MIDA 2 COURSE (2nd PART of MIDA)

5 CFU – 50 hours

LECTURER → SERGIO SAVARESI

EXERCISES → Ing. STEFANO DATTILO

GENERAL TOPIC of MIDA COURSE:

- Collect DIGITALLY DATA from REAL systems
- BUILD BLACK-BOX (GRAY-BOX) models from DATA, with EMPHASIS ON
  - [• DYNAMICAL systems
  - • CONTROL/AUTOMATION-ORIENTED applications]

PURPOSE of this MODELLING:

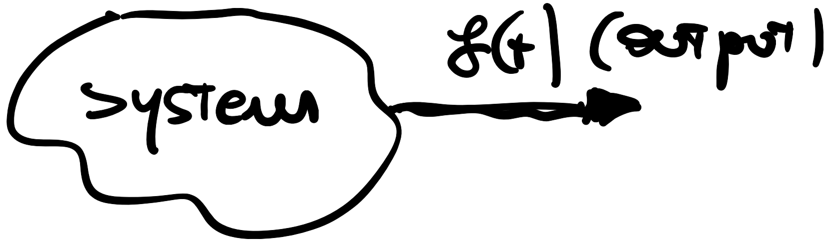
- PREDICTION
- SW-SENSING
- MODELLING for CONTROL DESIGN

AREA: MACHINE-LEARNING → FOCUS ON "CONTROL"

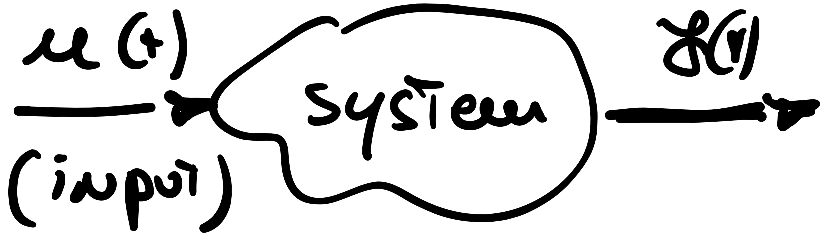
SUPER-SUMMARY of MIDA 1

Focus:

- TIME SERIES

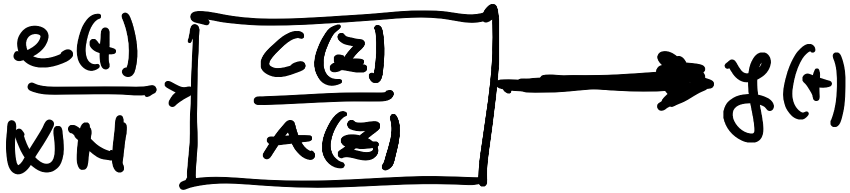


- INPUT/OUTPUT systems (I/O sy.)

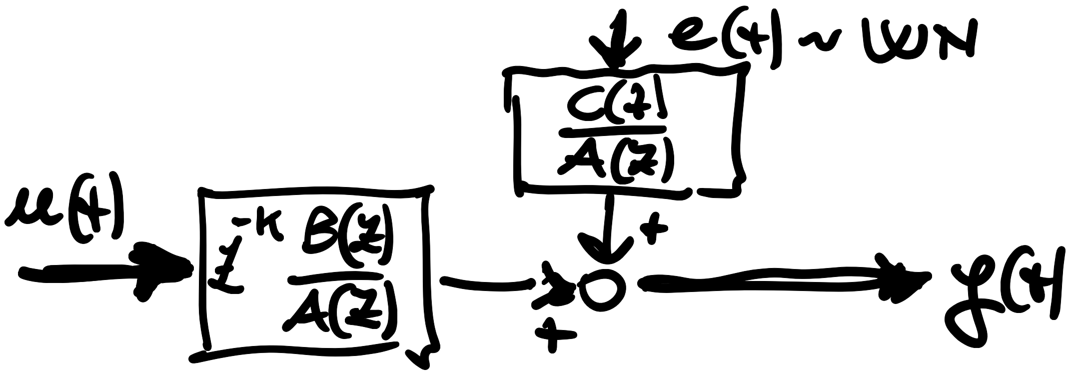


Model CLASSES USED in MIDA 1

ARMA models for T.S.



ARMAX models for I/O systems:



$\mathcal{M}(\theta)$ where θ is the parameters VECTOR (θ: coeff. of $A(z), B(z), C(z)$)

A PARAMETRIC IDENT. method has been USED →

Perf. index is DEFINED:

$$J(\theta)=\frac{1}{N}\sum_{t=1}^{N}\left(y(t)-\hat{y}(t|t-1;\theta)\right)^2$$

→ VARIANCE of the PREDICTION ERROR made by the model

$\Downarrow$ $\boxed{\text{PREDICTION ERROR METHOD (P.E.M.)}}$

$$\hat{\theta}_N=\underset{\theta}{\text{argmin}}\left\{J(\theta)\right\}$$

MIDA-2: Focus on I/O systems (more close to REAL Applications than T. series)

Chapter 1: NON-parametric ("DIRECT/CONSTRUCTIVE") B. Box identification of I/O systems using State-space models

Chapter 2: PARAMETRIC identification of B. Box I/O systems, with a frequency-domain approach

Chapter 3: KALMAN Filter for $\boxed{\text{SW-sensing}}$ using Feedback on W. Box models

Chapter 4: B. Box methods for $\boxed{\text{SW-SENSING}}$, without feedback

Chapter 5: $\boxed{\text{GRAY - BOX}}$ sy. identification:

- USING KALMAN FILTER
- USING "SIMULATION - ERROR - METHODS" (S.E.M.)

Chapter 6: MINIMUM - VARIANCE CONTROL (M.V.C.)

→ DESIGN of optimal feedback CONTROLLERS using the th. BACKGROUND of the MIDA COURSE

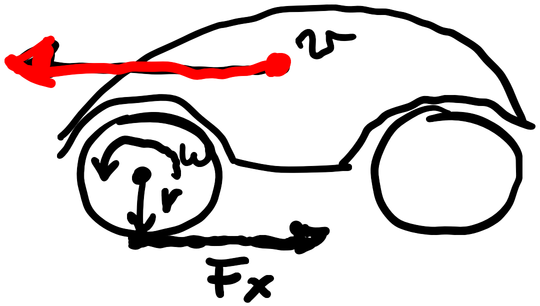
Appendix: RECURSIVE ("ON-LINE") implementation of ALGO. for sy. identification

---

some Application ex. / MATLAB / SIMULINK NUMER. ex

EXAM → $\boxed{\text{WRITTEN - ONLY EXAM}}$

MOTIVATION EXAMPLE for the COURSE:

