CHAPTER 2: PARAMETRIC B. BOX SYSTEM ID. of I/O SYSTEMS USING a FREQ.-DOMAIN APPROACH

So far we have seen:

(MIDA 1): Parametric B.B. identif. of I/O systems (ARMAX) and Time series (ARMA)

(MIDA 2 - ch. 1) → NON parametric BB ident. of I/O systems (4 SID → SS. REP)

FREQ. DOMAIN APPROACH is B.B and parametric (VERY USED in practice / ROBUST / RELIABLE)

PARAMETRIC SY. ID → classic 4 steps

1. Experiment DESIGN and DATA PRE-processing
2. SELECTION of parametric model class: $\mathcal{M}(\vartheta)$
3. DEFINITION of a PERF. INDEX: $J(\vartheta)$
4. optimization: $\hat{\vartheta} = \underset{\vartheta}{\arg\min}\{J(\vartheta)\}$

A special type of experiment and data pre-processing is NEEDED

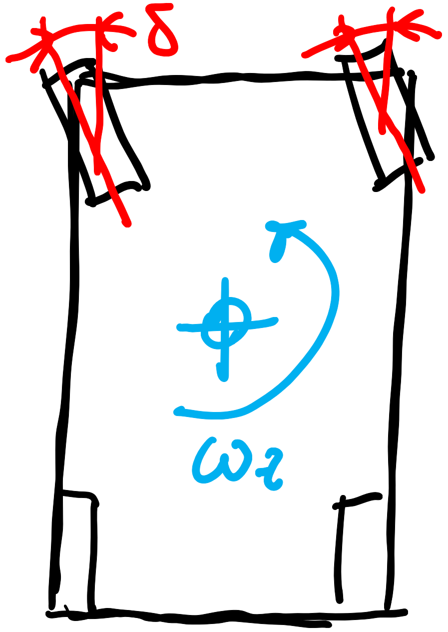
A NEW/special perf. INDEX is NEEDED.

The General idea of the method is:

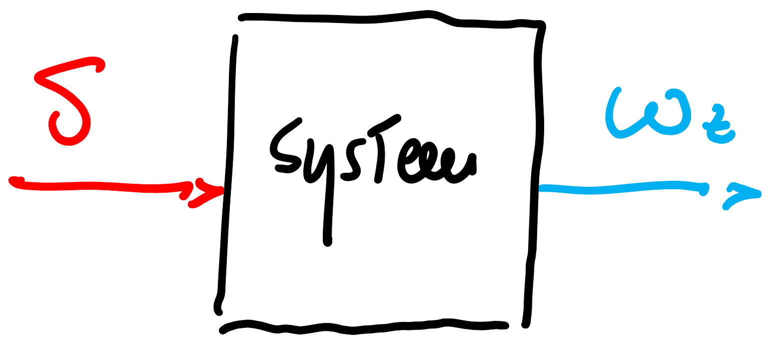
- $\boxed{\text{MAKE a set of "SINGLE SINUSOID" EXCITATION / "SINGLE-TUNE" experiments}}$

- from each experiment → estimate a SINGLE point of FREQUENCY-RESPONSE of system

- "fit" the estimated and modeled freq. response to obtain the optimal model

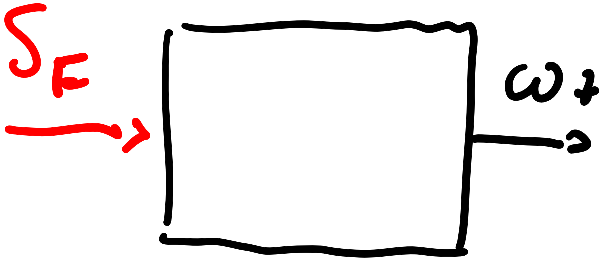


ω_z is the "YAW-RATE" (ROT. VELOCITY of car around its VERTICAL AXIS)



This is the dy. relationship linking the input (STEER) to the output (YAW-RATE)
→ VERY important for STABILITY-CONTROL DESIGN (ALSO AUTONOMOUS CAR)

