PRACTICE SESSION 4

KALMAN FILTER

EXERCISE 1

Given the system

$$\begin{cases} x(t+1) = 2/5\, x(t) + v_1(t) \\ y(t) = 3\, x(t) + v_2(t) \end{cases} \qquad \begin{matrix} v_1 \sim WN(0, \frac{123}{125}) \\ v_2 \sim WN(0, 1) \end{matrix} \qquad v_1 \perp v_2$$

ⓐ Compute the 1-step Kalman predictor

ONE-STEP KALMAN PREDICTOR

$$y(t) \longrightarrow \boxed{\text{1-STEP PREDICTOR}} \longrightarrow \hat{x}(t+1|t)$$

time varying systems

$$\begin{cases} \hat{x}(t+1|t) = F\,\hat{x}(t|t-1) + K(t)\, e(t) & \text{(STATE PREDICTION)} \\ \hat{y}(t|t-1) = H\,\hat{x}(t|t-1) & \text{(OUTPUT PREDICTION)} \\ e(t) = y(t) - \hat{y}(t|t-1) & \text{(INNOVATION)} \\ K(t) = (F P(t) H^T + V_{12})(H P(t) H^T + V_2)^{-1} & \text{(FILTER GAIN)} \\ P(t+1) = F P(t) F^T + V_1 - (F P(t) H^T + V_{12})(H P(t) H^T + V_2)^{-1}(F P(t) H^T + V_{12})^T & \text{(DRE)} \end{cases}$$

$\hat{x}(1|0) = X_0$ → guess of the initial state

$P(1) = P_0$ → Represents how much we trust the initial guess

$n = 1 \qquad F = 2/5 \qquad H = 3 \qquad V_1 = 123/125 \qquad V_2 = 1 \qquad V_{12} = 0$

STEP 1 : Compute the DRE

$$P(t+1) = F P(t) F^T + V_1 - (F P(t) H^T + V_{12})(H P(t) H^T + V_2)^{-1}(F P(t) H^T + V_{12})^T$$

$$P(t+1) = \frac{4}{25} P(t) + \frac{123}{125} - \frac{\left(\frac{6}{5} P(t)\right)^2}{9 P(t) + 1}$$

$$P(t+1) = \frac{\left(\frac{4}{25} P(t) + \frac{123}{125}\right)(9 P(t) + 1) - \frac{36}{25} P(t)^2}{9 P(t) + 1}$$

$$P(t+1) = \frac{\cancel{\frac{36}{25} P(t)^2} + \frac{4}{25} P(t) + 9\, \frac{123}{123} P(t) + \frac{123}{125} - \cancel{\frac{36}{25} P(t)^2}}{9 P(t) + 1}$$

$$P(t+1) = \frac{(20 + 9 \cdot 123) P(t) + 123}{125\,(9 P(t) + 1)} \qquad \text{DRE}$$

STEP 2: Compute the filter gain $K(t)$

$$K(t) = (FP(t)H^T + V_{12})(HP(t)H^T + V_2)^{-1}$$

$$K(t) = \frac{6/5\, P(t)}{9P(t)+1}$$

STEP 3: Write the systems of equations

$$\begin{cases} \hat{x}(t+1|t) = 2/5\, \hat{x}(t|t-1) + K(t)\, e(t) \\ \hat{y}(t|t-1) = 3\, \hat{x}(t|t-1) \\ e(t) = y(t) - \hat{y}(t|t-1) \\ K(t) = \dfrac{6/5\, P(t)}{9P(t)+1} \\ P(t+1) = \dfrac{(20 + 9 \cdot 123)P(t) + 123}{125\,(9P(t)+1)} \end{cases}$$

$$\begin{cases} \hat{x}(t+1|t) = 2/5\, \hat{x}(t|t-1) + \dfrac{6/5\, P(t)}{9P(t)+1}\left(y(t) - 3\hat{x}(t|t-1)\right) \\ P(t+1) = \dfrac{(20 + 9 \cdot 123)P(t) + 123}{125\,(9P(t)+1)} \\ \hat{y}(t|t-1) = 3\, \hat{x}(t|t-1) \end{cases}$$

