PRACTICE SESSION 1

SUBSPACE STATE SPACE IDENTIFICATION METHOD

EXERCISE 1

Given a 2^{nd} order system in state space with matrices

$$F = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} \qquad G = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad H = [1 \quad 0] \qquad D = [0]$$

a) Write the system of difference equation

State space representation

$$\begin{cases} x(t+1) = F x(t) + G u(t) & \leftarrow \text{STATE EQUATION} \\ y(t) = H x(t) + D u(t) & \leftarrow \text{OUTPUT EQUATION} \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \qquad n \text{ is the system order}$$

$y(t), u(t) \in \mathbb{R}^1$ (SISO system)

$$n = 2 \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u(t) \\ y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]\, u(t) \end{cases}$$

Performing the matrix multiplication

$$\boxed{\begin{cases} x_1(t+1) = 2\, x_2(t) + \frac{1}{2} u(t) \\ x_2(t+1) = \frac{1}{2} x_1(t) + 3\, x_2(t) + \frac{1}{2} u(t) \\ y(t) = x_1(t) \end{cases}}$$

(b) Compute the system transfer function

Transfer function

$y(t) = W(z)\, u(t)$ $\qquad u(t) \rightarrow \boxed{W(z)} \rightarrow y(t)$

$\hookrightarrow$ digital filter

I method: apply the z-transformation directly to the difference equations

$x(t+1) = z\, x(t)$

$$\begin{cases} x_1(t+1) = 2\, x_2(t) + \frac{1}{2} u(t) \\ x_2(t+1) = \frac{1}{2} x_1(t) + 3 x_2(t) + \frac{1}{2} u(t) \\ y(t) = x_1(t) \end{cases}$$

$$\begin{cases} (1)\quad z\, x_1(t) = 2\, x_2(t) + \frac{1}{2} u(t) \\ (2)\quad z\, x_2(t) = \frac{1}{2} x_1(t) + 3\, x_2(t) + \frac{1}{2} u(t) \\ (3)\quad y(t) = x_1(t) \end{cases}$$

(2) $\rightarrow x_2(t)$

$$(z-3)\, x_2(t) = \frac{1}{2} x_1(t) + \frac{1}{2} u(t)$$

$$x_2(t) = \frac{1/2}{z-3} x_1(t) + \frac{1/2}{z-3} u(t)$$

$x_2(t) \rightarrow$ (1) $\rightarrow x_1(t) \rightarrow$ (3)

$$z\, x_1(t) = \frac{1}{z-3} x_1(t) + \frac{1}{z-3} u(t) + \frac{1}{2} u(t)$$

$$\left(\frac{z^2-3z-1}{\cancel{z-3}}\right) x_1(t) = \frac{1 + \frac{1}{2}z - \frac{3}{2}}{\cancel{z-3}}\, u(t) \quad \rightarrow \quad \frac{1}{2}z - \frac{1}{2} = \frac{1}{2}(z-1)$$

$$x_1(t) = \frac{1/2\,(z-1)}{z^2-3z-1}\, u(t)$$

$$y(t) = \underbrace{\frac{\frac{1}{2}(z-1)}{z^2-3z-1}}_{W(z)}\, u(t)$$

$$\boxed{W(z) = \frac{\frac{1}{2}(z-1)}{z^2-3z-1}}$$

II Method: Transformation formula

$$W(z) = H(zI-F)^{-1}G + D$$

STEP 1: Compute $zI - F$

$$zI - F = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} = \begin{bmatrix} z & -2 \\ -\frac{1}{2} & z-3 \end{bmatrix}$$

STEP 2: Compute $\det(zI-F)$

$$\det(zI-F) = z(z-3) - 1 = z^2 - 3z - 1$$

STEP 3: Compute $(zI-F)^{-1}$

$$\begin{bmatrix} z & -2 \\ -\frac{1}{2} & z-3 \end{bmatrix}^{-1} = \frac{1}{z^2-3z-1}\begin{bmatrix} z-3 & 2 \\ \frac{1}{2} & z \end{bmatrix}$$

STEP 4: Matrix multiplication

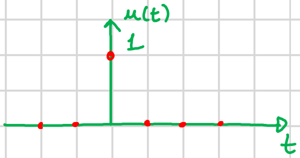
$$W(z) = \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\frac{1}{z^2-3z-1}\begin{bmatrix} z-3 & 2 \\ \frac{1}{2} & z \end{bmatrix}\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} =$$