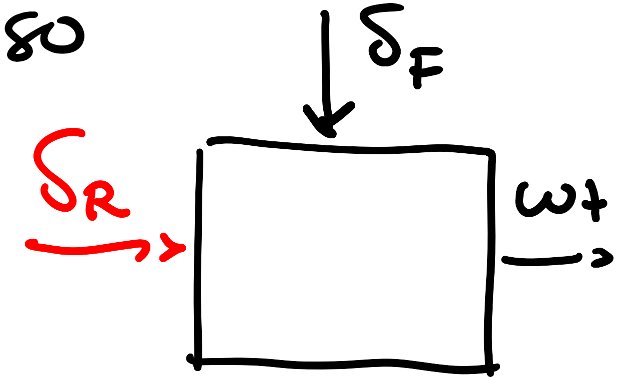
SISO



CONTROL VARIABLE is front STEER -->

Application is **AUTONOMOUS CAR**

SISO

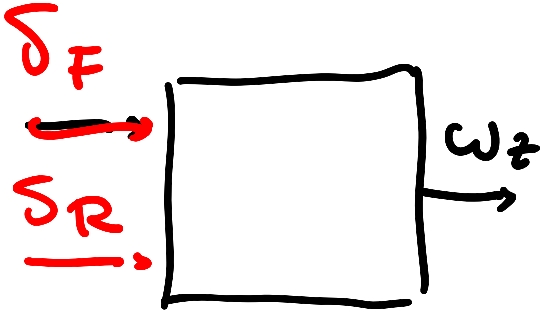


Today case in high. perf. cars

DRIVER --> δ_F (MEASURABLE DIST.)

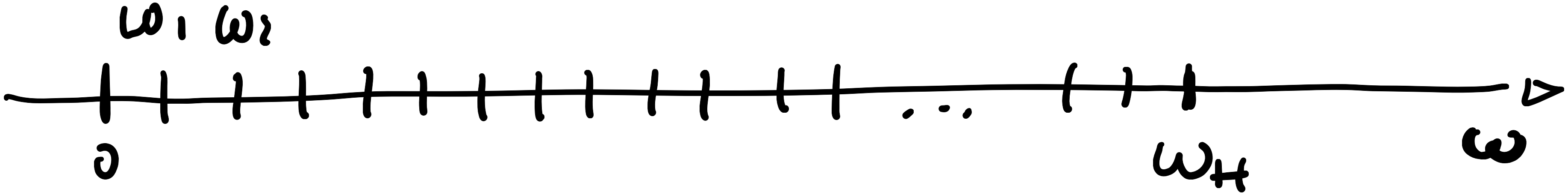
CONTROL system --> δ_F (CONTROL VARIABLE)

MISO



Both δ_F and δ_R are CONTROL VARIABLES (High perf. AUT. cars)

In the Exp. DESIGN STEP we first have to select a set of <u>EXCITATION FREQUENCIES</u>



MAX frequency $\rightarrow$ ω_H

$\{\omega_1, \omega_2, \omega_3 \dots \omega_i, \dots \omega_H\}$

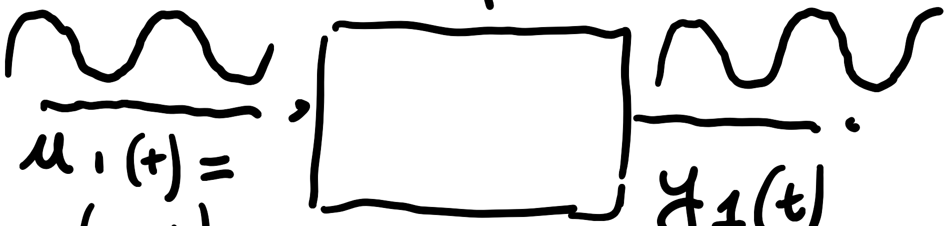
H frequencies RANGING from ω_1 to ω_H

USUALLY they are EVENLY SPACED "$\Delta\omega$" is CONSTANT

ω_H must be selected ACCORDING to the BANDWIDTH of the control system we make H experiments (INDEPENDENT)

exp #1

$\underline{\underline{\omega_1}}$

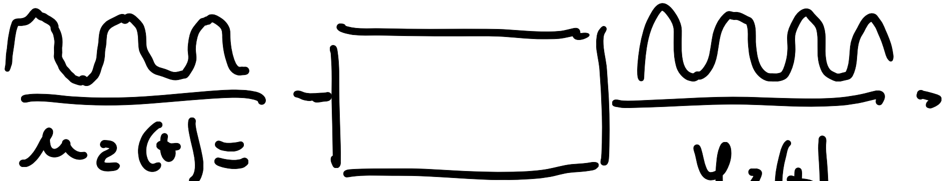


$u_1(t) = A_1 \sin(\omega_1 t)$, $y_1(t)$

$t = 1, 2, 3 \dots N$

exp #2

$\underline{\underline{\omega_2}}$



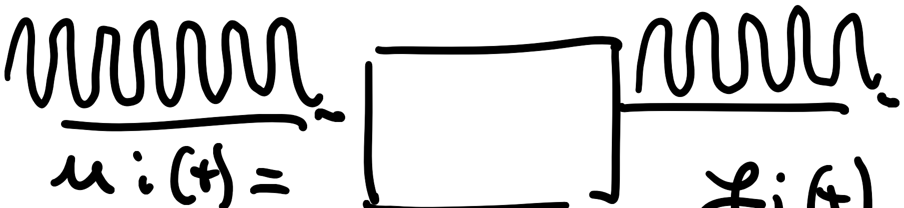
$u_2(t) = A_2 \sin(\omega_2 t)$, $y_2(t)$

