Fourier Transform

Arnaldo Spalvieri

Dipartimento di Elettronica, Informazione e Bioingegneria

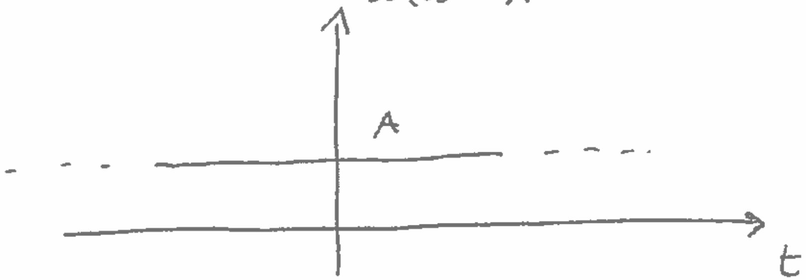
Politecnico di Milano, ITALY

Continuos-time Signals

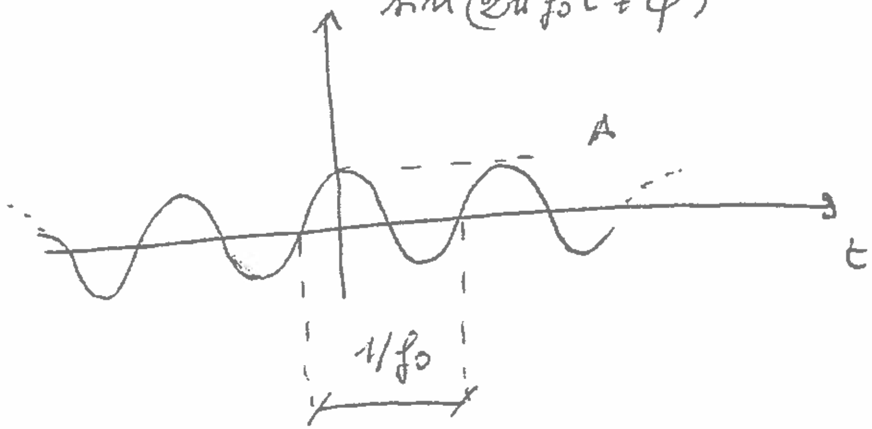
Examples of continuos-time signals are

- DC of amplitude A: $x(t) = A$
- AC (sinusoid) at frequency f_0 with amplitude A: $x(t) = A\sin(2\pi f_0 t + \phi)$
- complex sinusoid, that is two quadrature sinusoids on two separate couples of wires, at frequency f_0 with amplitude $A: x(t) = Ae^{j2\pi f_0 t} = A\cos(2\pi f_0 t) + jA\sin(2\pi f_0 t)$
- rectangular impulse between $-T/2$ and $T/2$ and amplitude A: $x(t) = A\, rect\left(\frac{t}{T}\right)$,
- cardinal sine wave with amplitude A and zeros spaced T seconds: $x(t) = A\frac{sin(\pi t/T)}{(\pi t/T)}$, will be denoted as $\frac{sin(\pi t/T)}{(\pi t/T)} = sinc\left(\frac{t}{T}\right)$

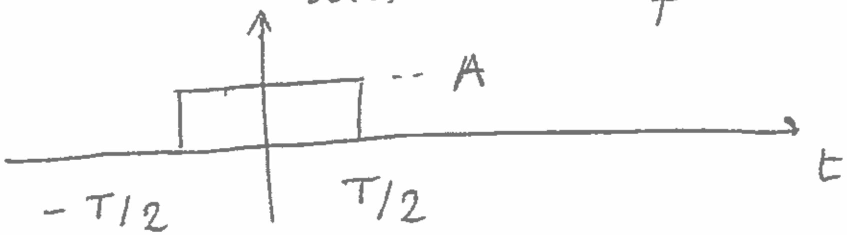
$x(t) = A$



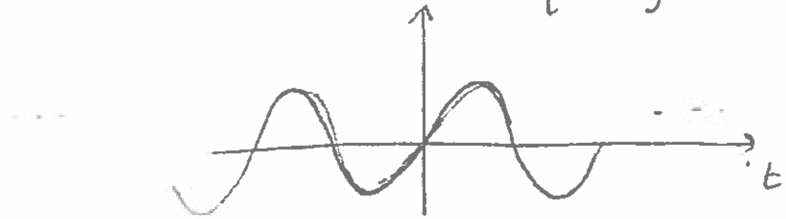
$\sin(2\pi f_0 t + \varphi)$



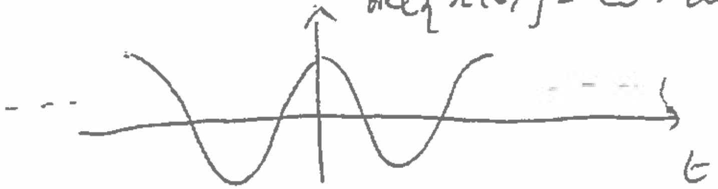
$x(t) = A \,\mathrm{rect}\, \frac{t}{T}$



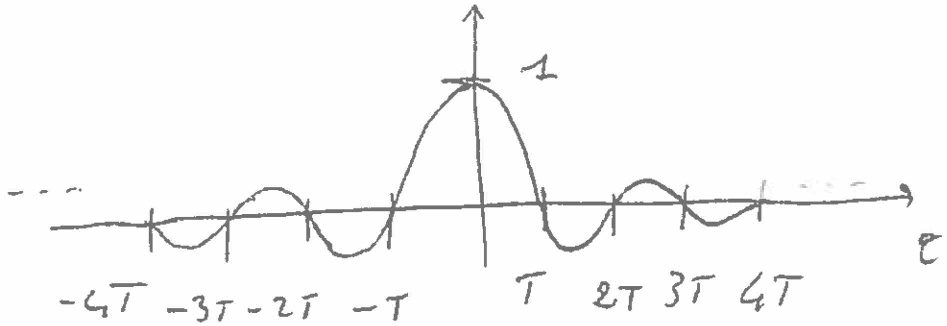
$\mathrm{Im}\{x(t)\} = \sin 2\pi f_0 t$



$\mathrm{Re}\{x(t)\} = \cos 2\pi f_0 t$



$x(t) = \mathrm{sinc}\, \frac{t}{T}$



Continuos-time Signals

Particular signals are

- unit step, $u(t)$, defined as $u(t) = 0,\ t < 0;\ \ u(t) = 1,\ t > 0;\ \ u(t) = 0.5,\ t = 0.$
  The unit step allows, for instance, to write the transient of the amplitude during the charge of an RC circuit:
  $$x(t) = (1 - e^{-\frac{t}{RC}})u(t).$$

- Dirac delta: $\delta(t) = \lim_{T\to 0} \frac{1}{T} rect\left(\frac{t}{T}\right)$. Note that, in the limit, the amplitude $\to \infty$, but the area is fixed because $\int_{-\infty}^{\infty} \frac{1}{T} rect\left(\frac{t}{T}\right) dt = 1$. Also,
  $$\delta(t) = \frac{du(t)}{dt}.$$

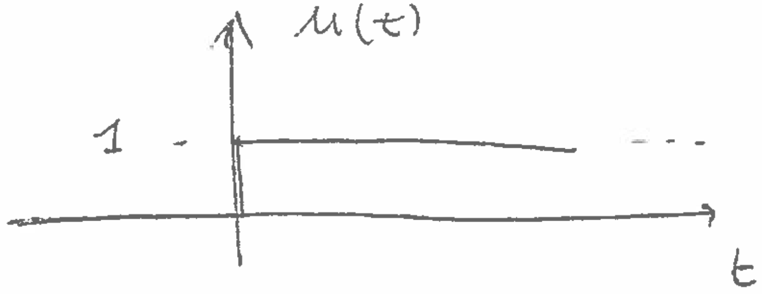
Energy and Power Signals

We divide continuous-time signals in energy signals and power signals depending on existence and positivity of the following limits

