NON COOPERATIVE GAMES

EXAMPLES:

- CHECKERS
- CHESS
- POKER
- ROCK - PAPER - SCISSORS

MULTIPLE PLAYERS $\longrightarrow$ EACH WITH A POTENTIALLY DIFFERENT OBJECTIVE FUNCTION

SELFISH

REPRESENTATIONS

$\swarrow$ EXTENSIVE FORM

$\searrow$ NORMAL FORM

NORMAL FORM

(N, A, U)

N → SET OF PLAYERS

A → SET OF ACTIONS

$A = \{A_1, A_2, \dots A_n\}$

A_1 → SET OF ACTIONS OF PLAYER 1

U → SET OF UTILITY FUNCTIONS

ROCK - PAPER - SCISSORS

$N = \{1, 2\}$

$A = \{A_1, A_2\}$

$A_1 = A_2 = \{$ROCK, PAPER, SCISSORS$\}$

$\{R, P, S\}$

$U = \{U_1, U_2\}$

PLAYER 2

$U_1 =$

	R	P	S
R	0	-1	1

$U = \{ U_1, U_2, \dots U_n \}$

UTILITY FUNCTION OF PLAYER 1

$U_1 : A_1 \times A_2 \times A_3 \times \dots A_n \longrightarrow \mathbb{R}$

PLAYER 1

P	1	0	-1
S	-1	1	0

VICTORY 1 → 1

IS better

TIES → 0

IS better

VICTORY 2 → -1

PLAYER 2

$U_2 =$

PLAYER 1	R	P	S
R	0	1	-1
P	-1	0	1
S	1	-1	0

V 2 → 1

IS better

TIES → 0

IS better

V 1 → -1

UTILITY OF PLAYER 1

PLAYER 2

UTILITY OF PLAYER 2

$U =$

PLAYER 1	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

BACH OR STRAVINSKY GAME (BATTLE OF SEXES)

PLAYER 2

PLAYER 1	B	S
B	2, 1	0, 0
S	0, 0	1, 2

PRISONER'S DILEMMA

PLAYER 2

PLAYER 1	NC	C
NC	-1, -1	-3, 0
C	0, -3	-2, -2

↓

PLAYER 2

PLAYER 1	NC	C
NC	2, 2	0, 3
C	3, 0	1, 1

MATCHING PENNIES

PLAYER 2

PLAYER 1	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

PLAYER 2

PLAYER 1	A	B
A	x_1, x_2, x_3	
B		

A

PLAYER 2

PLAYER 1	A	B
A		
B		

B

PLAYER 3

SOLUTION

↓

DEFINES THE BEHAVIOR OF THE PLAYERS IN THE GAME

NOTION OF STRATEGY

$\sigma_i \to$ STRATEGY OF PLAYER i

(s_i)

$$\sigma_i : A_i \to [0,1] \qquad \sigma_i \in \Delta(A_i)$$

SIMPLEX
(PROBABILITY DISTRIBUTION)

DEGENERATE CASE IN WHICH σ_i IS PURE

$A_1 = \{R, P, S\}$

$$\sigma_1 = \left\{ \begin{array}{ll} 1 & R \\ 0 & P \\ 0 & S \end{array} \right. \qquad \sigma_1 = \left\{ \begin{array}{ll} 0 & R \\ 1 & P \\ 0 & S \end{array} \right. \Bigg\} \text{ EXAMPLES OF PURE STRATEGIES}$$

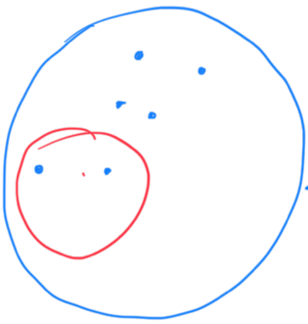
"MIXED"

$$\sigma_1 = \left\{ \begin{array}{ll} 1/3 & R \\ 1/3 & P \\ 1/3 & S \end{array} \right.$$

