Multidimensional Digital Modulation

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Error Probability of Multidimensional Constellations

With truly multidimensional constellations, the evaluation of error probability is less simple than with one-dimensional constellations, because the decision regions can be non-regular polygons, hence the probability of symbol error cannot be calculated as the integral of the tail of an one-dimensional Gaussian distribution. Let

$$\mathbf{x} = \mathbf{a} + \mathbf{w}$$

be the received vector made as the sum of the transmitted symbol $\mathbf{a}$ belonging to the alphabet (constellation) $\mathcal{A}$ and of the AWGN $\mathbf{w}$ with two-dimensional variance N_0.

Union Bound

Invoking the union bound, one writes an upper bound on the symbol error probability given $\mathbf{a}$ as

$$P_s(e|\mathbf{a}) \leq \sum_{d \geq d_{min}} k_{d,\mathbf{a}} \cdot Q\left(\sqrt{\frac{d^2}{4\sigma^2}}\right),$$

where $k_{d,\mathbf{a}}$ is the number of adversaries of $\mathbf{a}$ at distance d that dtermine the boundary of the decision region of $\mathbf{a}$, d_{min} is the distance between $\mathbf{a}$ and its nearest neighbor, and $\sigma^2 = N_0/2$ is the variance of the one-dimensional noise. Define now the average number of adversaries at distance d over the entire constellation as

$$k_d = \sum_{\mathbf{a} \in \mathcal{A}} k_{d,\mathbf{a}} \cdot P(\mathbf{a}),$$

leading to

$$P_s(e) = \sum_{\mathbf{a} \in \mathcal{A}} P_s(e|\mathbf{a}) \cdot P(\mathbf{a}) \leq \sum_{d \geq d_{min}} k_d \cdot Q\left(\sqrt{\frac{d^2}{4\sigma^2}}\right).$$

Approximation to the Symbol Error Probability

At intermediate-to-low error probability, the Q-function dominates over its coefficient, hence the union bound is closely approximated by taking into account only the nearest neighbors

$$P_s(e) \approx k_{d_{min}} \cdot Q\left(\sqrt{\frac{d_{min}^2}{4\sigma^2}}\right).$$

Union Bound for 32-Cross

For the cross-constellation with 32 points one has (consider one quadrant by rows):

$$k_{d_{min}} = \frac{1}{8} \cdot (4 + 4 + 3 + 4 + 4 + 2 + 3 + 2) = \frac{13}{4}.$$

Also, in 32-cross the decision region of the points with larger energy is determined by one non-nearest neighbor:

$$k_{\sqrt{2}d_{min}} = \frac{1}{8} \cdot (1 + 1) = \frac{1}{4}.$$

Union Bound for PSK

The M-PSK constellation is the set

$$\mathcal{A} = \{Ae^{j2k\pi/M}\}, \;\; k = 1, 2, \cdots, M.$$

The boundary of the decision region is determined by one nearest neighbor for $M = 2$, by two nearest neighbors for $M > 2$, therefore $k_{d_{min}} = 1$ for $M = 2$, $k_{d_{min}} = 2$ elsewhere. The bit error probability depends on how bits are mapped onto constellation symbols. For PSK Gray mapping is feasible. In Gray mapping, any two nearest neighbor symbols differ for only one bit, therefore, neglecting the probability of error between two non-nearest neighbor symbols, for M-PSK one has

$$P_b(e) \approx \frac{P_s(e)}{\log_2 M} \leq \frac{1}{\log_2 M} \cdot 2 \cdot Q\left(\sqrt{\frac{d_{min}^2}{4\sigma^2}}\right).$$

Symbol Error Probability of 2D Constellations Versus SNR

Considering the approximation based on nearest neighbors, we suggest to use the formula

$$P_s(e) \approx k_{d_{min}} \cdot Q\left(\sqrt{\frac{2 \cdot \text{SNR}}{k_c}}\right),$$

where k_c is the power of the 2D constellation when it is scaled in such a way that

$$d_{min} = 2.$$

Symbol Error Probability of 2D Constellations Versus SNR

For M-QAM one has

$$k_c = \frac{2 \cdot (M-1)}{3}.$$

For M-PSK one has

$$\frac{1}{k_c} = sin^2\left(\frac{\pi}{M}\right).$$

For general AM-PM constellations based on the grid of two-dimensional integers $\mathcal{Z}^2$, one calculates k_c as the variance of constellation symbols, by considering the constellation symbols drawn from the two-dimensional grid of odd integers.

