Free Grammars - II

Translated and adapted by L. Breveglieri

SYNTAX TREE AND CANONICAL DERIVATION

SYNTAX TREE: directed acyclic graph, such that for every pair of nodes there exists one and only one path (not necessarily directed) that connects them

THE SYNTAX TREE

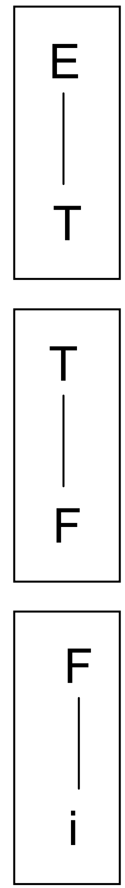
- represents graphically the derivation process
- gives a *parent-child* relation or a *root-node-leaf* relation
- the sequence of leaves, scanned from left to right, is the so-called *frontier*
- the *degree* of a node (so-called node *arity*) is the length of the rule

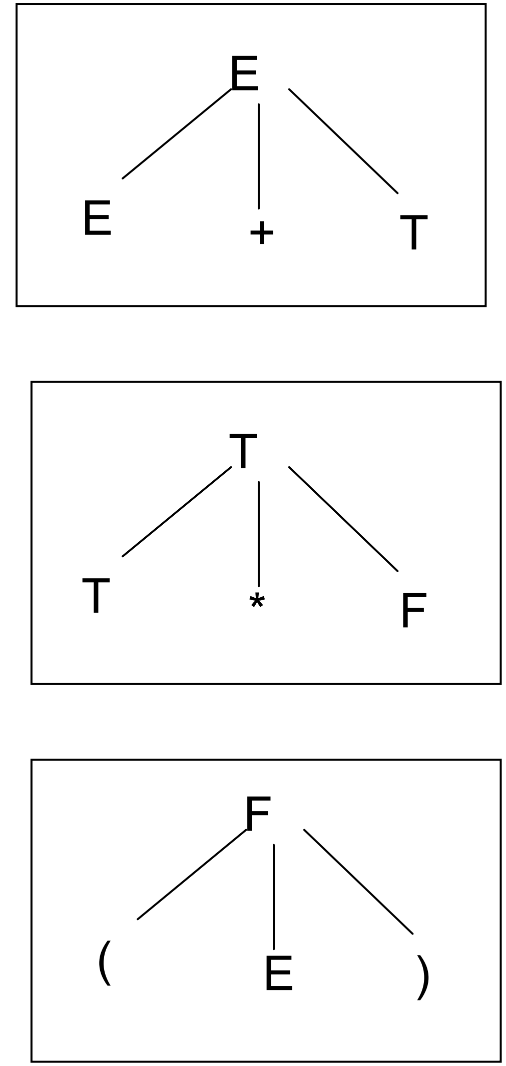
SUBTREE with root N: the tree that has root N and includes all the descendants of N (the immediate siblings of N, the siblings of the siblings, and so on)

SYNTAX TREE: the root is the axiom and the frontier is the generated phrase grammar

1. $E \rightarrow E + T$
2. $E \rightarrow T$
3. $T \rightarrow T * F$
4. $T \rightarrow F$
5. $F \rightarrow (E)$
6. $F \rightarrow i$

