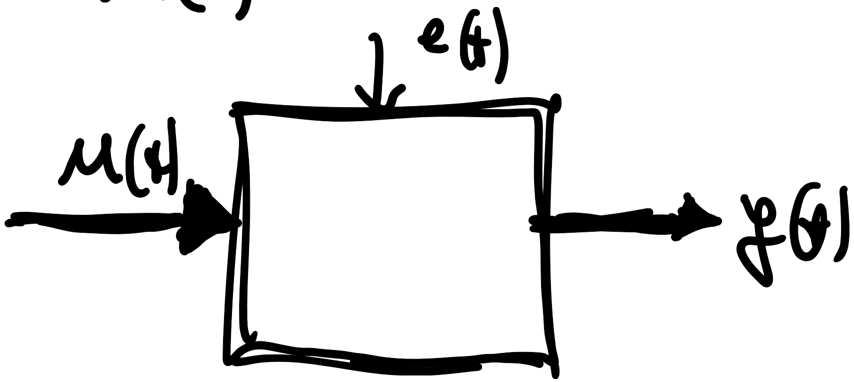
Chapter # 3 → KALMAN filter

(SW SENSING in FEEDBACK)

preliminary remark -- in MIDA 1 we have mostly used I/O (TF) representations:

$$y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \qquad e \sim WN$$

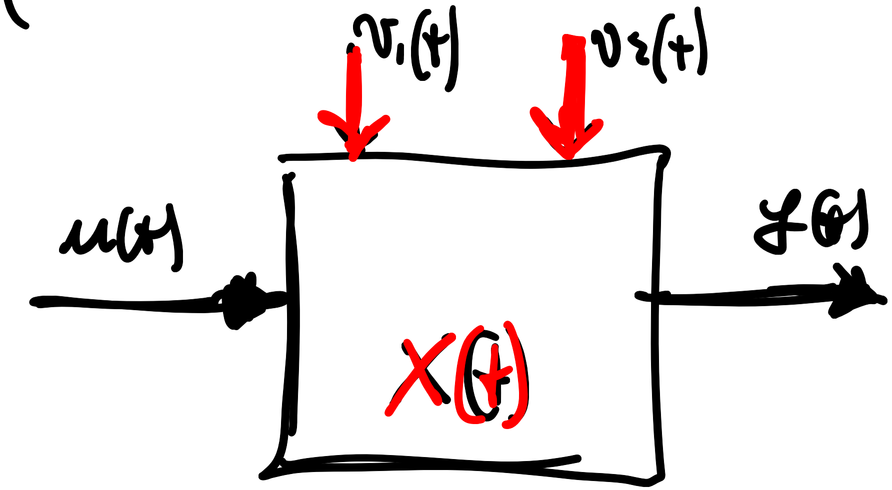


Kalman Filter theory is fully BASED on S.S. REPRESENT.

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) + \boxed{v_1(t)} \\ y(t) = Hx(t) + \cancel{Du(t)} + \boxed{v_2(t)} \end{cases}$$

$v_1 \sim WN$

$v_2 \sim WN$



- We are INTERESTED in STATES
- We use a 2-NOISES DESCRIPTION

MOTIVATIONS AND GOALS of K.F. theory

GIVEN a MODEL DESCRIPTION $\{F, G, H\}$ + NOISES VARIANCES

(NOT. A SY. ID. TECHNIQUE) WE DO NOT NEED RECORDED DATA

with K.F. theory we can ADDRESS the following problems:

#1 --> K-STEP Ahead prediction of output:

$$\hat{y}(t+k\,/\,t)$$

PRESENT TIME (NEWEST AVAILABLE DATA)

prediction horizon (K-STEPS AHEAD in the FUTURE)

this problem already solved in MIDA 1 - ARMAX

k-steps ahead prediction of state

$\hat{x}(t+k|t)$

$\hookrightarrow$ at time t we have available:

$y(t), y(t-1), y(t-2) \dots \rightarrow$

$u(t), u(t-1), u(t-2) \dots \rightarrow$

New problem (NOT possible with ARMAX models)

#3: find the FILTER of STATE

$$\hat{x}(t|t)$$

ESTIMATION MADE AT SAME TIME

given DATA ($y(t)$ and past + $u(t)$ and past) up to time t

Notice that $\hat{y}(t|t)$ is NONSENSE $\rightarrow \hat{y}(t|t) = y(t)$

In practice is "SW-SENSING" → most important problem solved by K.F. (INDEED K.F. is NAMED "FILTER")

#4 (SEE chapter 5) $\rightarrow$ GRAY BOX SYSTEM IDENTIFICATION

(SIDE-BENEFIT of KF, not the KEY objective of KF)

$\rightarrow$ we have a RECORDED DATA SET

$$\{u(1), u(2) \dots u(N)\}, \{y(1), y(2) \dots y(N)\}$$

AND the model STRUCTURE (F, G, H), but with some UN-KNOWN parameters:

$$S: \begin{cases} x(t+1) = F(\theta)\, x(t) + G(\theta)\, u(t) \\ y(t) = H(\theta)\, x(t) \end{cases}$$

θ is a vector of SOME UN-KNOWN parameters of the model

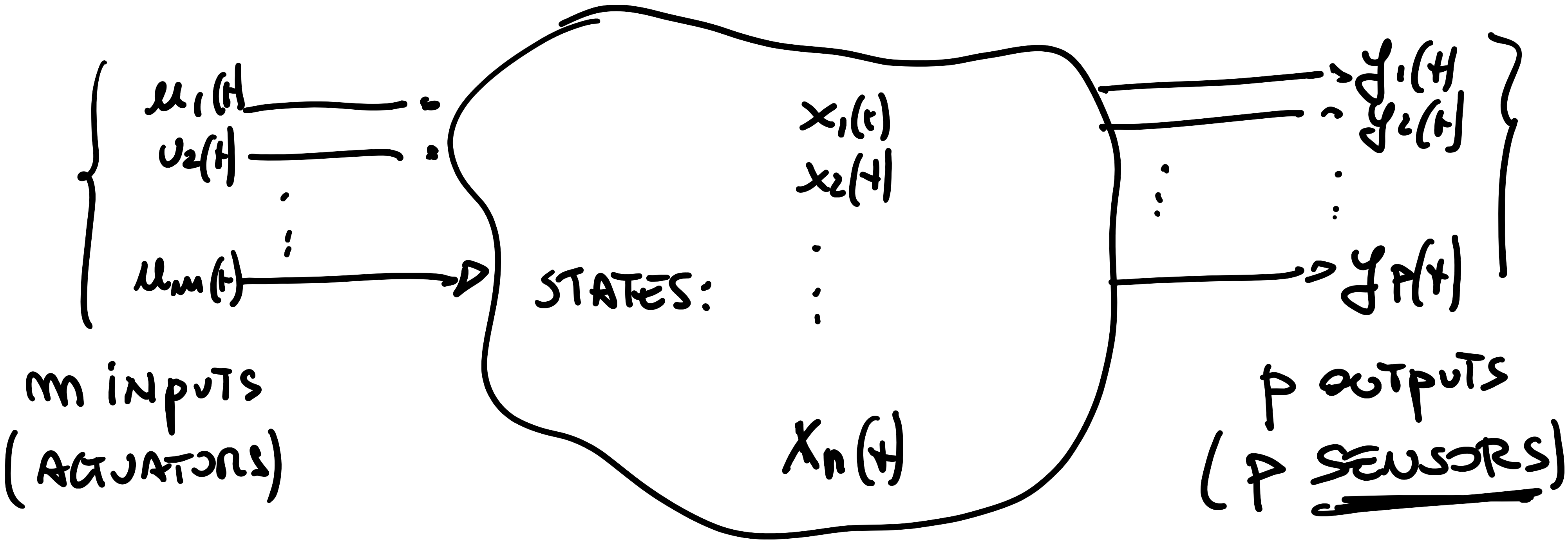
$$F = \begin{bmatrix} 1 & 1/2 \\ a & 1/4 \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ b \end{bmatrix} \qquad H = \begin{bmatrix} c & 1/2 \end{bmatrix}$$

$$\vartheta = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \underline{KF} \longrightarrow \text{ESTIMATION } \underline{\underline{\text{from data}}}$$

$$\hat{\vartheta}_N = \begin{bmatrix} \hat{a}_N \\ \hat{b}_N \\ \hat{c}_N \end{bmatrix}$$

Why K.F. is so USEFUL?

Dy. systems have this layout:



MIMO system with m inputs, p outputs and n STATES

Key problem is that usually $\boxed{p \ll n}$ ⟹ physical sensors are much less than sy. states

<u>why?</u>:
- <u>Cost</u>
- <u>Cables</u>, power supply..
- Maintenance (faults, degradation..)

⟹ usually not all the states are measured

It is useful to have full "<u>measurement</u>" (phy. or sw) of states because:

- Control design (state feedback)
- <u>Monitoring</u> (fault detection – <u>predictive</u> <u>maintenance</u>, ...)

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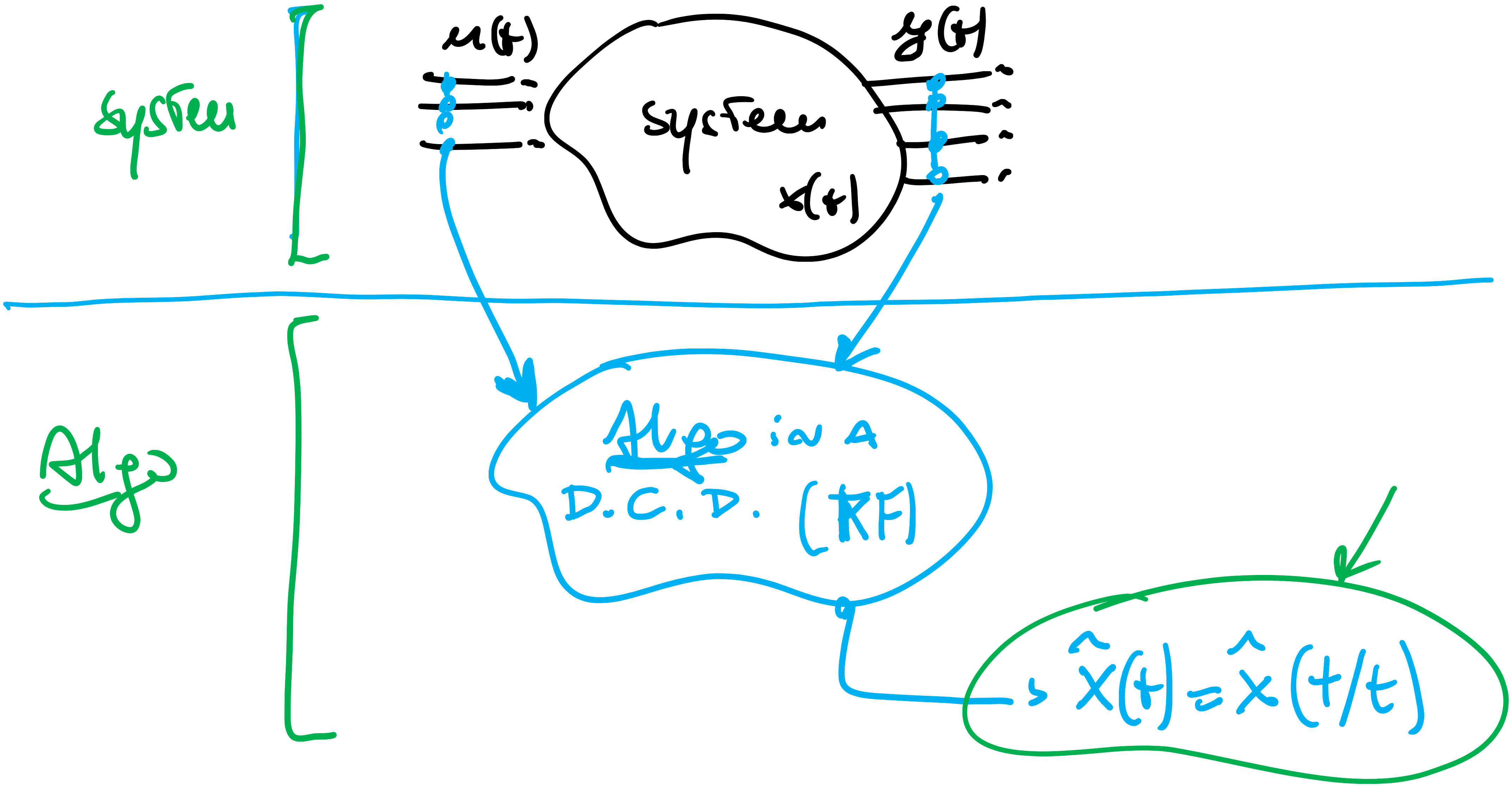
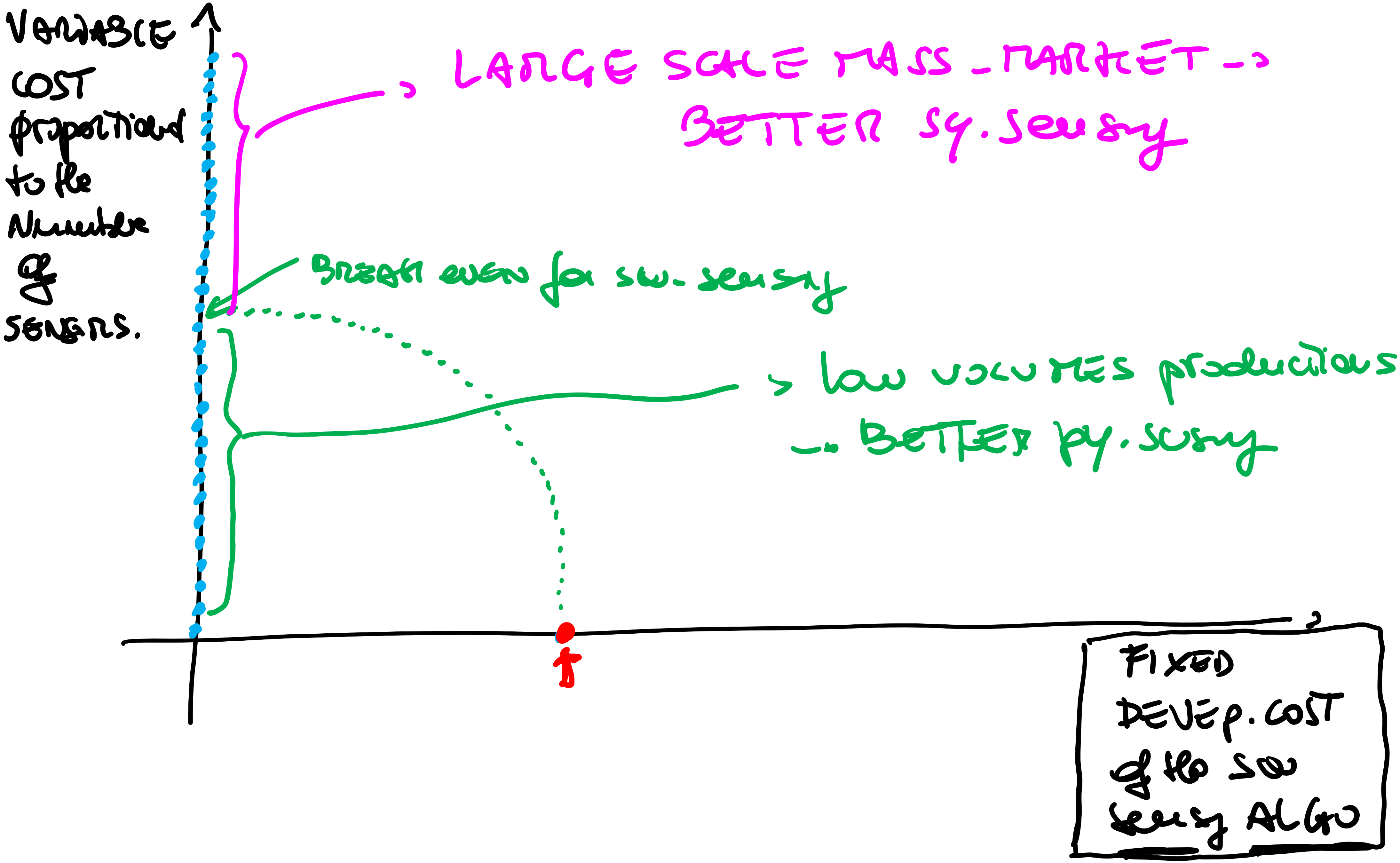
"SW. SENSING" (VIRTUAL SENSING)



Dilemma -> when using SW-sensing?

