PRACTICE SESSION 5

Method 2: Use the theorems

ASYMPTOTIC THEOREMS

THEOREM 1

If:
- The system is asymptotically stable
- $V_{12} = 0$

then:
- ARE has one and only one positive-semidefinite solution $\bar{P} \geq 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- The corresponding $\bar{K}$ is such that $F-\bar{K}H$ is asymptotically stable.

THEOREM 2

If:
- $V_{12} = 0$
- (F, H) is observable
- (F, Γ) is reachable $\quad (\Gamma\Gamma^T = V_1)$

then:
- ARE has one and only one positive-definite solution $\bar{P} > 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- The corresponding $\bar{K}$ is such that $F-\bar{K}H$ is asymptotically stable.

THEOREM 1

- S is stable $\Leftrightarrow$ all the eigenvalues of F are strictly inside the unitary circle

$F = 2/5 \rightarrow zI - F = 0 \qquad z = 2/5 \qquad |z| < 1$ (ok)

- $V_{12} = 0 \rightarrow v_1 \perp v_2 \Rightarrow V_{12} = 0$ (ok)

THEOREM 2

- (F, H) observable $\rightarrow \mathcal{O} = H = 3 \quad rank(\mathcal{O}) = n$ (ok)
- (F, Γ) reachable $\rightarrow \Gamma\Gamma^T = \frac{123}{125} \quad \Gamma = \sqrt{\frac{123}{125}} \quad R = \Gamma = \sqrt{\frac{123}{125}} \quad rank(R) = n$ (ok)
- $V_{12} = 0 \rightarrow v_1 \perp v_2 \quad V_{12} = 0$ (ok)

- ARE has one and only one positive definite solution $\bar{P} > 0$
- DRE converges to $\bar{P}$ $\forall P_0 \geq 0$
- $\bar{K}$ is such that all the eigenvalues of $F - \bar{K}H$ are strictly inside the U.C

$\bar{P} = \cdots = 1$
$\bar{K} = \cdots = 3/25$
$\rightarrow$

$$\boxed{\begin{cases} \hat{x}(t+1|t) = \frac{1}{25}\hat{x}(t|t-1) + \frac{3}{25} y(t) \\ \hat{y}(t|t-1) = 3\hat{x}(t|t-1) \end{cases}}$$

(d) Compute the transfer function from the measured output to the 1-STEP steady state predicted output

$$y(t) \longrightarrow \boxed{W(z)} \longrightarrow \hat{y}(t+1|t)$$

$$\begin{cases} \hat{x}(t+1|t) = \overset{\tilde{F}}{\boxed{1/25}}\, \hat{x}(t|t-1) + \overset{\tilde{G}}{\boxed{3/25}}\, y(t) \\ \hat{y}(t|t-1) = \underset{\tilde{H}}{\boxed{3}}\, \hat{x}(t|t-1) \end{cases}$$

$$\hat{y}(t|t-1) = \left(\tilde{H}\,(zI-\tilde{F})^{-1}\tilde{G} \right) y(t)$$

$$\hat{y}(t|t-1) = \frac{9/25}{z - \frac{1}{25}}\, y(t)$$

$$\hat{y}(t+1|t) = z\, \hat{y}(t|t-1)$$

$$\hat{y}(t+1|t) = \frac{9/25\, z}{z - \frac{1}{25}}\, y(t) \longrightarrow \boxed{W(z) = \frac{9/25\, z}{z - 1/25}}$$

(e) compute the variance of the steady state 1-STEP output predictor error $e(t) = y(t) - \hat{y}(t|t-1)$

$$var[e(t)] = var[y(t) - \hat{y}(t|t-1)] = E[(y(t) - \hat{y}(t|t-1))^2] =$$

$$= E[(\underbrace{3x(t) + v_2(t)}_{y(t)} - \underbrace{3\hat{x}(t|t-1)}_{\hat{y}(t|t-1)})^2] =$$

$$= E[(3(x(t) - \hat{x}(t|t-1)) + v_2(t))^2] =$$

$$= 9\, E[(x(t) - \hat{x}(t|t-1))^2] + E[v_2(t)^2] + 6\, E[(x(t) - \hat{x}(t|t-1))\, v_2(t)] =$$

$$= 9\, \underbrace{var[x(t) - \hat{x}(t|t-1)]}_{P(t)} + \underbrace{var[v_2(t)]}_{V_2} + 6\, \underbrace{\cancel{E[x(t)\, v_2(t)]}}_{0 \quad *} - 6\, \underbrace{\cancel{E[\hat{x}(t|t-1)\, v_2(t)]}}_{0 \quad **}$$