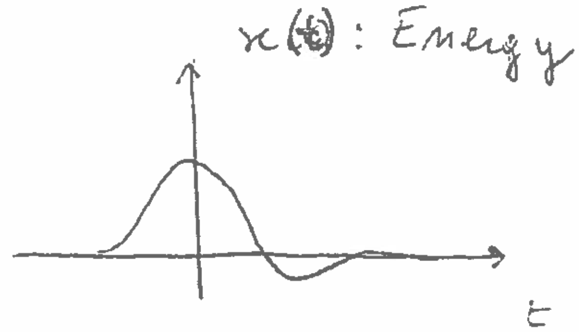
$$E_x = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{1}$$

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2}$$

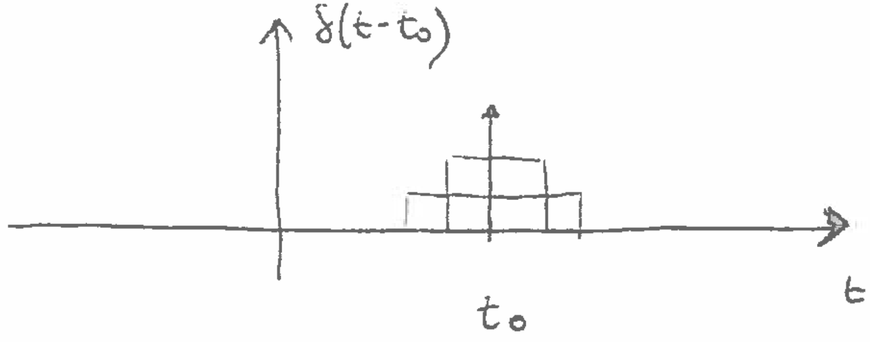
The cardinal sine, the decaying exponential, and the rectangular pulse are energy signals. Dirac delta is borderline, it is an impulsive signal, a third class of signals. The unit step, the AC and the DC are power signals. Random signals like noise and periodic signals are power signals, because, according to their mathematical model, their duration occupy all the time between $-\infty$ and ∞, hence only integral (2) converges.



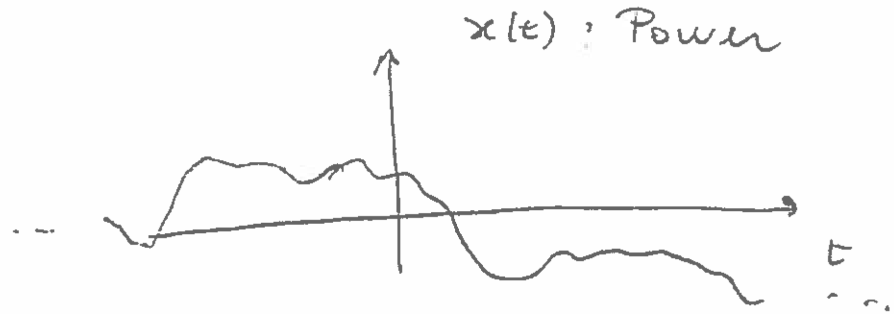
u(t)
1
t



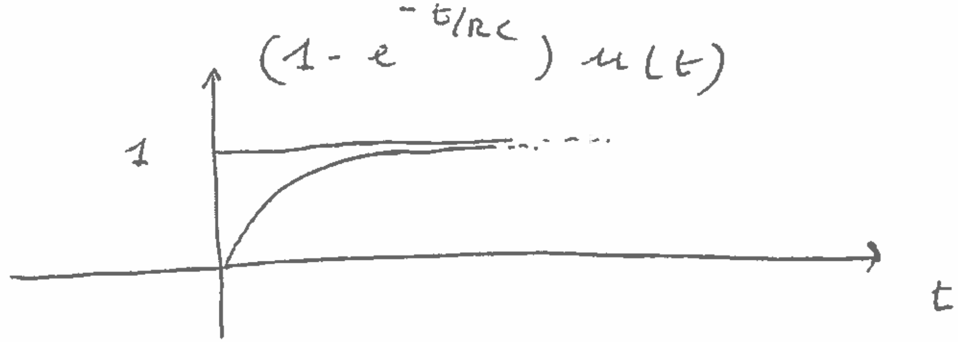
x(t) : Energy
t



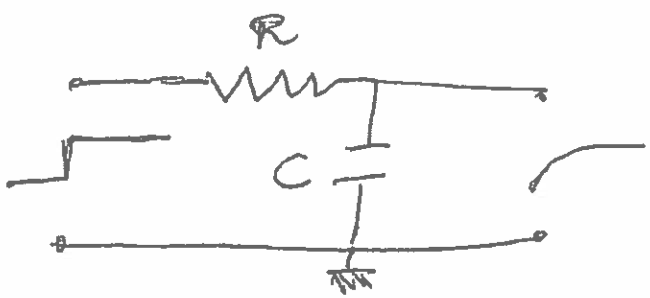
δ(t-t0)
t0
t



x(t) : Power
t



(1 - e^(-t/RC)) u(t)
1
t



R
C

Fourier Transform

The Fourier transform of a continuous-time signal is a linear transformation made on the signal. It is a mathematical operation that allows to represent the signal through its transform, that is defined as

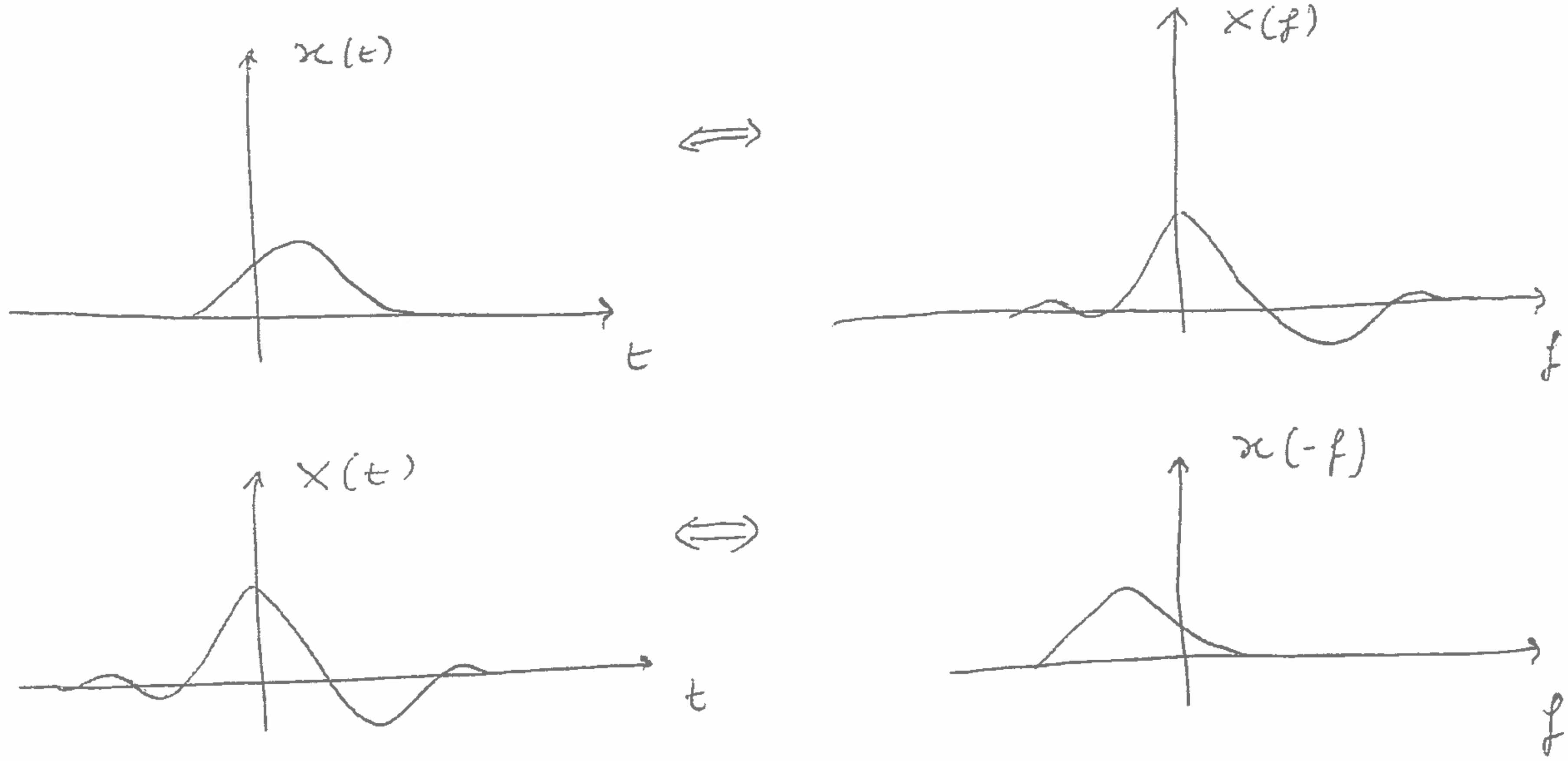
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt. \tag{3}$$