← collect $\hat{x}(t|t-1)$

$$\boxed{\begin{cases} \hat{x}(t+1|t) = \left(\overbrace{\left(\dfrac{2}{5}\right)}^{F} - \overbrace{\left(\dfrac{6/5\, P(t)}{9P(t)+1}\right)}^{K(t)}\overbrace{(3)}^{H}\right)\hat{x}(t|t-1) + \overbrace{\left(\dfrac{6/5\, P(t)}{9P(t)+1}\right)}^{K(t)} y(t) \\ P(t+1) = \dfrac{(20 + 9 \cdot 123)P(t) + 123}{125\,(9P(t)+1)} \\ \hat{y}(t|t-1) = 3\hat{x}(t|t-1) \end{cases}}$$

(b) Compute the steady-state 1-STEP Kalman predictor

ASYMPTOTIC KALMAN PREDICTOR

The Kalman predictor is time varying $(K(t))$

Asymptotic Kalman predictor exists if:

- $K(t) \xrightarrow{t\to\infty} \bar{K}$
- The Kalman predictor is Asymptotically stable (4.5.)

$$K(t) = f(P(t)) \quad \Rightarrow \quad K(t) \to \bar{K} \quad \text{if} \quad P(t) \to \bar{P}$$

$$P(t+1) = P(t) = \bar{P}$$

$$\bar{P} = F\bar{P}F^T + V_1 - (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}(F\bar{P}H^T + V_{12})^T \qquad \text{(ARE)}$$

$\bar{K} = (F\bar{P}H^T + V_{12})(H\bar{P}H^T + V_2)^{-1}$

We have to check if $P(t) \to \bar{P}$ & $(F - \bar{K}H)$ is (A.S)

Method 1: Graphical method

STEP 1: Compute the ARE solution

$$\bar{P} = \frac{(20 + 123 \cdot 9)\bar{P} + 123}{125\,(9\bar{P} + 1)} \longrightarrow \text{find } \bar{P}$$

$$\bar{P}\,125\,(9\bar{P} + 1) = (20 + 123 \cdot 9)\bar{P} + 123$$

$$1125\,\bar{P}^2 + 125\,\bar{P} - 1127\,\bar{P} - 123 = 0$$

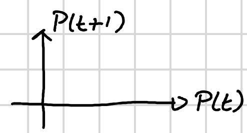
$$1125\,\bar{P}^2 - 1002\,\bar{P} - 123 = 0$$

$$\bar{P} = \frac{1002 \pm \sqrt{1002^2 + 4 \cdot 1125 \cdot 123}}{2 \cdot 1125} \left\langle \begin{array}{l} 1 \\ \cancel{-\frac{41}{375}} \end{array} \right.$$