* Represents the correlation between $x(t)$ and $v_2(t)$.
$x(t)$ does not depend on the output noise $\rightarrow x(t) \perp v_2(t)$

** Represents the correlation between $\hat{x}(t|t-1)$ and $v_2(t)$.
$v_2(t)$ affects $\hat{x}(t+1|t)$ not $\hat{x}(t|t-1) \rightarrow \hat{x}(t|t-1) \perp v_2(t)$

$$\boxed{var[e(t)] = 9P(t) + 1}$$

(f) Find the steady state 3-step predictor and compute the transfer function from $y(t)$ to $\hat{y}(t+3|t)$

K - STEP PREDICTOR

$$\hat{x}(t+1|t) \longrightarrow \boxed{F^{K-1}} \longrightarrow \hat{x}(t+k|t)$$

$$\begin{cases} \hat{x}(t+k|t) = F^{K-1}\, \hat{x}(t+1|t) \\ \hat{y}(t+k|t) = H\, \hat{x}(t+k|t) \end{cases}$$

$K = 3 \qquad F^2 = \frac{9}{25}$

$$\boxed{\begin{cases} \hat{x}(t+1|t) = \frac{1}{25}\,\hat{x}(t|t-1) + \frac{3}{25}\,y(t) \\ \hat{x}(t+3|t) = \frac{9}{25}\,\hat{x}(t+1|t) \\ \hat{y}(t+3|t) = 3\,\hat{x}(t+3|t) \end{cases}}$$

$$\overbrace{y(t) \to \boxed{\text{1-STEP PRED}} \xrightarrow{\hat{x}(t+1|t)} \boxed{F^2} \xrightarrow{\hat{x}(t+3|t)} \boxed{H}}^{W(z)} \to \hat{y}(t+3|t)$$

$$\hat{x}(t+1|t) = \frac{1}{25}\,\hat{x}(t|t-1) + \frac{3}{25}\,y(t)$$

$$\hat{x}(t+1|t) = z^{-1}\frac{1}{25}\,\hat{x}(t+1|t) + \frac{3}{25}\,y(t)$$

$$\hat{x}(t+1|t) = \frac{3/25}{1 - z^{-1}\frac{1}{25}}\,y(t)$$

$$\hat{x}(t+3|t) = \frac{9}{25}\,\frac{3/25}{1 - z^{-1}\frac{1}{25}}\,y(t)$$

$$\hat{y}(t+3|t) = 3\,\frac{9}{25}\,\frac{3/25}{1 - z^{-1}\frac{1}{25}}\,y(t) \longrightarrow \boxed{W(z) = \frac{36/625}{1 - z^{-1}\cdot 1/25}}$$

(9) Find the state equation of the kalman filter

KALMAN FILTER

$$y(t) \to \boxed{K.F.} \to \hat{x}(t|t)$$

if F is non singular : $\hat{x}(t|t) = F^{-1}\,\hat{x}(t+1|t)$

otherwise, if $V_{12} = 0$

$$\begin{aligned} \hat{x}(t|t) &= F\,\hat{x}(t-1,t-1) + K_F(t)\,e(t) \\ \hat{y}(t|t-1) &= H\,\hat{x}(t|t-1) \\ e(t) &= y(t) - \hat{y}(t|t-1) \\ K_F(t) &= P(t)H^T\,(H P(t) H^T + V_2)^{-1} \\ P(t+1) &= F P(t) F^T + V_1 - (F P(t) H^T)(H P(t) H^T + V_2)^{-1}(F P(t) H^T)^T \end{aligned}$$