From the frequency-domain representation $X(f)$ of the signal one can come back to the time-domain representation of the signal by the inverse Fourier transform:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df. \tag{4}$$

$X(f)$ and $x(t)$ are said to be a *Fourier pair*. This relationship is often denoted as

$$x(t) \Longleftrightarrow X(f).$$



The concept of duality

Properties of the Fourier Transform

Given the Fourier pairs

$$x(t) \Longleftrightarrow X(f), \;\; y(t) \Longleftrightarrow Y(f),$$

the complex scalars a and b, and the real scalars t_0, f_0, α, it can be shown that

- Linearity: $ax_1(t) + bx_2(t) \Longleftrightarrow aX_1(f) + bX_2(f)$
- Duality: $X(t) \Longleftrightarrow x(-f)$
- Time scaling: $x(\alpha t) \Longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$
- Time shifting: $x(t - t_0) \Longleftrightarrow X(f)e^{-j2\pi f t_0}$
- Frequency shifting (the dual of time shifting): $x(t)e^{j2\pi f_0 t} \Longleftrightarrow X(f - f_0)$

Properties of the Fourier Transform

- Differentiation in time: $\frac{dx(t)}{dt} \Longleftrightarrow j2\pi f X(f)$
- Integration in time: $\int_{-\infty}^{t} x(t)dt \Longleftrightarrow \frac{X(f)}{j2\pi f} + \frac{X(0)\delta(f)}{2}$
- Conjugation: $x^*(t) \Longleftrightarrow X^*(-f)$
- Multiplication in time: $x(t)y(t) \Longleftrightarrow \int_{-\infty}^{\infty} X(\nu)Y(f-\nu)d\nu$
- Convolution in time (dual of multiplication): $\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \Longleftrightarrow X(f)Y(f)$
- One important property of the delta function is

$$y(t) = x(t) \otimes \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(\tau - t_0)x(t-\tau)d\tau = x(t-t_0), \qquad (5)$$

where $\otimes$ is a shorthand notation for convolution. Actually, one proves (5) by the properties of convolution (6) and time shift (7):

$$Y(f) = X(f)e^{-j2\pi f t_0} \Longleftrightarrow x(t) \otimes \delta(t-t_0), \qquad (6)$$

$$Y(f) = X(f)e^{-j2\pi f t_0} \Longleftrightarrow x(t-t_0). \qquad (7)$$

Useful Equalities

Given the Fourier pair

$$x(t) \Longleftrightarrow X(f)$$

and the real scalar T, the following equalities can be demonstrated.

- Rayleigh's energy theorem: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$
- Area under $x(t)$ and its dual: $\int_{-\infty}^{\infty} x(t) dt = X(0), \quad \int_{-\infty}^{\infty} X(f) df = x(0)$
- Poisson's sum formula: $\sum_{i=-\infty}^{\infty} x(iT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{k}{T}\right)$
- When $x(t)$ is real, its Fourier transform is Hermitian: $X(f) = X^*(-f)$

Examples of Fourier Pairs

$$1 \Longleftrightarrow \delta(f), \;\; \delta(t) \Longleftrightarrow 1$$

$$u(t) \Longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$e^{j2\pi f_0 t} \Longleftrightarrow \delta(f - f_0), \;\; \delta(t-\tau) \Longleftrightarrow e^{-j2\pi f \tau}$$

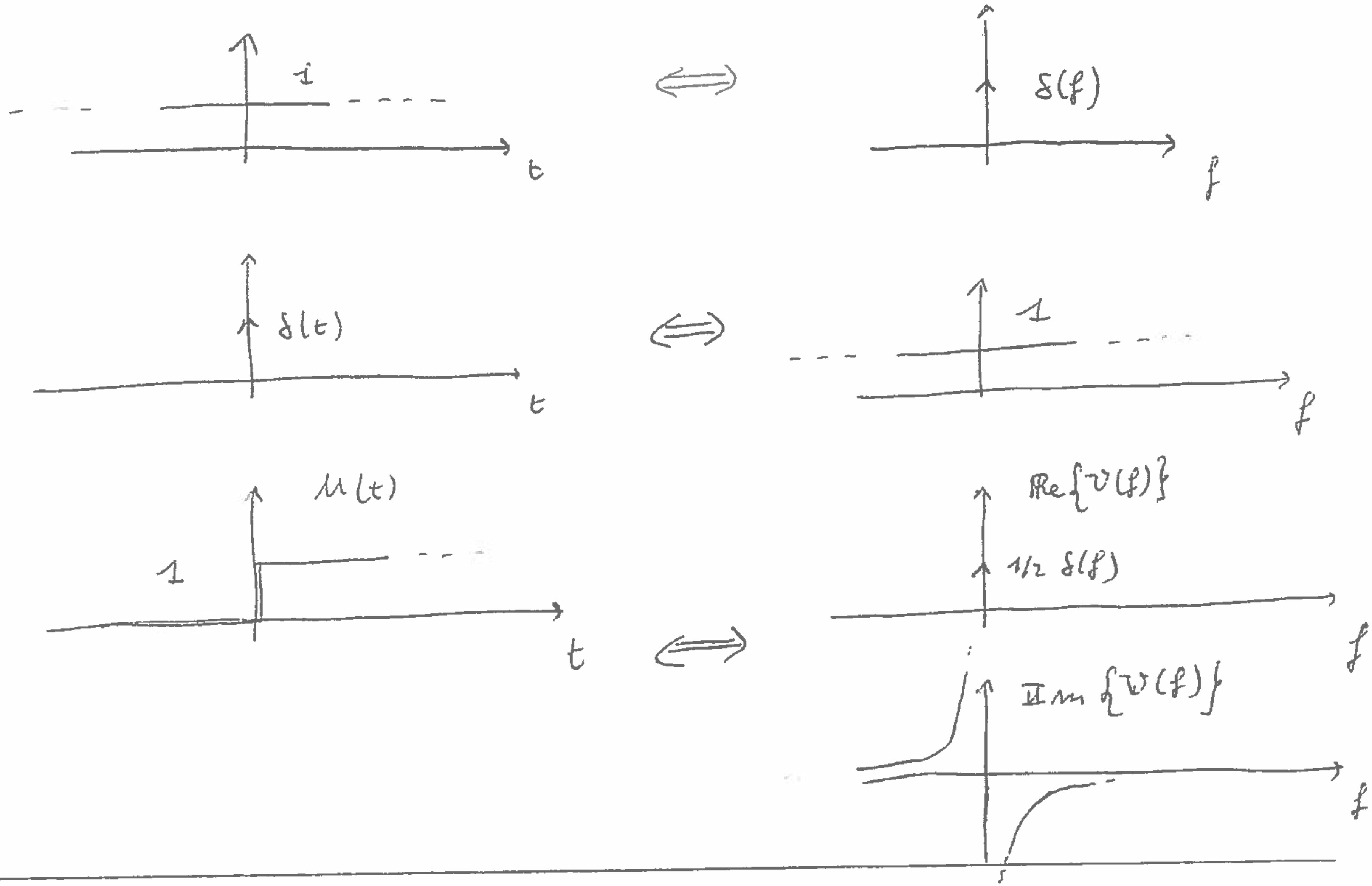
$$A\sin(2\pi f_0 t) \Longleftrightarrow -j\frac{A}{2}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

$$A\cos(2\pi f_0 t) \Longleftrightarrow \frac{A}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