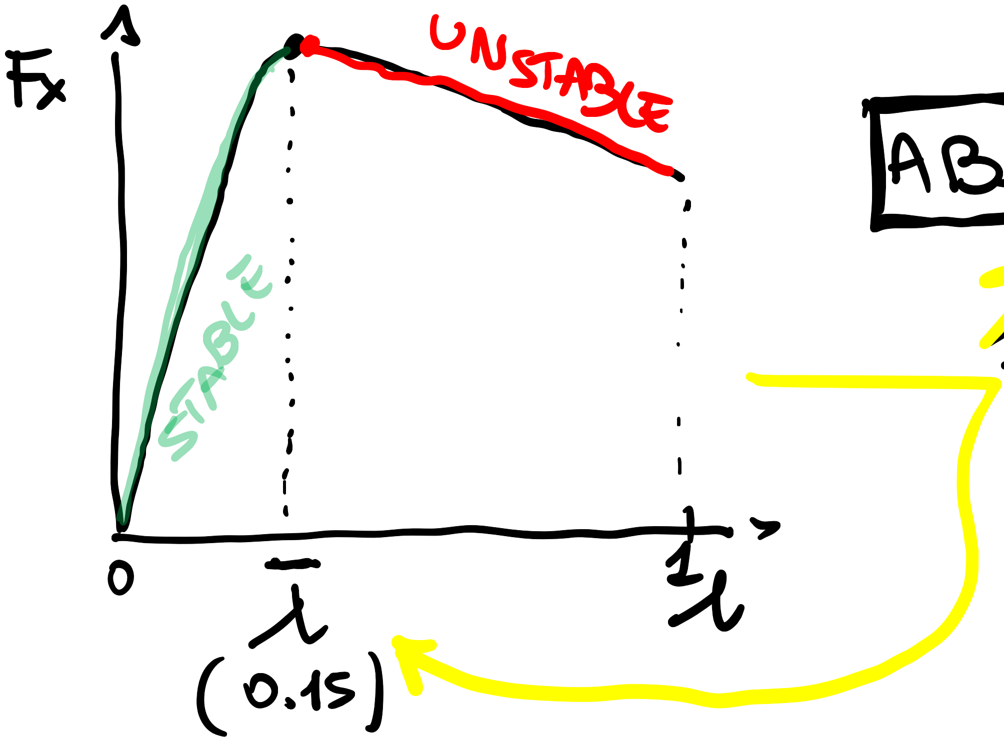


SLIP of the wheel: $\lambda = \dfrac{v - \omega r}{v}$

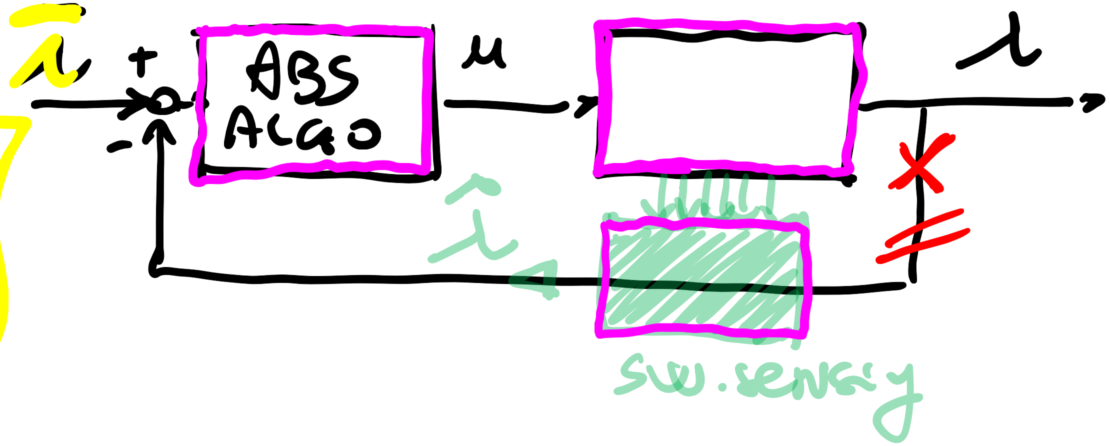
$0 \le \lambda \le 1$

locked wheel

free rolling wheel

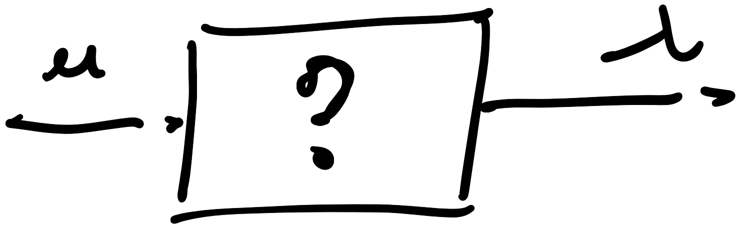


$\boxed{\text{ABS}}$ CONTROL PROBLEM :



SUB-PROBLEMS:

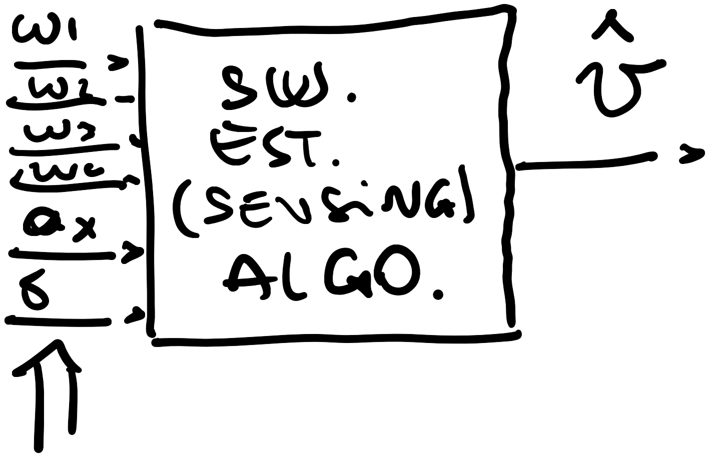
(1) MODEL the system :



② SW. ESTIMATION of λ:

$$\lambda = \frac{v - \omega r}{v}$$

NOT MEASURABLE (~~GPS~~ or ~~OPTICAL~~ SENSORS)

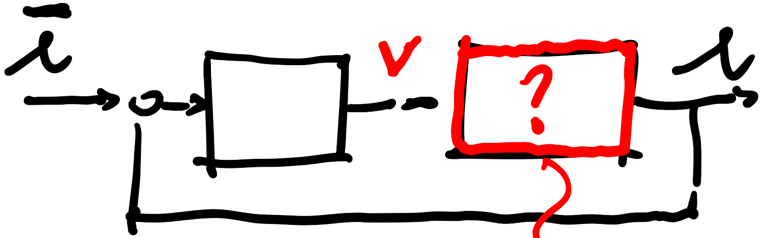


$\hat{v} \Rightarrow \hat{\lambda}$

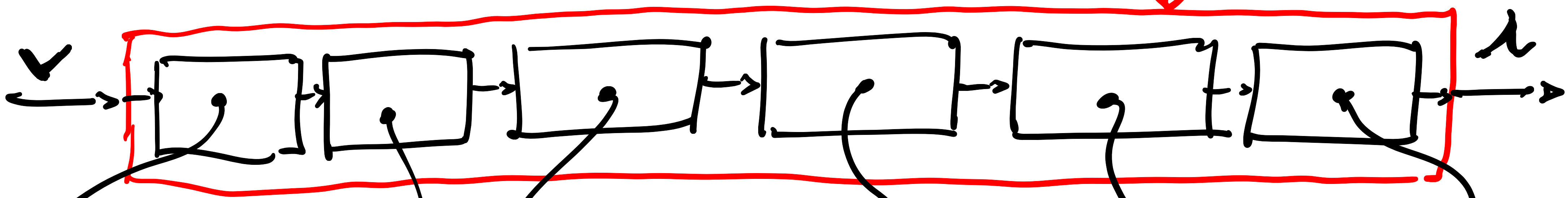
physical sensors

③ Design the ABS control ALGO.

why BLACK-BOX modelling?



V: voltage driving an electric motor of a BRAKE-BY wire ACTUATOR



Current dynamics of electric motor
E

Position dynamics of BBW actuator
M

Dynamics of the hydraulic circuit of brake system
H

Tire dynamics
M

Wheel rotational dynamics
M

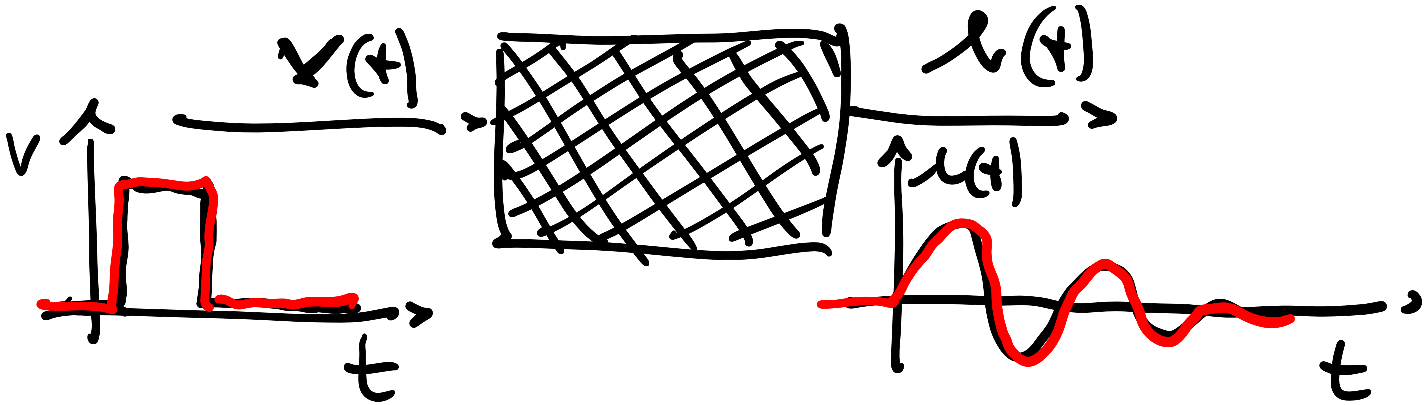
Vehicle full dynamics
M

Multi domain system → E, H, M

• White Box (physical) modelling → WRITE equations from "FIRST-principles"

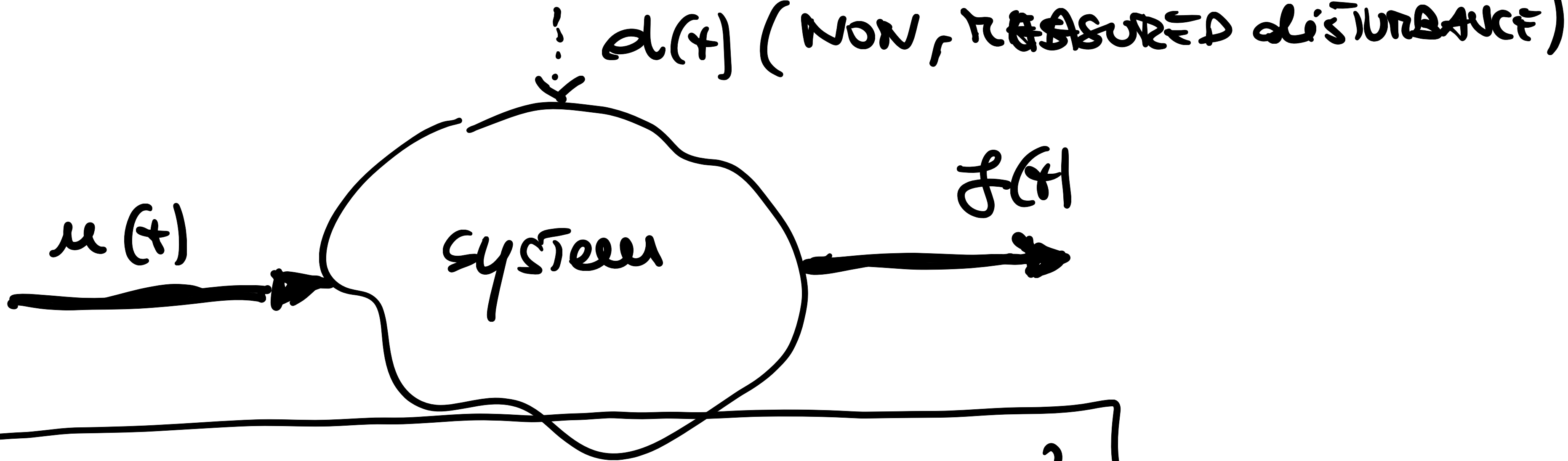
• Black-Box modelling: ("MACHINE-LEARNING")

Experiment → collect DATA → BUILD model



Using (just) I/O measured data → we can "LEARN" a mathematical model of the I/O behaviour of the system!

