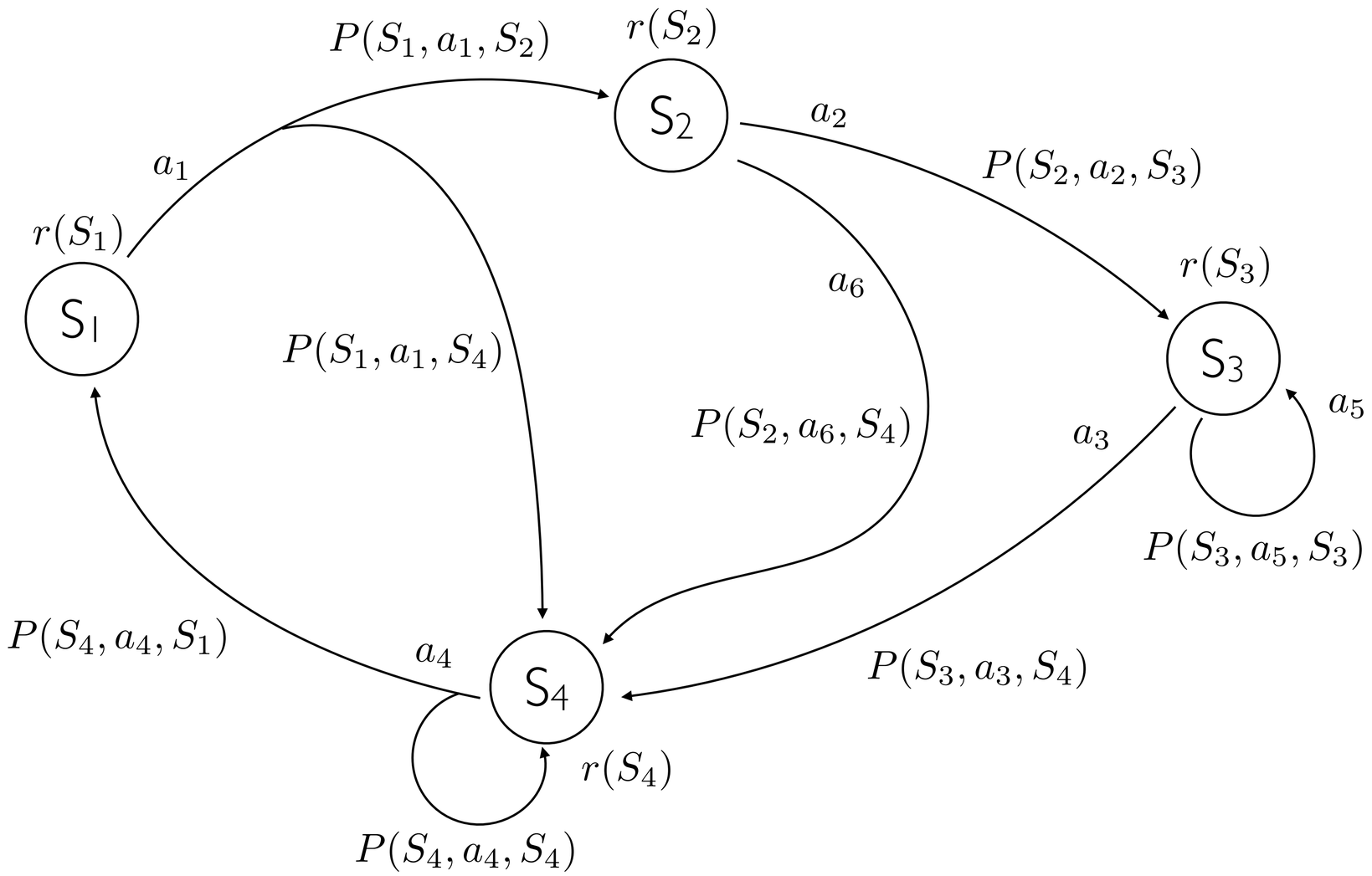
Stochastic games and multi-agent reinforcement learning (MARL) algorithms

POLITECNICO
MILANO 1863

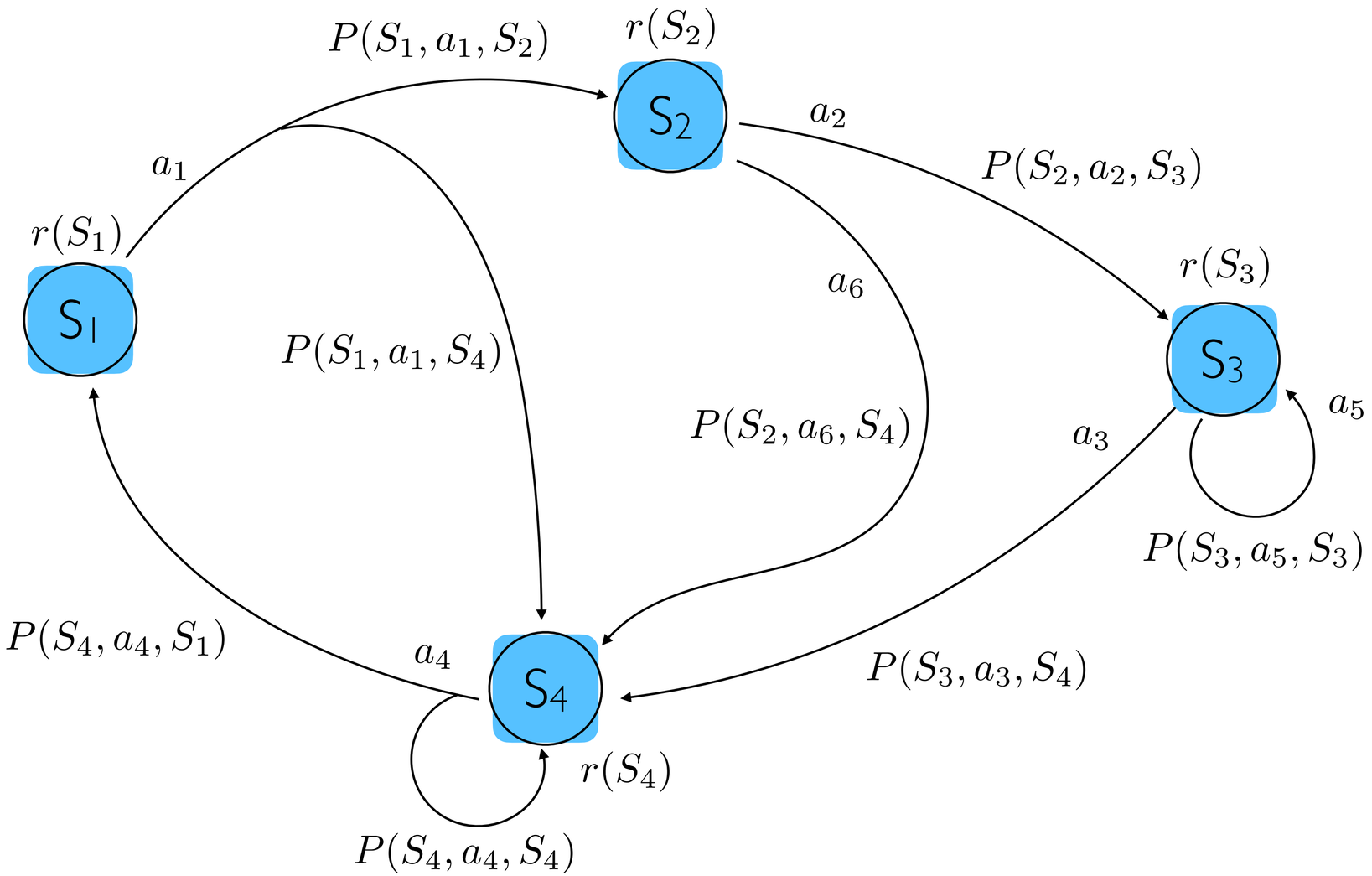
Markov Decision Processes (MDP)

MDP



States Actions Transitions Rewards

MDP



States Actions Transitions Rewards

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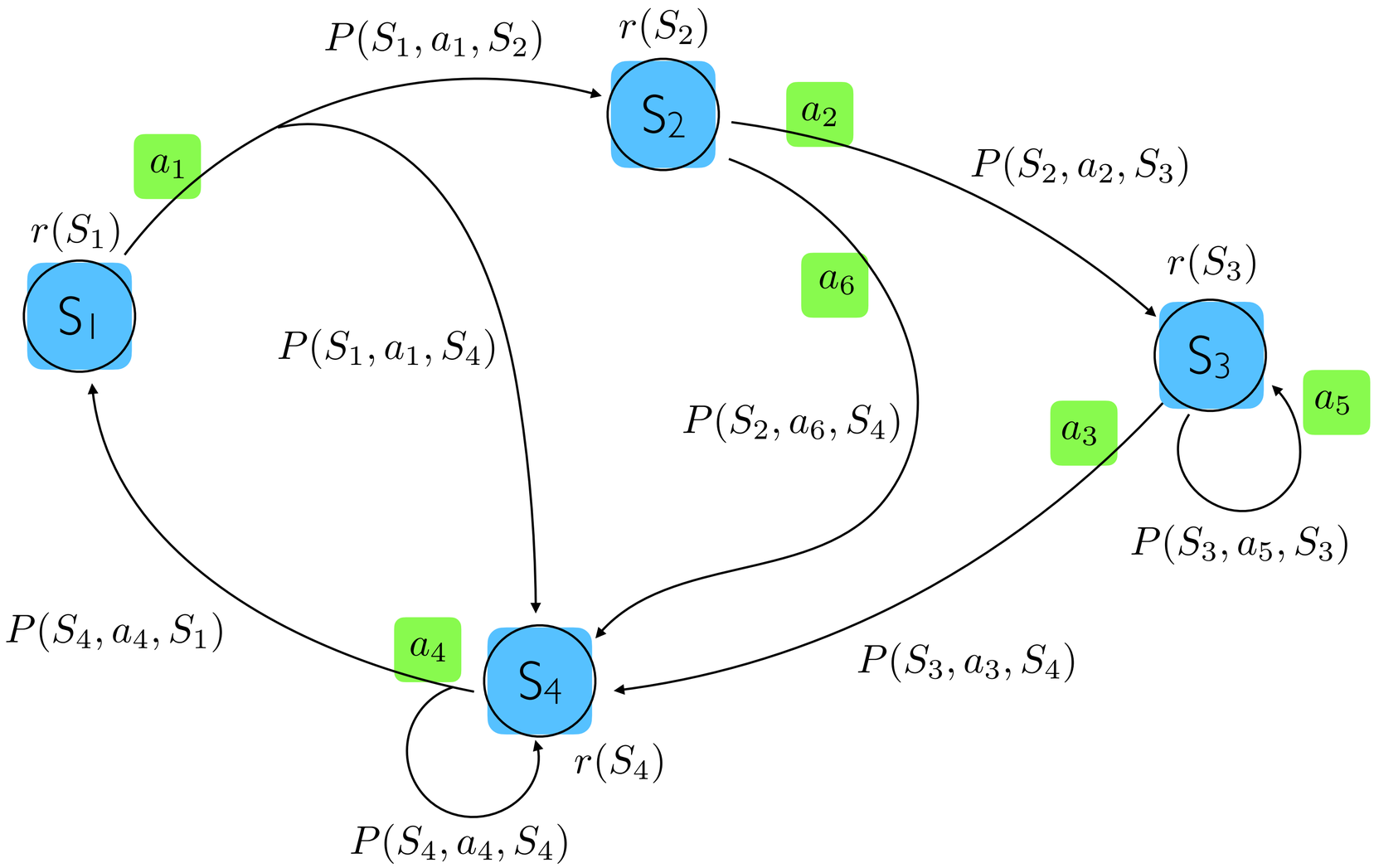
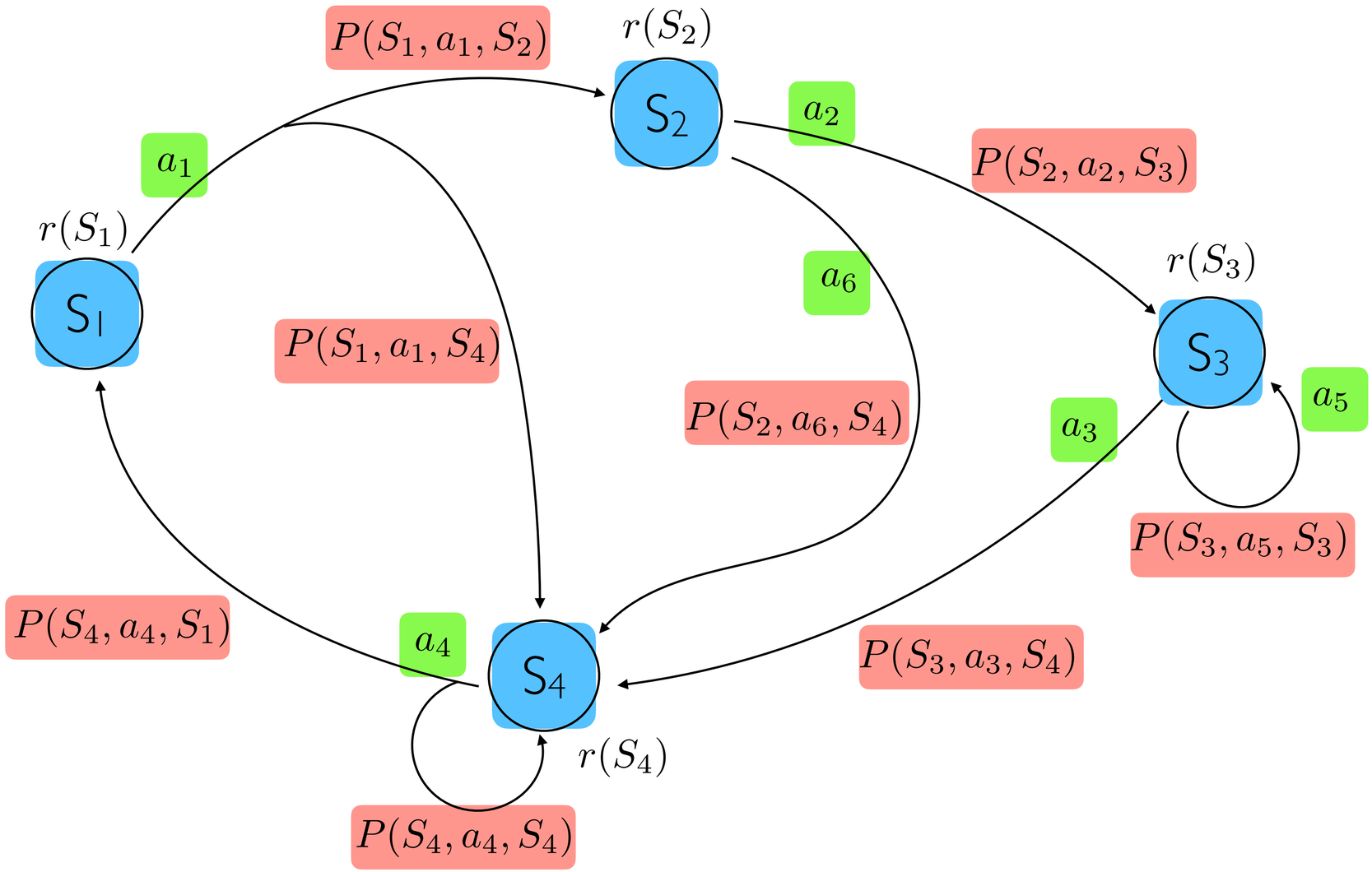


States Actions Transitions Rewards

MDP



States Actions Transitions Rewards

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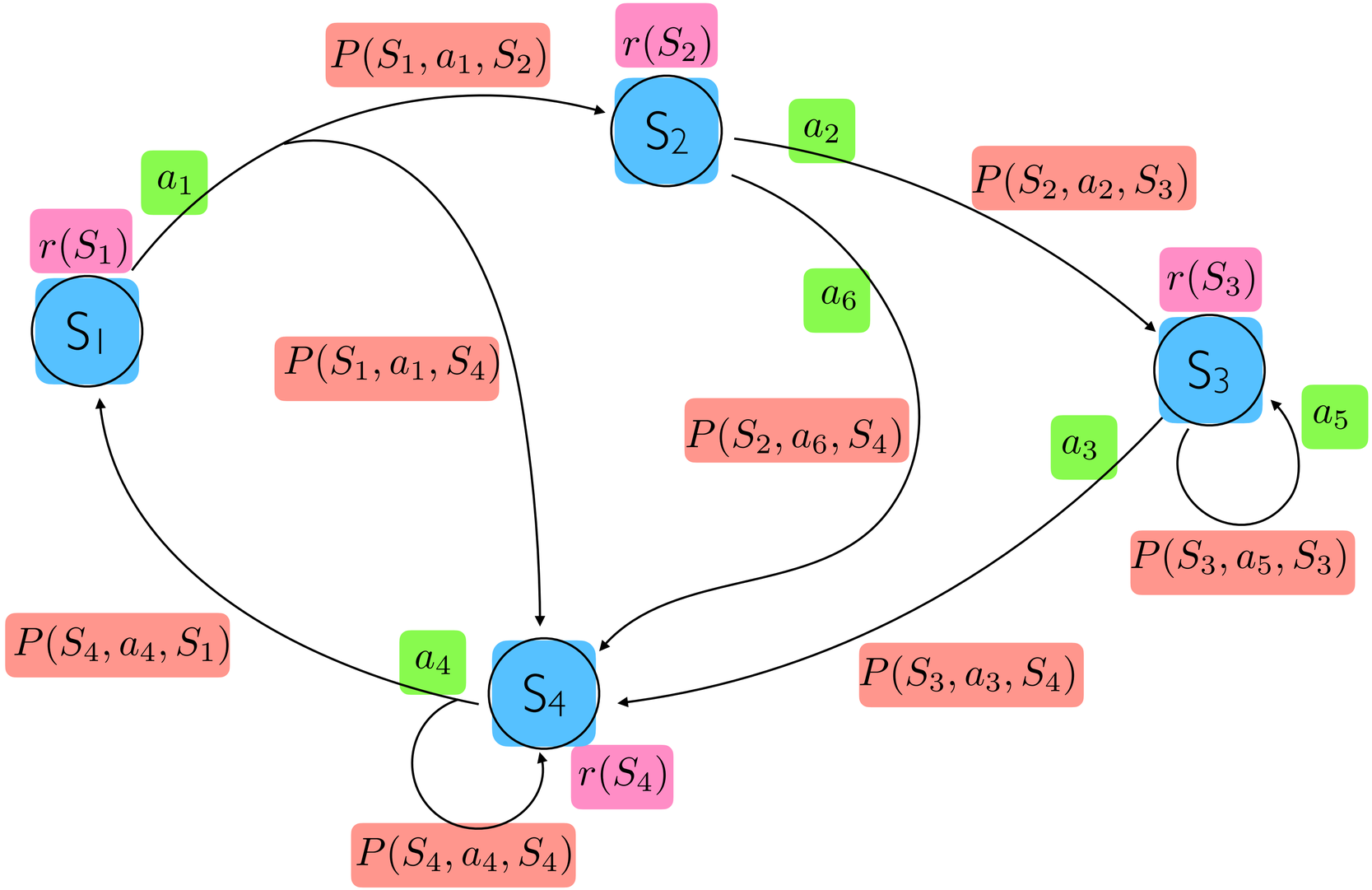
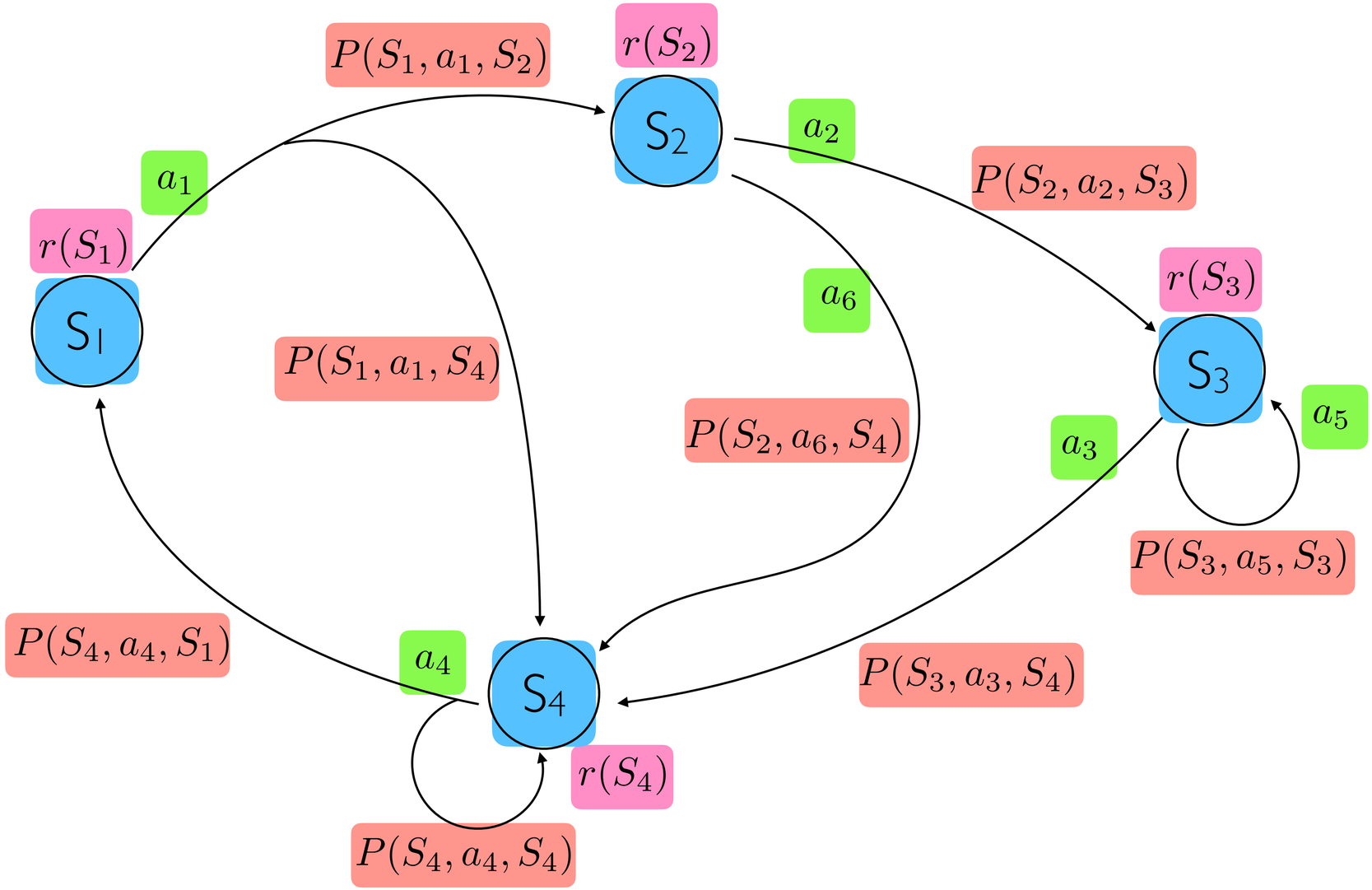


States Actions Transitions Rewards

MDP

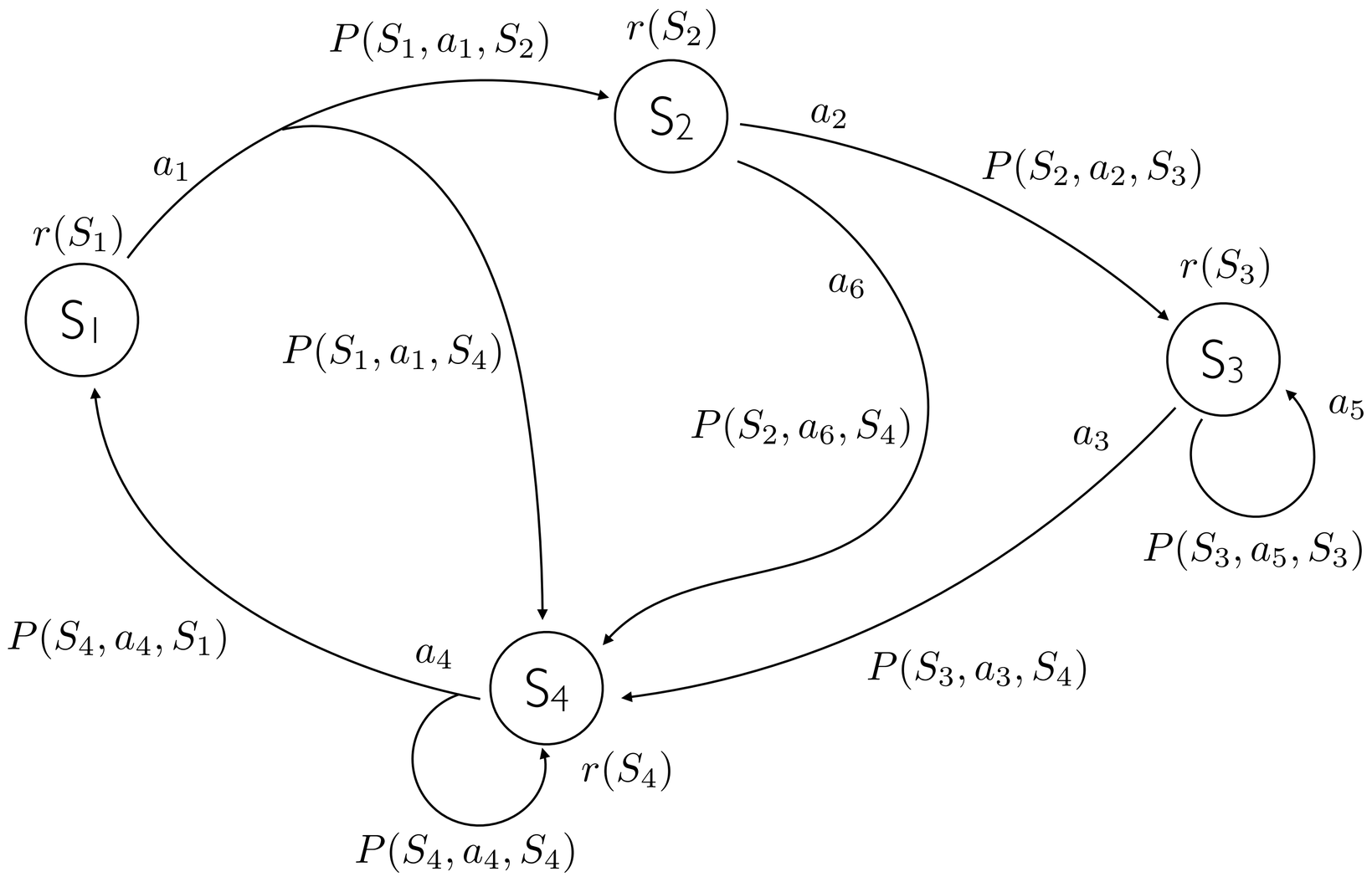
γ Discount factor



States Actions Transitions Rewards

MDP

γ


$P(S_1, a_1, S_2)$
$r(S_2)$
a_2
$P(S_2, a_2, S_3)$
a_1
$r(S_1)$
$r(S_3)$
a_6
$P(S_1, a_1, S_4)$
$P(S_2, a_6, S_4)$
a_5
a_3
$P(S_3, a_5, S_3)$
$P(S_4, a_4, S_1)$
a_4
$P(S_3, a_3, S_4)$
$r(S_4)$
$P(S_4, a_4, S_4)$
S1
S2
S3
S4

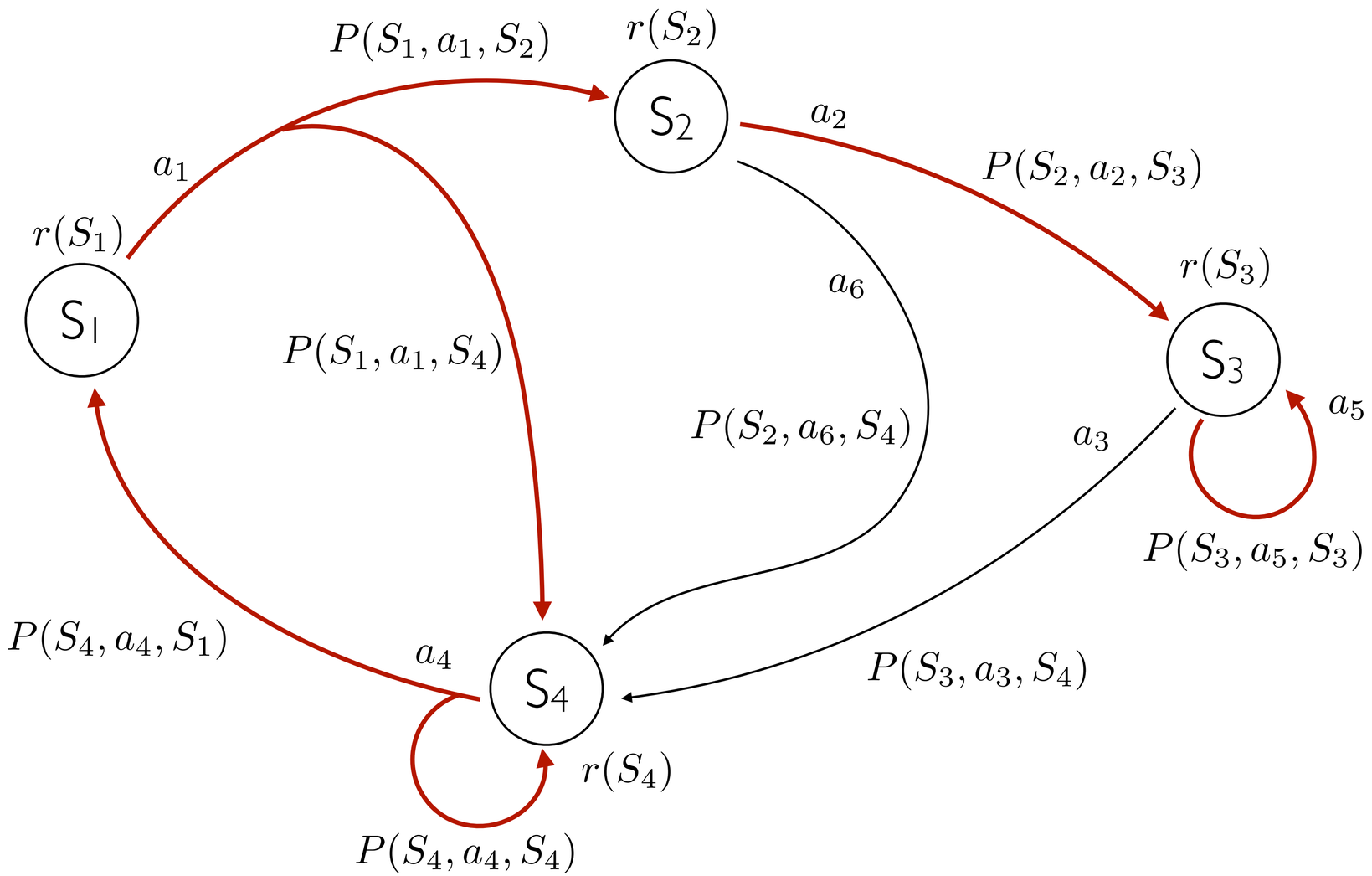
Q-learning

Solving an MDP without information

- Solving an MDP means finding the best policy, mapping each state to a specific action, and maximizing the revenue to the agent

MDP

γ


$P(S_1, a_1, S_2)$
$r(S_2)$
a_2
$P(S_2, a_2, S_3)$
a_1
$r(S_1)$
$r(S_3)$
a_6
$P(S_1, a_1, S_4)$
$P(S_2, a_6, S_4)$
a_5
a_3
$P(S_3, a_5, S_3)$
$P(S_4, a_4, S_1)$
a_4
$P(S_3, a_3, S_4)$
$r(S_4)$
$P(S_4, a_4, S_4)$
S1
S2
S3
S4

Solving an MDP without information

- Solving an MDP means finding the best policy, mapping each state to a specific action, and maximizing the revenue to the agent

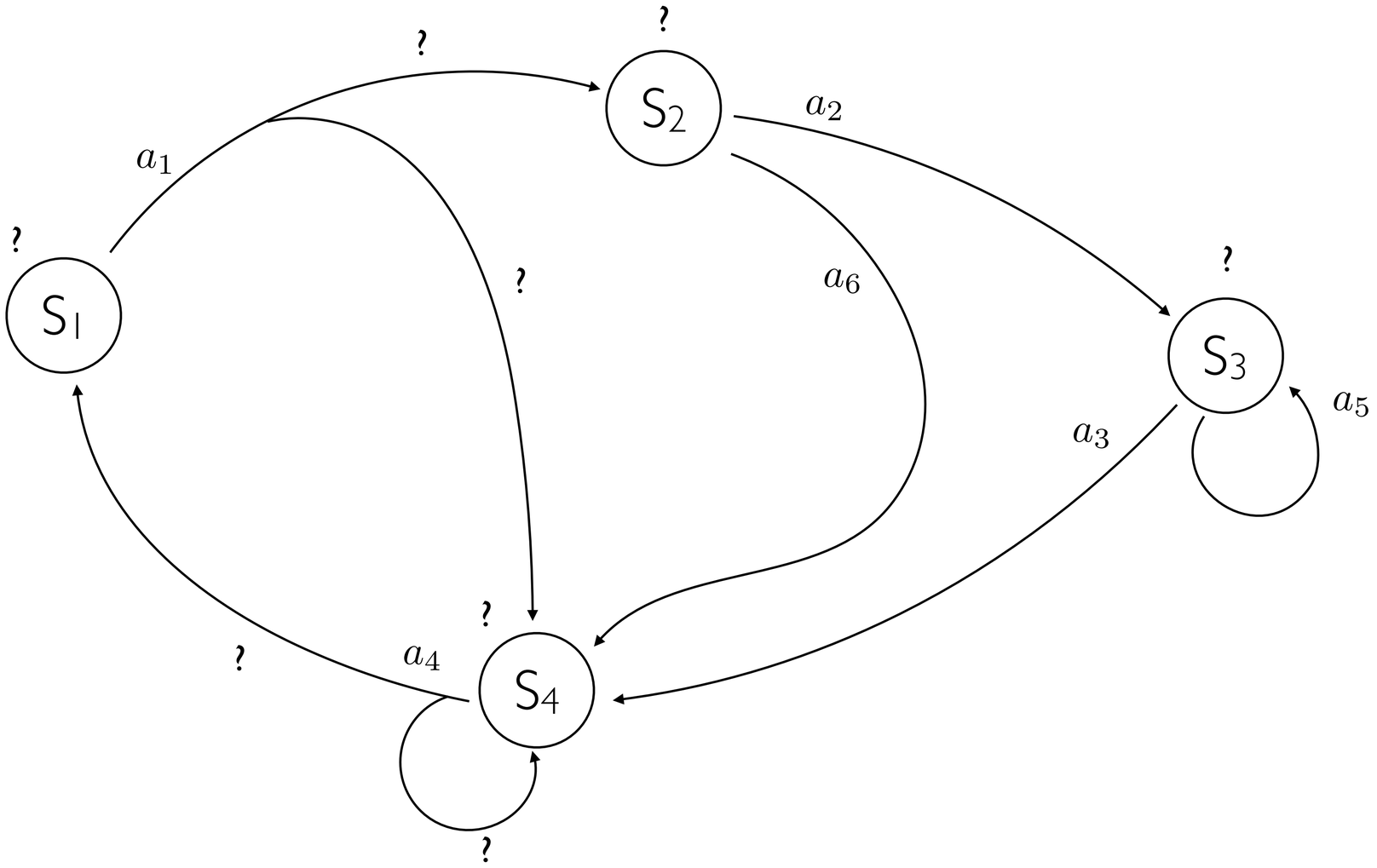
- When the value of all the parameters are known, MDP can be easily solved in polynomial time with linear programming or dynamic programming

Solving an MDP without information

- Solving an MDP means finding the best policy, mapping each state to a specific action, and maximizing the revenue to the agent
- When the value of all the parameters are known, MDP can be easily solved in polynomial time with linear programming or dynamic programming
- When some (or all) parameters are unknown (e.g., rewards and/or transition probabilities), the Q-learning algorithm can be used

MDP

γ



Q-learning algorithm (1)

- The Q-learning algorithm builds a Q-table in which storing the Q-values, corresponding to the estimate of the values the agent expects to gain from playing a given action in a given state