Note that

$$e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t) \Longleftrightarrow$$

$$\frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) + j\left(-j\frac{1}{2}\left(\delta(f-f_0) - \delta(f+f_0)\right)\right) = \delta(f-f_0)$$


1
t
δ(f)
f
δ(t)
t
1
f
u(t)
1
t
Re{V(f)}
1/2 δ(f)
f
Im{V(f)}
f

Examples of Fourier Pairs

Rectangular impulse

$$A rect\left(\frac{t}{T}\right) \Longleftrightarrow AT sinc(fT)$$

RF rectangular impulse

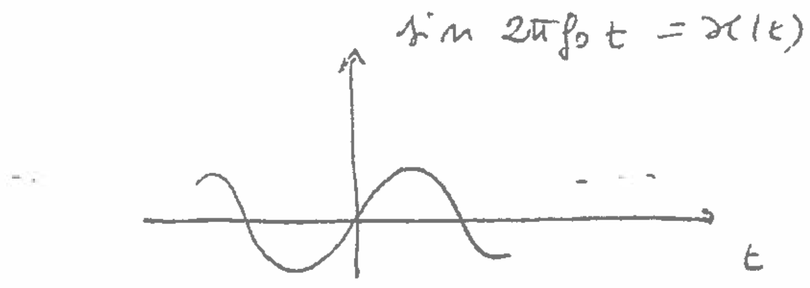
$$A\cos(2\pi f_0 t) rect\left(\frac{t}{T}\right) \Longleftrightarrow \frac{AT}{2}\left(sinc((f-f_0)T) + sinc((f+f_0)T)\right)$$

Cardinal sine impulse

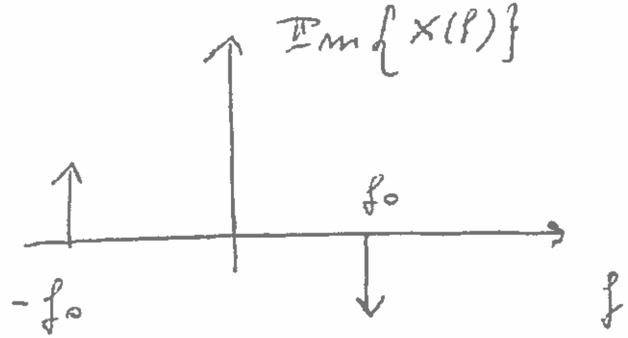
$$A sinc\left(\frac{t}{T}\right) \Longleftrightarrow AT rect(fT)$$

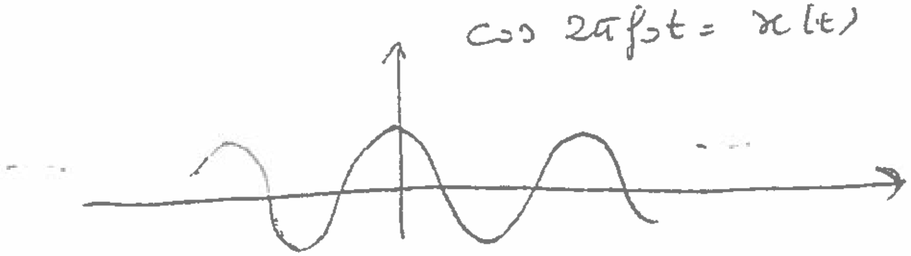
One-sided decaying exponential impulse (output of the RC circuit with $\delta(t)$ as input)

$$\frac{1}{RC} e^{\frac{-t}{RC}} u(t) \Longleftrightarrow \frac{1}{1 + j2\pi f RC}.$$

