# PRACTICE SESSION 6

(C) Compute the steady state 1-STEP predictors.

Method 1: Consider separately the 2 subsystems

SUBSYSTEM A

$$\begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + v_{11}(t) \\ y_A(t) = 0 \end{cases}$$

$$\bar{K}_A = (F_A \bar{P}_A \cancel{H_A^T} + \cancel{V_{12A}})(H_A \bar{P}_A H_A^T + V_{2A})^{-1} = 0$$

$$\hat{x}_1(t+1|t) = \frac{1}{2} \hat{x}_1(t|t-1)$$

$x_1(t)$ does not influence $y(t)$. Measuring the output it is not possible to get information about $x_1(t)$. The best we can do is to replicate the dynamic of the first subsystem

SUBSYSTEM B:

$$\begin{cases} x_2(t+1) = 2x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$$

(Same procedure of Ex 1)

$$\bar{P} = F\bar{P}F^T + V_1 - (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}(F\bar{P}H^T + V_{12})^T$$

$$\bar{P} = 4\bar{P} + 1 - \frac{4\bar{P}^2}{\bar{P}+1}$$

$$\bar{P}(\bar{P}+1) = (\bar{P}+1)(4\bar{P}+1) - 4\bar{P}^2$$

$$\bar{P}^2 + \cancel{\bar{P}} = \cancel{4\bar{P}^2} + \cancel{\bar{P}} + 4\bar{P} + 1 \cancel{- 4\bar{P}^2}$$

$$\bar{P}^2 - 4\bar{P} - 1 = 0$$

$$\bar{P} = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \frac{\sqrt{20}}{2} \begin{cases} 2+\sqrt{5} \\ \cancel{2-\sqrt{5}} < 0 \end{cases} \qquad \bar{P} = 2+\sqrt{5}$$

$$\bar{K}_B = (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}$$

$$\bar{K}_B = \frac{2(2+\sqrt{5})}{2+\sqrt{5}+1} = \frac{4+2\sqrt{5}}{3+\sqrt{5}}$$

$$\begin{cases} \hat{x}_2(t+1|t) = 2\hat{x}_2(t|t-1) + \frac{4+2\sqrt{5}}{3+\sqrt{5}}\left(y(t) - \hat{y}(t|t-1)\right) \\ \hat{y}(t|t-1) = \hat{x}_2(t|t-1) \end{cases}$$

TOTAL SYSTEM

$$\begin{cases} \hat{x}_1(t+1|t) = \frac{1}{2}\hat{x}_1(t|t-1) \\ \hat{x}_2(t+1|t) = 2\hat{x}_2(t|t-1) + \frac{4+2\sqrt{5}}{3+\sqrt{5}}\left(y(t) - \hat{y}(t|t-1)\right) \\ \hat{y}(t|t-1) = \hat{x}_2(t|t-1) \end{cases}$$

$$\begin{cases} \hat{x}(t+1|t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \hat{x}(t|t-1) + \begin{bmatrix} 0 \\ \frac{4+2\sqrt{5}}{3+\sqrt{5}} \end{bmatrix} \left(y(t) - \hat{y}(t|t-1)\right) \\ \hat{y}(t|t-1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}(t|t-1) \end{cases}$$

Method II : Consider the entire system

ARE: $\bar{P} = F\bar{P}F^T + V_1 - (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}(F\bar{P}H^T + V_{12})^T$

We have to find $\bar{P} \geq 0$ that satisfies the ARE

$$\bar{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \boxed{\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} +$$

$$- \underbrace{\left(\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}_{\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \frac12\beta \\ 2\gamma \end{bmatrix}} \underbrace{\left(\boxed{\begin{bmatrix} 0 & 1 \end{bmatrix}}\begin{bmatrix} \alpha & \beta \\ \beta & \boxed{\gamma} \end{bmatrix}\boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + 1\right)^{-1}}_{\frac{1}{\gamma+1}} \underbrace{\left(\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^T}_{\begin{bmatrix} \frac12\beta & 2\gamma \end{bmatrix}}$$

$$\frac{1}{\gamma+1}\begin{bmatrix} \frac12\beta \\ 2\gamma \end{bmatrix}\begin{bmatrix} \frac12\beta & 2\gamma \end{bmatrix} = \frac{1}{\gamma+1}\begin{bmatrix} \frac14\beta^2 & \beta\gamma \\ \beta\gamma & 4\gamma^2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} \frac12\alpha & \frac12\beta \\ 2\beta & 2\gamma \end{bmatrix}\begin{bmatrix} \frac12 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac14\alpha+1 & \beta \\ \beta & 4\gamma+1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} \frac14\alpha+1 & \beta \\ \beta & 4\gamma+1 \end{bmatrix} - \frac{1}{\gamma+1}\begin{bmatrix} \frac14\beta^2 & \beta\gamma \\ \beta\gamma & 4\gamma^2 \end{bmatrix} =$$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} \frac14\alpha - \frac{\frac14\beta^2}{\beta\gamma} + 1 & \beta - \frac{\beta\gamma}{\gamma+1} \\ \beta - \frac{\beta\gamma}{\gamma+1} & 4\gamma+1 - \frac{4\gamma^2}{\gamma+1} \end{bmatrix}$$

$$\begin{cases} \alpha = \frac14\alpha - \frac14\beta^2 + 1 & (1) \\ \beta = \beta - \frac{\beta\gamma}{\gamma+1} & (2) \\ \gamma = 4\gamma + 1 - \frac{4\gamma^2}{\gamma+1} & (3) \end{cases}$$

