Chapter 6: MINIMUM VARIANCE CONTROL

("Design and Analysis of feedback systems")

NOT SY. ID.
NOT SW. SENSING

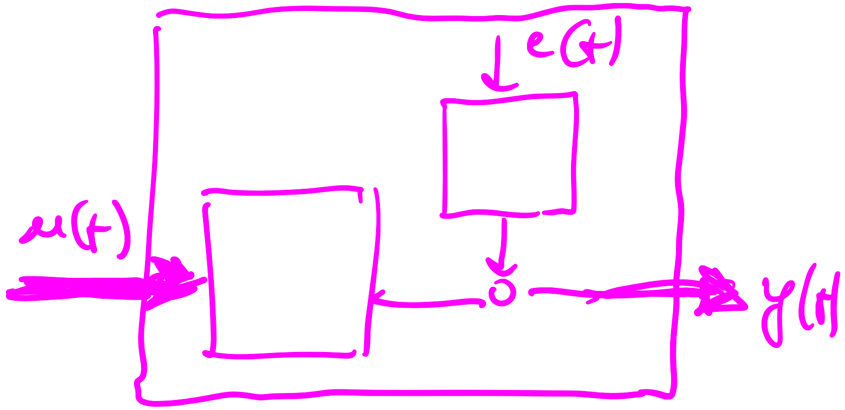
- CONTROL DESIGN is the main MOTIVATION TO SY ID AND SW. SENSING
- MVC is BASED on "MATHEMATICS" of SY. ID AND SW. SENSING.
  (PREDICTION theory)

Setup of the problem:

Consider a generic ARMAX model:

I/O part of model — noise model

$$y(t) = \boxed{\frac{B(z)}{A(z)}\, u(t-k)} + \boxed{\frac{C(z)}{A(z)}\, e(t)} \qquad e \sim WN(0, \lambda^2)$$



$$B(z) = b_0 + b_1 z^{-1} + \dots + b_p z^{-p}$$

$$A(z) = 1 + a_1 z^{-1} + \dots + a_m z^{-m}$$

$$C(z) = 1 + c_1 z^{-1} + \dots + c_n z^{-n}$$

- $C(z)/A(z)$ is in canonical form

Assumptions:

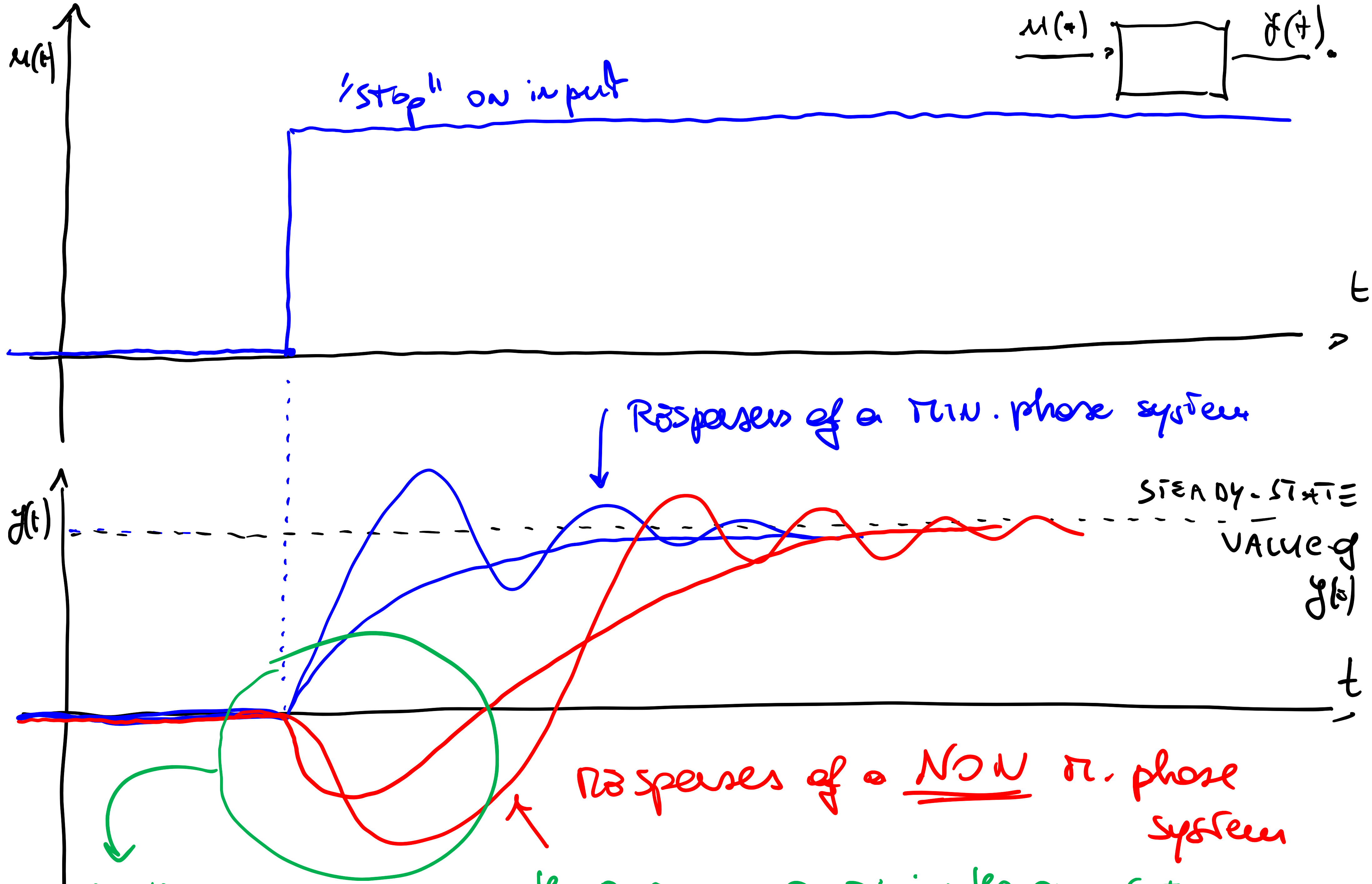
- $b_0 \neq 0$ ⇒ means that k is the actual delay of the system
- $\boxed{B(z)/A(z) \text{ is "minimum phase"}}$

$\frac{B(z)}{A(z)}$ is said to be "min. phase" iff all the roots of $B(z)$ (NUMERATOR) are strictly inside the unit circle

NUM → zeros → min. phase

Denom → poles → stability

FEATURE of a NON-min. phase system →



At the very beginning the response goes in the opposite direction w.r.t. the final VALUE !!!

Intuitively → very difficult to control non-min. phase systems ⇒ you can take the wrong decision if you react immediately!

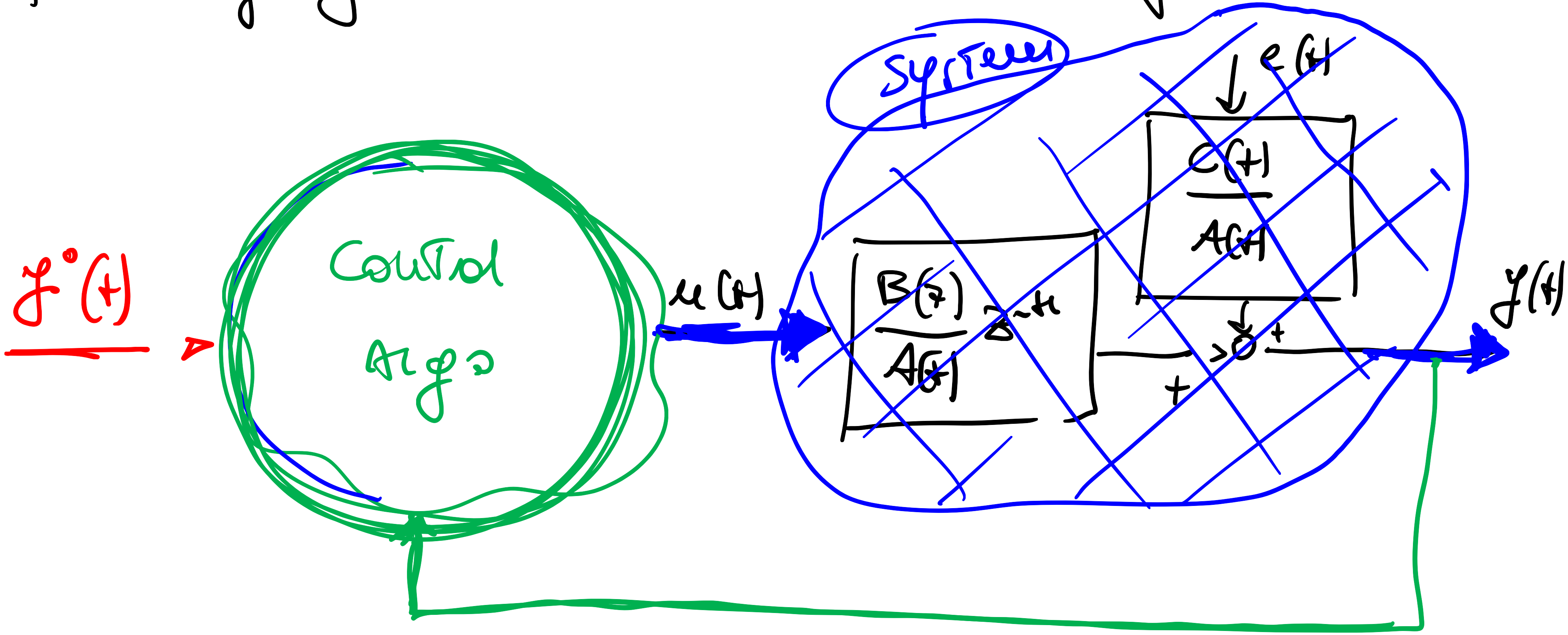
Also for a human is difficult →

ex: STEER → Roll dynamics in a bicycle

Design of controllers for non m. ph. is difficult → requires special Design Techniques

(NO ~~MVC~~ but GENERALIZED M.V.C)

The problem we wish to solve is optimal Tracking of the desired behaviour of output:



In a more formal way → M.V.C.

Tries to minimize this perf. index:

$$\boxed{J = E\left[\left(y(t) - y^\circ(t)\right)^2\right]}$$

→ VARIANCE of the TRACKING ERROR

↓

called

"MINIMUM VARIANCE CONTROL"

Some additional (small) technical assumptions:

- $y^\circ(t)$ and $e(t)$ are not correlated $\left(y^\circ(t) \perp e(t)\right)$ (usually easily fullfilled)

- we assume that $y^\circ(t)$ is known only up to time t (present time) ⇒ we have NO preview of future desired $y^\circ(t) \Rightarrow$

$y^\circ(t)$ is totally UN-PREDICTABLE $\Rightarrow$

$$\hat{y}^\circ(t+k/t) = y^\circ(t)$$

at present time t
best prediction
is simply $y^\circ(t)$

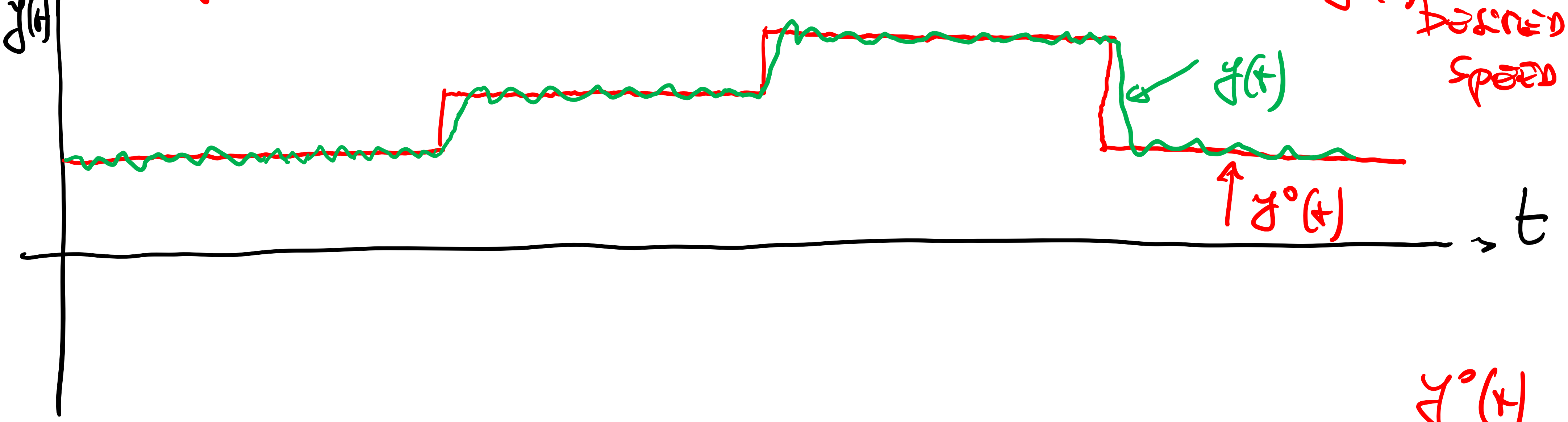
Remark: there are 2 sub-classes of control problems:

1) when $y^\circ(t)$ is CONSTANT OR STEP-wise $\Rightarrow$ "REGULATION problem"

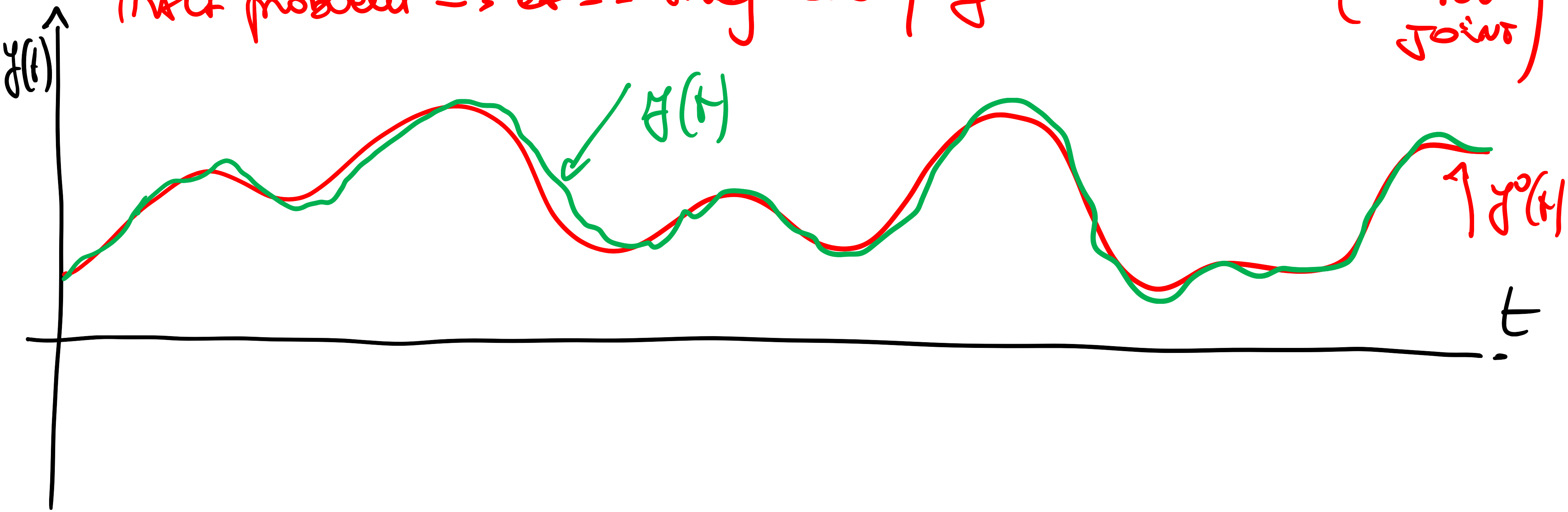
2) when $y^\circ(t)$ is varying $\rightarrow$ "Tracking problem"