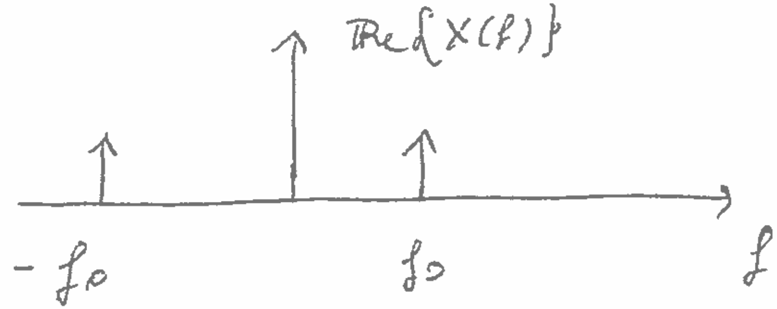


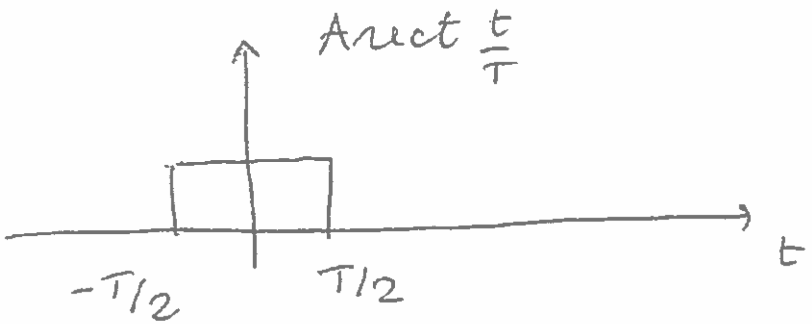
$\Longleftrightarrow$



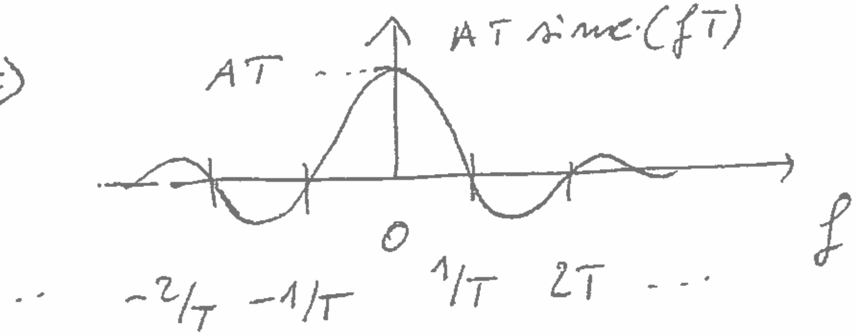


$\Longleftrightarrow$

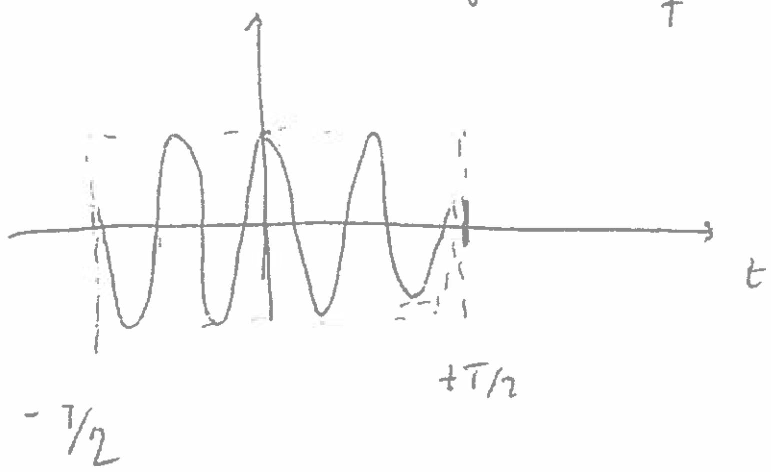




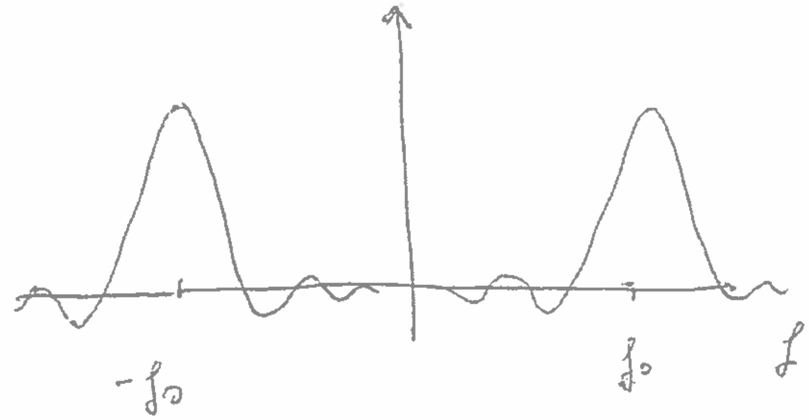
$\Longleftrightarrow$



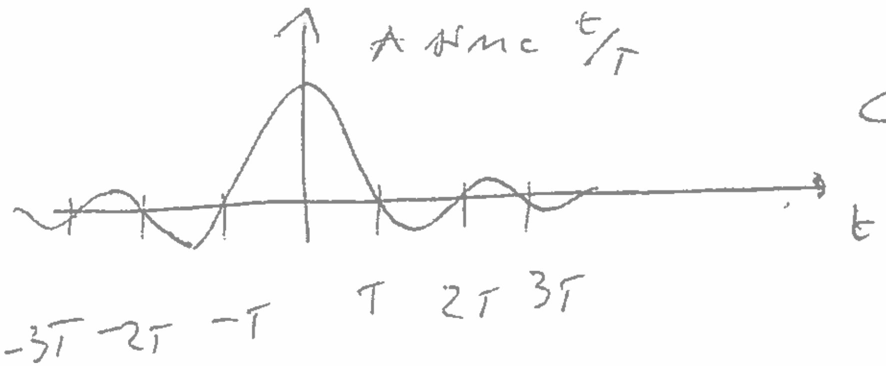
$A \cos 2\pi f_0 t \ \mathrm{rect}\frac{t}{T}$



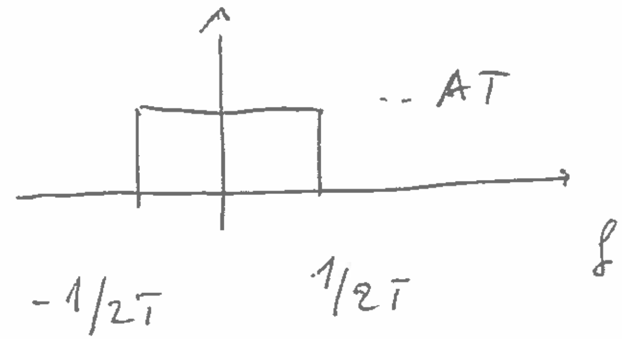
$\Longleftrightarrow$



$A \ \mathrm{sinc}\ t/T$



$\Longleftrightarrow$



Fourier Transform and Fourier Series of a Periodic Signal

Consider a periodic signal of period T obtained from the repetition of a basic waveform with repetition step equal to T seconds:

$$y(t) = \sum_{i=-\infty}^{\infty} x(t - iT) = \frac{1}{T} \sum_{i=-\infty}^{\infty} X\left(\frac{i}{T}\right) e^{j\frac{2\pi it}{T}} = \sum_{i=-\infty}^{\infty} Y_i e^{j\frac{2\pi it}{T}}, \tag{8}$$

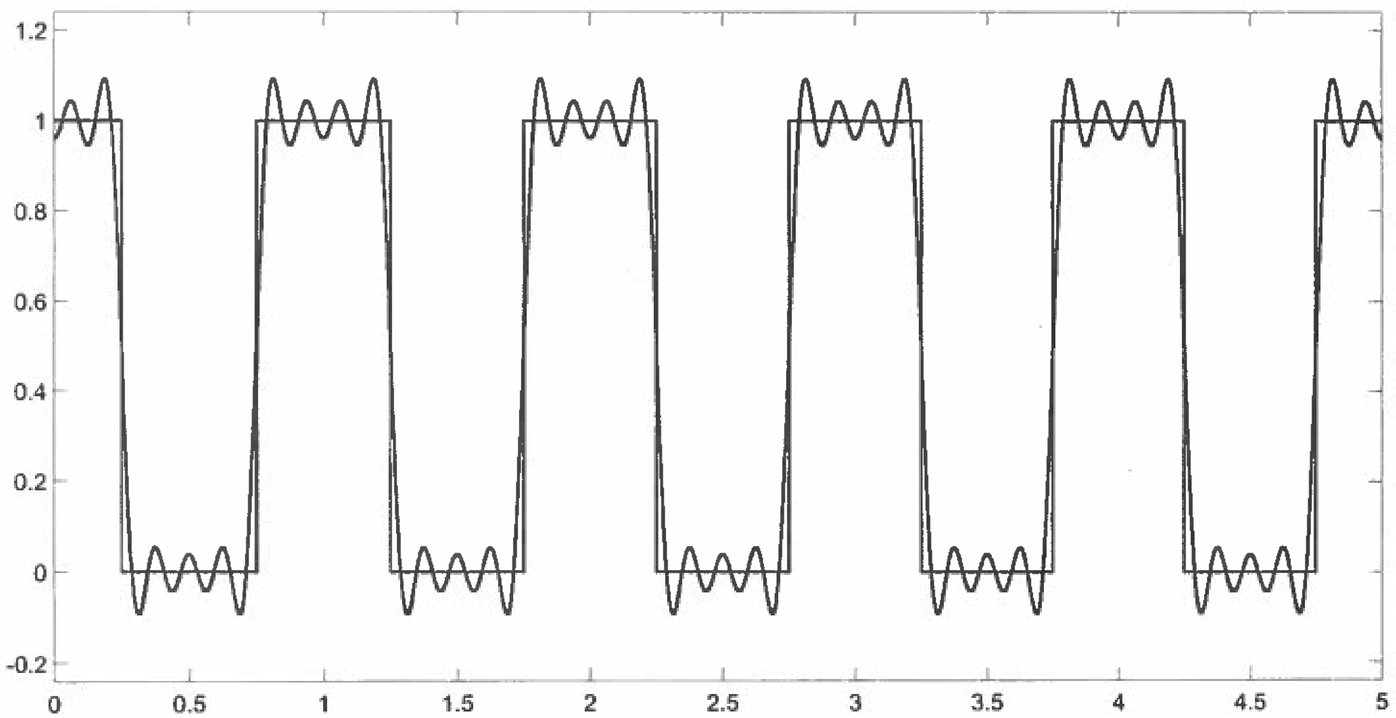
where the second equality is a form of Poisson's sum formula. The set of coefficients $\{Y_i\}$ of the complex exponentials are the *Fourier series* of $y(t)$ and can be calculated directly from $y(t)$ as

$$Y_i = \frac{1}{T} \int_0^T y(t) e^{-j\frac{2\pi it}{T}} dt. \tag{9}$$

Taking the Fourier transform of (8) one gets a weighted (the weights being $\{Y_i\}$) sum of *spectral lines* (the delta functions in frequency domain) at frequencies that are integer multiples of the *fundamental frequency* $1/T$:

$$Y(f) = \sum_{i=-\infty}^{\infty} Y_i \delta\left(f - \frac{i}{T}\right). \tag{10}$$

APPROXIMATION OF A SQUARE WAVE (CLOCK)
BY THE FIRST 7 SINUSOIDS OF ITS FOURIER SÉRIES



Fourier Transform of a Sampled Signal

A very important Fourier transform is that of the periodic repetition of delta functions:

$$\sum_{i=-\infty}^{\infty} \delta(t - iT) \Longleftrightarrow \frac{1}{T} \sum_{i=-\infty}^{\infty} \delta\left(f - \frac{i}{T}\right). \tag{11}$$

We can regard sampling as multiplication:

