PRACTICE SESSION 7

MINIMUM VARIANCE CONTROL

EXERCISE 1

Consider the ARMAX system

$$y(t) = \frac{1}{2} y(t-1) + u(t-1) + e(t) + \frac{1}{3} e(t-1) \qquad e(t) \sim WN(0,1)$$

ⓐ Check the Assumptions for the design of the Minimum variance controller

ASSUMPTION FOR MVC DESIG

Given a general ARMAX system

$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t) \qquad e(t) \sim WN(\mu, \lambda^2)$$

- $b_0 \neq 0$
- $B(z)$ is minimum phase (all the roots strictly inside the U.C.)
- $C(z)/A(z)$ is in canonical form

$$y(t) = \frac{1}{2} z^{-1} y(t) + u(t-1) + \left(1 + \frac{1}{3} z^{-1}\right) e(t)$$

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1}} u(t-1) + \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} e(t)$$

$A(z) = 1 - \frac{1}{2} z^{-1}$
$B(z) = 1$
$C(z) = 1 + \frac{1}{3} z^{-1}$
$k = 1$

- $b_0 = 1 \qquad b_0 \neq 0$ ✓
- $B(z)$ has no roots $\rightarrow$ $B(z)$ is minimum phase ✓
- $\frac{C(z)}{A(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} = \frac{z^{(1)} + 1/3}{z^{(1)} - \frac{1}{2}}$ $\qquad C(z) = 0 \rightarrow z = -1/3$, $\quad A(z) = 0 \rightarrow z = 1/2$
  - zero relative degree ✓
  - $C(z)$, $A(z)$ are coprime ✓
  - $C(z)$, $A(z)$ are monics ✓
  - $C(z)$, $A(z)$ have all the roots strictly inside the U.C. ✓

  $\Big\}$ $C(z)/A(z)$ is in canonical form ✓

ⓑ Compute the k-step predictor

in our case $k = 1$

K-STEP PREDICTOR

$$\hat{y}(t|t-k) = \frac{B(z)E(z)}{C(z)} u(t-k) + \frac{\tilde{R}(z)}{C(z)} y(t-k)$$

$$\begin{array}{l|l} C(z) & A(z) \\ \vdots & \hline \\ & E(z) \\ \hline z^{-k}\tilde{R}(z) & \end{array} \qquad \frac{C(z)}{A(z)} = E(z) + \frac{z^{-k}\tilde{R}(z)}{A(z)}$$

1- STEP PREDICTOR

$$\hat{y}(t|t-1) = \frac{B(z)}{A(z)} u(t-1) + \frac{C(z)-A(z)}{C(z)} y(t)$$

$$C(z) - A(z) = 1 + \tfrac{1}{3} z^{-1} - 1 + \tfrac{1}{2} z^{-1} = \tfrac{5}{6} z^{-1}$$

$$\hat{y}(t|t-1) = \frac{1}{1+\frac{1}{3}z^{-1}} u(t-1) + \frac{5/6}{1+\frac{1}{3}z^{-1}} y(t-1)$$

ⓒ Compute the MVC

MINIMUM VARIANCE CONTROL

general control objective: find $u(t)$ such that $y(t) \cong y^\circ(t)$

$$u(t) = \underset{u(t)}{\arg\min} \left\{ E\left[(y(t) - y^\circ(t))^2 \right] \right\} = \underset{u(t)}{\arg\min} \left\{ E\left[(\hat{y}(t|t-k) - y^\circ(t))^2 \right] \right\}$$

the solution can be obtained imposing $\hat{y}(t|t-k) = y^\circ(t)$

$$\frac{B(z)E(z)}{C(z)} u(t-k) + \frac{\tilde{R}(z)}{C(z)} y(t-k) = y^\circ(t)$$

does not depend on $u(t)$ $\rightarrow$ time shift

$$\frac{B(z)E(z)}{C(z)} u(t) + \frac{\tilde{R}(z)}{C(z)} y(t) = y^\circ(t+k)$$

$y^\circ(t)$ is completely unpredictable $\rightarrow$ $y^\circ(t+k) = y^\circ(t)$

$$u(t) = \frac{1}{B(z)E(z)} \left(C(z)\, y^\circ(t) - \tilde{R}(z)\, y(t) \right)$$

$$\hat{y}(t|t-1) = \frac{1}{1+\frac{1}{3}z^{-1}} u(t-1) + \frac{5/6}{1+\frac{1}{3}z^{-1}} y(t-1) \qquad \hat{y}(t+1|t) = y^\circ(t)$$