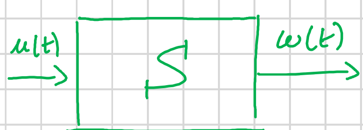
$$= \frac{1}{z^2-3z-1}\begin{bmatrix} z-3 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{\frac{1}{2}z - \frac{3}{2} + 1}{z^2-3z-1} = \frac{\frac{1}{2}(z-1)}{z^2-3z-1}$$

$$\boxed{W(z) = \frac{\frac{1}{2}(z-1)}{z^2-3z-1}}$$

c) Compute the first 4 samples of the impulse response

Impulse Response (IR)





the impulse response w(t) is the output of the system when the input is a UNITARY impulse.

$$u(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

The IR can be computed with 4 methods

$x(t)$, $y(t)$ — I) Directly from the system of difference equations

II) Matrix multiplication formula

(I and II: from state-space)

$y(t)$ — III) Long Division

IV) Geometric series trick

(III and IV: from transfer function)

In this exercise we will use method III

Convolution of the input with the impulse response

$$y(t) = w(0)u(t) + w(1)u(t-1) + w(2)u(t-2) + \dots = \sum_{k=0}^{\infty} w(k)u(t-k)$$

$$y(t) = \underbrace{\left(w(0) + w(1)z^{-1} + w(2)z^{-2} + \dots\right)}_{W(z)} u(t)$$

$$W(z) = \frac{B(z)}{A(z)} = w(0) + w(1)z^{-1} + \dots$$

Long Division

$$W(z) = \frac{\frac{1}{2}(z-1)}{z^2-3z-1} = \frac{\frac{1}{2}z - \frac{1}{2}}{z^2-3z-1} = \frac{\boxed{\frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}}{\boxed{1-3z^{-1}-z^{-2}}}$$

$$\begin{array}{l|l}
\frac{1}{2}z^{-1} - \frac{1}{2}z^{-2} & 1-3z^{-1}-z^{-2} \\ \cline{2-2}
-\frac{1}{2}z^{-1} + \frac{3}{2}z^{-2} - \frac{1}{2}z^{-2} & \underbrace{\frac{1}{2}}_{w(1)}z^{-1} + \underbrace{1}_{w(2)}z^{-2} + \underbrace{\frac{7}{2}}_{w(3)}z^{-3} + \underbrace{\frac{23}{2}}_{w(4)}z^{-4} \\ \hline
/ \quad z^{-2} - \frac{1}{2}z^{-2} & \\
-z^{-2} + 3z^{-3} + z^{-4} & \\ \hline
/ \quad \frac{7}{2}z^{-3} + z^{-4} & \\
-\frac{7}{2}z^{-3} + \frac{21}{2}z^{-4} + \frac{7}{2}z^{-5} & \\ \hline
/ \quad \frac{23}{2}z^{-4} + \frac{7}{2}z^{-5} &
\end{array}$$

$$W(z) = \left(\frac{1}{2}z^{-1} + z^{-2} + \frac{7}{2}z^{-3} + \frac{23}{2}z^{-4} + \dots\right)$$

$$\boxed{w(0) = 0 \qquad w(1) = \frac{1}{2} \qquad w(2) = 1 \qquad w(3) = \frac{7}{2} \qquad w(4) = \frac{23}{2}}$$

(d) Check the system observability and reachability

OBSERVABILITY

$\mathcal{S}$ is observable iff the observability matrix $\mathcal{O}$ is full-rank

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

REACHABILITY

$\mathcal{S}$ is reachable iff the reachability matrix $\mathcal{R}$ is full-rank

$$\mathcal{R} = [\,G \quad FG \quad F^2G \cdots F^{n-1}G\,]$$

$n = 2 \qquad F = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} \qquad G = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad H = [1 \quad 0]$

OBSERVABILITY

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix}$$

$$\mathcal{O}(1,:) = [\,1 \quad 0\,]$$

$$\mathcal{O}(2,:) = [\,1 \quad 0\,]\begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} = [\,0 \quad 2\,]$$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{rank}(\mathcal{O}) = 2 = n \rightarrow \mathcal{O} \text{ is full rank}$$

- $\det(\mathcal{O}) \neq 0$
- all the rows/columns of $\mathcal{O}$ are linearly indipendent.