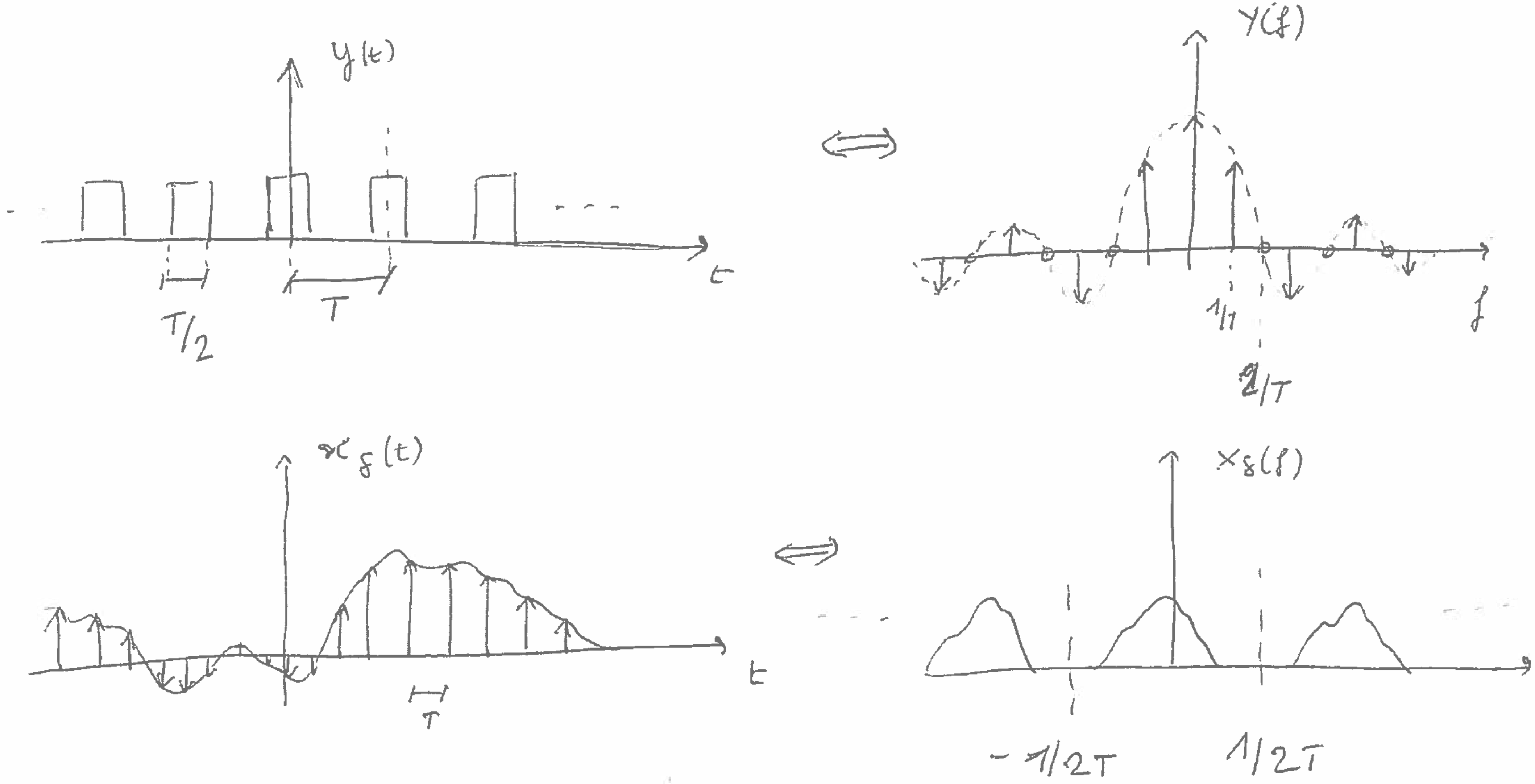
$$x_\delta(t) = x(t) \cdot \left(\sum_{i=-\infty}^{\infty} \delta(t - iT) \right) \Longleftrightarrow \frac{1}{T} \sum_{i=-\infty}^{\infty} X\left(f - \frac{i}{T}\right) = X_\delta(f) \tag{12}$$

where $1/T$ is the *sampling frequency*, the Fourier transform is obtained by noting that product in time domain is convolution in frequency domain, and the result of the convolution

$$\int_{-\infty}^{\infty} X(f - \nu)\delta\left(\nu - \frac{i}{T}\right) d\nu = X\left(f - \frac{i}{T}\right)$$

is a consequence of property (5) of the delta function. Note the relationship (12):

sampling $\Longleftrightarrow$ periodic repetition.


y(t)
T/2
T
t
Y(f)
1/T
2/T
f
$x_\delta(t)$
T
t
$X_\delta(f)$
-1/2T
1/2T

Discrete Fourier Transform

By Poisson's sum formula one writes

$$X_\delta(f) = \frac{1}{T}\sum_{i=-\infty}^{\infty} X\left(f - \frac{i}{T}\right) = \sum_{i=-\infty}^{\infty} x_i e^{-j2\pi i f T}, \tag{13}$$

where $x_i = x(iT)$. The rightmost side of the above equation is called the *discrete Fourier Transform* of the sequence $\{x_i\}$, and, with some abuse of notation, in the following will be denoted

$$X(e^{j2\pi f T}) = \sum_{i=-\infty}^{\infty} x_i e^{-j2\pi i f T}, \tag{14}$$

where the argument $e^{j2\pi fT}$ of $X(\cdot)$, that emphasizes the periodic nature of the spectrum, will help us in distinguishing between the Fourier transform, where the argument is f, and the discrete Fourier transform, where the argument is the complex exponential. The IDFT is

$$x_i = T\int_0^{T^{-1}} X(e^{j2\pi f T}) e^{j2\pi i f T} df. \tag{15}$$

Energy and Power Spectral Density

The Fourier transform is so important because it allows to see what are the frequencies where the time-domain signal has energy/power. To gain insight into this concept we have to introduce the notion of Energy Spectral Density (ESD), that is the right side of the following Fourier pair

$$\int_{-\infty}^{\infty} x^*(\tau)x(t+\tau)d\tau \Longleftrightarrow |X(f)|^2, \tag{16}$$

where the left side takes the name of *autocorrelation*. Rayleigh's energy theorem says that

$$E_x = \int_{-\infty}^{\infty} x^*(\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} |X(f)|^2 df. \tag{17}$$

Similar formulas hold for the Power Spectral density (PSD) of power signals, upon a limit operation for $T \to \infty$ and division by T in the left side of (16) similar to (2). Energy and power spectral densities are often expressed in decibels:

$$\left[|X(f)|^2\right]_{dB} = 10\log_{10}|X(f)|^2.$$

Energy Spectral Density

The energy/power spectral density is, for instance, the basis of transmission techniques as frequency division multiplexing, where multiple signals, each one occupying a different frequency range, are simultaneously transmitted through the same medium without interfering with each other, as it happens in video and audio broadcasting. Suppose that one wants to transmit two rectangular pulses at the same time over the same medium, then, using two frequencies f_0 and f_1 and two RF rectangular impulses $x_0(t)$ and $x_1(t)$, one gets the two energy spectral densities

$$|X_0(f)|^2 = \left|\frac{AT}{2}\left(sinc((f-f_0)T) + sinc((f+f_0)T)\right)\right|^2,$$

$$|X_1(f)|^2 = \left|\frac{AT}{2}\left(sinc((f-f_1)T) + sinc((f+f_1)T)\right)\right|^2.$$