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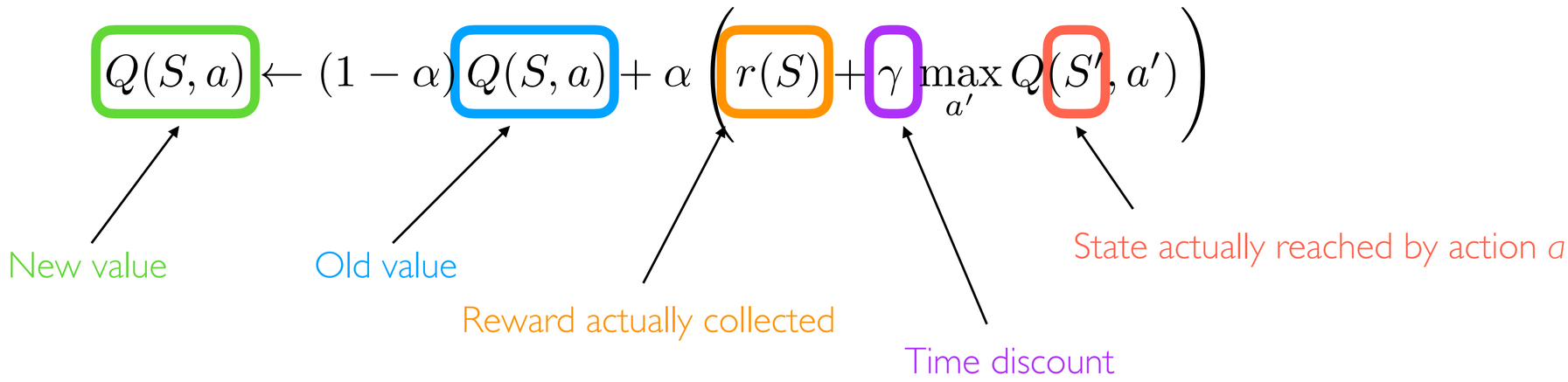
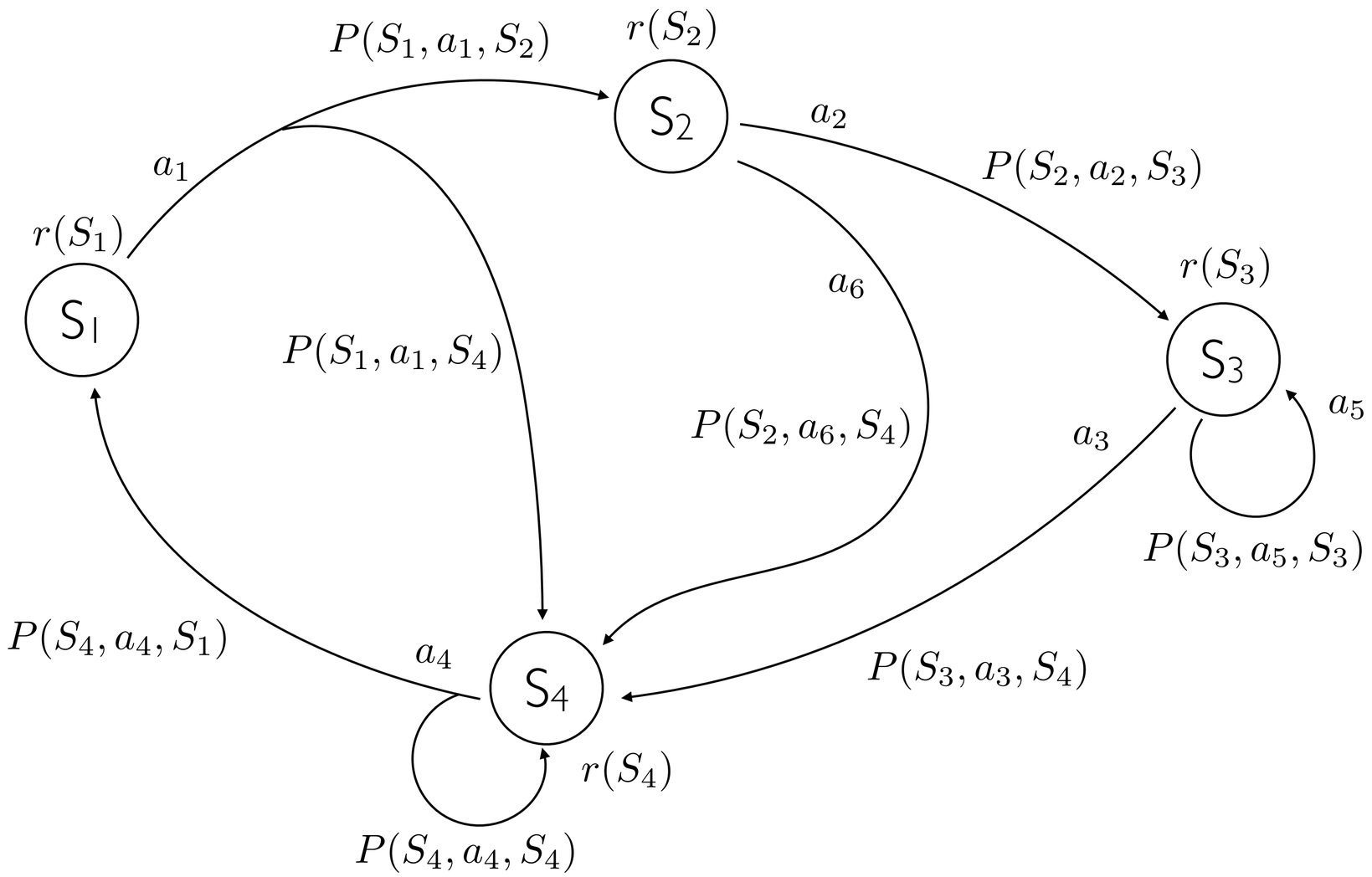
- The agent moves over the MDP and Q-values are updated according to its experience as follows:



MDP

γ


$P(S_1, a_1, S_2)$
$r(S_2)$
a_2
$P(S_2, a_2, S_3)$
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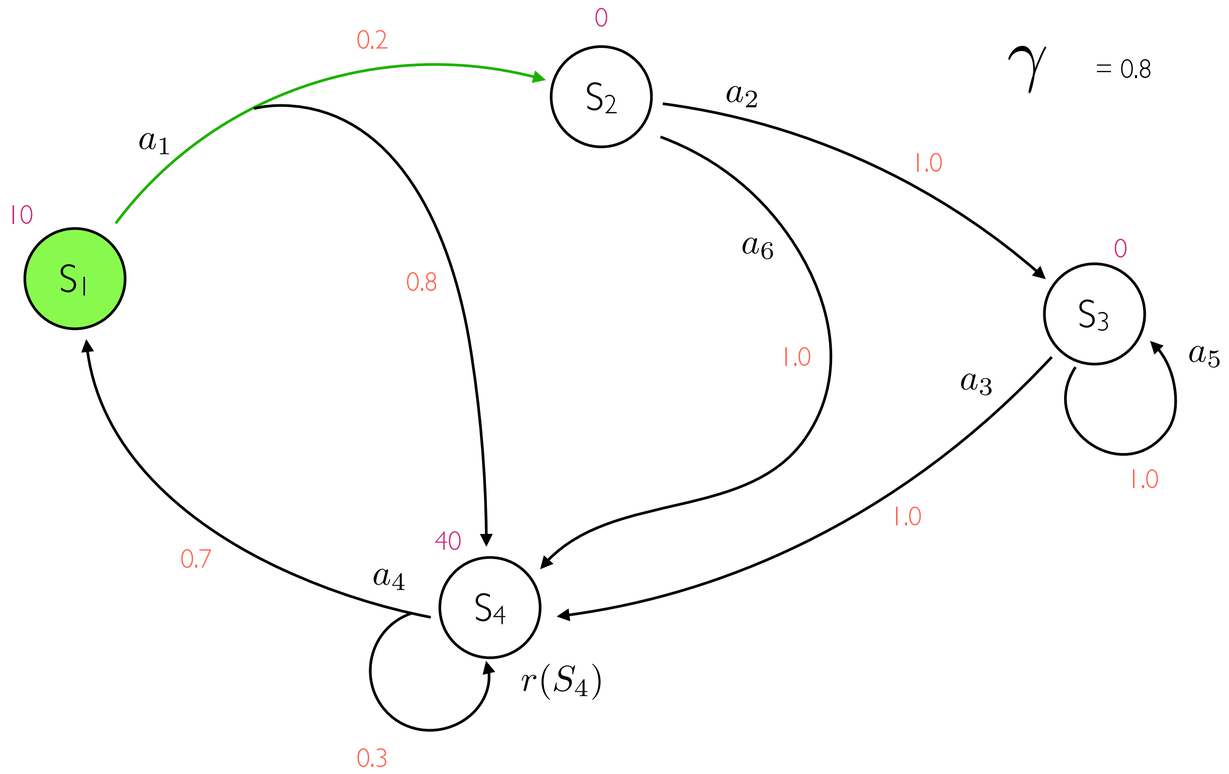
$r(S_3)$
a_6
$P(S_1, a_1, S_4)$
$P(S_2, a_6, S_4)$
a_5
a_3
$P(S_3, a_5, S_3)$
$P(S_4, a_4, S_1)$
a_4
$P(S_3, a_3, S_4)$
$r(S_4)$
$P(S_4, a_4, S_4)$
S1
S2
S3
S4

Q-table

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	0					
	S2		0				0
	S3			0		0	
	S4				0		

MDP



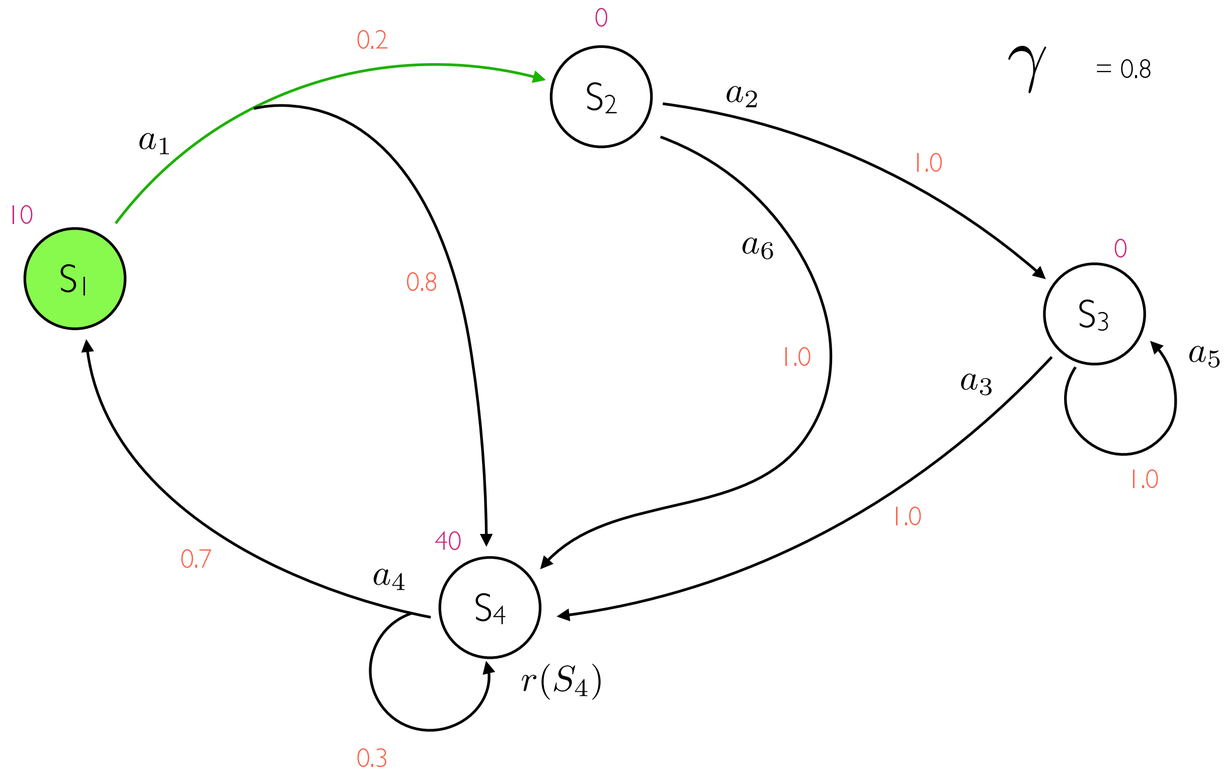
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	0					
	S2		0				0
	S3			0		0	
	S4				0		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

5 0.5 0.0 0.5 10 0.8 0.0

MDP



γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	**5**					
	S2		0				0
	S3			0		0	
	S4				0		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

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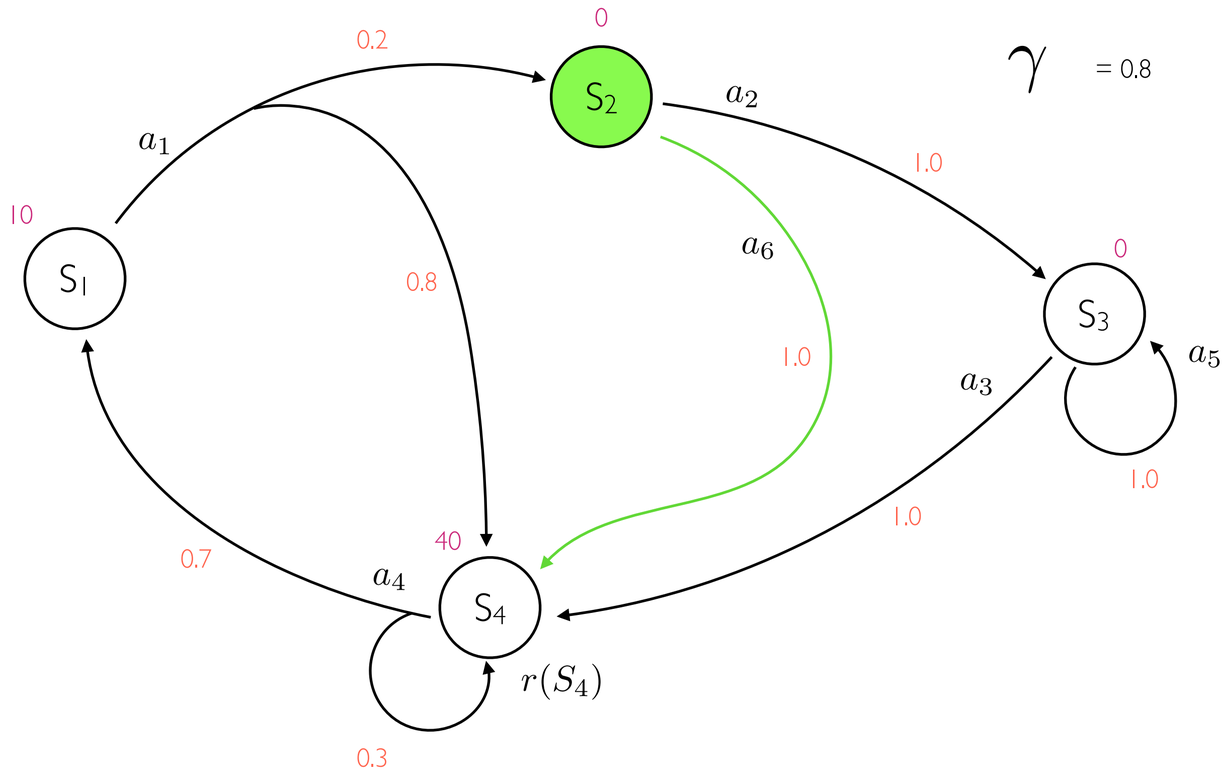
MDP



γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	5					
	S2		0				0
	S3			0		0	
	S4				0		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

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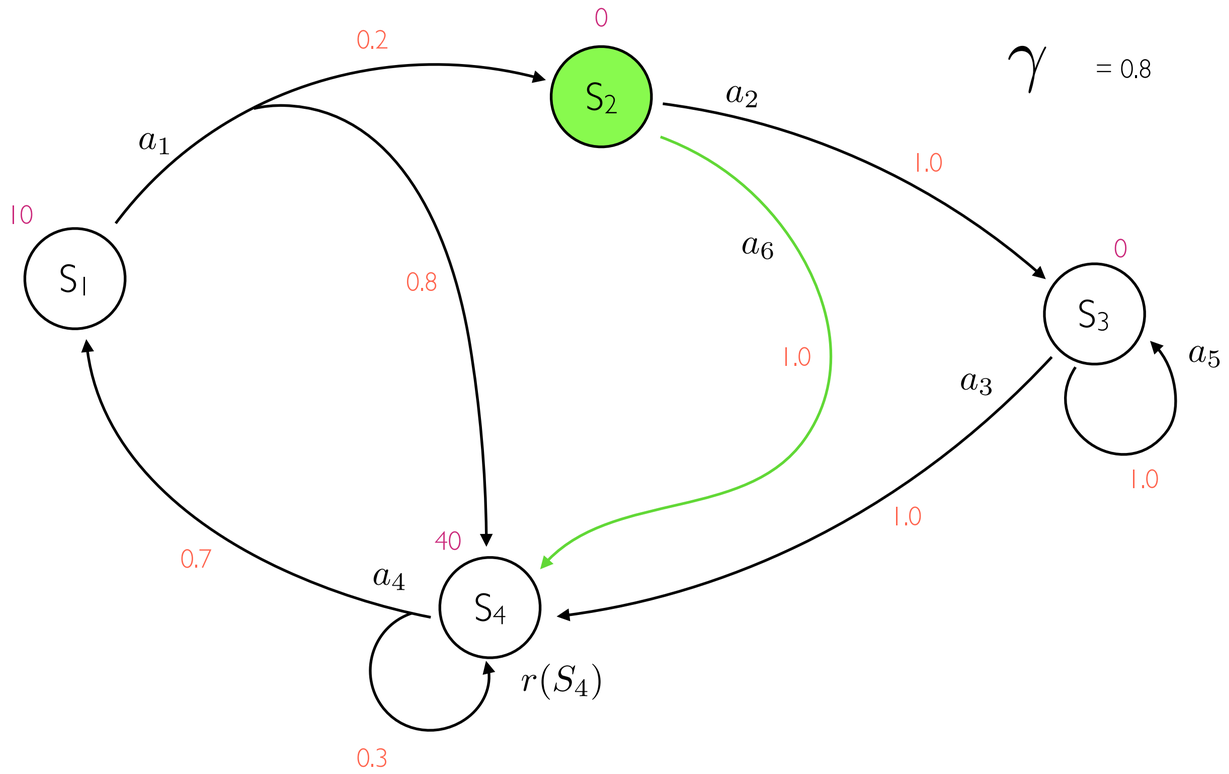
MDP



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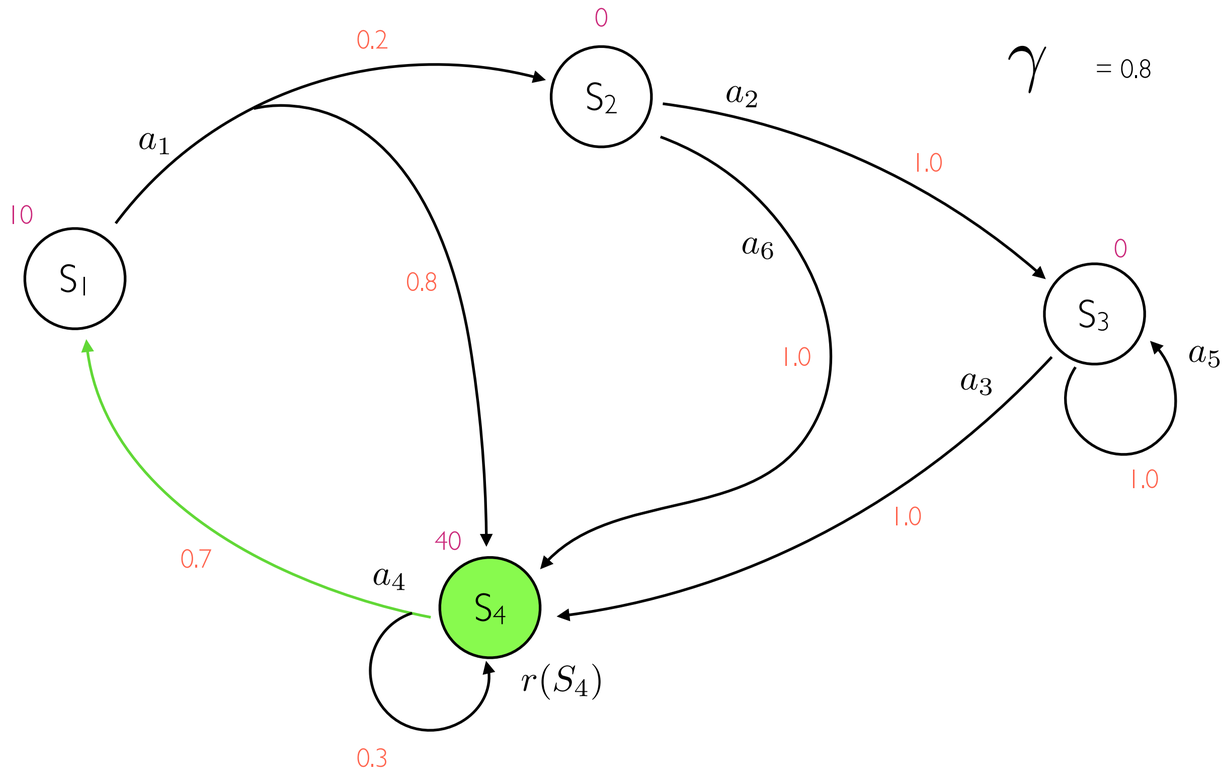
Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	5					
	S2		0				**0**
	S3			0		0	
	S4				0		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

0 0.5 0.0 0.5 0 0.8 0.0

MDP



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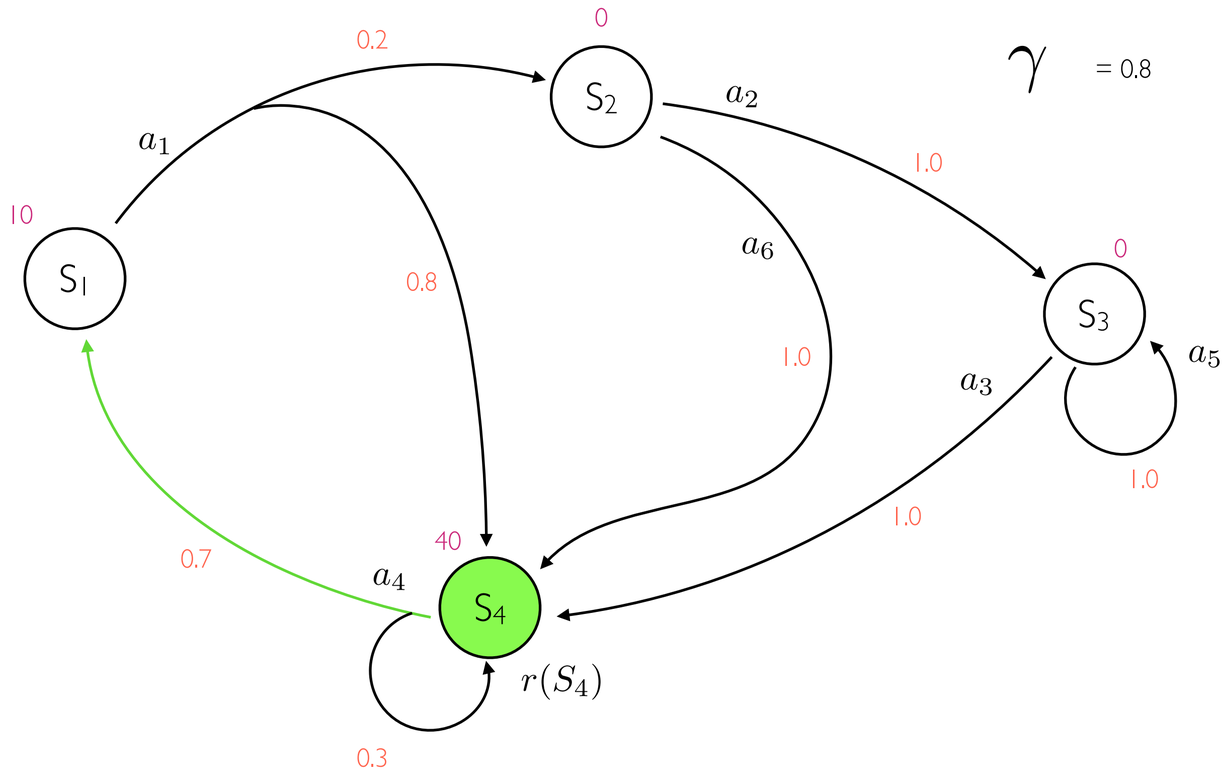
Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	5					
	S2		0				0
	S3			0		0	
	S4				0		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

22 0.5 0.0 0.5 40 0.8 5

MDP



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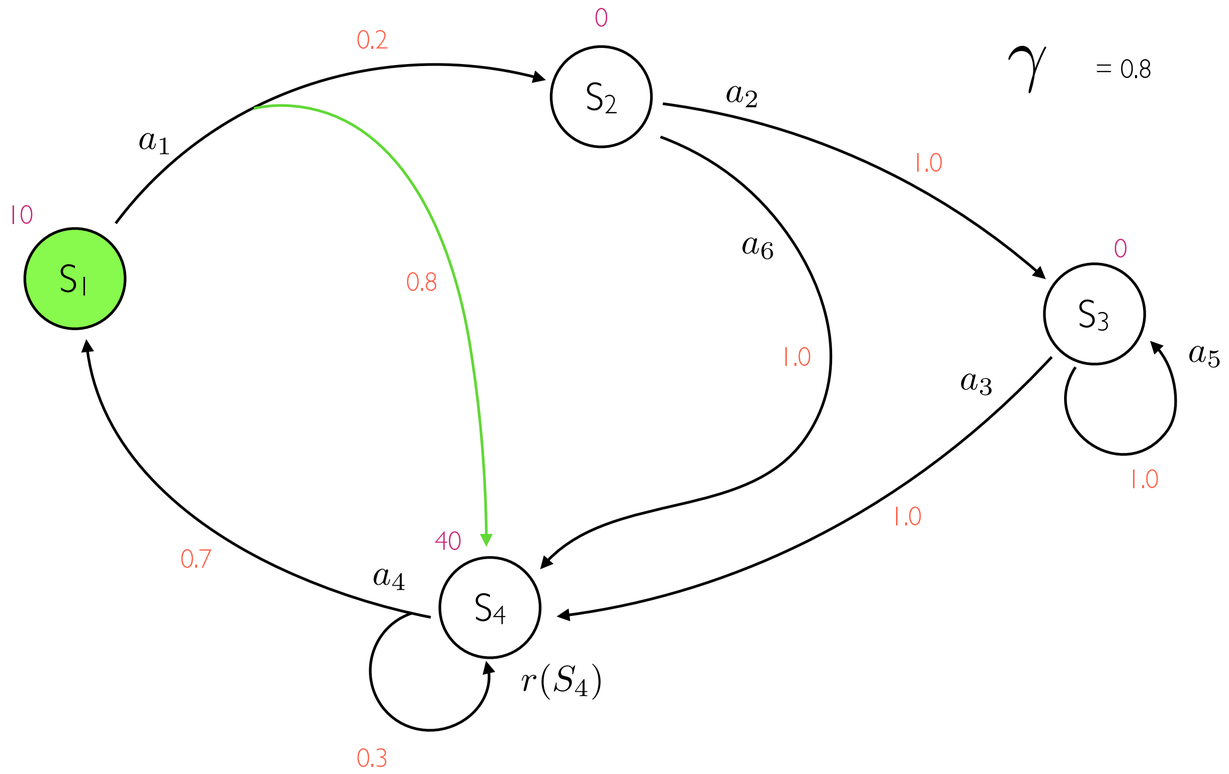
Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	5					
	S2		0				0
	S3			0		0	
	S4				**22**		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

22 0.5 0.0 0.5 40 0.8 5

MDP



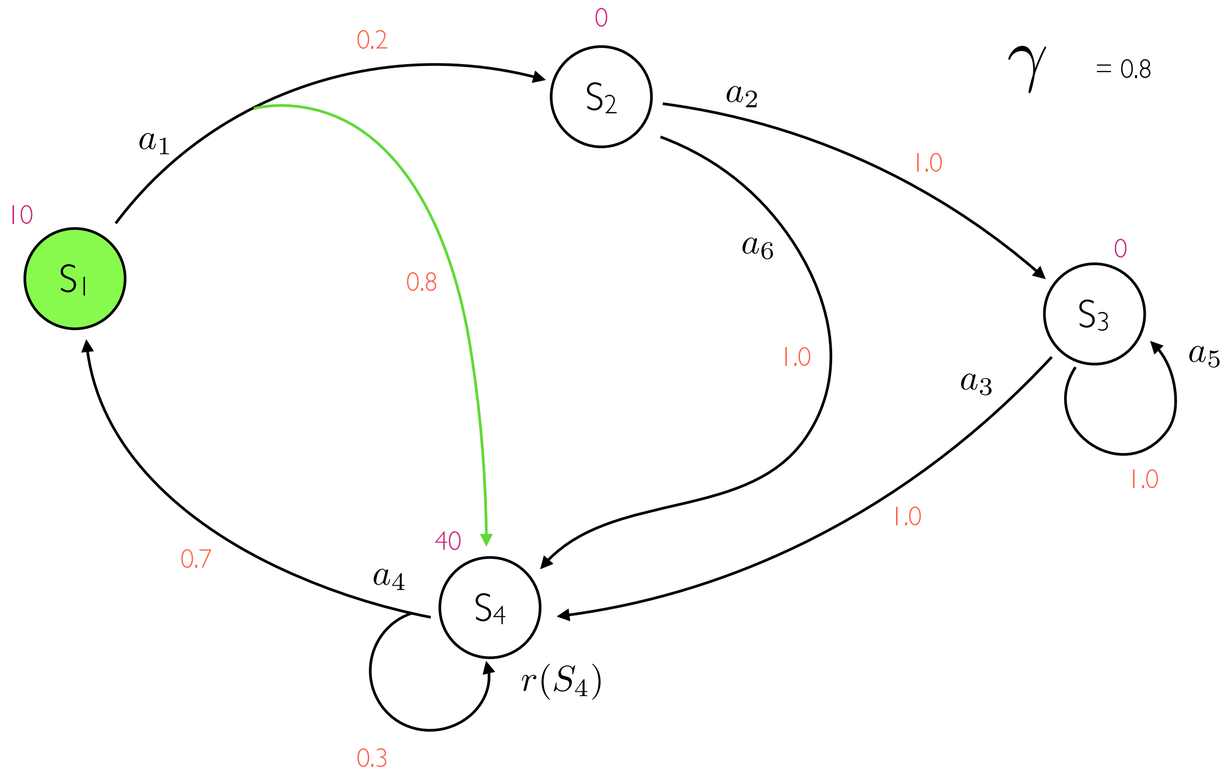
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	5					
	S2		0				0
	S3			0		0	
	S4				22		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

16.3 0.5 5 0.5 5 0.8 22

MDP



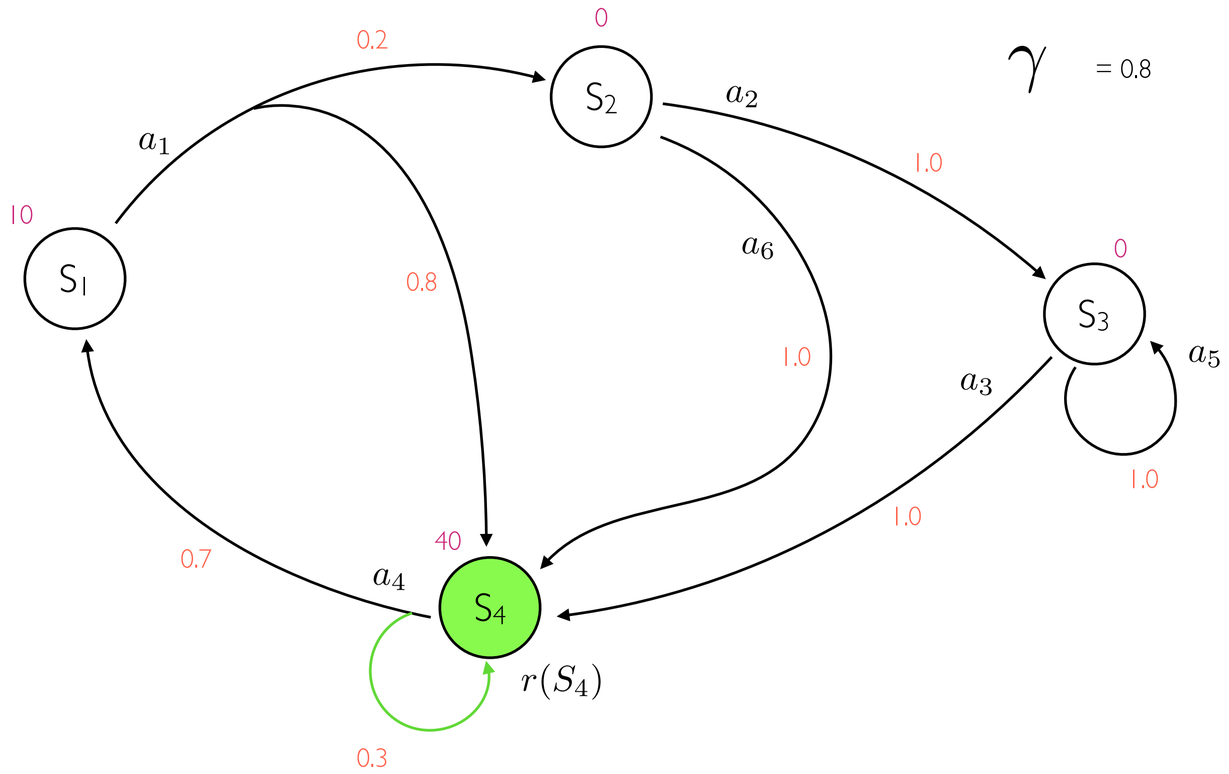
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	**16.3**					
	S2		0				0
	S3			0		0	
	S4				22		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

16.3 0.5 5 0.5 5 0.8 22

MDP



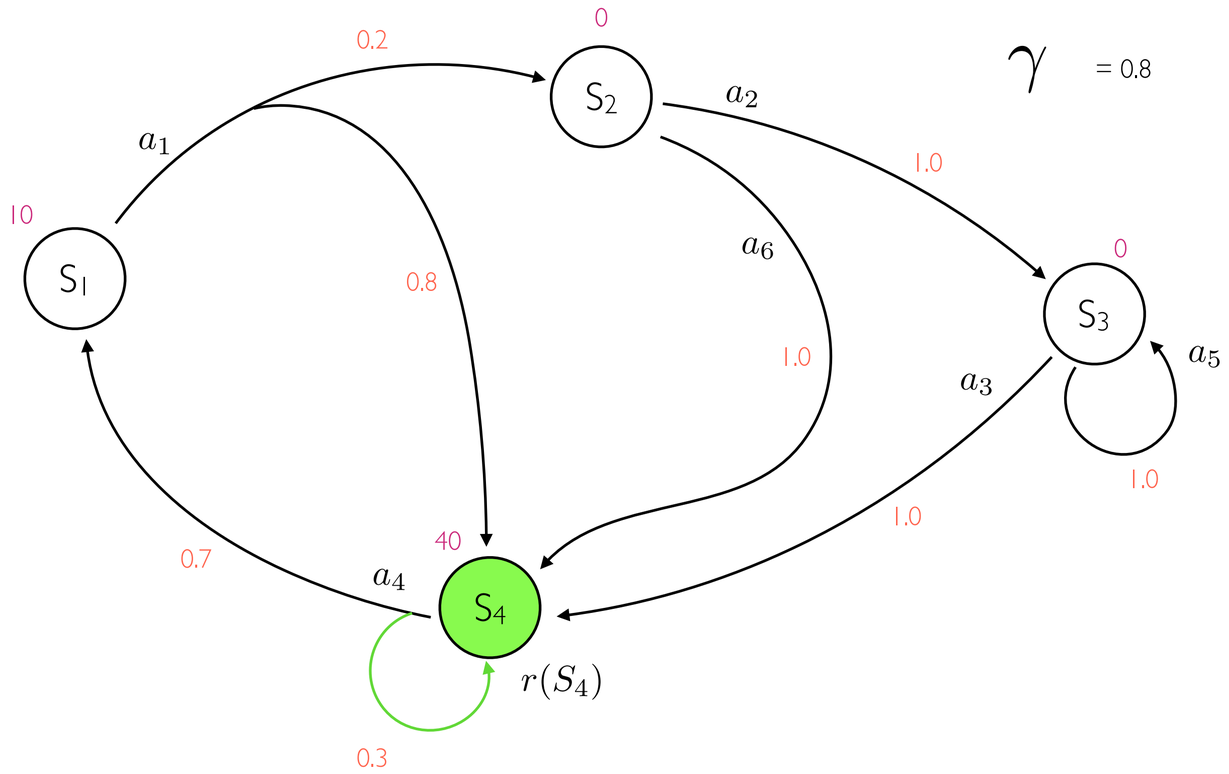
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	15.5					
	S2		0				0
	S3			0		0	
	S4				22		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