$t = 1, 2 \dots N$

$\vdots$

exp #i

$\underline{\underline{\omega_i}}$



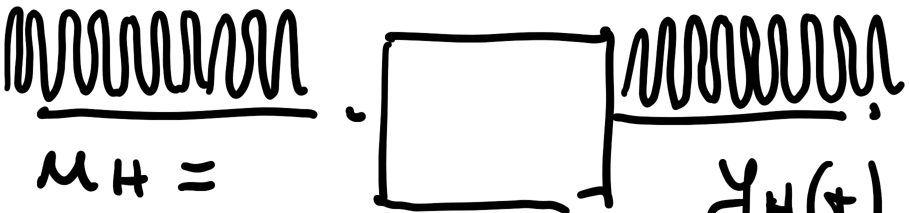
$u_i(t) = A_i \sin(\omega_i t)$, $y_i(t)$

$t = 1, \dots N$

$\vdots$

exp #H

$\underline{\underline{\omega_H}}$



$u_H = A_H \sin(\omega_H t)$, $y_H(t)$

$t = 1, 2 \dots N$

<u>Remark</u>: the AMPLITUDES $A_1, A_2 \dots A_i \dots A_H$ can be EQUAL (CONSTANT) or, more frequently they DECREASE as the frequency increase → To fulfill power CONSTRAINTS on the INPUT (ACTUATOR)

<u>Ex of STEER</u>:

$\delta(t)$ is the steering ANGLE (MOVED by an ACTUATOR)

The requested steering TORQUE is proportional to δ

$$T(t) = K\delta(t)$$

$\Rightarrow$ steering power → $T(t) \cdot \dot{\delta}(t) = K\delta(t)\dot{\delta}(t)$

if $\delta(t) = A_i \sin(\omega_i t) \Rightarrow$

steering power: $KA_i \sin(\omega_i t) \cdot \omega_i A_i \cos(\omega_i t)$

⇒ steering power is proportional to

$$\boxed{K A_i^2 \omega_i}$$ → if we have a limit to

this power, this power should be constant

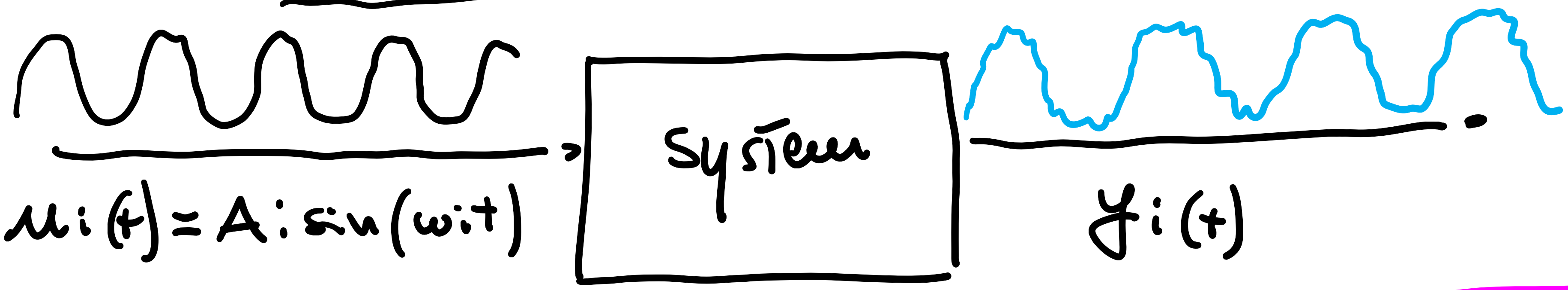
during #1 ... #H experiments:

$$K A_i^2 \omega_i = \text{CONSTANT}$$

$$\Downarrow$$

$$\boxed{A_i = \sqrt{\frac{\text{CONSTANT}}{K \omega_i}}}$$

Now let's focus on the i-th experiment:



Remember that, if the system is LTI (Linear, Time-Invariant) the Freq. Resp. theorem says that if the input is a sinusoid of freq. ω_i → also the output must be a pure sinusoid of freq. ω_i

However, $y_i(t)$, in REAL Applications is NOT a perfect sinusoid:

- Noise on output measurement
- Noise on the system (not directly on output)
- (Small) non-linear effects (that we neglect)