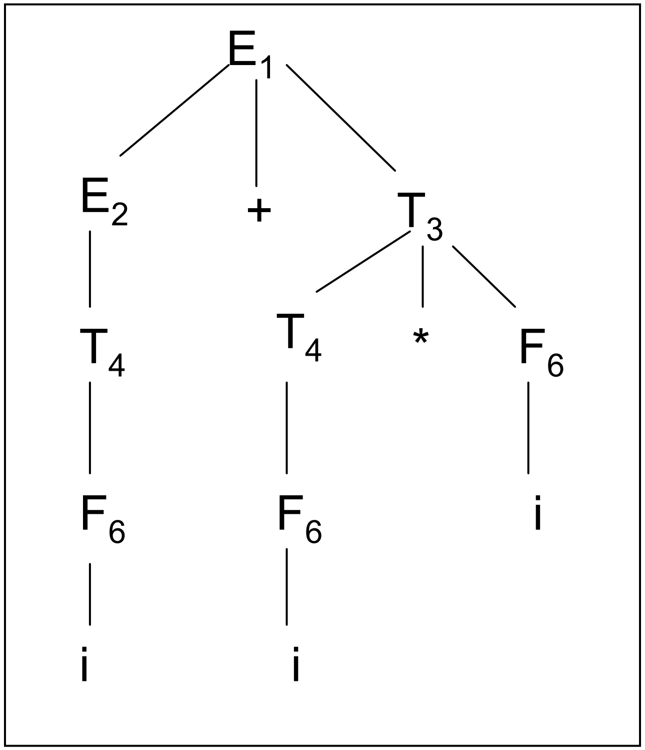
production subtrees





parenthesized representation of the syntax tree

$$[[[[i]_F\,]_T\,]_E + [[[i]_F\,]_T * [i]_F\,]_T\,]_E$$



syntax tree

LEFTMOST DERIVATION

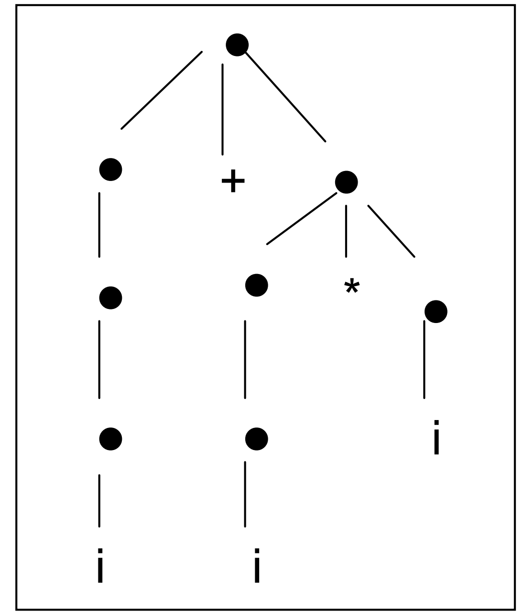
$$E \underset{1}{\Rightarrow} E+T \underset{2}{\Rightarrow} T+T \underset{4}{\Rightarrow} F+T \underset{6}{\Rightarrow} i+T \underset{3}{\Rightarrow} i+T*F \underset{4}{\Rightarrow}$$
$$\underset{4}{\Rightarrow} i+F*F \underset{6}{\Rightarrow} i+i*F \underset{6}{\Rightarrow} i+i*i$$

RIGHTMOST DERIVATION

$$E \underset{1}{\Rightarrow} E+T \underset{3}{\Rightarrow} E+T*F \underset{6}{\Rightarrow} E+T*i \underset{4}{\Rightarrow} E+F*i \underset{6}{\Rightarrow} E+i*i \underset{2}{\Rightarrow}$$
$$\underset{2}{\Rightarrow} T+i*i \underset{4}{\Rightarrow} F+i*i \underset{6}{\Rightarrow} i+i*i$$

SKELETON TREE (only the frontier and the connections)

skeleton syntax tree

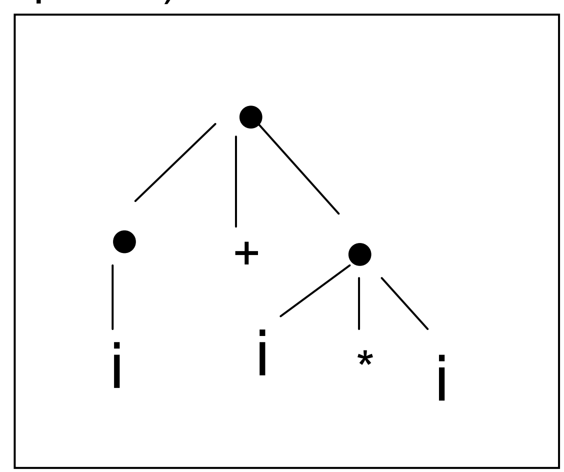


parenthesized representation of the syntax tree

$$\boxed{[[[[i]]]+[[[i]]*[i]]]}$$

CONDENSED SKELETON TREE (merge into one all the internal nodes aligned on a linear path, i.e., a path without branch points)