Chapter 1: BLACK-BOX NON-PARAMETRIC SYSTEM IDENTIFICATION of I/O SYSTEMS USING STATE-SPACE MODELS



DATA:

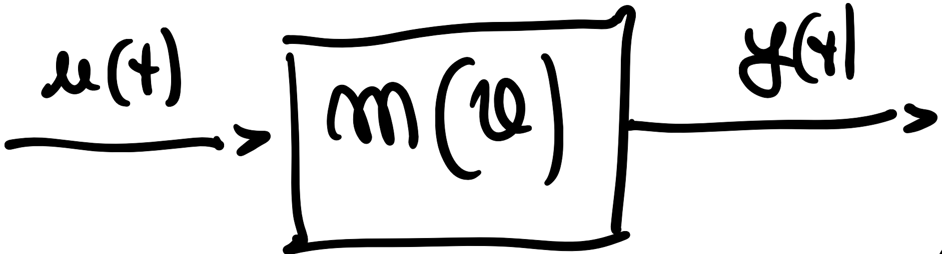
$$\{u(1), u(2) \ldots \quad u(N)\}$$

$$\{y(1), y(2) \ldots \quad y(N)\}$$

Recall of the GENERAL path of a PARAMETRIC IDENT. method:

① Collect DATA: $\{u(1), u(2), \dots u(N)\}, \{y(1), y(2), \dots y(N)\}$ ✓

② SELECT (A-PRIORI) of a class/family of PARAMETRIC models: $\mathcal{M}(\vartheta)$: ϑ is the parameters VECTOR ✓



③ SELECT (A-priori) a "performance INDEX" ⟶ ✓ gives an "ORDER" to the quality of models

n_ϑ ⟶ size of the parameter vector ϑ

$$J(\vartheta) : \mathbb{R}^{n_\vartheta} \longrightarrow \mathbb{R}^+$$

$\boxed{\mathcal{M}(\theta_1) \text{ is BETTER than } \mathcal{M}(\theta_2) \text{ if } J(\theta_1) < J(\theta_2)}$

Example: PEM

$$J(\theta) = \frac{1}{N}\sum_{t=1}^{N}\Big(y(t) - \underbrace{\hat{y}(t|t-1;\theta)}\Big)^2$$

TRUE OUTPUT

1 step PREDICTOR of the model

VARIANCE of ERROR made by the predictor BASED on model $\mathcal{M}(\theta)$

④ optimization step (MINIMIZE $J(\theta)$ W.R.T. θ):

$$\hat{\theta}_N = \arg\min_{\theta}\{J(\theta)\} \Rightarrow \text{optimal model} \dashrightarrow \boxed{\mathcal{M}(\hat{\theta}_N)}$$

In this chapter we are presenting a TOTALLY DIFFERENT sy. IDENT. approach:

NOT-PARAMETRIC:

- ~~NO A-priori model-class selection~~
- ~~NO perf.-INDEX DEFINITION~~
- ~~NO optimization TASK~~

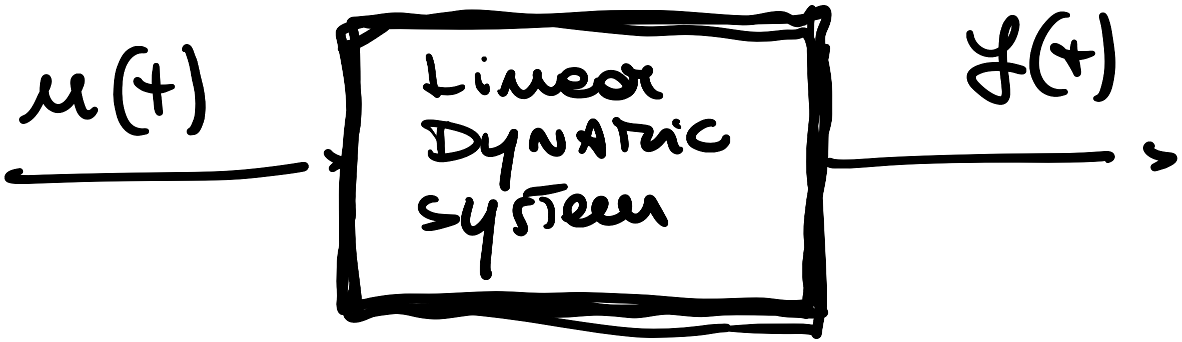
!

PROLOGUE (RECALL): <u>3 main REPRESENTATIONS</u>

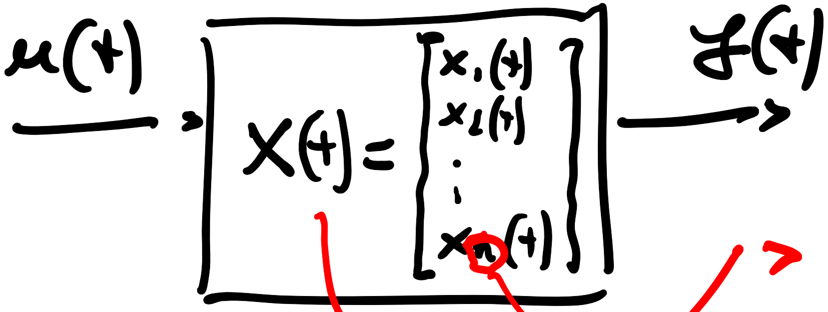
of a:

- DISCRETE-TIME
- LINEAR
- DYNAMIC

<u>SYSTEM</u>



Representation # 1: STATE-SPACE REPRESENTATION



$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

→ INTERNAL STATES of the system

n is the ORDER of the system

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) & \leftarrow \text{“STATE EQUATIONS”} \\ y(t) = Hx(t) + Du(t) & \leftarrow \text{“OUTPUT EQUATIONS”} \end{cases}$$

$F = \begin{bmatrix} n \times n \\ \text{(STATE MATRIX)} \end{bmatrix}$ $G = \begin{bmatrix} n \times 1 \\ \text{INPUT} \\ \text{MATRIX} \end{bmatrix}$

$H = \begin{bmatrix} 1 \times n \text{ OUTPUT M.} \end{bmatrix}$ $\cancel{D = \begin{bmatrix} 1 \times 1 \end{bmatrix}}$ (I/O MATRIX)

USUALLY IS ZERO for STRICTLY-PROPER SYSTEMS

we are assuming

1 input and 1 output

SISO

(can be extended for multiple inputs and/or outputs)

S.S. representation is NOT unique for a system:

$F \longrightarrow F_1 = TFT^{-1}$

$G \longrightarrow G_1 = TG$

$H \longrightarrow H_1 = HT^{-1}$

$D \longrightarrow D_1 = D$

For ANY invertible square matrix T

$\Rightarrow$ $\{F, G, H, D\}$ and $\{F_1, G_1, H_1, D_1\}$ ARE EQUIVALENT!

Ex.

$$\begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + 2u(t) \\ x_2(t+1) = x_1(t) + 2x_2(t) + u(t) \\ y(t) = \frac{1}{4} x_1(t) + \frac{1}{2} x_2(t) \end{cases}$$

$\boxed{n=2}$ $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

1 input → $u(t)$

1 output → $y(t)$

$$F = \begin{bmatrix} 1/2 & 0 \\ 1 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1/4 & 1/2 \end{bmatrix} \quad D = 0$$

Representation #2: TRANSFER-FUNCTION REP.

(I/O Representation)

$$u(t) \longrightarrow \boxed{\frac{B(z)}{A(z)} \cdot z^{-k}} \longrightarrow y(t)$$

"pure-DELAY"

$$W(z) = \frac{B(z)}{A(z)} z^{-k} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots b_p z^{-p}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots a_n z^{-n}} \, z^{-k}$$

order of system

$W(z)$ is a RATIONAL function of the "z" operator

↳ is a "DIGITAL FILTER"

Very easy to move from T.F. rep to a time-domain description of the system. → EX.

$$y(t) = \underbrace{\left[\frac{1+\frac{1}{2}z^{-1}}{2+\frac{1}{3}z^{-1}+\frac{1}{4}z^{-2}} z^{-1}\right]}_{W(z)} \cdot u(t)$$

$$\Downarrow$$

$$2y(t) + \frac{1}{3}y(t-1) + \frac{1}{4}y(t-2) = u(t-1) + \frac{1}{2}u(t-2)$$

DIFFERENCE EQUATION in $u(t)$ and $y(t)$ (NO STATES)