REACHABILITY

$$\mathcal{R} = [\,G \quad FG\,]$$

$$\mathcal{R}(:,1) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad \mathcal{R}(:,2) = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{7}{4} \end{bmatrix}$$

$$\mathcal{R} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{7}{4} \end{bmatrix} \rightarrow \text{rank}(\mathcal{R}) = 2 = n \rightarrow \mathcal{R} \text{ is full-rank}$$

The system is observable and reachable

ⓒ Compute the Hankel matrix of order n

1 - starting from the impulse response

2 - starting from $\mathcal{O}$ and $\mathcal{R}$

Hankel matrix

$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \omega(4) & \cdots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \cdots & \omega(n) & \omega(n+1) \\ \omega(3) & \omega(4) & \cdots & \omega(n) & \omega(n+1) & \omega(n+2) \\ \vdots & & & & & \\ \omega(n) & \omega(n+1) & \omega(n+2) & \cdots & \cdots & \omega(2n-1) \end{bmatrix} = \mathcal{O}_n \mathcal{R}_n$$

$n = 2$

$$\boxed{\begin{aligned} H_2 &= \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{7}{2} \end{bmatrix} \\ H_2 &= \mathcal{O}_2 \mathcal{R}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{7}{2} \end{bmatrix} \end{aligned}}$$

EXERCISE 2

Given the impulse response

$$w(t) = \begin{cases} 0 & t \leq 1 \\ (-2)^{2-t} & t > 1 \end{cases}$$

a) Compute the transfer function

Impulse response Analysis

$w(0) = 0$
$w(1) = 0$
$w(2) = (-2)^0 = 1$
$w(3) = (-2)^{-1} = -1/2$
$w(4) = (-2)^{-2} = 1/4$
$w(5) = (-2)^{-3} = -1/8$
$w(6) = (-2)^{-4} = 1/16$

$$y(t) = \cancel{w(0)} + \cancel{w(1)}\,u(t-1) + w(2)\,u(t-2) + \dots$$

$$y(t) = u(t-2) - \frac{1}{2}u(t-3) + \frac{1}{4}u(t-4) - \frac{1}{8}u(t-5) + \frac{1}{16}u(t-6) + \dots$$

$$y(t) = \left(z^{-2} - \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4} - \frac{1}{8}z^{-5} + \frac{1}{16}z^{-6} + \dots\right)u(t)$$

$$y(t) = \left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3} + \frac{1}{16}z^{-4} + \dots\right)z^{-2}u(t)$$

$$y(t) = \left(\sum_{k=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^k\right)z^{-2}u(t)$$

Geometric series

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \qquad |a| < 1$$

$$y(t) = z^{-2}\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)u(t)$$

$$\boxed{W(z) = \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}}}$$

(b) Write the state space system in control form

$$W(z) = \frac{b_0 z^{m-1} + b_1 z^{m-2} + \dots + b_{m-1}}{z^m + a_1 z^{m-1} + \dots + a_m} \quad \text{(strictly proper)}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_m & -a_{m-1} & \cdots & & -a_1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$H = [\, b_{m-1} \ \cdots\cdots \ b_0 \,] \qquad D = [\,0\,]$$

$$W(z) = \frac{z^{-2}}{1 + \frac{1}{2} z^{-1}} = \frac{1}{z^2 + \frac{1}{2} z}$$

$m = 2 \qquad a_1 = \frac{1}{2} \quad a_2 = 0 \qquad b_1 = 1 \qquad b_0 = 0$

$$\boxed{F = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad H = [\,1 \quad 0\,] \qquad D = [\,0\,]}$$

Possible check : $W(z) = H\,(zI - F)^{-1} G + D$

(c) Write the system of difference equations and apply the change of variable $\tilde{x} = Tx \qquad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$m = 2 \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\boxed{\begin{aligned} x_1(t+1) &= x_2(t) \\ x_2(t+2) &= -\tfrac{1}{2} x_2(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}}$$

CHANGE OF VARIABLES (STATE-TRANSFORMATION)

$$\begin{cases} x(t+1) = F\,x(t) + G\,u(t) \\ y(t) = H\,x(t) + D\,u(t) \end{cases}$$

$$\tilde{x}(t) = T\,x(t)$$

$$\begin{cases} \tilde{x}(t+1) = \tilde{F}\,\tilde{x}(t) + \tilde{G}\,u(t) \\ y(t) = \tilde{H}\,\tilde{x}(t) + \tilde{D}\,u(t) \end{cases}$$

equivalent systems
(same input/output relationship)

$$x(t) = T^{-1}\tilde{x}(t) \longrightarrow x(t+1) = T^{-1}\tilde{x}(t+1)$$

$$\begin{cases} T^{-1}\tilde{x}(t+1) = F\,T^{-1}\tilde{x}(t) + G\,u(t) \\ y(t) = H\,T^{-1}\tilde{x}(t) + D\,u(t) \end{cases}$$

$$\begin{cases} \tilde{x}(t+1) = TFT^{-1}\tilde{x}(t) + TG\,u(t) \\ y(t) = HT^{-1}\tilde{x}(t) + D\,u(t) \end{cases}$$

$$\tilde{F} = TFT^{-1} \qquad \tilde{G} = TG \qquad \tilde{H} = HT^{-1} \qquad \tilde{D} = D$$