39.8 0.5 5 0.5 40 0.8 22

MDP



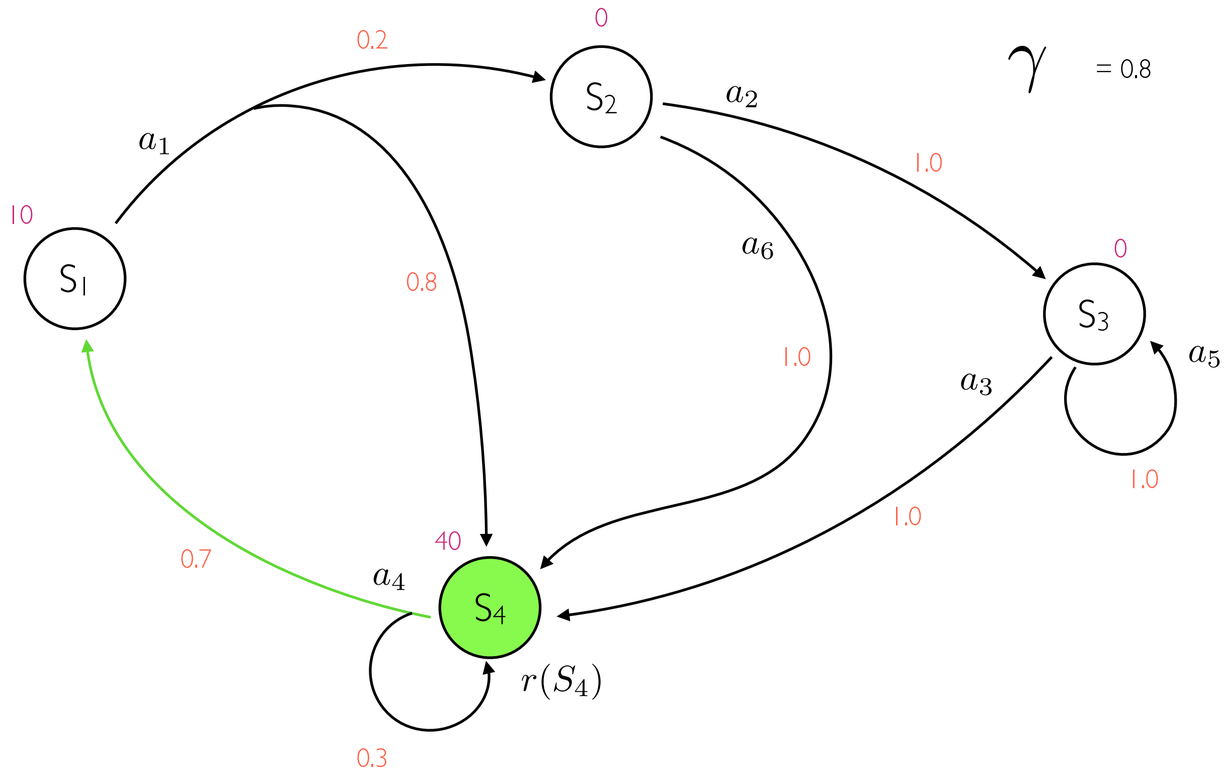
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	15.5					
	S2		0				0
	S3			0		0	
	S4				**39.8**		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

39.8 0.5 5 0.5 40 0.8 22

MDP



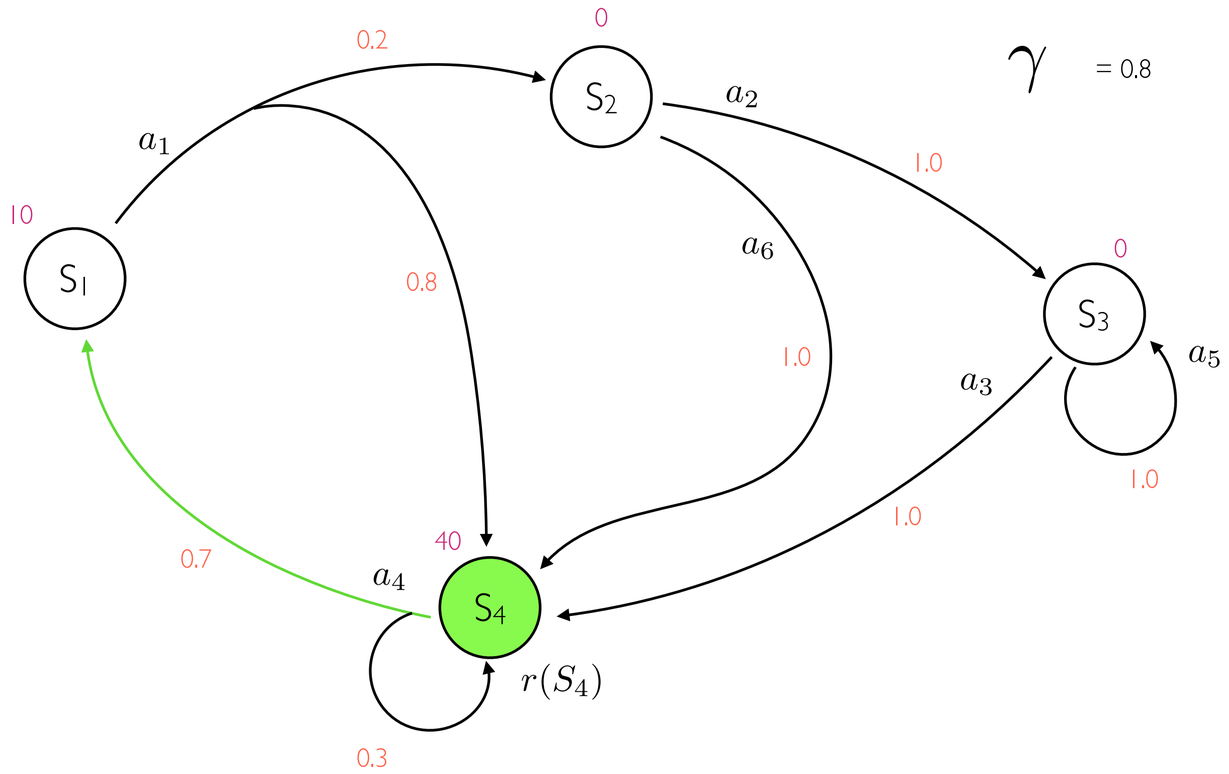
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	15.5					
	S2		0				0
	S3			0		0	
	S4				39.8		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

46.1 0.5 39.8 0.5 40 0.8 15.5

MDP



γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	15.5					
	S2		0				0
	S3			0		0	
	S4				**46.1**		

$$Q(S,a) \leftarrow (1-\alpha)\, Q(S,a) + \alpha \left(r(S) + \gamma \max_{a'} Q(S',a') \right)$$

46.1 0.5 39.8 0.5 40 0.8 15.5

Q-learning algorithm (2)

- We can follow several criteria to choose the action to play in every state multiple actions are available

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  - **Epsilon-greedy** $a \in \begin{cases} \arg\max_{a'} Q(S, a') & \text{with probability } 1 - \epsilon \\ \text{a random action} & \text{with probability } \epsilon \end{cases}$

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  - **Greedy** $a \in \arg\max_{a'} Q(S, a')$

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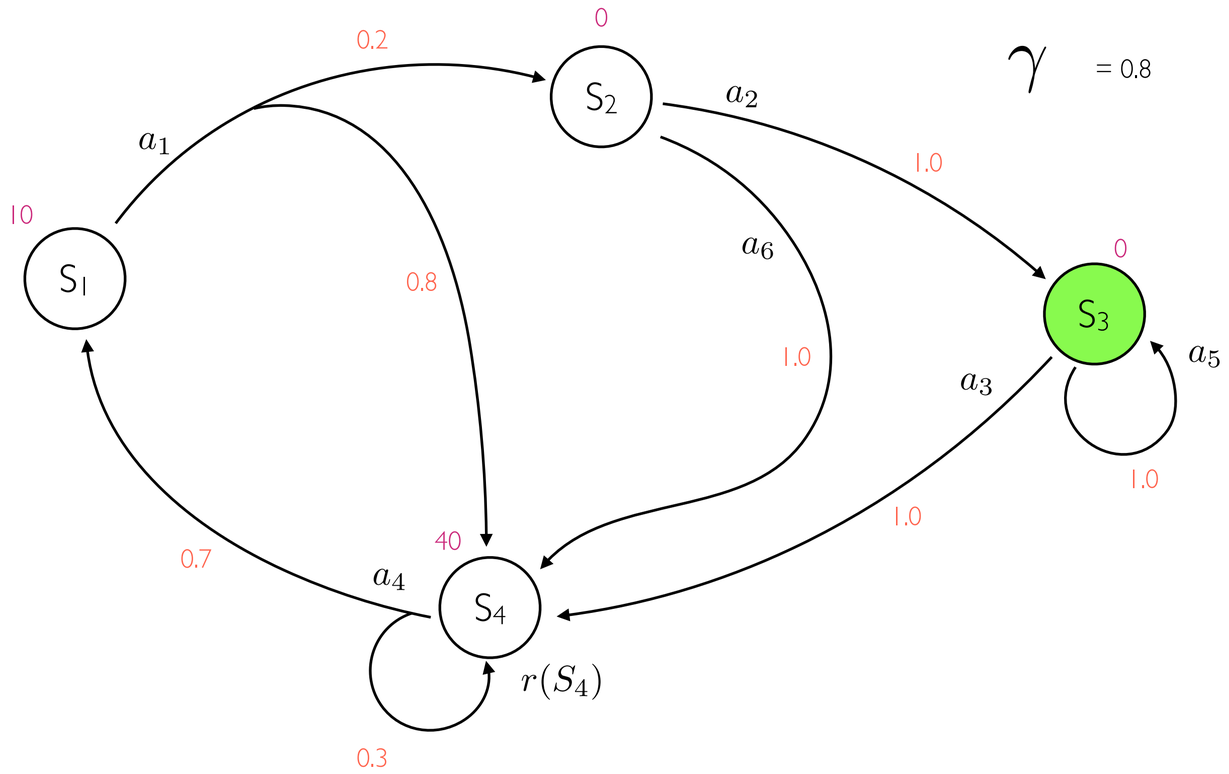
  - **Boltzmann exploration** $a \sim \frac{\exp\left(\frac{Q(s,a)}{\tau}\right)}{\sum_{a'} \exp\left(\frac{Q(s,a')}{\tau}\right)}$

    Tau is the temperature, zero corresponds to greedy, infinity to the random policy

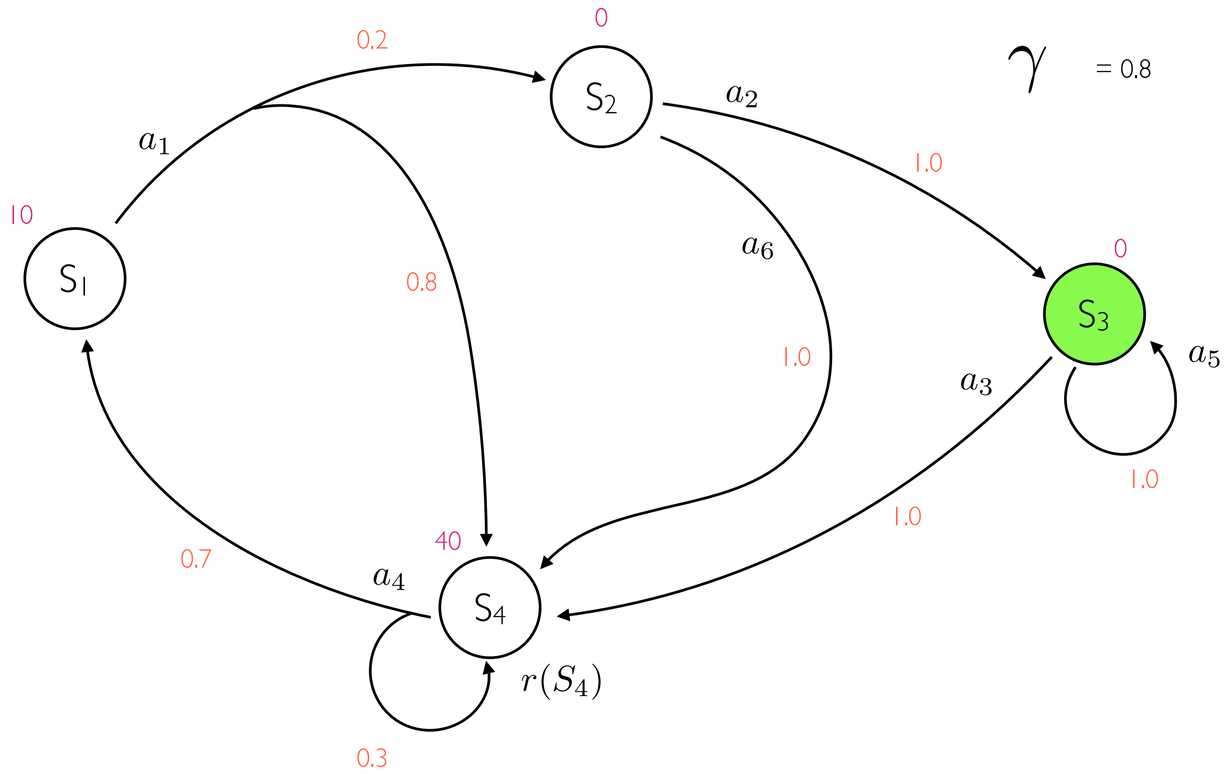
    Usually a schedule is adopted to reduce tau as the time increases

MDP



Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	25					
	S2		30				22
	S3			10		13	
	S4				21		

MDP

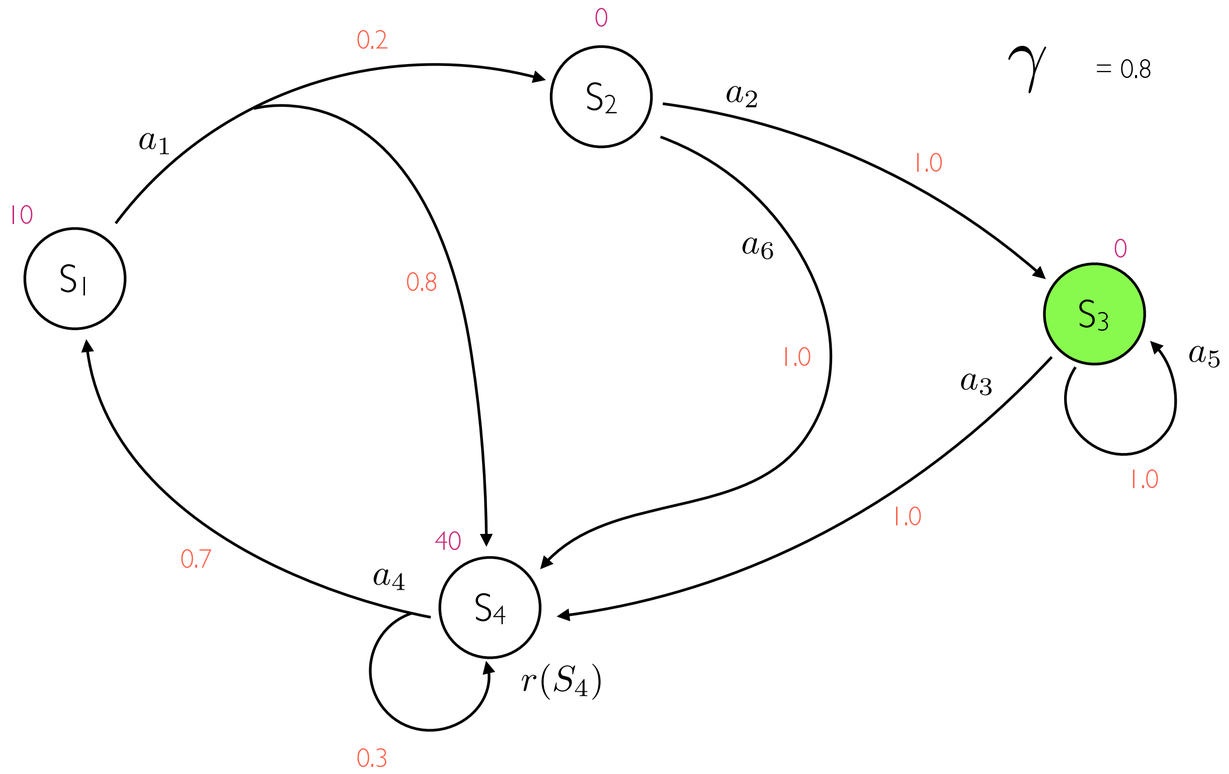


γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	25					
	S2		30				22
	S3			10		13	
	S4				21		

Greedy $\begin{cases} a_3 & 0.0 \\ a_5 & 1.0 \end{cases}$

MDP



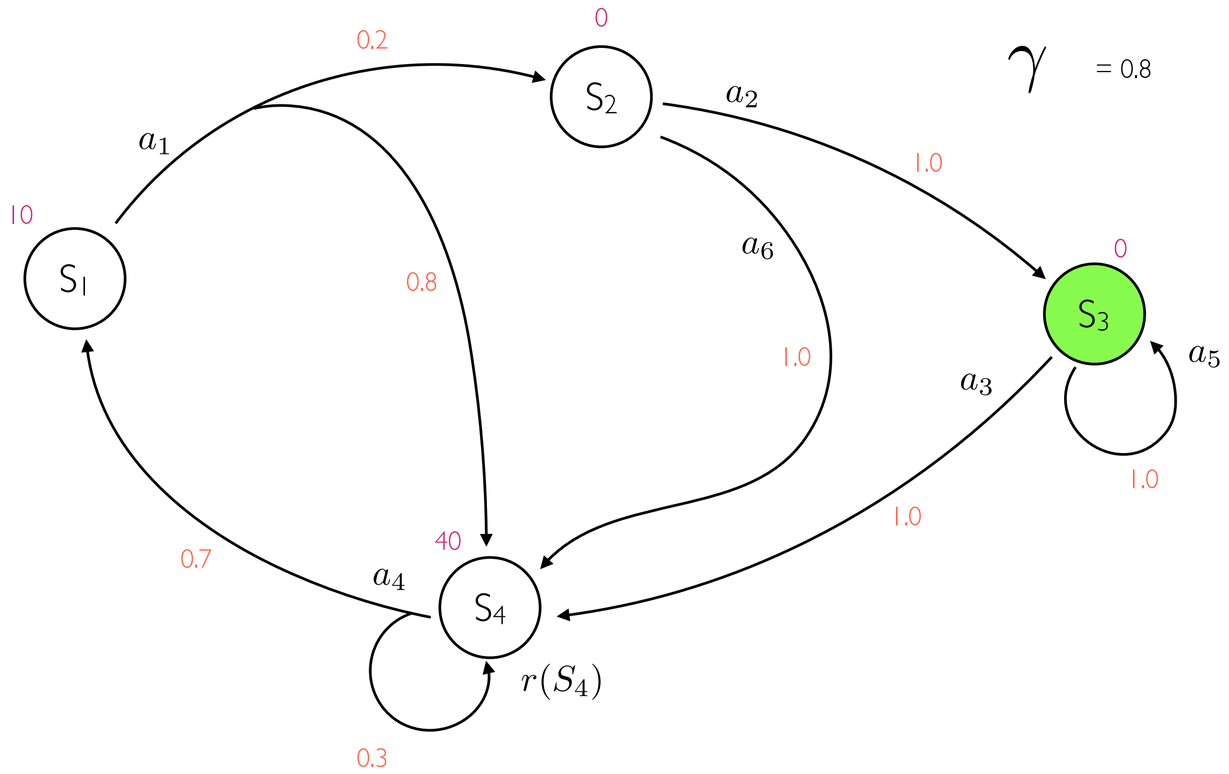
γ = 0.8

Q-Table		Actions					
		a1	a2	a3	a4	a5	a6
States	S1	25					
	S2		30				22
	S3			10		13	
	S4				21		

Greedy $\begin{cases} a_3 & 0.0 \\ a_5 & 1.0 \end{cases}$

Epsilon-Greedy $\begin{cases} a_3 & \epsilon \\ a_5 & 1.0 - \epsilon \end{cases}$

MDP



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		a1	a2	a3	a4	a5	a6
States	S1	25					
	S2		30				22
	S3			10		13	
	S4				21		

Greedy $\begin{cases} a_3 & 0.0 \\ a_5 & 1.0 \end{cases}$

Epsilon-Greedy $\begin{cases} a_3 & \epsilon \\ a_5 & 1.0 - \epsilon \end{cases}$

Boltzmann $\begin{cases} a_3 & \frac{\exp(10/\tau)}{\exp(10/\tau)+\exp(13/\tau)} \\ a_5 & \frac{\exp(13/\tau)}{\exp(10/\tau)+\exp(13/\tau)} \end{cases}$

Convergence (1)

- Q-leaning converges to the optimal policy when the environment is stationary and the policy is sufficiently explorative

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- For instance, when the greedy approach is used, there is no guarantee to converge, the policy not being sufficiently explorative
- Tradeoff
  - *Exploitation* to maximize the reward
  - *Exploration* to acquire information

Convergence (2)

- Sufficient conditions guaranteeing the convergence are:

Convergence (2)

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  - The learning rate is such that

$$\alpha = \alpha(t)$$

$$\sum_{t=1}^{\infty} \alpha(t) \rightarrow \infty$$

$$\sum_{t=1}^{\infty} \alpha(t)^2 \rightarrow \text{constant}$$

Convergence (2)

- Sufficient conditions guaranteeing the convergence are:
  - The learning rate is such that

$$\alpha = \alpha(t)$$

$$\sum_{t=1}^{\infty} \alpha(t) \rightarrow \infty$$

$$\sum_{t=1}^{\infty} \alpha(t)^2 \rightarrow \text{constant}$$

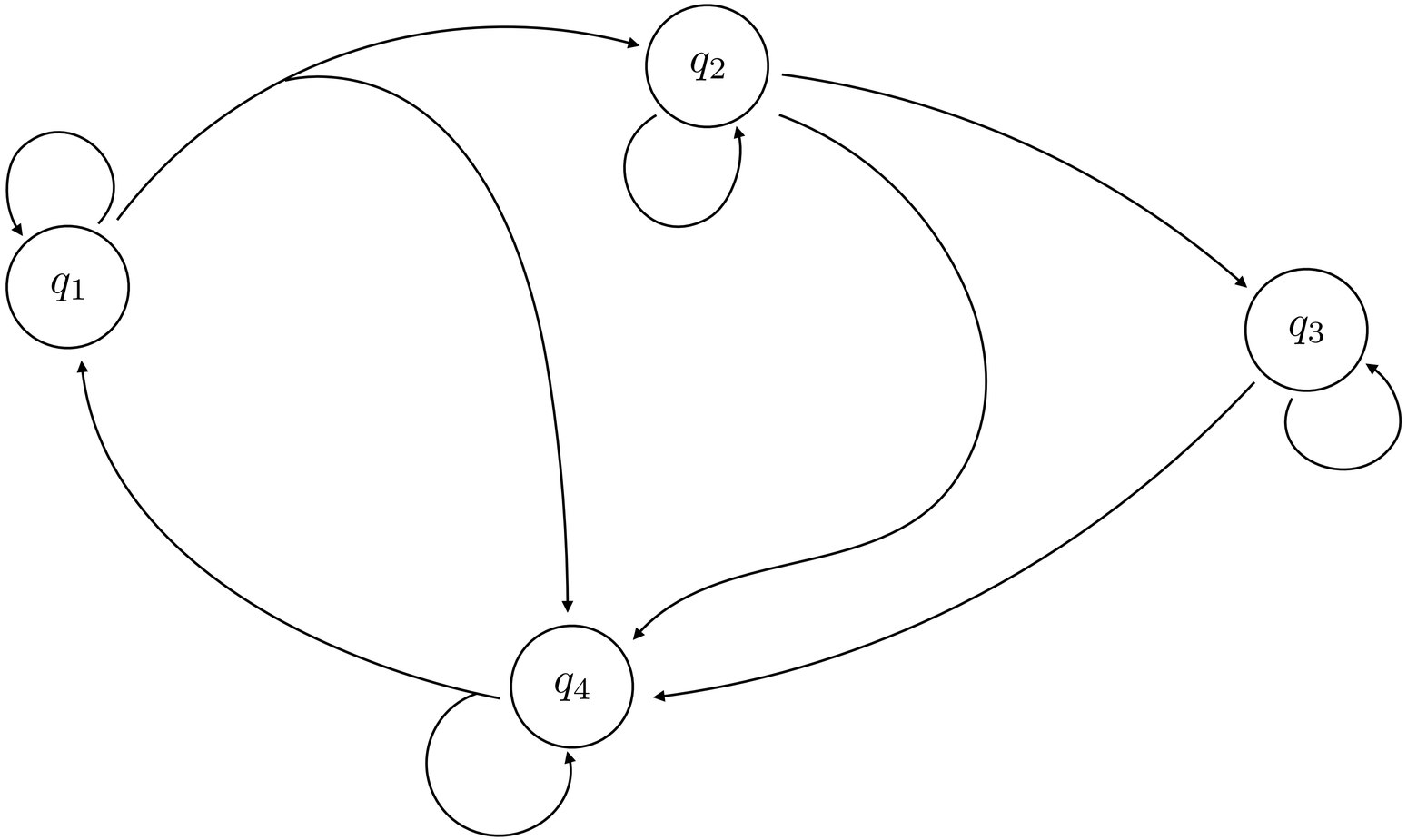
  - The exploration is such that

Epsilon-Greedy	Boltzmann
$\epsilon = \epsilon(t) > 0$	$\tau = \tau(t) > 0$
$\lim_{t \rightarrow \infty} \epsilon(t) = 0$	$\lim_{t \rightarrow \infty} \tau(t) = 0$

Stochastic Games (SG)

From MDP to Stochastic games

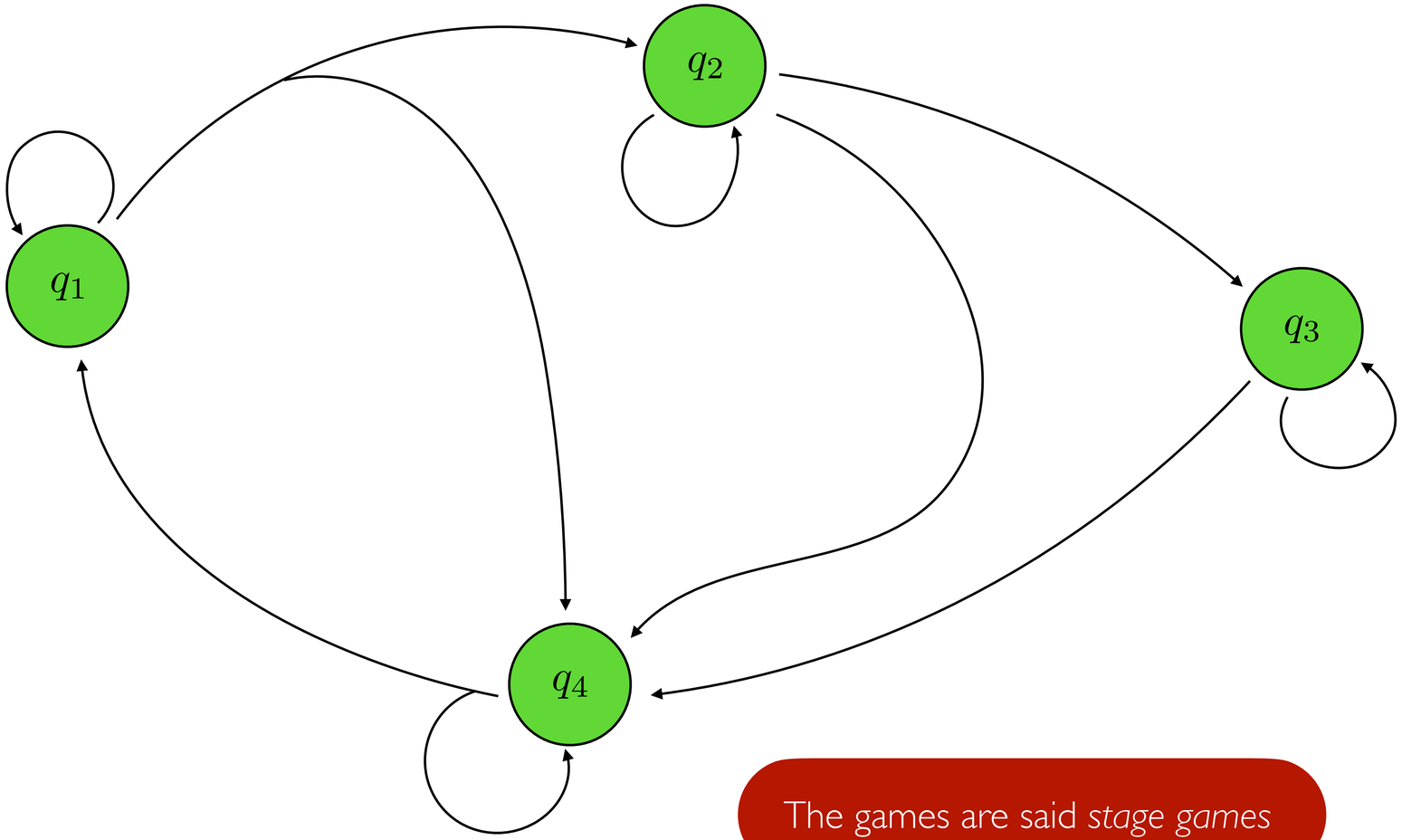
Players 1 and 2



γ

From MDP to Stochastic games

Players 1 and 2



q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

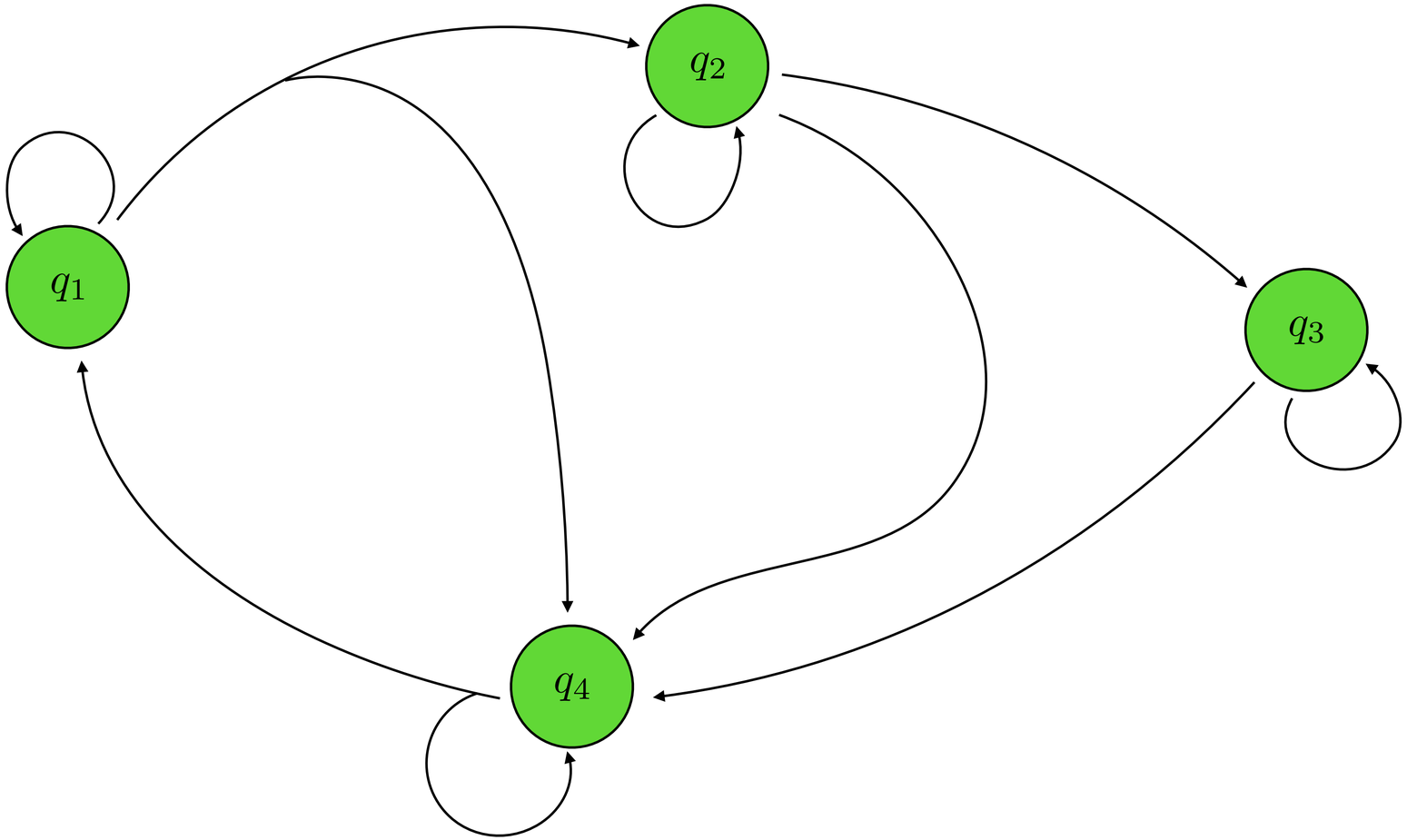
	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

The games are said *stage games*

γ

From MDP to Stochastic games

Players 1 and 2



q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

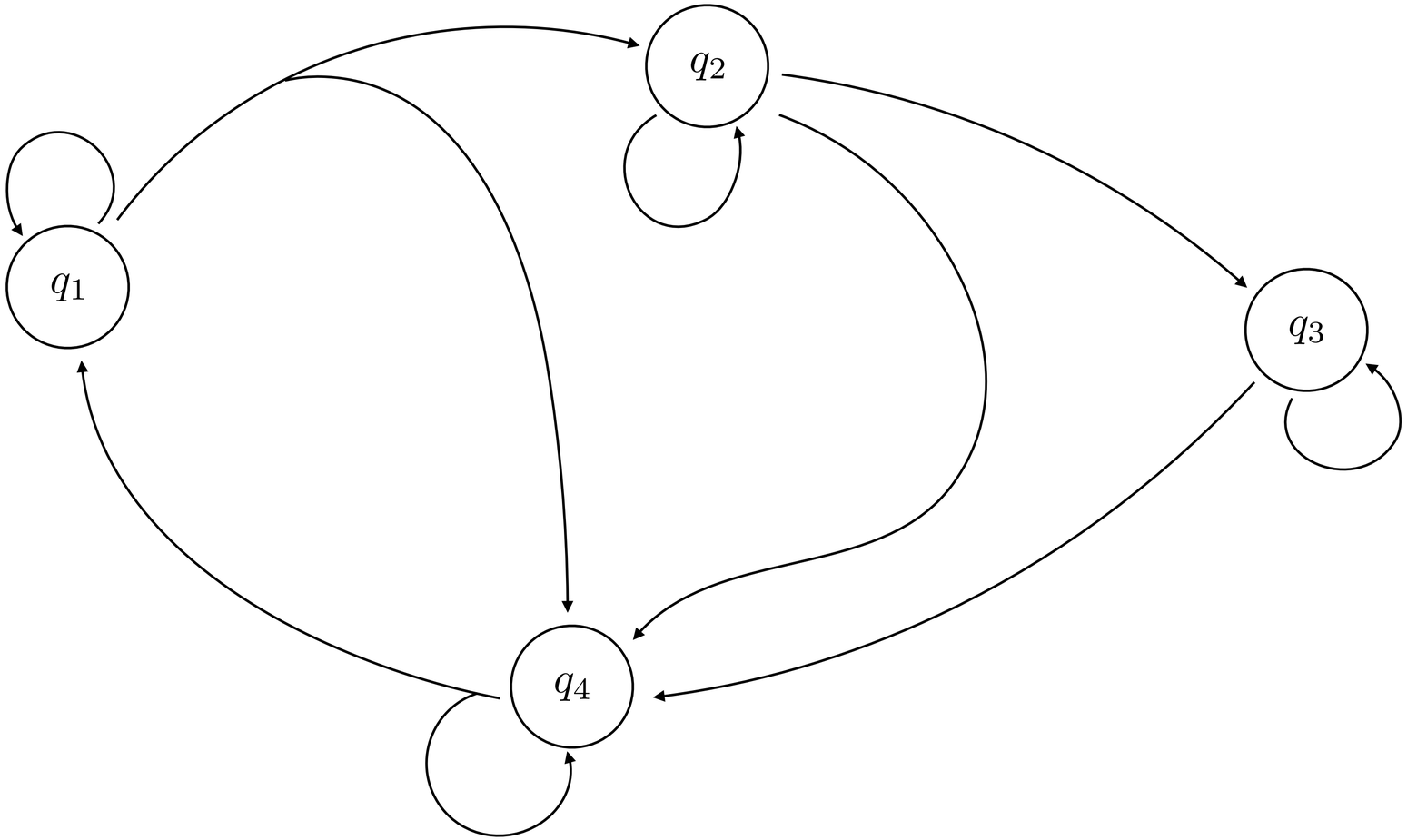
q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