- IN SOME CASES there is NO option => NOT feasible installation of a phy. sensor

- In MOST cases both options are VIABLE -> VARIABLE vs. FIXED COST


VARIABLE COST proportional to the Number of SENSORS.
LARGE SCALE MASS-MARKET →
BETTER SW. SENSING
BREAK EVEN for SW-SENSING
> low VOLUMES productions
→ BETTER HW. SENSING
FIXED DEVEP. COST of the SW sensing ALGO

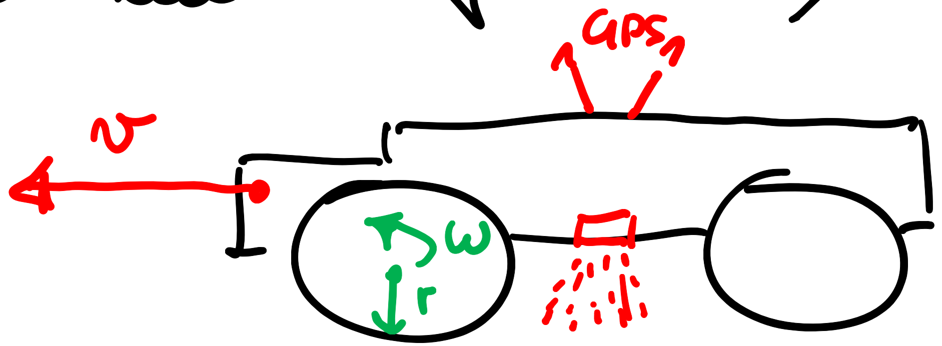
Key questions for SW. sensing:

1. Is SW. SENSING $\boxed{\text{FEASIBLE}}$? → TEST is the **OBSERVABILITY** of the STATES from MEASURED OUTPUTS

2. QUALITY of ESTIMATION ERROR ("NOISE" of "MEASUREMENT" for the SW. SENSOR)

Examples of SW. SENSING

Slip estimation for ABS / Traction control



$$\lambda = \frac{v - \omega r}{v}$$

problem →

ESTIMATION of v

⇓

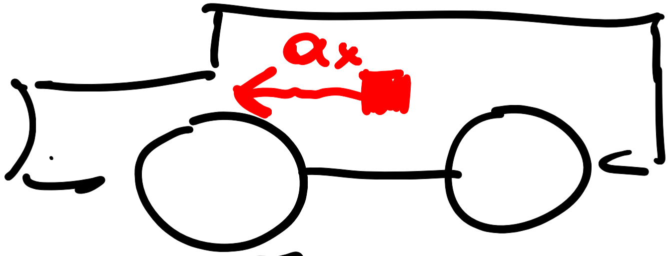
MEASURE of v:

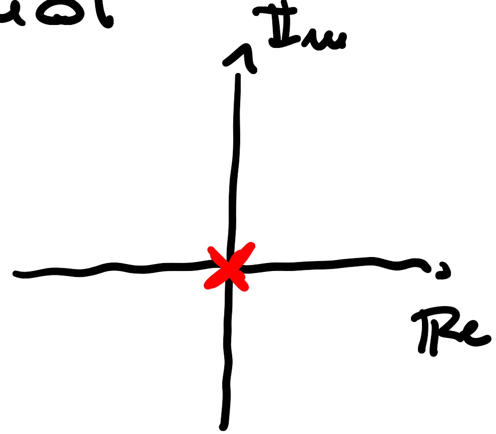
- optical sensors
- GPS-ESTIMATED SPEED

Both have a problem of AVAILABILITY (NOT GUARANTEED)

⇒ physical SENSING is NOT AN OPTION for INDUSTRIAL PRODUCTION

Intuitive solution → install a longitudinal accelerometer (a_x) → integrate it





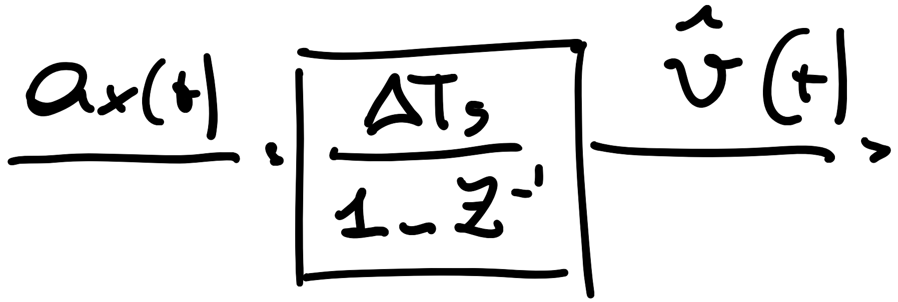
$$\boxed{\hat{v}(t) = \int a_x(t)\,dt} \iff a_x(t) \longrightarrow \boxed{\frac{1}{s}} \longrightarrow \hat{v}(t)$$

Discrete time domain → Discretization using approximation of derivative (Euler forward method

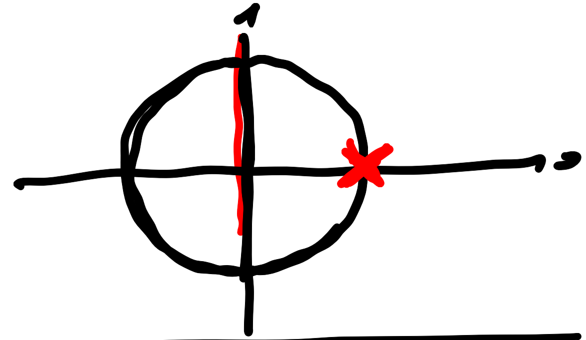
$$\frac{d}{dt}\hat{v}(t) = a_x(t) \qquad \frac{d\hat{v}(t)}{dt} \approx \boxed{\frac{\hat{v}(t+1) - \hat{v}(t)}{\Delta T_s} = a_x(t)}$$

ΔT_s = sampling interval (e.g. 10 ms)

$$\boxed{\hat{v}(t) = \hat{v}(t-1) + \Delta T_s \, a_x(t)}$$

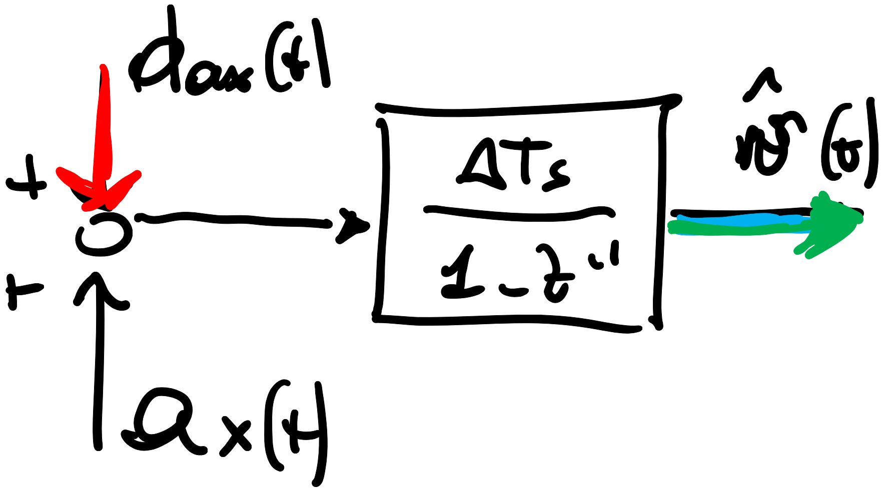


DISCRETE-TIME INTEGRATOR

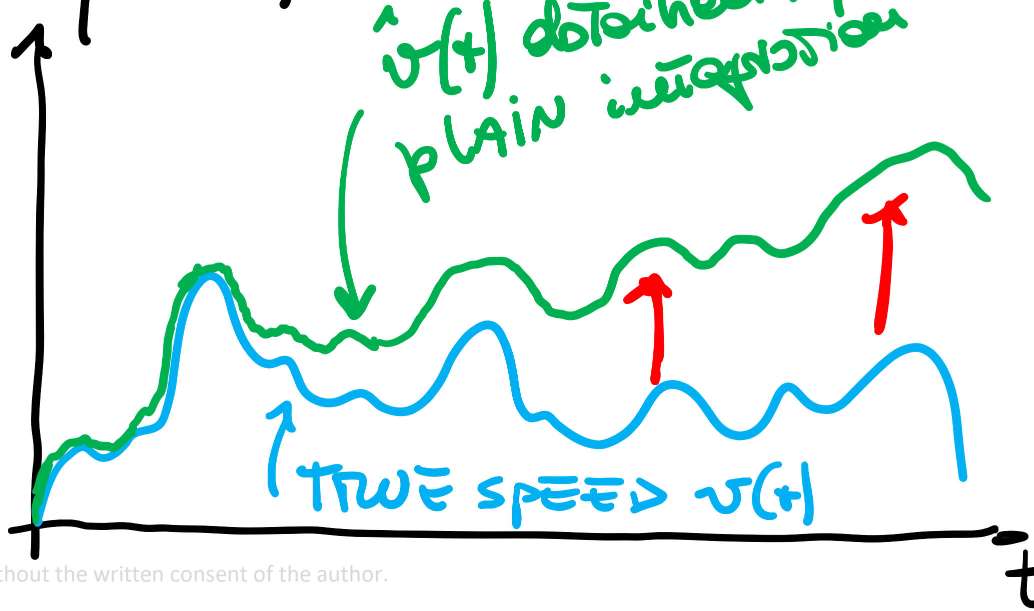


UNFORTUNATELY the MEASURED SIGNAL is NOT $a_x(t)$ but

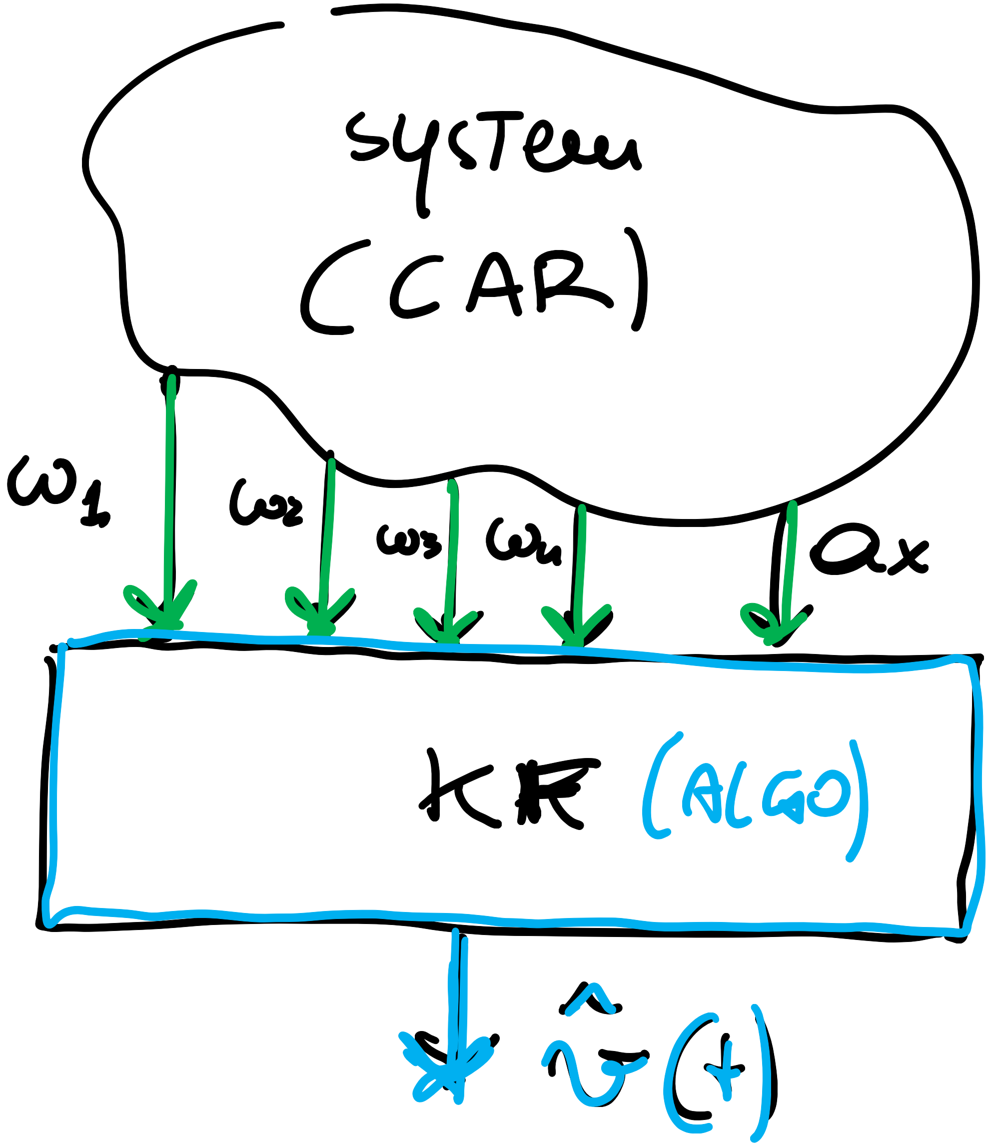
$a_x(t) + d_{a_x}(t)$ → MEASUREM. NOISE



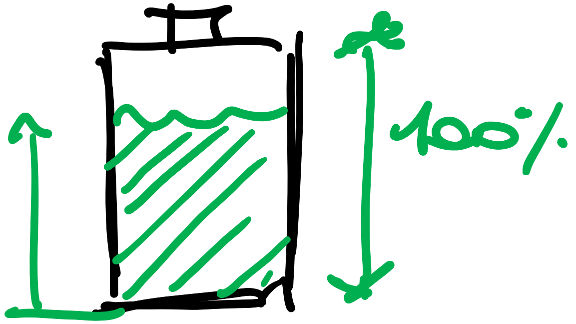
• Integrating noise generates a "drift" (Integrator is NOT As. STABLE system)

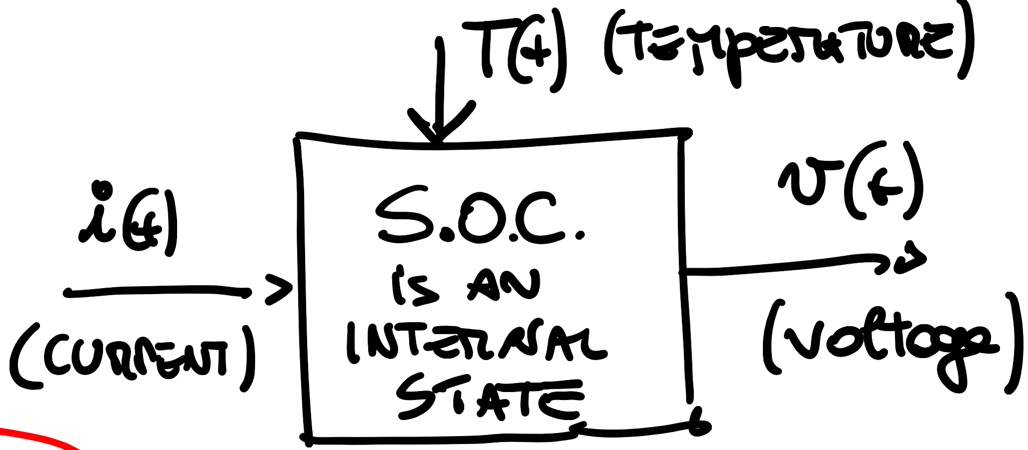


→ solution → use K.F.:



Example #1: State of charge estimation of a BATTERY:





State of charge = SoC

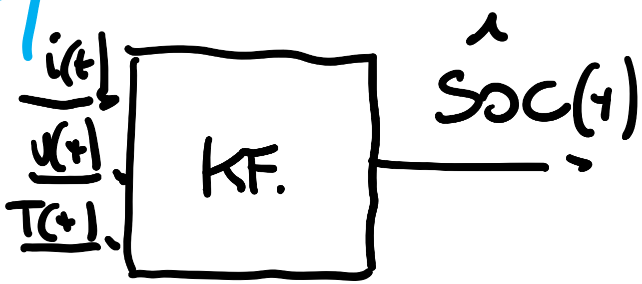
$$\underbrace{SOC(t)}_{0 \leq SOC \leq 1} = 1 - \frac{\int i(t)\,dt}{I}$$

Integral starts at 100% SoC.

