Matched Filter and Nyquist Criterion

Arnaldo Spalvieri

Dipartimento di Elettronica, Informazione e Bioingegneria

Politecnico di Milano, ITALY

Sampled Matched Filter

Consider a filter with possibly complex impulse response $h_1(t)$, use as a input to the filter

$$\frac{A}{\sqrt{2}}(\delta(t) + j\delta(t)),$$

where A is a real scalar, and regard filter's output, that is

$$x(t) = Ah_1(t)\frac{(1+j)}{\sqrt{2}},$$

whose energy is $E_x = A^2 E_1$, where E_1 is the energy of $h_1(t)$, as a wanted signal. Also, assume that complex additive white noise $w(t)$ with zero mean and psd N_0 is added to the wanted signal and that, after addition with the noise, the signal passes through a second filter with impulse response $h_2(t)$. The output of $h_2(t)$ is

$$y(t) = s(t) + n(t),$$

where

$$s(t) = h_2(t) \otimes x(t), \quad n(t) = h_2(t) \otimes w(t).$$

Sampled Matched Filter

Given a time instant t_0, deign the impulse response $h_2(t)$ of a filter in such a way that

$$\text{SNR} = \frac{|s(t_0)|^2}{E\{|n(t_0)|^2\}}$$

is maximized. This ratio, called the Signal-to-Noise Ratio (SNR), can be expressed by computing the numerator as the squared modulus of the inverse Fourier transform of filter's output at time t_0, while the denominator is the variance of the noise at filter's output, that is the integral of its psd:

$$\text{SNR} = \frac{A^2 \left| \int_{-\infty}^{\infty} H_1(f) H_2(f) e^{j2\pi f t_0} df \right|^2}{N_0 \int_{-\infty}^{\infty} |H_2(f)|^2 df},$$

where the following equality has been used in the numerator

$$\left| \frac{1+j}{\sqrt{2}} \right|^2 = 1.$$

Sampled Matched Filter

Schwartz inequality reads

$$\left|\int_{-\infty}^{\infty} H_1(f)H_2(f)e^{j2\pi f t_0}df\right|^2 \leq \left(\int_{-\infty}^{\infty} |H_1(f)e^{j2\pi f t_0}|^2 df\right)\left(\int_{-\infty}^{\infty} |H_2(f)|^2 df\right),$$

hence

$$\text{SNR} \leq \frac{A^2\left(\int_{-\infty}^{\infty} |H_1(f)e^{j2\pi f t_0}|^2 df\right)\left(\int_{-\infty}^{\infty} |H_2(f)|^2 df\right)}{N_0 \int_{-\infty}^{\infty} |H_2(f)|^2 df} = \frac{A^2 E_1}{N_0} = \frac{E_x}{N_0}, \qquad (1)$$

with equality when

$$H_2(f) = H_1^*(f)e^{-j2\pi f t_0} \Longleftrightarrow h_1^*(t_0 - t) = h_2(t).$$

The above $h_2(t)$ takes the name of Sampled Matched Filter (SMF). Note that the numerator of (1) is the energy of the wanted signal before the sum with the noise, therefore, with SMF, what determines the SNR at the sampling instant is the ratio between the energy of the wanted signal and the psd of the additive white noise before the SMF, while the specific shape of signal's spectrum has no impact on the SNR.

Sampled Matched Filter

Let the complex envelope of a passband signal plus additive white Gaussian noise be

$$s(t) = ah(t) + w(t),$$

where a ia s complex random variables with zero mean and variance σ_a^2, and that the power spectral density of the continuous-time complex noise is N_0. The signal $s(t)$ is passed through the matched filter and sampled at the time instant $t_0 = 0$. It is known that the frequency response of the matched filter and its impulse response are

$$H^*(f), \;\; h^*(-t).$$

A crucial property of the sampled matched filter is that its output contains all the information we need for optimal detection. Technically speaking, the matched filter output is a *sufficient statistics*.