$(2) \rightarrow \beta \quad \frac{\beta\gamma}{\gamma+1} = 0 \quad \rightarrow \quad \beta = 0 \quad \left(\text{if } \gamma = 0 \Rightarrow (3) \quad 1 = 0\right)$

$$\bar{P} = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} \Rightarrow \quad \alpha, \gamma \geq 0$$

③ $\longrightarrow \gamma$

$$\gamma^2+\gamma = (4\gamma+1)(\gamma+1) - 4\gamma^2$$

$$\gamma^2 + \cancel{\gamma} = \cancel{4\gamma^2} + 4\gamma + \cancel{\gamma} + 1 - \cancel{4\gamma^2}$$

$$\gamma^2 - 4\gamma - 1 = 0 \quad \begin{cases} 2+\sqrt{5} \\ \cancel{2-\sqrt{5}} \end{cases} \qquad \gamma = 2+\sqrt{5}$$

① $\longrightarrow \alpha$

$$\alpha = \frac{1}{4}\alpha + 1 \quad \longrightarrow \quad \alpha = 4/3$$

$$\bar{P} = \begin{bmatrix} 4/3 & 0 \\ 0 & 2+\sqrt{5} \end{bmatrix}$$

$$\bar{K} = \underbrace{(F\bar{P}H^T + V_{12})}_{A}\underbrace{(H\bar{P}H^T + V_2)^{-1}}_{B}$$

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4/3 & 0 \\ 0 & 2+\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2+\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 4+2\sqrt{5} \end{bmatrix}$$

$$B = \left( \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 4/3 & 0 \\ 0 & 2+\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \right)^{-1} = \frac{1}{3+\sqrt{5}}$$

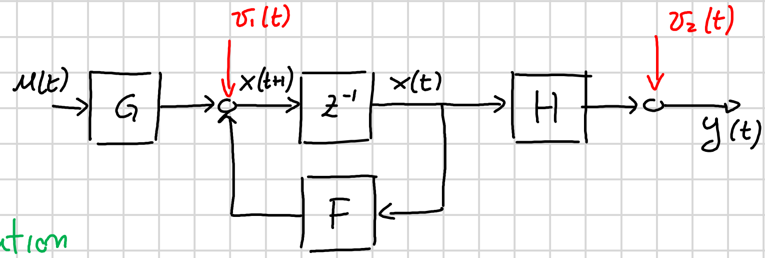
$$\bar{K} = \begin{bmatrix} 0 \\ \dfrac{4+2\sqrt{5}}{3+\sqrt{5}} \end{bmatrix}$$

$$\boxed{\begin{cases} \hat{x}(t+1|t) = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}\hat{x}(t|t-1) + \begin{bmatrix} 0 \\ \dfrac{2(2+\sqrt{5})}{3+\sqrt{5}} \end{bmatrix}\left(y(t) - \hat{y}(t|t-1)\right) \\ \hat{y}(t|t-1) = \begin{bmatrix} 0 & 1 \end{bmatrix}\hat{x}(t|t-1) \end{cases}}$$

EXERCISE 3

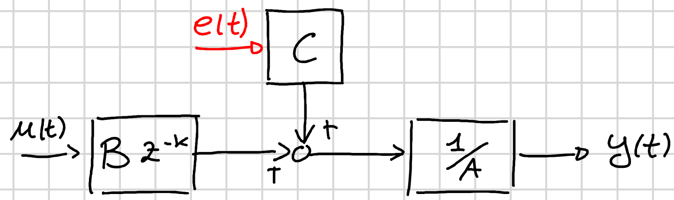
The Kalman Filter Theory is based on a "two noises" system representation.

$$\begin{aligned} x(t+1) &= F x(t) + G u(t) + v_1(t) \\ y(t) &= H x(t) + v_2(t) \end{aligned}$$




? equivalent representation

$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t)$$




Consider the system

$$\begin{cases} x(t+1) = \frac{1}{2} x(t) + \frac{1}{4} u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \qquad \begin{aligned} v_1 &\sim WN(0,1) \\ v_2 &\sim WN(0,1) \end{aligned} \qquad v_1 \perp v_2$$

a) Find the equivalent model in the form

$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \frac{C(z)}{A(z)} e(t) \qquad e \sim WN(\mu, \lambda^2)$$