POKER GAMES → BLUFF → MIXED STRATEGY
(LIBRATUS AND PLURIBUS)

OPTIMAL STRATEGIES

$\sigma_i \in \Sigma_i \qquad \Sigma_i = \Delta(A_i)$



$\Sigma_1 \times \Sigma_2$

(σ_1, σ_2) → STRATEGY PROFILE

BEST RESPONSE → $BR_i(\sigma_{-i})$

ALL THE PLAYERS EXCEPT PLAYER i

PLAYER 2

	H	T

PLAYER 2

	B	S

PLAYER 1		
H	1, -1	-1, 1
T	-1, 1	1, -1

PLAYER 1		
B	2, 1	0, 0
S	0, 0	1, 2

A SOLUTION (A STRATEGY PROFILE) IS "OPTIMAL" IF IS IN EQUILIBRIUM (NASH Equilibrium)

$(\sigma_1^*, \sigma_2^*, \dots \sigma_n^*)$ IS A NASH Equilibrium IF AND ONLY IF:

$\forall i \in N: \quad \sigma_i^* \in BR_i(\sigma_{-i}^*)$

EVERY FINITE GAME ADMITS AT LEAST ONE NASH EQUILIBRIUM (MIXED)

PLAYER 2

PLAYER 1	B	S
B	2, 1	0, 0
S	0, 0	1, 2

(B, B) → NASH
(S, S) → NASH
(B, S)
(S, B) → NOT NASH

PLAYER 2

PLAYER 1	NC	C
NC	-1, -1	-3, 0
C	0, -3	-2, -2

(C, C) → NASH
(NC, C)
(NC, NC)
(C, NC) → NOT NASH

PLAYER 2

PLAYER 1	B	S
B	2, 1	0, 0

MIXED STRATEGY

S	0, 0	1, 2

$$\sigma_1(B) > 0 \qquad \sigma_2(B) > 0$$
$$\sigma_1(S) > 0 \qquad \sigma_2(S) > 0$$

SELFISH → THE PLAYER WILL PLAY THE ACTIONS MAXIMIZING HIS/HER UTILITY

B AND S ARE BEST RESPONSE

$$\{B, S\} \in BR_1(\sigma_2)$$

$$\boxed{\mathbb{E}[U_1(B, \sigma_2)] = \mathbb{E}[U_1(S, \sigma_2)]}$$

$$\sigma_2(B) \cdot U_1(B,B) + \sigma_2(S) \cdot U_1(B,S) = \sigma_2(B) \cdot U_1(S,B) + \sigma_2(S) U_1(S,S$$

$$(\sigma_2(B) = 1 - \sigma_2(S))$$

$$\sigma_2(B) \underbrace{U_1(B,B)}_{2} + (1-\sigma_2(B)) \underbrace{U_1(B,S)}_{0} = \sigma_2(B) \underbrace{U_1(S,B)}_{0} + (1-\sigma_2(B)) \underbrace{U_1(S,}_{1}$$

$$\sigma_2(B) \cdot 2 = (1 - \sigma_2(B))$$

$$2\sigma_2(B) + \sigma_2(B) = 1 \Longrightarrow \boxed{\begin{cases} \sigma_2(B) = 1/3 \\ \sigma_2(S) = 2/3 \end{cases}}$$

$$\{B, S\} \in BR_2(\sigma_1)$$

$$\mathbb{E}[U_2(\sigma_1, B)] = \mathbb{E}[U_2(\sigma_1, S)]$$

$$U_2(B,B)\sigma_1(B) + U_2(S,B)\sigma_1(S) = U_2(B,S)\sigma_1(B) + U_2(S,S)\sigma_1(S$$

. . .

$$\boxed{\begin{aligned}\sigma_1(B) &= 2/3 \\ \sigma_1(S) &= 1/3\end{aligned}}$$

$$\sigma_1 = \begin{cases} 2/3 & B \\ 1/3 & S \end{cases} \qquad \sigma_2 = \begin{cases} 1/3 & B \\ 2/3 & S \end{cases}$$

$$\left.\begin{aligned} \sigma_1 &\in BR_1(\sigma_2) \\ \sigma_2 &\in BR_2(\sigma_1) \end{aligned}\right\} \Rightarrow (\sigma_1, \sigma_2) \text{ is a NASH Equilibrium}$$