Total AMOUNT of current that can be extracted by the user of the BATTERY

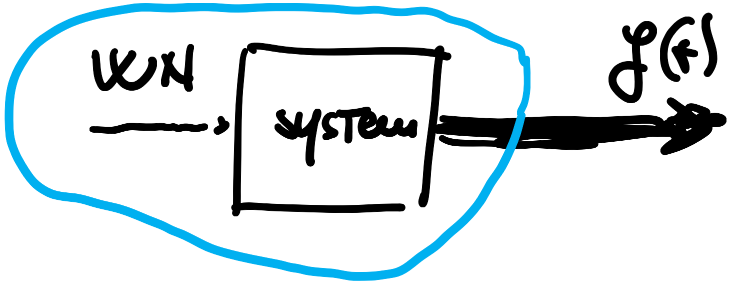
this solution is NOT feasible AGAIN BECAUSE WE INTEGRATE NOISE ON $i(t)$

$\Rightarrow$ K.F.



I'll present K.F. STARTING from a "BASIC" system:

- NO external inputs $(\cancel{G_u(t)}) \Rightarrow$ time series:
- LINEAR system
- T. INVARIANT system



then, after we describe the solution of K.F. for the BASIC system, we make the extensions to a more general system (in particular: $G_u(t)$)

Detailed description of the "BASIC system"

$$
S: \begin{cases} x(t+1) = Fx(t) + (Gu(t)) + v_1(t) \\ y(t) = Hx(t) + v_2(t) \end{cases}
$$

← STATE equation

← output equation

$$
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \left(u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \right) \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}
$$

System with n states, (m inputs) and p outputs (MIMO system) (if $m=1$ and $p=1$ => SISO system)

$v_1(t)$ is a <u>VECTOR WN</u>

$$\boxed{v_1(t) \sim WN(0, V_1)}$$

$$v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \\ \vdots \\ v_{1n}(t) \end{bmatrix}$$

$v_1(t)$ is CALLED <u>STATE NOISE</u> OR <u>MODEL NOISE</u> → It accounts for the <u>MODELING ERRORS</u> of the STATE EQUATION of the system

1) $E[v_1(t)] = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

2) $E[v_1(t) \cdot v_1(t)^T] = V_1$ → $n \times n$ matrix → COVARIANCE MATRIX ⇒ symmetric and s.d.p: $V_1 \geq 0$

$$\boxed{3)\ E[v_1(t) \cdot v_1(t-\tau)^T] = \emptyset \quad \forall t \quad \forall \tau \neq 0}$$

Key feature of a WN

$v_2(t)$ is a VECTOR WN

$$v_2(t) \approx WN(0, V_2)$$

$$v_2(t) = \begin{bmatrix} v_{21}(t) \\ v_{22}(t) \\ \vdots \\ v_{2p}(t) \end{bmatrix}$$

CALLED OUTPUT NOISE
OR MEASUREMENT (SENSOR) NOISE

1) $E[v_2(t)] = \vec{0}$

2) $E[v_2(t) v_2(t)^T] = V_2$

3) $E[v_2(t) \cdot v_2(t-\tau)^T] = \phi$
$\forall t \ \forall \tau \neq 0$

- Is a $p \times p$ symmetric covariance matrix, $V_2 \geq 0$ (s.d.p)
- we make the ADDITIONAL ASSUMPTION that V_2 is DEFINITE POSITIVE (d.p)

$$V_2 > 0$$

* we'll use this ASSUMPTION in RICCATI EQUATION of KALMAN FILTER

Now we NEED To make assumptions on the RELATIONSHIP between $v_1(t)$ and $v_2(t)$:

$$E\left[v_1(t) \cdot v_2(t-\tau)^T\right] = \underbrace{V_{12}}_{\substack{n \times p \\ \text{matrix} \\ \text{(CROSS-VARIANCE matrix)}}} = \begin{cases} \emptyset \text{ if } \tau \neq 0 \\ \rightarrow \text{can be NON-ZERO if } \tau = \emptyset \end{cases}$$

we assume that v_1 and v_2 CAN be correlated but only at the same time (IN practice, $V_{12} = 0$ is the most common assumption)

Since the system $\mathcal{S}$ is dynamic $\Rightarrow$ we NEED TO DEFINE the INITIAL CONDITIONS:

$$E[x(1)] = x_0 \quad (n \times 1 \text{ vector})$$

$$E[(x(1)-x_0)(x(1)-x_0)^T] = P_0 \geq 0 \quad \left(\begin{array}{c} n \times n \text{ symmetric} \\ \text{sdp matrix} \end{array}\right)$$

probabilistic description of the initial condition for $\mathcal{S}$

if $P_0 = \emptyset \Rightarrow$ INITIAL STATE IS perfectly KNOWN we finally observe the the two noises $v_1(t)$ and $v_2(t)$ are UNCORRELATED with the initial STATE:

$$X(1) \perp v_1(t)$$
$$X(1) \perp v_2(t)$$

Technical assumption NEEDED to guarantee the theoretical optimality of K.F.

Now we present the BASIC SOLUTION → 1-STEP AHEAD PREDICTION $\left(\hat{y}(t+1/t), \hat{x}(t+1/t)\right)$ for the BASIC SYSTEM

(LATER ON we'll SEE ALL the extensions)

$\boxed{\text{NO PROOF of KF}}$ → PRESENT the Solution

(BITTANTI BOOK - REFERENCES → PROOF
NOT in the EXAM)

K.F. for the BASIC SOLUTION of the BASIC SYSTEM

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + \boxed{K(t)}\,\boxed{e(t)}$$ STATE EQUATION

$$\hat{y}(t|t-1) = H\hat{x}(t|t-1)$$ OUTPUT EQUATION

$$\boxed{e(t) = y(t) - \hat{y}(t|t-1)}$$ OUTPUT PREDICTION ERROR

$$\boxed{K(t) = \left(FP(t)H^T + V_{12}\right)\left(HP(t)H^T + V_2\right)^{-1}}$$ EQUATION of the <u>GAIN</u> of the K.F.

$$\boxed{P(t+1) = \left(FP(t)F^T + V_1\right) - \left(FP(t)H^T + V_{12}\right)\left(HP(t)H^T + V_2\right)^{-1}\left(FP(t)H^T + V_{12}\right)^T}$$

↳ DIFFERENCE RICCATI EQUATION (DRE)

5 equations of K.F.

these equations must be completed with 2 initial conditions (2 equations are dynamic equation):

STATE EQUATION → $\boxed{\hat{x}(1|0) = E[x(1)] = x_0}$

D.R.E. → $\boxed{P(1) = var[x(1)] = P_0}$

END of solution of K.F.

Remark (STRUCTURE of $K(t)$ and D.R.E.):

Notice that $K(t)$ and DRE have a "BLOCKSET" STRUCTURE having this form: $\left(\square \cdot P(t) \cdot \diamond^T + \boxed{\text{NOISE MATRIX}} \right)$

3 $\neq$ types of "BLOCKS":

"STATE" $\rightarrow$ $F P(t) F^T + V_1$ $\rightarrow$ F, V_1 refer to STATE eq.

"OUTPUT" $\rightarrow$ $H P(t) H^T + V_2$ $\rightarrow$ H, V_2 refer to OUTPUT Eq.

"MIX" $\rightarrow$ $F P(t) H^T + V_{12}$ $\rightarrow$ $F \rightarrow$ STATE, $H \rightarrow$ OUTPUT

$V_{12} \rightarrow$ RELATIONSHIP BETWEEN V_1 and V_2

GAIN $K(t) = (\text{MIX}) \cdot (\text{OUTPUT})^{-1}$

DRE: $P(t+1) = (\text{STATE}) - (\text{MIX})(\text{OUTPUT})^{-1}(\text{MIX})^T$

Remark (Riccati equation)

Riccati EQATION is a special Type of NONLINEAR MATRIX DIFFERENCE EQUATION

(It was studied by COUNT RICCATI in 1700 → TURNED OUT TO BE USEful in '60 for K.F.)

Notice that DRE is AN AUTONOMOUS, NONLINEAR, DISCRETE TIME, MULTIVARIABLE SYSTEM, DESCRIBED BY a NONLINEAR DIFFERENCE matrix equation

$$\underline{DRE}: \begin{cases} P(t+1) = f(P(t)) + \cancel{\text{NO FORCING INPUTS}} \\ P(1) = P_0 \end{cases}$$

Remark (existence of DRE):

In order to GUARANTEE the existence of DRE $\forall t$ the only critical part is the <u>INVERSION</u> of the "OUTPUT" BLOCK:

$$\left(\boxed{HP(t)H^T} + V_2\right)^{-1}$$

→ we MADE the assumption that $\underline{V_2 > 0}$ (D.P.)

↓ $p \times p$ matrix always ≥ 0 (s.d.p) BUT NOT GUARANTEED to BE INVERTIBLE

→ thanks to $V_2 > 0$, the overall "BLOCK" is D.P. ⇒ INVERTIBLE!

Remark (meaning of $P(t)$)

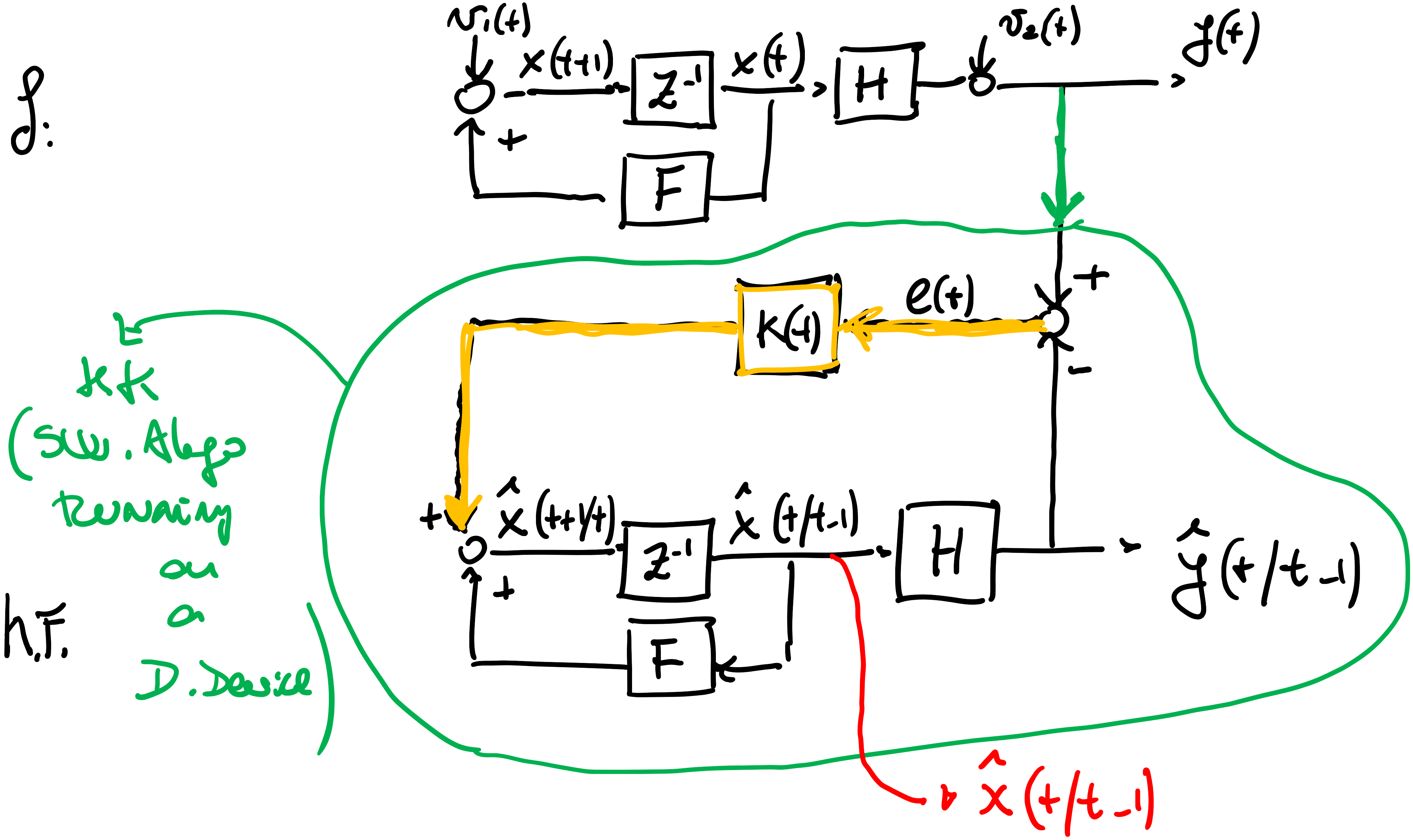
It can be proven that $P(t)$ has a very important meaning:

$$P(t) = \operatorname{var}\left[x(t) - \hat{x}(t|t-1)\right] = E\left[\left(x(t) - \hat{x}(t|t-1)\right)\left(x(t) - \hat{x}(t|t-1)\right)^T\right]$$

$P(t)$ is a square symmetric sdp (≥ 0) matrix (covariance matrix)

$P(t)$ is the covariance of the 1-step prediction error of the state

BLOCK-scheme Representation of K.F. →



The idea behind KF is simple and intuitive:

- MAKE A SIMULATED REPLICA of the system (without noises v_1 and v_2 NOT MEASURABLE)

- Compare the TRUE (measured) output with the ESTIMATED/SIMULATED output $\hat{y}(t|t-1)$

- MAKE CORRECTIONS ON K.F. main equation, proportional (with gain $K(t)$) to the output error, in order to keep K.F. as close as possible to S.