$\boxed{\text{(1) is a simplified problem 2)}}$

REG problem → es → "cruise control" in a car → $y^\circ(t)$ is the DESIRED SPEED



TRACK problem → es → Trajectory of a robot arm (ANGLE JOINT) $y^\circ(t)$



Bottom-up way of presenting M.V. theory →

simplified problem #1
simplified problem #2
-- $\boxed{\text{GENERAL SOLUTION}}$

Simplified problem #1:

ROOT of NUMERATOR must be inside UNIT circle

$$\mathcal{S}: y(t) = a\,y(t-1) + b_0 u(t-1) + b_1 u(t-2)$$

"noise free"

$$\hookrightarrow y(t) = \frac{\boxed{b_0 + b_1 z^{-1}}}{1 - a z^{-1}} u(t-1) + \cancel{\bigcirc}$$

Then we assume that $y^\circ(t) = \bar{y}^\circ$ (⇒ REGULATION problem)

$\boxed{b_0 \text{ must be } \neq 0}$

To design the M.V. controller we must minimize →

$$J = E\left[\left(y(t) - y^{\circ}(t)\right)^2\right]$$

There is no noise ⇒ we can remove "E"

$$J = \left(y(t) - y^{\circ}(t)\right)^2$$

since $y^{\circ}(t) = \bar{y}^{\circ}$

$$J = \left(y(t) - \bar{y}^{\circ}\right)^2$$

plugging in J expression

$$J = \left(a\,y(t-1) + b_0\,u(t-1) + b_1\,u(t-2) - \bar{y}^{\circ} \right)^2$$

time shift:

$$J = \left(a\,y(t) + \underbrace{b_0\,\boxed{u(t)} + b_1\,u(t-1)} - \bar{y}^{\circ} \right)^2$$

$$\longrightarrow b_0 u(t) + b_1 u(t-1) = \underbrace{\left(b_0 + b_1 z^{-1}\right) u(t)}$$

DERIVATIVE w.r.t. $u(t)$:

$$\frac{\partial J}{\partial u(t)} = 2\left(a\,y(t) + b_0\,u(t) + b_1\,u(t-1) - \bar{y}^{\circ} \right) \cdot \left(\overbrace{b_0}^{\text{this should be } \neq 0 \text{ hp}} \right)$$

? • b_0

• $b_0 \cancel{+ b_1 z^{-1}}$

why the derivative is just b_0 ?

→ we are at present time t

→ at time t the control algo must take a decision on the value of $u(t)$

→ at time t:

$y(t), y(t-1), y(t-2) \ldots$

$u(t-1), u(t-2), \ldots$

are no longer variables but numbers!

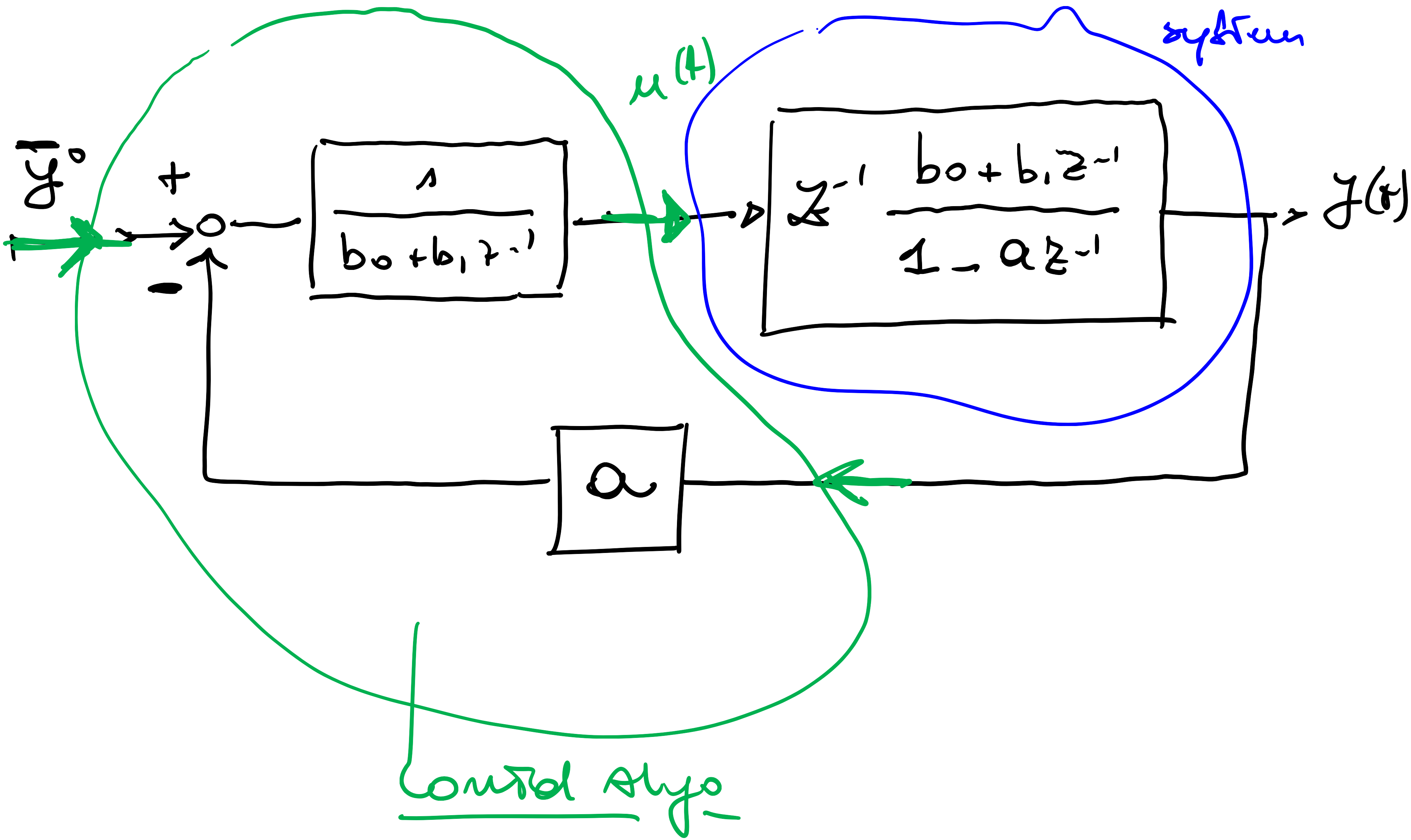
→ no longer the object of optimization

$$\frac{\partial J}{\partial u(t)} = 0 \Rightarrow a y(t) + b_0 u(t) \cancel{+ b_1 u(t-1)} - \bar{y}^o = 0$$

$$\Rightarrow u(t) = \boxed{\left(\bar{y}^o - a y(t)\right) \cdot \frac{1}{b_0 + b_1 z^{-1}}}$$

CONTROL ALGO

plot Block scheme



$\bar{y}^o$
+
-
$\frac{1}{b_0+b_1 z^{-1}}$
u(t)
system
$z^{-1}\frac{b_0+b_1 z^{-1}}{1-a z^{-1}}$
y(t)
a
Control Algo

Simplified ex. # 2

$$\mathcal{S}: y(t) = a\,y(t-1) + b_0\,u(t-1) + b_1\,u(t-2) + e(t) \quad e \sim WN(0,1^2)$$

$k=1$

the reference variance is $y^\circ(t)$ (Tracking problem)

Assumptions: $b_0 \neq 0$

$(b_0 + b_1 z^{-1})$ is min. phase

performance index $\longrightarrow$ $J = E\left[\left(y(t) - y^\circ(t)\right)^2\right]$

$\hookrightarrow$ the fundamental "TRICK" to solve this problem is to re-write $y(t)$ as:

$$\boxed{y(t) = \underbrace{\hat{y}(t/t-1)}_{\substack{\text{prediction of } y(t) \\ \text{at time } t-1}} + \underbrace{\varepsilon(t)}_{\substack{\text{corresponding} \\ \text{prediction error}}}}$$

since $\underline{k=1}$ $\rightarrow$ $\varepsilon(t) = e(t) \Rightarrow \underbrace{y(t) = \hat{y}(t/t-1) + e(t)}$

plug-in the perf. index

$$J = E\left[\left(\hat{y}(t/t-1) + e(t) - y^{o}(t)\right)^{2}\right]$$

$$= E\left[\left(\underline{\left(\hat{y}(t/t-1) - y^{o}(t)\right)} + \underline{e(t)}\right)^{2}\right]$$

Expand:

$$J = E\left[\left(\hat{y}(t|t-1) - y^{o}(t)\right)^2\right] + \underbrace{E\left[e(t)^2\right]}_{\lambda^2 \text{ (constant)}} + \underbrace{\cancel{2E\left[\overset{\perp \text{ by assumption}}{e(t)} \cdot \left(\overset{\perp \text{ construction}}{\hat{y}(t|t-1)} - y^{o}(t)\right)\right]}}_{=\,\emptyset}$$

Notice: $\underset{u(t)}{\operatorname{argmin}}\left\{\underbrace{E\left[\left(\hat{y}(t|t-1) - y^{o}(t)\right)^2\right] + \lambda^2}_{J}\right\} = \boxed{\underset{u(t)}{\operatorname{argmin}}\left\{E\left[\left(\hat{y}(t|t-1) - y^{o}(t)\right)^2\right]\right\}}$

BECAUSE λ^2 is a CONSTANT that does not DEPEND on $u(t)$

the best result is when $\boxed{\hat{y}(t|t-1) = y^{\circ}(t)}$

Force this relationship

Now we must compute the 1-step prediction for $\mathcal{S}$:

$$\mathcal{S}: \quad y(t) = \frac{b_0 + b_1 z^{-1}}{1 - a z^{-1}} u(t-1) + \frac{1}{1 - a z^{-1}} e(t)$$

$$ARMAX(1, \emptyset, 1+1) = ARX(1,2)$$

$K=1 \quad B(z) = b_0 + b_1 z^{-1} \quad A(z) = 1 - a z^{-1} \quad C(z) = 1$

General solution for 1-step prediction of ARMAX:

$$\boxed{\hat{y}(t|t-1) = \frac{B(z)}{C(z)}u(t-1) + \frac{C(z)-A(z)}{C(z)}y(t)}$$

apply this formula:

$$\hat{y}(t|t-1) = \frac{b_0+b_1z^{-1}}{1}u(t-1) + \frac{\cancel{1}-\cancel{1}+az^{-1}}{1}y(t)$$

$$\hat{y}(t|t-1) = \left(b_0+b_1z^{-1}\right)u(t-1) + a\,y(t-1)$$

Impose $\hat{y}(t|t-1) = y^{\circ}(t)$ or:

$$\boxed{\hat{y}(t+1|t) = y^{\circ}(t+1)}$$

$$\Downarrow$$

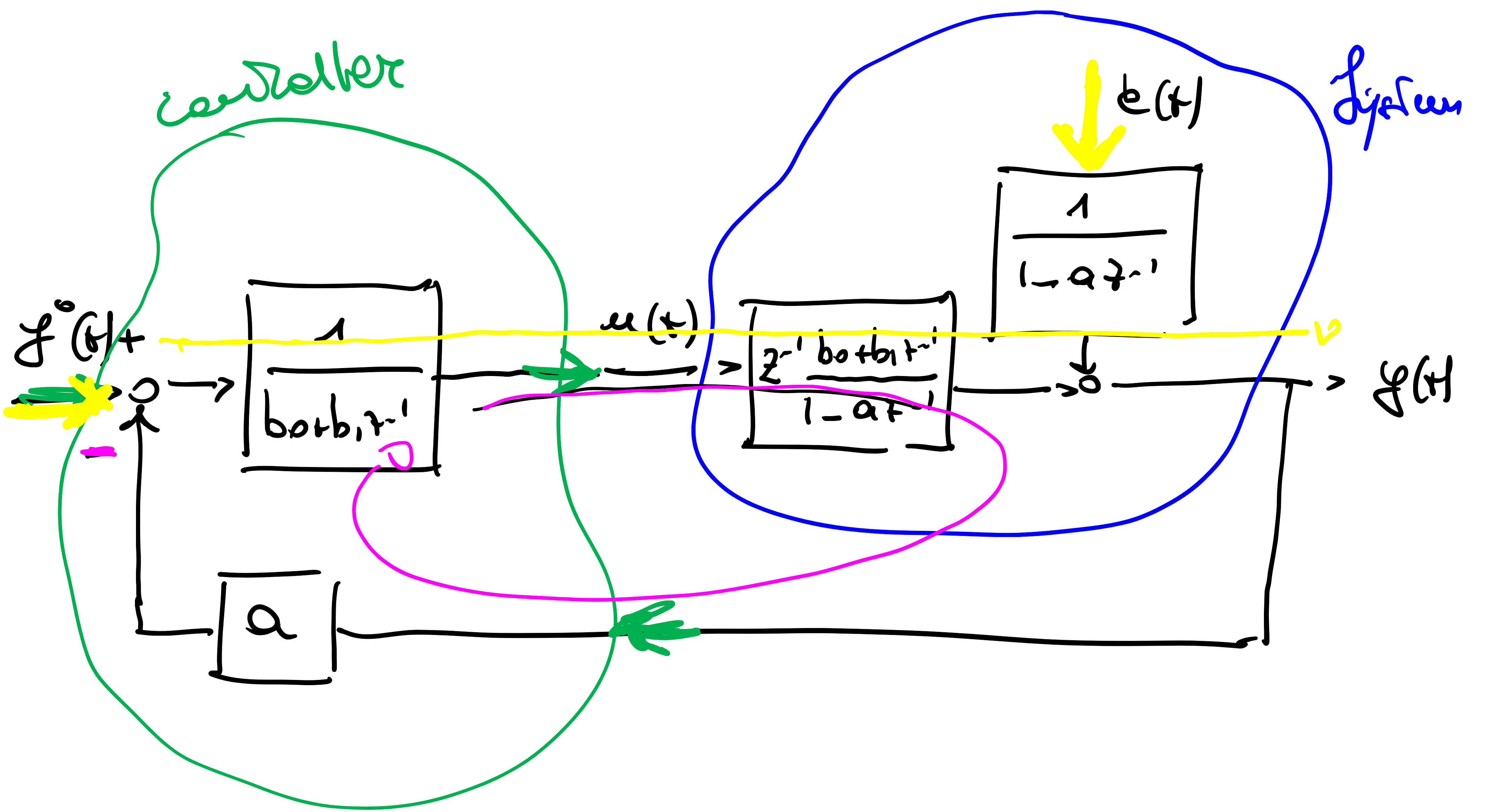
$$b_0 \boxed{u(t)} + b_1 u(t-1) + a y(t) = y^{\circ}(t+1)$$

$$\Downarrow$$

$$u(t) = \left(y^{\circ}(t+1) - a y(t)\right) \cdot \frac{1}{b_0 + b_1 z^{-1}}$$

NOT AVAILABLE at time t (No preview) $\rightarrow$ Replace with last AVAILABLE $\rightarrow y^{\circ}(t)$

$$\rightarrow u(t) = \left(y^{o}(t) - a\, y(t) \right) \frac{1}{b_0 + b_1 z^{-1}}$$



Analysis of this control system → [STABILITY? PERFORMANCE?]