PLAYER 2

PLAYER 1

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

$$\left.\begin{aligned} &(H, H) \\ &(T, T) \\ &(H, T) \\ &(T, H) \end{aligned}\right\} \Rightarrow \text{NOT NASH}$$

$$\mathbb{E}[U_1(H, \sigma_2)] = \mathbb{E}[U_1(T, \sigma_2)]$$

$$U_1(H,H)\,\sigma_2(H) + U_1(H,T)\,\sigma_2(T) = U_1(T,T)\,\sigma_2(T) + U_1(T,H)\,\sigma_2(H$$

$$(\sigma_2(T) = 1 - \sigma_2(H))$$

. . . .

$$\sigma_2 = \begin{cases} 1/2 & H \\ 1/2 & T \end{cases}$$

$$\mathbb{E}[U_2(\sigma_1, H)] = \mathbb{E}[U_2(\sigma_1, T)]$$

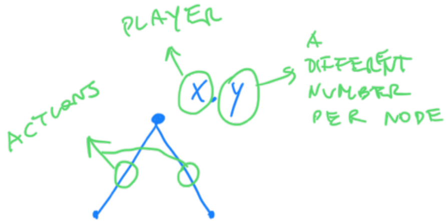
$$U_2(H,H)\,\sigma_1(H) + U_2(T,H)\,\sigma_1(T) = U_2(H,T)\,\sigma_1(H) + U_2(T,T)\,\sigma($$

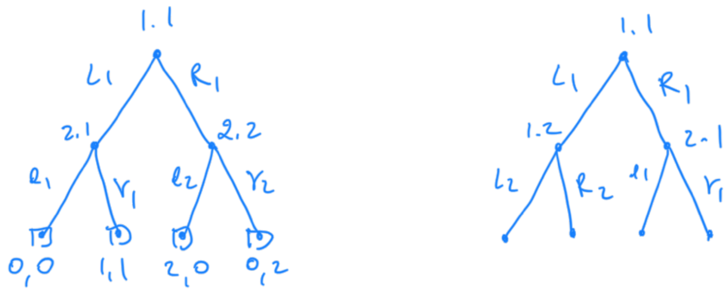
. . .

$$\sigma_1 = \begin{cases} 1/2 & H \end{cases}$$

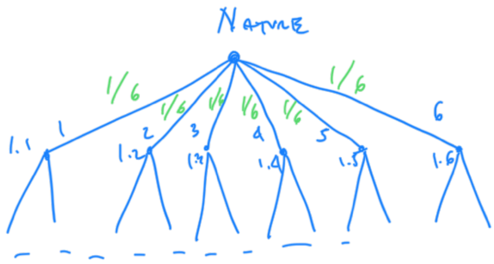
I II_L T

EXTENSIVE-FORM REPRESENTATION





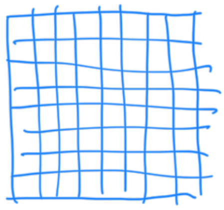
NATURE
(NON STRATEGIC PLAYER)



PERFECT INFORMATION → EVERY ACTION TACKLED BY ANY PLAYER CAN BE OBSERVED
↓
EVERY PLAYER PERFECTLY KNOWS THE STATE OF THE GAME

IMPERFECT INFORMATION → IF THE GAME IS NOT WITH PERFECT INFORMATION

PERFECT INFORMATION

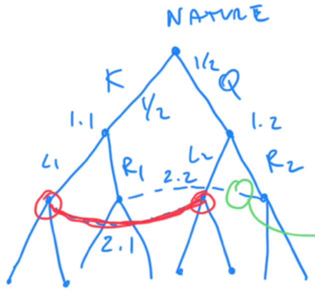


CHECKERS → EVERY PLAYER SEES ALL THE MOVES

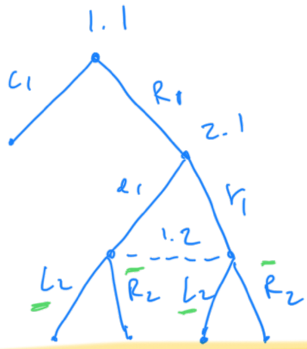
CHESS → THE SAME

IMPERFECT INFORMATION

POKER, . . .