condensed tree



$$\boxed{[[i]+[[i]*[i]]]}$$

condensed representation of the syntax tree

LEFTMOST AND RIGHTMOST DERIVATION / FREE GRAMMAR

$$E \overset{+}{\underset{s}{\Rightarrow}} i + i * i$$

$$E \overset{+}{\underset{d}{\Rightarrow}} i + i * i$$

legenda

s: left
d: right

$$E \overset{+}{\underset{s,d}{\Rightarrow}} i + i * i$$

$$E \underset{s,d}{\Rightarrow} E + T \underset{d}{\Rightarrow} E + T * F \underset{s}{\Rightarrow} T + T * F \Rightarrow T + F * F \underset{d}{\Rightarrow} T + F * i \underset{s}{\Rightarrow}$$
$$\underset{s}{\Rightarrow} F + F * i \underset{d}{\Rightarrow} F + i * i \underset{d}{\Rightarrow} i + i * i$$

EVERY PHRASE OF A FREE GRAMMAR CAN BE GENERATED BY MEANS OF A LEFTMOST DERIVATION (and by a RIGHTMOST DERIVATION as well).
This property is of great importance for the syntactic analysis algorithms.
It allows us to organize the derivation of the phrase in the most appropriate way

PARENTHESES LANGUAGE: artificial languages frequently contain nested structures, i.e., parentheses, where an element pair starts and ends some substructure (like for instance a substring), and where the pair may in turn contain a nested pair, and so on recursively down to arbitrary depth

Pascal:	begin ... end
C:	{ … }
XML:	< title > … < /title >
LaTeX:	\begin{equation} … \end{equation}

Do not consider the specific way the marker symbols are encoded

The paradigm of parenthesis languages is known as the *Dyck Language*

Alphabet $\Sigma = \{')', '(', ']', '['\}$ Phrases $()[[()[]]()]$

Parenthesis phrases can be equivalently defined by means of *cancellation rules*: repeatedly remove from the string any factor that consists of a pair of adjacent open and closed parentheses, as long as it is possible.
The original string is valid if and only if the final result is the *empty* string

$$[\ ] \Rightarrow \varepsilon \qquad (\ ) \Rightarrow \varepsilon$$

DYCK LANGUAGE: open parentheses *a, b, ...,* closed parentheses *a', b', ...*

$$\Sigma = \{a, a', b, b'\}$$

$$S \rightarrow aSa'S \mid bSb'S \mid \varepsilon$$

$$\underbrace{a\,\underbrace{a\,\underbrace{a\,a'}\,a'}\,\underbrace{a\,\underbrace{a\,\underbrace{a\,a'}\,a'}\,a'}\,a'}$$

LINEAR BUT NON-REGULAR LANGUAGE

$$L_1 = \{a^n c^n \mid n \geq 1\} = \{ac, aacc, ...\}$$

$$S \rightarrow aSc \mid ac$$

L_1 is a subset of the Dyck language.
It does not admit more than one parenthesis nest

REGULAR COMPOSITION OF FREE LANGUAGES

The basic regular operations (union, concatenation and star), applied to free languages, still yield free languages

The family of free languaegs is closed with respect to the language operations *union*, *concatenation* and *star*

$$G_1 = \left(\Sigma_1, V_{N_1}, P_1, S_1\right) \text{ e } G_2 = \left(\Sigma_2, V_{N_2}, P_2, S_2\right)$$

$$V_{N_1} \bigcap V_{N_2} = \varnothing \qquad S \notin (V_{N_1} \bigcup V_{N_2})$$

UNION

$$G = (\Sigma_1 \bigcup \Sigma_2, \{S\} \bigcup V_{N_1} \bigcup V_{N_2}, \{S \to S_1 \mid S_2\} \bigcup P_1 \bigcup P_2, S)$$

CONCATENATION

$$G = (\Sigma_1 \bigcup \Sigma_2, \{S\} \bigcup V_{N_1} \bigcup V_{N_2}, \{S \to S_1 S_2\} \bigcup P_1 \bigcup P_2, S)$$

STAR: G of $(L_1)^*$ is obtained by adding the rules $S \to S\ S_1 \mid \varepsilon$ to G_1