The ARE has one, and only one, positive definite solution

STEP 2: Draw $P(t+1) = f(P(t))$

$$P(t+1) = \frac{1127\,P(t) + 123}{1125\,P(t) + 125}$$



I) Horizontal asymptote

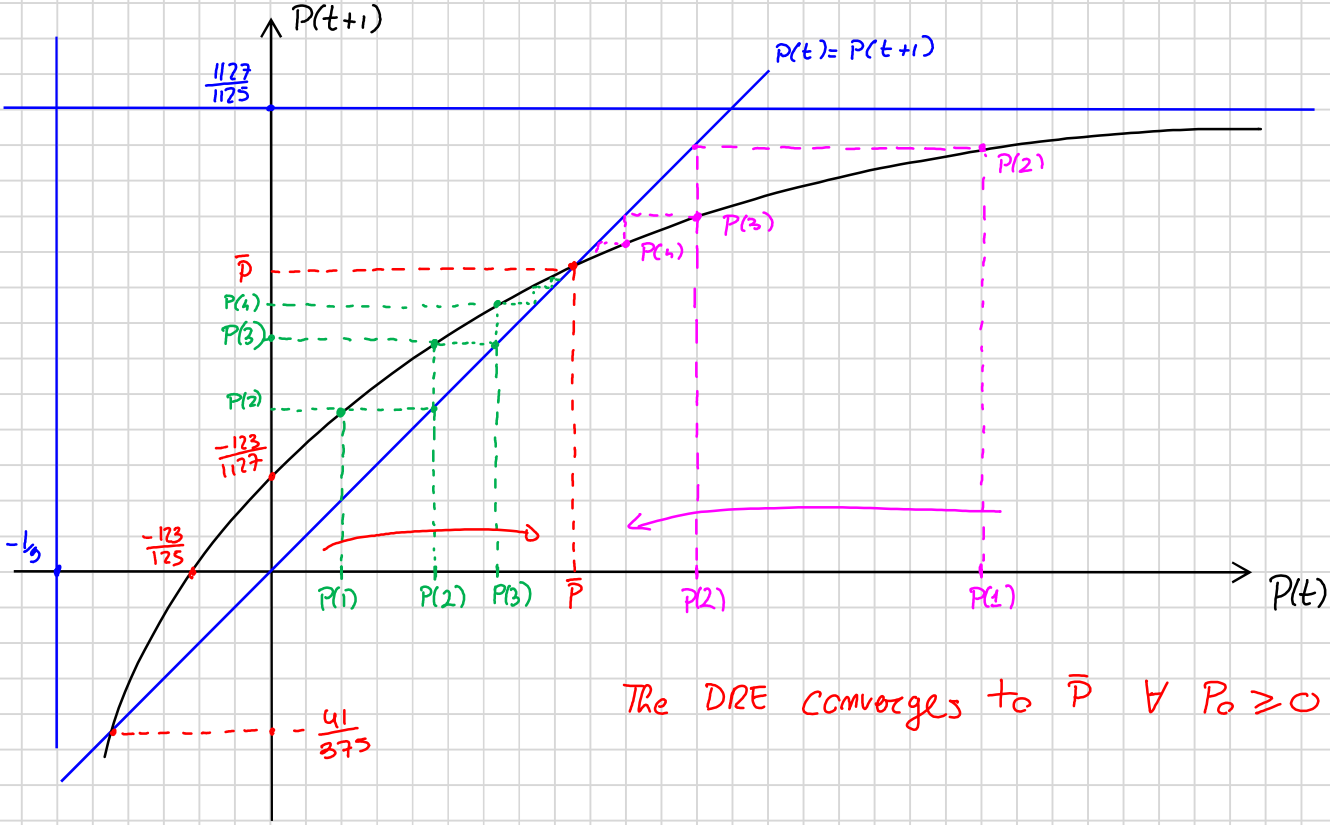
$P(t) \to \infty \qquad P(t+1) \to \frac{1127}{1125}$

II) Vertical Asymptote

$1125\,P(t) + 125 = 0 \qquad P(t) = -\frac{125}{1125} = -\frac{1}{9}$

III) Axis Intersection

$\begin{cases} P(t) = 0 \\ P(t+1) = 123/125 \end{cases} \qquad \begin{cases} P(t) = -123/1127 \\ P(t+1) = 0 \end{cases}$



STEP 3: Compute the gain and check stability

$$\bar{K} = \frac{6/5\,\bar{P}}{9\bar{P}+1} = \frac{6}{50} = 3/25$$

The asymptotic K.F. is asymptotically stable if, and only if, all the eigenvalues of $(F - \bar{K}H)$ are strictly inside the unitary circle.

$$F - \bar{K}H = 2/5 - \frac{3}{25}\cdot 3 = 2/5 - 9/25 = \frac{10-9}{25} = 1/25$$

$$\text{eig}(F - \bar{K}H) \rightarrow (zI - (F - \bar{K}H)) = 0$$

$$z - 1/25 = 0 \quad \rightarrow \quad z = 1/25 \qquad |z| < 1$$

$\bar{K}$ is such that $F - \bar{K}H$ has all the eigenvalues strictly inside the unitary circle

STEADY STATE KALMAN PREDICTOR

$$\boxed{\begin{aligned} \hat{x}(t+1|t) &= \frac{1}{25}\,\hat{x}(t|t-1) + \frac{3}{25}\,y(t) \\ \hat{y}(t|t-1) &= 3\,\hat{x}(t|t-1) \end{aligned}}$$