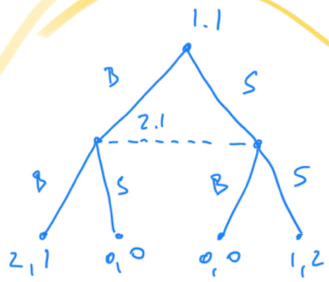


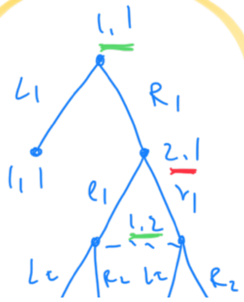
INFORMATION SET (IS A SUBSET OF DECISION NODES)



P 2

P1		B	S
	B	2, 1	0, 0
	S	0, 0	1, 2





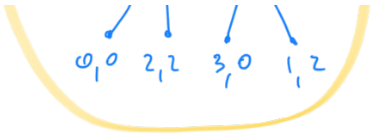
NORMAL FORM → EXTENSIVE FORM

ACTION → PLAN

$L_1 L_2$

$L_1 R_2$

< , >



R1 L2
R1 R2

1.1 $\{L_1, R_1\}$

1.2 $\{L_2, R_2\}$

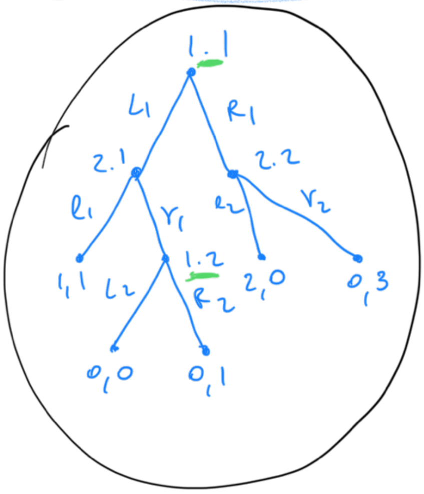
< , >

2.1 $\{\ell_1, r_1\}$

PLAYER 2

PLAYER 1

	ℓ_1	r_1
L_1L_2	1,1	1,1
L_1R_2	1,1	1,1
R_1L_2	0,0	3,0
R_1R_2	2,2	1,2



PLANS OF PLAYER 1

< , >

1.1 $\{L_1, R_1\}$

1.2 $\{L_2, R_2\}$

L_1L_2
L_1R_2
R_1L_2
R_1R_2

PLAYER 2

PLAYER 1

	$\ell_1\ell_2$	ℓ_1r_2	$r_1\ell_2$	r_1r_2
L_1L_2	1,1	1,1	0,0	0,0
L_1R_2	1,1	1,1	0,1	0,1
R_1L_2	2,0	0,3	2,0	0,3
R_1R_2	2,0	0,3	2,0	0,3

PLANS OF PLAYER 2

< , >

2.1 $\{\ell_1, r_1\}$

2.2 $\{\ell_2, r_2\}$

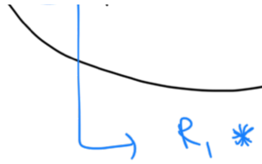
$\ell_1\ell_2$
ℓ_1r_2
$r_1\ell_2$
r_1r_2

