Category Theory

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A glossary reference for Category Theory and relevant terms which I have came across.

Glossary

2-Category is a category $\mathbf{C}$ with the following additional structure:

- A set of objects of $\mathbf{C}$ (called *0-cells*).
- $\forall A, B \in \mathbf{C}_0$ (for every pair of 0-cells), a small category $\mathbf{C}(A,B)$ whose objects are arrows from A to B in $\mathbf{C}$ (objects and arrows in $\mathbf{C}(A,B)$ are called *1-cells* and *2-cells* respectively).
- $\forall A, B, C \in \mathbf{C}_0$ (for every triple of 0-cells), there is a composition functor:

  $$comp : \mathbf{C}(B,C) \times \mathbf{C}(A,B) \longrightarrow \mathbf{C}(A,C)$$

  and that the following diagram commutes:

  $$\begin{tikzcd} \mathbf{C}(C,D) \times \mathbf{C}(B,C) \times \mathbf{C}(A,B) \arrow[r, "comp\times id"] \arrow[d, "id\times comp"'] & \mathbf{C}(B,D) \times \mathbf{C}(A,B) \arrow[d, "comp"] \\ \mathbf{C}(C,D) \times \mathbf{C}(A,C) \arrow[r, "comp"'] & \mathbf{C}(A,D) \end{tikzcd}$$

  ie: *comp* is associative

- For each object $A \in \mathbf{C}$, there is the identity functor:

  $$unit_A : 1 \to \mathbf{C}(A,A)$$

  where 1 is the terminal object in **Cat** (ie: the category with one object denoted as 0 and its identity arrow) defined as $unit_A(0) = id_A$ and that the following diagram commutes:

  $$\begin{tikzcd} & \mathbf{C}(A,B) \arrow[dl, "unit_B\times id"'] \arrow[d, "id"] \arrow[dr, "id\times unit_A"] & \\ \mathbf{C}(B,B) \times \mathbf{C}(A,B) \arrow[r, "comp"'] & \mathbf{C}(A,B) & \mathbf{C}(A,B) \times \mathbf{C}(A,A) \arrow[l, "comp"'] \end{tikzcd}$$

  ie: *unit* is the identity functor with respect to *comp*

The structure of a 2-category can be summarized as follows:

- A *horizontal category* whose objects are 0-cells, arrows are 2-cells.
- A *vertical category* whose objects are 1-cells, arrows are 2-cells.
- A *base category* (that can be derived from the horizontal and vertical categories) whose objects are 0-cells, arrows are 1-cells.

Visual Depictions:

0-cells x, y

1-cells $x \to y$

2-cells $\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$

Cat as a 2-category:

- 0-cells are small categories.
- 1-cells are functors.
- 2-cells are natural transformations.
- *comp* on 1-cells is composition of functors:

  $$comp(G, F) = G \circ F$$

  for arrows $F : A \to B, G : B \to C$ in **Cat**.
- *comp* on 2-cells is horizontal composition of natural transformations:

  $$comp(\beta, \alpha) = \beta * \alpha$$

  where $F, F' : A \to B$ and $G, G' : B \to C$ are functors $\alpha : F \to F', \beta : G \to G'$ are natural transformations between them.
- By the *Interchange Law*

  $$comp((\beta', \alpha') \circ (\beta, \alpha)) = comp(\beta', \alpha') \circ comp(\beta, \alpha)$$

2-Functor If **C** and **D** are 2-categories, a **2-functor** $F : \mathbf{C} \to \mathbf{D}$ consists of three maps of the form $F_i : \mathscr{C}_i \to \mathscr{D}_i$ where $i = b, v, h$ and $\mathscr{C}_i, \mathscr{D}_i$ are base, vertical or horizontal categories respectively.

Automorphism is an *endomorphism* that is *invertible*.