CROSS: G of $(L1)^+$ is obtained by adding the rules $S \to S\ S_1 \mid S_1$ to G_1

EXAMPLE: union of languages

$$\boxed{L=\left\{a^ib^ic^* \mid i\geq 0\right\}\bigcup\left\{a^*b^ic^i \mid i\geq 0\right\}=L_1\bigcup L_2}$$

G_1

$$S_1 \to XC$$
$$X \to aXb \mid \varepsilon$$
$$C \to cC \mid \varepsilon$$

G_2

$$S_2 \to AY$$
$$Y \to bYc \mid \varepsilon$$
$$A \to aA \mid \varepsilon$$

$$\boxed{\left\{S \to S_1 \mid S''\right\}\bigcup P_1 \bigcup P''}$$

G''

$$S'' \to AX$$
$$X \to bXc \mid \varepsilon$$
$$A \to aA \mid \varepsilon$$

CAUTION: if the hypothesis that the two non-terminal alphabets are disjoint does not hold, then the construction above yields a grammar that generates a superset of the real union language. For example, replacing G_2 with G'' would allow us to generate the invalid phrase shown aside

$$\boxed{abcbc}$$

The family LIB of free languages is closed with respect to STRING MIRRORING

Given the grammar G of the language, the grammar G_R that generates the mirror image of the language is obtained from G by mirroring the right member of every rule of G

REG and LIB are both closed with respect to union, concatenation and star

but later it will be proved that

they do not behave in the same way as for complement and intersection

AMBIGUITY: semantic versus syntactic

“Vedo un uomo in giardino con il cannocchiale”

(do I watch a man who has a spyglass or do I use a spyglass to watch a man ?)

“La pesca è bella” (the peach is fine or the sport of fishing is fine ?)

“half baked chicken” (a chicken that is half baked or a half chicken that is baked ?)

Natural languages are largely ambiguous and this phenomenon is unavoidable, as they aim at describing everything, but with only a finite lexicon (dictionary). Instead, in the artifical languages ambiguity is an undesirable phenomenon (think of a program, how could we tolerate ambiguity ?)

Therefore ambiguity must be eliminated (if possible) or kept under control

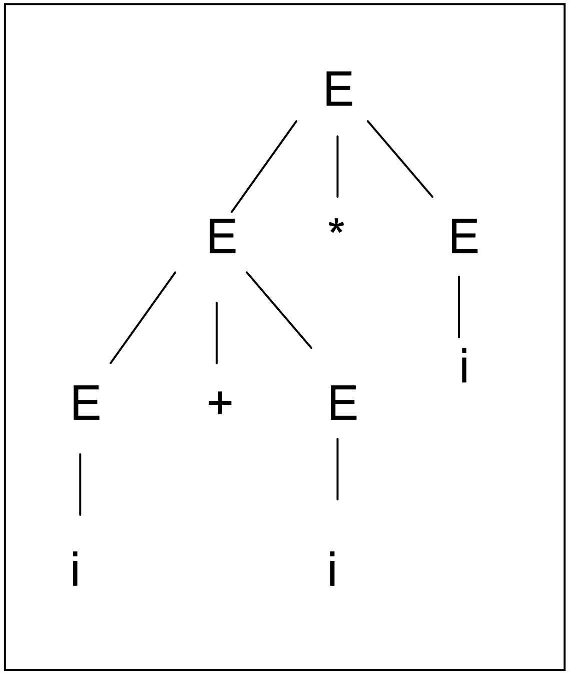
Consider SYNTACTIC AMBIGUITY. A phrase *x* of the language defined by the grammar G is said to be ambiguous if it admits two or more different syntactic trees (or, equivalently, two different derivations). A grammar G is said to be ambiguous if it generates (at least) an ambiguous string