PLAYER 2

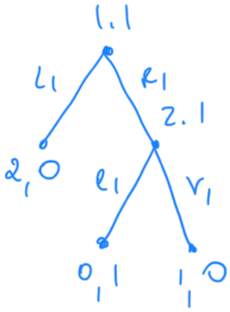
PLAYER 1

	$\ell_1\ell_2$	ℓ_1r_2	$r_1\ell_2$	r_1r_2
L_1L_2	1,1	1,1	0,0	0,0
L_1R_2	1,1	1,1	0,1	0,1
R_1L_2	2,0	0,3	2,0	0,3
R_1R_2	2,0	0,3	2,0	0,3

	$\ell_1\ell_2$	ℓ_1r_2	$r_1\ell_2$	r_1r_2
L_1L_2				
L_1R_2				
R_1*				



REDUCED NORMAL FORM

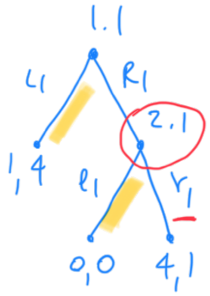


PLAYER 2

PLAYER 1	l_1	r_1
L_1	2,0	2,0
R_1	~~0,1~~	~~1,0~~

(L_1, l_1)
(L_1, r_1) } PURE NASH Equilibria

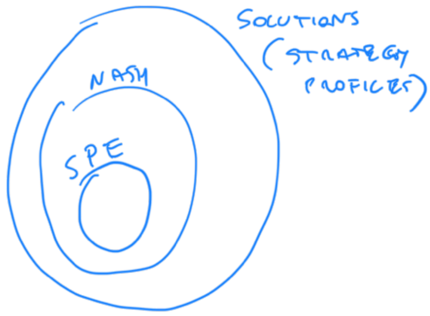
↑ NASH Equilibria



PLAYER 2

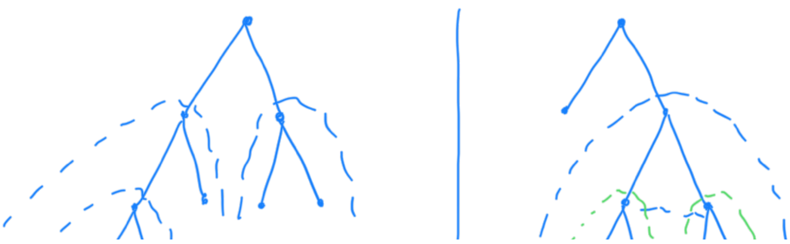
PLAYER 1	l_1	r_1
L_1	1,4	1,4
R_1	0,0	4,1

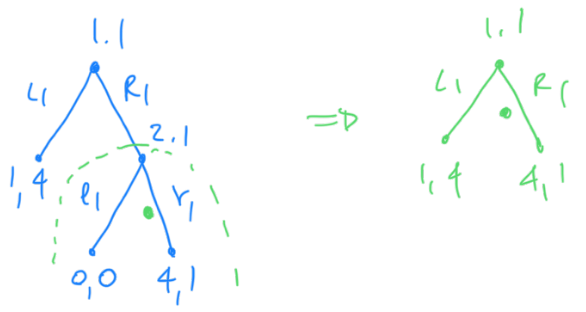
(L_1, l_1)
(R_1, r_1) } PURE NASH Equilibria

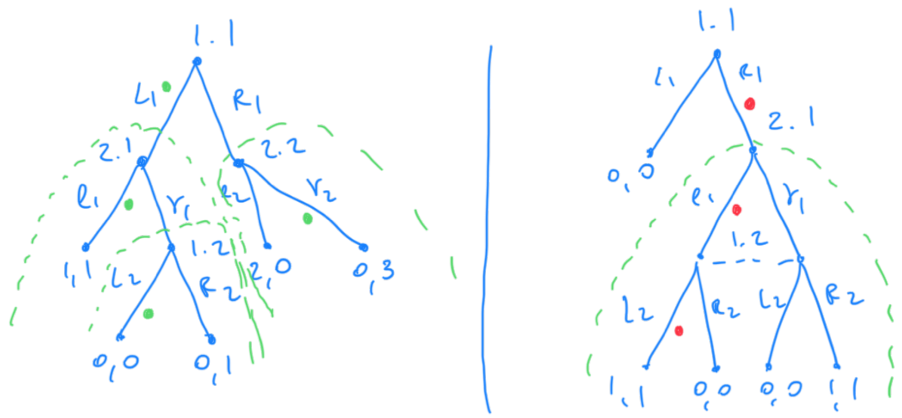


SUBGAME PERFECT Equilibria (SPE)