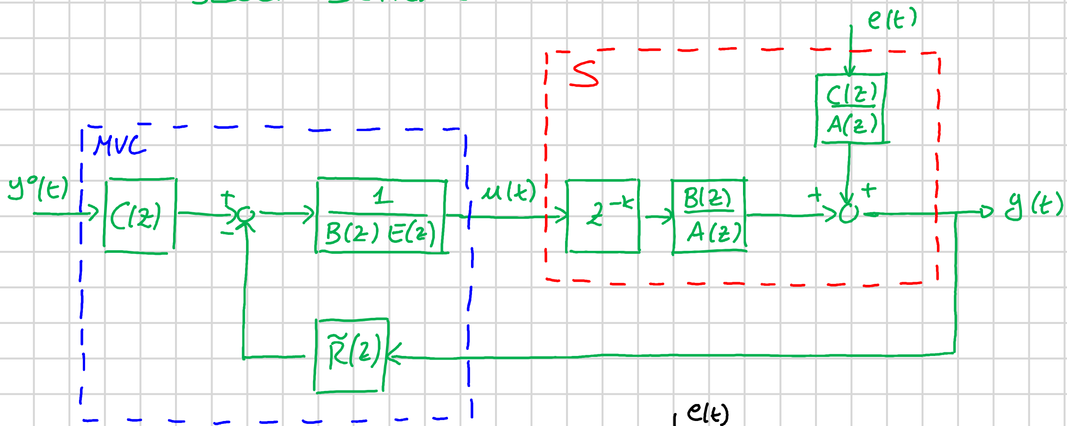
$$\frac{1}{1+\frac{1}{3}z^{-1}} u(t) + \frac{5/6}{1+\frac{1}{3}z^{-1}} y(t) = y^\circ(t)$$

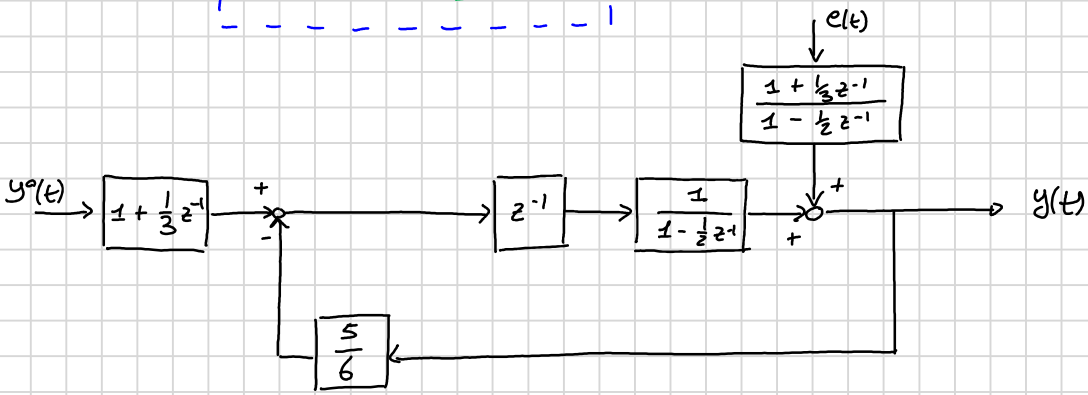
$$u(t) = (1+\tfrac{1}{3}z^{-1})\, y^\circ(t) - \tfrac{5}{6} y(t)$$

$$u(t) = -\tfrac{5}{6} y(t) + y^\circ(t) + \tfrac{1}{3} y^\circ(t-1)$$

d) Draw the closed loop block scheme

GENERAL BLOCK SCHEME





e) Find the transfer functions from

- $y^\circ(t)$ to $y(t)$
- $e(t)$ to $y(t)$

Method I: signal replace

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1}} u(t-1) + \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} e(t)$$

$$u(t) = (1 + 1/3\, z^{-1})\, y^\circ(t) - 5/6\, y(t)$$

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1}} \left[(1 + 1/3\, z^{-1}) y^\circ(t) - \frac{5}{6} y(t) \right] z^{-1} + \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} e(t)$$

$$y(t) = \frac{z^{-1}(1 + \frac{1}{3} z^{-1})}{1 - \frac{1}{2} z^{-1}} y^\circ(t) - \frac{5/6}{1 - \frac{1}{2} z^{-1}} z^{-1} y(t) + \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} e(t)$$