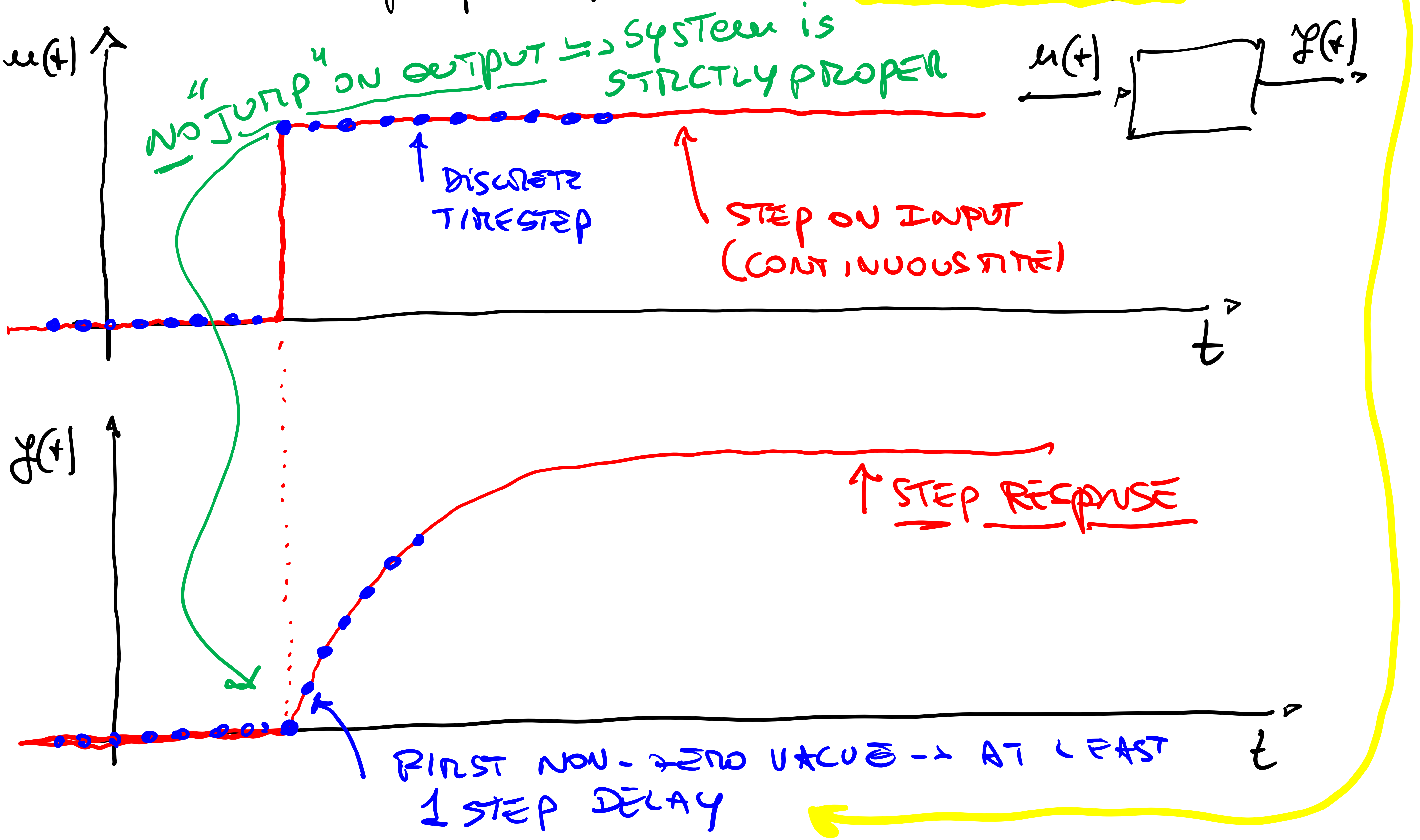
$$\Downarrow$$

$$y(t) = \underbrace{-\frac{1}{6}y(t-1) - \frac{1}{8}y(t-2)}_{\text{OLD VALUES of } y(t)} + \underbrace{\frac{1}{2}u(t-1) + \frac{1}{4}u(t-2)}_{\text{OLD VALUES of INPUT}}$$

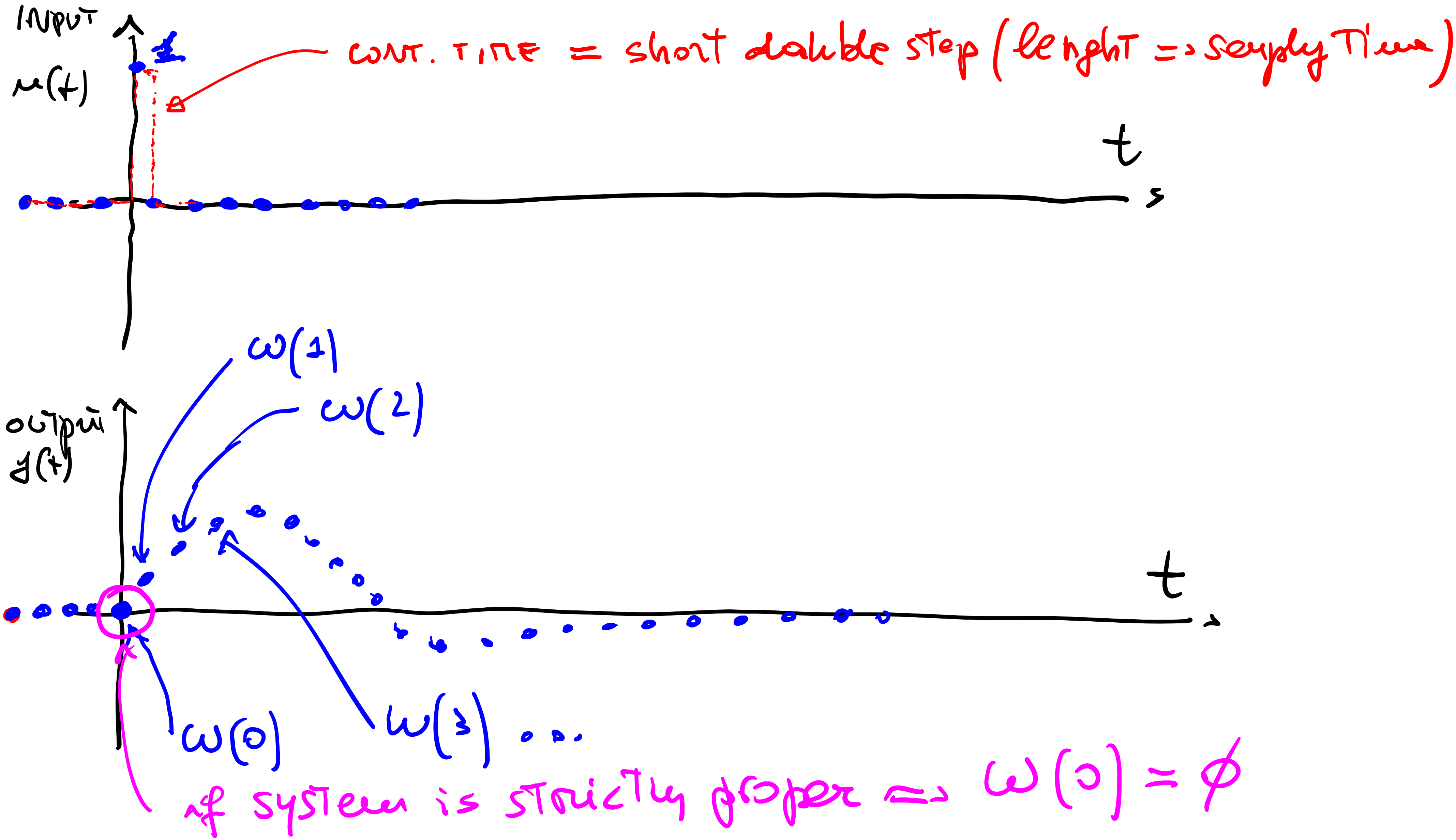
OLD VALUES of $y(t)$ (RECURSIVE or AUTO-REGRESSIVE part)

OLD VALUES of INPUT (EXOGENOUS PART)

Remark → STRICTLY-PROPER SYSTEMS → Notice that for a strictly proper system the $\boxed{\text{DELAY } k \geq 1}$



Representation #3 → convolution of the input with the IMPULSE RESPONSE (IR)



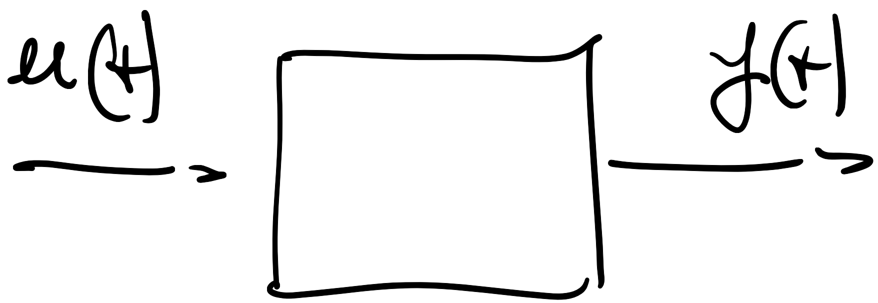
It can be proven that the I/O RELATIONSHIP from $u(t)$ and $y(t)$ can be written as:

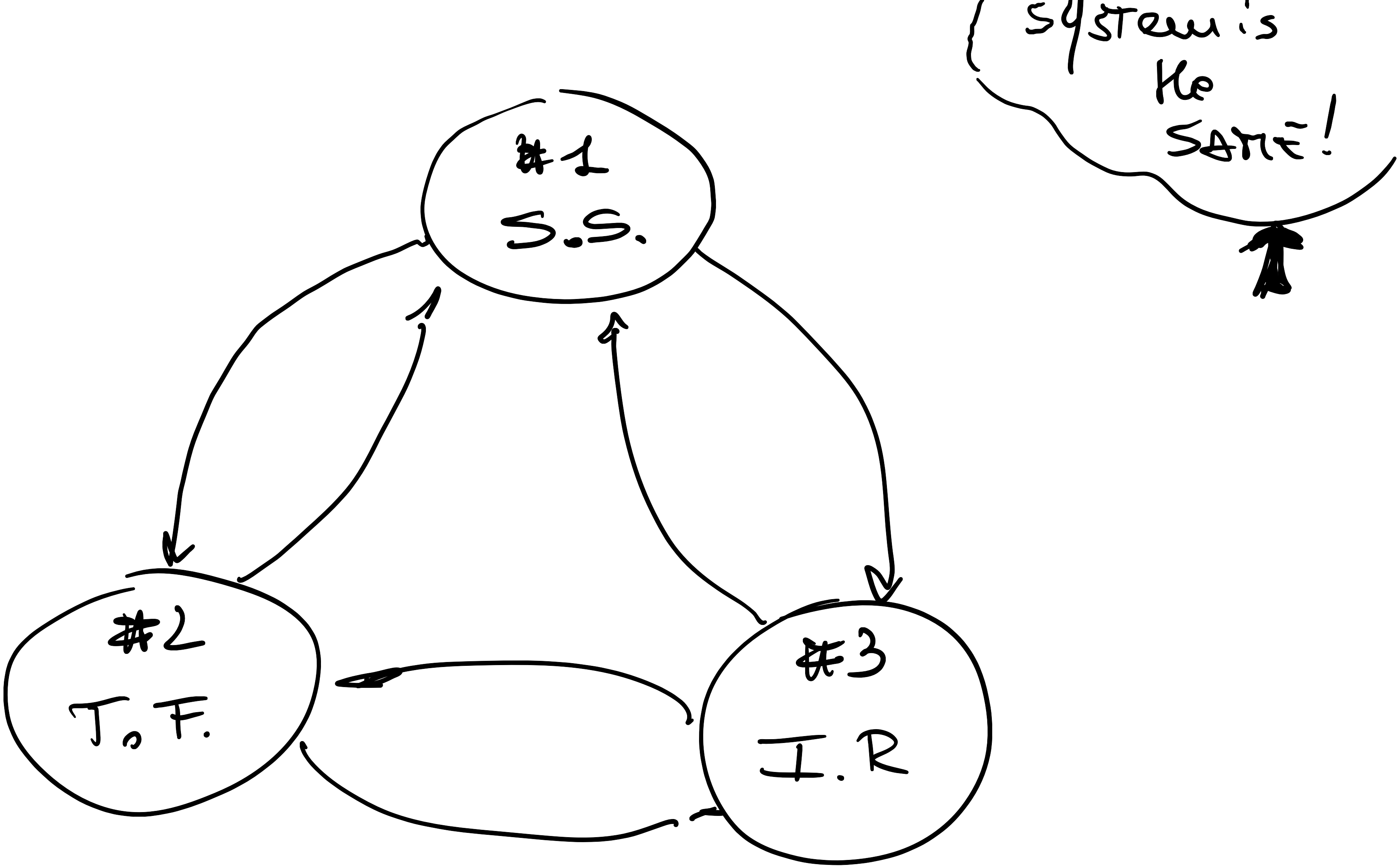
$$y(t) = w(0)\,u(t) + w(1)\,u(t-1) + w(2)\,u(t-2) + \dots$$

$$\boxed{y(t) = \sum_{k=0}^{+\infty} w(k)\,u(t-k)}$$

I.R. Representation of the system

CONVOLUTION of I.R. with the Input signal





Are we able to move from one REP. to another one?

Most classical transformation #1 → #2

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + \cancel{Du(t)} \end{cases}$$

we can use the "z-operator": $z\,x(t) = x(t+1)$

$z^{-1}x(t) = x(t-1)$

$$\rightarrow z\,x(t) = Fx(t) + Gu(t) \Rightarrow x(t) = (zI - F)^{-1}Gu(t)$$

$$y(t) = \boxed{H(zI-F)^{-1}G}\,u(t)$$

↓ T.F. from $u(t)$ to $y(t)$

$$W(z) = H(zI-F)^{-1}G$$

Exaple : $F = \underbrace{\begin{bmatrix} 1 & 0 \\ 1/2 & 2 \end{bmatrix}}_{n=2}$ $G = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{1\ \text{input}}$ $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ 1 output $D = \emptyset$ => strictly proper

$$W(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1/2 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ -1/2 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(z-1)(z-2)} \begin{bmatrix} z-2 & 0 \\ 1/2 & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(z-1)(z-2)} \begin{bmatrix} z-2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det[\ ]} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$W(z) = \frac{\cancel{(z-2)}}{(z-1)\cancel{(z-2)}} = \boxed{\frac{1}{z-1}} = \frac{1}{1-z^{-1}} z^{-1}$$

pos. pow

NEG pow.

we have only 1 pole "visible" from I/O REPRESENTATION

(BUT $n=2$)

(REMEMBER --> controllability and observability)

from #2 → #1 : T.F. → S.S. → this transformation
(NOT VERY USED in practice) → REALIZATION
(of a T.F. into a S.S. model)

PROBLEM: S.S. REP is NOT UNIQUE →
from a SINGLE T.F. we can get ∞ different
S.S. models (EQUIVALENT)

EXAMPLE → "CONTROL REALIZATION"

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \dots \quad b_{n-1}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots \quad a_n}$$

assumption --> strictly proper system

we assume MONIC denominator

Formula for control realization of $W(z)$:

$$F = \begin{bmatrix} 0 & 1 & & & & \emptyset \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & \emptyset & & & \ddots & \ddots \\ & & & & 0 & 1 \\ -a_n & -a_{n-1} & \dots & & & -a_1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & & b_0 \end{bmatrix} \qquad D = \emptyset$$

Numerical example:

$\to$ Strictly proper

$$W(z) = \frac{2z^{2} + \frac{1}{2}z + \frac{1}{4}}{1z^3 + \frac{1}{4}z^2 + \frac{1}{3}z + \frac{1}{5}}$$

$n = 3$

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{5} & -\frac{1}{3} & -\frac{1}{4} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & 2 \end{bmatrix} \quad D = \emptyset$$

$\to$ cross check $\to$ compute $\underline{W(z)} = H(zI - F)^{-1}G$

from #2 (T.F.) to #3 (I.R.) → EASY → JUST make the ∞-long division between NUM and DEN of $W(z)$ →

<u>Ex</u>:

$$W(z) = \frac{1}{z - \frac{1}{2}} = \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$
\begin{array}{l|l}
z^{-1} & 1 - \frac{1}{2} z^{-1} \\
\hline
-z^{-1} + \frac{1}{2} z^{-2} & z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{4} z^{-3} \\
\hline
// \quad \frac{1}{2} z^{-2} & \\
\quad -\frac{1}{2} z^{-2} + \frac{1}{4} z^{-3} & \\
\hline
// \quad \frac{1}{4} z^{-3} &
\end{array}
$$

$$W(z) = \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}} = \underset{\uparrow\, w(0)}{0}\, z^{-0} + \underset{\uparrow\, w(1)}{1}\, z^{-1} + \underset{\nwarrow\, w(2)}{\frac{1}{2}}\, z^{-2} + \underset{\nwarrow\, w(3)}{\frac{1}{4}}\, z^{-4} + \dots$$

$w(0)$ $w(1)$ $w(2)$ $w(3)$ ... I.R. VALUES

Quicker way:

$$y(t) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}u(t) = \left(z^{-1}\cdot\frac{1}{1-\frac{1}{2}z^{-1}}\right)u(t)$$

$$\boxed{\text{GEOM. SERIES: } \sum_{k=0}^{+\infty} a^k = \frac{1}{1-a} \quad (\text{if } |a|<1)}$$

$$y(t) = \left(z^{-1}\sum_{k=0}^{+\infty}\left(\frac{1}{2}z^{-1}\right)^k\right)u(t)$$

$$y(t) = \left(\underbrace{0}_{\omega(0)} + \underbrace{1}_{\omega(1)}z^{-1} + \underbrace{\frac{1}{2}}_{\omega(2)}z^{-2} + \underbrace{\frac{1}{4}}_{\omega(3)}z^{-3} + \frac{1}{8}z^{-4}\ldots\right)u(t)$$

$\omega(3)$ ..

Remark (NOTATIONAL REMARK):

$$W(z) = \boxed{\frac{z^{-1}}{1+\frac{1}{3}z^{-1}}}$$ → called an "IIR filter"

Infinite Impulse Response

→ T.F. of DIGITAL FILTER

$$W(z) = \boxed{z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}}$$ → "F.I.R. filter"

Finite Impulse Response

$w(0)=0$, $w(1)=1$, $w(2)=\frac{1}{2}$, $w(3)=\frac{1}{4}$

$\boxed{w(k)=0 \ \forall \ k \geq 4}$

Transf. from #3 (I.R.) $\longrightarrow$ #2 (T.F.)

we need to recall a fundamental theoretical definition and result:

<u>DEF</u>: Given a Discrete-Time signal $s(t)$ $\left(s(t) = \emptyset \ \forall t < 0\right)$, its "<u>Z-Transform</u>" is defined as:

$$\boxed{\mathcal{Z}\left(s(t)\right) = \sum_{t=0}^{+\infty} s(t) z^{-t}}$$

$\rightarrow$ Transform of a <u>SIGNAL</u>

$\rightarrow$ function of z

Given this def. it can be proven that:

$$\boxed{\boxed{W(z)} = \mathcal{Z}(w(t)) = \sum_{t=0}^{+\infty} w(t) z^{-t}}$$

→ is the TRANSf. formula from I.R to T.F.

the T.F. of a SYSTEM is the

Z transform of a special signal - this signal is the Impulse Response of the system

Remark → can this formula be used IN PRACTICE to move from IR → TF?