In pre-processing of I/O data → we want to extract from $y_i(t)$ a perfect sinusoid of frequency ω_i → we force the assumption that system is $\boxed{\text{LTI}}$ → the output MUST BE a pure sinusoid of frequency ω_i (All the remaining signal is noise)

The "MODEL" of the output signal is:

$$\hat{y}_i(t) = B_i \sin(\omega_i t + \varphi_i)$$

OR:

PARAMETER HERE ARE "LINEAR"

we use this

$$\boxed{\hat{y}(t) = a_i \sin(\omega_i t) + b_i \cos(\omega_i t)}$$

B_i, φ_i or a_i, b_i are to be estimated the unknown parameters are a_i and b_i

we can find them by parametric identification

$$\{\hat{a}_i, \hat{b}_i\} = \underset{\{a_i, b_i\}}{\arg\min} \{ J_N(a_i, b_i) \}$$

$$J_N(a_i, b_i) = \frac{1}{N} \sum_{t=1}^{N} \Big(\overbrace{\underbrace{y_i(t)}_{\text{MEASURED (NOISY) OUTPUT}} - \underbrace{a_i \sin(\omega_i t) - b_i \cos(\omega_i t)}_{\text{MODELED OUTPUT}}}^{\text{MODELING ERROR}} \Big)^2$$

sample variance of MODELLING ERROR

(NOTE: it is a QUADRATIC function of a_i and b_i)

$\Rightarrow$ we can solve "one-shot" and explicitly the optimization problem $\rightarrow$

$$\begin{cases} \dfrac{\partial J_N}{\partial a_i} = \dfrac{2}{N}\displaystyle\sum_{t=1}^{N}\left(-\sin(\omega_i t)\right)\left(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)\right) = 0 \\[2ex] \dfrac{\partial J_N}{\partial b_i} = \dfrac{2}{N}\displaystyle\sum_{t=1}^{N}\left(-\cos(\omega_i)\right)\left(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)\right) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} \displaystyle\sum_{t=1}^{N} \sin(\omega_i t)^2 & \displaystyle\sum_{t=1}^{N} \sin(\omega_i t)\cos(\omega_i t) \\[2ex] \displaystyle\sum_{t=1}^{N} \sin(\omega_i t)\cos(\omega_i t) & \displaystyle\sum_{t=1}^{N} \cos(\omega_i t)^2 \end{bmatrix} \begin{bmatrix} a_i \\[2ex] b_i \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{t=1}^{N} y_i \sin(\omega_i t) \\[2ex] \displaystyle\sum_{t=1}^{N} y_i \cos(\omega_i t) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \hat{a}_i \\ \hat{b}_i \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}^{-1} \cdot \begin{bmatrix} \; \end{bmatrix}}$$

LEAST-SQUARES estimation of parameters a_i and b_i

At this point we prefer to "go back" to a "sin-only" form (B_i, φ_i) of the estimated sinusoid:

$$\underbrace{\hat{B}_i \sin(\omega_i t + \hat{\varphi}_i)}_{\text{EXPAND}} = \boxed{\hat{a}_i \sin(\omega_i t) + \hat{b}_i \cos(\omega_i t)}$$

$$\Updownarrow$$

$$\boxed{\hat{B}_i \sin(\omega_i t) \cdot \cos(\hat{\varphi}_i) + \hat{B}_i \cos(\omega_i t) \cdot \sin(\hat{\varphi}_i)}$$

$$\boxed{\begin{aligned}\hat{B}_i \cos\hat{\varphi}_i &= \hat{a}_i \\ \hat{B}_i \sin\hat{\varphi}_i &= \hat{b}_i\end{aligned}} \longrightarrow \frac{\hat{b}_i}{\hat{a}_i} = \frac{\sin\hat{\varphi}_i}{\cos\hat{\varphi}_i} = \tan(\hat{\varphi}_i) \Longrightarrow$$

$$\boxed{\hat{\varphi}_i} = \arctan\left(\frac{\hat{b}_i}{\hat{a}_i}\right)$$

$$\boxed{\hat{B}_i} = \frac{\hat{a}_i/\cos\hat{\varphi}_i + \hat{b}_i/\sin\hat{\varphi}_i}{2}$$

$$y_i(t) \approx \boxed{\boxed{\hat{B}_i} \sin\left(\omega_i t + \boxed{\hat{\varphi}_i}\right)}$$