$F = 2/5 \quad \rightarrow \quad F^{-1} = 5/2$

$\hat{x}(t|t) = \frac{5}{2}\,\hat{x}(t+1|t) \leftarrow$ we need to manipulate the equation in to obtain a difference equation : $\hat{x}(t|t) = f(\hat{x}(t-1|t-1))$

$$\hat{x}(t+1|t) = \frac{1}{25} \underbrace{\hat{x}(t|t-1)}_{F\,\hat{x}(t-1|t-1)} + \frac{3}{25} y(t)$$

$$\hat{x}(t|t) = \frac{5}{2} \left[\frac{1}{25} \cdot \frac{2}{5} \hat{x}(t-1|t-1) + \frac{3}{25} y(t) \right]$$

$$\boxed{\hat{x}(t|t) = \frac{1}{25} \hat{x}(t-1|t-1) + \frac{3}{10} y(t)}$$

(b) Compute the variance of the state estimation error using the filter, at steady state

$$\operatorname{var}[x(t) - \hat{x}(t|t)]$$

$$P(t) = \operatorname{var}[x(t) - \hat{x}(t|t-1)]$$

$$P(t+1) = \operatorname{var}[x(t+1) - \hat{x}(t+1|t)]$$

$$P(t+1) = \operatorname{var}[x(t+1) - \hat{x}(t+1|t)] = E\Big[(\underbrace{x(t+1)}_{\frac{2}{5}x(t)+v_1(t)} - \underbrace{\hat{x}(t+1|t)}_{\frac{2}{5}\hat{x}(t|t)})^2 \Big] =$$

$$= E\left[\left(\frac{2}{5} x(t) + v_1(t) - \frac{2}{5} \hat{x}(t|t) \right)^2 \right] =$$

$$= \frac{4}{25} \underbrace{E[(x(t) - \hat{x}(t|t))^2]}_{\operatorname{var}[x(t)-\hat{x}(t|t)]} + \underbrace{E[v_1(t)^2]}_{\frac{123}{125}} + \frac{4}{5} \underbrace{\cancel{E[(x(t) - \hat{x}(t|t)) v_1(t)]}}_{0\ *}$$

* $x(t)$ and $\hat{x}(t|t)$ are influenced by past values of $v_1(t)$

$$\operatorname{var}[x(t) - \hat{x}(t|t)] = \frac{25}{4} \left(P(t+1) - \frac{123}{125} \right)$$

at steady state

$$\operatorname{var}[x(t) - \hat{x}(t|t)] = \frac{25}{4} \left(\bar{P} - \frac{123}{125} \right) = \frac{1}{10}$$

$$\operatorname{var}[x(t) - \hat{x}(t|t)] \leq \operatorname{var}[x(t) - \hat{x}(t|t-1)]$$

EXERCISE 2

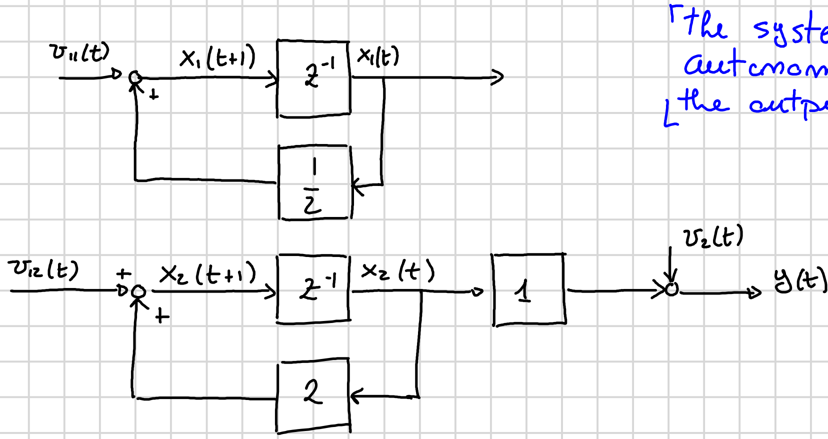
Given the system

$$\begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + v_{11}(t) \\ x_2(t+1) = 2 x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$$

$v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \end{bmatrix}$ $\quad v_1 \sim W\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ $\quad v_2 \sim WN(0,1)$ $\quad v_1 \perp v_2$

$F = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad H = [0 \quad 1]$ $\quad V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad V_2 = 1$ $\quad V_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) Draw the system block scheme



The system is composed by two autonomous subsystems and the output is influenced only by x_2

(b) Is the 1-step asymptotic state prediction error bounded?

