PRACTICE SESSION 3

EXERCISE 6

Given $y(t) = \dfrac{z^{-1} + \frac{1}{5} z^{-2}}{1 + \frac{1}{2} z^{-1}} u(t)$

ⓐ Identify the system order using the impulse response

$$W(z) = \left(z^{-1} + \tfrac{1}{5} z^{-2}\right)\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) = \left(z^{-1} + \tfrac{1}{5} z^{-2}\right)\left(\sum_{k=0}^{\infty} \left(-\tfrac{1}{2} z^{-1}\right)^k\right) =$$

$$= \left(z^{-1} + \tfrac{1}{5} z^{-2}\right)\left(1 - \tfrac{1}{2} z^{-1} + \tfrac{1}{4} z^{-2} - \tfrac{1}{8} z^{-3} + \tfrac{1}{16} z^{-4} - \tfrac{1}{32} z^{-5} + \dots\right) =$$

$$= z^{-1} + \tfrac{1}{5} z^{-2} - \tfrac{1}{2} z^{-2} - \tfrac{1}{10} z^{-3} + \tfrac{1}{4} z^{-3} + \tfrac{1}{20} z^{-4} - \tfrac{1}{8} z^{-4} - \tfrac{1}{40} z^{-5} + \tfrac{1}{16} z^{-5} + \dots$$

$$= z^{-1} + \underbrace{\left(\tfrac{1}{5} - \tfrac{1}{2}\right)}_{-3/10} z^{-2} + \underbrace{\left(\tfrac{1}{4} - \tfrac{1}{10}\right)}_{3/20} z^{-3} + \underbrace{\left(\tfrac{1}{20} - \tfrac{1}{8}\right)}_{-3/40} z^{-4} + \underbrace{\left(\tfrac{1}{16} - \tfrac{1}{40}\right)}_{3/80} z^{-5} + \dots$$

$w(0) = 0 \quad w(1) = 1 \quad w(2) = -3/10 \quad w(3) = 3/20 \quad w(4) = -3/40 \quad w(5) = 3/80$

$H_1 = 1 \qquad \text{rank}(H_1) = 1$

$H_2 = \begin{bmatrix} 1 & -3/10 \\ -3/10 & 3/20 \end{bmatrix} \qquad \text{rank}(H_2) = 2$

$H_3 = \begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \\ 3/20 & -3/40 & 3/80 \end{bmatrix} \qquad H_3(3,:) = -\frac{1}{2} H_3(2,:) \rightarrow \text{rank}(H_3) = 2$

$\text{rank}(H_2) = 2$
$\text{rank}(H_3) = 2$ $\rightarrow$ $\boxed{n = 2}$

ⓑ Find the state space representation

In this case it is possible to use the

- Control form
- 4SID method ← used in this exercise

$n = 2$ from ⓐ

$$H_{n+1} = H_3 = \begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \\ 3/20 & -3/40 & 3/80 \end{bmatrix}$$

factorization

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}}_{\mathcal{O}_3} \underbrace{\begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \end{bmatrix}}_{\mathcal{Q}_3} = \begin{bmatrix} 1 & -3/10 & 3/20 \\ -3/10 & 3/20 & -3/40 \\ 3/20 & -3/40 & 3/80 \end{bmatrix}$$

$$\hat{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\hat{G} = \begin{bmatrix} 1 \\ -3/10 \end{bmatrix}$$

$$\hat{H} = [\,1 \quad 0\,]$$

$\hat{D} = 0$ strictly proper system $(w(0) = 0)$

$$\boxed{\hat{F} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} 1 \\ -3/10 \end{bmatrix} \qquad \hat{H} = [\,1 \quad 0\,] \qquad \hat{D} = [\,0\,]}$$

EXERCISE 7

given $\quad W(z) = \dfrac{z}{z-a} \qquad a \in \mathbb{R} \setminus \{0\}$

ⓐ Find the first 7 samples of the IR

$$W(z) = \frac{1}{1-az^{-1}} = \sum_{k=0}^{\infty} (az^{-1})^k = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots$$

$$\omega(k) = a^k$$

$\omega(0) = 1$	$\omega(1) = a$	$\omega(2) = a^2$	$\omega(3) = a^3$
$\omega(4) = a^4$	$\omega(5) = a^5$	$\omega(6) = a^6$	$\omega(7) = a^7$

$\omega(0) \neq 0 \implies$ The system is not strictly proper

ⓑ Identify the system order starting from the impulse response

$$H_1 = a$$

$$H_2 = \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix} \qquad H_2(2,:) = a\, H_2(1,:)$$

$$\begin{aligned} rank(H_1) &= 1 \\ rank(H_2) &= 1 \end{aligned} \rightarrow \boxed{n = 1}$$

ⓒ Find a state space representation

$n = 1$

$$H_2 = \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 \\ a \end{bmatrix}}_{O_2} \underbrace{\begin{bmatrix} a & a^2 \end{bmatrix}}_{R_2} = \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix}$$

$$\hat{F} = a \qquad \hat{G} = a$$

$$\hat{H} = 1 \qquad \hat{D} = ? \qquad \hat{D} \neq 0$$

Matrix D identification

consider a general state space system (SISO)

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Du(t) \end{cases}$$

Consider $x(0) = 0$ and $u(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$

$t = 0$ $\quad y(0) = H\,x(0) + D\,u(0) = D$

D is the first sample of the impulse response

$D = \omega(0)$

$\hat{D} = 1$

$$\boxed{\begin{array}{ll} \hat{F} = a & \hat{G} = a \\ \hat{H} = 1 & \hat{D} = 1 \end{array}}$$

d) Compute the transfer function

$$W(z) = \hat{H}\,(zI - \hat{F})^{-1}\,\hat{G} + \hat{D} =$$

$$= \frac{a}{z-a} + 1 = \frac{a + z - a}{z - a} = \frac{z}{z-a}$$

$$\boxed{W(z) = \frac{z}{z-a}}$$