SUBGAME: SUBTREE, WITH A SINGLE ROOT AND EVERY INFORMATION SET IS SELF CONTAINED IN THE SUBTREE







	l_1	r_1
L_2	1,1	0,0
R_2	0,0	1,1

DOMINANCE

$a \in A_i$ IS DOMINATED WHEN THERE EXISTS $a' \in A_i$:

$$U_i(a, \sigma_{-i}) \le U_i(a', \sigma_{-i}) \qquad \forall \sigma_{-i} \in \Sigma_{-i}$$

$<$ STRICTLY DOMINATED

$\le$ WEAKLY DOMINATED

1 \ 2	C	D
A	2, 0	0, 1
B	1, 1	-1, 0

IS A DOMINATED BY B → NO, IT IS NOT

IS B DOMINATED BY A → YES, IT IS

IS C DOMINATED BY D → NO, IT IS NOT

IS D DOMINATED BY C → NO, IT IS NOT

1 \ 2	NC	C
NC	-1, -1	-3, 0
C	0, -3	-2, -2

(P1) IS NC DOMINATED BY C → YES, IT IS

(P2) IS NC DOMINATED BY C → YES, IT IS

1 \ 2	C
C	-2, -2

IF AN ACTION IS DOMINATED ⇒ DISCARD IT

REDUCING THE GAME

IF THE REDUCED GAME HAS A SINGLE ACTION PER PLAYER, THEN YOU FOUND A DOMINANT STRATEGY EQUILIBRIUM

1 \ 2	C	D
A	2, 0	0, 1
B	1, 1	-1, 0

ITERATED DOMINANCE

NON ITERATED DOMINANCE

PLAYER 1 → she/he DISCARDS HER/HIS ACTIONS

PLAYER 2 → she/he DISCARDS HER/HIS ACTIONS

ITERATED DOMINANCE

PLAYER 1
DISCARDS HIS/HER
DOMINATED ACTIONS

PLAYER 2
DOES THE SAME

PLAYER 1

....

1 \ 2	C	D
A	2, 0	0, 1
B	1, 1	-1, 0

PLAYER 1

	C	D
A	2	0
B	~~1~~	~~-1~~

PLAYER 2

	C	D
A	0	1
B	1	0

1 \ 2	C	D
A	2, 0	0, 1
B	1, 1	-1, 0

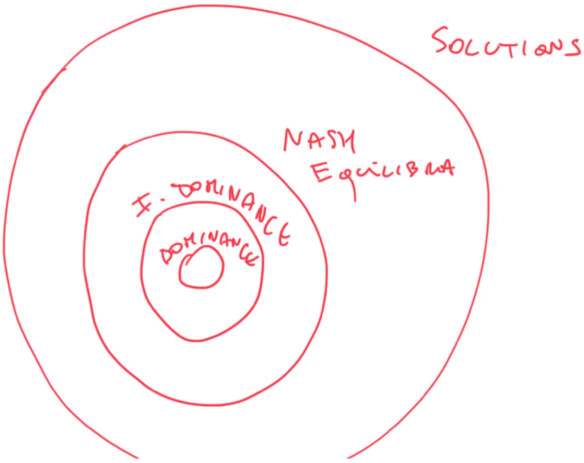
1 \ 2	D
A	0, 1

2

1		A	B	C
	D	2, 1	1, −2	3, 0
	E	1, 0	0, 0	2, 2
	F	0, 1	1, 0	4, 1

$\longrightarrow$ ITERATED DOMINANCE

		2
		A
1	D	2, 1



		2	
		B	S
1	B	2, 1	0, 0
	S	0, 0	1, 2