Categorical Dual (^{op}) Given any category $\mathscr{C}$, $\mathscr{C}^{op}$ is the **dual category** of $\mathscr{C}$ where all the arrows are reversed. $\mathscr{C}^{op}$ is defined as follows:

- The objects and arrows of $\mathscr{C}^{op}$ are the objects and arrows of $\mathscr{C}$.
- If $f : A \to B$ in $\mathscr{C}$, then $f : B \to A$ in $\mathscr{C}^{op}$.
- If $h = g \circ f$ in $\mathscr{C}$, then $h = f \circ g$ in $\mathscr{C}^{op}$.

Since identity arrows have the same source and target objects, identity arrows in $\mathscr{C}$ are the same as in $\mathscr{C}^{op}$ and thus $\mathscr{C}^{op}$ is also a category.
Category A **category** $\mathscr{C}$ contains a collection of *objects* (denoted as $\mathscr{C}_0$) and a collection of *morphisms* (denoted as $\mathscr{C}_1$) where every morphism has $source, target : \mathscr{C}_1 \to \mathscr{C}_0$ functions mapping each arrow to its *source* and *target* objects respectively into the collection of objects.

For any two arrows f, g where the $target(f) = source(g)$, the *composition* of f and g are written as $g \circ f : source(f) \to target(g) \in \mathscr{C}_1$. Composition is associative, $(h \circ g) \circ f = h \circ (g \circ f)$ when either side is defined.

$\forall A \in \mathscr{C}_0$, there exists the *identity* arrow, $id_A : A \to A$ and for any $f : A \to B$, $f \circ id_A = id_B \circ f = f$.

A category is *small* if its objects and arrows constitute sets; otherwise it is *large*.
Category Equivalence A functor $F : \mathscr{C} \to \mathscr{D}$ is an **equivalence of categories** if there is a functor $G : \mathscr{D} \to \mathscr{C}$ such that:

- A family $u_C : C \to G(F(C))$ of isomorphisms of $\mathscr{C}$ indexed by the objects of $\mathscr{C}$ with the property that for every arrow $f : C \to C'$ in $\mathscr{C}$, $G(F(f)) = u_{C'} \circ f \circ u_C^{-1}$.
- A family $v_D : D \to F(G(D))$ of isomorphisms of $\mathscr{D}$ indexed by the objects of $\mathscr{D}$ with the property that for every arrow $g : D \to D'$ in $\mathscr{D}$, $F(G(g)) = v_{D'} \circ g \circ v_D^{-1}$.

Note: In this case, the functor G is called a *pseudo-inverse* of F and u, v are *natural isomorphisms*.
Note: F and G are both full and faithful.

Congruence Relation ($\sim$) is an equivalence relation $\sim$ on the arrows of a category $\mathscr{C}$ if:

- Whenever $f \sim g$, then f and g have the same domain and the same codomain.
- In the diagram below:

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "k"] & D \end{tikzcd}$$

  If $f \sim g$, then $f \circ h \sim g \circ h$ and $k \circ f \sim k \circ g$ for arrows h, k where $target(h) = source(f)$ and $source(k) = target(f)$.

Note: The congruence class containing the arrow f is denoted by $[f]$.

Constant/Global Element is an arrow from a terminal object. Thus, the **constant** A is an arrow $1 \to A$ for some object A. In **Set**, such an arrow is precisely a function from a singleton set to the set A, where each element $x \in A$ can be uniquely determined by a constant arrow $x : 1 \to A$.
Contravariant Functor is a functor $F : \mathscr{C}^{op} \to \mathscr{D}$ for categories $\mathscr{C}$ and $\mathscr{D}$. In contrast to *covariant* functors, for an arrow f in $\mathscr{C}$

$$F(f : B \to A) = F(A) \to F(B)$$

is an arrow in $\mathscr{D}$. This follows from the fact that f is an arrow from A to B in $\mathscr{C}^{op}$.

Discrete A category in which all arrows are identity arrows is called **discrete**.