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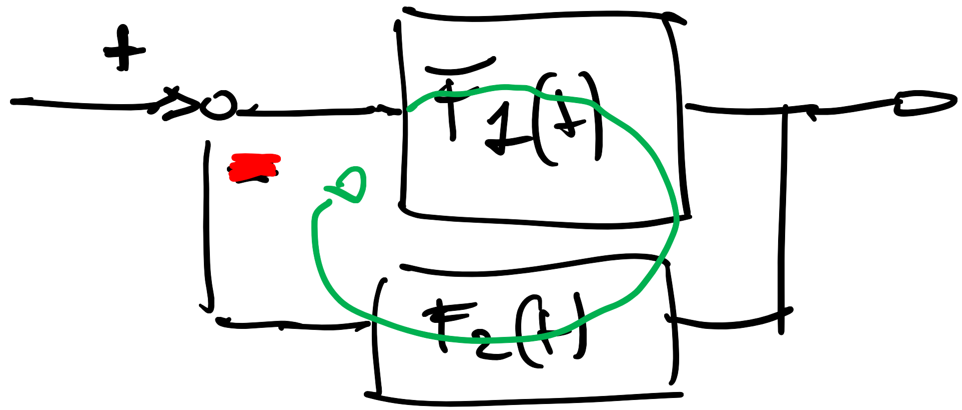
for stability LET'S RECALL a result on feedback system:



Assumption: NEG. feedback

To check the closed-loop stability:

- compute the "LOOP function": $L(z) = F_1(z) \cdot F_2(z)$

- Build the "characteristic polynomial":

$$\chi(z) = L_N(z) + L_D(z)$$

$\uparrow$ Num $\qquad \uparrow$ Den

- Find the roots of $\chi(z)$ $\Rightarrow$ closed loop system is Asy. STABLE iff All the roots of $\chi(z)$ are STRICTLY inside u. circle

$\downarrow$ Apply to our system:

$$L(z) = \frac{1}{b_0 + b_1 z^{-1}} \cdot \frac{z^{-1}(b_0 + b_1 z^{-1})}{1 - a z^{-1}} \cdot a$$

DO NOT simplify!!

$$\cancel{X}(z) = a z^{-1}(b_0 + b_1 z^{-1}) + (1 - a z^{-1})(b_0 + b_1 z^{-1})$$

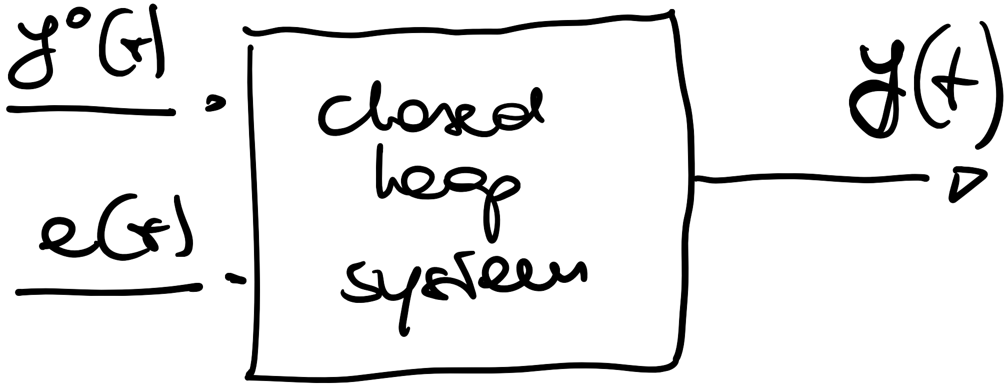
$$= (b_0 + b_1 z^{-1})(\cancel{a z^{-1}} + 1 - \cancel{a z^{-1}}) = \underline{\underline{(b_0 + b_1 z^{-1})}}$$

$$B(z)$$

$\Rightarrow$ Closed loop system is As. STABLE thanks to $\underline{\text{MIN. phase}}$

$\underline{\underline{\text{Assumption}}}$!

performance analysis:



Since the system is L.T.I. we can use the "super-position principle"

$$y(t) = \underbrace{F_{y^\circ y}(z)}_{\text{T.F. from } y^\circ \text{ to } y}\, y^\circ(t) + \underbrace{F_{ey}(z)}_{\text{T.F. from } e \text{ to } y}\, e(t)$$

Compute:

DIRECT LINE from input to output

$$F_{y\hat{y}}(z) = \frac{\boxed{\dfrac{1}{b_0+b_1 z^{-1}} \cdot \dfrac{z^{-1}\left(b_0+b_1 z^{-1}\right)}{1-a_1 z^{-1}}}}{1 + \boxed{\dfrac{1}{b_0+b_1 z^{-1}}\, z^{-1}\, \dfrac{b_0+b_1 z^{-1}}{1-a z^{-1}} \cdot a}} = \dots = z^{-1}$$

Loop function

NEG feedback

$$F_{ey}(z) = \frac{\dfrac{1}{1-a z^{-1}}}{1 + \boxed{\text{Loop function}}} = \dots = 1$$

Notice that the closed loop system has a very simple closed-loop behaviour:

$$y(t) = z^{-1} y^o(t) + 1 \cdot e(t)$$

$$\boxed{y(t) = y^o(t-1) + e(t)}$$

→ $y(t)$ follows exactly $y^o(t)$ (but with 1-step delay), disturbed by noise $e(t)$

→ BEST possible solution! (optimal control)

General solution to M.V control problem

$$\mathcal{S}: \; y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t) \qquad e \sim WN(0, \lambda^2)$$

- $b_0 \neq 0$
- $B(z)$ has all roots strictly inside U.C. ($\mathcal{S}$ is min phase)
- $C(z)/A(z)$ is in canonical representation
- $y^0(t) \perp e(t)$
- $y^0(t)$ is un-predictable $\dashrightarrow \hat{y}^0(t+k|t) = \boxed{y^0(t)}$
  (NO-PREVIEW)

GOAL → minimize $J = E\left[\left(\underbrace{y(t) - y^o(t)}_{\text{Tracking error}}\right)^2\right]$

"Trick" → $\underbrace{y(t) = \hat{y}(t/t-k) + \varepsilon(t)}$

$$J = E\left[\left(\hat{y}(t/t-k) + \varepsilon(t) - y^o(t)\right)^2\right] =$$

$$= E\left[\left(\left(\hat{y}(t/t-k) - y^o(t)\right) + \varepsilon(t)\right)^2\right]$$

x by ASS.

x by constr.

$$= E\left[\left(\hat{y}(t/t-k) - y^o(t)\right)^2\right] + E\left[\varepsilon(t)^2\right] + \cancel{2E\left[\varepsilon(t)\left(\hat{y}(t/t-k) - y^o(t)\right)\right]}$$

$$= E\left[\left(\hat{y}(t|t-k) - \hat{y}^{o}(t)\right)^2\right] + \underbrace{\text{Function of (only) } e(t) \Rightarrow \text{ DOES NOT depend on } \underline{\underline{u(t)}}}$$

$\Rightarrow$ optimal solution $\rightarrow$

$$\boxed{\hat{y}(t|t-k) = \hat{y}^{o}(t)}$$

we need to compute $\underline{\underline{\hat{y}(t|t-k)}}$

General formula for ARMAX predictor:

RE_WRITE:

$$\boxed{\frac{C(z)}{A(z)} = E(z) + \frac{\overbrace{\tilde{R}(z) z^{-k}}^{R(z)}}{A(z)}}$$

solution residual

$\underbrace{\qquad\qquad\qquad\qquad}$

L. long Division $C(z)/A(z)$ of k steps

$$\boxed{\hat{y}(t|t-k) = \frac{B(z)E(z)}{C(z)} u(t-k) + \frac{\tilde{R}(z)}{C(z)} y(t-k)}$$

Shift ahead k steps:

$$\hat{y}(t+k|t) = \frac{B(z)\tilde{E}(z)}{C(z)}\boxed{u(t)} + \frac{\tilde{R}(z)}{C(z)} \cdot y(t)$$

Suppose $\hat{y}(t+k|t) = y^{\circ}(t+k)$

At time t we don't know $y^{\circ}(t+k)$ => Replace with $y^{\circ}(t)$

$$\Downarrow$$

$$\hat{y}(t+k|t) = y^{\circ}(\theta)$$

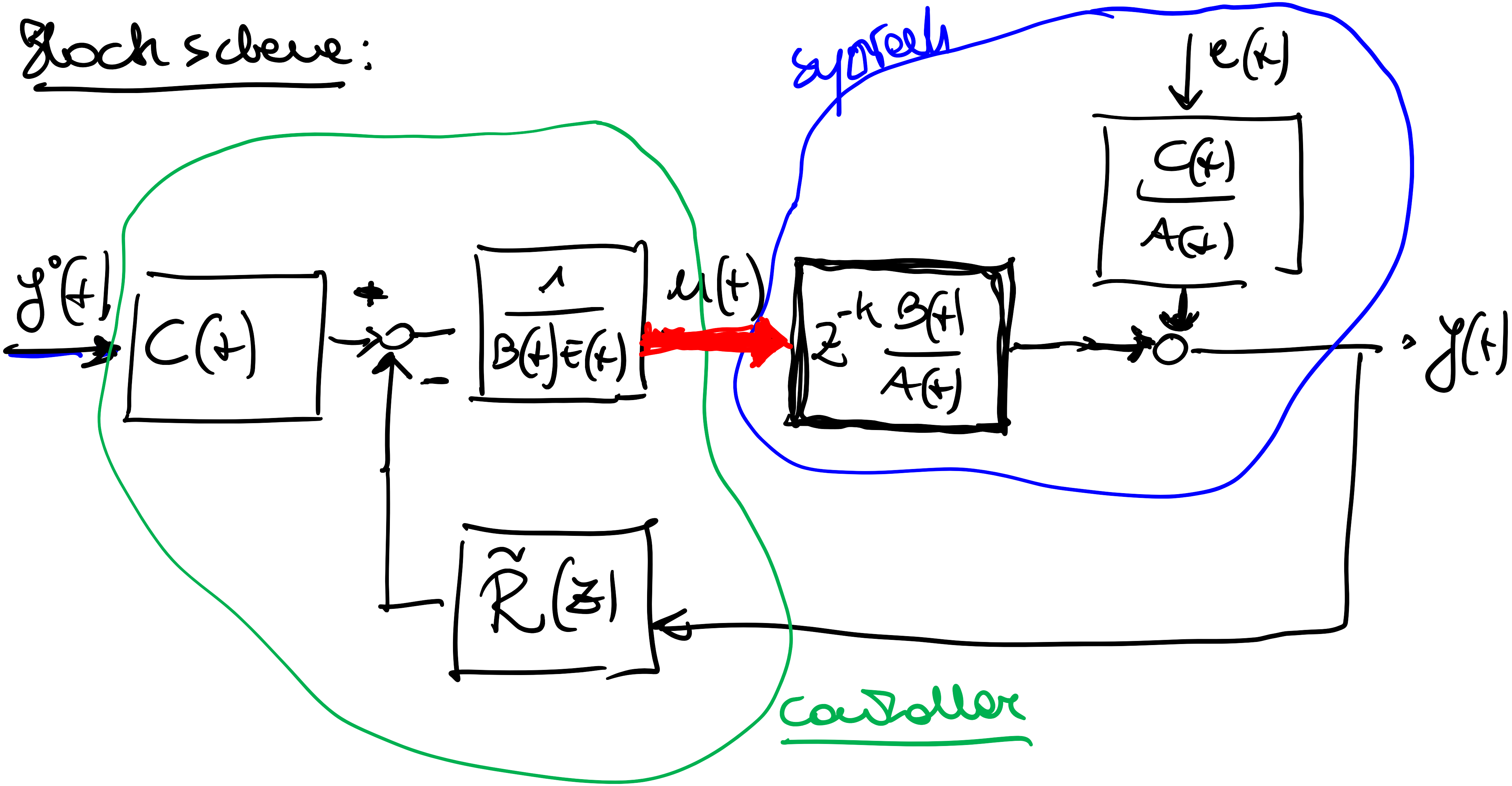
$$\frac{B(z)E(z)}{C(z)}\boxed{u(t)} + \frac{\tilde{R}(z)}{C(z)}y(t) = y^{\circ}(t)$$

make $u(t)$ explicit $\Rightarrow$ <u>CONTROL ALGO:</u>

$$\boxed{u(t) = \frac{1}{B(z)E(z)}\left(C(z)y^{\circ}(t) - \tilde{R}(z)y(t)\right)}$$

MV. control general formula

Block scheme:



Stability check:

No cancellations!!

$$L(z) = \frac{1}{B(z)E(z)} \cdot \frac{z^{-k} B(z)}{A(z)} \cdot \tilde{R}(z)$$

$$\chi(z) = L_N(z) + L_D(z) = z^{-k} B(z) \tilde{R}(z) + B(z)E(z)A(z) =$$

$$= B(z)\underbrace{\left(z^{-k} \cdot \tilde{R}(z) + E(z)A(z)\right)}_{C(z)} = \boxed{B(z) \cdot C(z)} \Longrightarrow$$

System in closed loop is Asy. stable iff:

- All roots of $B(z)$ are stable $\Rightarrow$ Sy is Min. phase ✓
- All roots of $C(z)$ are stable $\Rightarrow$ Sy is in canonical rep. ✓

perf. analysis $\rightarrow$ $y(t) = F_{y^o y}(z)\, y^o(t) + F_{ey}(z)\, e(t)$

$$F_{y^o y}(z) = \frac{C(z) \cdot \dfrac{1}{B(z)E(z)} \cdot \dfrac{z^{-k}B(z)}{A(z)}}{1 + \left[\dfrac{1}{B(z)E(z)} \dfrac{z^{-k}B(z)}{A(z)} \cdot \tilde{R}(z)\right]} = \dots = z^{-k}$$

$$F_{ey}(z) = \frac{\dfrac{C(z)}{A(z)}}{1 + \left[\qquad\qquad\right]} = \dots = E(z)$$

loop function

At closed-loop the behaviour is very simple ⟶

$$\boxed{y(t) = \underbrace{y^o(t-k)} + \underbrace{E(z) \cdot e(t)}}$$

This is the BEST possible solution!!