$\rightarrow$ "EXTRACT" the state estimation $\hat{x}(t|t-1)$

KF is a FEEDBACK system; feedback here is NOT USED for CONTROL but for ESTIMATION!

this general structure / idea was known before K.F. developement → called "<u>STATE OBSERVER</u>"

Fundamental contribution of Kalman was to find the <u>OPTIMAL GAIN $K(t)$</u>

$K(t)$ is not a simple scalar gain (cannot be tuned empirically) but is a (maybe very large) $\boxed{n \times p \text{ matrix}}$

→ matrix of gains that can be very large and difficult to be tuned

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Selection of gain matrix $K(t)$ is very critical ↓

- if $K(t)$ is "too small" → estimation is not optimal because we are "UNDER-exploiting" the information in $y(t)$
- if $K(t)$ is "too big" → Risk of OVER-Exploiting $y(t)$ → Noise amplification → EVEN Risk of INSTABILITY

KALMAN provided a theoretically OPTIMAL matrix gain $K(t)$ with a constructive / direct design from system matrices $\begin{pmatrix} F, G, H \\ V_1, V_2 \end{pmatrix}$

→ DESIGN of KF does NOT REQUIRE a "TRAINING DATASET" BUT a complete model of the system:

→ $\boxed{F, G, H \text{ matrices}}$ → usually obtained with a white-Box physical modeling of S

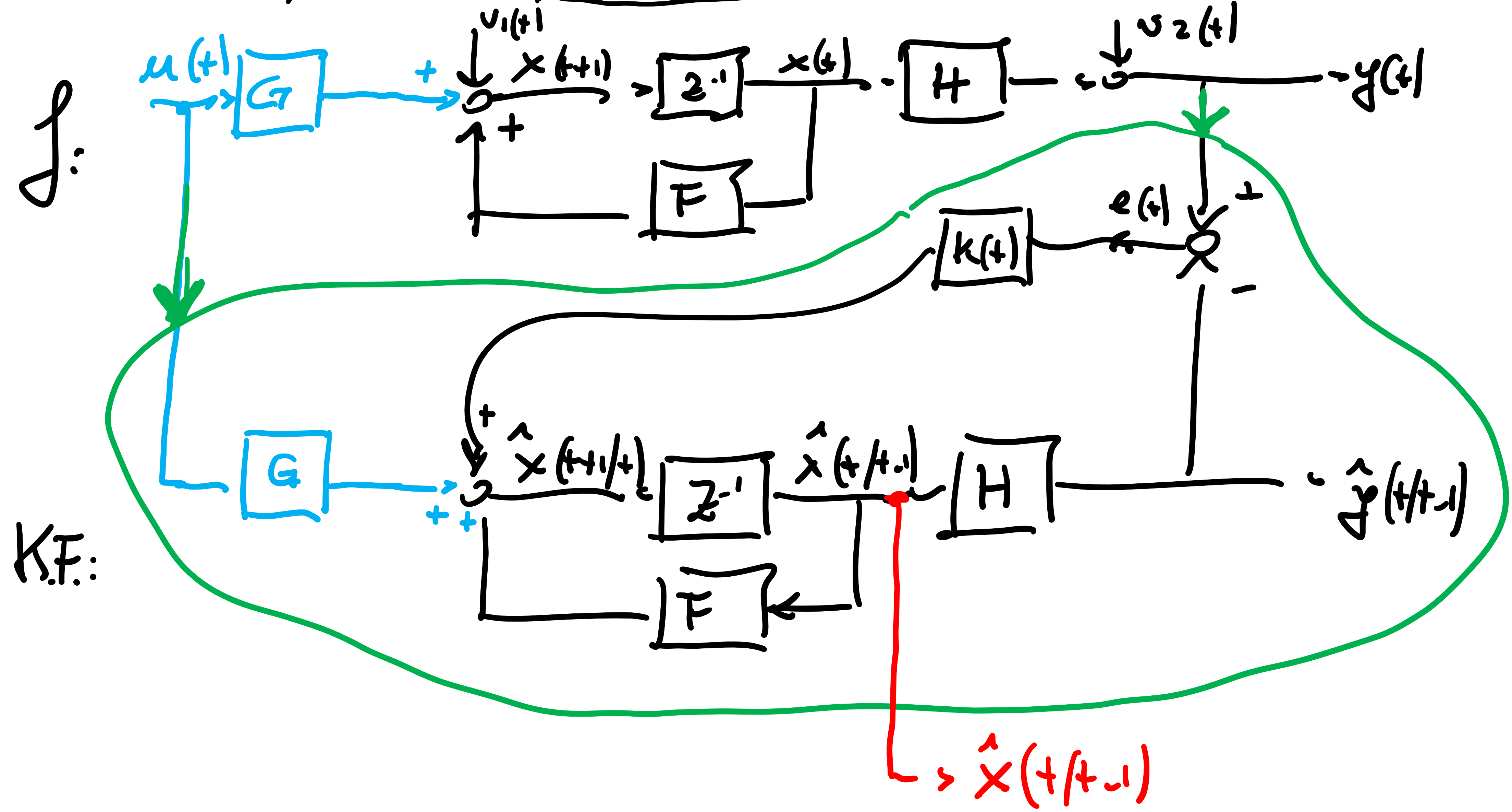
$\boxed{\boxed{V_1}, \boxed{V_2} \; (V_{12})}$ → NOISE MATRICES

V_2 → EASILY BUILT FROM SENSORS specifications

V_1 → "MODEL NOISE" → much more difficult to be DESIGNED (V_1 is the most critical DESIGN parameter of K.F)

EXTENSIONS of BASIC problem of BASIC system

extension # 1 → EXOGENOUS input:



Notice that $\boxed{K(t)}$ remains the same because $\boxed{P(t)}$ is the covariance of the estimation (prediction) error on $x(t)$ and remains the same because $G\,u(t)$ does NOT introduce ANY additional noise or UNCERTAINTIES to the system

($G\,u(t)$ is a totally known "deterministic" signal)

Extension #2 → Multi-step prediction

Assume that $\hat{x}(t+1/t)$ is known (from basic solution)

we can simply obtain a multi-step prediction as:

$$\hat{x}(t+2/t) = F \cdot \hat{x}(t+1/t)$$

$$\hat{x}(t+3/t) = F \cdot \hat{x}(t+2/t) = F^2 \hat{x}(t+1/t)$$

$$\vdots$$

$$\boxed{\begin{aligned} \hat{x}(t+k/t) &= F^{k-1} \underbrace{\hat{x}(t+1/t)} \\ \hat{y}(t+k/t) &= H \underbrace{\hat{x}(t+k/t)} \end{aligned}}$$

Valid also with exogenous system (Ass. → $u(t)$ is zero-mean) ($u(t)$ is known up to t)

Extension #3 → Filter ⇝ $\hat{x}(t|t)$ [we consider filter only with no ex. input]

Basic solution

$$\boxed{\hat{x}(t+1|t)}$$

$$\hat{x}(t+1|t) \xrightarrow{\times F} \hat{x}(t+2|t)$$

$$\hat{x}(t+1|t) \xrightarrow{\times F^{-1}} \hat{x}(t|t)$$

$$\Downarrow$$

$$\boxed{\hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)}$$

this formula is VALID only if F is INVERTIBLE

If F is NOT INVERTIBLE the filter can be obtained with a specific "FILTER" formulation of K.F. →

$\boxed{\text{KF in the FILTER FORM}}$

(Not in the prediction form)

$$\begin{cases} \hat{x}(t|t) = F\hat{x}(t-1|t-1) + \cancel{G u(t)} + K_0(t)e(t) \\ \hat{y}(t|t-1) = H\hat{x}(t|t-1) \\ e(t) = y(t) - \hat{y}(t|t-1) \\ K_0(t) = \left(P(t)H^T\right)\left(HP(t)H^T + V_2\right)^{-1} \\ + \text{DRE (un-changed)} \end{cases}$$

"FILTER GAIN"

I.C. → $\hat{x}(1|1) = X_0$

these equations are VALID UNDER the RESTRICTIVE assumption of $V_{12} = 0$

Remark on $K(t)$ and $K_0(t)$ (when $V_{12}=0$)

Gain of KF in prediction form $\rightarrow$ $K(t) = \left(\boxed{F}\, P(t) H^T\right)\left(H P(t) H^T + V_2\right)^{-1}$

Gain of KF in filter form $\rightarrow$ $K_0(t) = \left(\cancel{\boxed{}}\, P(t) H^T\right)\left(H P(t) H^T + V_2\right)^{-1}$

Extension #4 → TIME-VARYING SYSTEMS →

$$S: \begin{cases} x(t+1) = F(t)\,x(t) + G(t)\,u(t) + v_1(t) \\ y(t) = H(t)\,x(t) + v_2(t) \end{cases}$$

KF, AS IS, is VALID ALSO for LTV syst.

KF equations are exactly the same! Just replace $F \to F(t)$, $G \to G(t)$, $H \to H(t)$ in the 5 equations of KF.

$$\begin{array}{ccc} F & \longrightarrow & F(t) \\ G & \longrightarrow & G(t) \\ H & \longrightarrow & H(t) \\ \downarrow & & \downarrow \\ \text{LTI} & & \text{LTV} \end{array}$$

"AGING of the system"

Extension #5 → Extension to N.L. systems

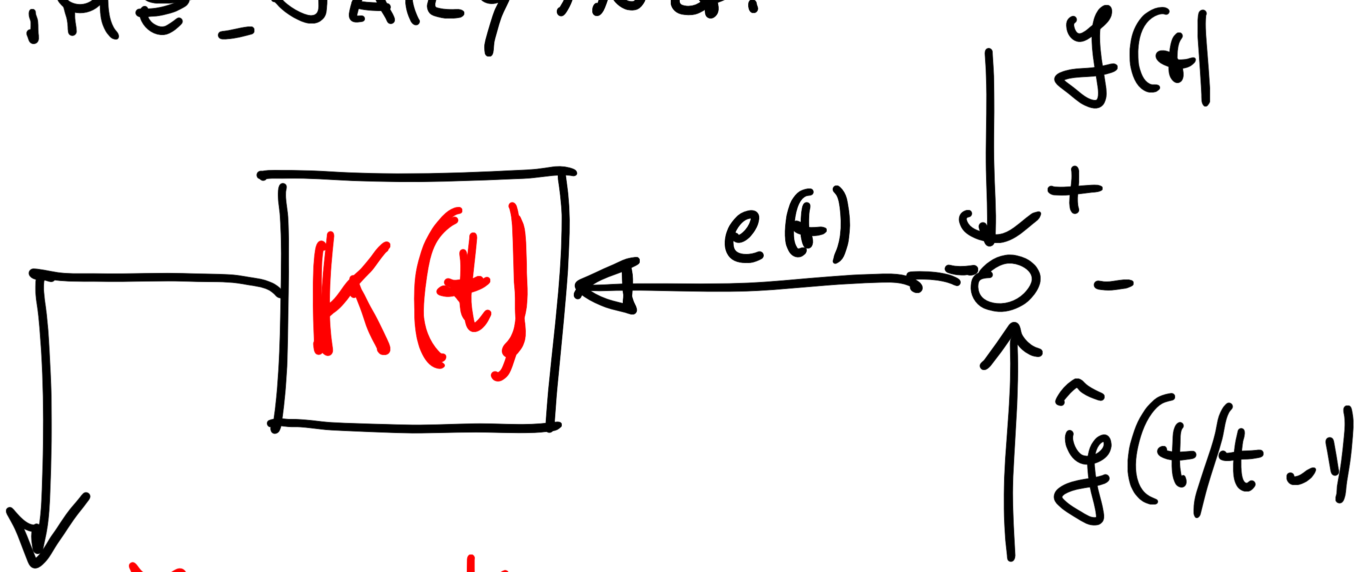
→ It is an extension much more complicated →

$\boxed{\text{Extended Kalman Filter (EKF)}}$

↓ see later

Now we consider the issue of "ASYMPTOTIC solution of K.F."

Observe that KF is NOT itself an LTI system BUT is a LTV system, because the gain $K(t)$ is TIME_VARYING:



Matrix K is Time VARYing ⇒ the whole K.F. is a LTV system

The fact that K.F. is a LTV system is the source of TWO PROBLEMS:

1. Checking the STABILITY of K.F. Algorithm is very DIFFICULT!!

The STABILITY check of an LTV system is not simple as the STAB. check of an LTI system →

NOTE:

LTI: $x(t+1) = \boxed{F} x(t) + Gu(t)$ → STABILITY check → $\boxed{EIG(F)}$

LTV: $x(t+1) = F(t)x(t) + G(t)u(t)$ → EVEN IF ALL $EIG(F(t))$ are strictly inside U.CIRCLE AT $\forall t$ ⇒ the system is NOT GUARANTEED to be ASY. STABLE (in practice it is if time-variations are "slow" like in aging)

② COMPUTATIONAL PROBLEM → $K(t)$ must be computed at each sampling time (e.g. every 5 ms) → inversion of matrix $\boxed{\left(HP(t)H^T+V_2\right)}$ → $\underline{p \times p}$ matrix

---

Because of ① (AND ②) in REAL / PRACTICAL applications → the ASYMPTOTIC VERSION of K.F. is preferred!

BASIC IDEA:

STEADY-STATE VALUE of $P(t)$

if $P(t) \xrightarrow[\text{CONVERGES}]{} \bar{P}$

$\Downarrow$

STEADY-STATE ("ASYMPTOTIC") VALUE of $K(t)$

Also $K(t) \xrightarrow[\text{CONVERGES}]{} \boxed{\bar{K}}$ → USING $\bar{K}$ ⇒ K.F. BECOMES AN LTI SYSTEM!

FIRST, LET'S ANALYZE the ASY. STABILITY of K.F.
when $\bar{K}$ is USED (K.F. is LTI)
(LET'S ASSUME that $\bar{K}$ does exsist)

→ ANALYSIS of dynamics of STATE EQUATION of K.F.

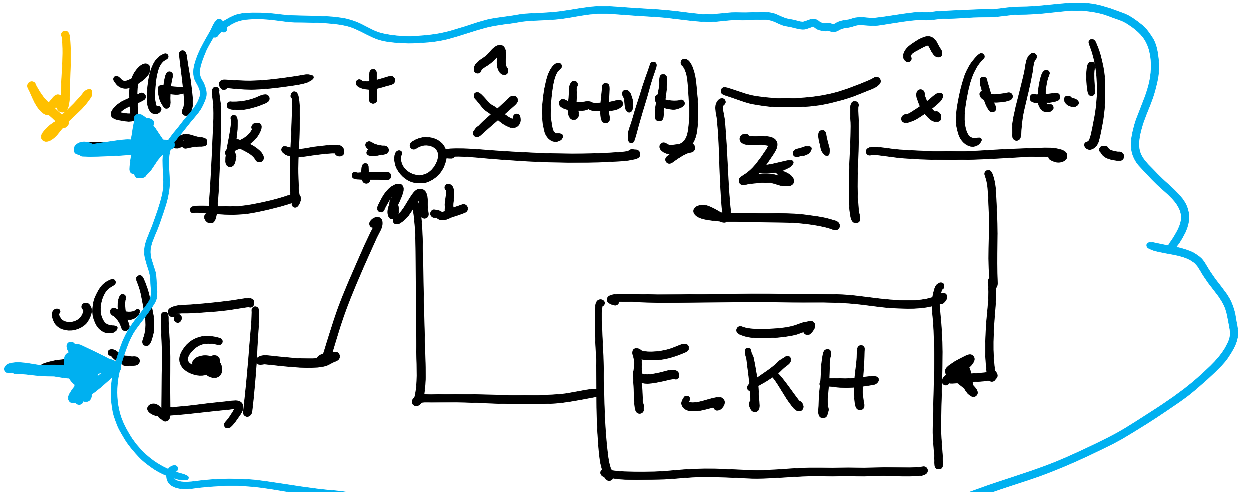
$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + Gu(t) + \bar{K}e(t)$$

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + Gu(t) + \bar{K}\left(y(t) - \hat{y}(t|t-1)\right)$$

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + Gu(t) + \bar{K}\left(y(t) - H\hat{x}(t|t-1)\right)$$

$$\boxed{\hat{x}(t+1|t) = \underbrace{(F - \bar{K}H)}_{\text{NEW STATE MATRIX of K.F.}}\hat{x}(t|t-1) + Gu(t) + \bar{K}y(t)}$$

NEW STATE MATRIX of K.F.