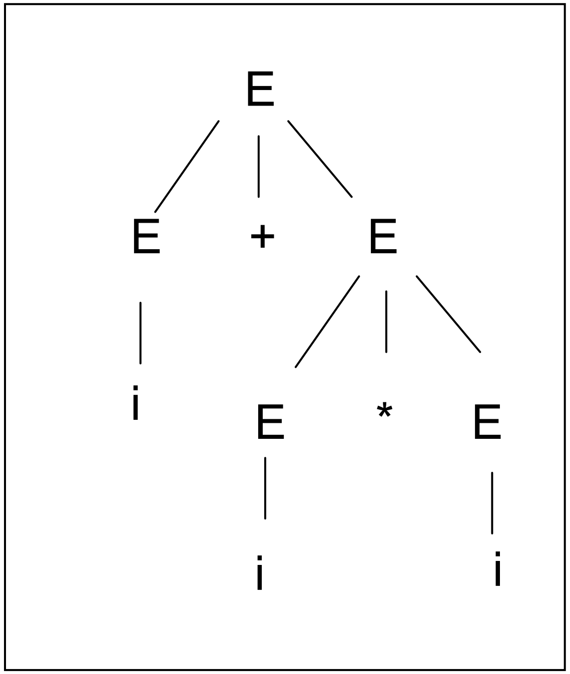
EXAMPLE – grammar G’ of the arithmetic expressions

$$E \to E + E \mid E * E \mid (E) \mid i$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow i + E * E \Rightarrow i + i * E \Rightarrow i + i * i$$

$$E \Rightarrow E + E \Rightarrow i + E \Rightarrow i + E * E \Rightarrow i + i * E \Rightarrow i + i * i$$





The phrase *i + i * i* is ambiguous. Therefore grammar G' is itself ambiguous

In fact, grammar G' is not compliant with the standard convention that multiplication must precede addition

Previously, an unambiguous grammar G that generates the arithmetic expressions has been shown. G' is equivalent to G (that is, L(G) = L(G')), but is smaller than G. However, grammar G' is ambiguous. This is often the case: simplifying the grammar frequently ends up with having an ambiguous behaviour, in general

The AMBIGUITY DEGREE of a phrase x in the language L(G) is the number of different syntax trees that x admits. The ambiguity degree of a string may be infinite (only if there are nullable non-terminal symbols)
The AMBIGUITY DEGREE of a grammar G is the maximum ambiguity degree of the strings generated by G. While every string of L(G) may have a finite ambiguity degree, the ambiguity degree of G may still turn out to be unbounded

RELEVANT PROBLEM: decide whether a grammar G is ambiguous or not.
UNFORTUNATELY, THIS PROBLEM IS UNDECIDABLE: there does not exist any algorithm to decide whether a given grammar G is ambiguous or not.
This can be done for some grammars, but in general not for every grammar

By using the methods of theoretical information science, it is possible to prove that any potential decision algorithm should examine increasingly long derivations, which is unfeasible and hence leads to undecidability. However, the inexistence of a general algorithm does not preclude to decide for a specific grammar, by resorting to *ad hoc* methods depending on the structure of the grammar. From a practical point of view, examining all the derivations of G up to a given length is often sufficient to get convinced that the grammar is unambiguous, though it is not a rigorous proof

AS AMBIGUITY MAY BE DIFFICULT TO IDENTIFY A POSTERIORI
IT IS ADVISABLE TO AVOID IT BY CONSTRUCTION (A FORTIORI)

EXAMPLE: resume the previous example

The phrase *i* + *i* + *i* has ambiguity degree equal to 2

The phrase *i* + *i* * i + *i* has ambiguity degree equal to 5 (see below)

$$\underbrace{\underbrace{i+i}*\underbrace{i+i}}, \quad \underbrace{\underbrace{i+\underbrace{i*i}}+i}, \quad \underbrace{i+\underbrace{\underbrace{i*i}+i}}, \quad \underbrace{\mathrm{i}+\underbrace{\mathrm{i}*\underbrace{\mathrm{i}+\mathrm{i}}}}, \quad \underbrace{\underbrace{\underbrace{i+i}*i}+i}$$

Typically, as the derived phrase gets longer, its ambiguity degree grows as well

DIRECTORY OF AMBIGUOUS MODELS AND HOW TO REMOVE AMBIGUITY

AMBIGUITY OF TWO-SIDED RECURSION: a non-terminal symbol that is recursive both on the left and on the right (two-sided recursion) always allows two or more derivations, and thus gives rise to an ambiguous behaviour (see below)

$$\boxed{A \stackrel{+}{\Rightarrow} A\ldots \quad A \stackrel{+}{\Rightarrow} \ldots A}$$