$\boxed{\text{NO!}}$ → we NEED:

- $\boxed{\infty \text{ points of the I.R.}}$
- I.R. must be NOISE FREE

make this formula/transformation only theoretical

Transformation from #1 (SS) to #3 (IR)

Start from

$$\boxed{\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + \emptyset \end{cases} \quad \text{Initial conditions:} \quad \begin{aligned} x(0) &= \emptyset \\ y(0) &= \emptyset \end{aligned}}$$

$$x(1) = F\cancel{x(0)} + Gu(0) = Gu(0)$$

$$y(1) = Hx(1) = HGu(0)$$

$$x(2) = Fx(1) + Gu(1) = FGu(0) + Gu(1)$$

$$y(2) = Hx(2) = HFGu(0) + HGu(1)$$

$$x(3) = Fx(2) + Gu(2) = F^2Gu(0) + FGu(1) + Gu(2)$$

$$y(3) = Hx(3) = HF^2Gu(0) + HFGu(1) + HGu(2)$$

⋮

$$y(t) = \underbrace{\emptyset}_{\omega(0)} u(t) + \underbrace{HG}_{\omega(1)} u(t-1) + \underbrace{HFG}_{\omega(2)} u(t-2) + \underbrace{HF^2G}_{\omega(3)} u(t-3) + \dots$$

$\dots$

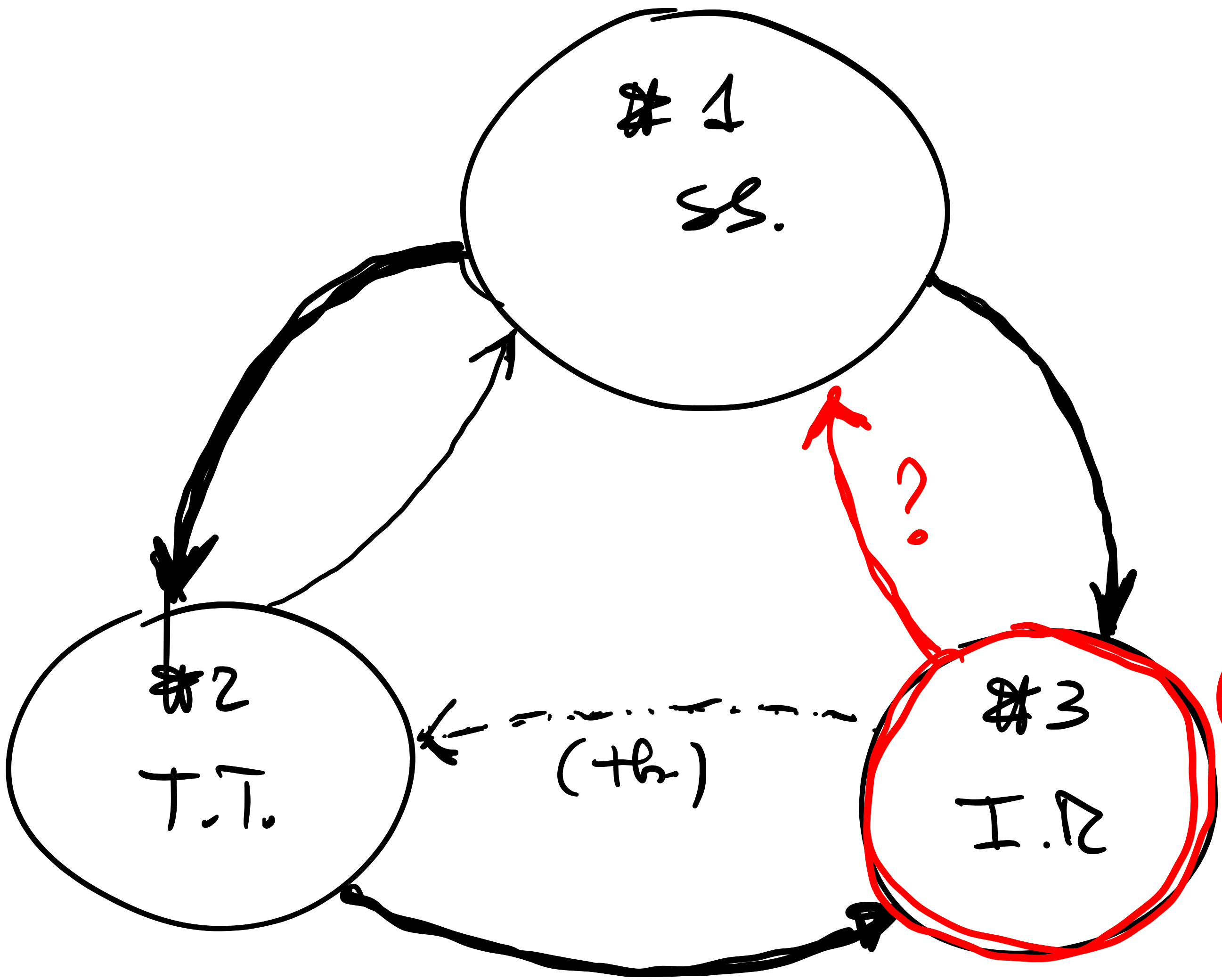
$$\boxed{\omega(t) = \begin{cases} \emptyset & \text{if } t = \emptyset \quad (\text{if } D = 0) \\ HF^{t-1}G & \text{if } t > \emptyset \end{cases}}$$

※1 (SS)

↓

※3 (I.R.)

↓ SUMMARY OF TRANSFORMATIONS:



Easy to obtain Experimentally

Difficult to "get out"

Moving from I.R. to S.S. is the key task of:

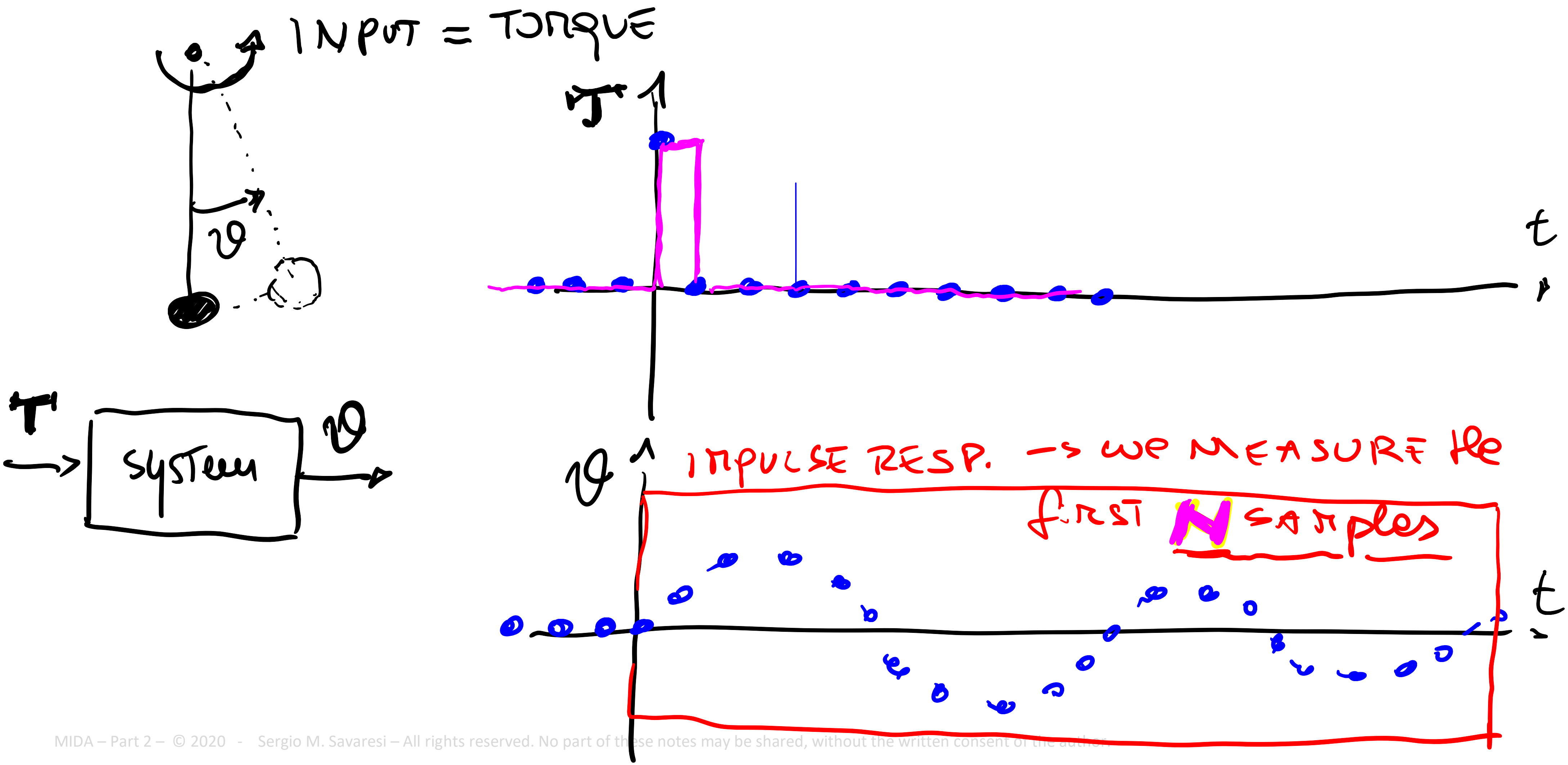
SUBSPACE-BASED STATE SPACE SYSTEM IDENTIFICATION

$\boxed{\text{4SID methods}}$

Remark -> in MATLAB System ID. Toolbox:

>> n4sid

MOTIVATION → the original (first developed) 4SID method STARTS from the MEASUREMENT of the system output in a VERY simple EXPERIMENT → "IMPULSE EXPERIMENT"



The idea is that it is experimentally very easy to make this experiment →

problem: how to IDENTIFY a model $\{\hat{F}, \hat{G}, \hat{H}\}$?