$\Rightarrow$ if $\bar{K}$ does exist, the K.F. is Asy. STABLE if and only if All the EIG$(F-\bar{K}H)$ are strictly inside the u. circle!

↓ notice that:

STABILITY of S is RELATED TO matrix $\boxed{F}$

$\neq$

STABILITY of K.F. is RELATED TO matrix $\boxed{F-\bar{K}H}$

$\Rightarrow$ K.F. can be Asy. STABLE EVEN if the system is UNSTABLE!

Now → what about the EXISTENCE of $\bar{K}$??

$$\bar{K} = \left(F\bar{P}H^T + V_{12}\right)\left(H\bar{P}H^T + V_2\right)^{-1}$$

<u>$\bar{K}$ exists</u> if <u>$\bar{P}$ exists</u>! ⟹

we NEED to check the <u>CONVERGENCE</u>

properties of <u>D.R.E.</u>

↓

To find the equilibrium of a dynamical

AUTONOMOUS system

↓

CONT. TIME

$$\dot{x} = f(x)$$

→ equilibrium ⇒ $\dot{x} = 0$

$$\Rightarrow \boxed{f(\bar{x}) = 0}$$

DISCRETE TIME:

$$x(t+1) = f(x(t))$$

→ equilibrium ⇒ $x(t+1) = x(t)$

$$\Rightarrow \boxed{f(\bar{x}) = \bar{x}}$$

↳ DRE IS AN AUTONOMOUS DISCRETE-TIME system

$$\Rightarrow \boxed{\bar{P} = f(\bar{P})}$$

$$\boxed{\bar{P} = \left(F\bar{P}F^T + V_1\right) - \left(F\bar{P}H^T + V_{12}\right)\left(H\bar{P}H^T + V_2\right)^{-1}\left(F\bar{P}H^T + V_{12}\right)^T}$$

↳ this is a NONLINEAR ALGEBRAIC EQUATION

↳ ALGEBRAIC RICCATI EQUATION (A.R.E)

if a STEADY STATE $\boxed{\bar{P}}$ solution of DRE does exsist, it MUST BE A SOLUTION of ARE →

3 QUESTIONS:
- → EXSISTENCE?
- → CONVERGENCE?
- → STABILITY?

---

① (EXSISTENCE) → DOES A.R.E. have S.d.p. solutions?

② (CONVERGENCE) → if ① is YES; does the DRE converge to $\bar{P}$? (in principle $\bar{P}$ can be an equilibrium point for DRE, but NOT an "ATTRACTOR")

③ (STABILITY) → if ① YES; if ② YES; is the corresponding $\bar{K}$ such that the K.F. is Asy. STABLE?
(ALL EIG$(F-\bar{K}H)$ are strictly inside the u. circle)

→ Answer to these questions is difficult → we need two fundamental theorems (K.F. Asy. theorems)
→ they provide SUFFICIENT conditions for 1 2 3

1st Asymptotic theorem:

Assumptions:
- $V_{12} = \emptyset$
- F is Asy. stable ⇒ All EIG(F) are strictly inside U.C.

→ then:
- ARE has one and only one sdp solution: $\bar{P} \geq 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- the corresponding $\bar{K}$ is such that KF is Asy. stable

for 2nd th. we need to recall Obs./Controll. properties

observability: the pair (F, H) is observable iff:

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \text{ is full rank } (n)$$

↳ observability of the STATE (x) from the output (y)

for 2nd theorem we need controllability FROM NOISE

$$x(t+1) = Fx(t) + (Gu(t)) + \boxed{v_1(t)} \qquad v_1 \sim WN(0, V_1)$$

is an input → we NEED controllab. from this input

$$x(t+1) = Fx(t) + v_1(t)$$ → it is always possible to factorize $V_1 = \Gamma \cdot \Gamma^T$

⇓ RE-WRITE:

$$x(t+1) = Fx(t) + \boxed{\Gamma} w(t) \quad \rightarrow \quad \boxed{w(t) \sim WN(0, \boxed{I})}$$

check ↓

$$\boxed{var[\Gamma\omega(t)]} = E\left[\Gamma\omega(t)\cdot\omega(t)^T\Gamma^T\right] =$$

$$= \Gamma E[\omega(t)\omega(t)^T]\Gamma^T = \Gamma \underbrace{var[\omega(t)]}_{I}\Gamma^T =$$

$$= \Gamma I \Gamma^T = \Gamma\Gamma^T = V_1 = \boxed{var[v_1]}$$

Ex: $x(t+1) = \frac{1}{2}x(t) + v_1(t) \qquad v_1 \sim WN(0,4)$

$\Updownarrow$

$x(t+1) = \frac{1}{2}x(t) + 2\omega(t) \qquad \omega \sim WN(0,1)$

$\Gamma = 2 \qquad \Gamma^2 = 4 = V_1$

we can say that the STATE x is controllable
(REACHABLE)

from input NOISE $v_1(t)$ iff:

$$R = \left[\Gamma \quad F\Gamma \quad F^2\Gamma \quad \dots \quad F^{n-1}\Gamma \right] \text{ is full rank } (n)$$

2nd Asy. theorem

Assumptions:
- $V_{12} = \emptyset$
- (F, H) is observable
- (F, Γ) is controllable $\left(V_1 = \Gamma \Gamma^T\right)$

then:
- ARE has 1! d.p solution $\boxed{\bar{P} > 0}$ → stronger than $\bar{P} \geq 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- the corresponding $\bar{K}$ is such that the K.F. is Asy. stable

these theorems are VERY usefull in practice because we can fully avoid the (VERY difficult) CONVERGENCE ANALYSIS of DRE

Remember that th 1 and th 2 provide (just) SUFFICIENT CONDITIONS

Example (Numerical Exercise on KF)

$$S: \begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) \\ y(t) = 2x(t) + v_2(t) \end{cases}$$

$v_1 \sim WN\left(0, \frac{19}{20}\right)$ $\quad v_1 \perp v_2$ $\quad (\Rightarrow V_{12} = 0)$

$v_2 \sim WN(0, 1)$

$(v_1 \perp x(1), \; v_2 \perp x(1))$

QUESTIONS: find (if exists) the STEADY STATE (Asy.) KF: $\begin{cases} \hat{x}(t+1|t) \\ \hat{x}(t|t) \end{cases}$

$\boxed{n=1} \rightarrow x(t) = x_1(t)$

$V_1 = 19/20$

$F = \frac{1}{2} \quad (G=0) \quad H = 2$

$V_2 = 1$

$V_{12} = 0$ — we can try to use Asy. theorem.

1st step → compute $\boxed{D.R.E}$:

$$P(t+1) = \left(F P(t) F^T + V_1\right) - \left(F P(t) H^T + V_{12}\right)\left(H P(t) H^T + V_2\right)^{-1}\left(F P(t) H^T + V_{12}\right)^T$$

↓ plug in values:

$$P(t+1) = \frac{1}{4} P(t) + \frac{19}{20} - \frac{\left(\frac{1}{\cancel{2}} P(t) \cdot \cancel{2}\right)^2}{4P(t) + 1}$$

$$P(t+1) = \frac{\boxed{\cancel{P(t)^2}} + \frac{1}{4} P(t) + \frac{19}{5} P(t) + \frac{19}{20} \boxed{\cancel{- P(t)^2}}}{4P(t) + 1}$$

→ the "P^2" elements must disappear

DRE: →
$$\boxed{P(t+1) = \frac{81\, P(t) + 19}{80\, P(t) + 20}}$$
→ $\boxed{P(t+1) = f\left(P(t)\right)}$

2nd step → Compute and SOLVE A.R.E:

$$\bar{P} = \frac{81\bar{P}+19}{80\bar{P}+20} \Leftrightarrow 80\bar{P}^2 + 20\bar{P} - 81\bar{P} - 19 = 0$$

$$80\bar{P}^2 - 61\bar{P} - 19 = 0 \Rightarrow (\bar{P}-1)(80\bar{P}+19) = 0 \begin{cases} \boxed{\bar{P}_1 = 1} \\ \cancel{\bar{P}_2 = -\frac{19}{80}} < 0 \end{cases}$$

1! d.p. solution of ARE

→ we found the only d.p. solution of ARE;

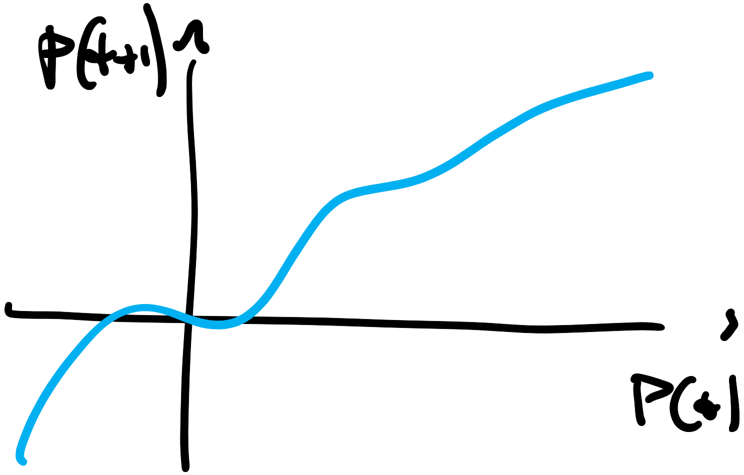
question → $\boxed{\text{does DRE converge to } \bar{P}=1 \ \forall P_0 \geq 0?}$

2 methods to address this question:

- DIRECT ANALYSIS of DRE
- USE Asy. theorems.

DIRECT DRE CONVERG. ANALYSIS

DRE: $P(t+1) = f(P(t))$ => we NEED TO plot $f(\cdot)$ in the $P(t) \rightarrow P(t+1)$ plane



$$P(t+1) = \frac{81 P(t) + 19}{80 P(t) + 20}$$

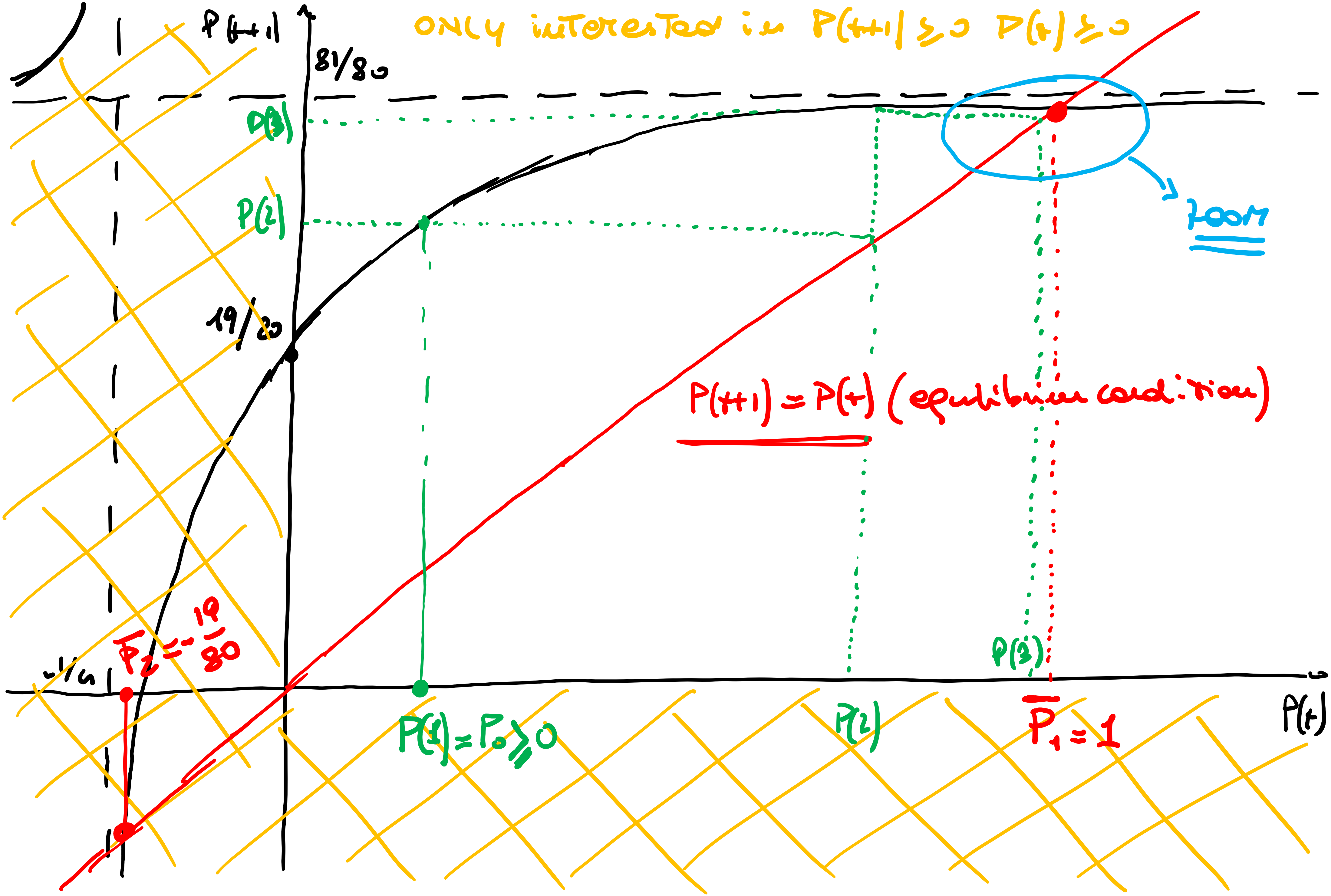
→ VERTICAL Asy. VALUE: $P(t) = -\frac{20}{80} = -\frac{1}{4}$

→ Horizontal Asy. VALUE: $P(t+1) = \frac{81}{80}$

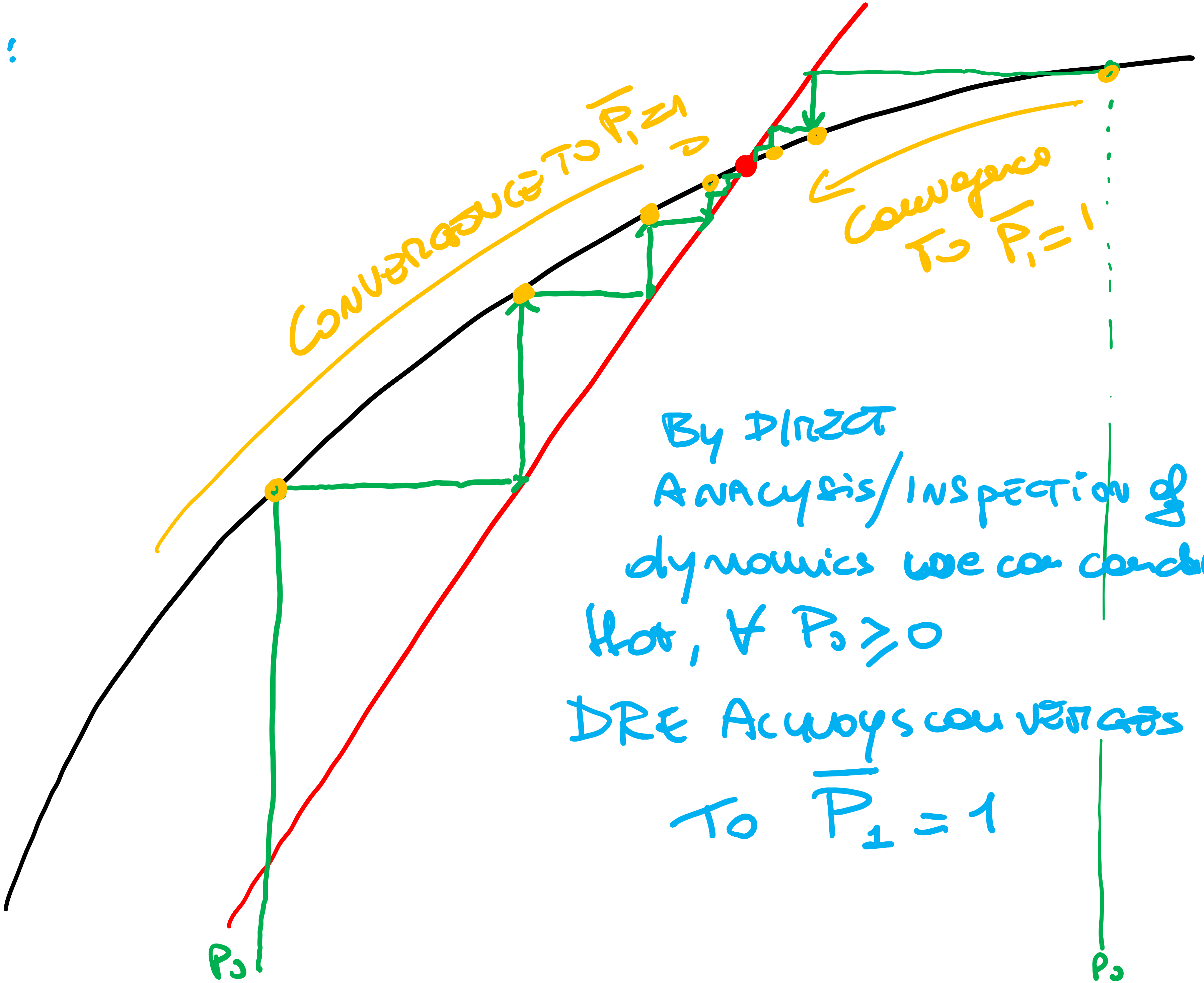
if $P(t) = 0 \Rightarrow P(t+1) = \frac{19}{20}$