EXAMPLE 1: grammar G_1 generates the phrase *i + i + i* in two different ways (by means of two different leftmost derivations)

ambiguous grammar

$$\boxed{G_1: \quad E \to E + E \mid i}$$

notice that L is regular

non-ambiguous version

$$L(G_1) = i(+i)^*$$

$$E \to i + E \mid i$$

$$E \to E + i \mid i$$

EXAMPLE 2: ambiguity that originates form left and right recursion, separately

ambiguous grammar

$$G_2: \quad A \to aA \mid Ab \mid c$$

ambiguity can be removed by separately generating the two lists

$$L(G_2) = a^* c b^*$$

$$S \to AcB$$

$$A \to aA \mid \varepsilon$$

$$B \to bB \mid \varepsilon$$

or one may choose to orderly generate the two lists (first *a* then *b*, or viceversa)

$$L(G_2) = a^* c b^*$$

$$S \to aS \mid X$$

$$X \to Xb \mid c$$

EXAMPLE 3: the grammar that generates Polish postfix expressions that contain addition and multiplication, has a left recursion (but not right) and is not ambiguous

$$S \to +SS \mid \times SS \mid i$$

AMBIGUITY OF UNION (maybe the simplest to understand)

If two languages $L_1(G_1)$ and $L_2(G_2)$ share some phrases (i.e., their intersection is not empty), the grammar G of the union language, constructed as shown before, is surely ambiguous

CAUTION: one needs to assume that the two non-terminal sets are disjoint, otherwise, as seen before, the union grammar would generate a superlanguage that strictly contains both languages

A phrase *x* that belongs to the union of L_1 and L_2, and admits two different derivations, the first only using rules of G_1 and the second only using rules of G_2, is ambiguous for the grammar G, as G contains both sets of rules. Only the phrases that belong to $L_1 \setminus L_2$ or to $L_2 \setminus L_1$, if any, would be generated in a non-ambiguous way

EXAMPLE 1: ambiguity of union may arise when in a programming language a special case, out of a general one, is treated by means of separate rules

$$\boxed{\begin{array}{l} E \rightarrow E+1 \\ E \rightarrow inc\ E \end{array}}$$

separate rules

$$E \rightarrow E+T \mid T \quad T \rightarrow V \mid C \quad V \rightarrow \ldots$$
$$C \rightarrow 0 \mid 1B \mid \ldots \mid 9B \quad B \rightarrow 0B \mid 1B \mid \ldots \mid 9B \mid \varepsilon$$

basic grammar

EXAMPLE 2: ambiguity of union may arise when in a programming language the same operator has two different meanings. For example, in Pascal the operator “+” can indicate both integer addition and set-theoretic union

ambiguous grammar

$$E \rightarrow E+T \mid T \quad T \rightarrow V \quad V \rightarrow \ldots$$
$$E_{set} \rightarrow E_{set}+T_{set} \mid T_{set} \quad T_{set} \rightarrow V$$

If one keeps on overloading the “+” operator but also wants to remove ambiguity, one must give up with pretending to have separate rules to generate arithmetic or set-theoretic expressions. Otherwise, one may split the operator “+” and use the symbol “+” only to denote integer addition, while a new operator, say for example “U”, is introduced to denote set-theoretic union. The latter solution, however, changes the language

EXAMPLE 3

$$\boxed{S \to bS \mid cS \mid \varepsilon}$$

non-ambiguous version

ambiguous grammar

$$G: \quad S \to bS \mid cS \mid D \quad D \to bD \mid cD \mid \varepsilon$$

$$L(G) = L_S(G) \bigcup L_D(G)$$

$$L_S(G) = \{b, c\}^* = L_D(G)$$

$$S \overset{+}{\Rightarrow} bbcD \Rightarrow bbc \quad S \Rightarrow D \overset{+}{\Rightarrow} bbcD \Rightarrow bbc$$