↳ $y(t)$ EXACTLY TRACKS/follows $y^o(t)$ but with k steps delay

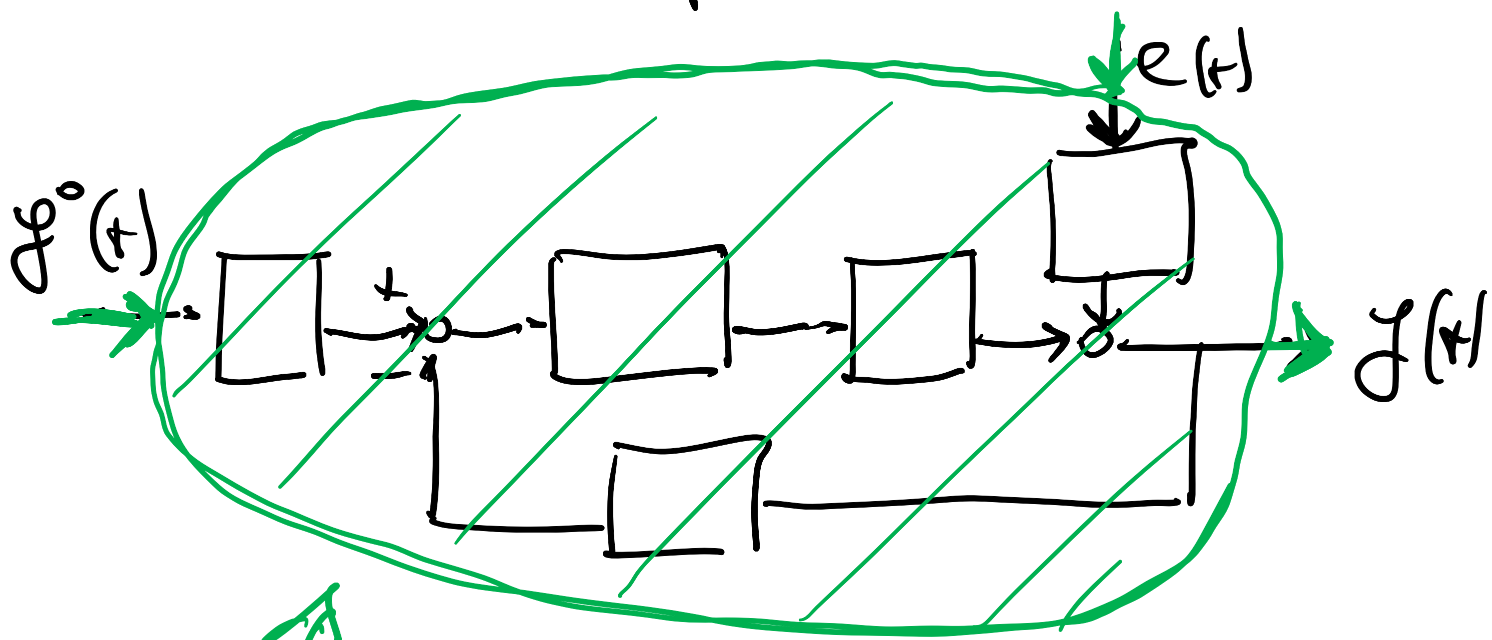
$y(t)$ is DISTURBED by noise $\underbrace{E(t) \cdot e(t)}$

↓ PREDICTION error k-steps ahead

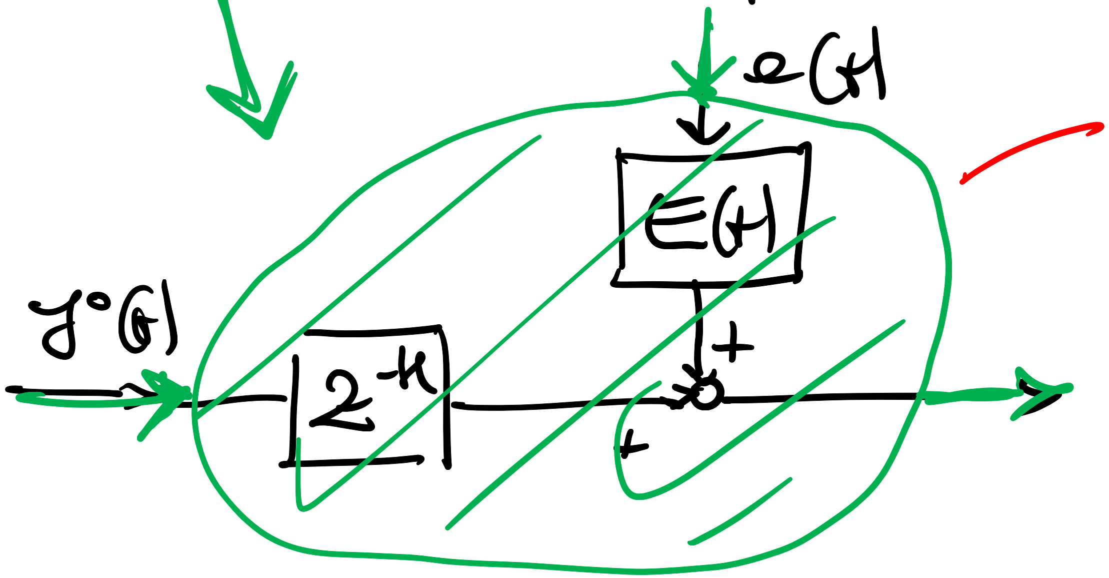
Ideal "perfect behaviour" would be:

$$y(t) = \overbrace{y^o(t)} + \overbrace{\emptyset}$$

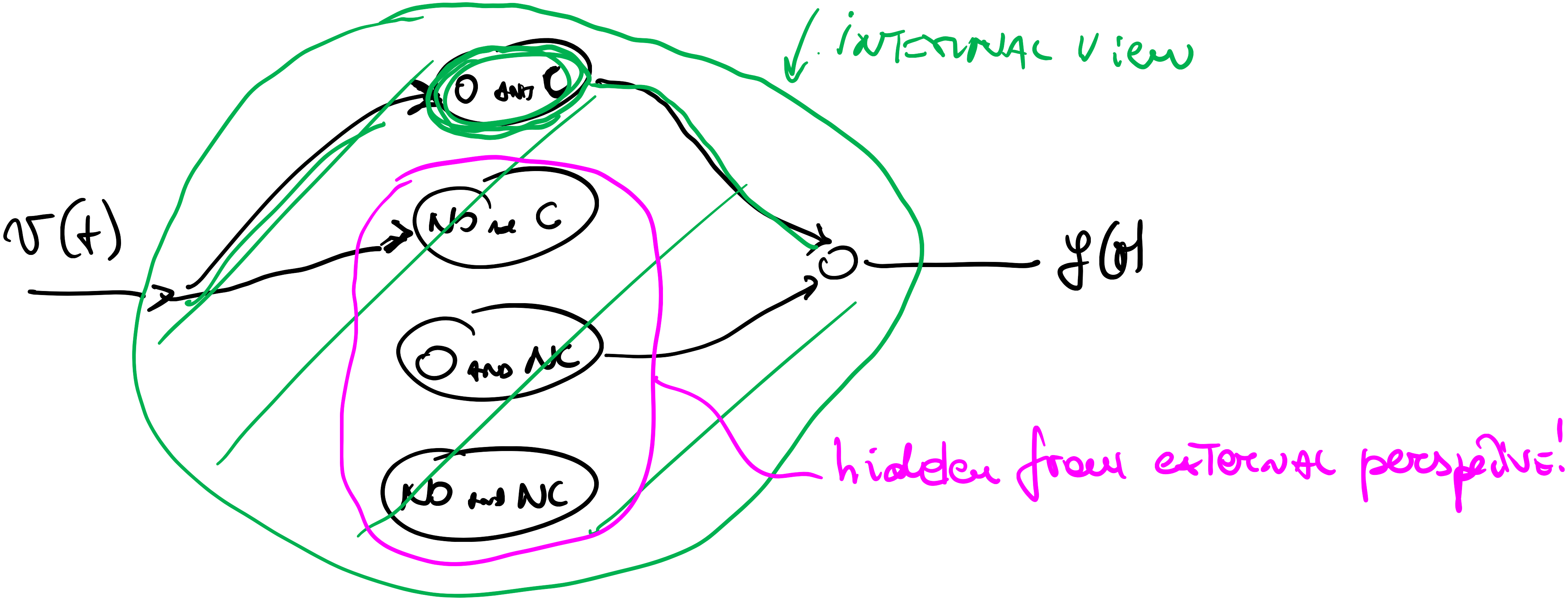
Remark : closed loop behaviour is very simple (?)



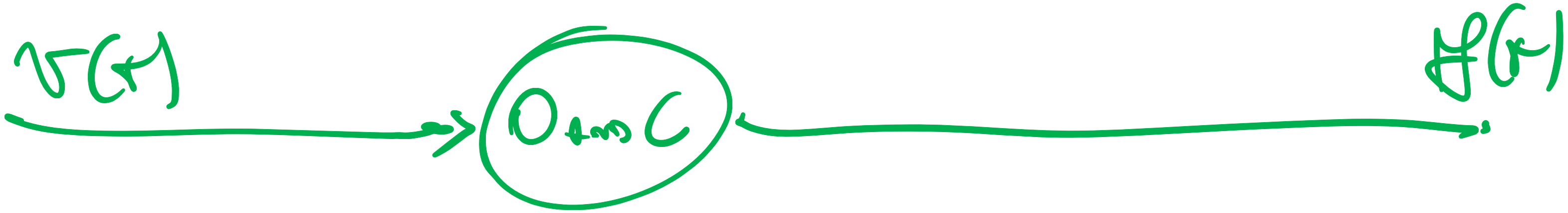
||| equivalent INPUT-OUTPUT



→ No poles!! where are all the poles of the original system?



equivalent from an EXTERNAL view to:



M.V. controller "pushes" all the system poles onto the N.O AND/OR N.C. parts of the system (it makes internal "CANCELLATIONS")

problem? NO → verified that it is INTERNALLY ASY. STABLE!!

MAIN limits of M.V.C ??

$\Longrightarrow$ Can be applied only to MIN. phase systems

$\Longrightarrow$ we cannot "MODERATE" the control/ACTUATION effort

$\Longrightarrow$ we cannot design a specific behaviour from $y^{o}(t) \dashrightarrow$ TO $\dashrightarrow y(t)$

To overcome these limits $\longrightarrow$ Extension of MVC called $\boxed{\text{GMVC}}$ (Generalized M.V.C.)

$\rightarrow$ MAIN Difference: Extension of the perf. INDEX:

$P(t)=1$

$$\text{MVC: } J = E\left[\left(y(t) - y^{o}(t)\right)^2\right]$$

$\Rightarrow Q(z) = 0$ [MVC is a special case of GMVC]

$$\text{GMVC: } J = E\left[\left(\boxed{P(z)}\,y(t) - y^{o}(t) + \boxed{Q(z)\,u(t)}\right)^2\right]$$

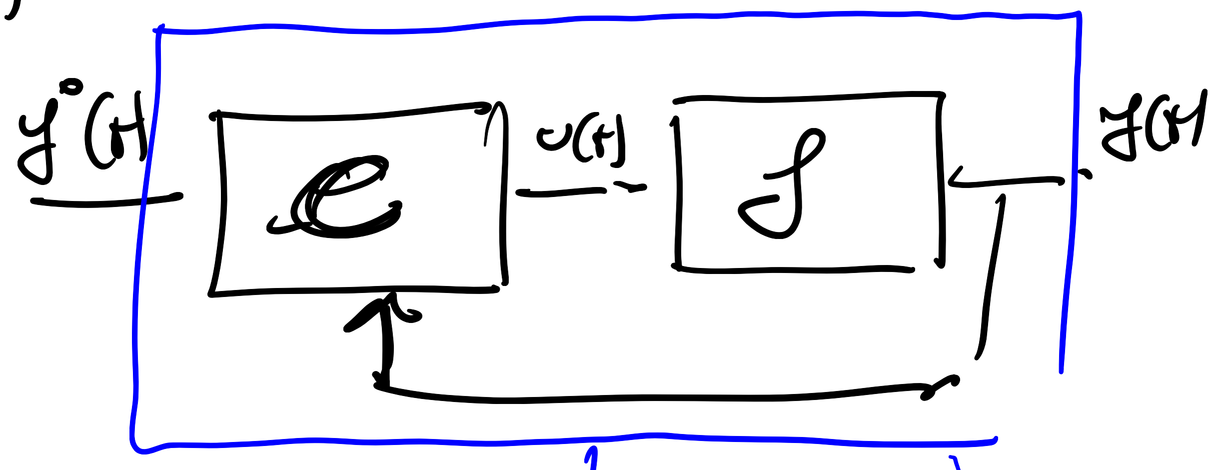
$P(z)$ is a transfer function called "Reference model"

$Q(t)$ is a T.F. that, multiplied by $u(t)$ works as a penalty to big values of $u(t)$