↓

plot:


ONLY INTERESTED in P(t+1) ≥ 0 P(t) ≥ 0
P(t+1)
81/80
P(3)
P(2)
19/80
-1/4
$\bar{P}_2 = -\frac{19}{80}$
P(t+1) = P(t) (equilibrium condition)
ZOOM
P(1) = P₀ ≥ 0
P(2)
P(3)
$\bar{P}_1 = 1$
P(t)

Dom:



By direct analysis/inspection of DRE dynamics we can conclude that, $\forall\, P_0 \geq 0$ DRE always converges to $\bar{P}_1 = 1$

if $n=1$ direct inspection feasible / very efficient

for $n \geq 2$

with theorems we can skip this very challenging step.

2nd method --> use theorems -->

$$\boxed{\begin{array}{l} \gamma_{12} = 0 \\ F = \frac{1}{2} \Rightarrow \mathcal{S} \text{ is ASY. STABLE} \end{array}}$$

=> theorem #1 is fulfilled

observability matrix of $\{F, H\}$ --> $\Theta = [2] \Rightarrow \text{RANK} = 1$

=> system is obs. from output

Controllability from noise $v_1(t)$

$V_1 = \frac{19}{20} \Rightarrow \Gamma = \sqrt{\frac{19}{20}}$ $\quad \Gamma^2 = V_1$

Controllability of pair $\{F, \Gamma\}$ $\rightarrow R = \left[\sqrt{\frac{19}{20}}\right] =$ Rank is 1

$\Rightarrow$ system is controllable from noise $v_1(t)$

$\boxed{\begin{array}{l} V_{12} = 0 \\ \{F, H\} \text{ obs.} \\ \{F, \Gamma\} \text{ controll.} \end{array}}$ $\Rightarrow$ Also th. 2 is fulfilled

$\Rightarrow$

ARE has 1! solution $\bar{P} > 0$ (d.p.)

DRE $\xrightarrow[t \to \infty]{} \bar{P} \quad \forall P_0 \geq 0$

the corresponding $\bar{K}$ makes the K.F. Asy. STABLE

Compute $\bar{K}$

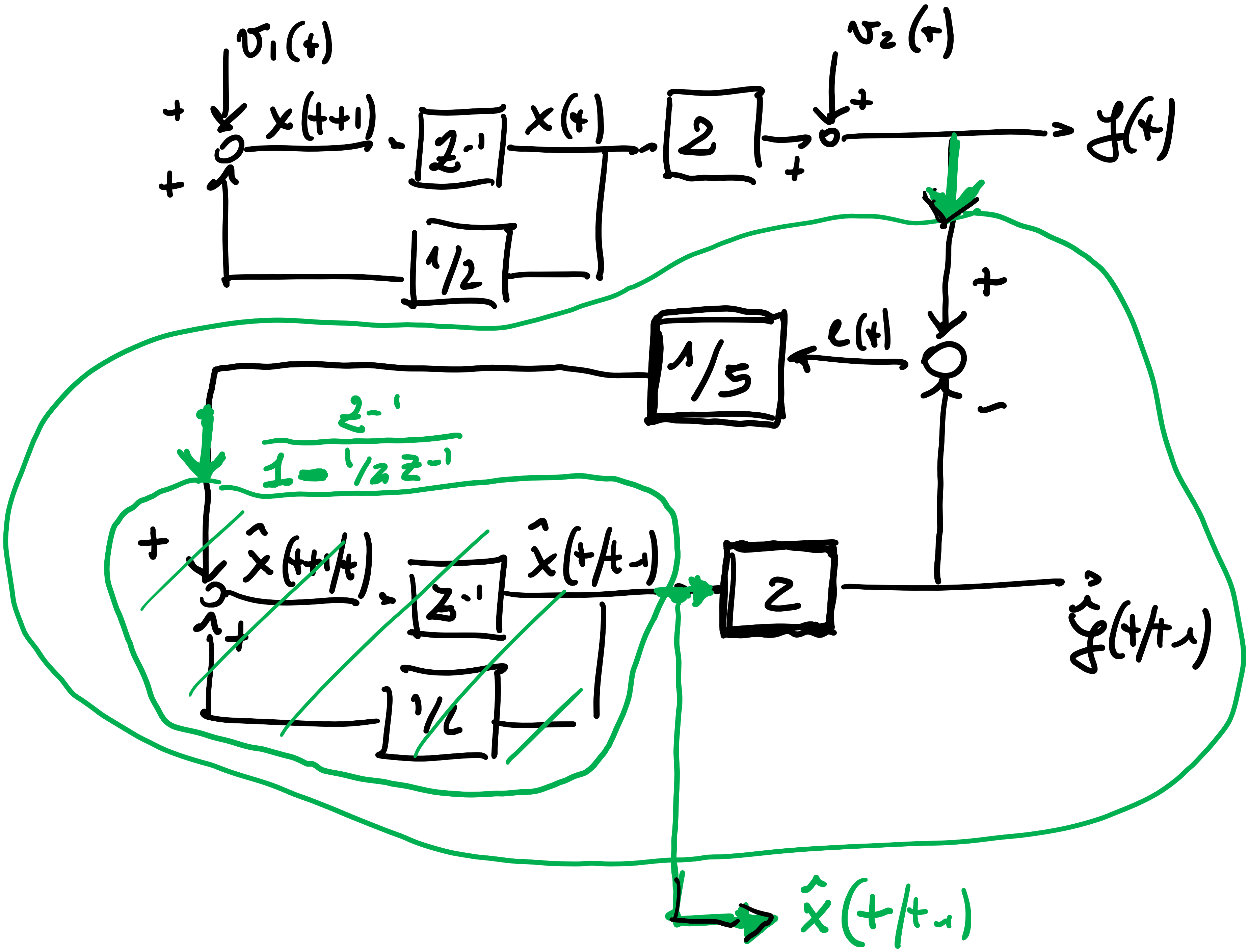
$$\bar{K} = (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1} = \left(\frac{1}{2}\cdot 1\cdot 2 + 0\right)(2\cdot 1\cdot 2 + 1)^{-1} = \boxed{\frac{1}{5}}$$

→ RE-check the Asy. STABILity of KF (Already guaranteed by theorems)

$$F - \bar{K}H = \frac{1}{2} - \frac{1}{5}\cdot 2 = \frac{5-4}{10} = \boxed{\frac{1}{10}}$$ – this EIG/pole of KF is Asy. STABLE

↓ Plot Block scheme of S and KF:

S:

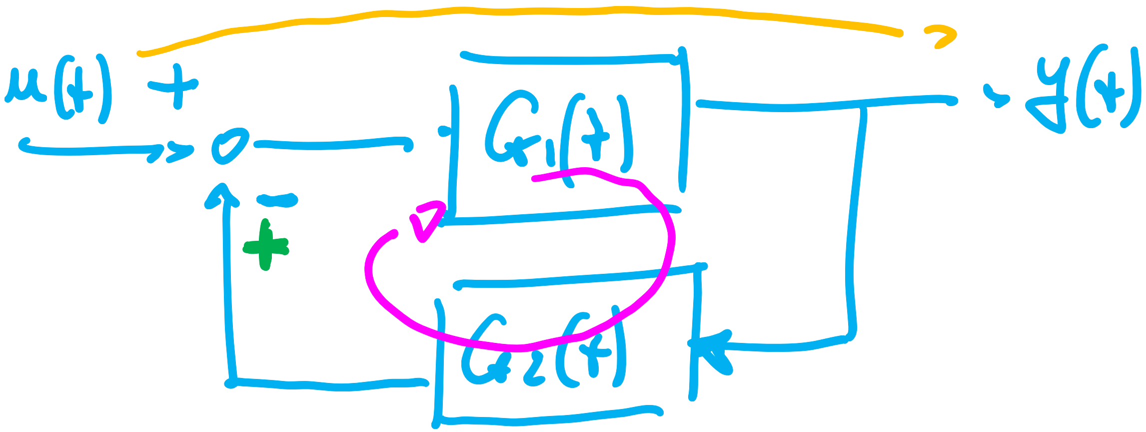


K.F.

2 NESTED Loops

find the T.F. from $y(t)$ to $\hat{x}(t|t-1)$:

Recall -- T.F. from Block schemes of feedback systems.



T.F. from $u(t)$ to $y(t)$?

$$y(t) = G_1(z)\cdot\left(u(t) \pm G_2(z)\cdot y(t)\right)$$

$$y(t)\left[1 \mp G_1(z)G_2(z)\right] = G_1(z)\,u(t) \rightarrow y(t) = \frac{\boxed{G_1(z)}}{1 \pm \boxed{G_1(z)G_2(z)}}\,u(t)$$

"DIRECT" line from input to output

loop function

$$\hat{x}(t|t-1) = \frac{\frac{1}{5}\,\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}}{1+\boxed{\frac{1}{5}\,\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}\cdot 2}}\,y(t)$$

$$\ldots\ \hat{x}(t|t-1) = \frac{\frac{1}{5}z^{-1}}{\boxed{1-\frac{1}{10}z^{-1}}}\,y(t)$$

→ Asy. STABLE
pole in $z=\frac{1}{10}$

in time domain:

$$\hat{x}(t|t-1) = \frac{1}{10}\hat{x}(t-1|t-2) + \frac{1}{5}y(t-1)$$

PREDICTOR (1 step) of STATE

Predictor of output:

$$\hat{y}(t/t-1) = H\,\hat{x}(t/t-1)$$

$$\hat{y}(t/t-1) = 2 \cdot \frac{1/5}{1-\frac{1}{10}z^{-1}}\,y(t-1)$$

Filter of state: $\hat{x}(t/t) \rightarrow \hat{x}(t/t) = F^{-1}\hat{x}(t+1/t)$

$F = 1/2 \rightarrow$ invertible $F^{-1} = 2$

$$\boxed{\hat{x}(t/t) = 2 \cdot \frac{1/5}{1-\frac{1}{10}z^{-1}}\,y(t)}$$

filter of state

"<u>Remark</u>: (on white noise)

In the formulas of K.F. there is a requirement that $v_1(t)$ and $v_2(t)$ <u>MUST</u> BE white noises

In many practical application this assumption can be <u>TOO DEMANDING</u>

=> WE NEED A "<u>WORKAROUND</u>" TO DEAL with practical Applications where this assumption <u>is NOT VALID</u>

=> "Trick" of "EXTENDED system" → <u>presentation</u> <u>with an EXAMPLE</u>

Ex:

$$S: \begin{cases} x(t+1) = a\,x(t) + \boxed{\eta(t)} \\ y(t) = b\,x(t) + v_2(t) \end{cases}$$

where:

$v_2(t) \sim WN(0,1)$

BUT $\eta(t)$ is <u>NOT</u> A white noise:

→ A <u>model</u> of $\eta(t)$ is <u>given</u>

$$\underbrace{\eta(t) = \frac{1}{1 - cz^{-1}}\, e(t)}$$

$e \sim WN(0,1)$

$e \perp v_2$

↓ $\eta(t)$ is NOT WN but is AN AR(1) stoch. process

$\Rightarrow$ we cannot apply K.F. formulas to this system.

→ we can proceed as follows:

$$\eta(t) = c\,\eta(t-1) + e(t)$$

$\downarrow$

$$\eta(t+1) = c\,\eta(t) + e(t+1)$$

• DEFINE

$$v(t) = e(t+1)$$

$v \sim WN(0,1)$

$v \perp v_2$

$\downarrow$

$$\boxed{\eta(t+1) = c\,\eta(t) + v(t)}$$

<u>Trick</u> → "EXTENSION" of the state vector:

$x(t) \rightarrow x_1(t)$

$\boxed{\eta(t) \rightarrow x_2(t)}$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

EXTENDED system

$$\begin{cases} x_1(t+1) = a x_1(t) + x_2(t) \\ \boxed{x_2(t+1) = c x_2(t) + v(t)} \\ y(t) = b x_1(t) + v_2(t) \end{cases}$$