γ

From MDP to Stochastic games

Players 1 and 2



Actions available at game q_1

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

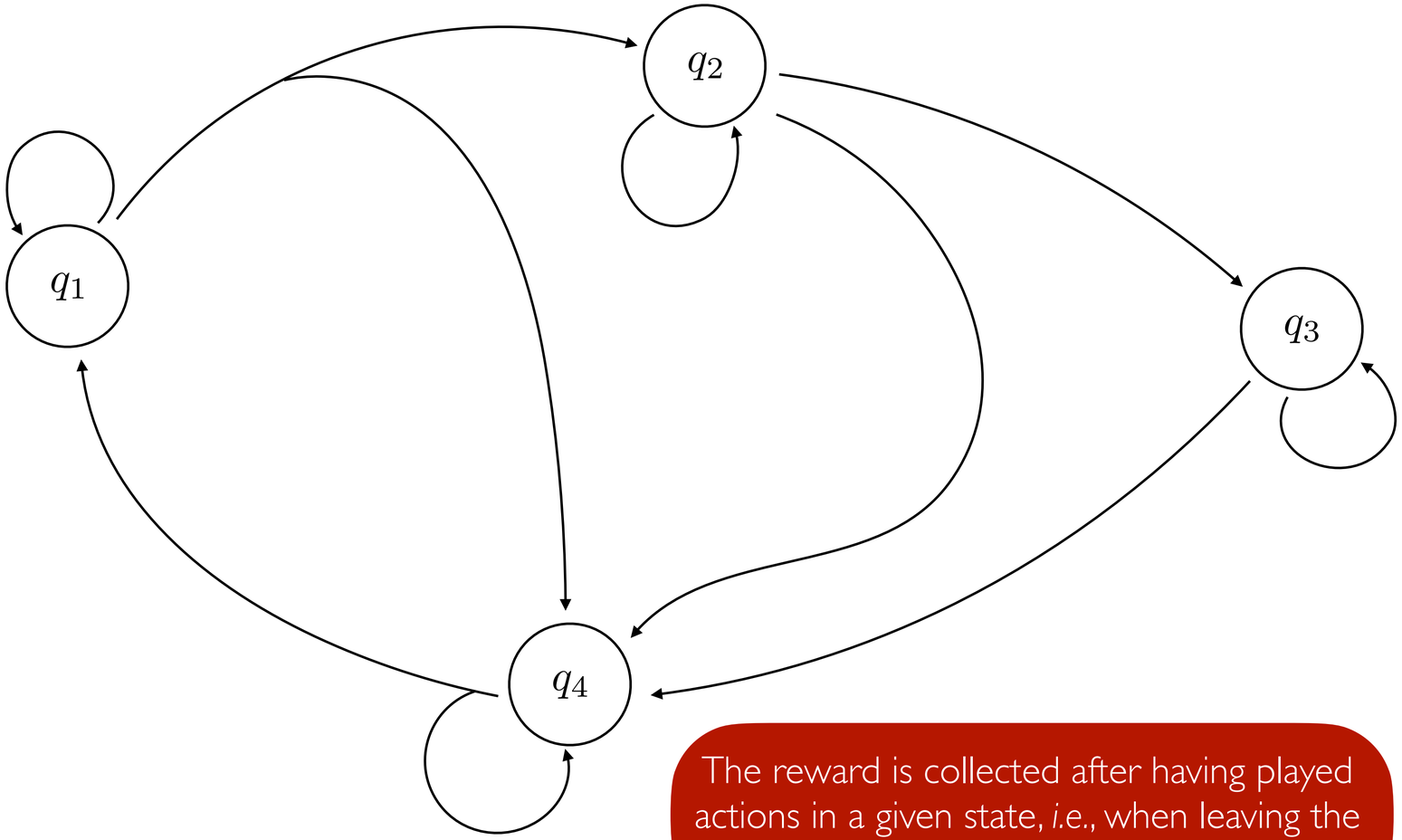
q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

γ

From MDP to Stochastic games

Players 1 and 2



Players' rewards given a pair of actions

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

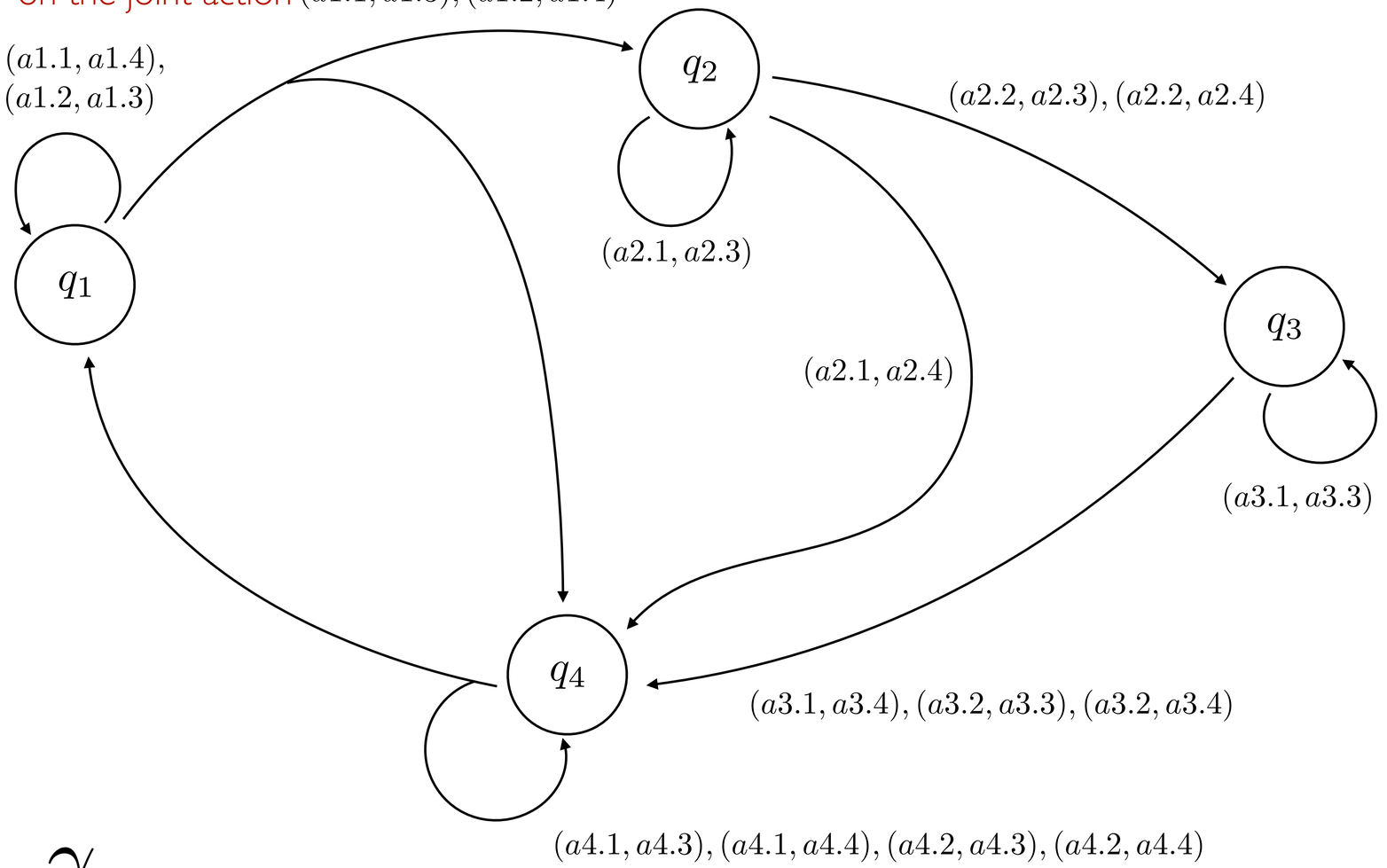
The reward is collected after having played actions in a given state, *i.e.*, when leaving the state, not when entering the state

γ

From MDP to Stochastic games

Players 1 and 2

Transitions defined
on the joint action $(a1.1, a1.3), (a1.2, a1.4)$



q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

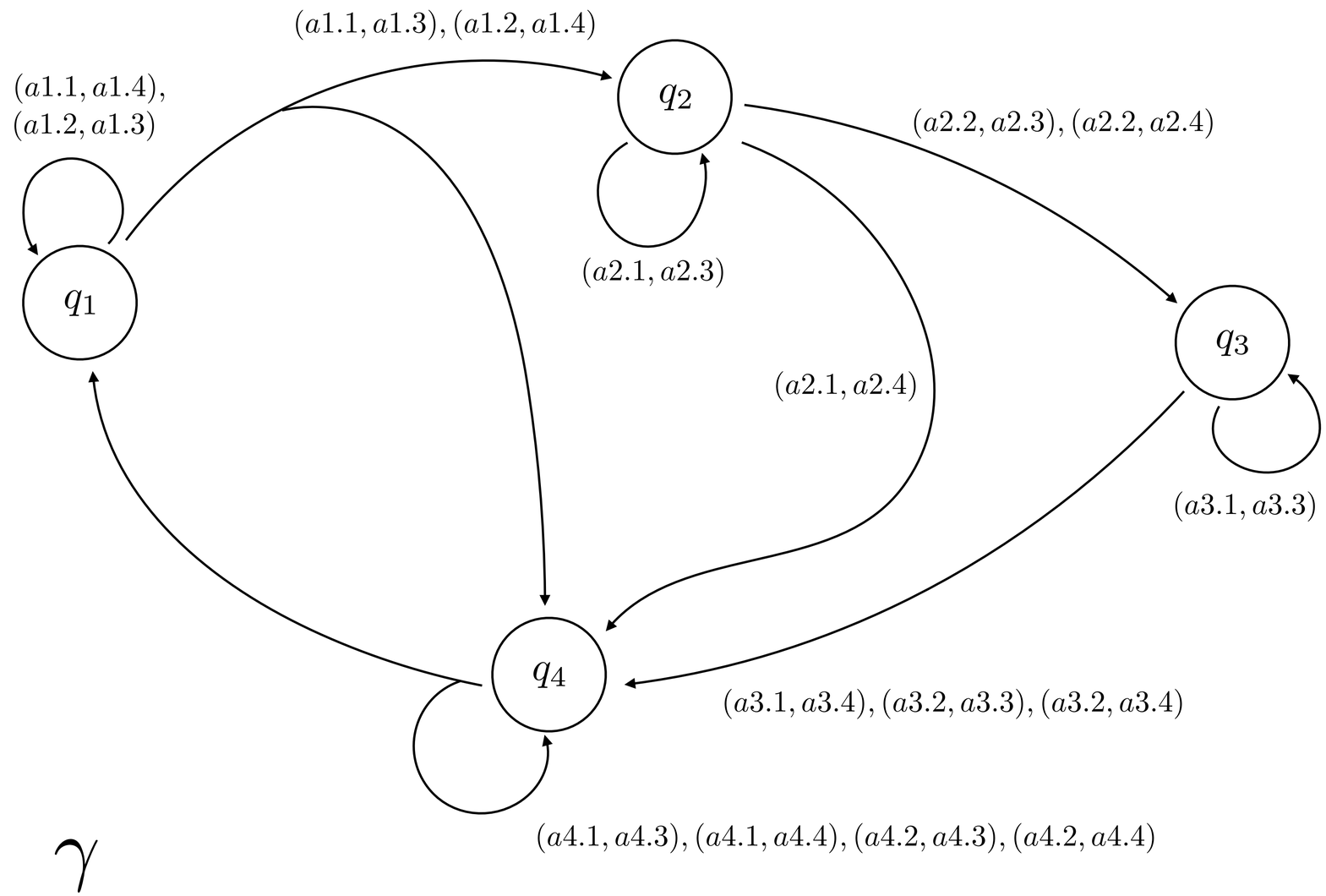
q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

γ

An example

Players 1 and 2



γ

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

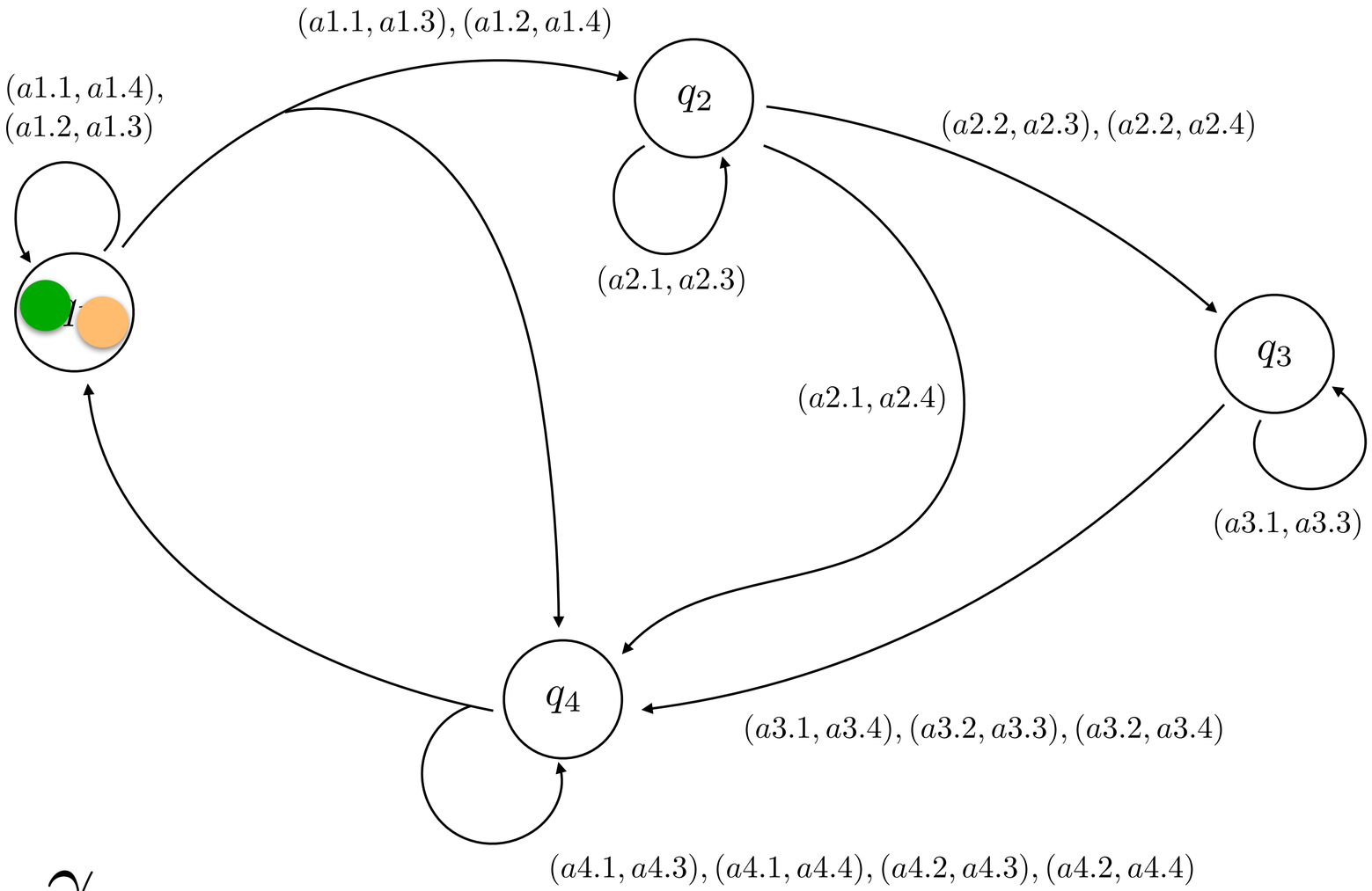
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

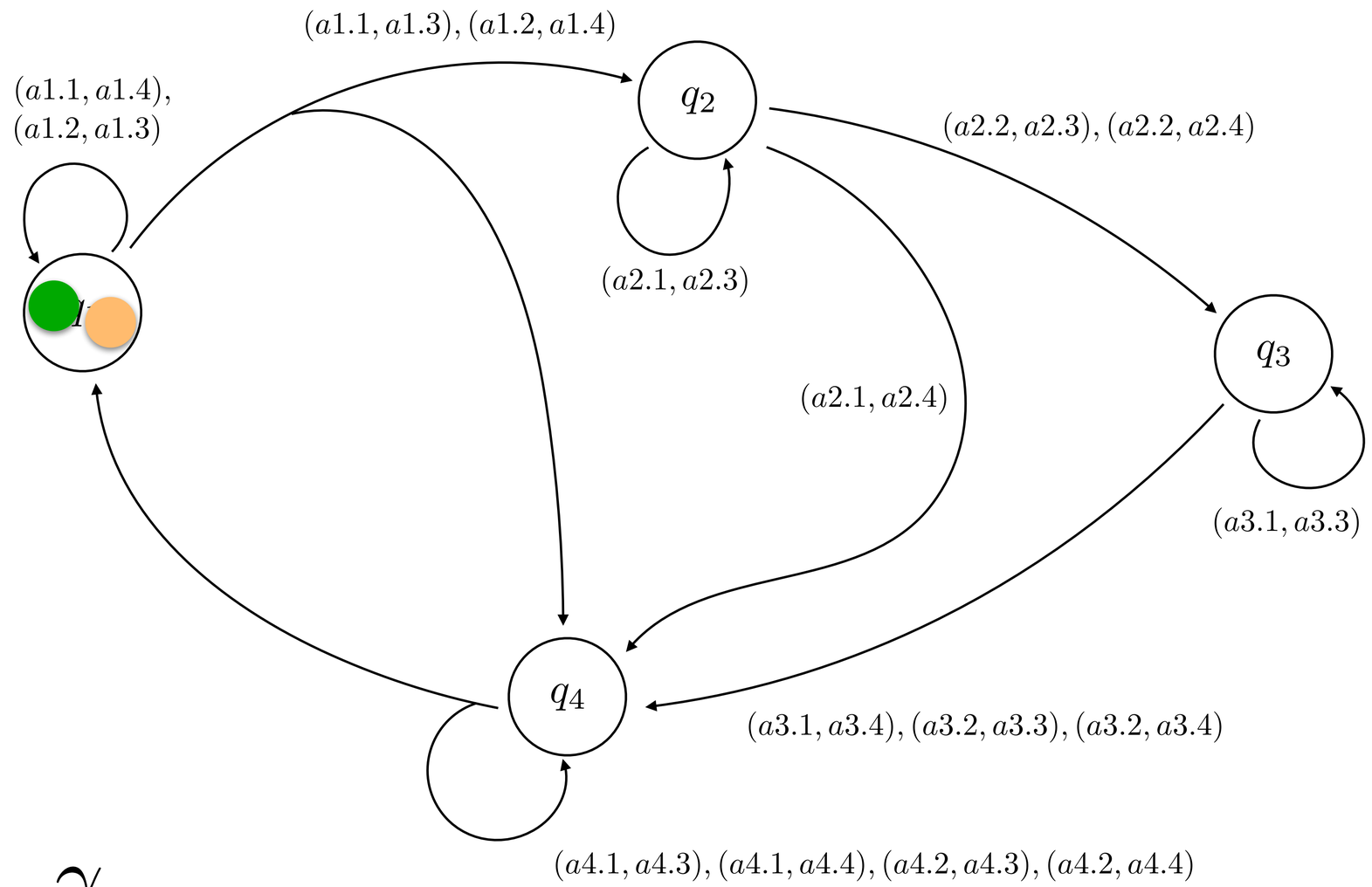
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

	a1.2	
	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

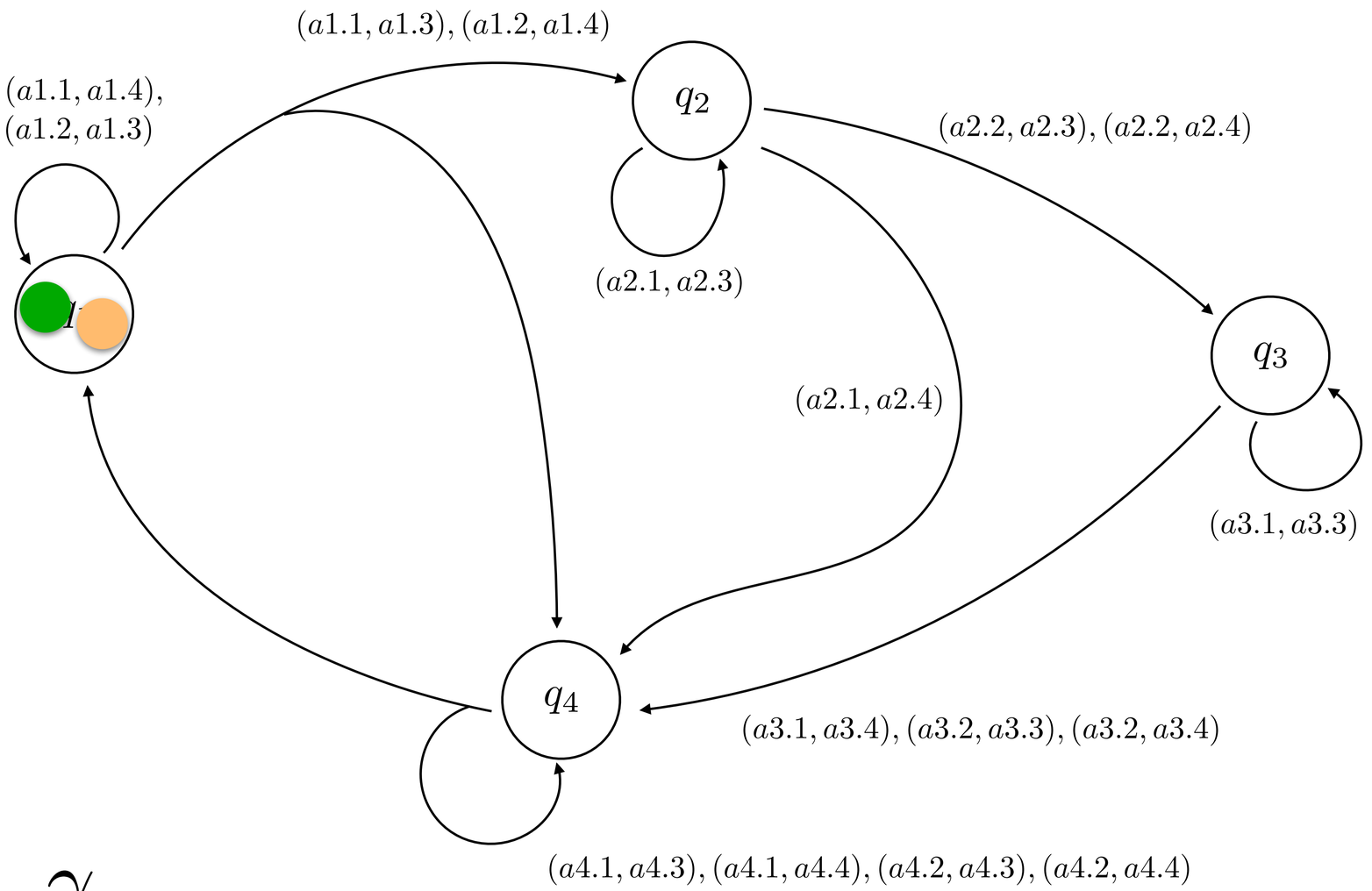
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

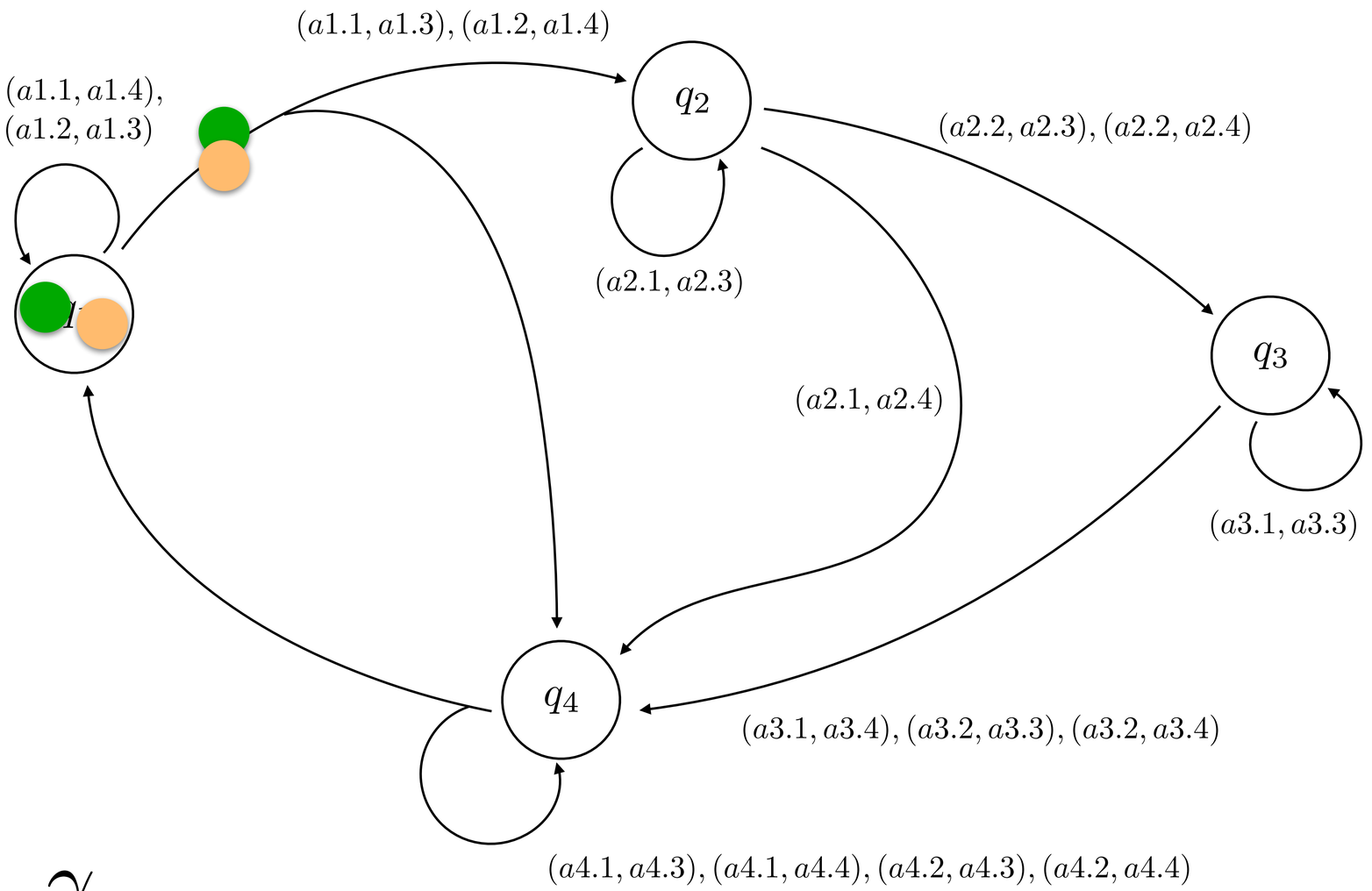
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

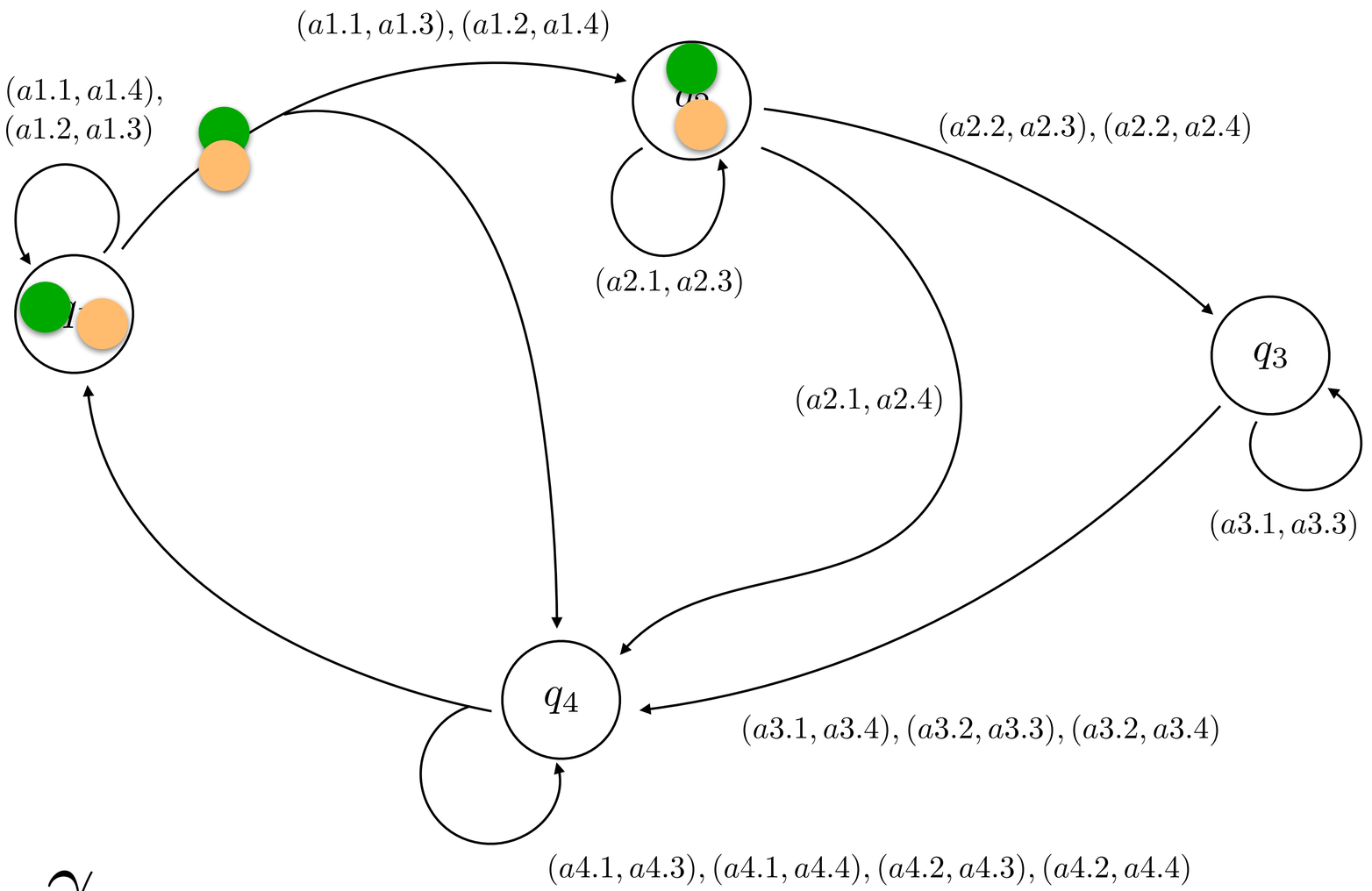
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

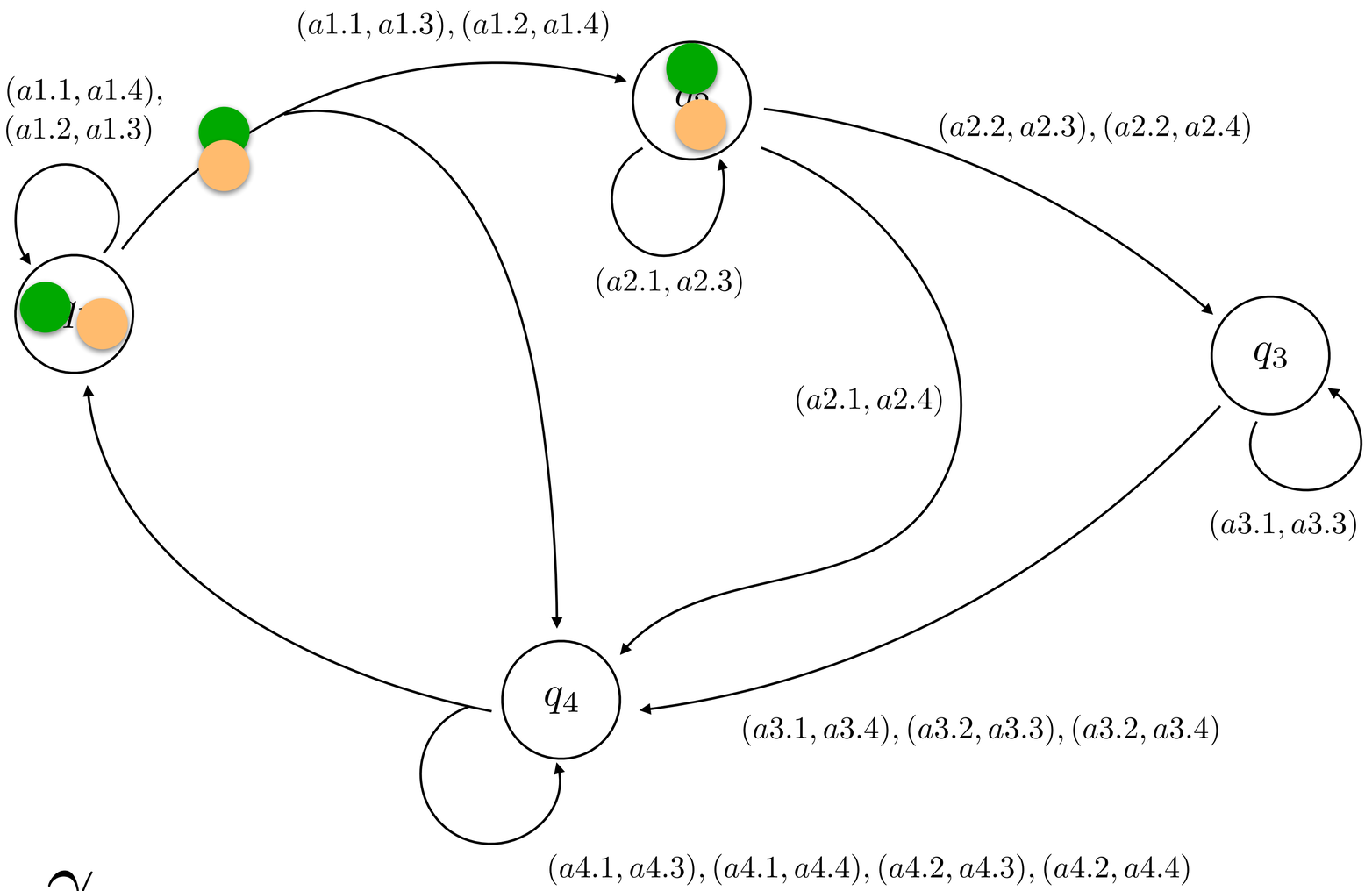
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

	a1.2	
	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.2	
a2.3	1, 1	3, 0
	0, 3	2, 2

q_3

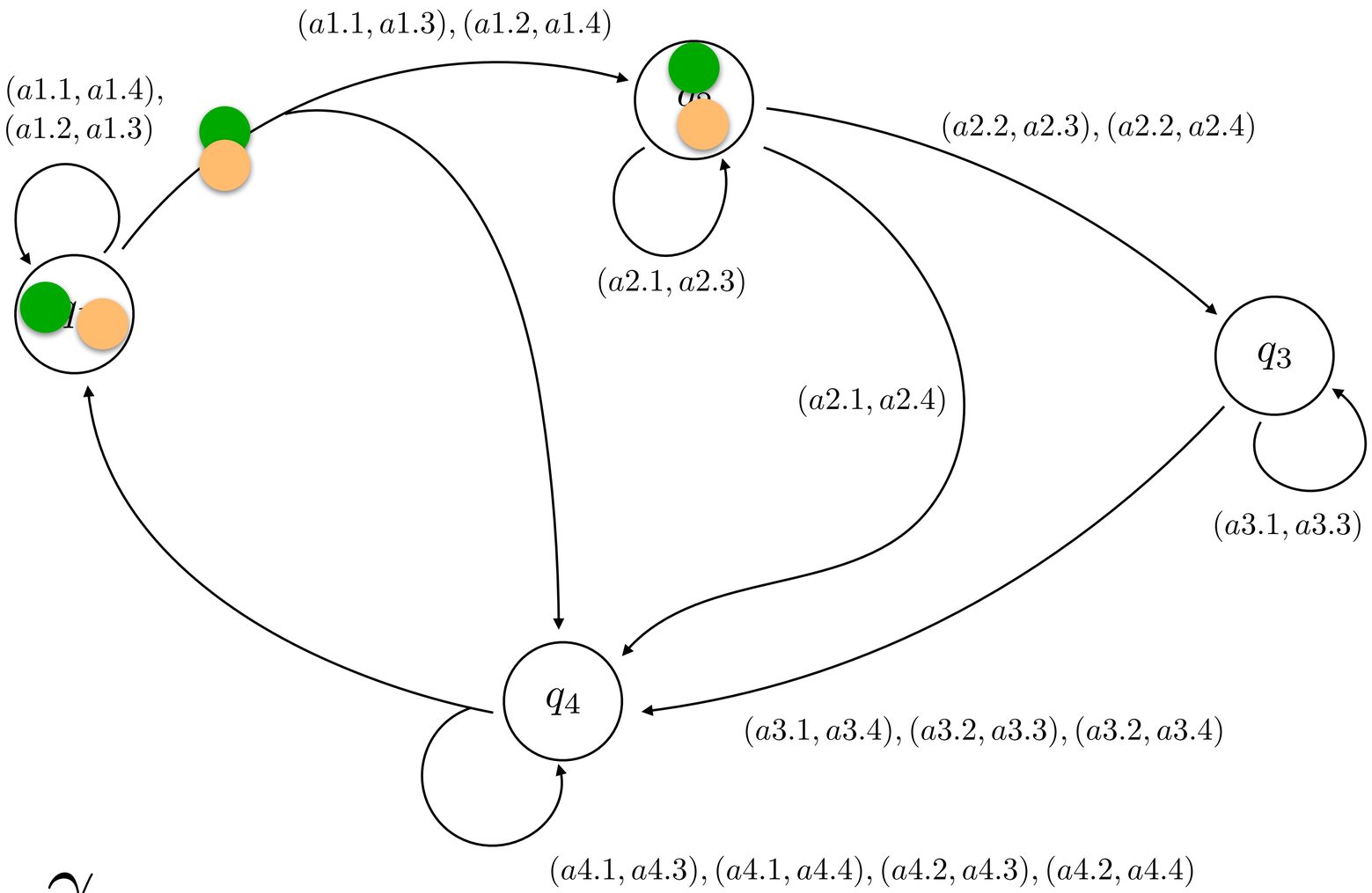
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