First of all shift in the z-domain

$$\begin{cases} z\, x(t) = \frac{1}{2} x(t) + \frac{1}{4} u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases}$$

$$x(t) = \frac{1/4}{z - \frac{1}{2}} u(t) + \frac{1}{z - \frac{1}{2}} v_1(t)$$

$$y(t) = \frac{1/4}{z - \frac{1}{2}} u(t) + \frac{1}{z - \frac{1}{2}} v_1(t) + v_2(t)$$

$$y(t) = \frac{1/4\, z^{-1}}{1 - \frac{1}{2} z^{-1}} u(t) + \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}} v_1(t) + v_2(t)$$

$$y(t) = \frac{1/4}{1 - \frac{1}{2} z^{-1}} u(t-1) + \boxed{\frac{z^{-1}}{1 - \frac{1}{2} z^{-1}} v_1(t) + v_2(t)} \; d(t)$$

$$y(t) = \frac{B(z)}{A(z)} u(t-k) + \boxed{\frac{C(z)}{A(z)} e(t)} \; \eta(t)$$

We want to find an equivalent "noise" representation

in frequency domain $\rightarrow \Gamma_d(\omega) = \Gamma_\eta(\omega)$

$$d(t) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}} v_1(t) + v_2(t) = \frac{1}{1-\frac{1}{2}z^{-1}} \tilde{v}_1(t) + v_2(t)$$

$\tilde{v}_1(t) = v_1(t-1)$

$\tilde{v}_1(t) \sim WN(0,1)$

$$\Gamma_d(\omega) = \left| \left( \frac{1}{1-\frac{1}{2}z^{-1}} \right)_{z=e^{j\omega}} \right|^2 \cdot \underbrace{\Gamma_{\tilde{v}_1}(\omega)}_{1} + \underbrace{\Gamma_{v_2}(\omega)}_{1}$$

$$|a+be^{j\omega}|^2 = (a+be^{j\omega})(a+be^{-j\omega})$$

$$\Gamma_d(\omega) = \left( \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} \right)_{z=e^{j\omega}} \quad +1$$

$$\Gamma_d(\omega) = \left( \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} + 1 \right)_{z=e^{j\omega}}$$

$$\Gamma_d(\omega) = \left( \frac{1+1-\frac{1}{2}z^{-1}-\frac{1}{2}z+\frac{1}{4}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} \right)_{z=e^{j\omega}}$$

$$\Gamma_d(\omega) = \left( \frac{\frac{9}{4}-\frac{1}{2}(z^{-1}+z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} \right)_{z=e^{j\omega}}$$

$$\eta(t) = \frac{C(z)}{A(z)} e(t)$$

$A(z) = 1-\frac{1}{2}z^{-1}$ $\quad$ $C(z)$ is unknown $\rightarrow$ consider a parametric polynomials

$$\eta(t) = \frac{1+c_0 z^{-1}}{1-\frac{1}{2}z^{-1}} e(t) \qquad e \sim WN(0,\lambda^2)$$

$c_0$ and $\lambda$ are free parameters.

$$\Gamma_\eta(\omega) = \left| \left( \frac{1+c_0 z^{-1}}{1-\frac{1}{2}z} \right)_{z=e^{j\omega}} \right|^2 \lambda^2$$

$$\Gamma_\eta(\omega) = \left( \frac{(1+c_0 z^{-1})(1+c_0 z)}{(1-\frac{1}{2}z)(1-\frac{1}{2}z^{-1})} \lambda^2 \right)_{z=e^{j\omega}} =$$

$$\Gamma_\eta(\omega) = \left( \frac{1+c_0(z^{-1}+z)+c_0^2}{(1-\frac{1}{2}z)(1-\frac{1}{2}z^{-1})} \lambda^2 \right)_{z=e^{j\omega}}$$

$$\underbrace{\left(\frac{(1+c_o^2)\lambda^2 + c_o\lambda^2(z^{-1}+z)}{(1-\frac{1}{2}z)(1-\frac{1}{2}z^{-1})}\right)_{z=e^{j\omega}}}_{\Gamma_\eta(\omega)} = \underbrace{\left(\frac{9/4 - \frac{1}{2}(z^{-1}+z)}{(1-\frac{1}{2}z)(1-\frac{1}{2}z^{-1})}\right)_{z=e^{j\omega}}}_{\Gamma_d(\omega)}$$

$$\begin{cases} (1+c_o^2)\lambda^2 = 9/4 & (1) \\ c_o\lambda^2 = -\frac{1}{2} & (2) \end{cases}$$

$$(2) \longrightarrow \lambda^2 = -\frac{1}{2c_o}$$

$$(1) \longrightarrow (1+c_o^2)\frac{1}{2c_o} = -\,9/4$$

$$\frac{1}{2c_o} + \frac{c_o}{2} = -\frac{9}{4}$$

$$1 + c_o^2 = -\frac{9}{2}c_o$$

$$c_o^2 + \frac{9}{2}c_o + 1 = \begin{cases} \cancel{4.236} \\ -0.236 \longrightarrow \lambda^2 = 2.13 \end{cases}$$

(not in canonical form)

$$\boxed{y(t) = \frac{1/4}{1-\frac{1}{2}z^{-1}}u(t-1) + \frac{1-0.236z^{-1}}{1-\frac{1}{2}z^{-1}}e(t) \qquad e(t) \sim WN(0, 2.13)}$$