Bandwidth

The bandwidth is the extension of positive frequencies occupied by the energy/power spectral density of a signal. For instance, given $x(t) = sinc(\frac{t}{T})$ one has

$$|X(f)|^2 = |Trect(fT)|^2 \tag{18}$$

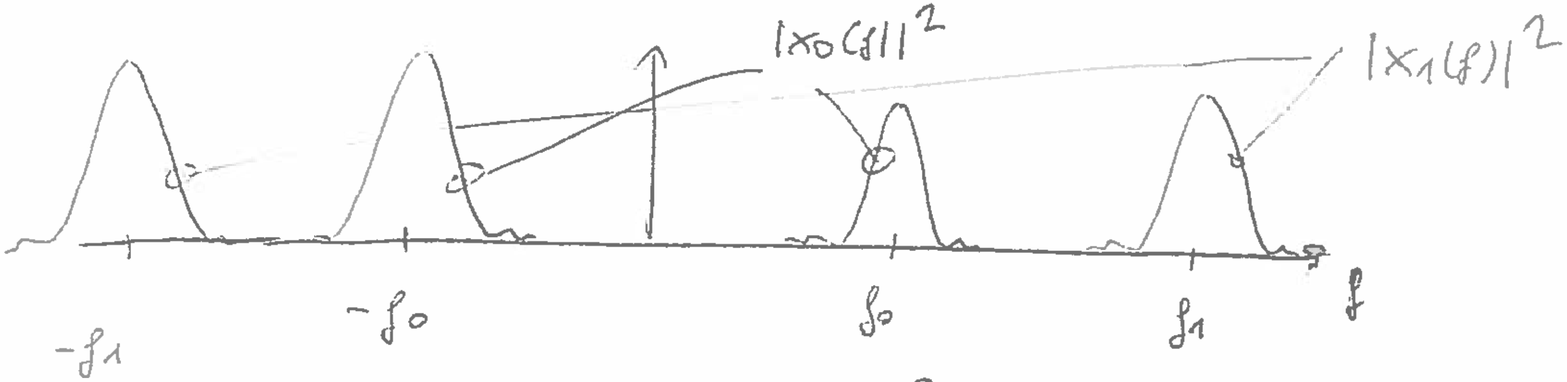
whose bandwidth is $1/2T$. A signal whose ESD is flat over its bandwidth, as the cardinal sine time wave, or, in the limit, the Dirac delta, is said to have white spectrum (over its bandwidth). For non-white signals, the notion of bandwidth should be better specified. A common one is the so-called noise equivalent bandwidth, that is

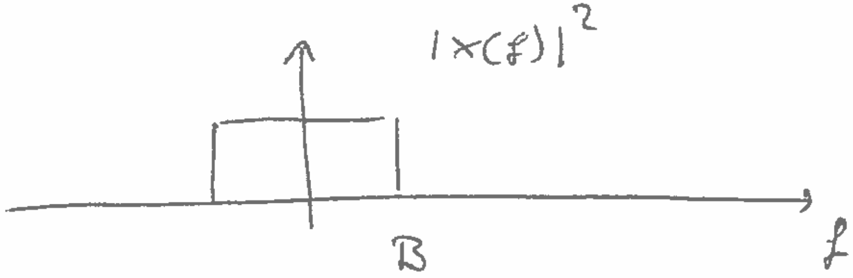
$$B_n = \frac{1}{\max_{f\geq 0}\{|X(f)|^2\}} \int_0^\infty |X(f)|^2 df.$$

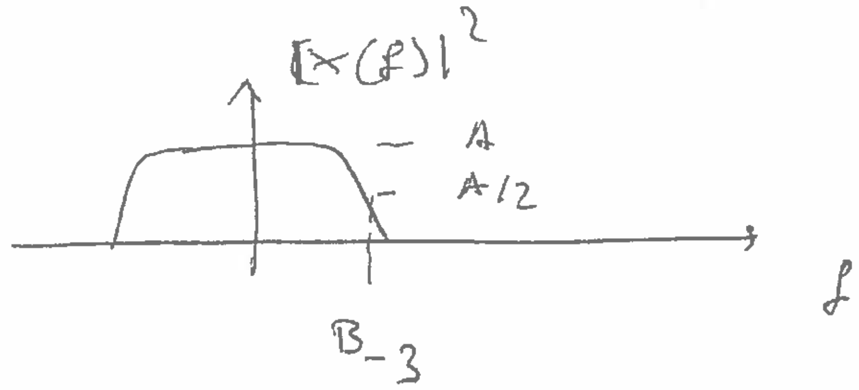
Often the -3 dB bandwidth B_{-3} is used. For the one-sided decaying exponential one has

$$\left|\frac{1}{1 + j2\pi fRC}\right|^2 = \frac{1}{1 + (f/B_{-3})^2},$$

with $B_{-3} = (2\pi RC)^{-1}$.


$|X_0(f)|^2$
$|X_1(f)|^2$
$-f_1$
$-f_0$
f_0
f_1
f


$|X(f)|^2$
B
f


$|X(f)|^2$
A
$A/2$
B_{-3}
f

Duration-Bandwidth Product

Consider a low-pass signal and define the RMS bandwidth and RMS duration as

$$B_{rms} = \frac{1}{2}\sqrt{\frac{\int_{-\infty}^{\infty} f^2|X(f)|^2 df}{E_x}}, \quad D_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty}(t-t_0)^2|x(t)|^2 dt}{E_x}},$$

where t_0 is equal to the mean value of $|x(t)|^2$. The uncertainty principle holds:

$$B_{rms}D_{rms} \geq \frac{1}{8\pi},$$

which means

long duration $\Longleftrightarrow$ narrowband, short duration $\Longleftrightarrow$ wideband.

The two limit cases are the Dirac delta in time domain, that has white (constant) spectrum, and the DC (constant) in time domain, whose Fourier transform is the Dirac delta at frequency zero.

Filtering

A filter is a linear and time invariant system characterized by the impulse response $h(t)$. $h(t)$ is the output of the filter when the filter is excited by $\delta(t)$. Given the input $x(t)$, the output $y(t)$ of the filter is the *convolution* between the input and the impulse response, that is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{19}$$

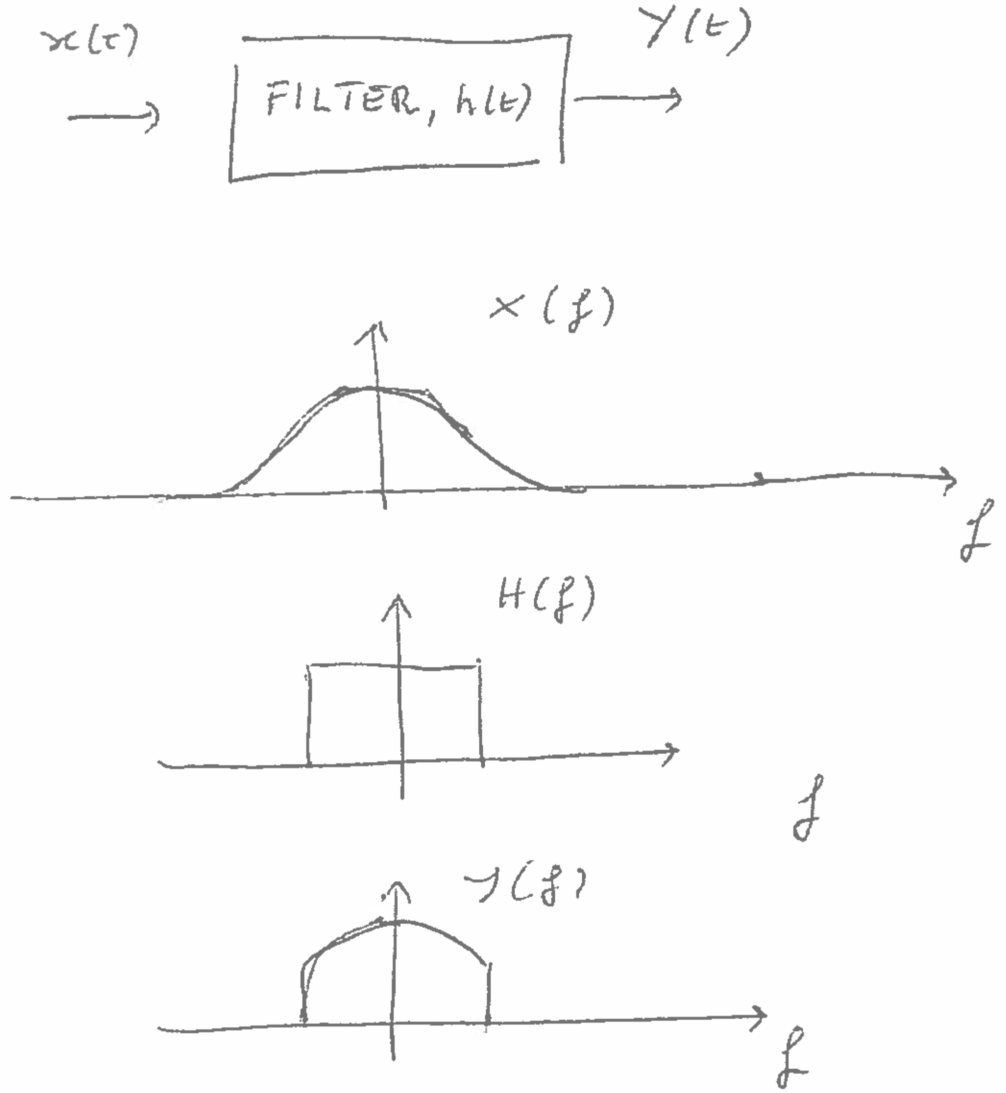
The Fourier transform is a powerful tool to analyze filters, because

$$y(t) \Longleftrightarrow Y(f) = X(f)H(f), \tag{20}$$

where $H(f)$ is called *frequency response* of the filter, therefore

$$|Y(f)|^2 = |X(f)|^2|H(f)|^2. \tag{21}$$

The above equation means that the filter can be used to reject those signals whose ESD lies in a frequency range where the modulus of filter's frequency response is low.