$e_x(t) = x(t) - \hat{x}(t|t-1)$ is bounded?

e_x is bounded if the kalman predictor exists and it is stable.

THEOREM 1 ✗

$\mathcal{S}$ is A.S ? $\quad F = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ $\rightarrow$ we have to find the eigenvalues of F

$\det(zI - F) = 0 \quad \det\left(\begin{bmatrix} z - 1/2 & 0 \\ 0 & z-2 \end{bmatrix}\right) = 0 \quad (z - \frac{1}{2})(z-2) = 0 \quad eig(F) \begin{cases} z_1 = \frac{1}{2} \\ z_2 = 2 \end{cases}$

$|z_2| > 1 \rightarrow$ $\mathcal{S}$ is not A.S.

THEOREM 2: ✗

(F, H) Observable ? $\quad \mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix}$

$$\mathcal{O}(1,:) = H = [0 \quad 1]$$
$$\mathcal{O}(2,:) = HF = [0 \quad 1]\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} = [0 \quad 2] \quad \Bigg\} \quad \mathcal{O} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \qquad rank(\mathcal{O}) = 1$$

$\mathcal{S}$ is not observable.

Considering the whole system neither THM 1 nor THM 2 is valid.

Since the system is composed by two autonomous subsystems we can try to analyze one subsystems at a time.

$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + v_{11}(t) \quad] \; A \\ x_2(t+1) = 2x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases} \Bigg] \; B$$

SUBSYSTEM A

$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + v_{11}(t) \\ y_A(t) = 0 \end{cases} \leftarrow \text{FICTITIOUS OUTPUT}$$

$F_A = \frac{1}{2}$ $\quad H_A = 0$ $\quad V_{1A} = 1$ $\quad V_{2A} = 0$ $\quad V_{12A} = 0$ $\quad v_{2A} \perp v_1$

free paramet

$\det(zI - F_A) = 0 \quad z - \frac{1}{2} = 0 \quad z = \frac{1}{2} \quad |z| < 1 \quad \mathcal{S}$ is A.S.

$$\left.\begin{matrix} \mathcal{S} \text{ is A.S.} \\ V_{12A} = 0 \end{matrix}\right\} \rightarrow \text{THM 1 is valid}$$

SUBSYSTEM B

$$\begin{cases} x_2(t+1) = 2x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$$

$F_B = 2 \quad H_B = 1 \quad V_{1B} = 1 \quad V_{2B} = 1 \quad V_{12B} = 0 \qquad \Gamma_B^T \Gamma_B = V_{1_B} \rightarrow \Gamma_B = 1$

(F, H) observable? $\quad \mathcal{O} = H_B = 1 \quad \rightarrow rank(\mathcal{O}) = 1$ ✓

(F, Γ) reachable? $\quad \mathcal{R} = \Gamma_B = 1 \quad \rightarrow rank(\mathcal{R}) = 1$ ✓ $\quad \Bigg\}$ THM 2 is valid

$V_{12} = 0$ ✓

For THM 1 it is possible to build an A.S. asymptotic Kalman Predictor of $x_1(t)$

$$x_1(t) - \hat{x}_1(t|t-1) < \infty \qquad \forall t$$

For THM 2 it is possible to build an A.S. asymptotic Kalman predictor of $x_2(t)$

$$x_2(t) - \hat{x}_2(t|t-1) < \infty \quad \forall t$$

$$e_x(t) = x(t) - \hat{x}(t|t-1) = \begin{bmatrix} x_1(t) - \hat{x}_1(t|t-1) \\ x_2(t) - \hat{x}_2(t|t-1) \end{bmatrix} < \infty \qquad \forall t$$

The Asymptotic state prediction error is bounded