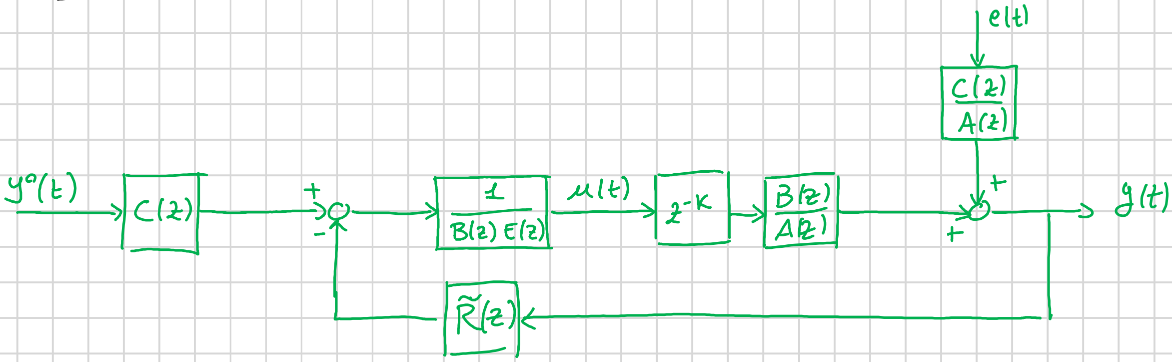
$$y(t) \left(1 + \frac{5/6\, z^{-1}}{1 - \frac{1}{2} z^{-1}} \right) = \frac{z^{-1} + \frac{1}{3} z^{-2}}{1 - \frac{1}{2} z^{-1}} y^\circ(t) + \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} e(t)$$

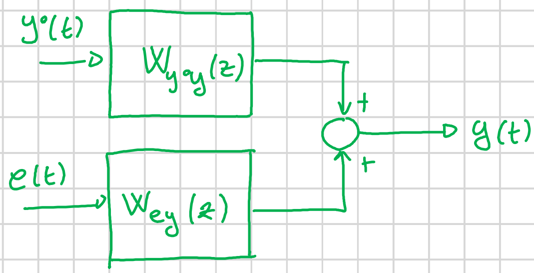
$$y(t) \left(\frac{\cancel{1 + \frac{1}{3} z^{-1}}}{\cancel{1 - \frac{1}{2} z^{-1}}} \right) = \frac{z^{-1}\, \cancel{(1 + 1/3\, z^{-1})}}{\cancel{(1 - \frac{1}{2} z^{-1})}} y^\circ(t) + \frac{\cancel{(1 + 1/3\, z^{-1})}}{\cancel{(1 - \frac{1}{2} z^{-1})}} e(t)$$

$$y(t) = z^{-1} y^\circ(t) + e(t)$$

$$\boxed{W_{y^\circ y}(z) = z^{-1} \qquad W_{ey}(z) = 1}$$

Method II : Block Scheme





$$W(z) = \frac{\text{DIRECT PATH}}{1 + \text{LOOP PATH}}$$

$$W_{ey}(z) = \frac{C(z)/A(z)}{1 + \frac{1}{\cancel{B(z)}E(z)} z^{-k} \frac{\cancel{B(z)}}{A(z)} \tilde{R}(z)} =$$

$$= \frac{\cancel{C(z)}/\cancel{A(z)}}{\frac{\cancel{E(z)A(z) + z^{-k}\tilde{R}(z)}}{E(z)\,\cancel{A(z)}}} = E(z)$$

$$\frac{C(z)}{A(z)} = E(z) + \frac{z^{-k}\tilde{R}(z)}{A(z)}$$

$$W_{y^\circ y}(z) = \frac{\cancel{C(z)} \frac{1}{\cancel{B(z)}\cancel{E(z)}} z^{-k} \frac{\cancel{B(z)}}{\cancel{A(z)}}}{\frac{\cancel{E(z)A(z) + z^{-k}\tilde{R}(z)}}{\cancel{E(z)}\,\cancel{A(z)}}} = z^{-k}$$

$$\boxed{\begin{array}{lcl} K = 1 & & W_{y^\circ y}(z) = z^{-1} \\ & \longrightarrow & \\ E(z) = 1 & & W_{ey}(z) = 1 \end{array}}$$

(f) Check the closed loop asymptotic stability

ASYMPTOTIC STABILITY

- Loop function: $L(z) = \frac{1}{B(z)E(z)} \cdot \frac{z^{-k} B(z)}{A(z)} \tilde{R}(z)$

with negative feedback

- characteristic polynomial: $\chi(z) = L_D(z) + L_N(z)$

$$\chi(z) = B(z)E(z)A(z) + z^{-k} B(z)\tilde{R}(z) =$$
$$= B(z)\left(E(z)A(z) + z^{-k}\tilde{R}(z)\right) =$$
$$= B(z)\,C(z)$$