x(t)
FILTER, h(t)
y(t)
X(f)
f
H(f)
f
Y(f)
f

Sampling Theorem

From the first equality of (13) one realizes that, if $X(f)$ is bandlimited with $B \leq 1/2T$, then shifted spectra $X(f - i/T)$ do not overlap, hence $X(f)$ can be recovered from $X_\delta(f)$ by passing $X_\delta(f)$ through a low-pass filter of bandwidth $1/2T$:

$$x(t) \Longleftrightarrow X(f) = TX_\delta(f)rect(fT). \tag{22}$$

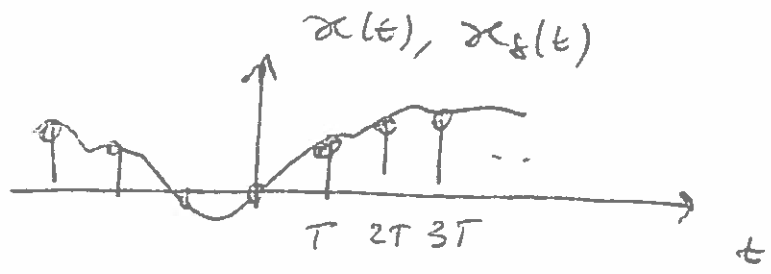
Passing from frequency to time, hence from product to convolution,

$$\int_{-\infty}^{\infty} sinc\left(\frac{t-\tau}{T}\right) x_\delta(\tau)d\tau = \sum_{i=-\infty}^{\infty} x_i sinc\left(\frac{t-iT}{T}\right) \Longleftrightarrow TX_\delta(f)rect(fT). \tag{23}$$

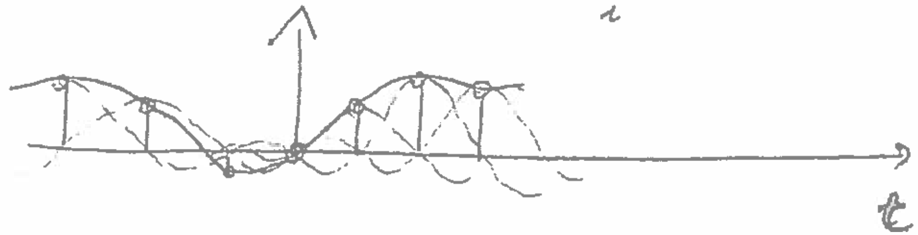
Putting together (22) and (23) one gets

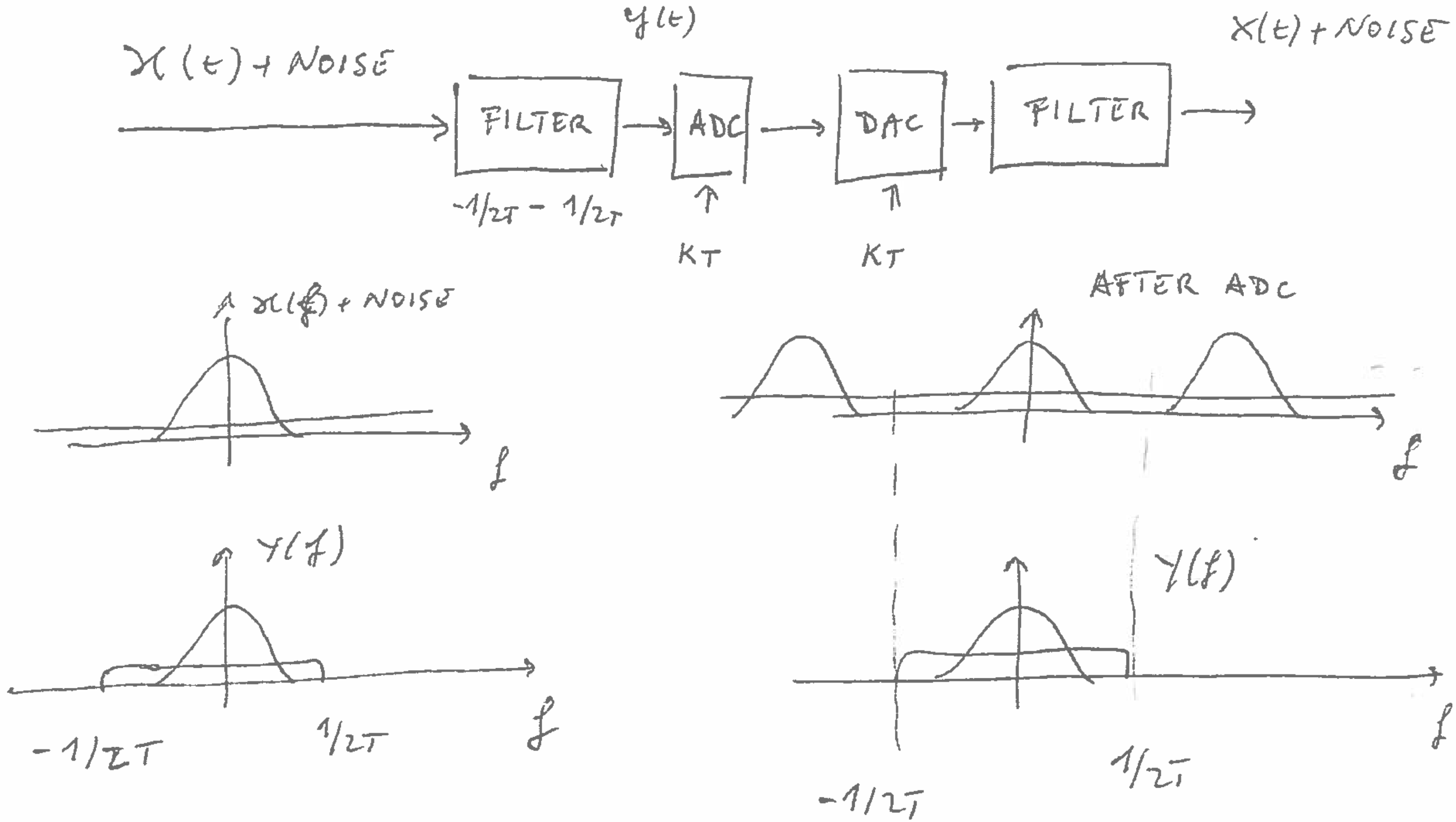
$$x(t) = \sum_{i=-\infty}^{\infty} x_i sinc\left(\frac{t-iT}{T}\right), \tag{24}$$

that is the celebrated sampling theorem: *given a signal bandlimited to B Hz it can be recovered by (24) from its samples taken at frequency $1/T > 2B$.*



$$x(t) = \sum_i x_i \operatorname{sinc}\left(\frac{t-iT}{T}\right)$$




$x(t)$ + NOISE
FILTER
$-1/2T$ – $1/2T$
$y(t)$
ADC
KT
DAC
KT
FILTER
$x(t)$ + NOISE
$x(f)$ + NOISE
f
$Y(f)$
$-1/2T$
$1/2T$
f
AFTER ADC
f
$Y(f)$
$-1/2T$
$1/2T$
f