Repeating the experiment and pre-processing for #1, #2 ... #H

we obtain:

$$\{\hat{B}_1, \hat{\varphi}_1\} \longrightarrow \frac{\hat{B}_1}{A_1} e^{j\hat{\varphi}_1}$$

$$\{\hat{B}_2, \hat{\varphi}_2\} \longrightarrow \frac{\hat{B}_2}{A_2} \cdot e^{j\hat{\varphi}_2}$$

$$\vdots$$

$$\{\hat{B}_i, \hat{\varphi}_i\} \longrightarrow \frac{\hat{B}_i}{A_i} e^{j\hat{\varphi}_i}$$

$$\vdots$$

$$\{\hat{B}_H, \hat{\varphi}_H\} \longrightarrow \frac{\hat{B}_H}{A_H} e^{j\hat{\varphi}_H}$$

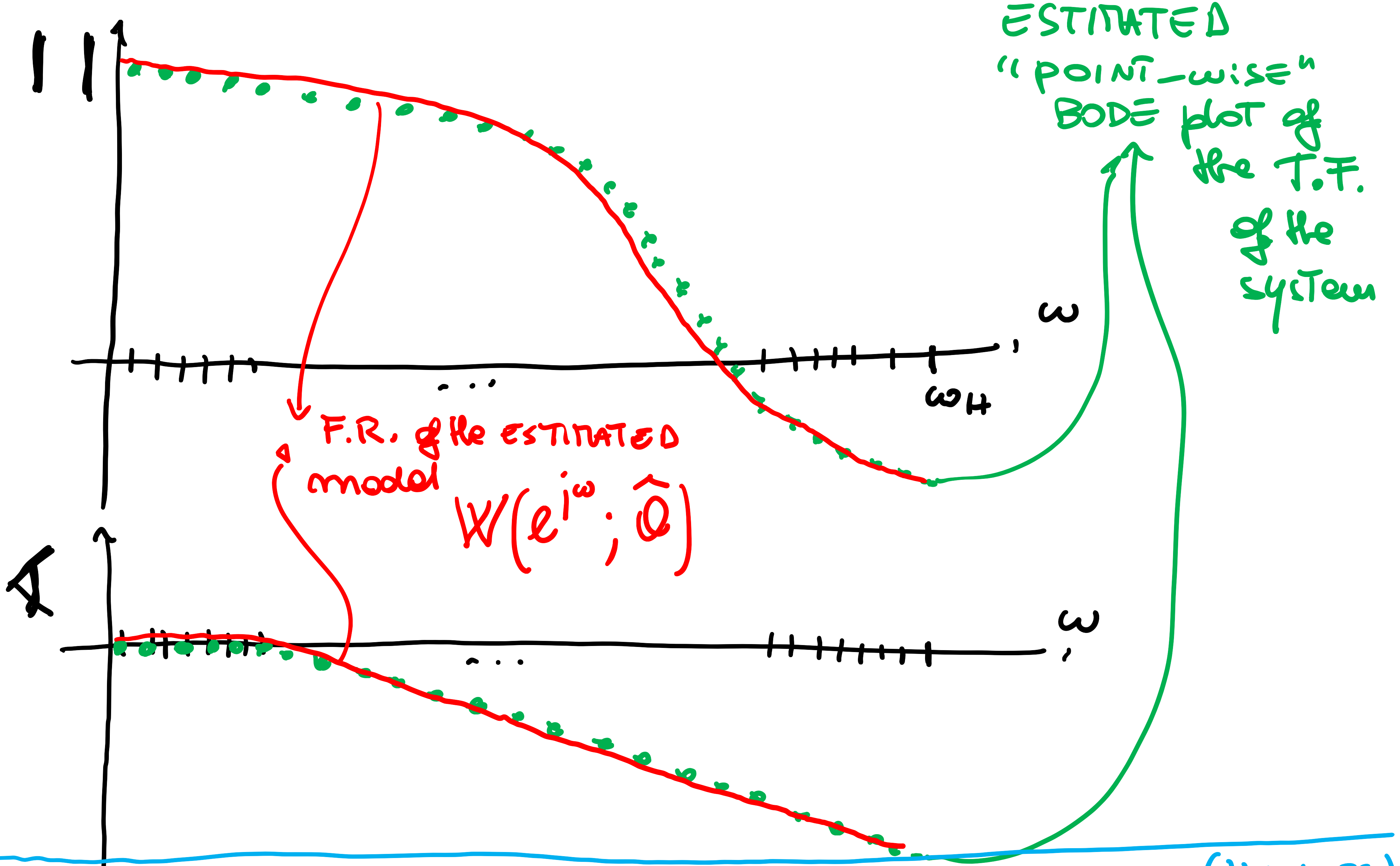
$\rightarrow$ $\boxed{\text{H complex numbers}}$

$\downarrow$

ESTIMATED H points of the frequency response of the transfer function $W(z)$ from input $u(t)$ to output $y(t)$ of the system

$$u(t) \longrightarrow \boxed{W(z)} \longrightarrow y(t)$$

$\downarrow$ graphically:



At the end of STEP 1 we have a freq.-domain DATASET (H VALUES) representing H ESTIMATED points of the FREQ. RESP. of the system

STEP 2: Model class selection (T. F.)

$$\mathcal{M}(\vartheta):\ W(z;\vartheta)=\boxed{\frac{b_0+b_1z^{-1}+\dots\ b_pz^{-p}}{1+a_1z^{-1}+\dots\ +a_nz^{-n}}\cdot z^{-1}} \qquad \vartheta=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\\b_0\\b_1\\\vdots\\b_p\end{bmatrix}$$

Remark: as usual, we have the problem of ORDER SELECTION (n and p) → USE CROSS-VALIDATION APPROACH (OR, more simply - VISUAL inspection of the fitting of BODE - PLOTS)

STEP # 3 -- we NEED a NEW perf. iNDEX

(freq. DOMAIN – NOT in Time Domain)

$$J_H(\theta) = \boxed{\frac{1}{H}\sum_{i=1}^{H}\left(\underbrace{W(e^{j\omega_i};\theta)}_{\text{"MODELED" F.RESP.}} - \underbrace{\frac{\hat{B}_i}{A_i}e^{j\hat{\varphi}_i}}_{\text{"MEASURED" F.RESP.}}\right)^2}$$

Function: $\mathbb{R}^{n+p} \rightarrow \mathbb{R}^{+}$

1. ESTIMATED error VARIANCE of FREQ. RESPONSE FITTING

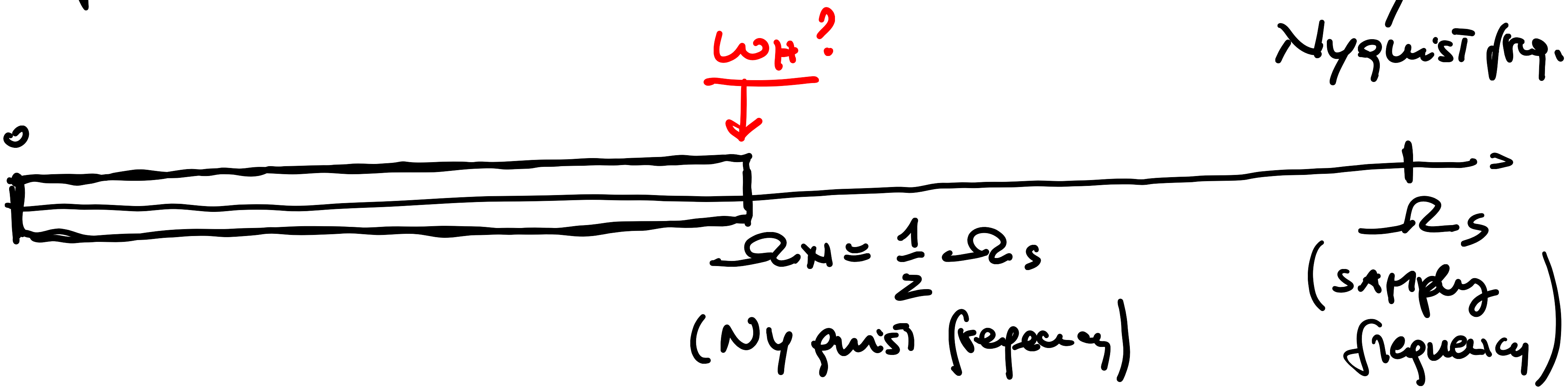
STEP 4 --> optimization:

$$\hat{\theta} = \underset{\theta}{\mathrm{argmin}}\left\{J_H(\theta)\right\}$$

USUALLY $J_H(\theta)$ is a NON QUADRATIC and NON-CONVEX function => ITERATIVE optimization methods are NEEDED (GRADIENT-DESC. QUASI-NEWT.)

Remark: FREQUENCY BANDwidth selection ($\omega_H = ?$)

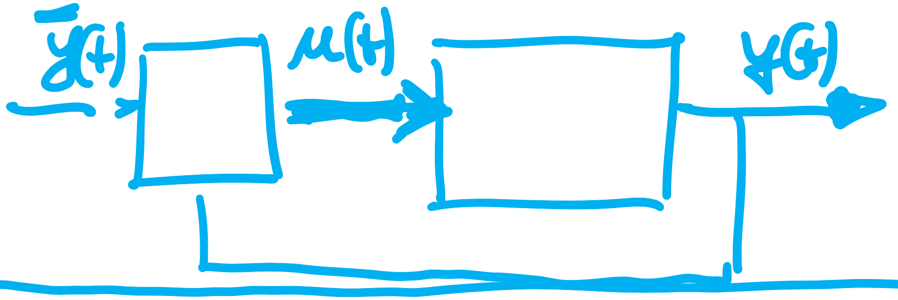
theoretically the standard "BEST" solution should be → H point DISTRIBUTED UNIFORMLY from $\emptyset$ to Ω_N