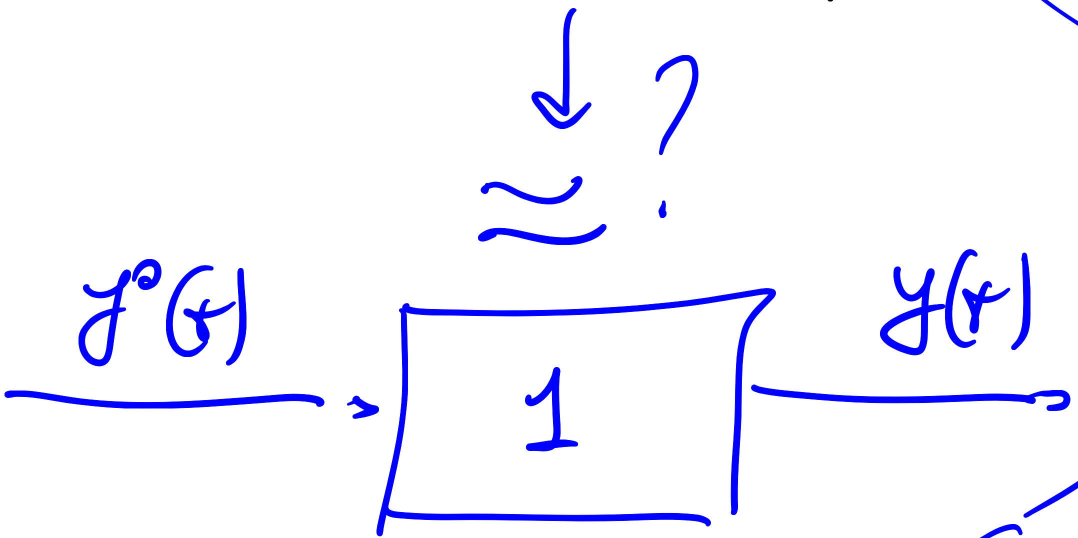
$Q(t)$ Big $\Rightarrow$ control action "moderates" the use of $u(t)$

Remark on reference model $P(t)$

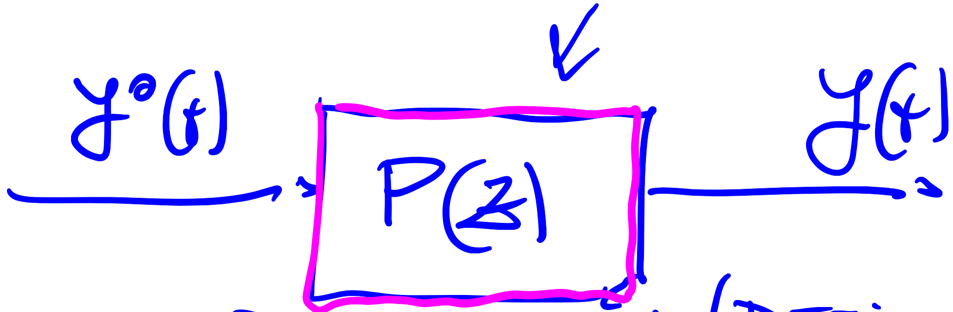
General feedback system:



Typical goal $\rightarrow$ obtain the best possible tracking:

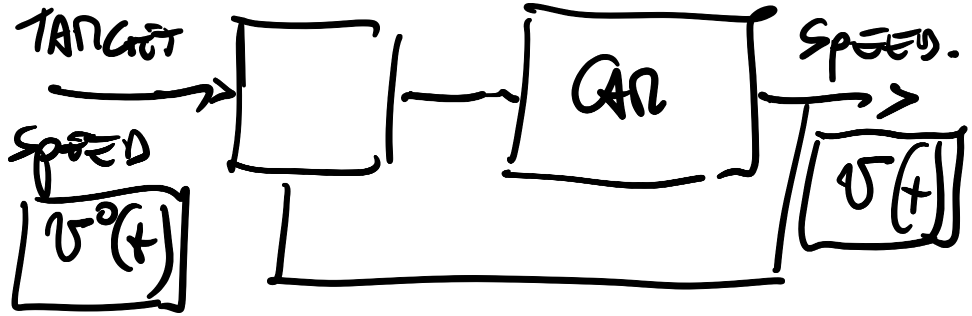


However, in many case fast/perfect tracking is not the best solution, but the best solution is to track a "reference model"

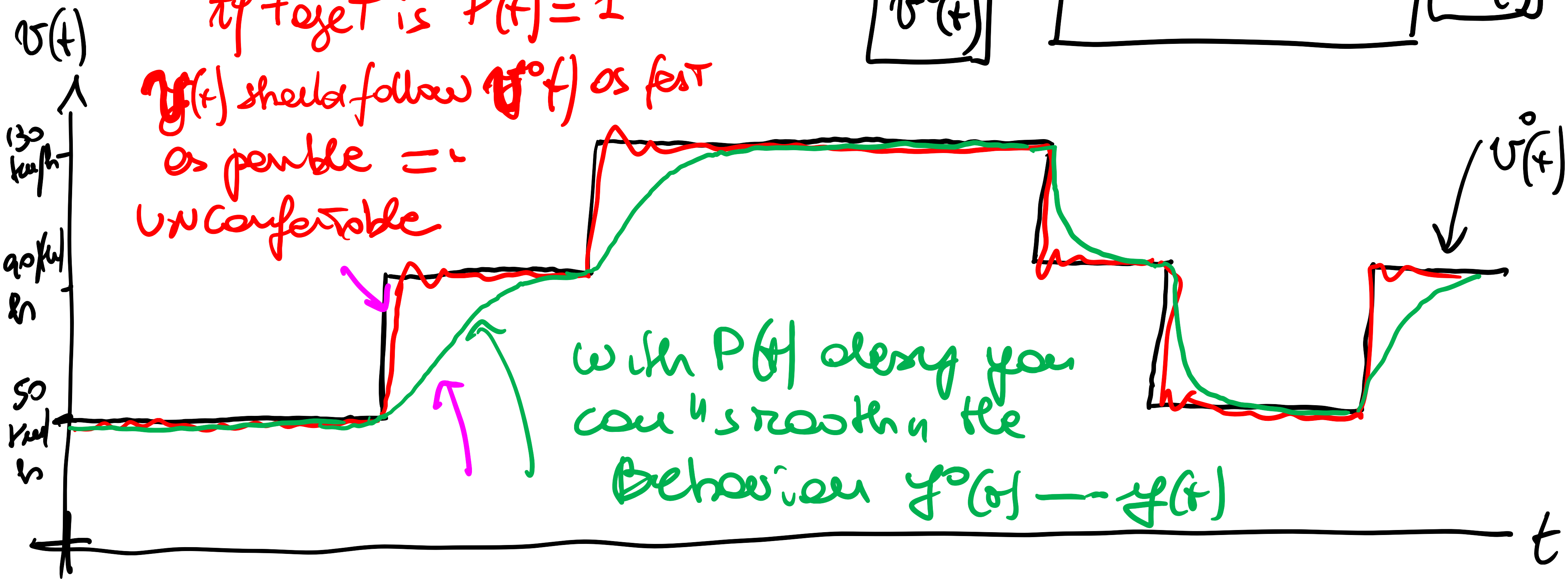


REF. model (DESIGN parameter)

Example: cruise control in a car:

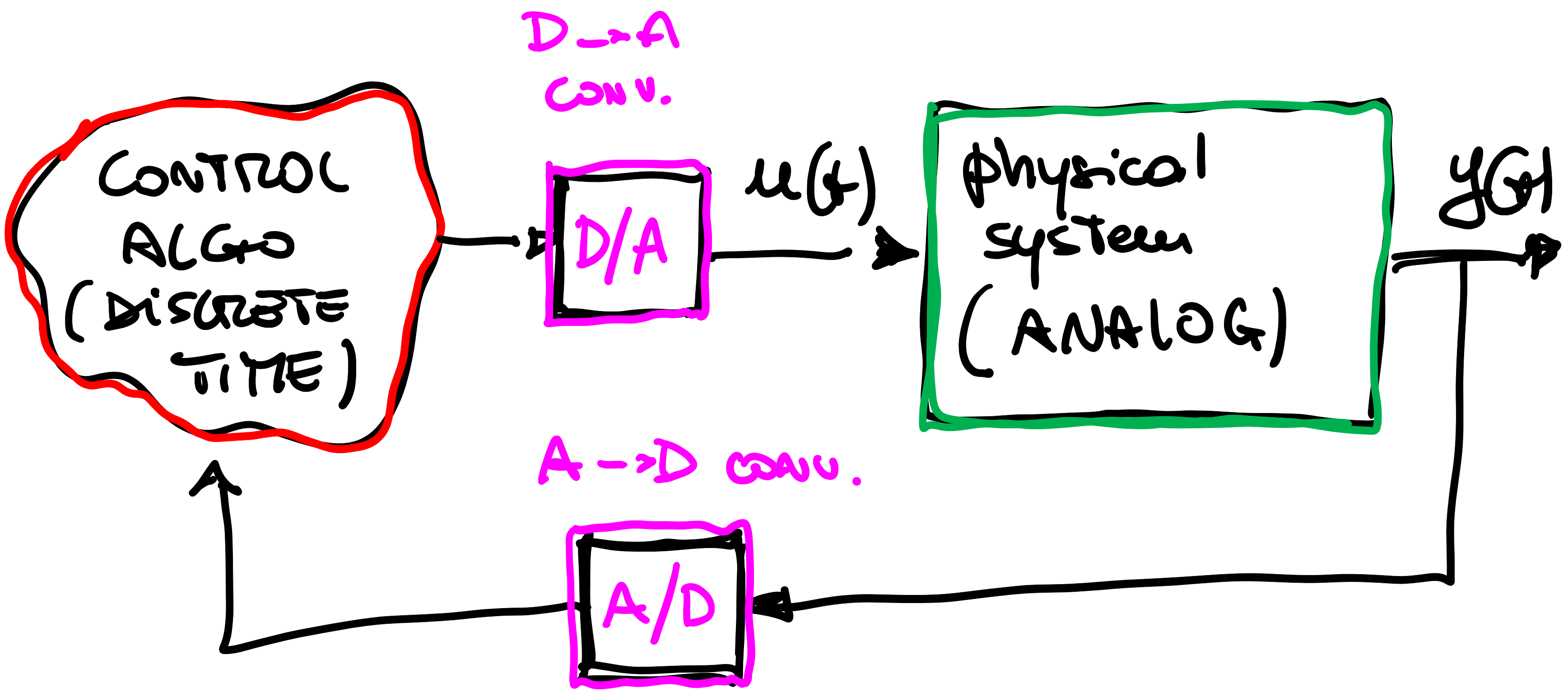


If target is $P(z)=1$
$y(t)$ should follow $y^o(t)$ as fast as possible ⇒ UNCOMFORTABLE

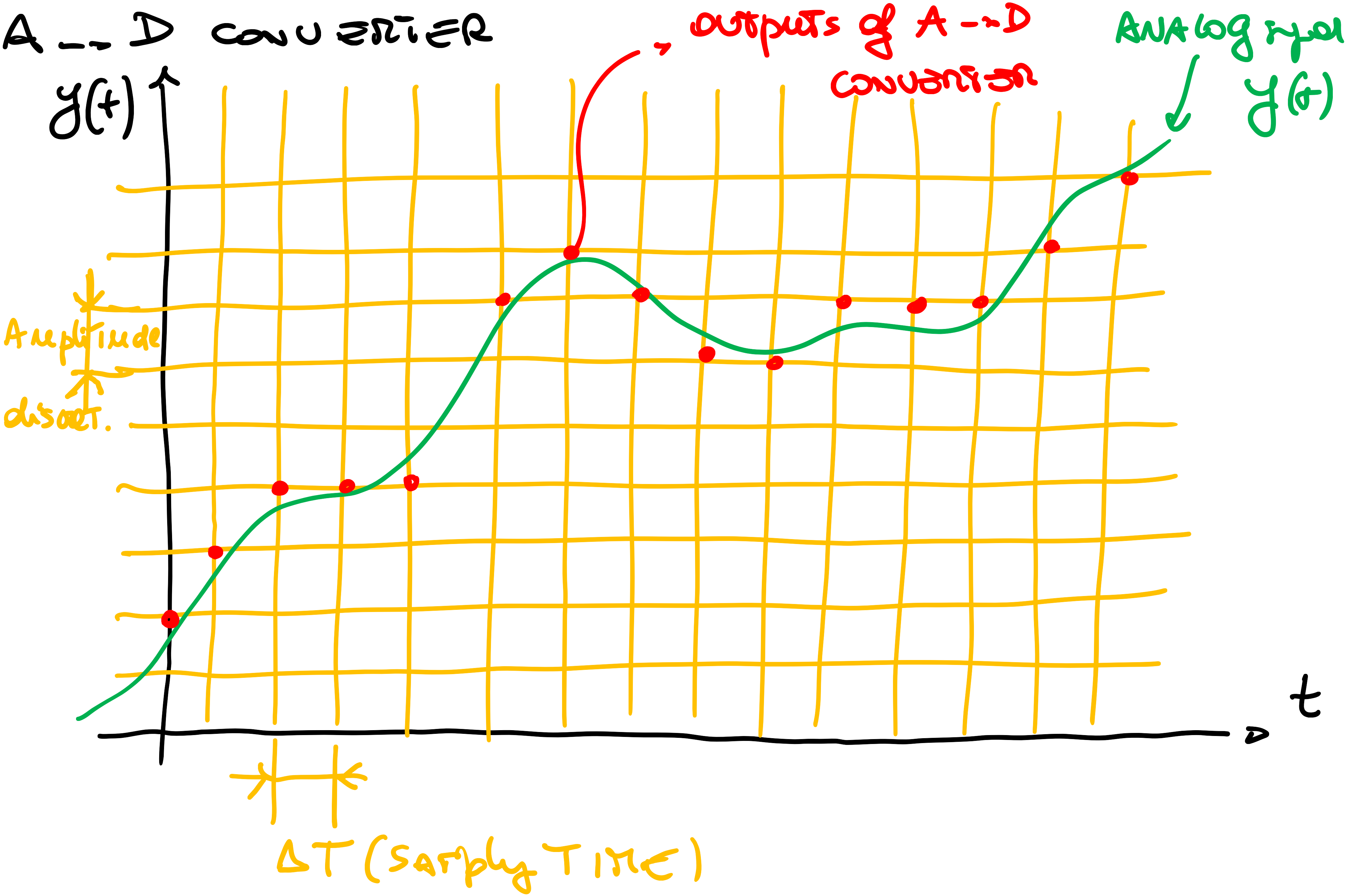


with $P(z)$ along you can "smooth" the behaviour $y^o(t) \longrightarrow y(t)$

Appendix to chapter 6 → Discretization of an ANALOG system (VALID for both identif. and control)