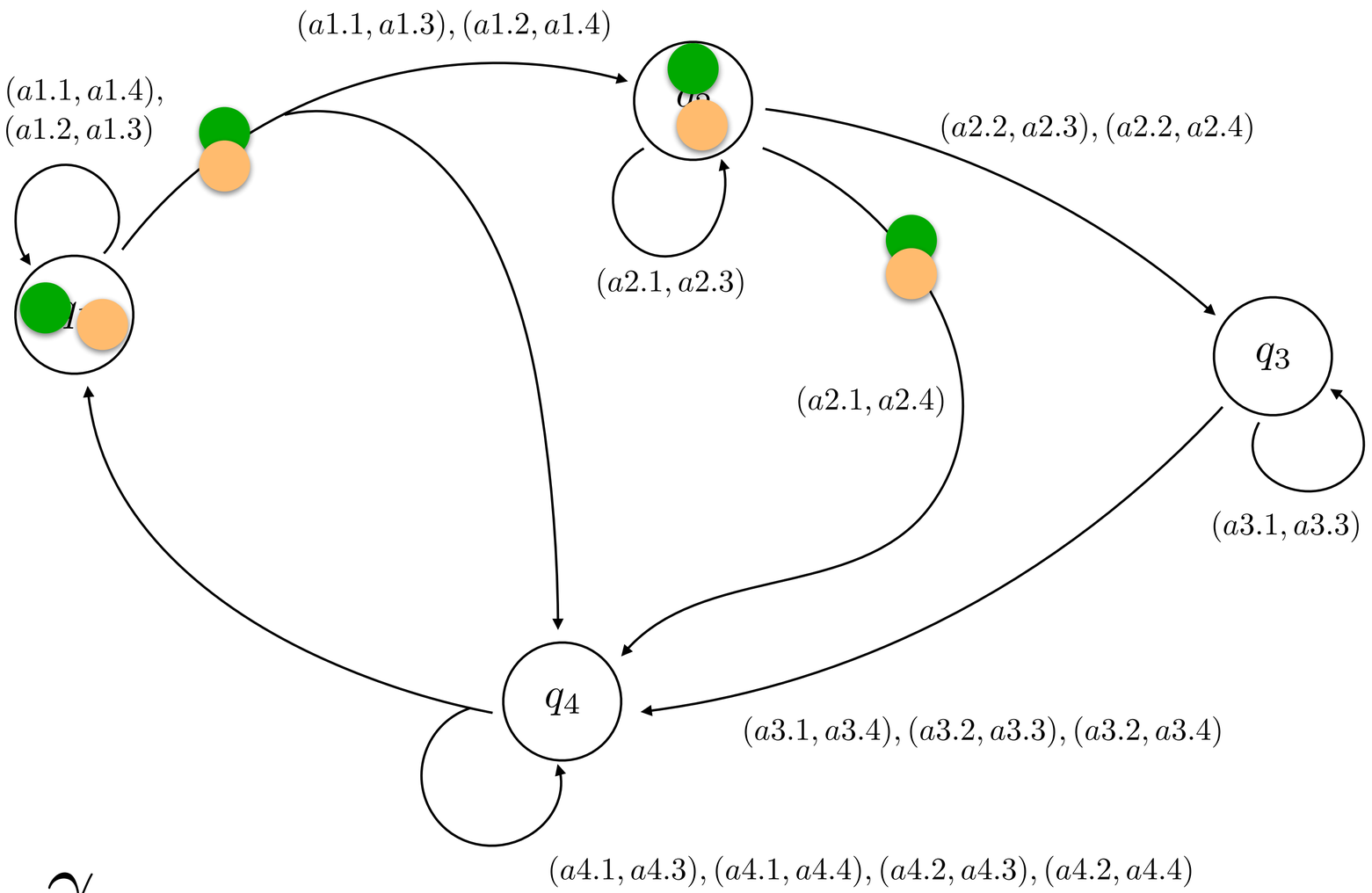
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



γ

q_1

		a1.2
	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

		a2.2
a2.3	1, 1	3, 0
	0, 3	2, 2

q_3

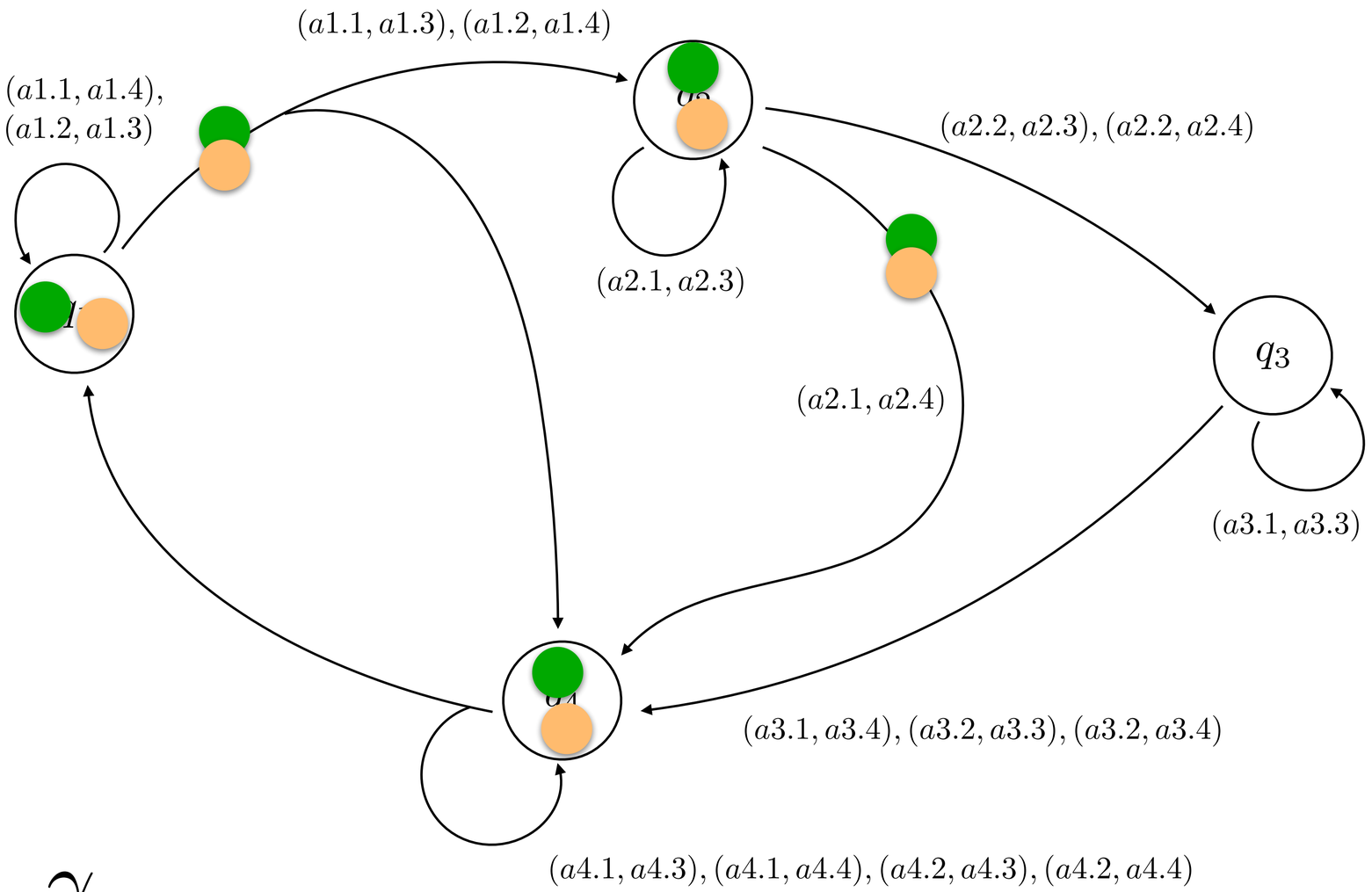
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

An example

Players 1 and 2



q_1

	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

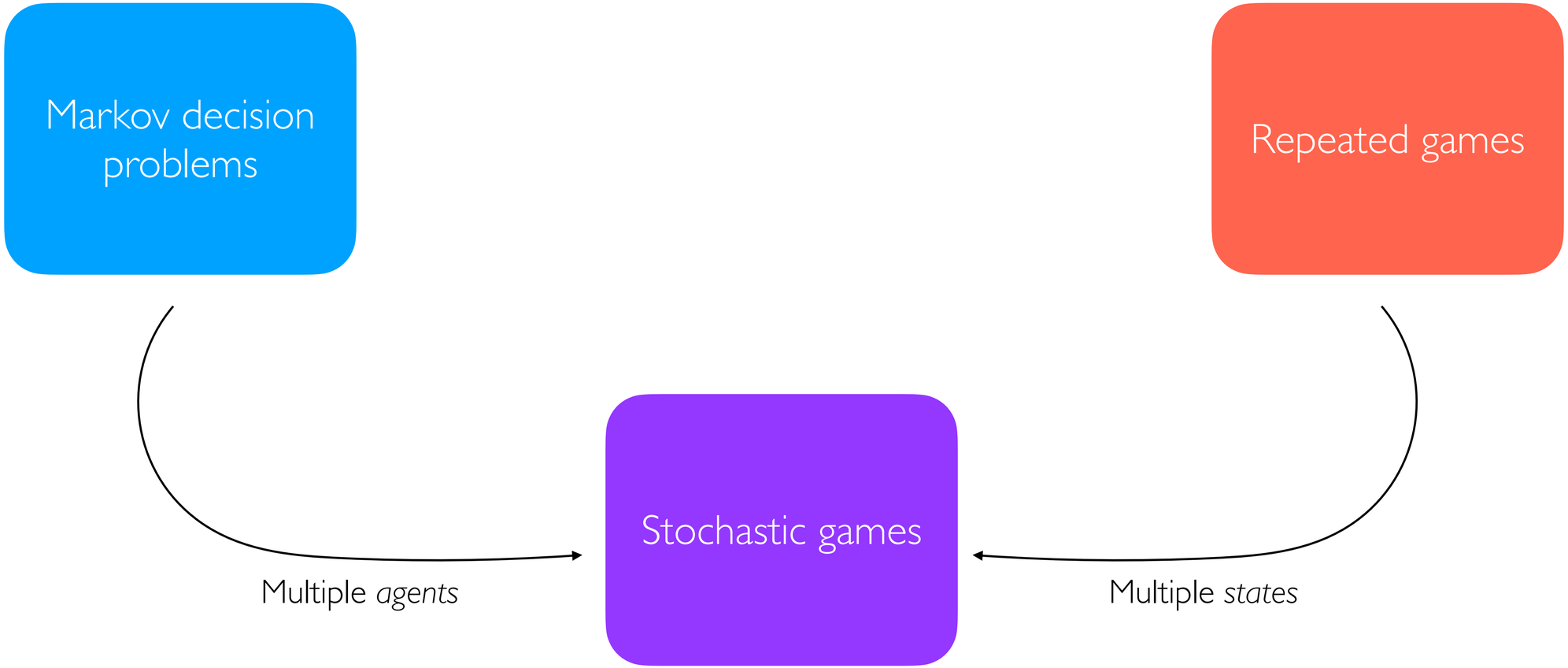
q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

Formal model

- A stochastic game (also known as a Markov game) is a tuple **(Q, N, A, P, r)**, where:
  - **Q** is a finite set of games
  - **N** is a finite set of n players
  - $\mathbf{A} = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, where $\mathbf{A}_i$ is a finite set of actions available to player ***i***
  - $\mathbf{P} : \mathbf{Q} \times \mathbf{A} \times \mathbf{Q} \rightarrow [0, 1]$ is the transition probability function; $\mathbf{P}(\mathbf{q}, \mathbf{a}, \mathbf{q}')$ is the probability of transitioning from game **q** to state **q'** after action profile **a**
  - $\mathbf{R} = \mathbf{r}_1, \ldots, \mathbf{r}_n$, where $\mathbf{r}_i : \mathbf{Q} \times \mathbf{A} \rightarrow \mathbf{R}$ is a real-valued payoff function for player **i**

Generalization



Markov Nash equilibrium

- A strategy profile is a Nash equilibrium of the Stochastic game if, for every state, following the strategy profile is a best response to the opponents' strategies for every player

Markov Nash equilibrium

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    - A Nash equilibrium of the Stochastic game, may prescribe, at every state, a strategy different from the Nash equilibrium of the single game

Markov Nash equilibrium

- A strategy profile is a Nash equilibrium of the Stochastic game if, for every state, following the strategy profile is a best response to the opponents' strategies for every player
    - A Nash equilibrium of the Stochastic game, may prescribe, at every state, a strategy different from the Nash equilibrium of the single game
- Every *n*-player discounted stochastic game possesses at least one Nash equilibrium point in stationary strategies

Markov Nash equilibrium

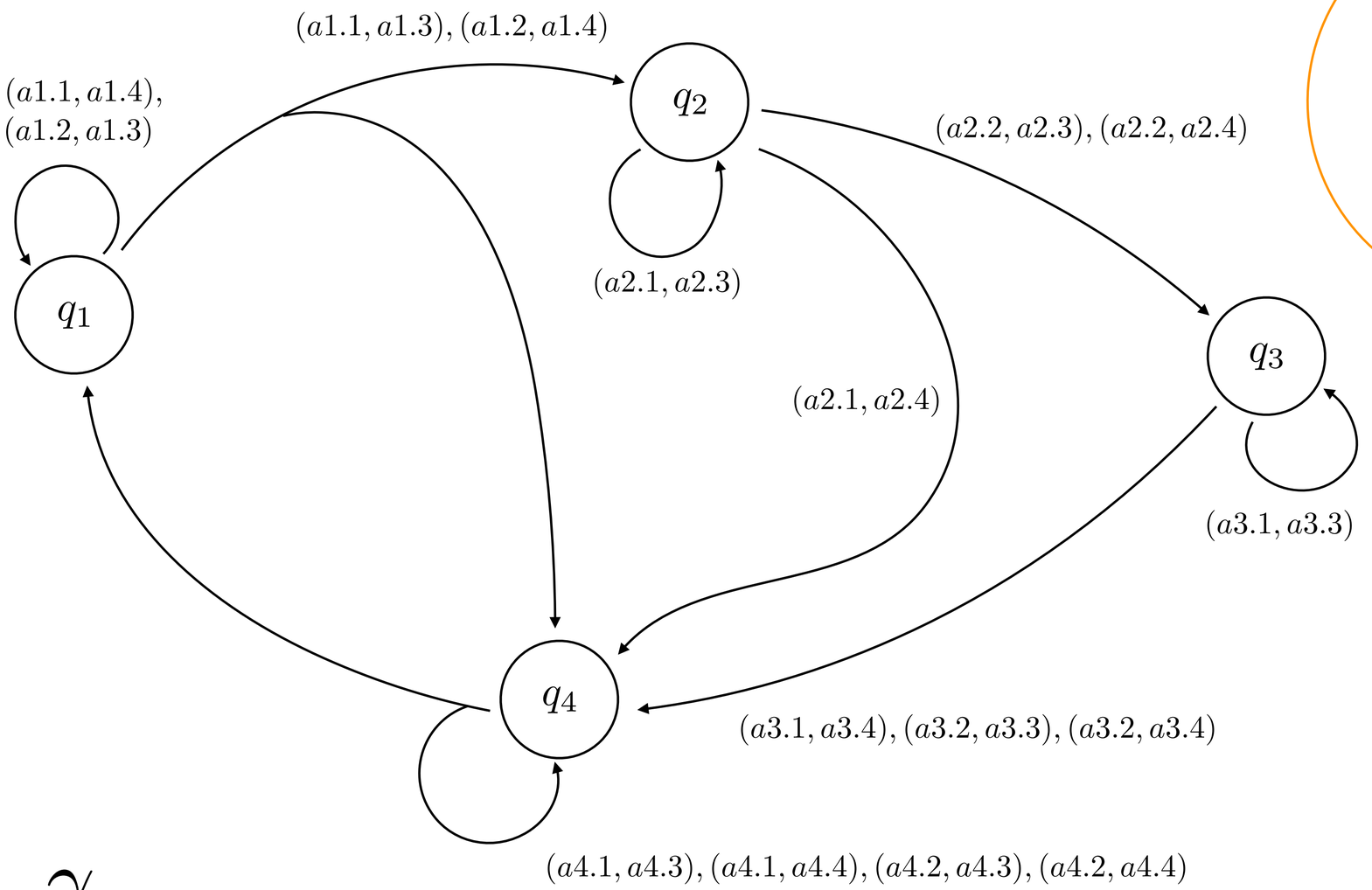
- A strategy profile is a Nash equilibrium of the Stochastic game if, for every state, following the strategy profile is a best response to the opponents' strategies for every player
    - A Nash equilibrium of the Stochastic game, may prescribe, at every state, a strategy different from the Nash equilibrium of the single game
- Every *n*-player discounted stochastic game possesses at least one Nash equilibrium point in stationary strategies
- Stationary (Markov) strategies do not span the whole space of equilibria (e.g., in repeated games, non-stationary equilibria may exist)

An example

Nash equilibrium of the single-stage game

Nash equilibrium of the Stochastic game

Players 1 and 2



q_1

		a1.2
a1.3	2, 1	0, 0
4	0, 0	1, 2

q_2

	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

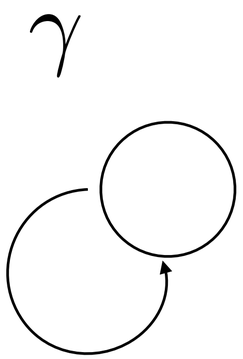
	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

γ

An example (repeated game)



	a2.1	a2.2
a2.3	6, 6	2, 9
a2.4	9, 2	3, 3

Non-stationary strategies are, for instance, to play (a2.2, a2.4) and to switch to play a2.1 or a2.3, respectively, as soon as the opponent deviates from (a2.2, a2.4)

Nash-Q learning

Assumptions

- The players do not know the rewards and the transition probabilities
- In particular,
  - Every player does not know his/her own rewards
  - Every player does not know the rewards of the opponents

Nash-Q learning (1)

- In every state, the Q-table stores the Q-values defined on the joint actions of the players for every player

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3		
a1.4		

q_2

Nash Q values	a2.1	a2.2
a2.3		
a2.4		

q_3

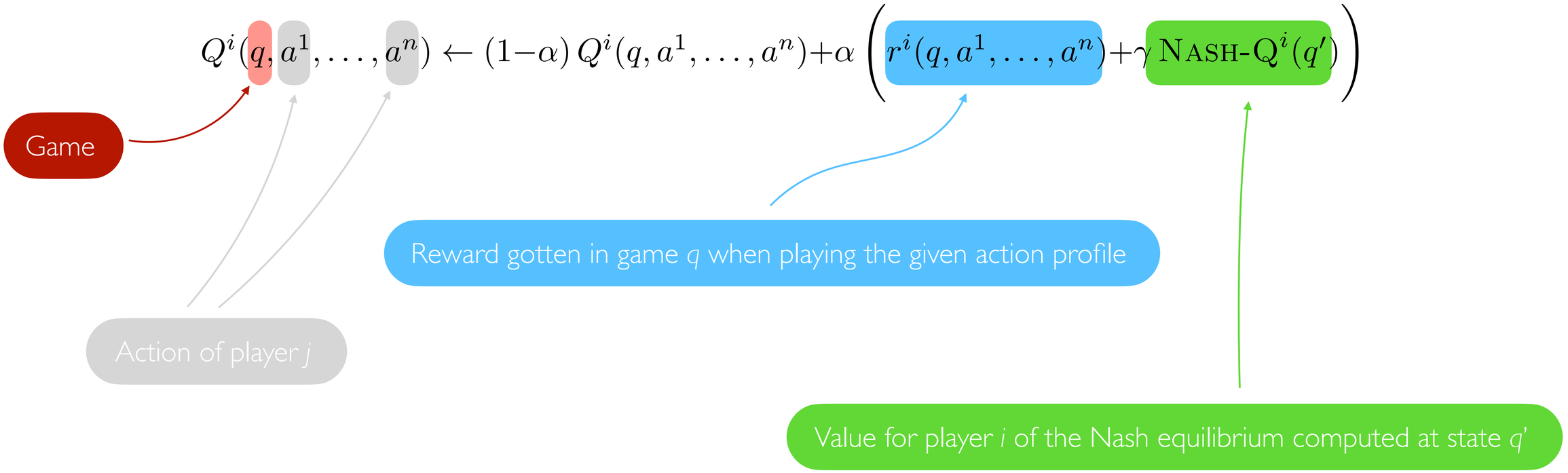
Nash Q values	a3.1	a3.2
a3.3		
a3.4		

q_4

Nash Q values	a4.1	a4.2
a4.3		
a4.4		

Nash-Q learning (2)

- The update of the Nash-Q values of every player is as follows:

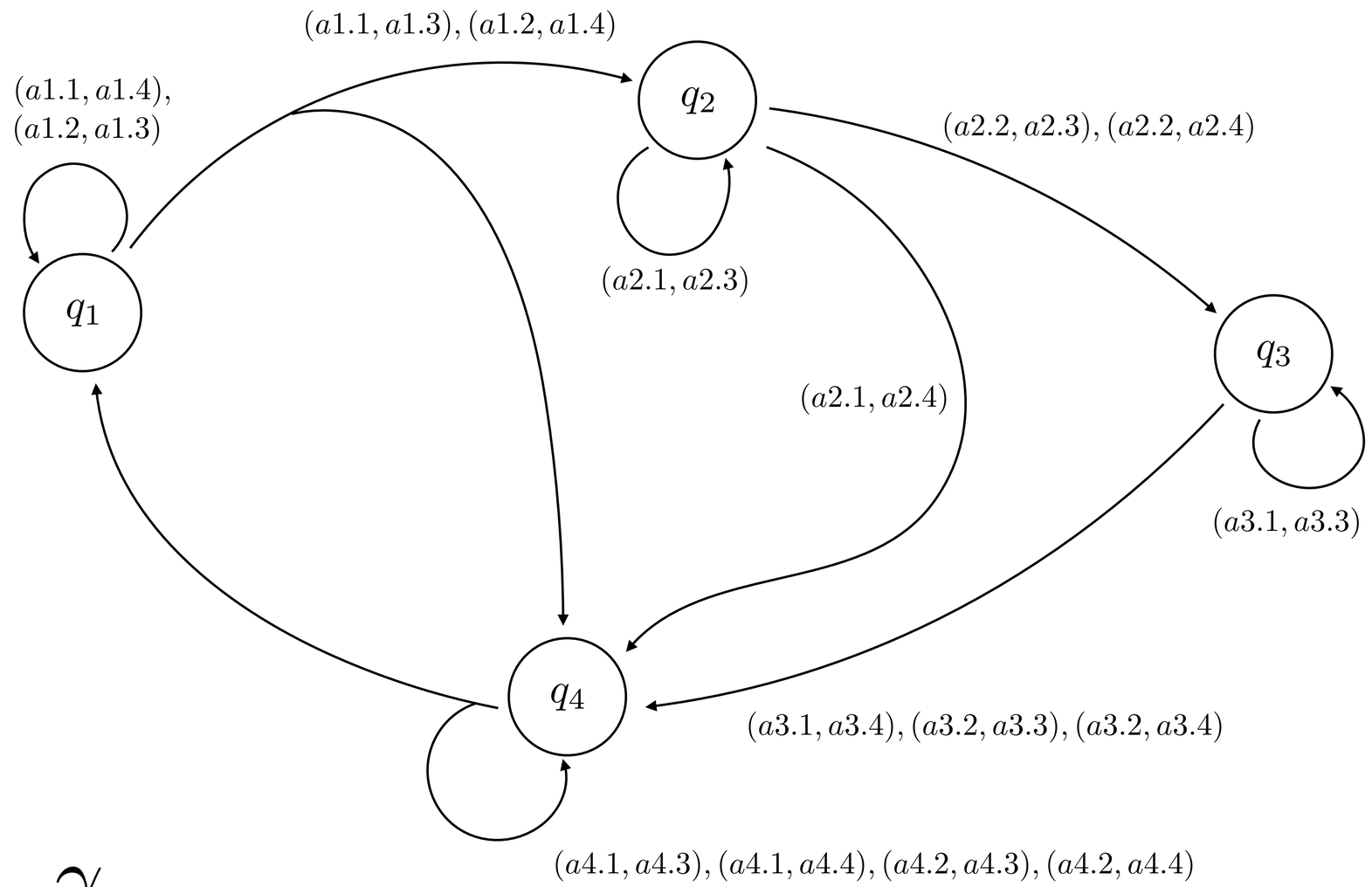


Nash-Q learning (3)

- In every game, the strategies of the players is a Nash equilibrium computed from the Nash-Q values

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	0, 0	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

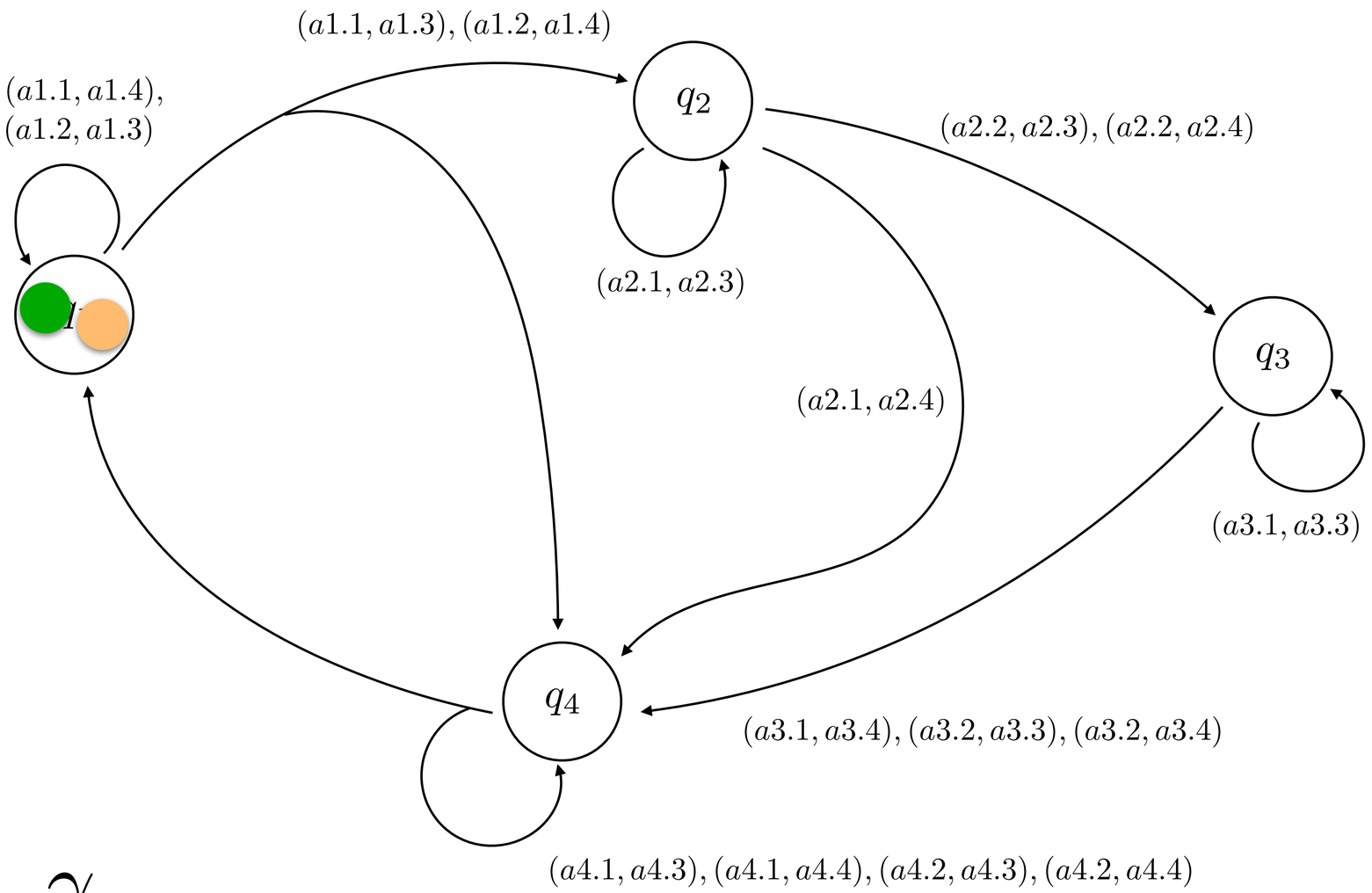
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	0, 0	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

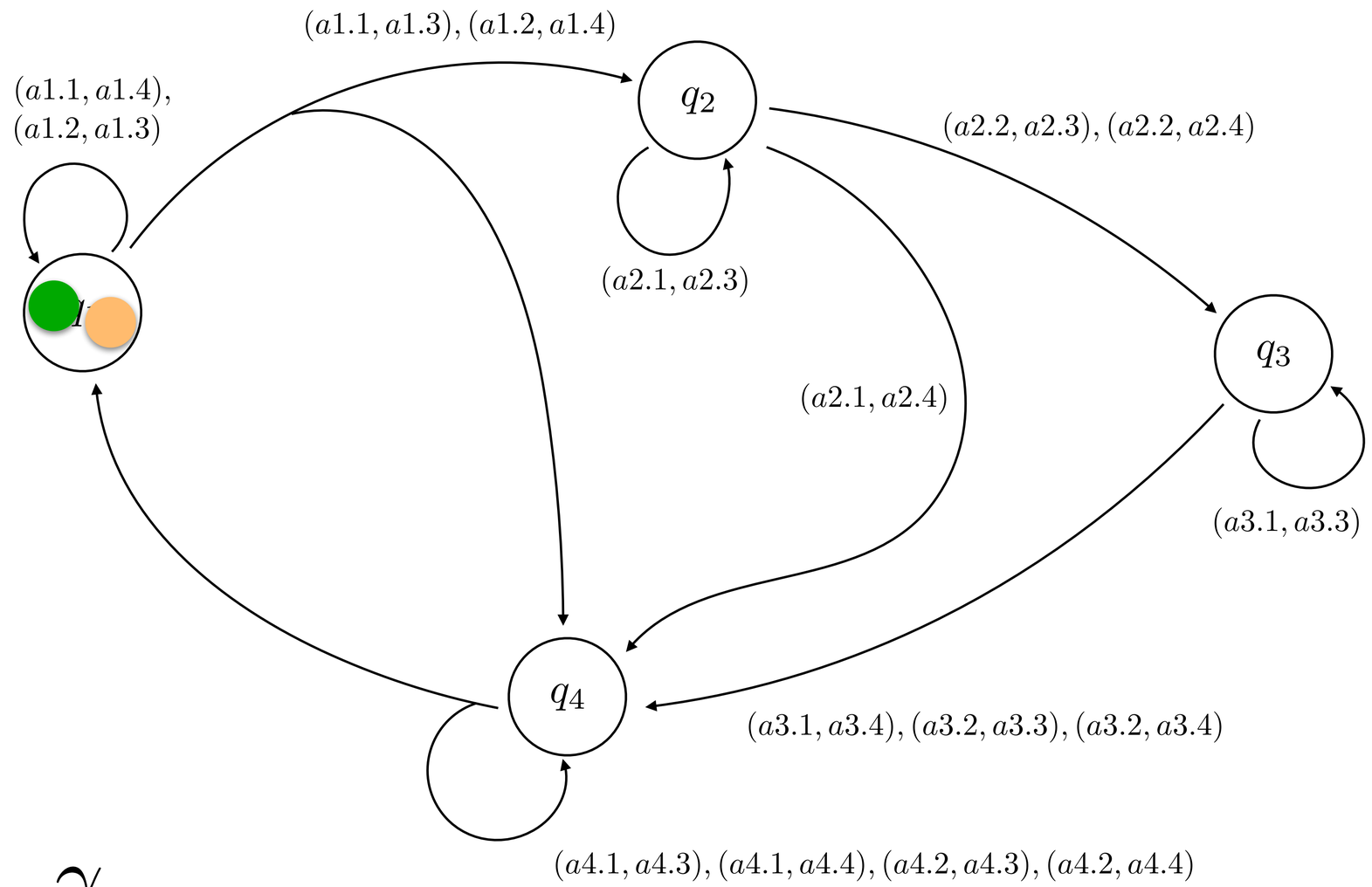
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	0, 0	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