STARTING from

$$\boxed{\{ \omega(0), \omega(1), \omega(2) \; \dots \; \omega(N) \}}$$

Remark: for TUTORIAL REASONS we'll SEE the solution of this problem in TWO STEPS:

① → I.R. measurement is ASSUMED "NOISE-FREE"

easier but NOT REALISTIC

REAL PROBLEM

② I.R. is measured with NOISE

$$\underbrace{\tilde{\omega}(t)}_{\text{MEASURED NOISY I.R.}} = \underbrace{\omega(t)}_{\text{TRUE (NOISE-FREE) I.R.}} + \underbrace{\eta(t)}_{\text{MEASUREM. NOISE (e.g. WN)}} \qquad t = 0, 1, 2 \ldots N$$

Remark → For tutorial reasons we see in detail only GSID when the experiment is an impulse-experiment (1st and original version of GSID)

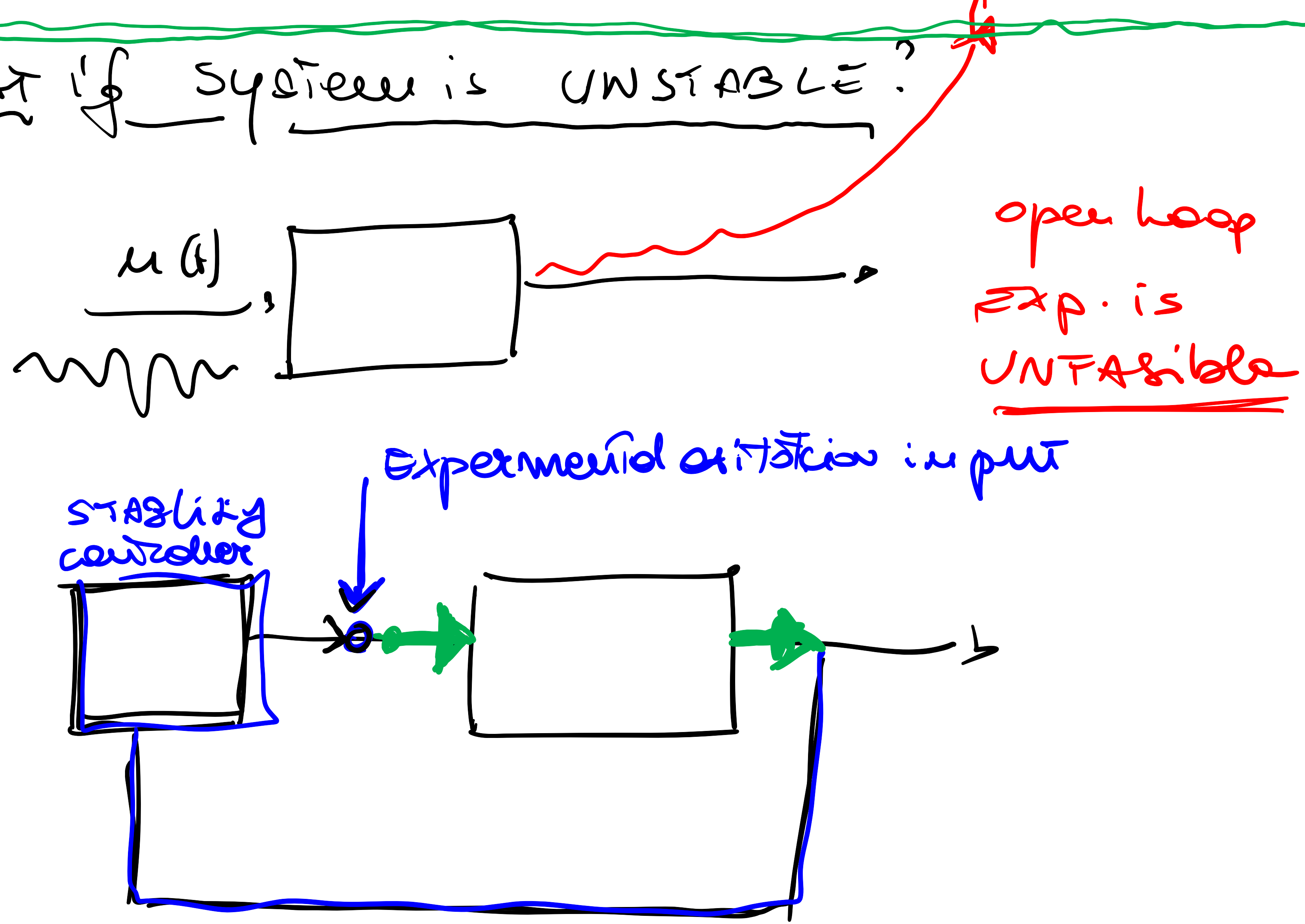
HOWEVER GSID can be EXTENDED to ANY input signal

$$\boxed{\{u(1), u(2) \dots u(N)\}}$$

GENERIC INPUT (sufficiently exciting)

$$\{y(1), y(2) \dots y(N)\}$$

what if system is UNSTABLE ?



I/O DATA ARE collected in a closed-loop experiment

Before presenting the algorithm we need to recall the fundamental concepts of:

- OBSERVABILITY AND
- CONTROLLABILITY (REACHABILITY)

of a dynamical system

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

The system is fully <u>observable from the output</u> if and only if the observability matrix

$$O = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}$$ is full-rank

$$\text{RANK}(O) = n$$

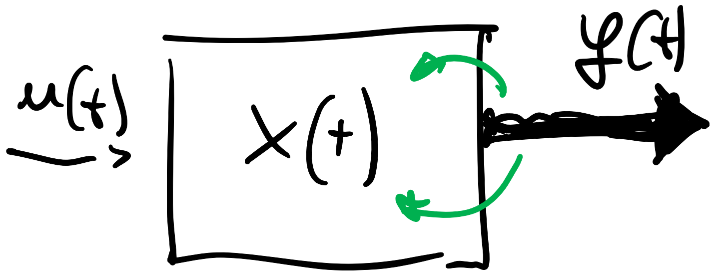
The system is fully CONTROLLABLE (REACHABLE) from the INPUT if and only if the CONTROLLABILITY MATRIX

$$R = [G \; FG \; F^2G \ldots F^{n-1}G]$$

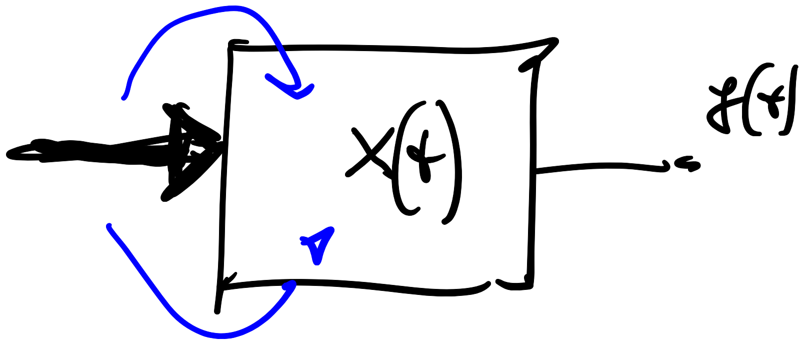
is full rank

$$\text{RANK}(R) = n$$

OBSERVABILITY of the STATE from OUTPUT = we can "SEE" the state from the output SENSORS:



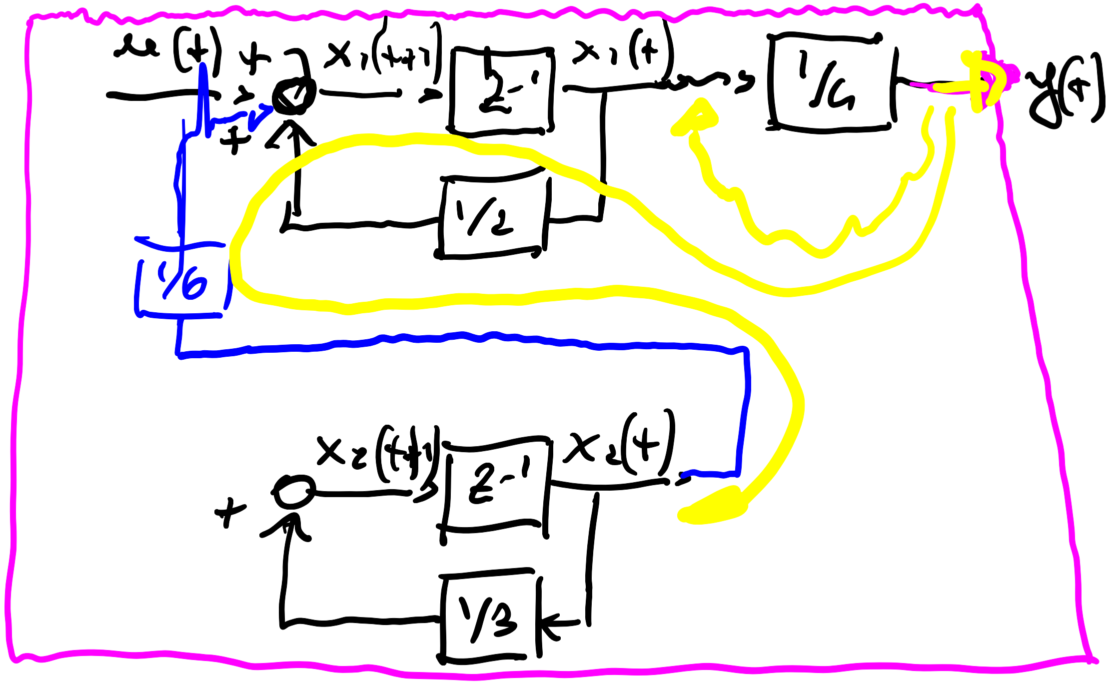
CONTROLLABILITY of the STATE from the INPUT = we can "CONTROL" ("MOVE") the state using input