Endomorphism is an arrow $f : A \to A$ in a category where the source and target objects are the same.
Epimorphism ($\twoheadrightarrow$) is the dual to monomorphisms. An arrow $f : S \to T$ is an **epimorphism** if for any arrows $g, h : T \to X$, $g \circ f = h \circ f$ implies $g = h$. Epimorphisms can be denoted by $f : S \twoheadrightarrow T$.
Properties: Let $f : A \to B$ and $g : B \to C$ be arrows:

- If f and g are epimorphisms, so is $g \circ f$.
- If $g \circ f$ is an epimorphism, so is g.

Note: In **Set**, a function is surjective if and only if it is an epimorphism.

Note: In the category determined by a Monoid, an epic element is right cancellable.

Examples of Categories:

Set is the category whose objects are sets and arrows are functions between sets. Composition is function composition and the identity arrow is the identity function id_S for a set S.

Fin is the category whose objects are finite sets and arrows are all the functions between finite sets.

Pfn is the category whose objects are sets but all arrows are all partial functions. If $f : S \to T, g : T \to V$ are partial functions with f defined on $S_0 \subseteq S$ and g defined on $T_0 \subseteq T$, then the composite $g \circ f : S \to V$ is the partial function from S to V defined on the subset $\{x \in S_0 | f(x) \in T_0\}$ of S by the requirement $(g \circ f)(x) = g(f(x))$.

Rel is the category whose objects are sets and arrows are relations relations between sets. The composite $\beta \circ \alpha$ of relations α from sets S to T and β from T to U is a relation from S to U defined as follows:

If $x \in S$ and $z \in U$, $(x, z) \in \beta \circ \alpha$ if and only if there is an element $y \in T$ for which $(x, y) \in \alpha$ and $(y, z) \in \beta$.

The identity arrow for a set S is the *diagonal relation* $\Delta_S = \{(x, x) | x \in S\}$

Poset is the category whose objects are elements of a poset P and arrows are the partially ordered relation between those elements. The reflexive and transitive properties provides the identity and composition of arrows.

Note: Every hom-set in this category has at most ONE element. Equivalently, any category where every hom-set has at most one element is a poset.

Monoid is the category whose only object is the Monoid M itself and arrows are the elements of the monoid (this category is denoted as $C(M)$). The identity arrow is the identity element and composition being the monoid binary operation.

Note: Any category with one object is a monoid.

Group is the category whose only object is the Group itself and arrows are elements of the group, same as the category determined by a Monoid.

Note: All arrows in this category are isomorphisms.

Grf is the category whose objects are graphs and arrows are graph homomorphisms. Identity arrow being the identity homormophism and composition being composition of homomorphisms.

Posets and Monotone Functions is the category whose objects are posets and arrows are monotone functions between posets.

Sem is the category whose objects are semigroups and arrows are semigroup homomorphisms.

Note: This can be extended to monoids, denoted as **Mon**.

ω**-CPOs and Scott-continuous Functions** is the category whose objects are ω-CPOs and arrows are Scott-continuous functions.

Note: A category is also formed with *strict* ω-CPOs and *strict* Scott-continous functions.

Path Category is the free category $F(\mathscr{G})$ generated by a graph $\mathscr{G}$ whose objects are the nodes of $\mathscr{G}$ and arrows are paths in $\mathscr{G}$. For each node/object A, the identity arrow is the empty path from A to A. Composition is defined by $(f_1, \ldots, f_k) \circ (f_{k+1}, \ldots, f_n) = (f_1, \ldots, f_n)$

Cat is the category whose objects are small categories and arrows are functors between small categories.

M-**Act** is the category whose objects are monoid actions and arrows are equivariant maps. Since equivariant maps are functions, the identity arrow is the identity equivariant map and composition is function composition.

FBool is the category whose objects are finite Boolean algebras and arrows are homormophisms between them.

Pts is the category whose objects are pointed sets and arrows are functions that preserves the pointed object.

Func$(\mathscr{C}, \mathscr{D})$ is the category whose objects are functors from categories $\mathscr{C}$ to $\mathscr{D}$ and arrows are natural transformations.