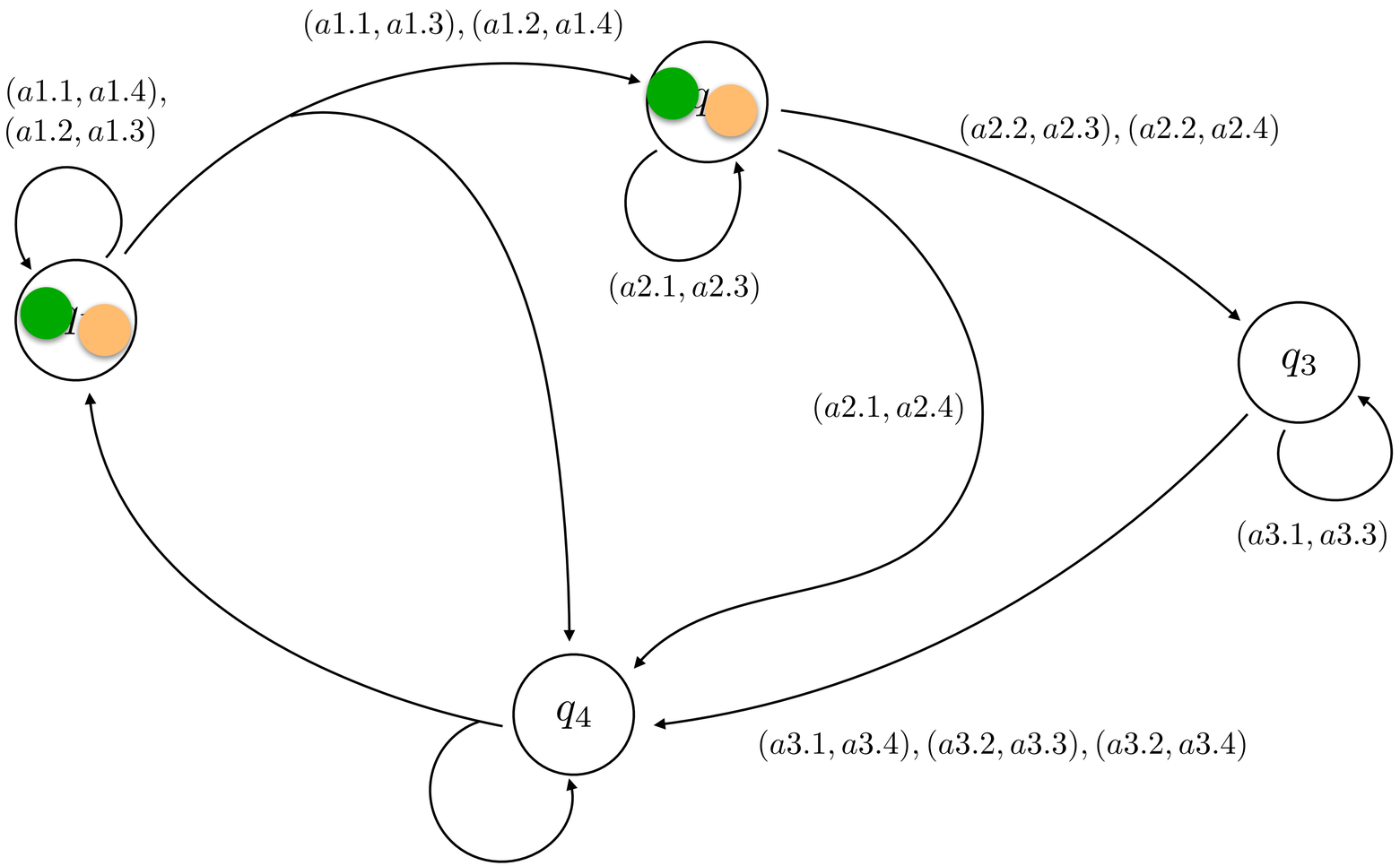
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	0, 0	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

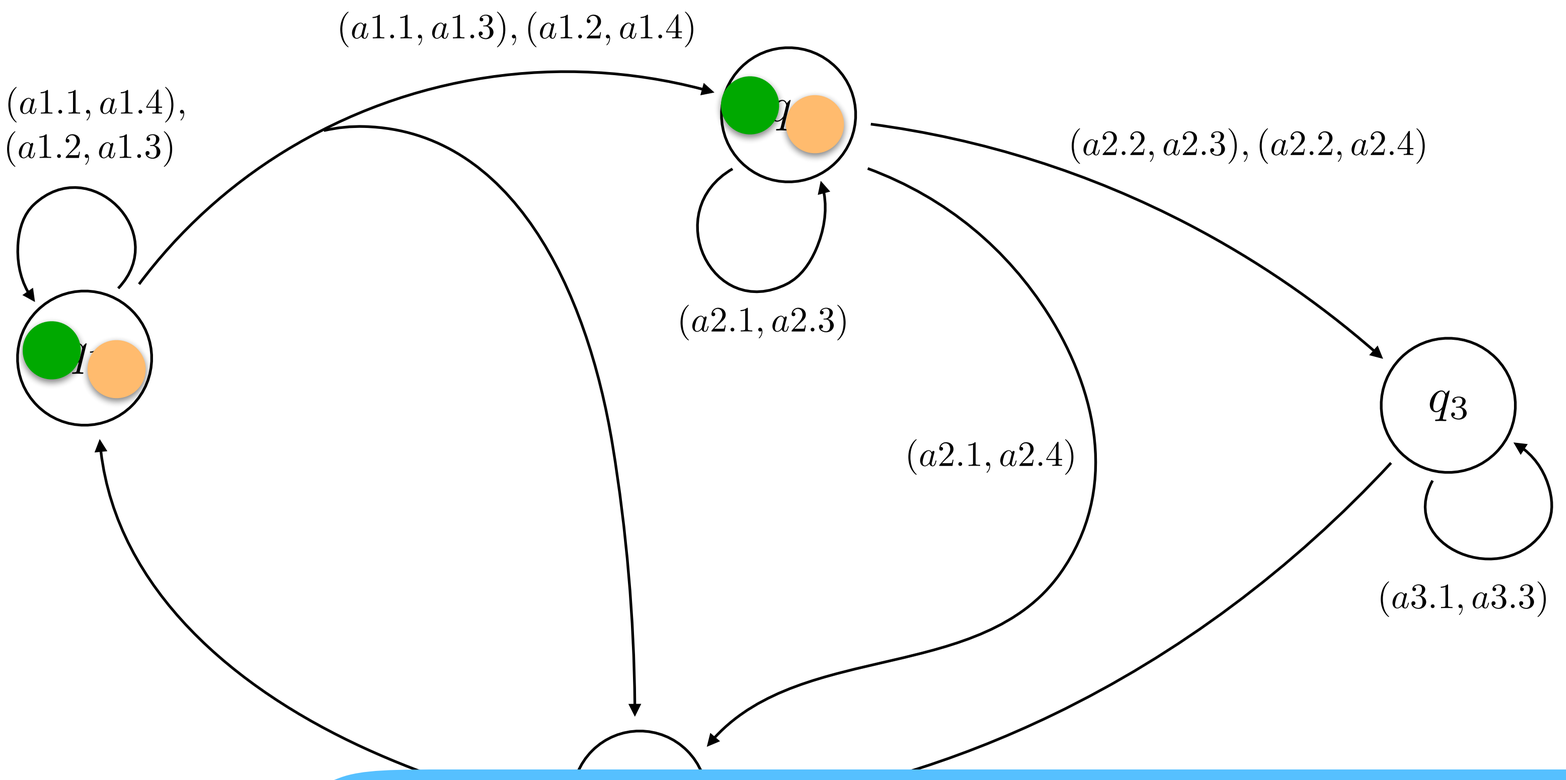
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	0, 0	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_4

	a4.1	a4.2
	0	0, 0
	0, 0	0, 0

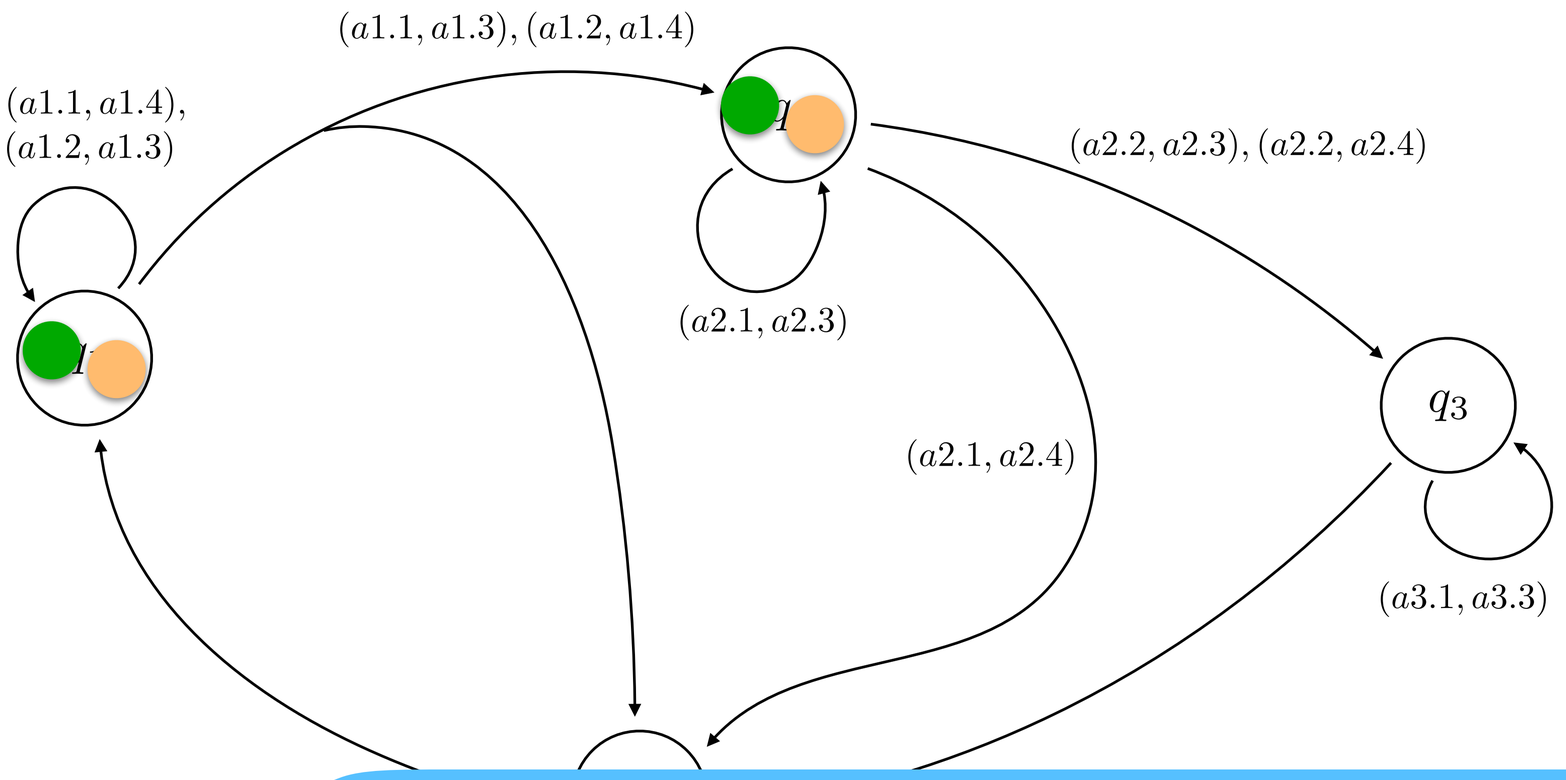
$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(1, .5) 0.5 (0, 0) 0.5 (2, 1) 0.8 (0, 0)

γ

An example

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	**1, .5**	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_4

	a4.1	a4.2
	0	0, 0
	0, 0	0, 0

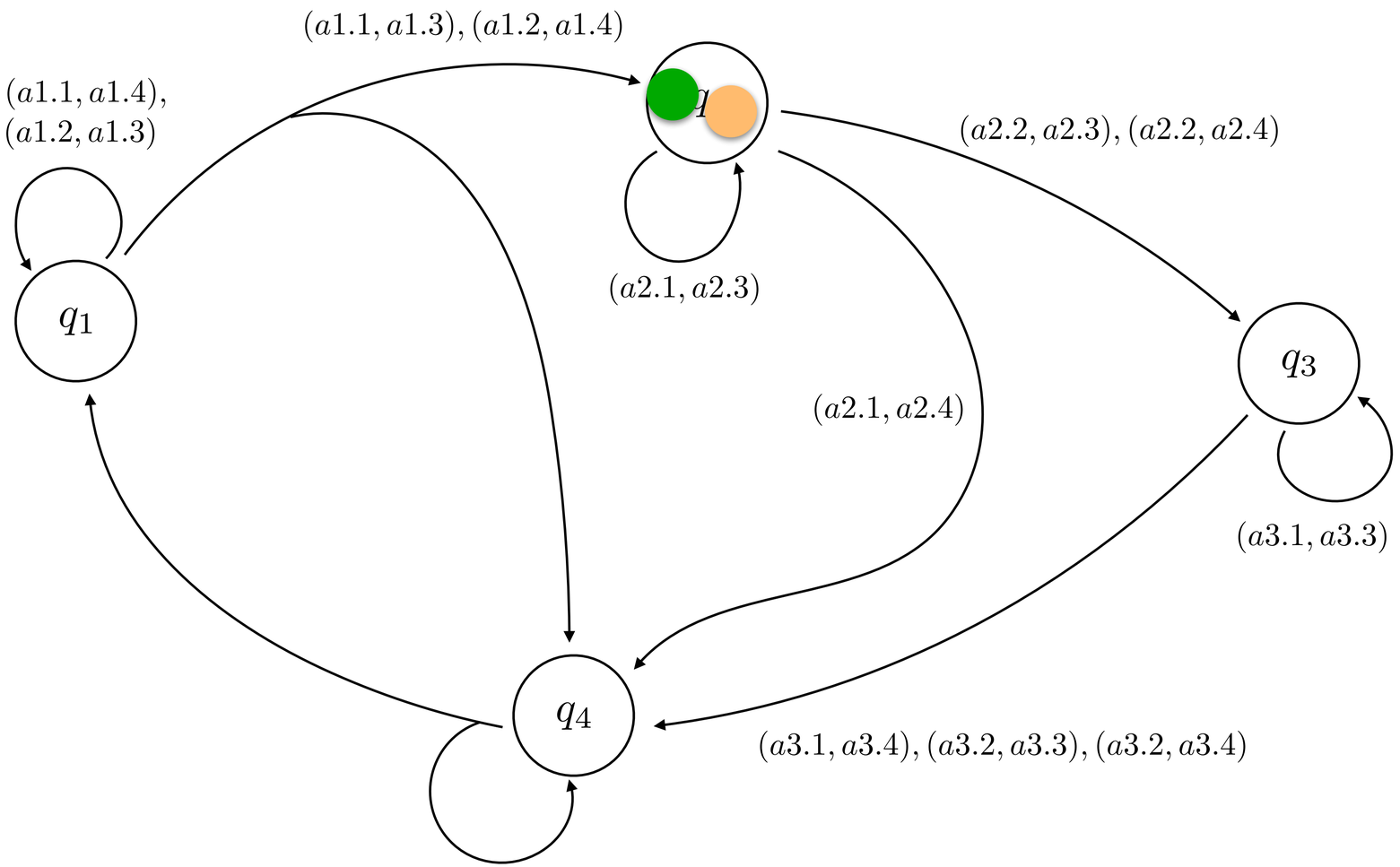
$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(1, .5) 0.5 (0, 0) 0.5 (2, 1) 0.8 (0, 0)

γ

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

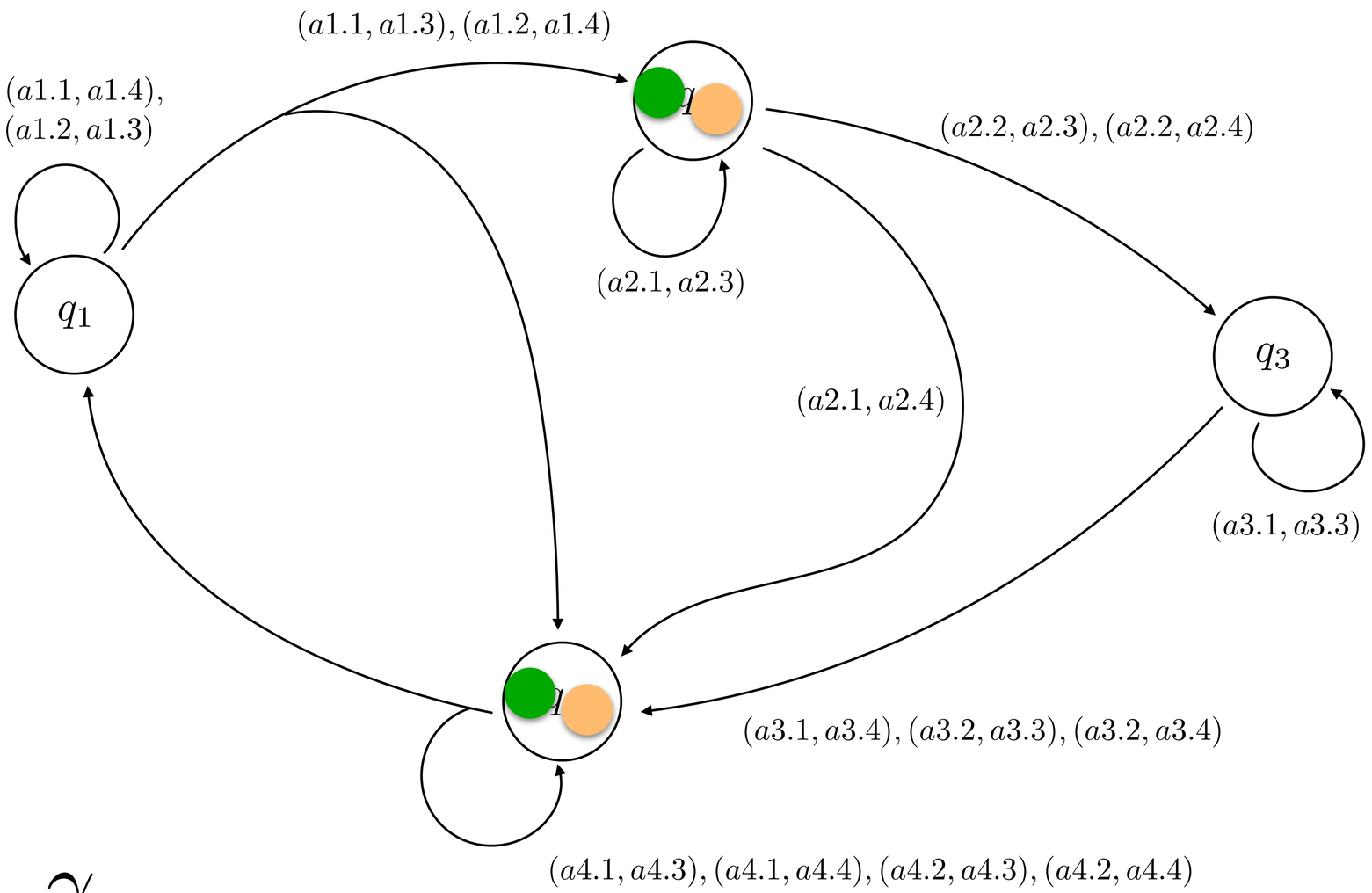
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a	a2.2
a2.3	1, 1	3, 0
a	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

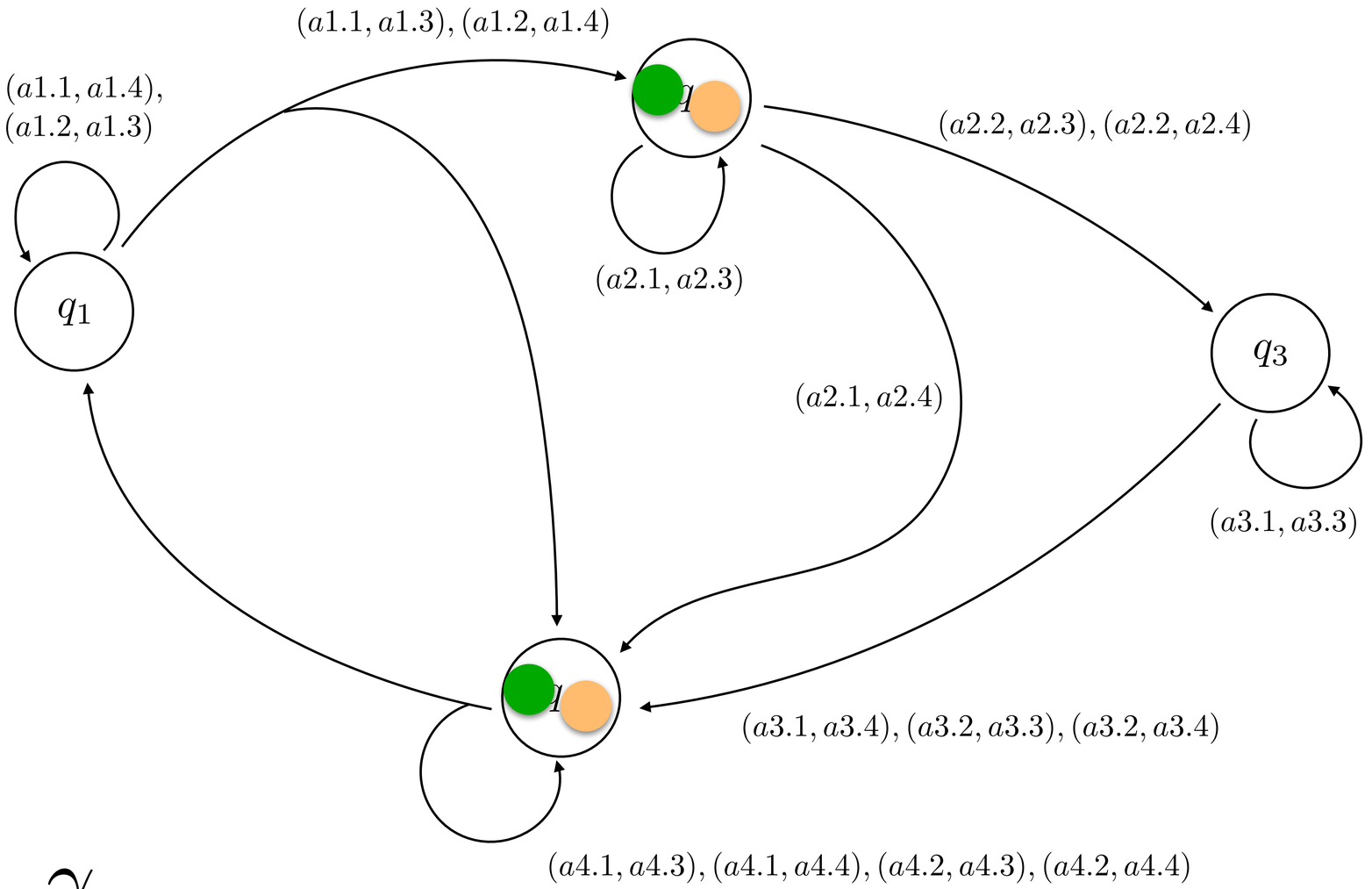
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(0, 1.5) 0.5 (0, 0) 0.5 (0, 3) 0.8 (0, 0)



γ

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 0	0, 0

q_3

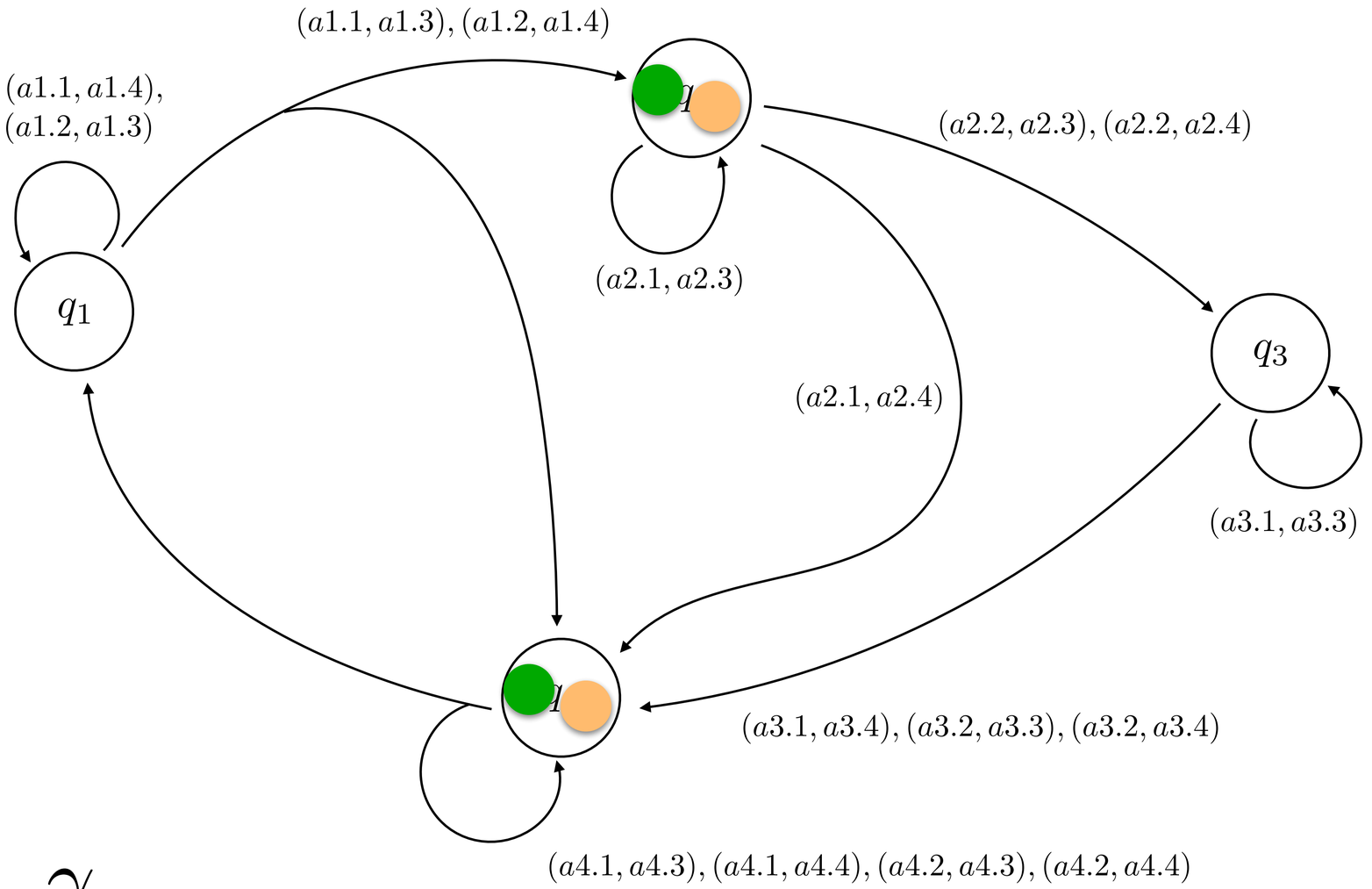
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(0, 1.5) 0.5 (0, 0) 0.5 (0, 3) 0.8 (0, 0)



γ

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	**0, 1.5**	0, 0

q_3

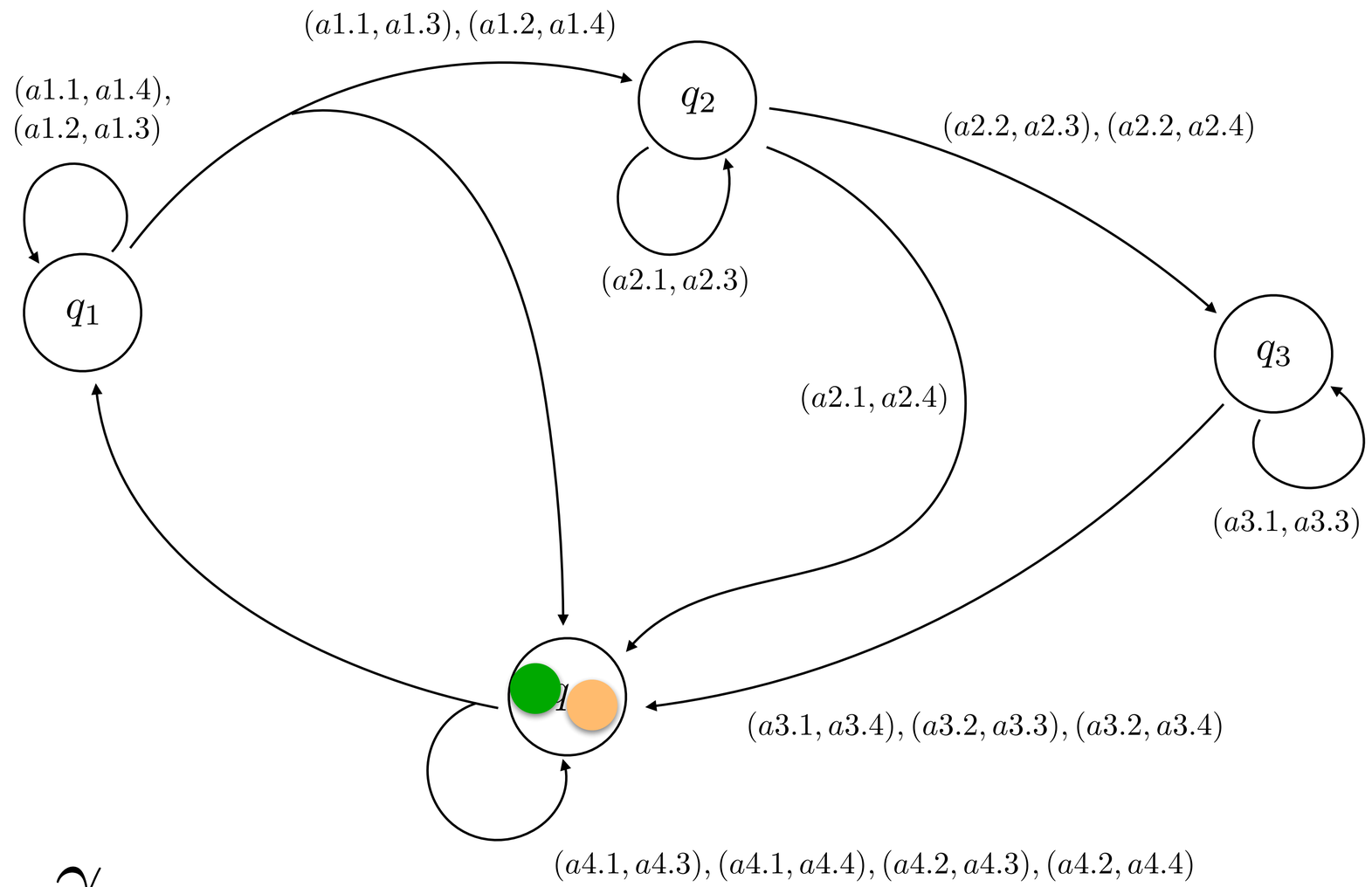
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

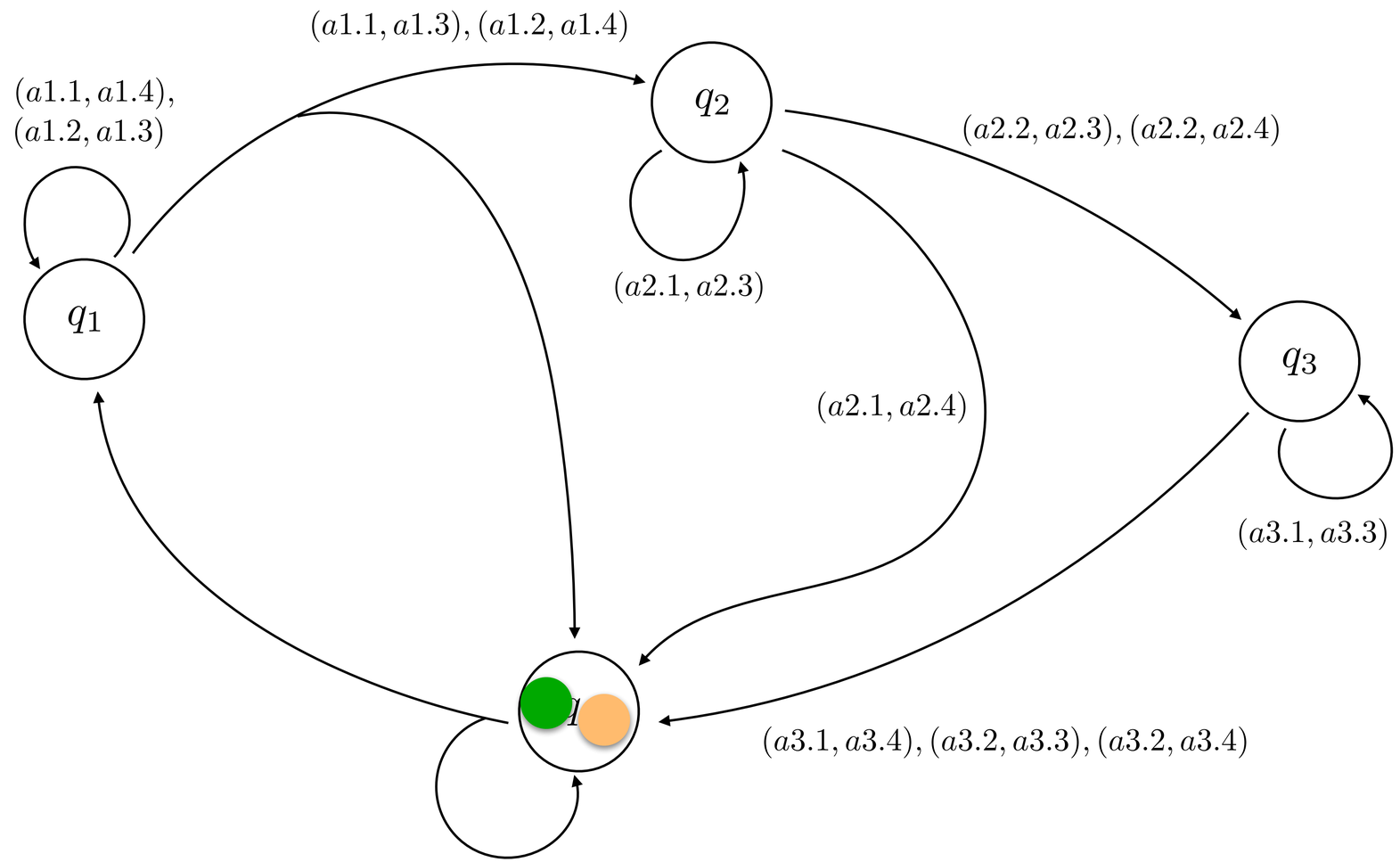
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 1.5	0, 0
a2.4	0, 0	0, 0

q_3

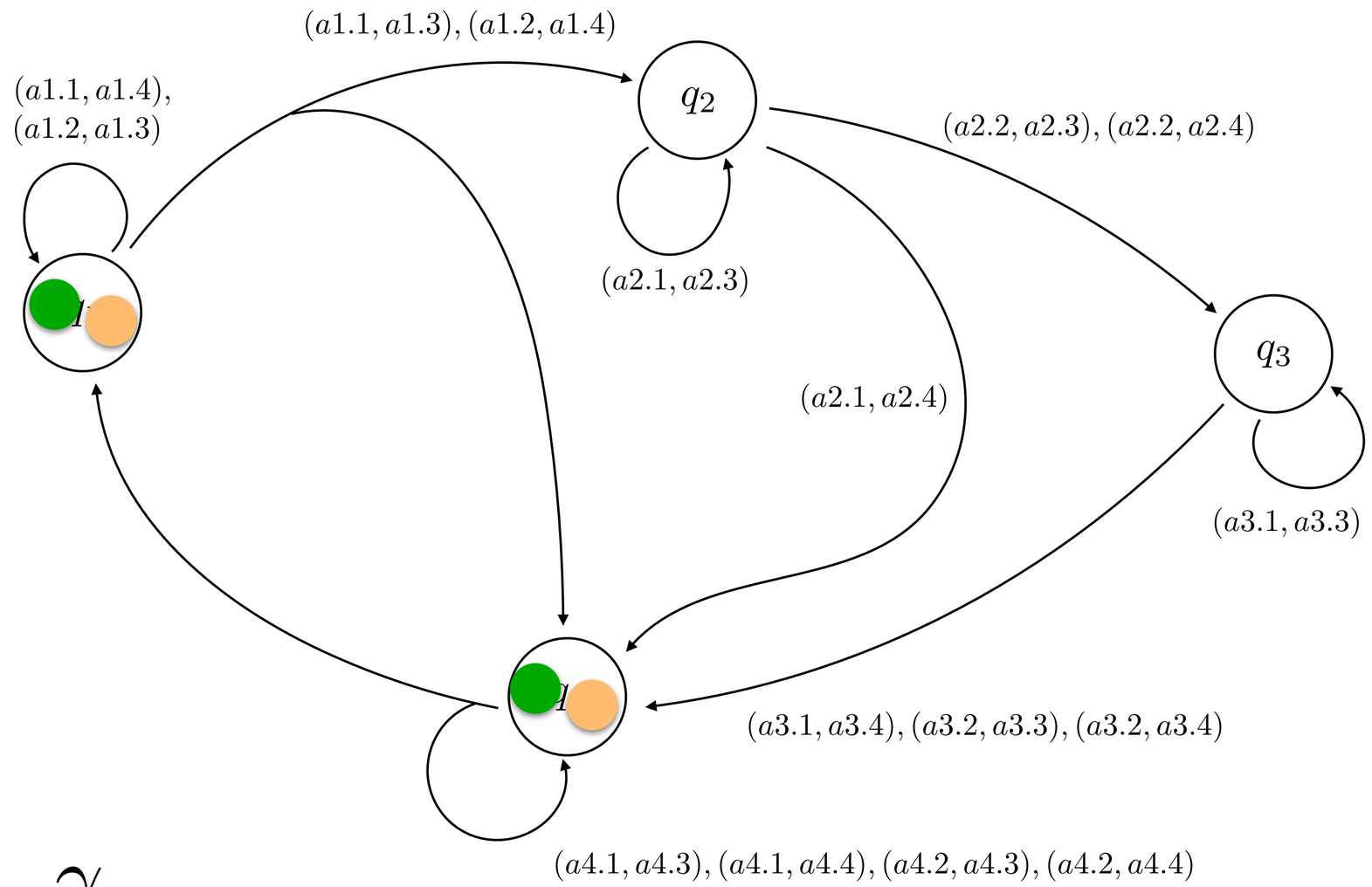
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