A --> D CONVERTER



Time discretization --> ΔT = sampling time

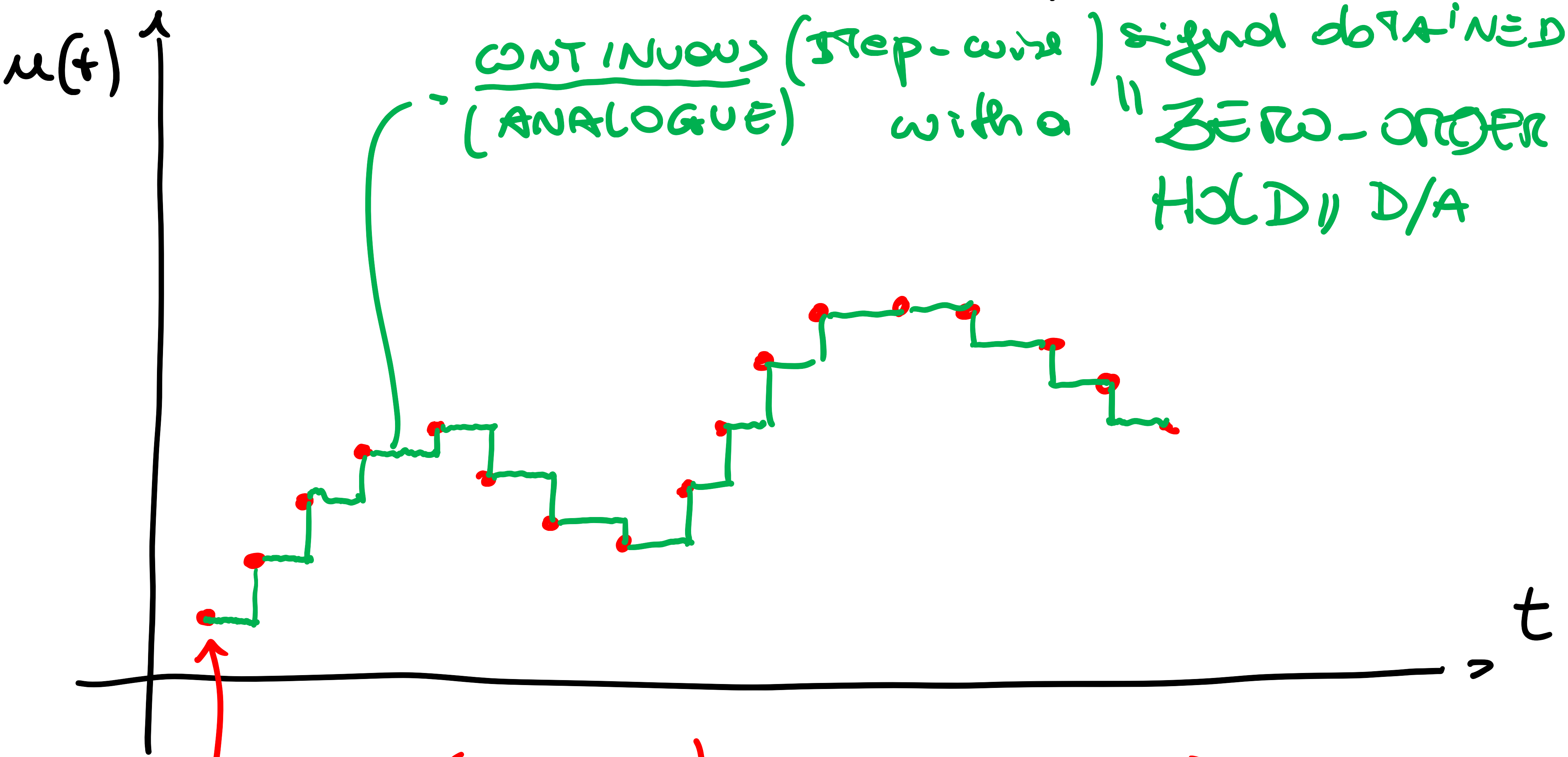
Amplitude discretization --> NUMBER (TOTAL) of LEVELS USED for discretiz.

ex: "10_bits discretization" --> 2^{10} --> 1024 levels of amplitude

--> "High quality" A/D CONVERTER -->

- can use small ΔT
- high number of levels (16 bits..)

D → A CONVERTER ("holder")

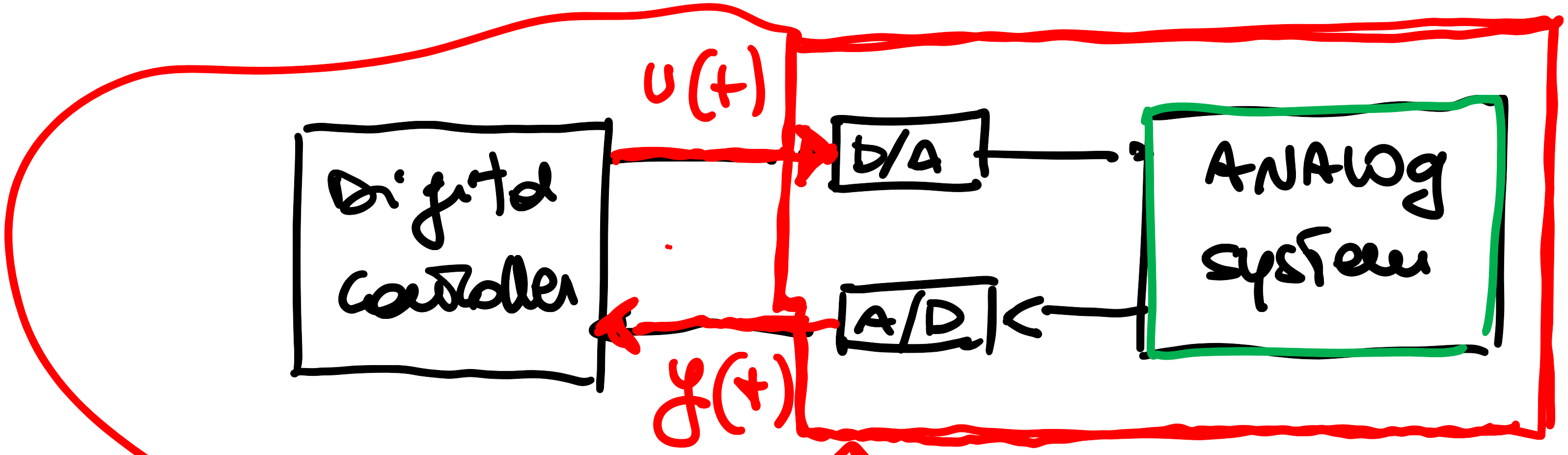


CONTINUOUS (step-wise) signal obtained (ANALOGUE) with a "ZERO-ORDER HOLD" D/A

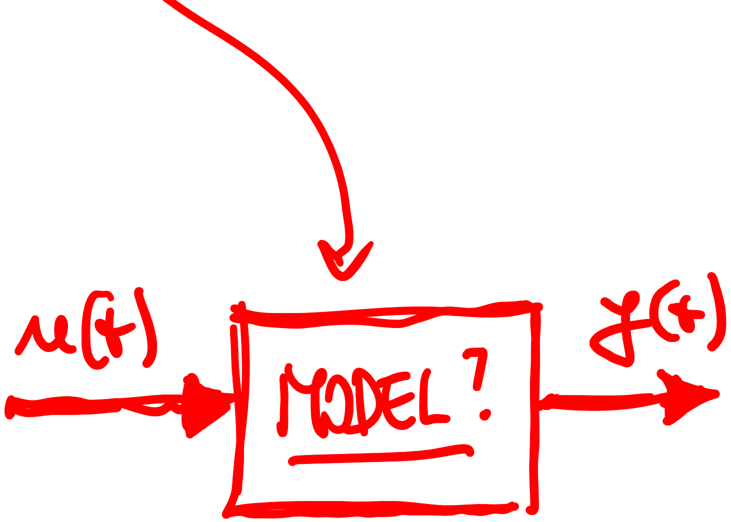
output (DIGITAL) of the digital controller

⇓

if ΔT is sufficiently SMALL → the step-wise ANALOG signal is very similar to a smooth ANALOG signal →



In the RED BOX we have the system from a DIGITAL perspective



MODEL? → 2 cases

- we make BB sys. identification from measured data → DIRECTLY ESTIMATE A D.T. model
- we have a physical W.B. model (CONT. TIME) → we NEED TO DISCRETIZE IT !!

The most used approach is the $\boxed{\text{STATE-SPACE TRANSFORMATION}}$

$$
\underbrace{\mathcal{S}: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + (Du) \end{cases} \quad A, B, C, D}_{\text{CONT. TIME}}
\quad \xrightarrow[\Delta T]{\text{sampling time}} \quad
\underbrace{\mathcal{S}: \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + (Du(t)) \end{cases} \quad F, G, H, D}_{\text{DISCRETE-TIME}}
$$

TRANSFOR. FORMULAS: $\longrightarrow$

$$
\begin{aligned}
F &= e^{A\Delta T} \\
G &= \int_0^{\Delta T} e^{A\delta} B \, d\delta \\
H &= C \\
D &= D
\end{aligned}
$$

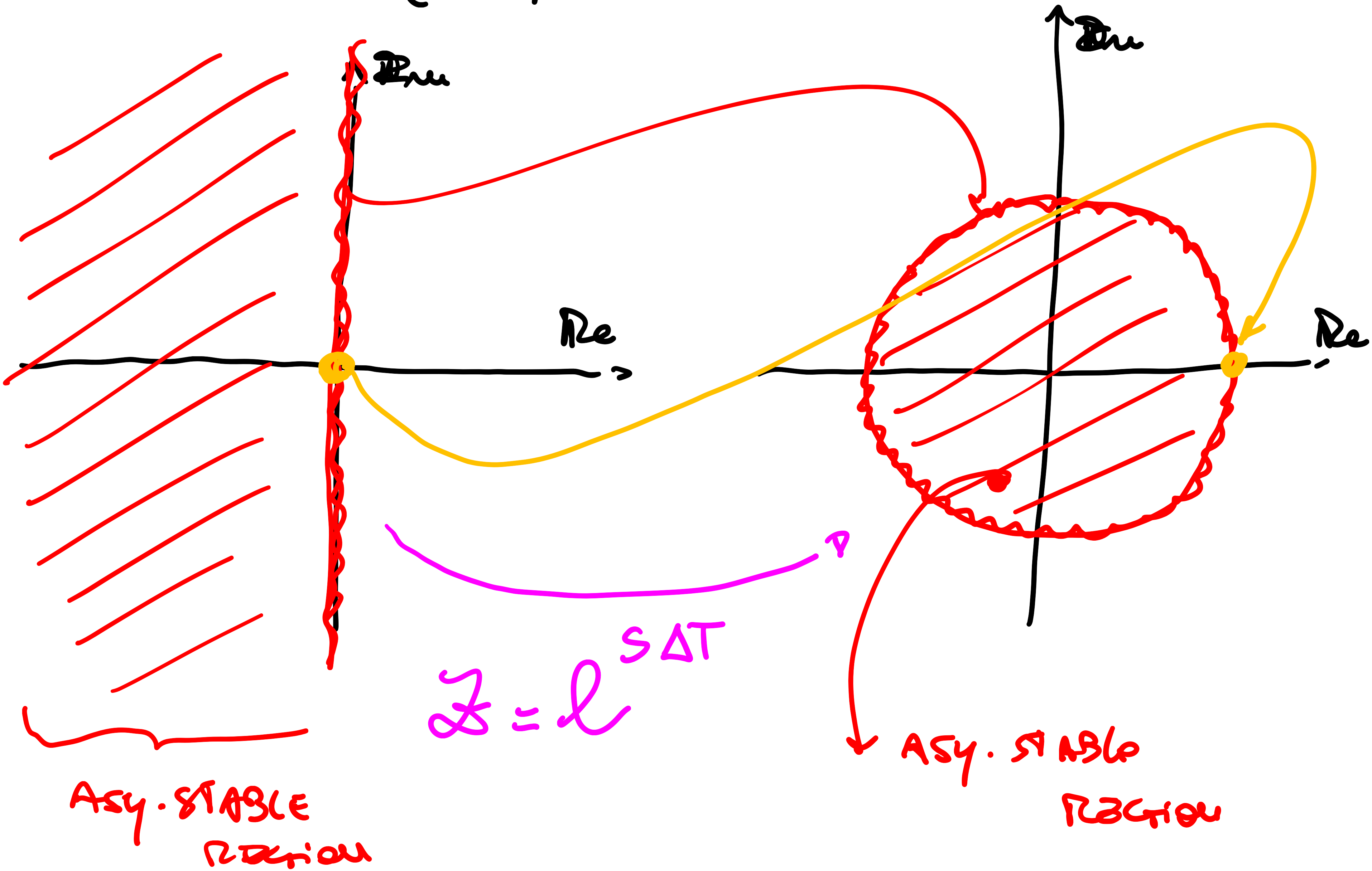
How the POLES of the C.Time system are Transformed?

$$EIG(A) \xrightarrow{?} EIG(F)$$

Can be proven that the EIGENVALUES (poles) follow the "sampling Transformation rule"

$$\underset{\substack{\uparrow \\ \text{Z-domain} \\ \text{(D.T.)}}}{\underset{\sim}{z}} = e^{\overset{\text{s.domain (C.T.)}}{s \cdot \Delta T}} \Longrightarrow \underbrace{\lambda_F}_{\text{EIG. of } F} = e^{\overbrace{\lambda_A}^{\text{EIG of } A}\Delta T}$$

S-domain (C.T.) | Z-domain (D.T.)