Sampled Matched Filter

The output of the sampled matched filter is

$$y = ag_0 + n_0,$$

where g_0 is the cascade of transmit filter and matched receive filter evaluated at $t = 0$, and n_0 is the filtered noise at $t = 0$. The SNR at the sampling instant is:

$$\text{SNR} = \frac{E\{|ag_0|^2\}}{E\{|n_0|^2\}} = \frac{\sigma_a^2|g_0|^2}{N_0 g_0} = \frac{\sigma_a^2 g_0}{N_0} = \frac{\sigma_a^2 E_h}{N_0}.$$

Recall that by Schwarz inequality one can prove that the sampled matched filter optimizes the SNR.

Nyquist Criterion

Let the complex envelope of a passband data signal plus additive white Gaussian noise be

$$s(t) = \sum_{i=-\infty}^{\infty} a_i h(t - iT) + w(t),$$

where T is the symbol repetition interval. Suppose we filter it through the receive filter and sample the output at $t = kT$, getting:

$$y_k = \sum_{i=-\infty}^{\infty} a_i g_{k-i} + n_k$$

where $g(t)$ is the impulse response of the cascade of transmit filter and receive filter and $g_k = g(kT)$ and n_k is the k-th sample of the filtered noise.

Nyquist Criterion

The data signal is InterSymbol Interference (ISI) free if the sampled impulse response between the source and the output of the sampler at kT satisfies the Nyquist criterion:

$$g_k = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

In the frequency domain, the Nyquist criterion reads

$$G(e^{j2\pi fT}) = \frac{1}{T} \sum_k G\left(f - \frac{k}{T}\right) = 1. \tag{2}$$

When the above condition is satisfied, the frequency response $G(f)$ is said to be a Nyquist frequency response.

Nyquist Criterion

The Nyquist condition is satisfied by infinitely many frequency responses. A popular one is the so-called *raised cosine*:

$$G(f) = RC(f) = \begin{cases} T, & |f| \leq \frac{1-\alpha}{2T}, \\ T cos^2\left(\frac{\pi T}{2\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right), & \frac{1-\alpha}{2T} < |f| \leq \frac{1+\alpha}{2T}, \\ 0, & |f| > \frac{1+\alpha}{2T}, \end{cases}$$

where the scalar α, which determines the excess bandwidth in the region $|f| > 1/2T$, is called *roll-off*. The impulse response of the raised cosine is

$$rc(t) = \left(\frac{sin(\pi t/T)}{\pi t/T}\right)\left(\frac{cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2}\right).$$

From the frequency response, it is easy to see that the Nyquist condition is satisfied:

$$\frac{1}{T}\sum_k RC\left(f - \frac{k}{T}\right) = 1.$$

Nyquist Criterion and Matched Filter

For complex white channel noise with p.s.d. N_0, the two conditions of ISI free transmission and matched receive filter can be obtained by suitable design of transmit and receive filters. Specifically, the receive filter must be matched to the transmit filter, and the transmit filter must be designed in such a way that the cascade of the two is Nyquist. let $H(f)$ be the frequency response of the transmit filter. The sampled impulse response after the matched filter must be such that

$$G(e^{j2\pi fT}) = \frac{1}{T}\sum_k |H(f-\frac{k}{T})|^2 = 1,$$

where the first equality imposes that the receive filter is the sampled matched filter while the second equality imposes that the cascade of transmit and receive filters is Nyquist. The above equation is fulfilled when the frequency response of the transmit filter is the squared root of a Nyquist frequency response. For instance, the inverse Fourier transform of the squared root of the raised cosine frequency response $RC(f)$ is

$$\mathcal{F}^{-1}\{\sqrt{RC(f)}\} = \sqrt{\frac{1}{T}}\left(\frac{sin(\pi(1-\alpha)t/T)}{\pi t(1-16\alpha^2t^2/T)/T} + \frac{4\alpha cos(\pi(1+\alpha)t/T)}{\pi(1-16\alpha^2t^2/T^2)}\right).$$

Nyquist Criterion and Matched Filter

The p.s.d. of the noise after the matched filter results

$$S_n(f) = N_0 |H(f)|^2,$$

therefore, after sampling

$$S_n(e^{j2\pi fT}) = \frac{N_0}{T} \sum_k |H(f - \frac{k}{T})|^2 = N_0.$$

Since $E_h = g_0 = 1$, for the SNR one has

$$\text{SNR} = \frac{\sigma_a^2}{N_0}.$$