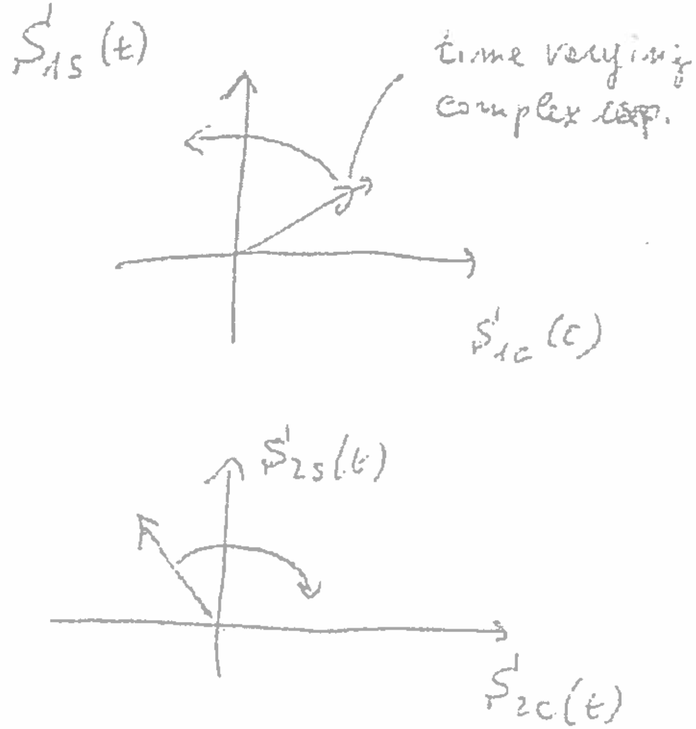
The complex envelope is the sum of two rotating phasors:

$$s(t) = \sqrt{P_1}e^{j(2\pi f_1 t + \phi_1)} + \sqrt{P_2}e^{j(-2\pi f_2 t + \phi_2)}.$$

This example is very useful to catch the meaning of negative frequencies in the Fourier transform of the complex envelope: they represent the frequencies below $f_0$ of the passband signal.




Positive frequency
rotation counterclockwise
Negative frequency
rotation clockwise



## Passband Signal and Complex Envelope: Example 3

Consider complex upconversion around $f_0$

$$s_{pb}(t) = \sqrt{2}s_c(t)cos(2\pi f_0 t) - \sqrt{2}s_s(t)sin(2\pi f_0 t),$$

and successive complex downconversion with a small frequency error $\Delta f$, such that the frequency shift of equation (4) is $f_0 + \Delta f$ towards left, leading to

$$\tilde{S}(f) = \sqrt{2}S_{pb}(f+(f_0+\Delta f)), \quad f_{min}-(f_0+\Delta f) \leq f \leq f_{max}-(f_0+\Delta f), \quad (6)$$

$$\tilde{s}(t) = \text{lowpass portion of } \{\sqrt{2}s_{pb}(t)e^{-j2\pi(f_0+\Delta f)t}\}$$
$$= (s_c(t) + js_s(t))e^{-j2\pi\Delta f t},$$

where $\tilde{s}(t)$ ($\tilde{S}(f)$) indicates the recovered $s(t)$ ($S(f)$) affected by the frequency error in downconversion.

## Passband Signal and Complex Envelope: Example 3

For instance, consider $s(t) = A$. One has

$$\tilde{s}(t) = Ae^{-j2\pi\Delta f t},$$

that is a rotating phaser with rotation frequency $\Delta f$. If $\Delta f > 0$, then the frequency shift from passband to baseband is too large as it is clear from the argument of the right side of (6). The result is that the continuous component of $s(t)$ is mapped onto a negative frequency (clockwise rotating phasor). If $\Delta f < 0$, then the frequency shift from passband to baseband is too small, and the continuous component of $s(t)$ is mapped onto a positive frequency (counterclockwise rotating phasor).