How the ZEROES of G in C.T. are Transformed into zeros in D.T.? $\longrightarrow$

UNFORTUNATELY $\longrightarrow$ NO simple RULE like poles

we can only say this:

$$G(s) = \frac{\text{poly. in "s" with } h \text{ zeros}}{\text{poly in "s" with } K \text{ poles}}$$

if $G(s)$ is strictly proper $\Rightarrow$ $K > h$

$\downarrow$ DISCRETIZATION RULE (s.s.)

$$G(z) = \frac{\boxed{\text{poly in "z" with } K-1 \text{ zeros}}}{\text{poly in "z" with } K \text{ poles}}$$

$\longrightarrow$ $G(z)$ with RELATIVE DEGREE $= 1$

$\Rightarrow$ we have "NEW" $k-h-1$ zeros that are generated by the discretization $\rightarrow$ they are called "**HIDDEN ZEROS**"

Unfortunately these hidden zeroes are frequently OUTSIDE the U.C. $\Rightarrow$ $G(z)$ is **NON-MIN. phase** even if $G(s)$ is min. phase

$\Rightarrow$ we NEED for instance **GMVC** to design the control system

Another simple discretization technique frequently used → Discretization of time derivative $\dot{x}$

EULER BACKWARD :

D.T. "TIMING"

$$\underbrace{\dot{x}(t)}_{C.T.} \simeq \frac{x(t) - x(t-1)}{\Delta T} = \frac{x(t) - z^{-1}x(t)}{\Delta T} = \frac{z-1}{z\Delta T} \cdot x(t)$$

EULER FORWARD :

$$\dot{x}(t) \simeq \frac{x(t+1) - x(t)}{\Delta T} = \frac{z-1}{\Delta T} \cdot x(t)$$

General formula for this approach:

$$\dot{x}(t) = \left[\frac{z-1}{\Delta T} \cdot \frac{1}{\alpha z + (1-\alpha)} \right] x(t)$$

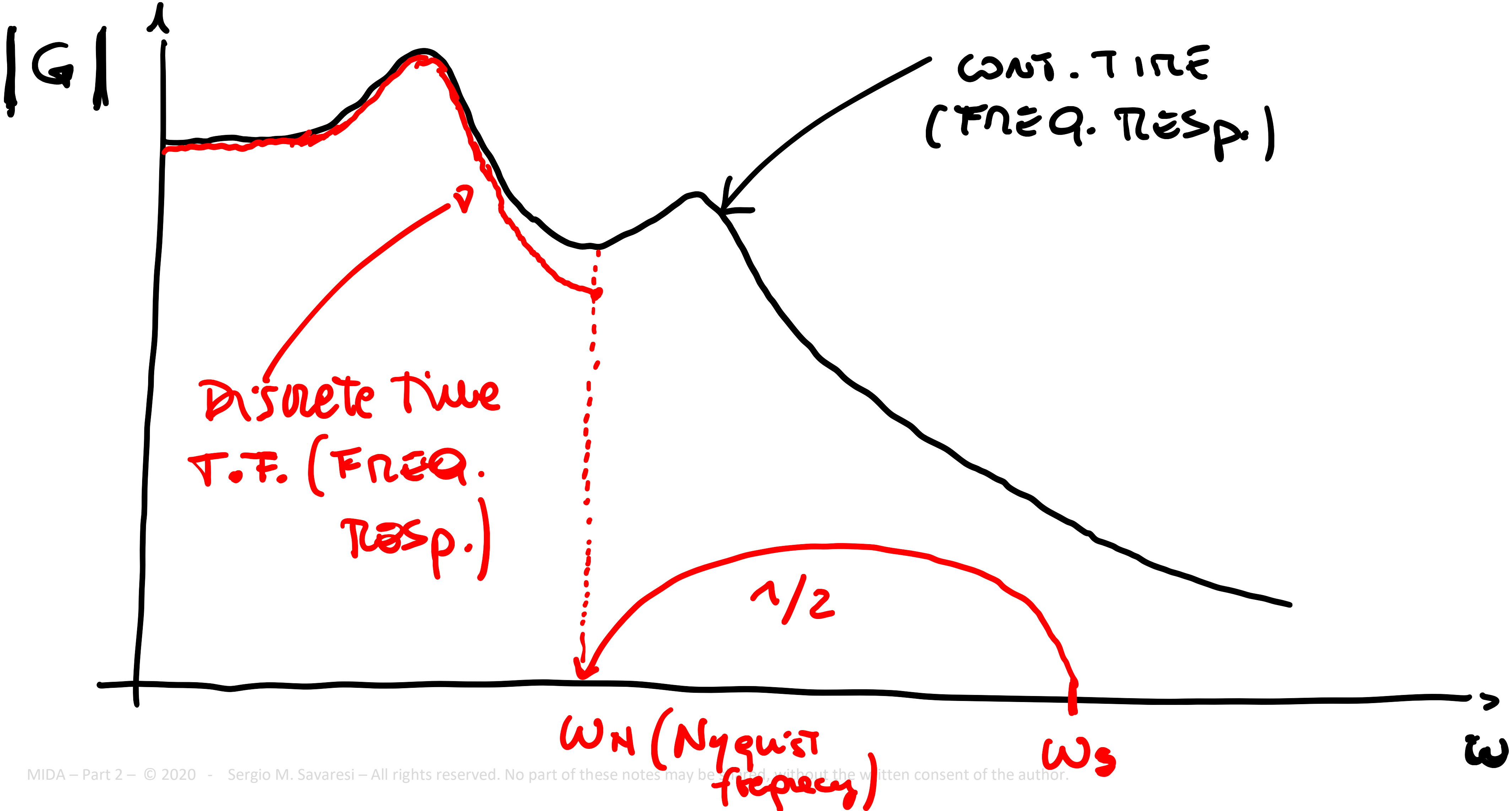
$0 \leq \alpha \leq 1$ → special cases:

- if $\alpha = 0$ ⇒ Euler F.
- if $\alpha = 1$ ⇒ Euler B.
- $\boxed{\text{if } \alpha = \frac{1}{2} \Rightarrow \text{"\underline{TUSTIN Method}"}}$

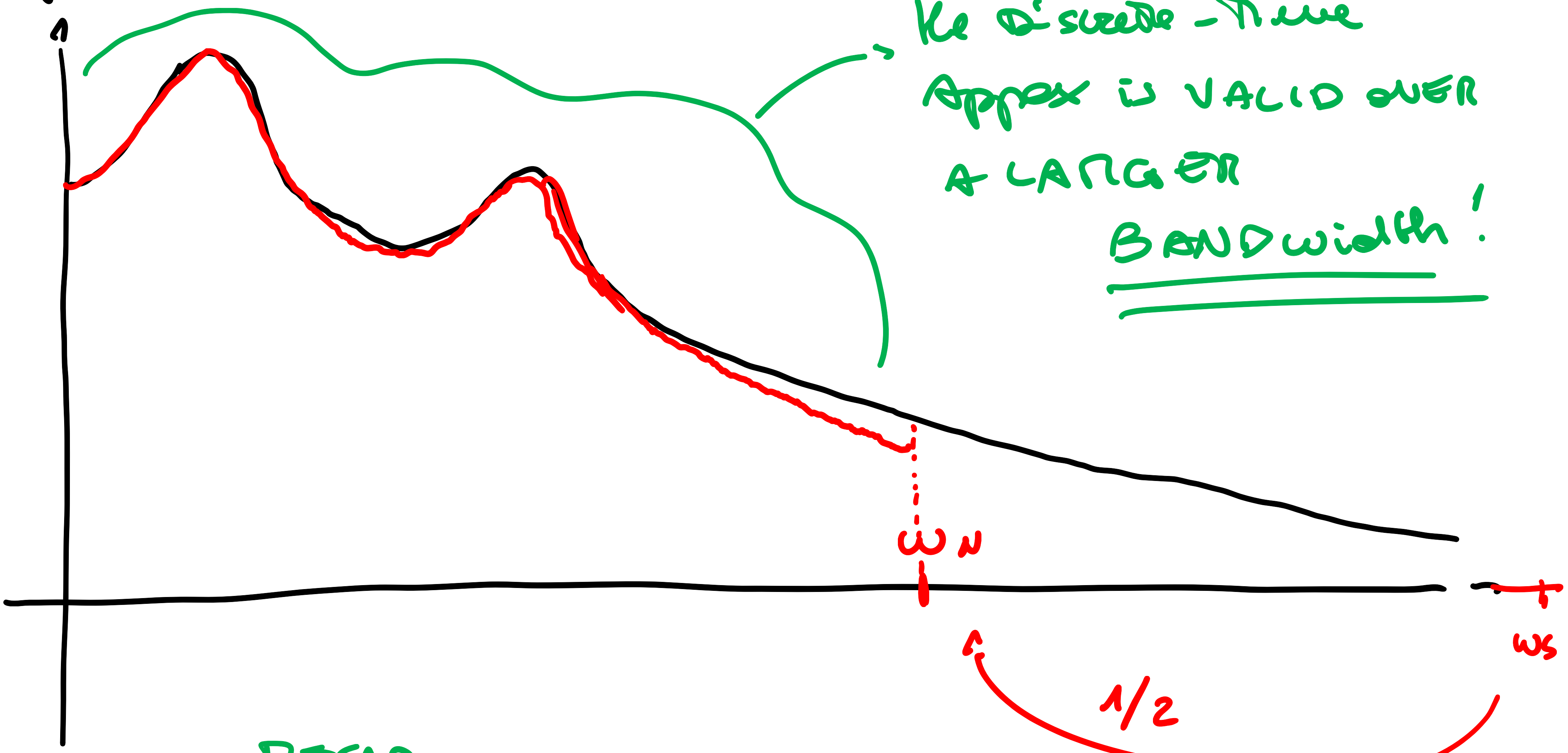
Critical choice is ΔT (sampling time) $\longleftrightarrow$ $\underbrace{f_s = \frac{1}{\Delta T}}_{\text{sampling FREQUENCY}}$

GENERAL INTUITIVE RULE $\rightarrow$

the smaller ΔT is, the BETTER!



if ΔT is smaller $\rightarrow$ ω_s larger



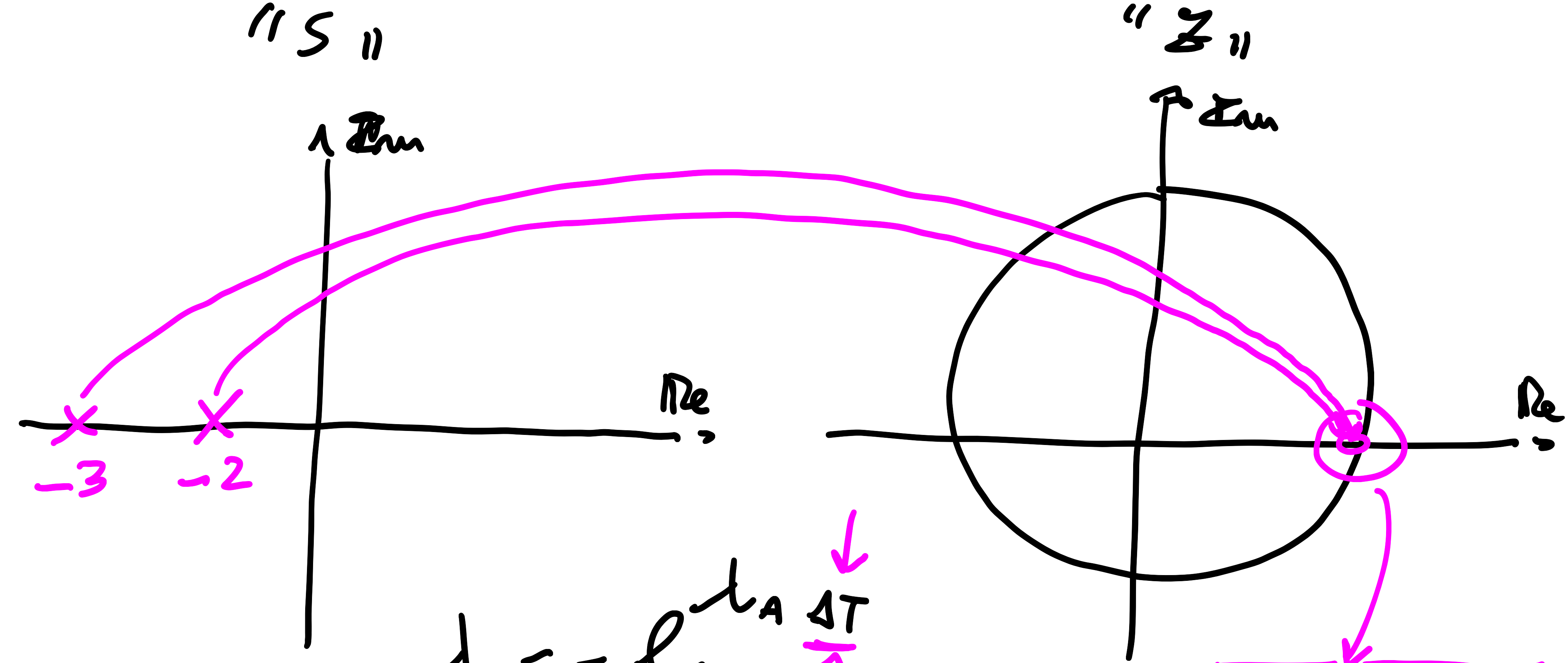
RECAP.

$$\Delta T \rightarrow f_s = \frac{1}{\Delta T} \rightarrow \omega_s = \frac{2\pi}{\Delta T} \quad \begin{cases} f_N = \frac{1}{2} f_s \\ \omega_N = \frac{1}{2} \omega_s \end{cases}$$

HIDDEN problems of a "TOO-SMALL" ΔT:

- sampling DEVICES (A/D and D/A) COST
- computational cost → update an algorithm every 1 μs is much heavier than updating at 1 ms
- cost of memory (if DATA-logging is NEEDED)
- NUMERICAL PRECISION "COST"

↑ AGAIN is a hidden computational cost



$$G(s) \xrightarrow[\text{poles transformation}]{\lambda_F = e^{\lambda_A \Delta T}} G(z)$$

$\boxed{\begin{matrix} 0{,}999995 \\ 0{,}999996 \end{matrix}}$

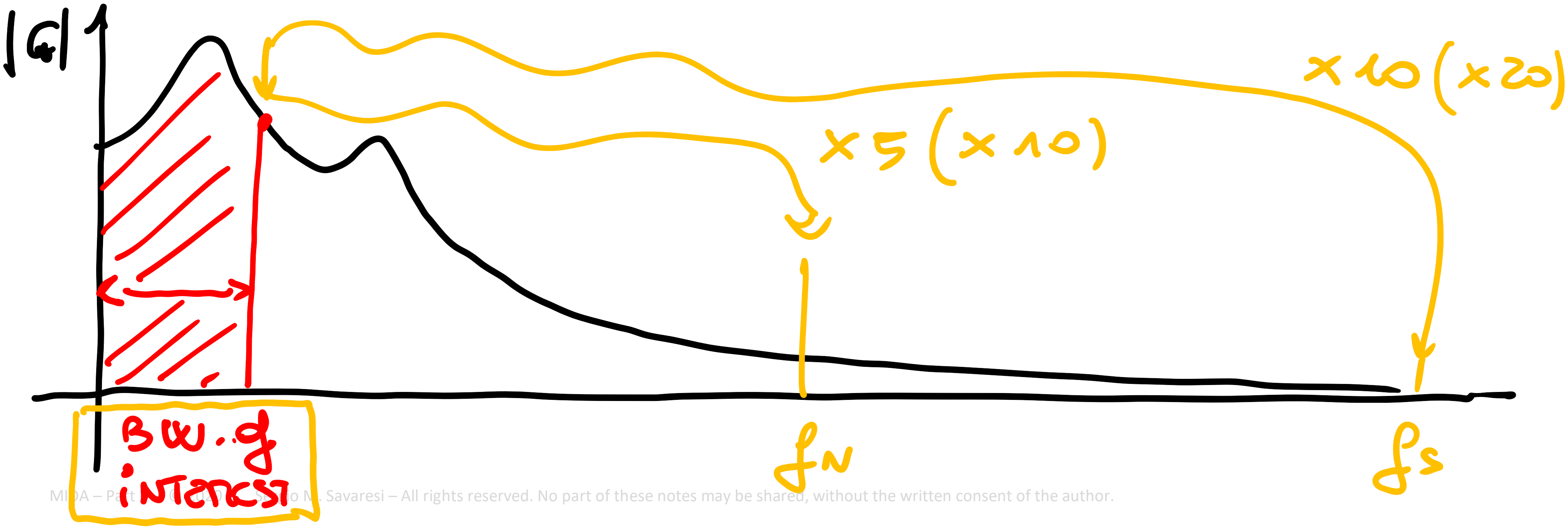
if ΔT very small ($\to 0$) we "squeeze" all the poles very close to $(1,0)$