Example

$$S:\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + u(t) + \frac{1}{6}x_2(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) \\ y(t) = \frac{1}{4}x_1(t) \end{cases}$$

$$n = 2 \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$



$$F = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix} \Rightarrow O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 1/8 & 0 \end{bmatrix} \rightarrow \text{RANK} = 1 < 2$$

$\Rightarrow S$ NOT fully observable!

$$F = \begin{bmatrix} 1/2 & 1/6 \\ 0 & 1/3 \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix} \Rightarrow O = \begin{bmatrix} 1/4 & 0 \\ 1/8 & 1/24 \end{bmatrix} \rightarrow \text{RANK} = 2 \Rightarrow$$

S fully observable!

$$
S: \begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + \frac{1}{6} x_2(t) \\ x_2(t+1) = \frac{1}{3} x_2(t) + u(t) \\ y(t) = \frac{1}{4} x_1(t) \end{cases}
$$

$$
F = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 0 & 0 \\ 1 & 1/3 \end{bmatrix} \rightarrow \text{RANK} = 1 \Rightarrow
$$

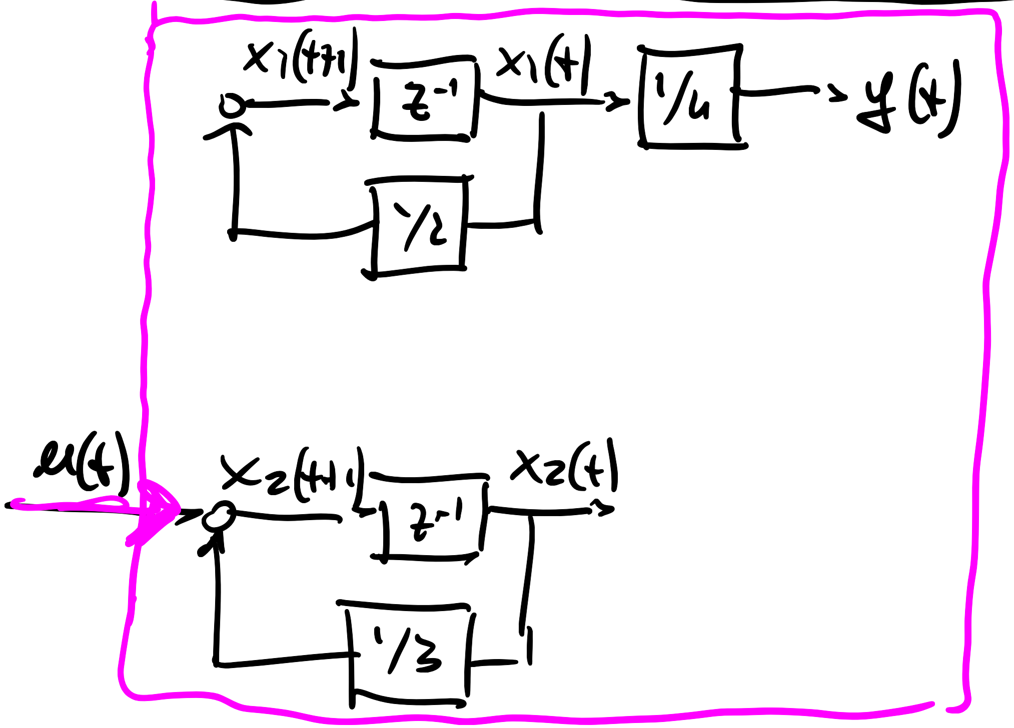
NOT controllable

$$
F = \begin{bmatrix} 1/2 & 1/6 \\ 0 & 1/3 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

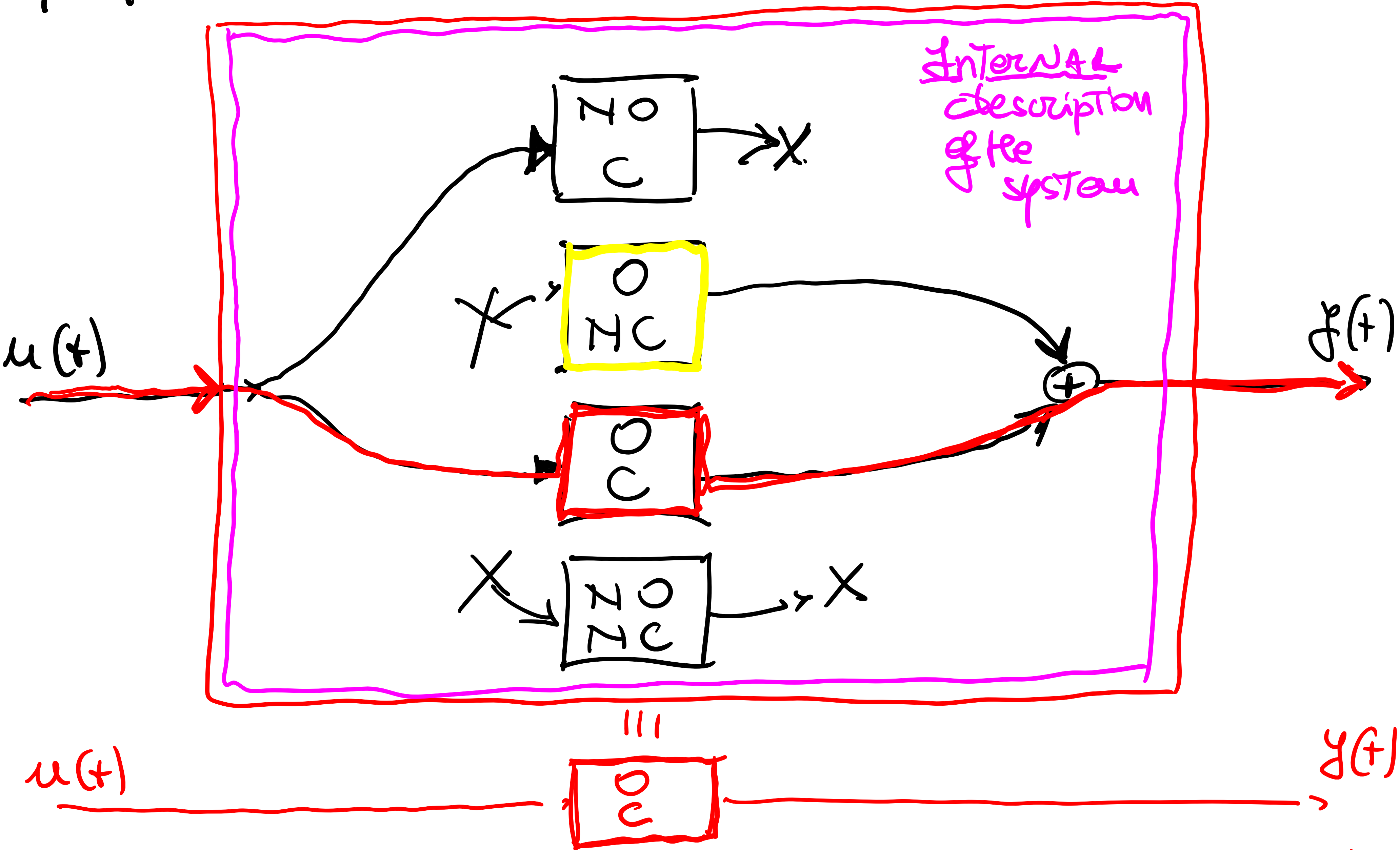
$$
R = \begin{bmatrix} 0 & 1/6 \\ 1 & 1/3 \end{bmatrix} \rightarrow
$$

RANK = 2 $\Rightarrow$

Fully controllable!



Any system can be divided into 4 sub-systems:



external (I/O) description → only O AND C part is VISIBLE!

FINAL DEFINITION (BEFORE GRID ALGO.):

HANKEL MATRIX of order n built from I.R. $\{\omega(0), \omega(1), \omega(2) \dots \omega(N)\}$

we start from $\omega(1)$ (not from $\omega(0)$; $\omega(0)=D$)

$$\underbrace{H_n}_{\text{square matrix "}n \times n\text{"}} = \underbrace{\begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \dots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \dots & \omega(n+1) \\ \omega(3) & \omega(4) & \omega(5) & \dots & \omega(n+2) \\ \vdots & & & & \\ \omega(n) & \omega(n+1) & \dots & & \omega(2n-1) \end{bmatrix}}_{n} \Big\} n$$

we need I.R. up to time $2n-1$

Now recall Transformation #1 (SS) $\longrightarrow$ #3 (IR):

$$\{F, G, H\} \longrightarrow \omega(t) = \begin{cases} 0 & \text{if } t = 0 \quad (\text{if } D = 0) \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$$

we can re-write H_n

$$H_n = \begin{bmatrix} HG & HFG & \cdots & HF^{n-1}G \\ HFG & HF^2G & \cdots & HF^{n}G \\ \vdots & & & \\ HF^{n-1}G & & \cdots & HF^{2n-2}G \end{bmatrix} = \underbrace{\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}}_{O} \underbrace{\begin{bmatrix} G & FG & F^2G & \cdots & F^{n-1}G \end{bmatrix}}_{R}$$

$$\Rightarrow \boxed{H_n = O \cdot R}$$

H_n can be factorized into observ. × controllability matrix