EXAMPLE 4

$$\boxed{\begin{array}{l} S \to B \mid D \mid \varepsilon \\ B \to bBc \mid bc \\ D \to dDe \mid de \end{array}}$$

non-ambiguous version

ambiguous grammar

$$S \to B \mid D \quad B \to bBc \mid \varepsilon$$

$$D \to dDe \mid \varepsilon$$

B generates $b^n c^n, n \geq 0$

D generates $d^n e^n, n \geq 0$

ε is the only ambiguous phrase

LANGUAGES THAT ARE INHERENTLY AMBIGUOUS

A language is said to be INHERENTLY AMBIGUOUS if every grammar that generates it is necessarily ambiguous, i.e., if all the equivalent grammars of the language happen to be ambiguous

EXAMPLE: here is a classical inherently ambiguous language

$$\boxed{L = \left\{ a^i b^j c^k \mid (i, j, k \geq 0) \wedge ((i = j) \vee (j = k)) \right\}}$$

$$\boxed{L = \left\{ a^i b^i c^* \mid i \geq 0 \right\} \bigcup \left\{ a^* b^i c^i \mid i \geq 0 \right\} = L_1 \bigcup L_2}$$

ambiguous union

Whatever grammar G is designed to generate L, the phrases ε, *abc*, $a^2b^2c^2$, ... ALWAYS HAPPEN TO BE DERIVABLE IN TWO OR MORE WAYS.
This behaviour is caused by the very nature of the language L and does not depend on the specific grammar G that generates L

In fact, such phrases can be generated by G_1 to check that $|x|_a = |x|_b$

$$\underbrace{a\ldots\underbrace{a\,\underbrace{ab}\,b}\ldots b}\,bcc\ldots c$$

generated by G_1

or can be generated by G_2 to check that $|x|_b=|x|_c$

$$a\ldots aa\,\underbrace{b\ldots\underbrace{b\,\underbrace{bc}\,c}\ldots c}$$

generated by G_2

But clearly, since both sub-grammars can generate such phrases, ambiguity arises.
In whatever way the two sub-grammars are modified, both checks are unavoidable, as they belong to the structure of the language, and therefore ambiguity is unavoidable as well

AMBIGUITY OF CONCATENATION

Concatenating two languages may give rise to ambiguity if there exists a suffix of a phrase of the former language that is a prefix of a phrase of the latter language

$$\boxed{G = (\Sigma_1 \bigcup \Sigma_2, \{S\} \bigcup V_{N_1} \bigcup V_{N_2}, \{S \to S_1 S_2\} \bigcup P_1 \bigcup P_2, S)}$$

Assume that G_1 and G_2 are not ambiguous, then G is ambiguous if there exist two phrases $x' \in L_1$ and $x'' \in L_2$, and a non-empty string *v* such that

$$x' = u'v \wedge u' \in L_1 \quad x'' = vz'' \wedge z'' \in L_2$$

$u'vz'' \in L_1.L_2$ and is ambiguous

$$S \Rightarrow S_1 S_2 \overset{+}{\Rightarrow} u' S_2 \overset{+}{\Rightarrow} u'vz''$$

$$S \Rightarrow S_1 S_2 \overset{+}{\Rightarrow} u'v S_2 \overset{+}{\Rightarrow} u'vz''$$

EXAMPLE 1 – concatenation of Dyck languages

$$\Sigma_1 = \{a, a', b, b'\} \quad \Sigma_2 = \{b, b', c, c'\}$$

$$aa'bb'cc' \in L = L_1 L_2$$

$$G(L): \quad S \to S_1 S_2$$

$$S_1 \to aS_1a'S_1 \mid bS_1b'S_1 \mid \varepsilon$$

$$S_2 \to bS_2b'S_2 \mid cS_2c'S_2 \mid \varepsilon$$

$$\overbrace{aa'bb'}^{S_1}\overbrace{cc'}^{S_2} \qquad \overbrace{aa'}^{S_1}\overbrace{bb'cc'}^{S_2}$$

In order to eliminate ambiguity, it is necessary to exclude the case when a suffix moves from the former language to the latter one. A solution consists of inserting a separator between the two languages. Such a separator may not belong to the alphabets of the two languages. The concatenation language $L_1 \# L_2$ is generated by the additional axiomatic rule $S \to S_1 \# S_2$

EXAMPLE 2 – encoding – the relationship between ambiguity and the uniqueness of encoding in information theory