We have to compute matrices $\tilde{F}$, $\tilde{G}$, $\tilde{H}$, $\tilde{D}$

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{F} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 3/2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ 3/4 & -\frac{3}{4} \end{bmatrix}$$

$$\tilde{G} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\tilde{H} = [\,1 \quad 0\,]\begin{bmatrix} 1/2 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = [\,1/2 \quad \frac{1}{2}\,]$$

$$\tilde{D} = D = 0$$

$$\boxed{\begin{cases} \tilde{x}_1(t+1) = \frac{1}{4}\tilde{x}_1(t) - \frac{1}{4}\tilde{x}_2(t) + u(t) \\ \tilde{x}_2(t+1) = \frac{3}{4}\tilde{x}_2(t) - \frac{3}{4}\tilde{x}_2(t) - u(t) \\ y(t) = \frac{1}{2}\tilde{x}_1(t) + \frac{1}{2}\tilde{x}_2(t) \end{cases}}$$

(d) Compute the transfer function starting from the new representation

$$W(z) = \tilde{H}\,(zI - \tilde{F})^{-1}\,\tilde{G} + \tilde{D}$$

$$(zI - \tilde{F}) = \begin{bmatrix} z - \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & z + \frac{3}{4} \end{bmatrix}$$

$$\det(zI - \tilde{F}) = \left(z - \frac{1}{4}\right)\left(z + \frac{3}{4}\right) + \frac{3}{16} = z^2 + \frac{3}{4}z - \frac{1}{4}z - \cancel{\frac{3}{16}} + \cancel{\frac{3}{16}} = z^2 + \frac{1}{2}z$$

$$(zI - \tilde{F})^{-1} = \frac{1}{z^2 + \frac{1}{2}z}\begin{bmatrix} z + \frac{3}{4} & -1/4 \\ +\,3/4 & z - \frac{1}{4} \end{bmatrix}$$

$$W(z) = \frac{1}{z^2 + \frac{1}{2}z}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} z + 3/4 & -1/4 \\ 3/4 & z - \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$= \frac{1}{z^2 + \frac{1}{2}z}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} z + 1 \\ -z + 1 \end{bmatrix} =$$

$$= \frac{1}{z^2 + \frac{1}{2}}\left(\frac{1}{2}z + \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}\right) = \frac{1}{z^2 + \frac{1}{2}} = \frac{z^{-2}}{1 + \frac{1}{2}z^{-2}}$$

$$\boxed{W(z) = \frac{z^{-2}}{1 + \frac{1}{2}z^{-2}}}$$

Same result of question (a)

(e) Compute H_3 and check that $\mathcal{O}_3 \mathcal{R}_3 = H_3$

$$\boxed{H_3 = \begin{bmatrix} w(1) & w(2) & w(3) \\ w(2) & w(3) & w(4) \\ w(3) & w(4) & w(5) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \end{bmatrix}}$$

$$\mathcal{O}_3 = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix}$$

$$\mathcal{O}_3(1,:) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathcal{O}_3(2,:) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} = \overbrace{\boxed{\begin{bmatrix} 0 & 1 \end{bmatrix}}}^{HF}$$

$$\mathcal{O}_3(3,:) = HF \cdot F = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}$$

$$\mathcal{O}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\mathcal{R}_3 = \begin{bmatrix} G & FG & F^2G \end{bmatrix}$$

$$\mathcal{R}_3(:,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathcal{R}_3(:,2) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overset{FG}{\boxed{\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}}}$$

$$\mathcal{R}_3(:,3) = F \cdot FG = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

$$\mathcal{R}_3 = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\boxed{H_3 = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}}_{\mathcal{O}_3} \underbrace{\begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}}_{\mathcal{R}_3} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \quad \checkmark}$$

(f) Check that $F = \mathcal{O}_3(1:m,:)^{-1}\mathcal{O}_3(2:m+1,:)$

$$\mathcal{O}_3(1:m,:) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{O}_3^{-1}(1:m,:)$$

$$\mathcal{O}_3(2:m,:) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\boxed{\mathcal{O}_3(1:m,:)^{-1}\mathcal{O}_3(2:m+1,:) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} = F}$$