$\Downarrow$ Now we can apply K.F. formulas to this EXTENDED system

$n = 2$

$$F = \begin{bmatrix} a & 1 \\ 0 & c \end{bmatrix} \quad H = \begin{bmatrix} b & 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 0 \\ v(t) \end{bmatrix} \rightarrow v_1 \sim WN(0, V_1)$$

$$V_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v_2 \sim WN(0, 1)$$

$$V_{12} = 0$$

EXTENSION #5 → EXTENSION of K.F. TO NONLINEAR SYSTEMS

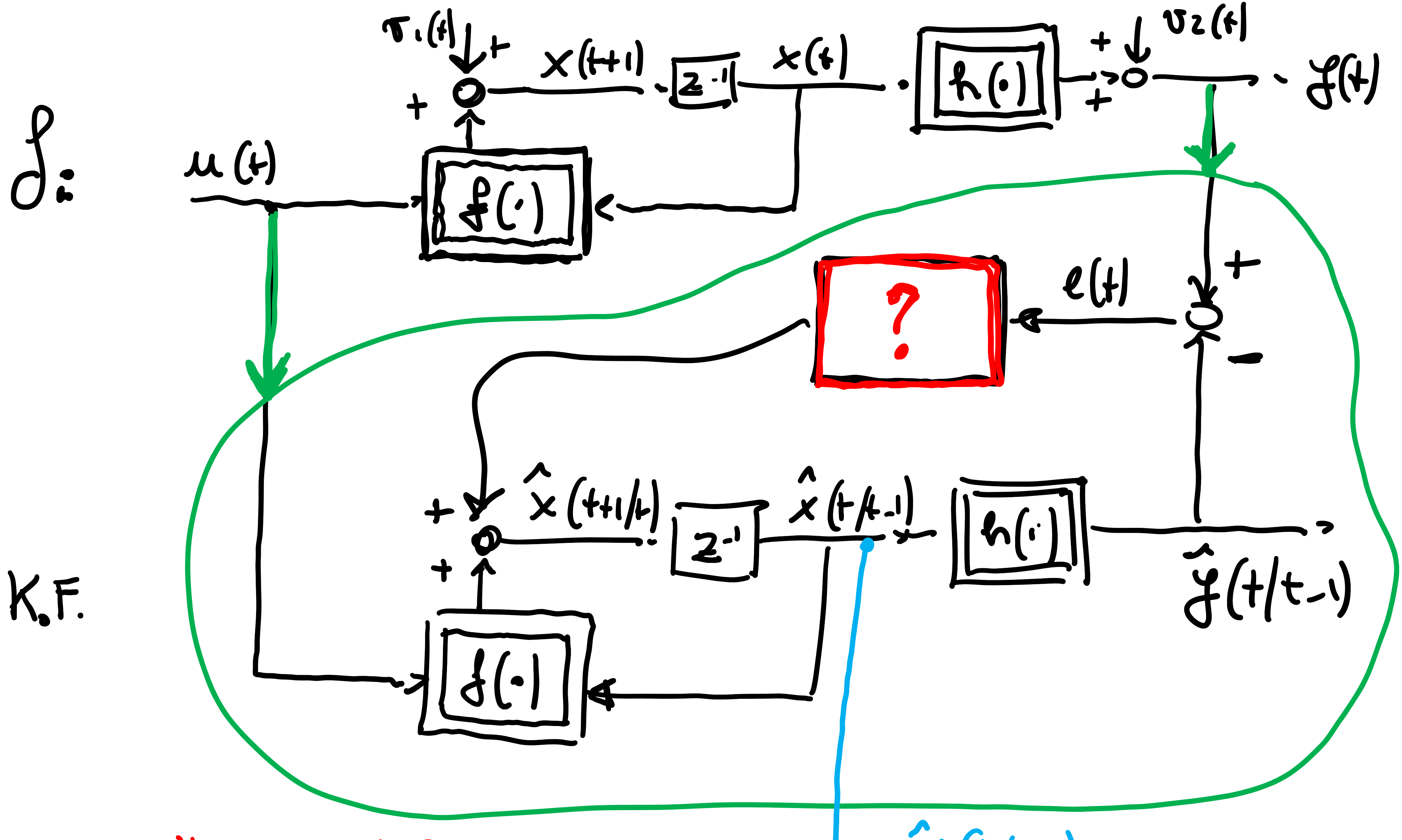
Consider a system with nonlinear dynamics:

$$S: \begin{cases} x(t+1) = f\big(x(t); u(t)\big) + v_1(t) \\ y(t) = h\big(x(t)\big) + v_2(t) \end{cases}$$

where $f(\cdot)$ and $h(\cdot)$ are NONLINEAR functions of $x(t)$ and $u(t)$ (smoothness cond: C^1 or higher)

Example: $S: \begin{cases} x(t+1) = \frac{1}{2} x(t)^5 + u(t)^3 + v_1(t) \\ y(t) = e^{x(t)} + v_2(t) \end{cases}$

How can we design K.F. in this case? => Follow basic IDEA of OBSERVER



K.F. "GAIN"?

For the "GAIN BLOCK" of K.F. we have two ≠ types of solution:

① (MOST NATURAL AND INTUITIVE) → the GAIN is a nonlinear function

$$\xleftarrow{\quad} \boxed{K(\cdot)} \xleftarrow{\ e(t)\ }$$

(output of gain block is a nonlinear function of $e(t)$: $K(e(t))$

② the gain is a LINEAR TIME-VARYING (LTV) function

$$\xleftarrow{\quad} \boxed{K(t)} \xleftarrow{\ e(t)\ } \qquad \boxed{K(t) \cdot e(t)}$$

$$K(e(t)) \neq K(t) \cdot e(t)$$

The 2nd solution is less "INTUITIVE" but is the most effective → we can RE_USE in full K.F. formulas (with small adjustments) → Remember K.F. can be applied to L.T. V systems

⇒ In practice EXTENDED KAL FILTER

IDEA is to make a TIME-VARYING linear local approximation of a NON-linear Time invariant system

$\boxed{K(t)}$ in EKF can be computed as:

$$K(t) = \left(F(t)P(t)H(t)^T + V_{12}\right)\left(H(t)P(t)H(t)^T + V_2\right)^{-1}$$

and $P(t)$ can be computed from DRE:

$$P(t+1) = \left(F(t)P(t)F(t)^T + V_1\right) - \left(F(t)P(t)H(t)^T + V_{12}\right)\left(H(t)P(t)H(t)^T + V_2\right)^{-1}\left(F(t)P(t)H(t)^T + V_{12}\right)^T$$

→ equations of $K(t)$ and DRE are the usual formulas of K.F.

→ the only difference → $F(t)$ and $H(t)$ are TIME-VARYING MATRICES computed as follows: ↓

$$F(t) = \left. \frac{\partial f(x(t), u(t))}{\partial x(t)} \right|_{x(t) = \hat{x}(t|t-1)}$$

$$H(t) = \left. \frac{\partial h(x(t))}{\partial x(t)} \right|_{x(t) = \hat{x}(t|t-1)}$$

$\hookrightarrow$ EKF is the time-varying solution of KF where $F(t)$ and $H(t)$ are local linearized matrices computed around the LAST AVAILABLE STATE prediction $\hat{x}(t|t-1)$

procedure to implement EKF:

At time t:

- TAKE LAST AVAILABLE STATE prediction $\hat{x}(t|t-1)$ (computed at previous step)
- USING $\hat{x}(t|t-1)$, compute $F(t)$ and $H(t)$
- Compute $K(t)$ and update DRE equation
- Compute $\hat{x}(t+1|t)$

→ ITERATE the same procedure at each sampling time

Remarks:

- EKF is very powerful since can be applied to N.L. systems (major extension of KF)
- Obviously EKF does NOT have a "STEADY-STATE" (ASYMPTOTIC) solution
- MAIN problems/issues of EKF? → same of LTV K.F.:

→ VERY DIFFICULT (ALMOST impossible) to have a theoretical guarantee of EKF STABILITY
(in practice → extensive empirical tests to check its stability ⇝ NEVER 100% guarantee of stability)

→ computational load (at each time $H(t)$, $F(t)$, $K(t)$ and $P(t)$ must be computed "RUN-TIME")

EKF is largely used today with some limitations in:

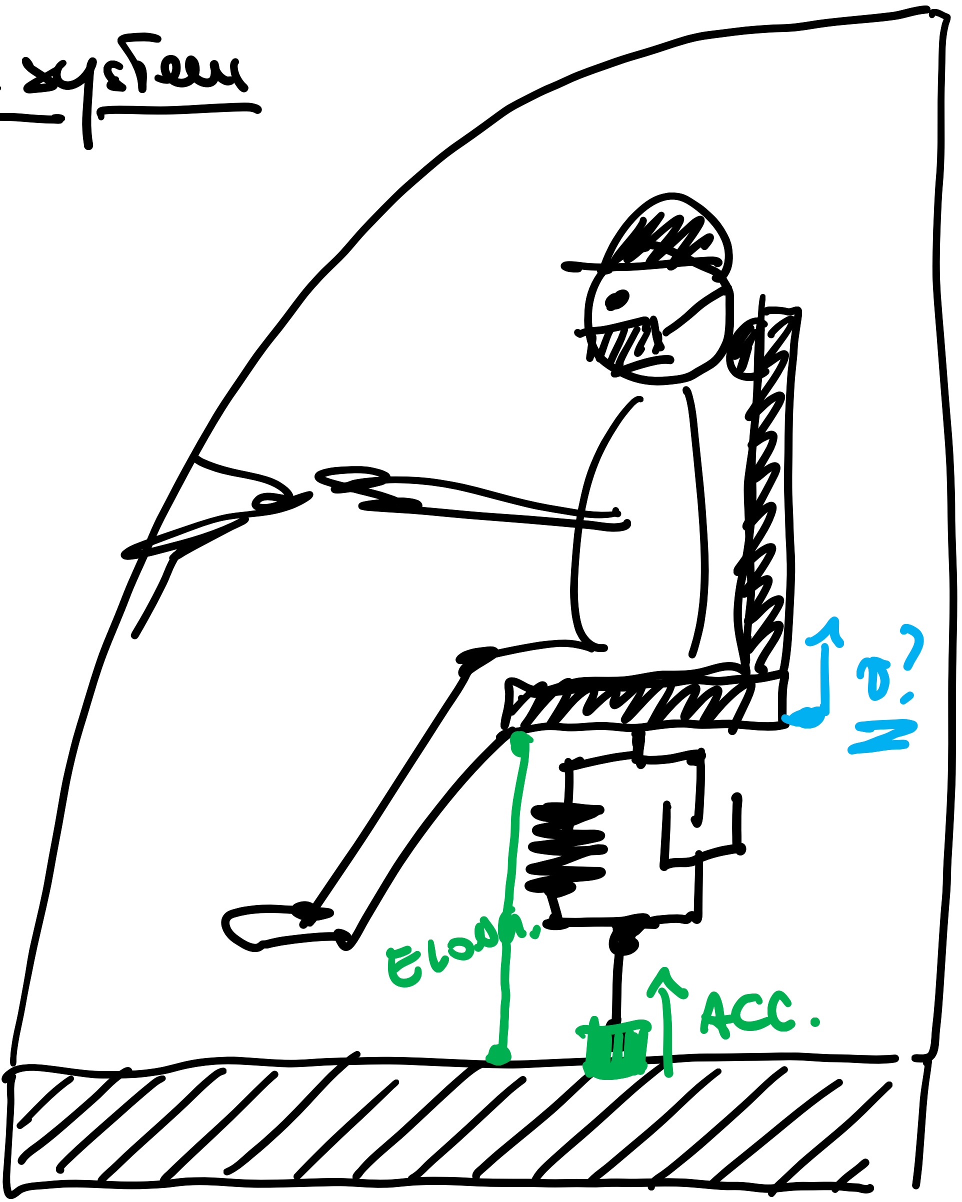
- SAFETY-CRITICAL Applications

AND / OR

- MISSION CRITICAL Applications

Example (~ almost real problem) -> Exemplification of K.F. full procedure (full "set-up" of a K.F.)

The system



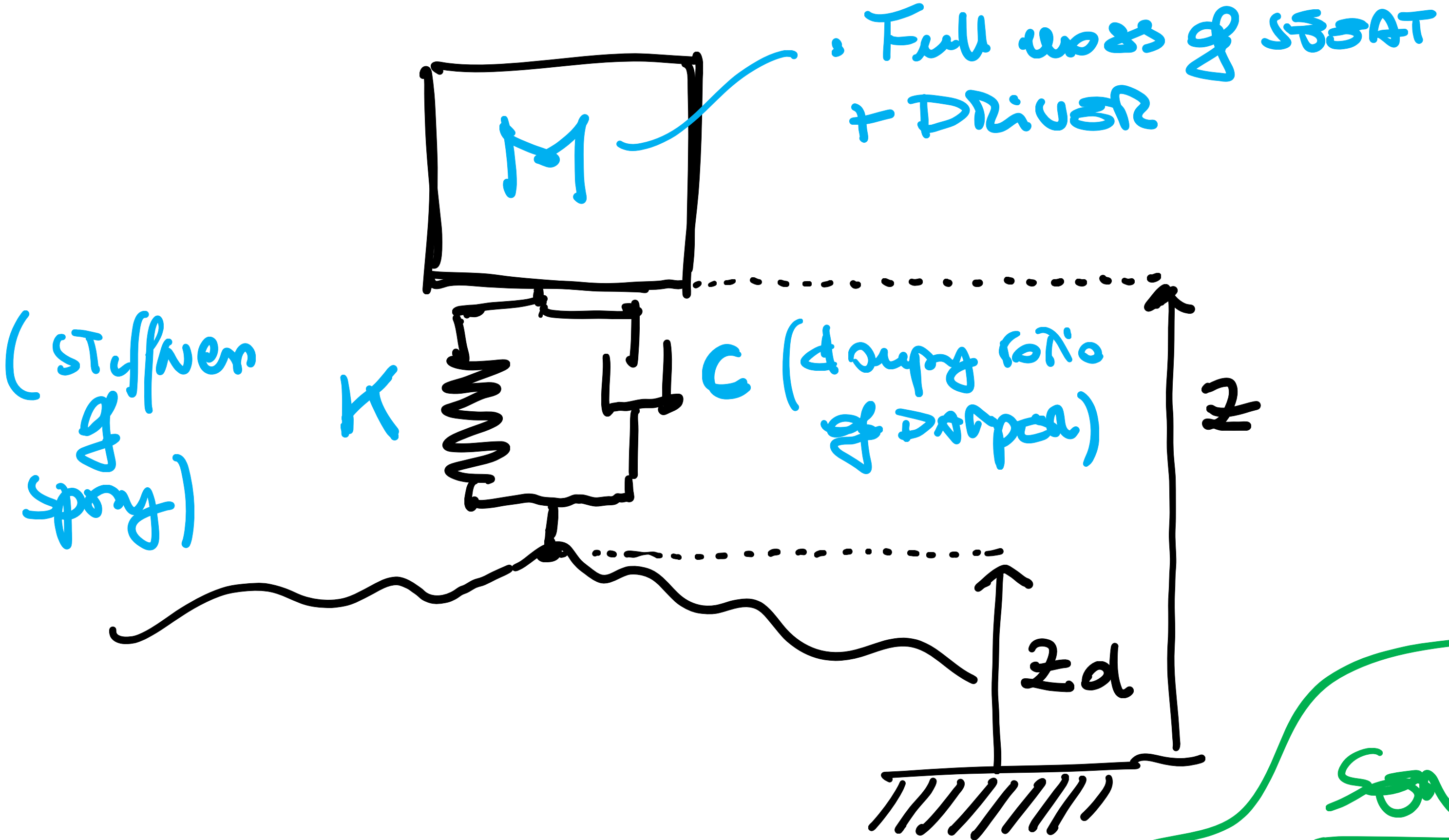
Suspended SEAT in the cabin of an OFF-highway vehicle (Agr. TRACTOR, earth-moving machine, etc.)