- Roots of $\chi(z) \to \chi(z) = 0$

  - $B(z)$ is minimum phase
  - $C(z)/A(z)$ is in canonical form

thanks to the MVC design Assumptions the closed loop system is asymptotically stable

$$\chi(z) = B(z)\,C(z) = 1\left(1 + \tfrac{1}{3} z^{-1}\right) = 0 \to z = -\tfrac{1}{3}$$

$|z| < 1 \to$ $\boxed{\text{the C.L. system is asymptotically stable}}$

EXERCISE 2

Consider the system

$$y(t) = \frac{1}{2} y(t-1) + u(t-2) + e(t-1) + 2e(t-2) \qquad e(t) \sim WN(0,1)$$

a) Find the MVC law

$$y(t) = \frac{1}{1-\frac{1}{2}z^{-1}} u(t-2) + \frac{z^{-1}+2z^{-2}}{1-\frac{1}{2}z^{-1}} e(t)$$

$A(z) = 1 - \frac{1}{2} z^{-1}$
$B(z) = 1$
$C(z) = z^{-1} + 2z^{-2}$
$K = 2$

I) check the assumption

- $b_0 = 1 \rightarrow b_0 \neq 0$ ✓
- $B(z)$ has no roots $\rightarrow$ $B(z)$ is minimum phase ✓
- $\frac{C(z)}{A(z)} = \frac{z^{-1}+2z^{-2}}{1-\frac{1}{2}z^{-1}} = \frac{z+2}{z^2-\frac{1}{2}z}$ $\qquad C(z)=0 \rightarrow z=-2$, $A(z)=0 \rightarrow z=+\frac{1}{2}$
  - relative degree is zero ✗ $\rightarrow \tilde{e}(t) = e(t-1) \rightarrow \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{z+2}{z-\frac{1}{2}}$ ✓
  - $C(z), A(z)$ are monic ✓
  - $C(z), A(z)$ are coprime ✓
  - $C(z), A(z)$ have all the root strictly inside the U.C ✗ we need to manipulate $C(z)/A(z)$

All pass filter trick

$$\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}} \tilde{e}(t) = \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \underbrace{\frac{1+az^{-1}}{1+\frac{1}{a}z^{-1}} \frac{1}{a}}_{\text{ALL PASS FILTER}} \tilde{e}(t) \qquad a = \frac{1}{2}$$

$$\frac{\cancel{1+2z^{-1}}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1+\frac{1}{2}z^{-1}}{\cancel{1+2z^{-1}}} \underbrace{2\tilde{e}(t)}_{\eta(t)} = \underbrace{\frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}}_{\text{is in canonical form}} \eta(t) \qquad \eta(t) \sim WN(0,4)$$

- $\frac{C(z)}{A(z)} = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$ is in canonical form ✓

II) Compute the k-step predictor

$$
\begin{array}{l|l}
1+\frac{1}{2}z^{-1} & 1-\frac{1}{2}z^{-1} \\
1+\frac{1}{2}z^{-1} & \overline{1+z^{-1}} \quad E(z) \\
\hline
/ \quad z^{-1} & \\
\quad -z^{-1}+\frac{1}{2}z^{-2} & \\
\hline
/ \quad \frac{1}{2}z^{-2} \quad z^{-k}\tilde{R}(z) &
\end{array}
$$

$E(z) = 1 + z^{-1}$

$\tilde{R}(z) = \frac{1}{2}$

$$\hat{y}(t+2 \mid t) = \frac{(1+z^{-1})}{1+\frac{1}{2}z^{-1}} u(t) + \frac{\frac{1}{2}}{1+\frac{1}{2}z^{-1}} y(t)$$

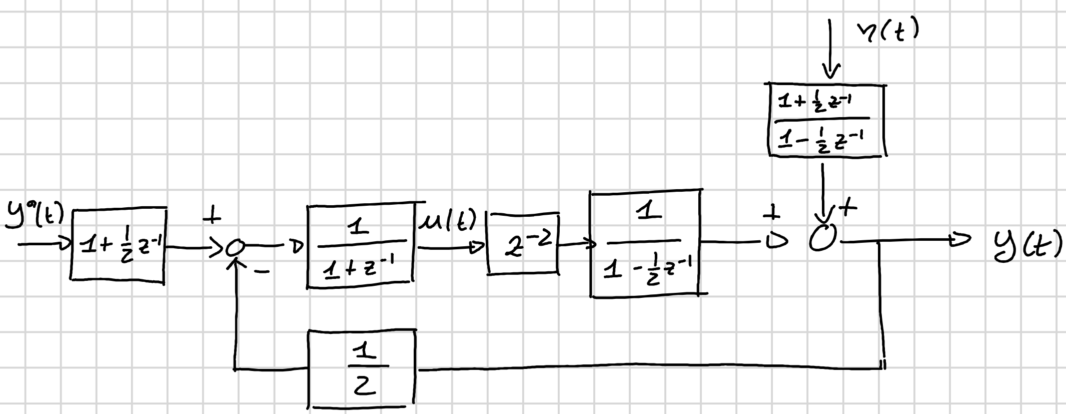
III) Impose $\hat{y}(t+k \mid t) = y^{o}(t)$

