PAM and QAM Modulation

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PAM

Let the baseband real data signal before the matched filter be

$$x(t) = \sum_i a_i h(t - iT) + w(t)$$

where $H(f)$ is the square root of a Nyquist frequency response ($E_h = 1$), the real symbols $\{a_i\}$ are i.i.d. random variables drawn with uniform probability from the set

$$\mathcal{A}_{PAM} = \{\pm A, \pm 3A, \cdots, \pm A(L-1)\}$$

with $L = 2^n$, where n is the number of bits per PAM symbol, and $w(t)$ is AWGN with power spectral density $N_0/2$. It is easy to see that $E\{a\} = 0$ and that

$$\sigma_a^2 = \frac{L^2 - 1}{3} \cdot A^2.$$

The output of the sampled matched filter is the sequence

$$y_k = a_k + n_k$$

where n_k is AWGN with variance $\sigma_n^2 = N_0/2$.

PAM

The Signal-to-Noise Ratio after the sampled matched filter is

$$\text{SNR} = \frac{2E\{a^2\}}{N_0} = \frac{2\sigma_a^2}{N_0}.$$

The symbol rate is

$$R_s = T^{-1},$$

the bit rate is

$$R_b = R_s \log_2 L = nR_s.$$

Note that the SNR after the sampled matched filter is the same as the SNR before the sampled matched filter if the power of the noise before the sampled matched filter is computed in a bandwidth $B = R_s/2$, that is the minimum bandwidth for transmission at symbol frequency R_s:

$$P_s = R_s \sigma_a^2 E_h, \quad P_n = N_0 B = N_0 R_s/2,$$

that , with $E_h = 1$ (Nyquist condition), leads to SNR $= 2\sigma_a^2/N_0$.

Symbol Error Probability of PAM

Consider threshold detection and write the symbol error probability of L-PAM as

$$P_{s,pam}(e) = \sum_{a \in \mathcal{A}_{PAM}} P(a)P(e|a) = k_n Q\left(\frac{d_{min}}{2\sigma_n}\right)$$

where

$$k_n = \frac{2 \cdot (L-1)}{L}$$

is the average number of nearest neighbor of L-PAM and $d_{min} = 2A$ is the minimum distance between two nearest neighbor in the noiseless PAM after sampling. Note that

$$Q\left(\frac{d_{min}}{2\sigma_n}\right) = Q\left(\sqrt{\frac{2A^2}{N_0}}\right) = Q\left(\sqrt{\frac{3 \cdot \mathsf{SNR}}{L^2-1}}\right).$$

Q-function

The function $Q(y)$ is the probability that a Gaussian random variable with zero mean and unit variance exceeds y:

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^{\infty} \exp(-x^2/2)dx, \quad Q\left(\frac{d}{2\sigma_n}\right) = \frac{1}{\sigma_n\sqrt{2\pi}} \int_{d/2}^{\infty} \exp(-x^2/2\sigma_n^2)dx.$$

For $y > 3$, the Q-function is closely approximated as

$$Q(y) \approx \frac{1}{y \cdot \sqrt{2\pi}} \exp(-y^2/2).$$

Someone prefers to use the complementary error function

$$erfc(y) = 2 \cdot Q(y\sqrt{2}),$$

$$Q(y) = \frac{1}{2} \cdot erfc(y/\sqrt{2}).$$

Bit Error Probability of PAM

When Gray mapping is adopted, with high probability one symbol error produces only bit error among the $log_2\sqrt{M}$ bits per dimension. Neglecting the case where multiple bit errors occur one writes

$$P_b(e) \approx \frac{P_{s,pam}(e)}{log_2 L}.$$

For binary modulation ($L = 2$), the above formula is exact because in case of symbol error on PAM one has always one bit error, therefore

$$P_b(e) = Q\left(\frac{d_{min}}{2\sigma}\right).$$

Bit Error Probability of 2-PAM versus SNR

For 2-PAM one has

$$P_b(e) = Q\left(\sqrt{\text{SNR}}\right).$$

Some values of the bit error probability versus SNR are:

$$P_b(e) = 10^{-3}, \;\; \text{SNR} = 9.79 \text{ dB},$$

$$P_b(e) = 10^{-5}, \;\; \text{SNR} = 12.59 \text{ dB},$$

$$P_b(e) = 10^{-7}, \;\; \text{SNR} = 14.31 \text{ dB},$$

$$P_b(e) = 10^{-10}, \;\; \text{SNR} = 16.06 \text{ dB},$$

$$P_b(e) = 10^{-13}, \;\; \text{SNR} = 17.31 \text{ dB}.$$

QAM

Let the complex symbols be i.i.d. random variables drawn with uniform probability from the alphabet (constellation)

$$\mathcal{A}_{QAM} = (\{\pm A, \pm 3A, \cdots, \pm A(\sqrt{M}-1)\}) \times (\{\pm jA, \pm j3A, \cdots, \pm jA(\sqrt{M}-1)\}).$$

It is easy to see that the variance of the complex symbols is

$$\sigma_a^2 = \frac{M-1}{3} \cdot 2A^2.$$

The number of bits per complex channel use, that is per QAM symbol, is

$$n = \frac{R_b}{R_s} = \log_2 M \;\; b/2D.$$

Considering the case of zero rolloff, where the bandwidth of passband QAM is $B = R_s \;\; Hz$, the *spectral efficiency*, that is the number of bits per second with a signal of bandwidth equal to one Hertz which is usually indicated with η, is just the above n:

$$\eta = \frac{R_b}{B} = n \;\; b/(Hz \cdot s).$$

Bit Error Probability of QAM

The bit error probability of M-QAM with Gray mapping on both rails is the same as the bit error probability of the two constituent $\sqrt{M}$-PAM:

$$P_b(e) \approx \frac{2 \cdot (\sqrt{M} - 1)}{\log_2(\sqrt{M}) \cdot \sqrt{M}} \cdot Q\left(\sqrt{\frac{3 \cdot \text{SNR}}{M-1}}\right).$$

As in the case of baseband transmission, the SNR after the sampled matched filter is the same as the SNR before the sampled matched filter provided that the noise power is evaluated in a bandwidth equal to the minimum Nyquist bandwidth for transmission at rate R_s. However, the minimum bandwidth of the passband signal is two times the minimum bandwidth of the baseband signal, therefore, with passband transmission and $E_h = 1$,

$$P_s = R_s \sigma_a^2, \;\; P_n = N_0 B = N_0 R_s, \;\; \text{SNR} = \frac{P_s}{P_n} = \frac{\sigma_a^2}{N_0}.$$

The above formula shows that, to achieve the same SNR of baseband transmission with the same N_0, hence the same error rate on the same channel when $M = L^2$, the power of the passband signal must be two times the power of the baseband signal.