Exercise: show that, for the popular cross-constellations, one has $k_c = 20$ for the 32-points constellation, while $k_c = 82$ for the 128-points constellation.

The Law of 6 dB per Bit with PSK

Suppose we want to increase the number of bits per channel use while maintaining a virtually error free transmission. This means that we want an argument of the Q-function that corresponds to an error probability that is acceptable for the application at hand. Recall that, for M-PSK , the symbol error probability is

$$P_s(e) \leq 2 \cdot Q\left(\sqrt{2 \cdot \text{SNR} \cdot sin^2\left(\frac{\pi}{M}\right)}\right),$$

with

$$2 \cdot \text{SNR} \cdot sin^2\left(\frac{\pi}{M}\right) = \frac{d_{min}^2}{4\sigma^2}.$$

The Law of 6 dB per Bit with PSK

Suppose of being satisfied with the symbol error probability

$$P_s(e) \approx 2 \cdot 10^{-5},$$

corresponding to

$$\frac{d_{min}^2}{4\sigma^2} \approx 18.$$

The SNR required for transmitting

$$\eta = \log_2 M \;\; b/2D$$

for large M is obtained by substituting the argument in place of the sin function:

$$2 \cdot \text{SNR} \cdot \left(\frac{\pi}{M}\right)^2 = \frac{d_{min}^2}{4\sigma^2} = 18.$$

The Law of 6 dB per Bit with PSK

From the above equation, for the SNR one gets

$$\text{SNR} = 9 \cdot \left(\frac{M}{\pi}\right)^2 = 9 \cdot \left(\frac{2^\eta}{\pi}\right)^2.$$

Therefore

$$\text{SNR} = 9 \cdot \frac{2^{2\eta}}{\pi^2},$$

$$\eta = \frac{1}{2}\log_2\left(\frac{\pi^2\text{SNR}}{9}\right)\ b/2D.$$

We conclude that, with large M, the cost of adding one bit per symbol is 6 dB of SNR.

The Law of 3 dB per Bit with QAM

For M-QAM the one-dimensional symbol error probability is

$$P_{s,pam}(e) \leq 2 \cdot Q\left(\sqrt{\frac{3 \cdot \text{SNR}}{M-1}}\right),$$

$$P_{s,qam}(e) \leq 2P_{s,pam}(e).$$

Suppose again of being satisfied with

$$P_{s,pam}(e) \approx 2 \cdot 10^{-5},$$

that corresponds to

$$\frac{3 \cdot \text{SNR}}{M-1} = \frac{d_{min}^2}{4\sigma^2} \approx 18.$$

The Law of 3 dB per Bit with QAM

The SNR required for transmitting $\eta = \log_2 M$ b/2D is

$$\text{SNR} = \frac{(M-1)\cdot d_{min}^2}{3\cdot 4\sigma^2} = \frac{(2^\eta - 1)\cdot 18}{3},$$

$$\eta = \log_2\left(1 + \frac{\text{SNR}}{6}\right) \;\; b/2D.$$

We conclude that, at SNR high enough to make negligible the term +1 inside the log, the cost of adding one bit per symbol is 3 dB of SNR.

Improving QAM by the Hexagonal Lattice

An example of 2D channel coding is a two-dimensional constellation whose decision regions are hexagons. For large M, the area occupied by the constellation is

$$A_{hexa} = M d_{hexa}^2 \frac{\sqrt{3}}{2},$$

where d_{hexa} is the minimum distance between two points in the hexagonal lattice. The area occupied by QAM is

$$A_{qam} = M d_{qam}^2.$$

With equal M and area (hence equal energy),

$$d_{qam}^2 = d_{hexa}^2 \frac{\sqrt{3}}{2}.$$

Improving QAM by the Hexagonal Lattice

For a fixed argument of the Q-function, hence fixed error probability, and for a fixed and large number of points,

$$\text{SNR}_{qam} = \text{SNR}_{hexa} \frac{2}{\sqrt{3}}.$$

With the hexagonal lattice and for large M, the cost of adding one bit per symbol is 3 dB of SNR as with QAM, but the hexagonal lattice achieves an advantage of

$$10 \log_{10} \frac{2}{\sqrt{3}} = 0.62 \text{ dB}$$

in SNR over QAM.