In practice/In real applications is BETTER to CONCENTRATE the experimental EFFORT in a SMALLER AND MORE FOCUSED BANDWIDTH

Rule of thumb --> $\approx$ 3 times the control cut off frequency

ω_c --> is the expected "cut off" frequency of the control system



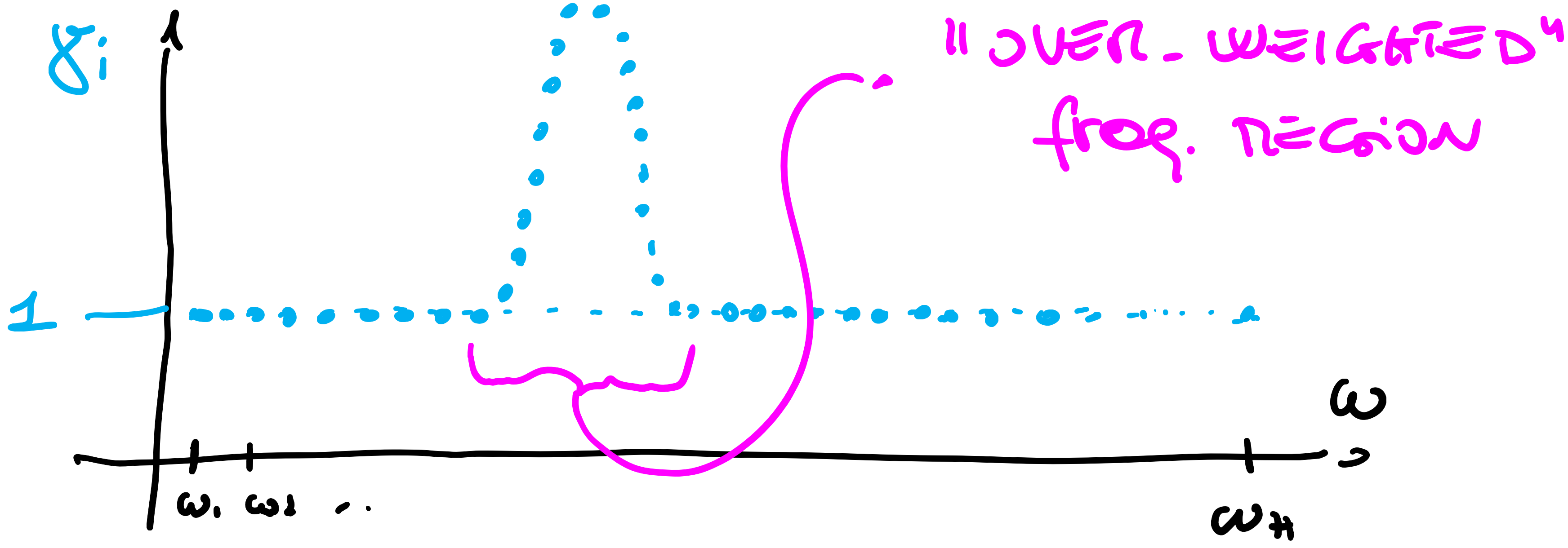
<u>ex</u>: $\boxed{\text{ESC --> expected bandwidth} \approx 2Hz}$

↳ we can limit our exploration to <u>$\omega_H \approx 6Hz$</u>

Remark #2 → Emphasis on special freq. RANGE

In some cases, between ω_1 and ω_H, we want to be MORE ACCURATE in sy. IDENT. is some frequency regions (Typically: AROUND CUT-OFF-FREQUENCY; AROUND RESONANCES)