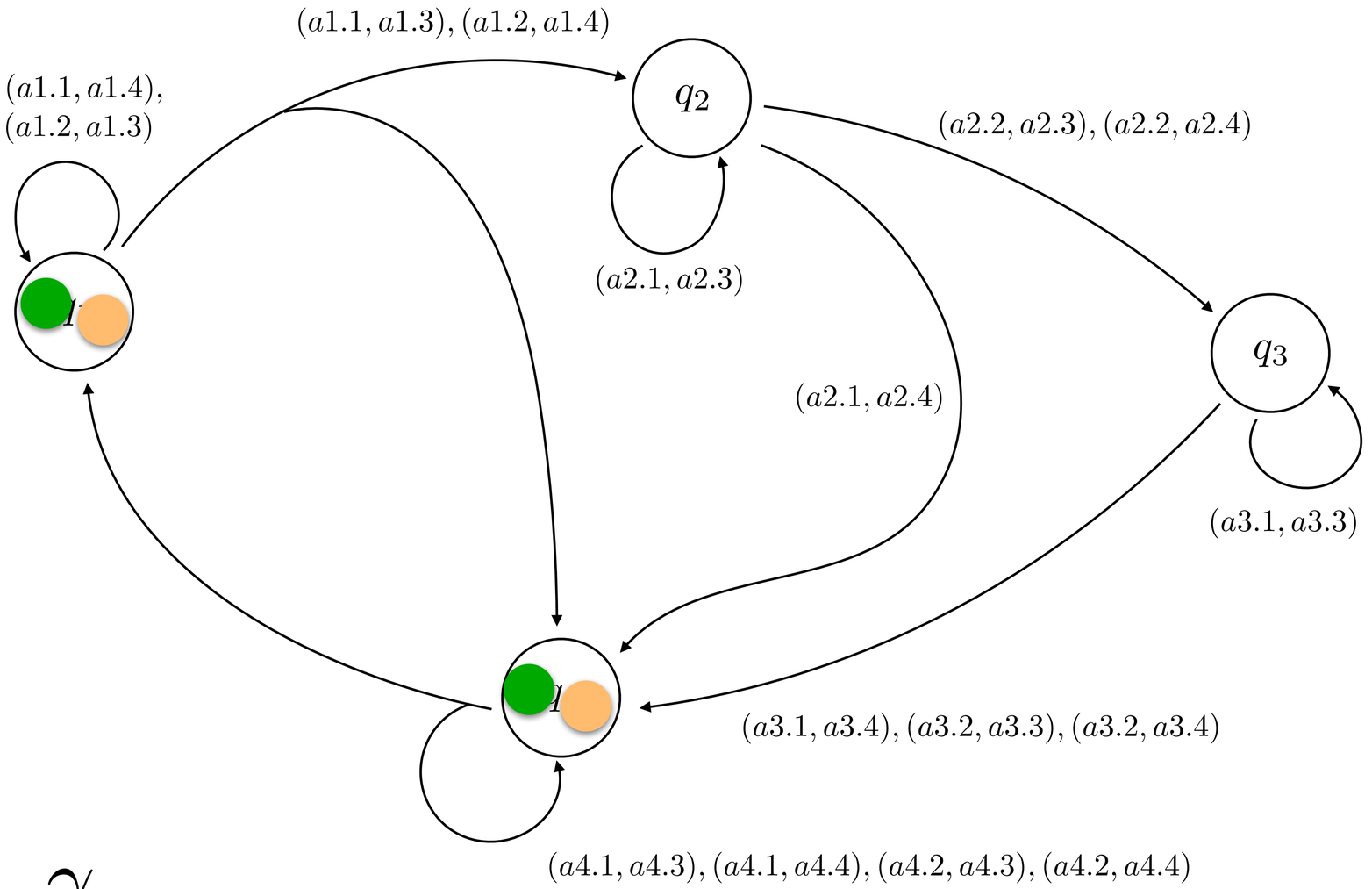
q_4

Nash Q values	a4.1	a4.2
a4.3	0, 0	0, 0
a4.4	0, 0	0, 0

$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(0.4, 0.6) 0.5 (0, 0) 0.5 (0, 0) 0.8 (1, .5)

P



γ

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a	a4.2
a	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

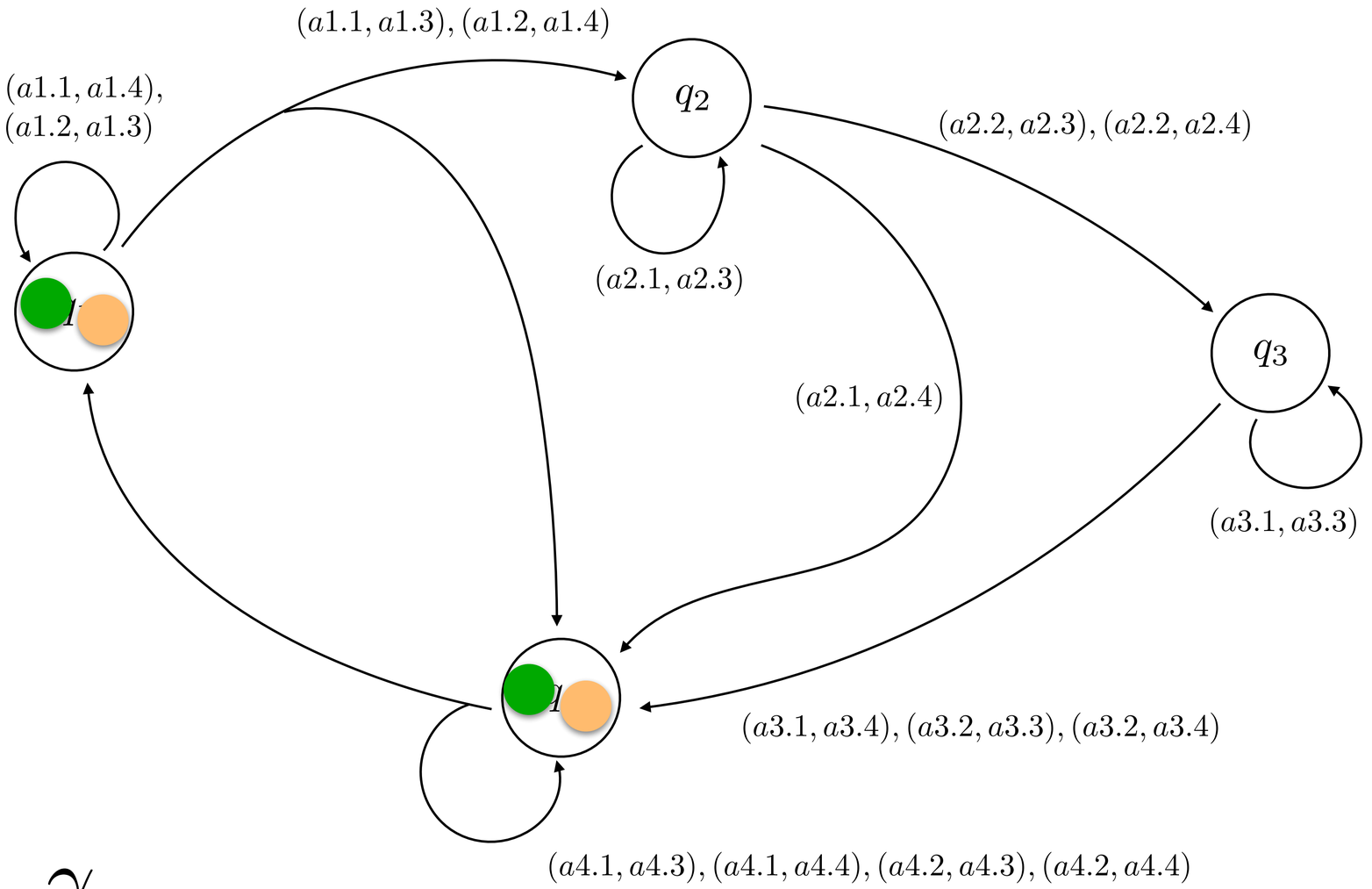
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a	a4.2
a	0, 0	0, 0
a4.4	0, 0	0, 0

$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(0.4, 0.6) 0.5 (0, 0) 0.5 (0, 0) 0.8 (1, .5)



γ

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

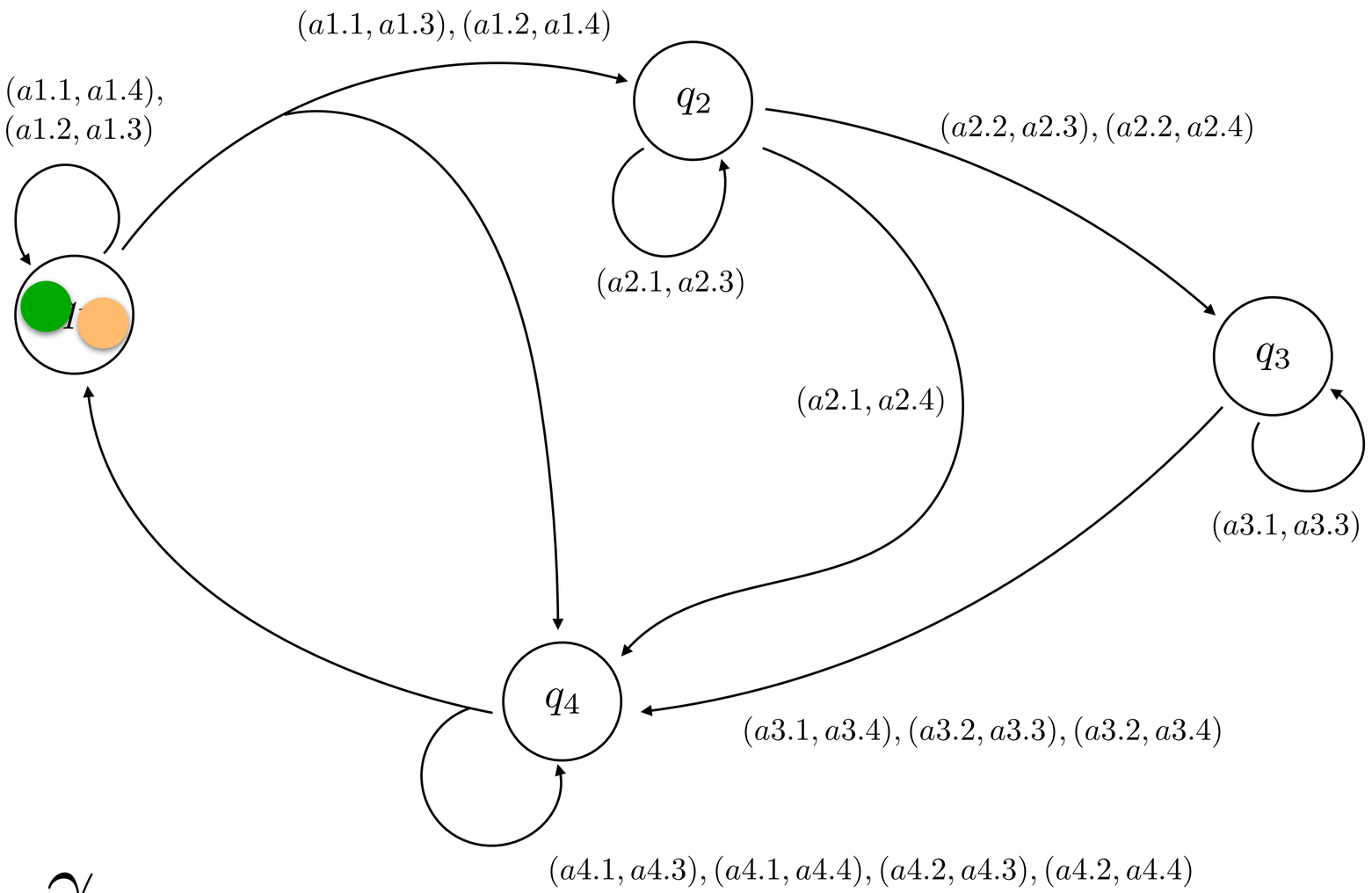
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	**.4, .6**	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

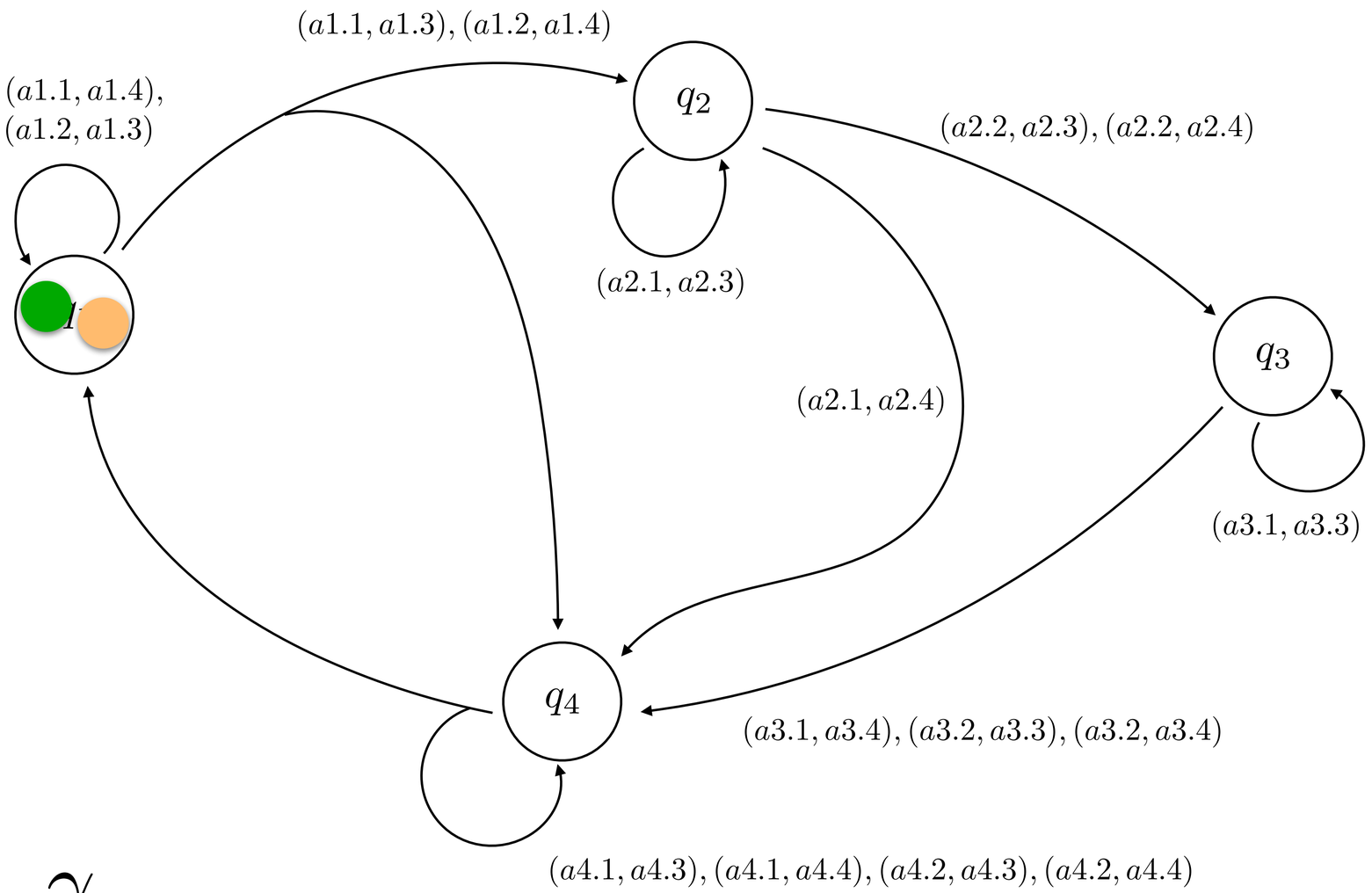
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	.4, .6	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

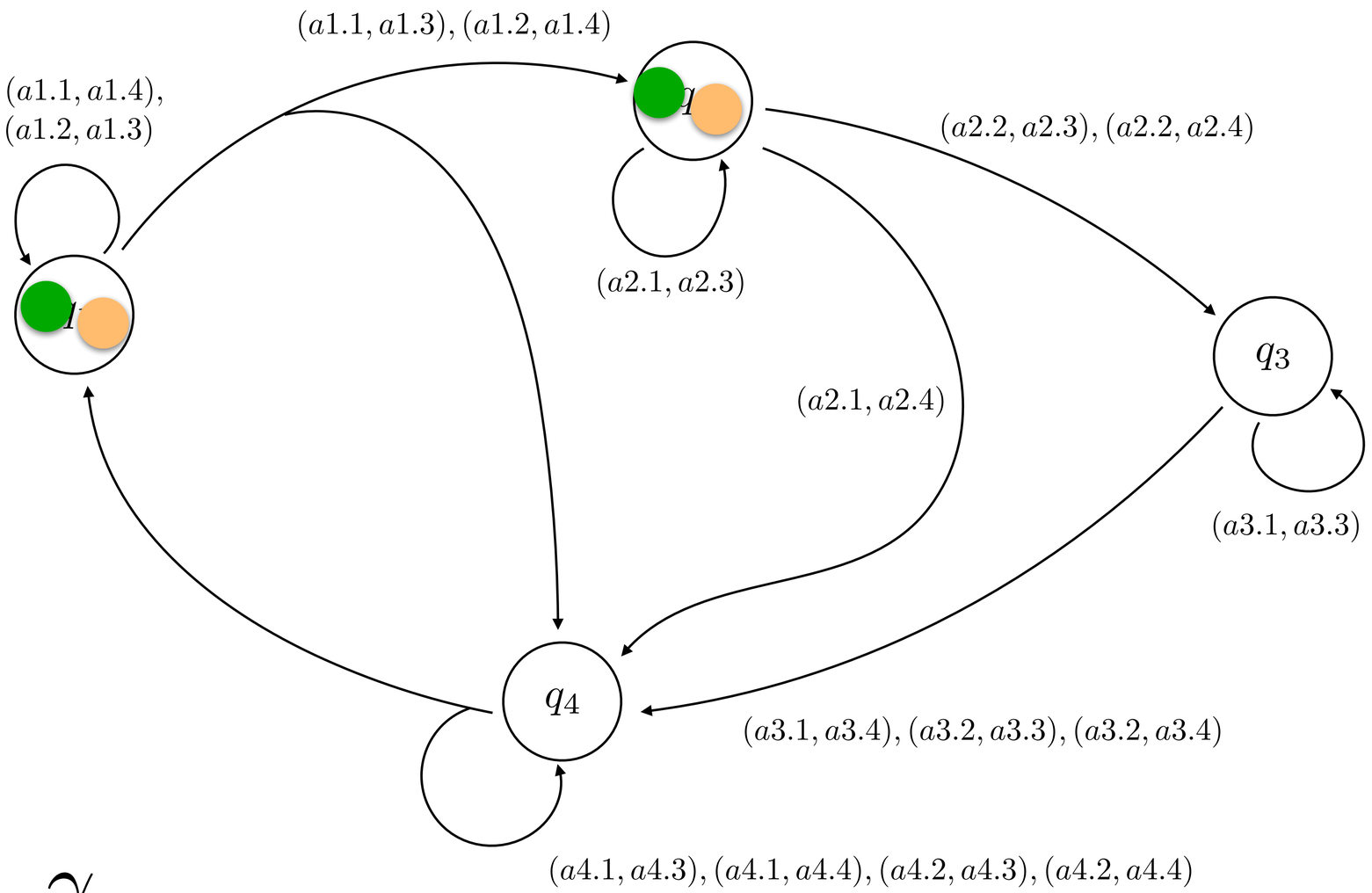
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	.4, .6	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_3

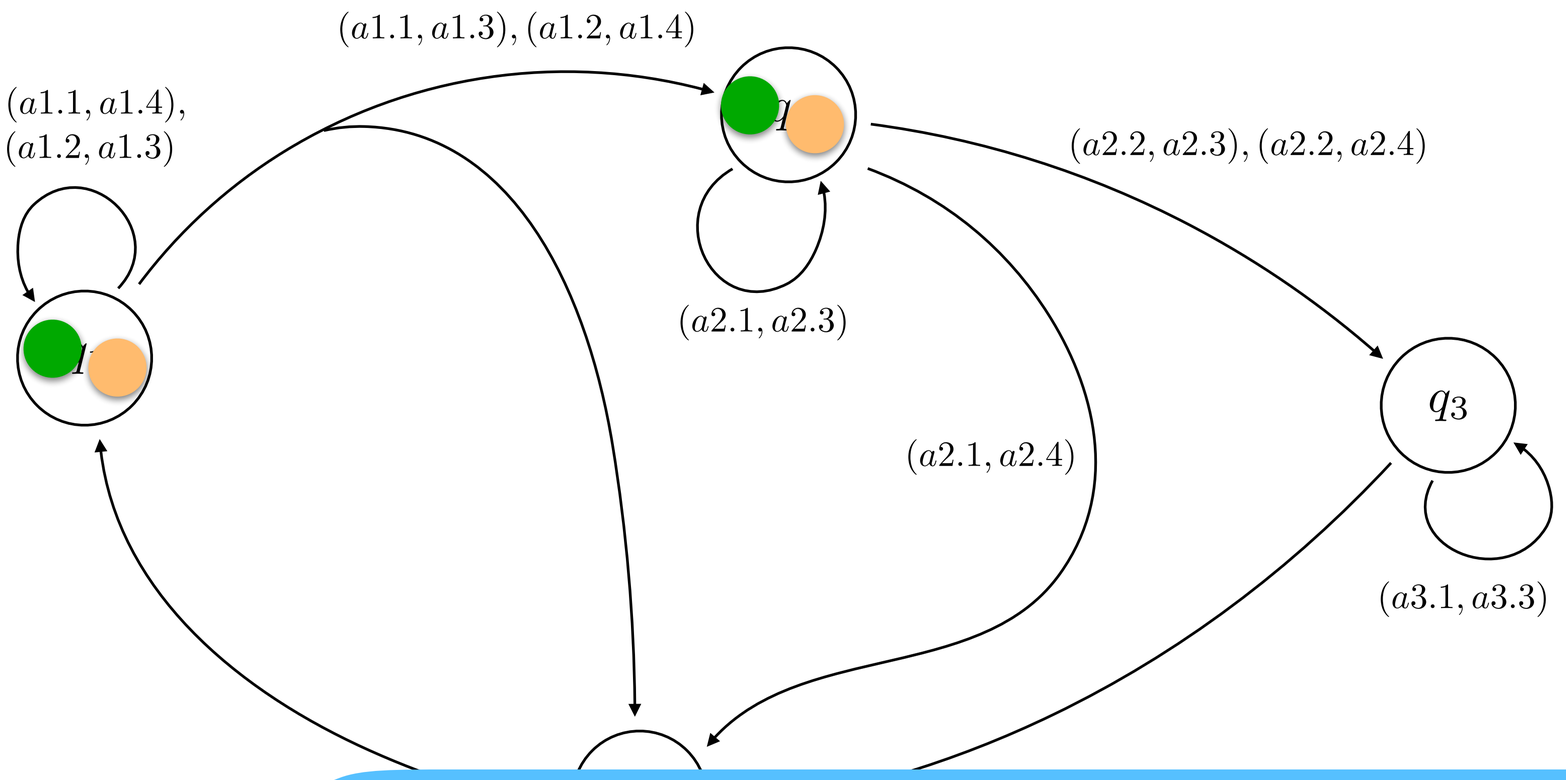
Nash Q values	a3.1	a3.2
a3.3	0, 0	0, 0
a3.4	0, 0	0, 0

q_4

Nash Q values	a4.1	a4.2
a4.3	.4, .6	0, 0
a4.4	0, 0	0, 0

An example

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1, .5	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_4

	a4.1	a4.2
	.6	0, 0
	0, 0	0, 0

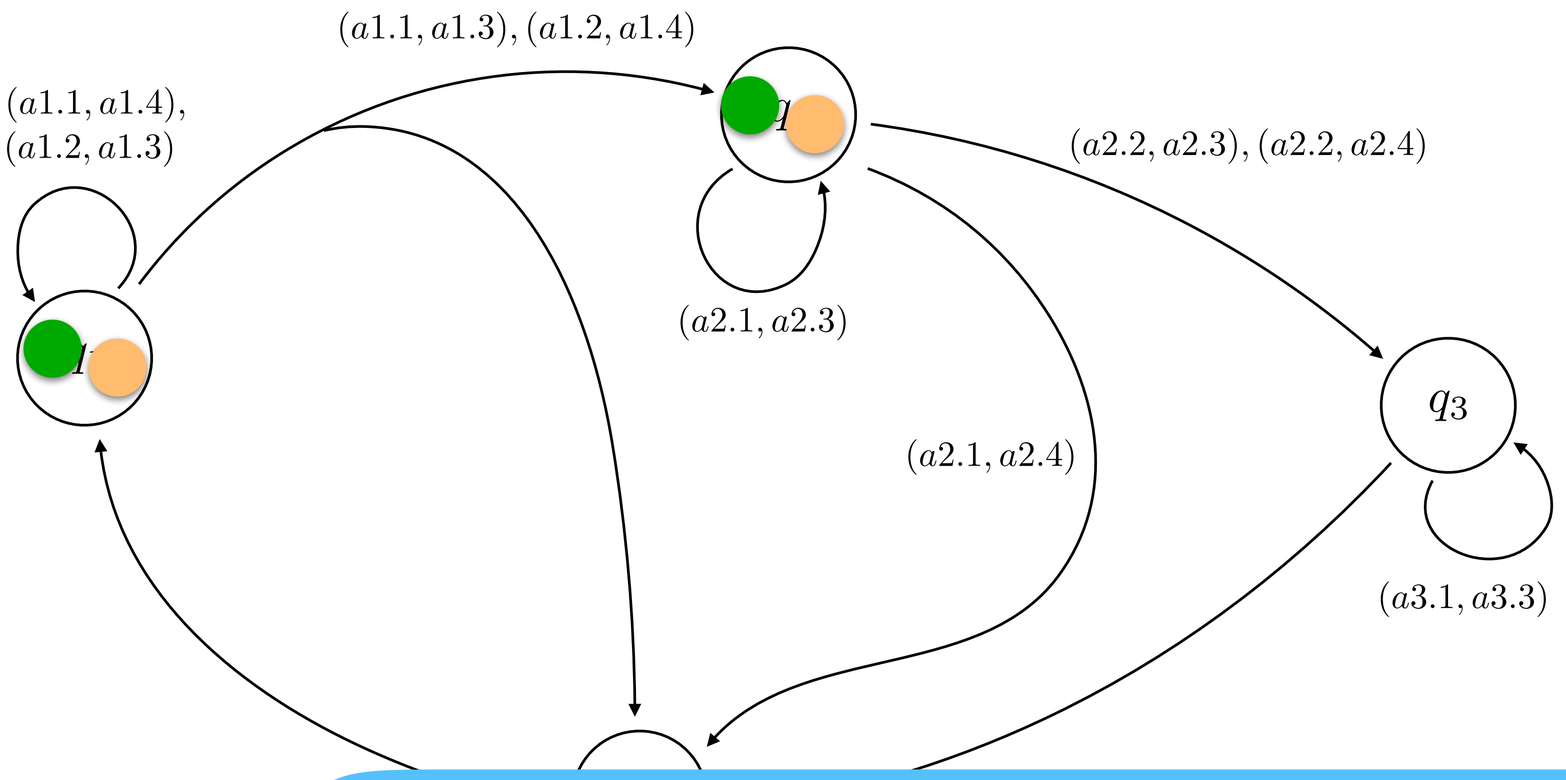
$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(1.5, 1.35) 0.5 (1, 0.5) 0.5 (2, 1) 0.8 (0, 1.5)

γ

An example

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	2, 1	0, 0
a1.4	0, 0	1, 2

q_2

Reward	a2.1	a2.2
a2.3	1, 1	3, 0
a2.4	0, 3	2, 2

q_3

Reward	a3.1	a3.2
a3.3	2, 0	0, 2
a3.4	0, 1	1, 0

q_4

Reward	a4.1	a4.2
a4.3	1, 1	0, 0
a4.4	0, 0	2, 2

q_1

Nash Q values	a1.1	a1.2
a1.3	1.5, 1.35	0, 0
a1.4	0, 0	0, 0

q_2

Nash Q values	a2.1	a2.2
a2.3	0, 0	0, 0
a2.4	0, 1.5	0, 0

q_4

	a4.1	a4.2
	.6	0, 0
	0, 0	0, 0

$$Q^i(q, a^1, \ldots, a^n) \leftarrow (1-\alpha)\, Q^i(q, a^1, \ldots, a^n) + \alpha \left(r^i(q, a^1, \ldots, a^n) + \gamma\, \text{NASH-Q}^i(q') \right)$$

(1.5, 1.35) 0.5 (1, 0.5) 0.5 (2, 1) 0.8 (0, 1.5)

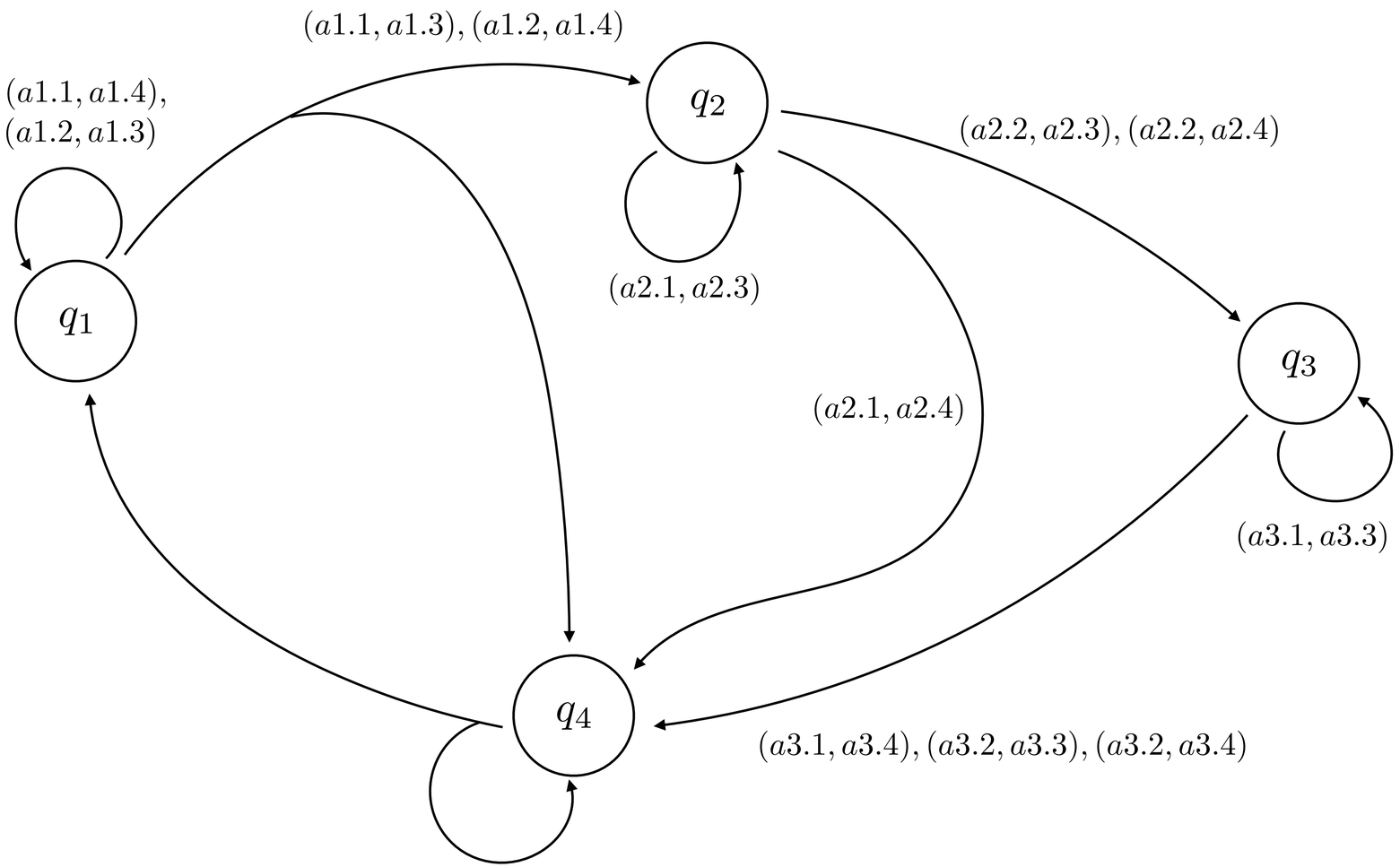
γ

Strategies Played

- **At every stage**
  - The players calculate a Nash equilibrium
  - If multiple Nash equilibria exist, they must agree on a specific Nash equilibrium
  - The actions of the players are drawn from the strategies
- **Exploration**
  - Epsilon-Nash strategy
    - With a probability of 1- epsilon, the Nash equilibrium is played
    - With a probability of epsilon, any joint strategy is played
  - Boltzmann exploration
    - Quantal equilibrium is computed in place of Nash equilibrium