A message is a sequence of symbols out of the set Γ = { A, B,…, Z }, with possible repetitions. Such symbols are then encoded into strings of letters out of the set of terminal symbols Σ. Most frequently Σ is the binary alphabet, Σ = { 0, 1 }

$$\Gamma = \left\{ \overbrace{A}^{01}, \overbrace{C}^{10}, \overbrace{E}^{11}, \overbrace{R}^{001}, \right\}$$

$$ARRECA: 01\ 001\ 001\ 11\ 10\ 01$$

$$Testo \rightarrow ATesto \mid CTesto \mid ETesto \mid RTesto \mid A \mid C \mid E \mid R$$

$$A \rightarrow 01 \quad C \rightarrow 10 \quad \mathrm{E} \rightarrow 11 \quad \mathrm{R} \rightarrow 001$$

unambiguous grammar

This grammar is not ambiguous: every message encoded onto Σ admits only one syntax tree, and therefore it can be decoded in a unique way

$$\overbrace{\underbrace{01}_{A}\ \underbrace{10}_{\overbrace{C}^{Testo}}}^{Testo}$$

A bad choice of the set of encoding strings causes the grammar to be ambiguous

ambiguous grammar

$$\Gamma = \left\{ \overbrace{A}^{00}, \overbrace{C}^{01}, \overbrace{E}^{10}, \overbrace{R}^{010}, \right\}$$

$$Testo \rightarrow ATesto \mid CTesto \mid ETesto \mid RTesto \mid A \mid C \mid E \mid R$$

$$A \rightarrow 00 \quad C \rightarrow 01 \quad \mathrm{E} \rightarrow 10 \quad \mathrm{R} \rightarrow 010$$

$$ARRECA = 00\ 010\ 010\ 10\ 01\ 00$$

$$ACAEECA = 00\ 01\ 00\ 10\ 10\ 01\ 00$$

The problem originates from two aspects:

01 00 10 = 010 010, and

01 is a prefix of 010

The *theory of codes* studies these and also other aspects, to identify conditions ensuring that a code (= a set of encoding strings) is decodable in a unique way

OTHER AMBIGUOUS SITUATIONS – regexps may be themselves ambiguous

ambiguous grammar

EXAMPLE 1

$$S \to DcD \quad D \to bD \mid cD \mid \varepsilon$$

$$\{b, c\}^* c \{b, c\}^*$$

Every phase that contains two or more c is ambiguous. To remove ambiguity, one can impose that the mandatory c is the leftmost one

unambiguous version

$$\boxed{S \to BcD \quad D \to bD \mid cD \mid \varepsilon \quad B \to bB \mid \varepsilon}$$

EXAMPLE 2 – Fixing the order of application of the rules – The rule to generate two b can be applied before or after the rule that generates one b

$$S \to bSc \mid bbSc \mid \varepsilon$$

$$S \Rightarrow bbSc \Rightarrow bbbScc \Rightarrow bbbcc$$

$$S \Rightarrow bSc \Rightarrow bbbScc \Rightarrow bbbcc$$

ambiguous grammar

$$\boxed{S \to bSc \mid D \quad D \to bbDc \mid \varepsilon}$$

unambiguous version

AMBIGUITY IN CONDITIONAL PHRASES

$$S \rightarrow if\ \ b\ then\ S\ else\ S \mid if\ \ b\ then\ S \mid a$$

$$\overbrace{if\ \ b\ then\ \overbrace{if\ \ b\ then\ a\ else\ a}}$$

$$\overbrace{if\ \ b\ then\ \ \overbrace{if\ \ b\ then\ a}\ \ else\ a}$$

so-called problem of the "dangling else"

$$S \rightarrow S_E \mid S_T \quad S_E \rightarrow if\ \ b\ then\ S_E\ else\ S_E \mid a$$
$$S_T \rightarrow if\ \ b\ then\ S_E\ else\ S_T \mid if\ \ b\ then\ S$$

Only S_E can precede *else*

unambiguous version 1

$$S \rightarrow if\ \ b\ then\ S\ else\ S\ end\_if \mid$$
$$\mid if\ \ b\ then\ S\ end\_if \mid a$$

unambiguous version 2

Use the additional keyword *end_if* to mark the end of the conditional phrase