⇒ we can use different weights for different frequencies →



→ The perf. INDEX can be RE-DEFINED →

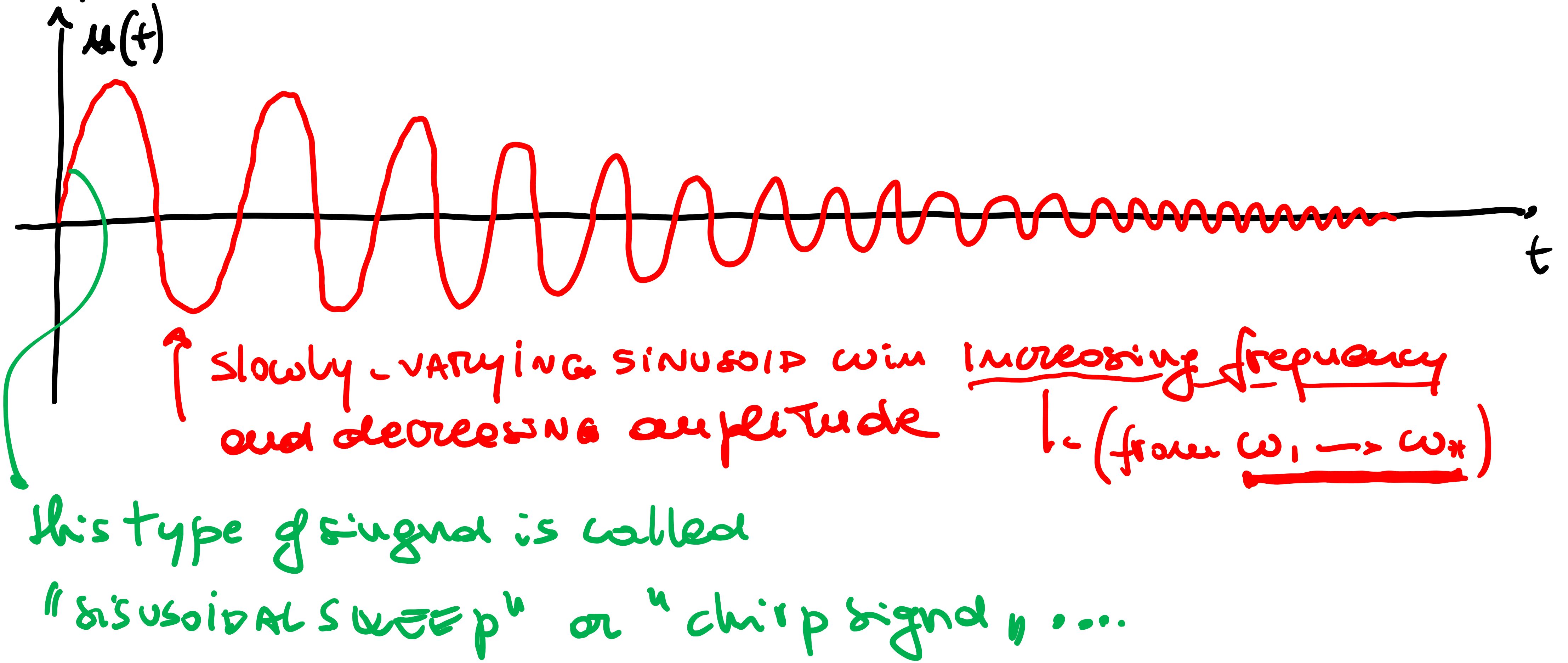
$$\tilde{J}_H(\theta) = \frac{1}{H} \sum_{i=1}^{H} \underbrace{\gamma_i}_{} \left(W(e^{j\omega_i}; \theta) - \frac{\hat{B}_i}{A_i} e^{j\hat{\varphi}_i} \right)^2$$

"ERRORS ARE MORE important" in the freq. REGION we WANT TO EMphasize

(Another "TRICK" → more DENSE ω_i spacing in the freq. REGION of special interest)

Remark #3 (single experiment)

Sometimes, the set of H independent/separated single-sinusoid experiments can be replaced by a long single "sine-sweep" experiment



We can cut A-posteriori the signal into H pieces → Back to the standard procedure

or → directly compute an estimation of $\boxed{\hat{W}(e^{j\omega})}$ as a ratio of the output and input spectra →

$$\hat{W}(e^{j\omega}) \simeq \frac{\hat{\Gamma}_y(e^{j\omega})}{\hat{\Gamma}_u(e^{j\omega})}$$

→ we can fit the estimated $\hat{W}(e^{j\omega})$ with the model freq. resp $W(e^{j\omega};\theta)$ in the perf. index (step 3)

this experiment is quicker but has usually a lower signal-to-noise-ratio

Comparison (PROS and CONS) between Time domain ("ARMAX") and freq. DOMAIN parametric methods

F.D. methods

- PROS:
  - ROBUST and VERY RELIABLE (we put a lot of ENERGY on each SINUSOID - F, RES poste ESTIMATION is VERY RELIABLE)
  - INTUITIVE (EASY TO UNDERST.)
  - CONSISTENT with CONTROL-DESIGN method (USUALLY in freq. DOMAIN)
- CONS: MORE DEMANDING for experiments

(NO NOISE model is ESTIMATED, UNLIKE ARMAX model)

Notice that T.D. and F.D methods should provide the SAME RESULT if DONE CORRECTLY

END ch. 2