⟹ we need very high NUMERICAL PRECISION (⟹ use a lot of DIGITS) to avoid INSTABILITY

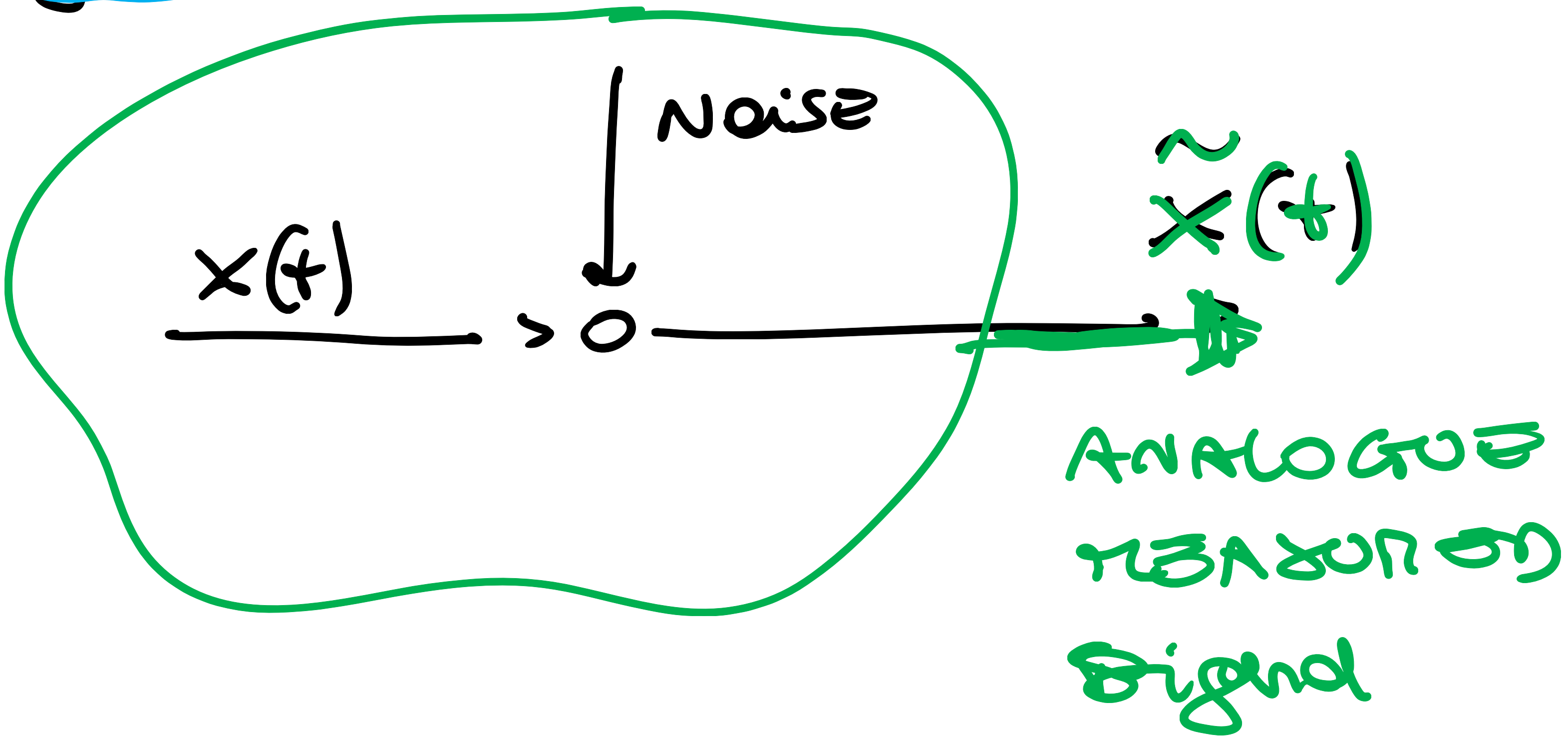
Rule of thumb of control engineers ⟶

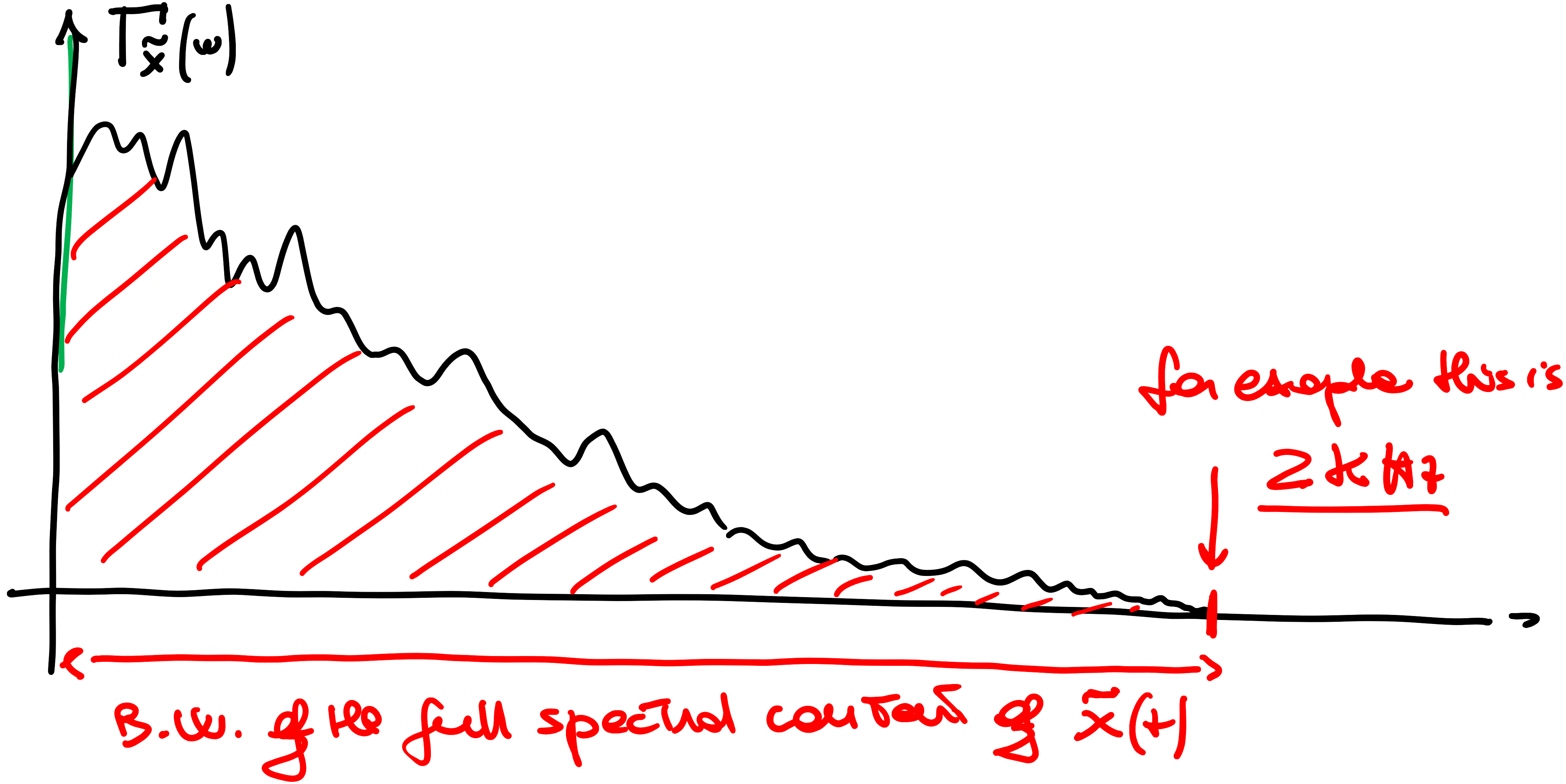
f_s is between $\boxed{10}$ and $\boxed{20}$ times the system BANDW. we are interested in

min max

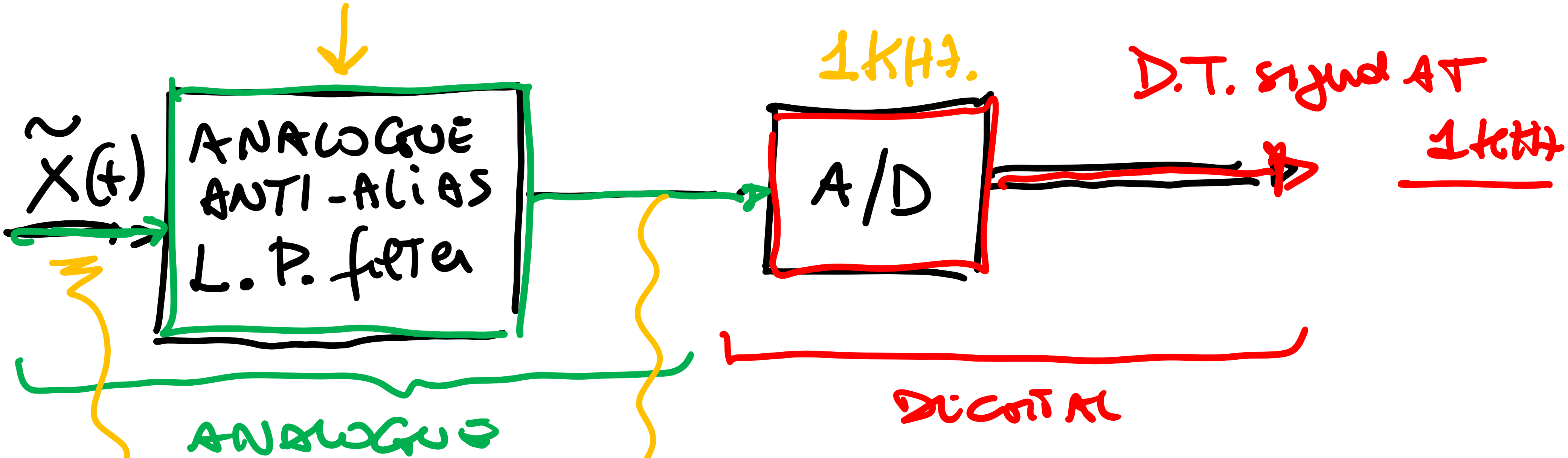


Final remark → another way of managing the choice of ΔT w.r.t. the ALIASING problem (signal-processing perspective)

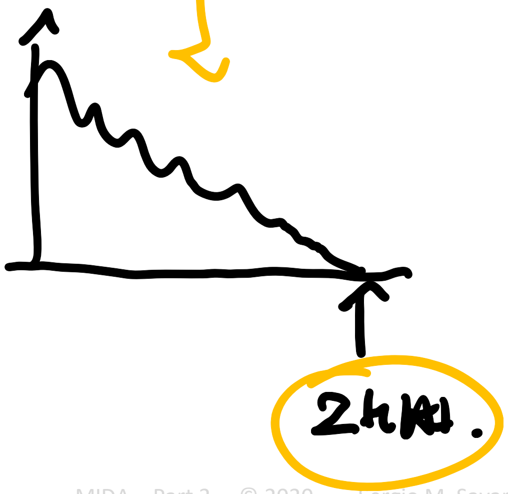


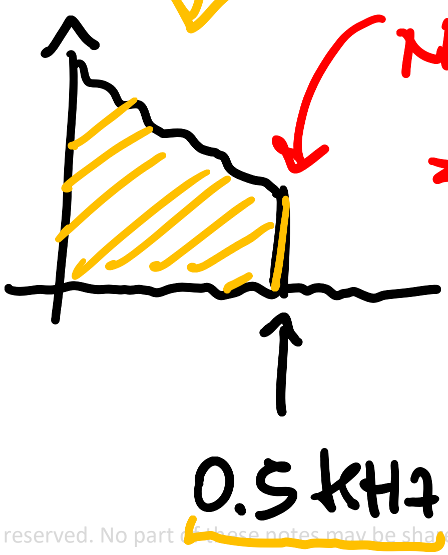


Classical way to deal with ALIASING is to use ANTI-ALIAS ANALOG filters:



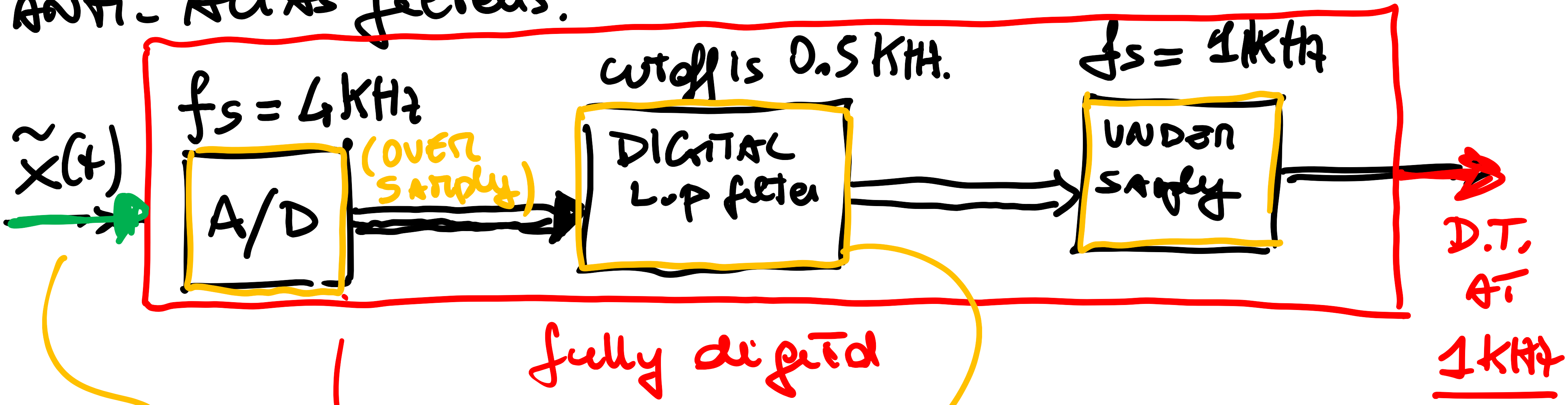
for example, if A/D is at 1 KHz ($\Delta T = 1\,ms$)





Nyquist frequency = 0.5 kHz → sampling time 1 KHz guarantees NO ALIAS

DIGITAL (full - DIGITAL) Approach without ANALOGUE ANTI-ALIAS filters:



ANTI ALIAS (D.T.)

END & MIDA 2 !!