SENSORS:

- VERTICAL accelerometer placed at the BASIS of the cabin
- ELONGATION SENSOR between the basis of cabin and SEAT

problem -> ESTIMATION of the SEAT VERTICAL SPEED (USED for suspension control)

Schematic representation of the system/parameters/variables:



z_d: height of the basis of cabin

z: height of the seat

Sensors:

Acceleration: $\ddot{z}_d$ (+noise)

Elongation: $z - z_d$ (+noise)

Model of the system dynamics (physical, w. box model, CONT.-TIME domain)

"CORE" model equation: (FORCE balance in VERTICAL direction):

$$M\ddot{z} = \underbrace{-C\frac{d}{dt}\left(z-z_d\right)}_{\substack{\text{Damping} \\ \text{"DISSIPATIVE"} \\ \text{force}}} \underbrace{-K\left(z-z_d\right)}_{\substack{\text{spring (RESTORING)} \\ \text{CONSERVATIVE} \\ \text{FORCE}}}$$

~~+ gravity force $-Mg$ => we NEGLECT SINCE CONSTANT~~

$$M\ddot{z} = -C\left(\dot{z}-\dot{z}_d\right) - K\left(z-z_d\right)$$

2nd order system

Moreover, since we measure $\ddot{z}_d$, the overall dimension of the system is 4 $\rightarrow$ vector of state variables:

$$X(t) = \begin{bmatrix} z \\ \dot{z} \\ z_d \\ \dot{z}_d \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

INPUT:

$$u(t) = \ddot{z}_d$$

(MEASURABLE DISTURBANCE, given as independent exogenous input)

output $y(t) = \underline{z - z_d}$ (SENSOR)

$\Downarrow$ full model in S.S. form:

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\frac{C}{M}(x_2 - x_4) - \frac{K}{M}(x_1 - x_3) \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = u \\
y(t) = x_1 - x_3
\end{cases}
$$

$$\Downarrow$$

$$
\boxed{
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\frac{K}{M}x_1 - \frac{C}{M}x_2 + \frac{K}{M}x_3 + \frac{C}{M}x_4 \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = u \\
y = x_1 - x_3
\end{cases}
}
$$

NORMAL form of SS. REP:
$$
\begin{cases}
\dot{x} = Ax + Bu \\
y = Cx
\end{cases}
$$

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M} & -\frac{C}{M} & \frac{K}{M} & \frac{C}{M} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad
B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}
$$

NEXT is $\rightarrow$ DISCRETIZATION (we use DIGITAL systems)

choice of SAMPLING TIME $\rightarrow$ Δ (for this application can be $\approx$ 5 ms)

EULERO FORWARD approx of TIME DERIVATIVE $\longrightarrow$ $\dot{x}(t) \approx \frac{x(t+1) - x(t)}{\Delta}$

ex. for 1ST equation $\rightarrow$

$$\frac{x_1(t+1) - x_1(t)}{\Delta} = x_2(t)$$

$$\boxed{x_1(t+1) = x_1(t) + \Delta x_2(t)}$$

$$\begin{cases} x_1(t+1) = x_1(t) + \Delta x_2(t) + v_{11}(t) \\ x_2(t+1) = -\frac{\Delta K}{M} x_1(t) + \left(-\frac{\Delta C}{M} + 1\right) x_2(t) + \frac{\Delta K}{M} x_3(t) + \frac{\Delta C}{M} x_4(t) + v_{12}(t) \\ x_3(t+1) = x_3(t) + \Delta x_4(t) + v_{13}(t) \\ x_4(t+1) = x_4(t) + \Delta u(t) + v_{14}(t) \\ y(t) = x_1(t) - x_3(t) + v_2(t) \end{cases}$$

$$F = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ -\frac{\Delta K}{M} & -\frac{\Delta C}{M}+1 & \frac{\Delta K}{M} & \frac{\Delta C}{M} \\ 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta \end{bmatrix}$$

1. normal s.s. form in Discrete-T.

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases} \longrightarrow H = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$$

$\downarrow$ ADD NOISES !

Noises on state equations:

$$v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \\ v_{13}(t) \\ v_{14}(t) \end{bmatrix} \qquad v_2(t)$$

Assumptions:

- All white noises
- All uncorrelated

$v_2(t) \approx WN(0, V_2)$ → can be estimated by data sheet of elongation sensor

$$v_1(t) \approx WN(0, V_1) \rightarrow V_1 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & 0 \\ 0 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & \lambda_3^2 & 0 \\ 0 & 0 & 0 & \lambda_4^2 \end{bmatrix}$$

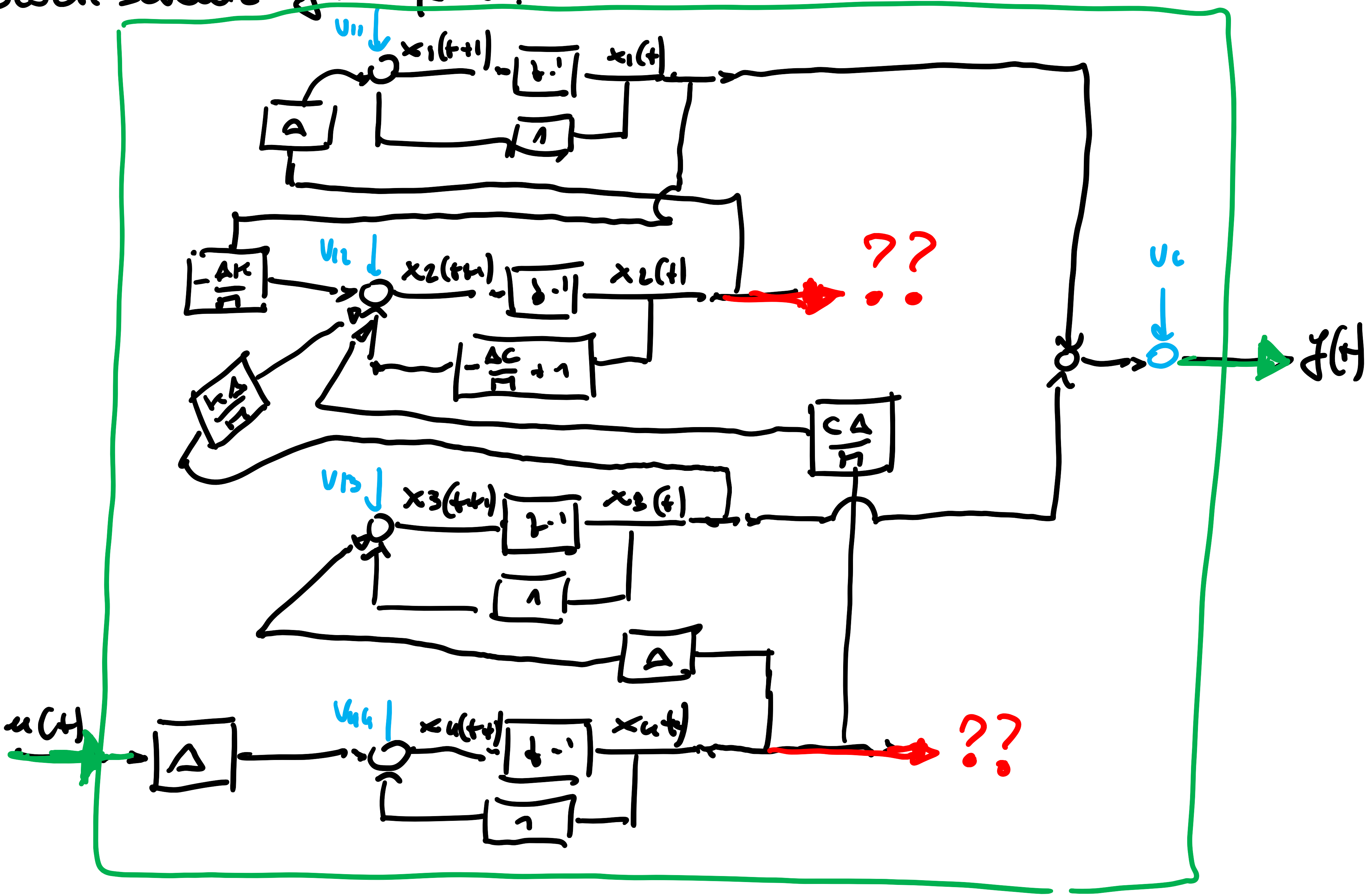
→ we expect small model errors

→ can be estimated by data sheet of acceler. sensor

We can even make the simplifying assumption $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda^2$

TUNED

Block scheme of the system:



we can compute:

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^3 \end{bmatrix} \text{ and } \mathcal{R} = \begin{bmatrix} G & FG & F^2G & F^3G \end{bmatrix}$$

From visual inspection of block scheme we expect full observability from output and full controll. from input

→ compute T.F from $u(t)$ to $y(t)$ we expect 4 poles

(→ check stability → system is Asy. STABLE)

K.F. at this point can be applied

th 1 and th 2 ARE VALID → we can directly jump to ARE solution → $\bar{P}$ → $\bar{K}$ ←

Numerical example : Direct optimization of gain K

$$S: \begin{cases} x(t+1) = 2x(t) + \cancel{0} \\ y(t) = x(t) + v(t) \end{cases} \quad v \sim WN(0,1)$$

→ System is UNSTABLE

→ STATE-EQ. is noise free ($V_1 = 0$)

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problem → compute the STEADY-STATE predictor of state: $\hat{x}(t+1|t)$

Using 2 ≠ methods → ① Direct optimization

② K.F. theory

1) Direct solution $\rightarrow$ let's start from the standard observer structure:

$$\begin{cases} \hat{x}(t+1/t) = 2\,\hat{x}(t/t-1) + \boxed{K\left(y(t) - \hat{y}(t/t-1)\right)} \\ \hat{y}(t/t-1) = \hat{x}(t/t-1) \end{cases}$$

feedback correction

optimal K? $\Rightarrow$ we can

Try by directly minimizing the variance of the state prediction error

$$\boxed{\text{minimize} \rightarrow \text{Var}\left[x(t) - \hat{x}(t/t-1)\right]}$$

Write state prediction error expression:

$$\underbrace{x(t+1)}_{\text{S. equation}} - \hat{x}(t+1|t) = 2x(t) - \left[2\hat{x}(t|t-1) + K\left(y(t) - \hat{y}(t|t-1)\right)\right] =$$

S. equation

observer equation

$$= 2x(t) - 2\hat{x}(t|t-1) - K\left(x(t) + v(t) - \hat{x}(t|t-1)\right) =$$

$$= (2-K)\left(x(t) - \hat{x}(t|t-1)\right) - Kv(t)$$

Definition: $\eta(t) = x(t) - \hat{x}(t|t-1)$

$$\boxed{\eta(t+1) = (2-k)\,\eta(t) - k\,v(t) \qquad v \sim WN(0,1)}$$

dy. eq. of state prediction error

$\rightarrow$ AR(1) process

$$e(t) = \underbrace{-k\,v(t)}$$

$$\eta(t) = \frac{1}{1-(2-k)z^{-1}}\,e(t) \qquad e \sim WN(0,k^2)$$

AR(1) in canonical form

Easy to find the variance of $\eta(t)$

$$\gamma_\eta(0) = var[\eta(t)] = var[x(t) - \hat{x}(t|t-1)] = \boxed{\frac{K^2}{1-(2-K)^2}}$$

By minimizing this function w.r.t. K:

$$\frac{\partial\, var[\eta(t)]}{\partial K} = 0 \;\dots\; = \begin{cases} (\underline{K_1 = \phi}) \\ \boxed{K_2 = \frac{3}{2}} \end{cases}$$

(?) NO Feedback correction

Both are Acceptable solutions

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(2) we can RE-DO the problem solution using K.F. theory:

from f: $\longrightarrow$

$F = 2 \quad V_1 = \phi \implies \Gamma = \phi$

$H = 1 \quad V_2 = 1 \quad (V_{12} = 0)$

check theorems:

th 1 : ✓ $V_{12}=0$ ✗ F is Asy. STABLE] NOT. Applicable

th 2 : ✓ $V_{12}=0$ ✗ (F,Γ) NOT REAchable ✓ (F,H) is observable] Not Applicable

$\Rightarrow$ we cannot skip the ANALYSIS of DRE
(conditions are only sufficient $\Rightarrow$ we can have a solution of STEADY STATE K.F. anyway)

DRE: $$P(t+1) = 4P(t) - \frac{(2P(t))^2}{P(t)+1} \dots$$

$$\boxed{P(t+1) = \frac{4P(t)}{P(t)+1}}$$ DRE

find A.R.E solutions (≥ 0)

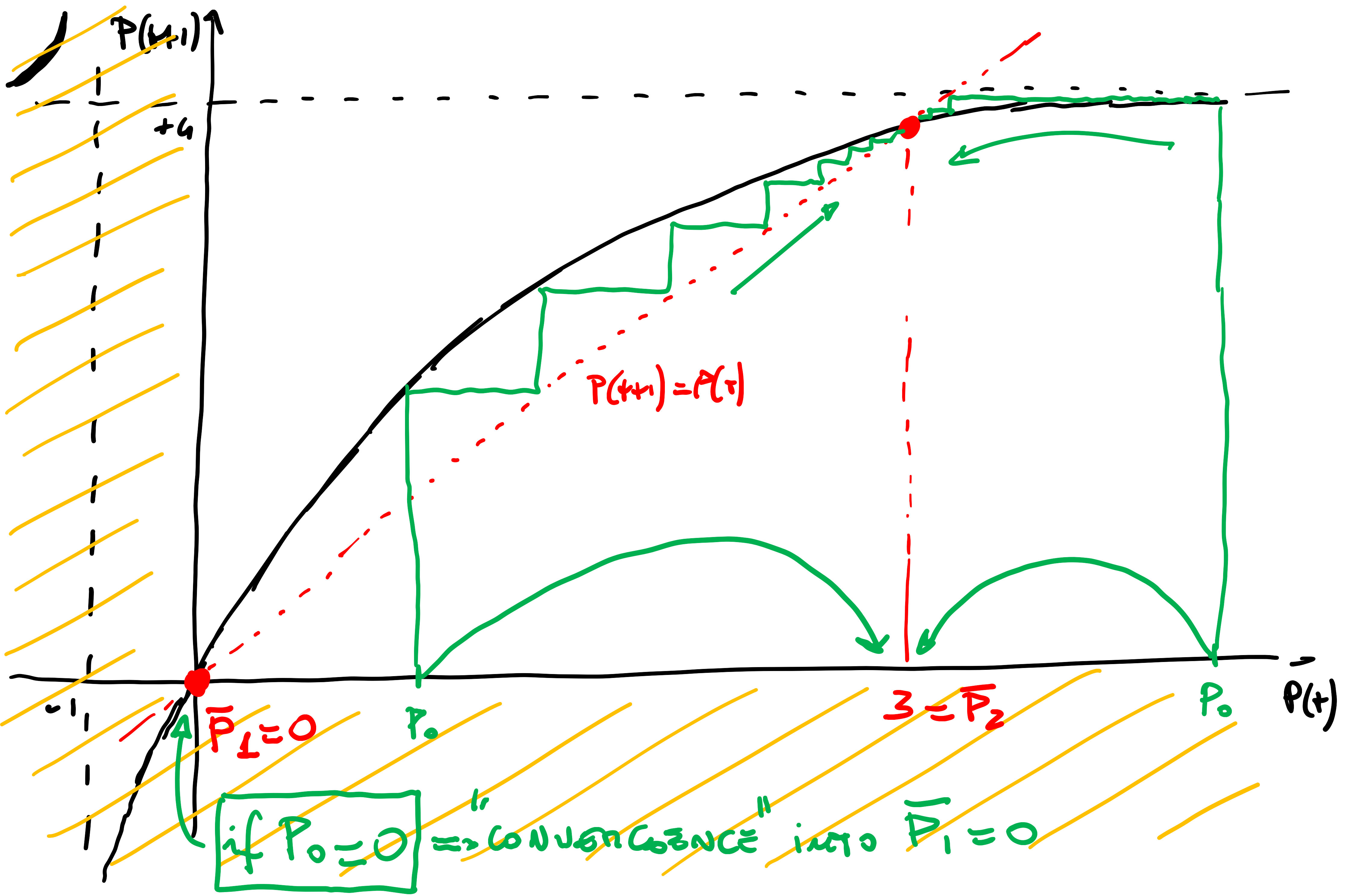
$$\bar{P} = \frac{4\bar{P}}{\bar{P}+1} \longrightarrow \dots \begin{cases} \bar{P}_1 = 0 \\ \bar{P}_2 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} \boxed{\bar{K}_1 = 0} \\ \boxed{\bar{K}_2 = \frac{3}{2}} \end{cases}$$

(No feedback ?)

same solutions as before

DRE CONVERGENCE ANALYSIS


P(t+1)
+4
-1
P(t+1) = P(t)
P̄₁ = 0
P₀
3 = P̄₂
P₀
P(t)
if P₀ = 0 ⇒ "CONVERGENCE" into P̄₁ = 0

if we start from $P_0 = 0$ => means that we have NO UNCERTAINTY in the knowledge of $x(1)$

Moreover STATE EQUATION is NOISE FREE =>

WE DO NOT NEED FEEDBACK; prediction is perfect:

$$\boxed{\hat{x}(t+1|t) = 2\hat{x}(t|t-1)}$$

OPEN-loop solution

perfect initial cond.

NO NOISE

feasible but "IDEAL" situation

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the "STANDARD" solution (NO perfect i.c.) ⟶

$K = \frac{3}{2}$ (CLOSED-loop. solution)