$$\frac{1+z^{-1}}{1+\frac{1}{2}z^{-1}} u(t) + \frac{\frac{1}{2}}{1+\frac{1}{2}z^{-1}} y(t) = y^{o}(t)$$

$$u(t) = \frac{1}{(1+z^{-1})}\left(\left(1+\frac{1}{2}z^{-1}\right) y^{o}(t) - \frac{1}{2} y(t)\right)$$

$$u(t) = -u(t-1) - \frac{1}{2} y(t) + y^{o}(t) + \frac{1}{2} y^{o}(t-1)$$

(b) Draw the C.L. block scheme



(c) Check the closed loop asymptotic stability

Loop function: $L(z) = \frac{\frac{1}{2}z^{-2}}{(1+z^{-1})(1-\frac{1}{2}z^{-1})}$

$$\chi(z) = (1+z^{-1})(1-\tfrac{1}{2}z^{-1}) + \tfrac{1}{2}z^{-2}$$

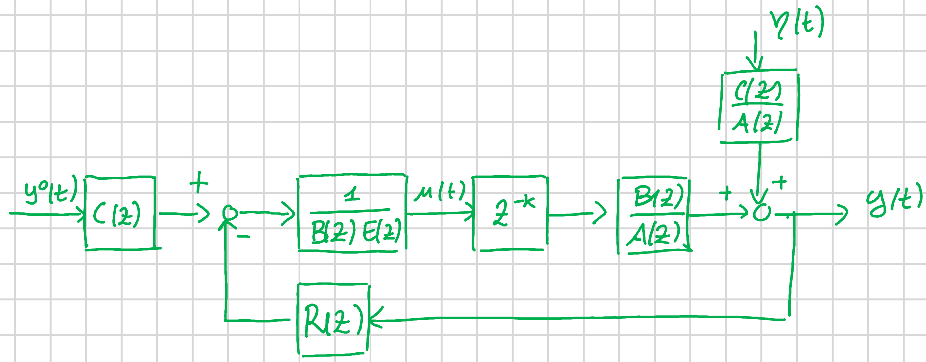
$$x(z) = 1 - \frac{1}{2}z^{-1} + z^{-1} - \cancel{\frac{1}{2}z^{-2}} + \cancel{\frac{1}{2}z^{-2}} = 1 + \frac{1}{2}z^{-1}$$

$$x(z) = 0 \quad \to \quad 1 + \frac{1}{2}z^{-1} = 0 \qquad z = -\frac{1}{2}$$

$|z| < 1 \quad \to \quad$ $\boxed{\text{the closed loop system is asymptotically stable}}$

(d) Compute the transfer function from

- y^o to u
- η to u



$$W_{y^o u}(z) = \frac{C(z)\,\frac{1}{B(z)E(z)}}{1 + \frac{\cancel{B(z)}\,\tilde{R}(z)\,z^{-k}}{\cancel{B(z)}\,E(z)A(z)}} =$$

$$= \frac{\cancel{C(z)}/B(z)\cancel{E(z)}}{\frac{\cancel{E(z)A(z) + \tilde{R}(z)z^{-k}}}{\cancel{E(z)}A(z)}} = \frac{A(z)}{B(z)}$$

$$W_{\eta u}(z) = \frac{-\frac{\cancel{C(z)}\,\tilde{R}(z)}{B(z)\cancel{E(z)A(z)}}}{\frac{\cancel{E(z)A(z) + \tilde{R}(z)z^{-k}}}{\cancel{E(z)A(z)}}} = -\frac{\tilde{R}(z)}{A(z)}$$

$$\boxed{W_{y^o u}(z) = \frac{A(z)}{B(z)} = \left(1 - \frac{1}{2}z^{-1}\right) \qquad W_{eu}(z) = -\frac{\tilde{R}(z)}{B(z)} = -\frac{1}{2}}$$