Nash equilibrium

Players 1 and 2



γ

$$\sigma_1 = \begin{cases} a1.1 & 1.0 \\ a1.2 & 0.0 \end{cases} \quad \sigma_2 = \begin{cases} a1.3 & 1.0 \\ a1.4 & 0.0 \end{cases}$$

q_1

Nash Q values	a1.1	a1.2
a1.3	5, 4	2, 3
a1.4	3, 2	4, 5

$$\sigma_1 = \begin{cases} a2.1 & 0.5 \\ a2.2 & 0.5 \end{cases} \quad \sigma_2 = \begin{cases} a2.3 & 0.5 \\ a2.4 & 0.5 \end{cases}$$

q_2

Nash Q values	a2.1	a2.2
a2.3	3, 4	6, 3
a2.4	4, 5	5, 6

$$\sigma_1 = \begin{cases} a3.1 & 0.5 \\ a3.2 & 0.5 \end{cases} \quad \sigma_2 = \begin{cases} a3.3 & 0.5 \\ a3.4 & 0.5 \end{cases}$$

q_3

Nash Q values	a3.1	a3.2
a3.3	3, 4	6, 3
a3.4	4, 5	5, 6

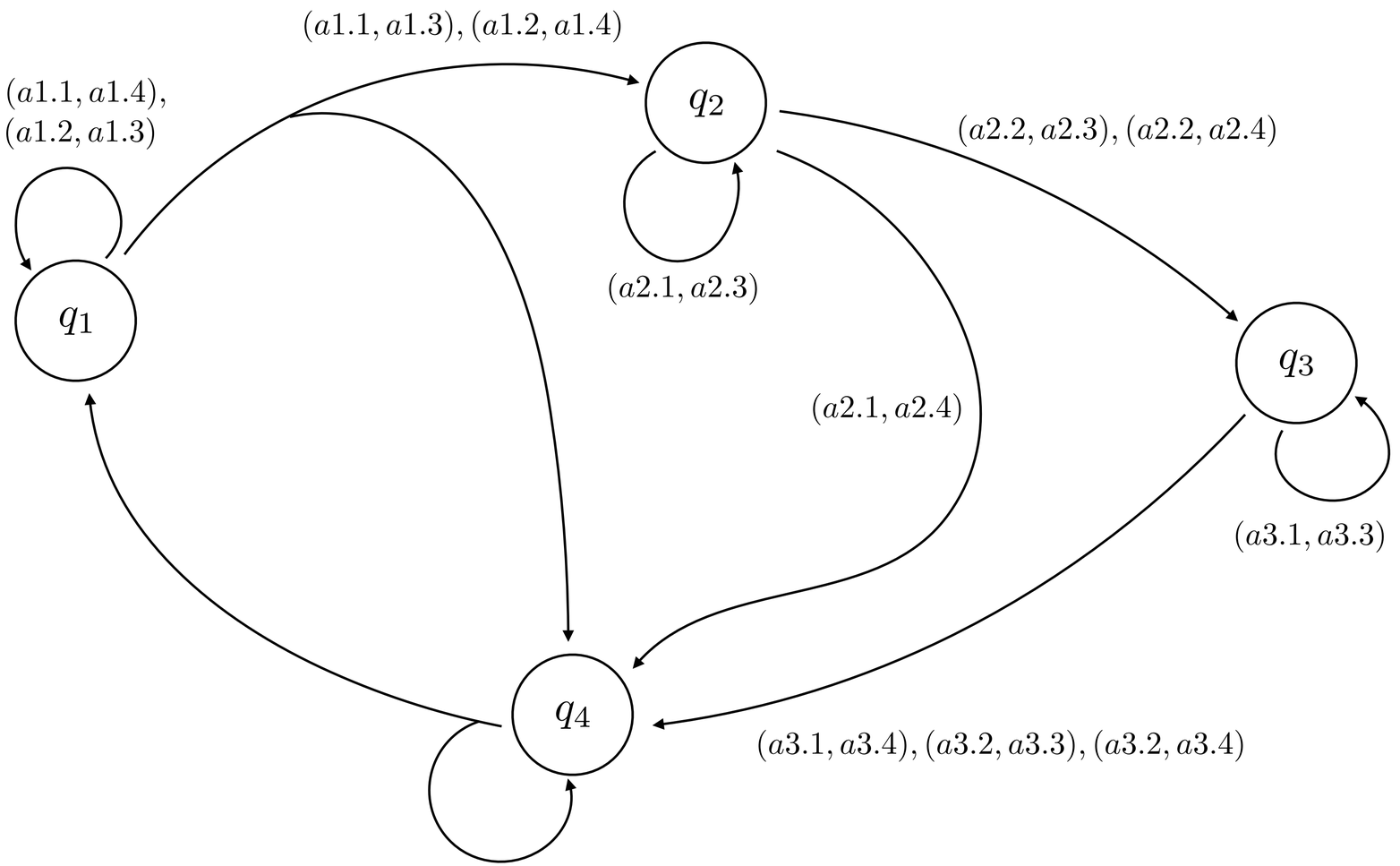
$$\sigma = \begin{cases} a4.1 & 1.0 \\ a4.2 & 0.0 \end{cases} \quad \sigma_2 = \begin{cases} a4.3 & 1.0 \\ a4.4 & 0.0 \end{cases}$$

q_4

Nash Q values	a4.1	a4.2
a4.3	5, 4	2, 3
a4.4	3, 2	5, 4

Epsilon-Nash strategy (exploration)

Players 1 and 2



γ

$$\sigma = \begin{cases} & a1.1 & a1.2 \\ a1.3 & 1-\epsilon\,3/4 & \epsilon/4 \\ a1.4 & \epsilon/4 & \epsilon/4 \end{cases}$$

$$\sigma = \begin{cases} & a2.1 & a2.2 \\ a2.3 & 0.5 & 0.5 \\ a2.4 & 0.5 & 0.5 \end{cases}$$

q_1

Nash Q values	a1.1	a1.2
a1.3	5, 4	2, 3
a1.4	3, 2	4, 5

q_2

Nash Q values	a2.1	a2.2
a2.3	3, 4	6, 3
a2.4	4, 5	5, 6

q_3

Nash Q values	a3.1	a3.2
a3.3	3, 4	6, 3
a3.4	4, 5	5, 6

q_4

Nash Q values	a4.1	a4.2
a4.3	5, 4	2, 3
a4.4	3, 2	5, 4

$$\sigma = \begin{cases} & a3.1 & a3.2 \\ a3.3 & 0.5 & 0.5 \\ a3.4 & 0.5 & 0.5 \end{cases}$$

$$\sigma = \begin{cases} & a4.1 & a4.2 \\ a4.3 & 1-\epsilon\,3/4 & \epsilon/4 \\ a4.4 & \epsilon/4 & \epsilon/4 \end{cases}$$

Convergence (1)

- Nash-Q values may not converge to the values of the Markov Nash equilibrium in general Stochastic games, even when the conditions assuring the convergence of the Q-learning are satisfied (learning rate and explorative strategies)

- A critical issue concerns the existence of multiple Nash equilibria in the stages

Convergence (2)

- In addition to the sufficient conditions for the convergence of the Q-learning algorithm, we require, for the convergence of the Nash-Q learning algorithm, one of the following
  - **Case 1**
    - For every game, there is a global optimum: a joint strategy providing the maximum possible reward (of that game) to all the players (this global optimum is a Nash equilibrium)
    - The global optima must be chosen as Nash equilibria during the whole learning dynamics
  - **Case 2**
    - For every game, there is a saddle point: a joint strategy that is a Nash equilibrium and is such that the reward of every player increases if at least an opponent deviates
    - The saddle points must be chosen as Nash equilibria during the whole learning dynamics

Examples

Reward	a1.1	a1.2
a1.3	3, 2	0, 1
a1.4	0, 0	1, 0

Global optimum

Reward	a1.1	a1.2
a1.3	5, 5	0, 6
a1.4	6, 0	2, 2

Saddle point

Reward	a1.1	a1.2
a1.3	6, 6	0, 3
a1.4	3, 0	2, 2

Global optimum

Saddle point

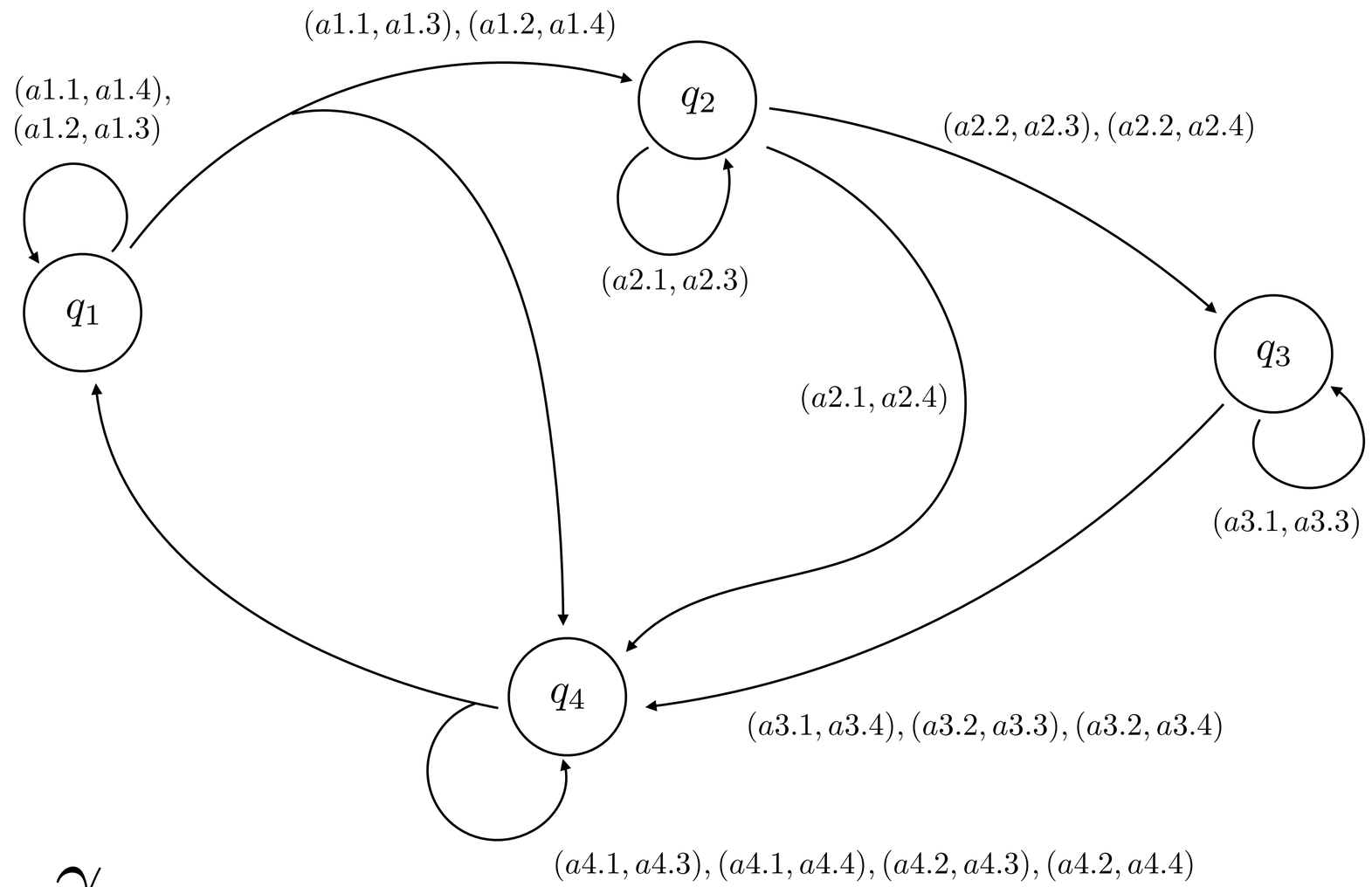
Convergence (3)

- In practice, it is sufficient that, in every stage game the same Nash equilibrium is played by all the players from a given time point on

Minimax-Q learning

2-player zero-sum stochastic game

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

γ

Minimax-Q learning (1)

- In every state, the Q-table stores the Q-values defined on the joint actions of player max

q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3		
a1.4		

q_2

Nash Q values	a2.1	a2.2
a2.3		
a2.4		

q_3

Nash Q values	a3.1	a3.2
a3.3		
a3.4		

q_4

Nash Q values	a4.1	a4.2
a4.3		
a4.4		

Minimax-Q learning (2)

- The update of the Nash-Q values of every player is as follows:

$$Q(q, a^1, a^2) \leftarrow (1-\alpha)\, Q(q, a^1, a^2) + \alpha \left(r(q, a^1, a^2) + \gamma\, \textsc{Maxmin-Q}^i(q') \right)$$

Game

Action of player *j*

Reward gotten in game *q* when playing the given action profile

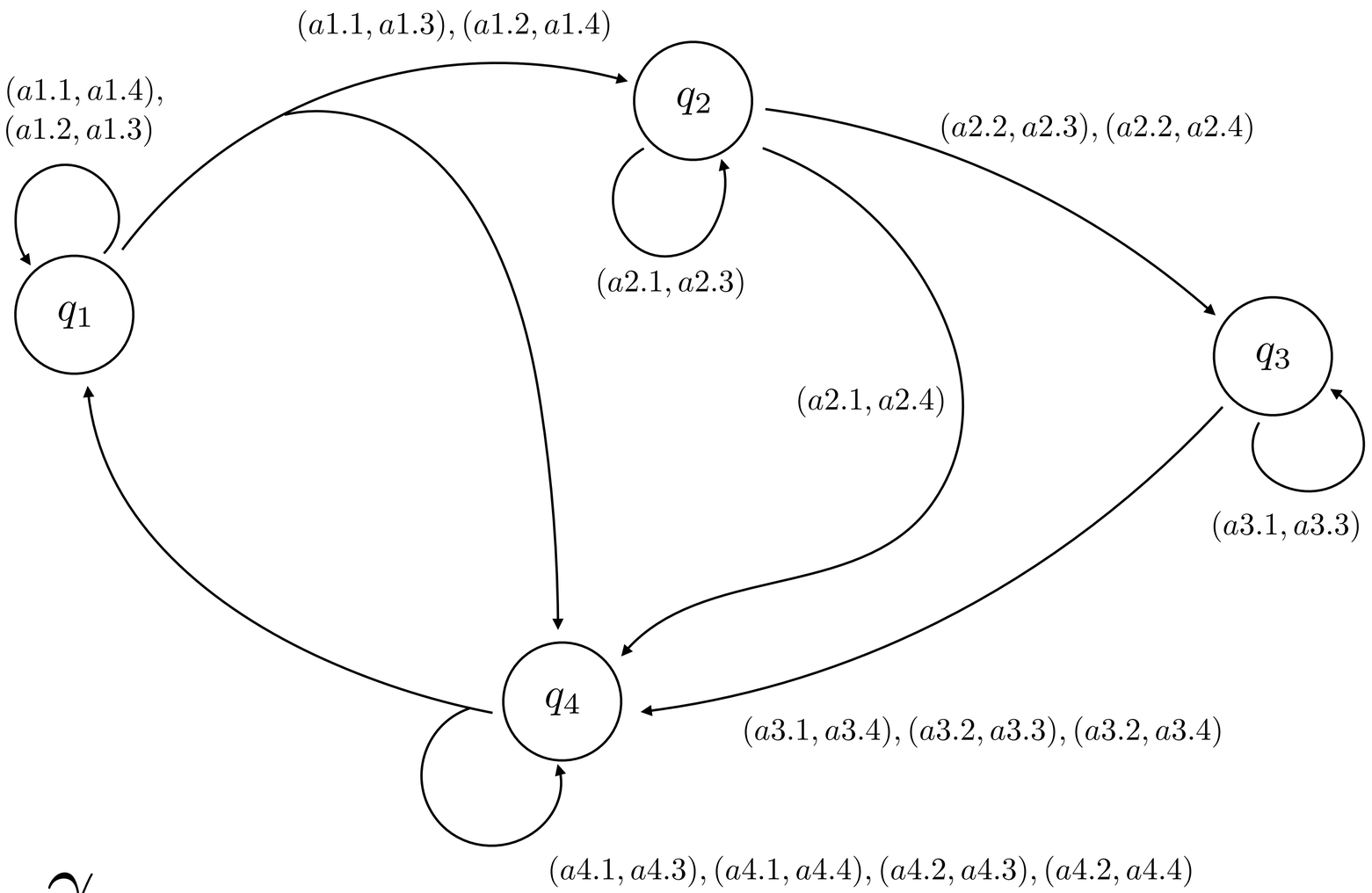
Value for player *i* of the MaxMin equilibrium computed at state *q'*

Convergence

- Minimax-Q converges the Markov Nash equilibrium in every zero-sum Stochastic game under the same assumptions for the convergence of Q-learning in stationary MDP games

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3	0	0
a1.4	0	0

q_2

Nash Q values	a2.1	a2.2
a2.3	0	0
a2.4	0	0

q_3

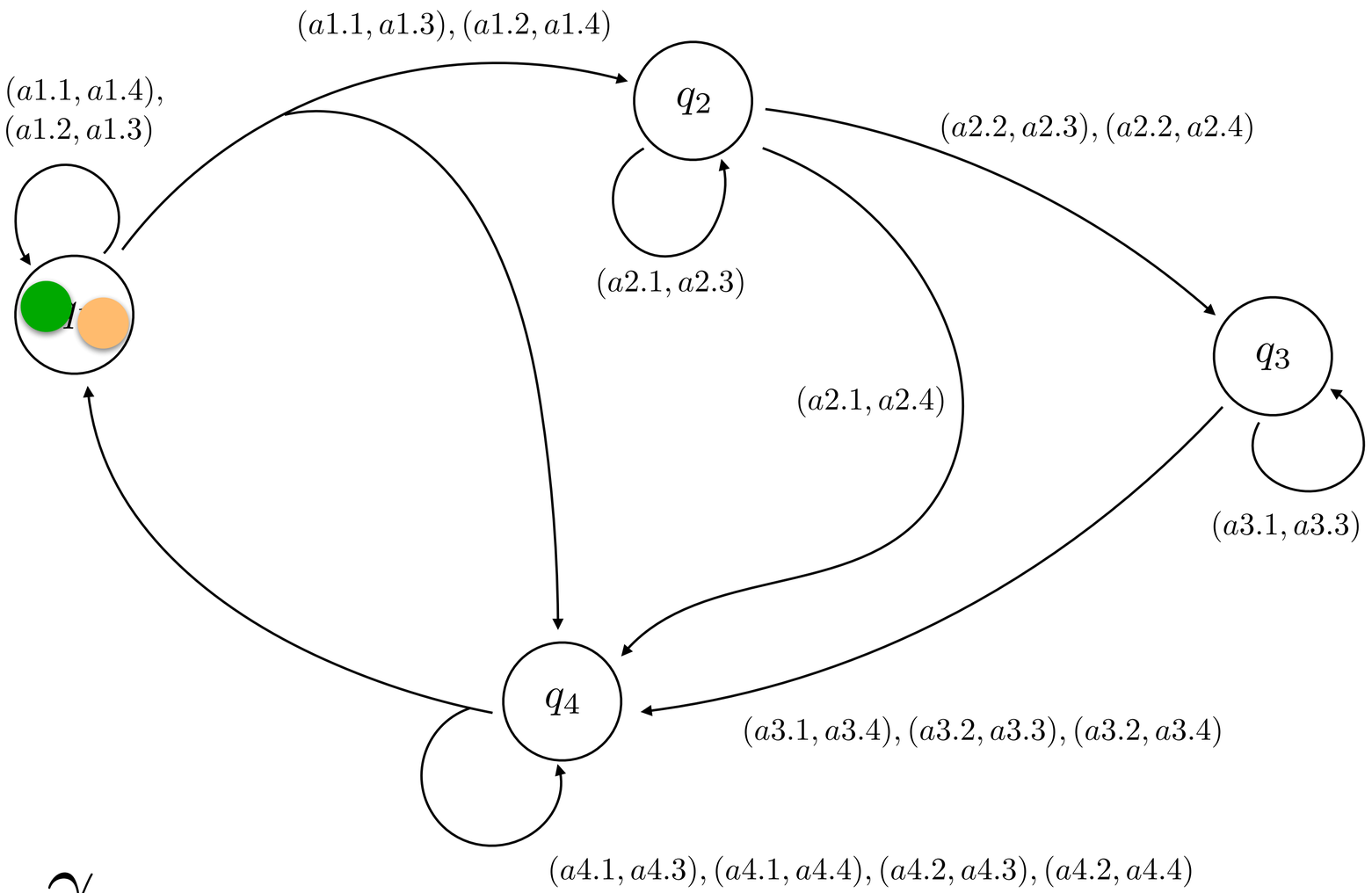
Nash Q values	a3.1	a3.2
a3.3	0	0
a3.4	0	0

q_4

Nash Q values	a4.1	a4.2
a4.3	0	0
a4.4	0	0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3	0	0
a1.4	0	0

q_2

Nash Q values	a2.1	a2.2
a2.3	0	0
a2.4	0	0

q_3

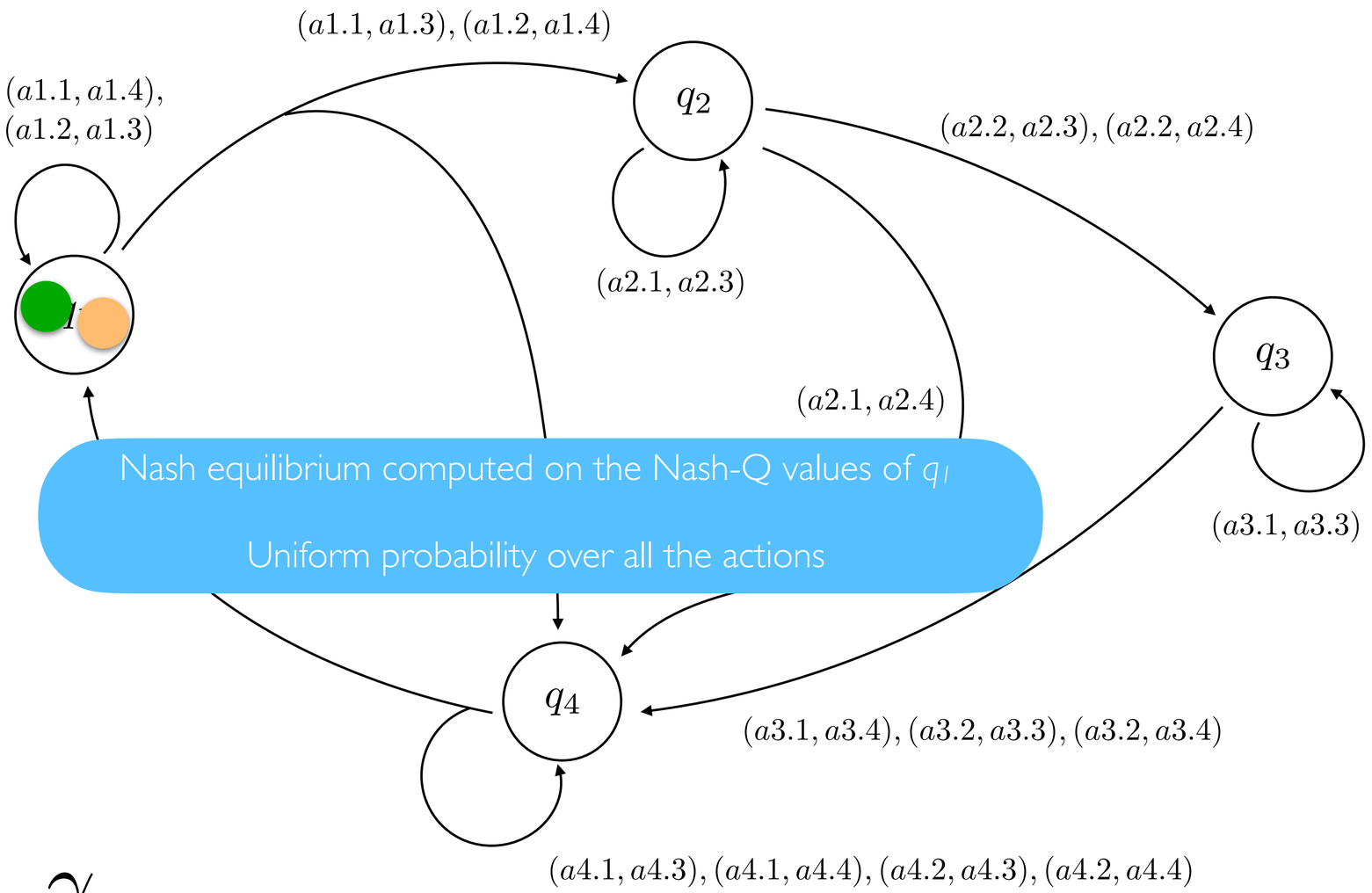
Nash Q values	a3.1	a3.2
a3.3	0	0
a3.4	0	0

q_4

Nash Q values	a4.1	a4.2
a4.3	0	0
a4.4	0	0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3	0	0
a1.4	0	0

q_2

Nash Q values	a2.1	a2.2
a2.3	0	0
a2.4	0	0

q_3

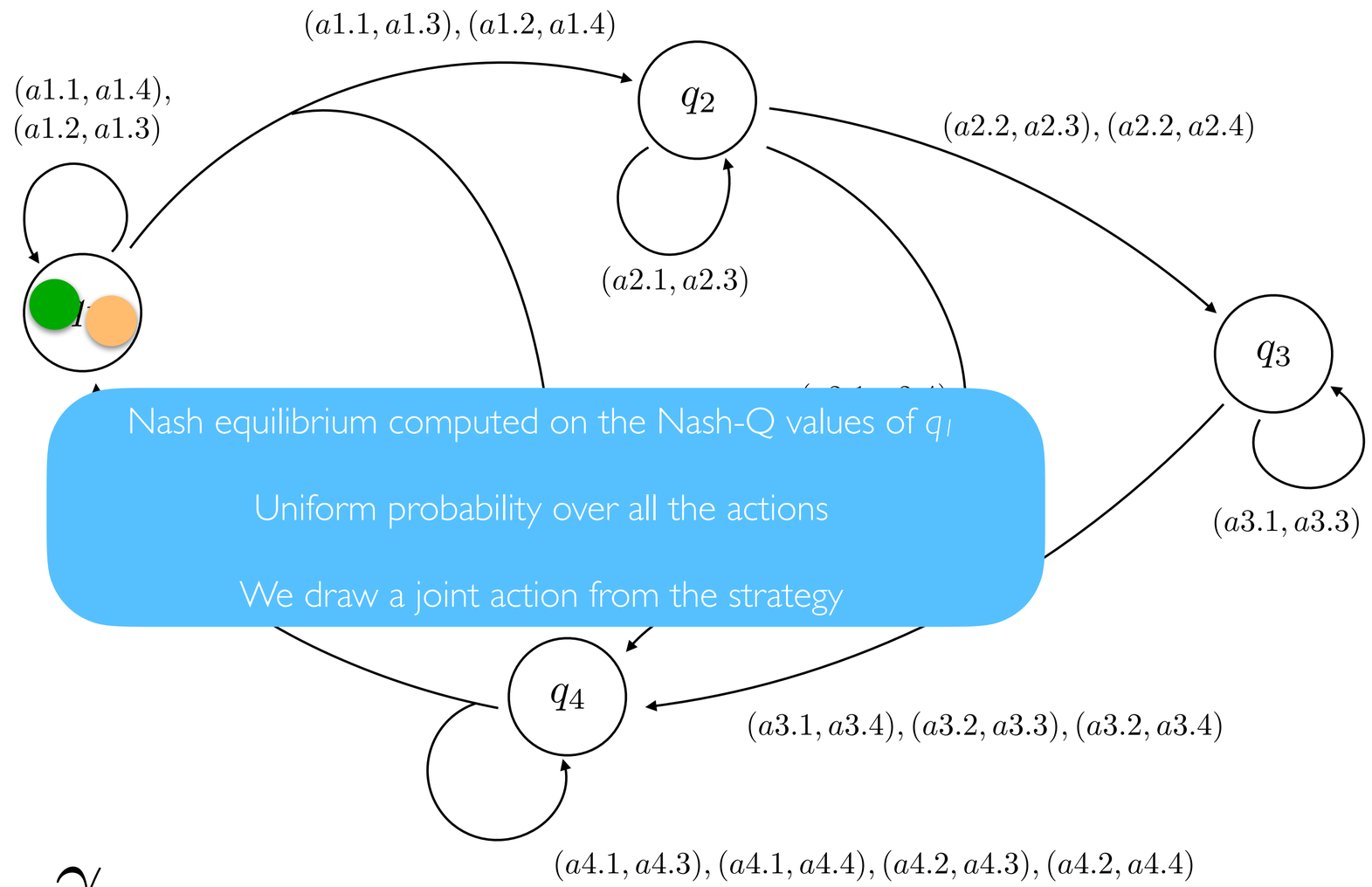
Nash Q values	a3.1	a3.2
a3.3	0	0
a3.4	0	0

q_4

Nash Q values	a4.1	a4.2
a4.3	0	0
a4.4	0	0

An example

Players 1 and 2



γ

q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3	0	0
a1.4	0	0

q_2

Nash Q values	a2.1	a2.2
a2.3	0	0
a2.4	0	0

q_3

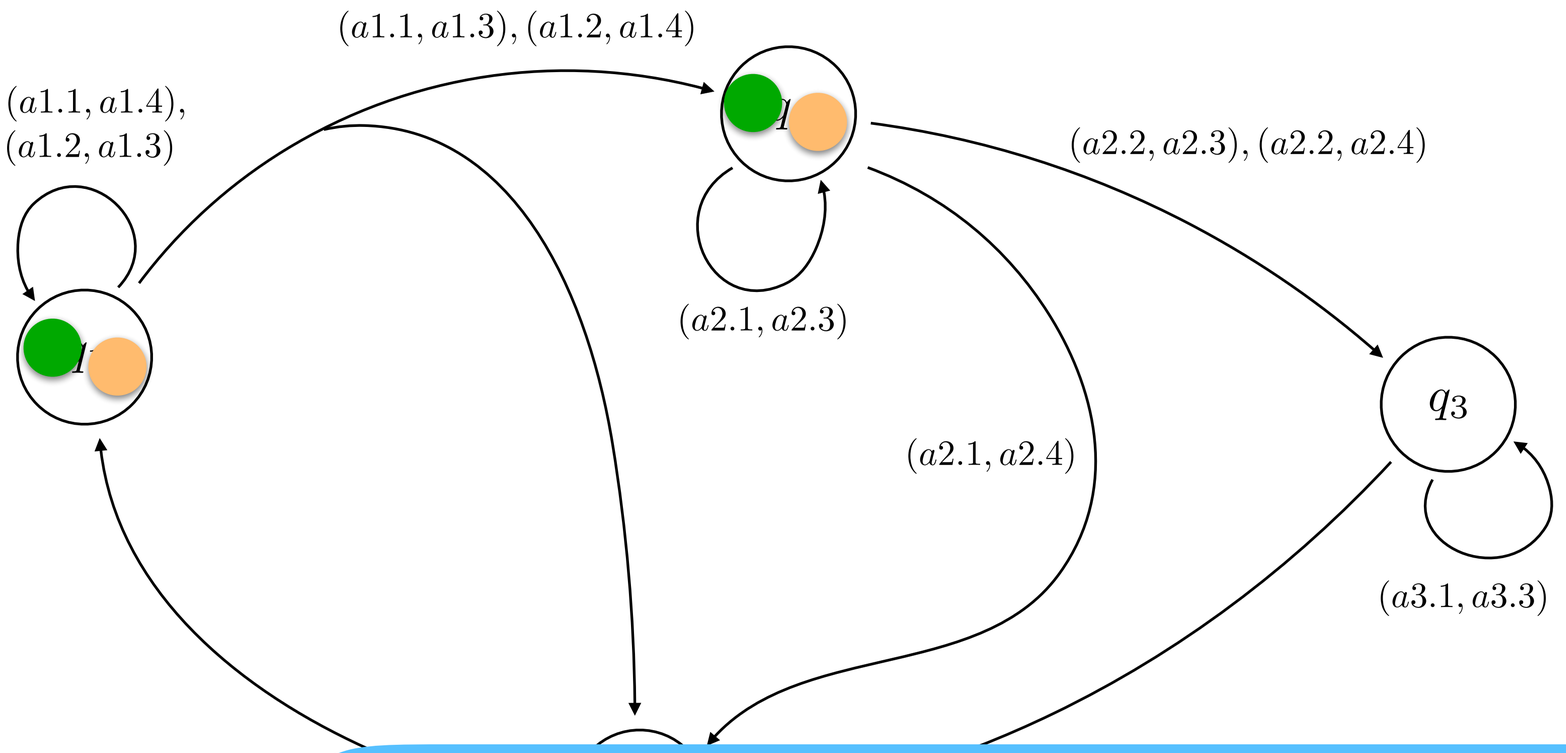
Nash Q values	a3.1	a3.2
a3.3	0	0
a3.4	0	0

q_4

Nash Q values	a4.1	a4.2
a4.3	0	0
a4.4	0	0

An example

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3	0	0
a1.4	0	0

q_2

Nash Q values	a2.1	a2.2
a2.3	0	0
a2.4	0	0

q_4

	a4.1	a4.2
	0	0
	0	0

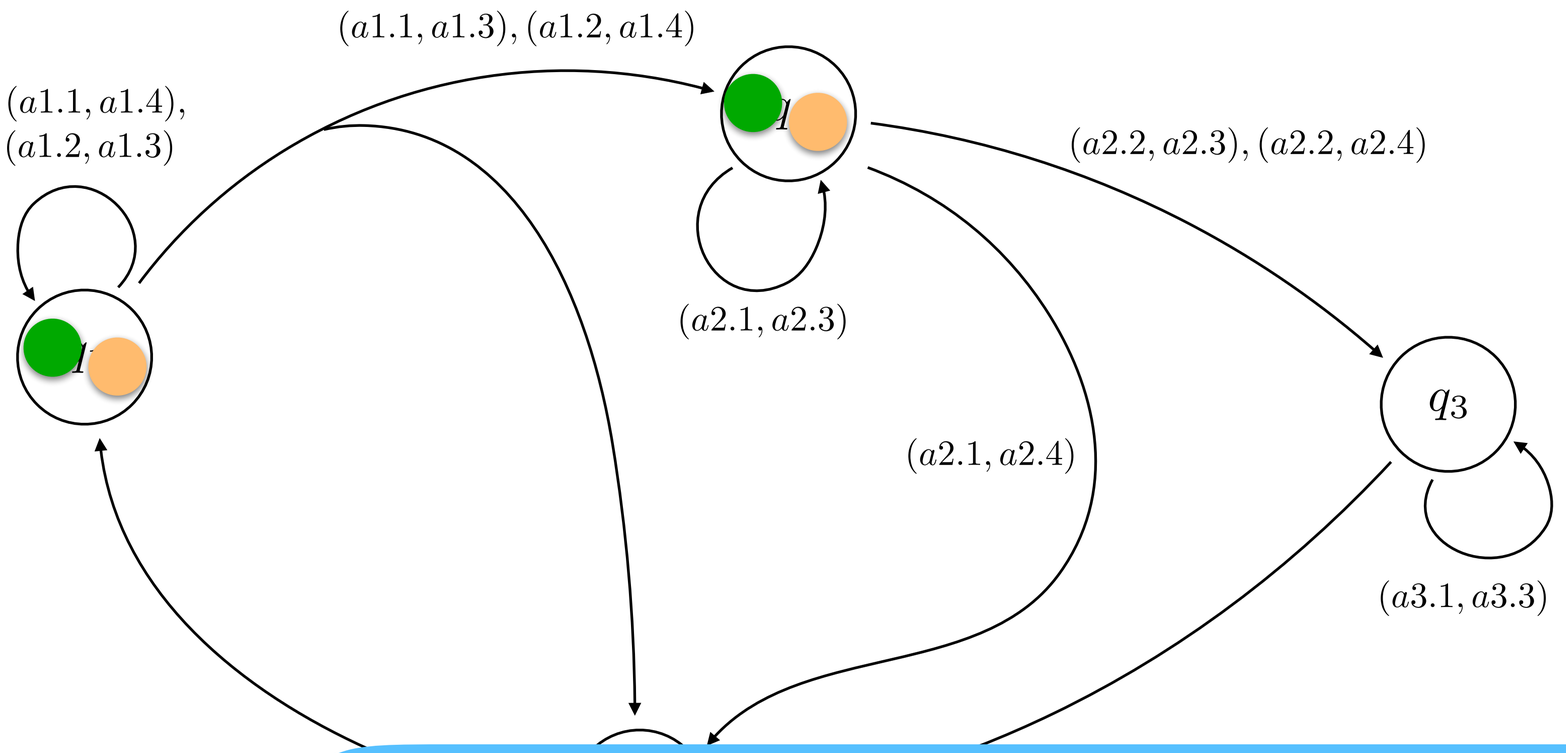
$$Q(q, a^1, a^2) \leftarrow (1-\alpha)\, Q(q, a^1, a^2) + \alpha \left(r(q, a^1, a^2) + \gamma\, \textsc{Maxmin-Q}^i(q') \right)$$

0.5 0.5 0 0.5 1 0.8 0

γ

An example

Players 1 and 2



q_1

Reward	a1.1	a1.2
a1.3	1	-1
a1.4	-1	1

q_2

Reward	a2.1	a2.2
a2.3	2	-1
a2.4	3	2

q_3

Reward	a3.1	a3.2
a3.3	2	-1
a3.4	-3	2

q_4

Reward	a4.1	a4.2
a4.3	1	-1
a4.4	-1	1

q_1

Nash Q values	a1.1	a1.2
a1.3	**0.5**	0
a1.4	0	0

q_2

Nash Q values	a2.1	a2.2
a2.3	0	0
a2.4	0	0

q_4

	a4.1	a4.2
	0	0
	0	0

$$Q(q, a^1, a^2) \leftarrow (1-\alpha)\, Q(q, a^1, a^2) + \alpha \left(r(q, a^1, a^2) + \gamma\, \textsc{Maxmin-Q}^i(q') \right)$$

0.5 0.5 0 0.5 1 0.8 0

γ