Examples of Functors:

Monoid Homomorphism $f : M \to N$ where M and N are monoids is a functor from the category $C(M)$ to the category $C(N)$.

Monotone Map between two posets is a functor between the category determined by the posets.

Projection The first and second projections from the product category to its first or second parts, $P_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $P_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ for categories $\mathscr{C}$ and $\mathscr{D}$ are functors.

Inclusion Map of a subcategory is a functor.

Note: By the definition of *subcategory*, objects and arrows of the subcategory need not be objects and arrows in the bigger category.

Example: **Set** is a subcategory of **Rel** and the functor is defined by taking sets to the sets as objects in **Rel** and set functions $f : S \to T$ to $\{(s,t) | t = f(s)\}$.

Monoid Actions For an action α of a monoid M acting on a set S, let $C(M)$ be the category determined by M and the action α determines a functor $F_\alpha : C(M) \to$ **Set** defined as:

- $F_\alpha(*) = S$ where $*$ denotes the single object in $C(M)$.
- $F_\alpha(m) = s \mapsto \alpha(m, s)$ for $m \in M$, $s \in S$.

Quotient Functor $Q : \mathscr{C} \to \mathscr{C}/\sim$

is the functor thats takes a category to its quotient category by where $Q(A) = A$ for any object A and $Q(f) = [f]$ for arrows f in the category $\mathscr{C}$.

Factor Let $f : A \to B$ and $g : C \to B$ be arrows. If there is an arrow $h : A \to C$ such that $f = g \circ h$, then f **factors** through g.

Factorization System A **factorization system** in a category $\mathscr{C}$ consists of two subclasses $\mathscr{E}$ and $\mathscr{M}$ of the arrows of $\mathscr{C}$ such that:

- If $\mathscr{I}$ is the class of isomorphisms, then $\mathscr{M} \circ \mathscr{I} \subseteq \mathscr{M}$ and $\mathscr{I} \circ \mathscr{E} \subseteq \mathscr{E}$.
- Every arrow $f \in \mathscr{C}$ factors as $f = m \circ e$ where $m \in \mathscr{M}$ and $e \in \mathscr{E}$.
- In any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there is a unique $h : B \to C$ such that $h \circ e = f$ and $m \circ h = g$.

Faithful A functor $F : \mathscr{C} \to \mathscr{D}$ is **faithful** if the set mapping:

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

for each pair of objects A and B in $\mathscr{C}$ induced by F is *injective*.

Note: If $f : A \to B$ and $g : C \to D$, as long as either $A \neq C$ or $C \neq D$, $F(f) = F(g)$ may hold and F can still be faithful.

Note: A faithful functor need not be injective on objects or arrows.

Forgetful/Underlying Functor is the functor which "forgets" some of the structure in a category of structures and structure-preserving functions.

Examples:

- U : **Mon** $\to$ **Sem**
- U : **Sem** $\to$ **Set**
- U : **Grf** $\to$ **Set** $\times$ **Set**

  that takes a graph $\mathscr{G}$ to the set of nodes and set of arrows, $U(\mathscr{G}) = (\mathscr{G}_0, \mathscr{G}_1)$. It follows that there is also an arrowset functor A : **Grf** $\to$ **Set** and nodeset functor N : **Grf** $\to$ **Set**.
- U : **Cat** $\to$ **Grf**

  by forgetting the identity arrows and composition, the remains of a category becomes a graph.

  Note: If F is a functor, then F is a graph homomorphism. Note that the converse is not true.

  Simiarily, there is the functor A : **Cat** $\to$ **Set** and O : **Cat** $\to$ **Set** which takes a category to its set of arrows and objects respectively.

– $U : \mathscr{C}/A \to \mathscr{C}$
for an object A in the category $\mathscr{C}$.
Note: If $\mathscr{C}$ is **Set**, the underlying functor $U : \mathbf{Set}/S \to \mathbf{Set}$ forgets the indexing of S-indexed sets.

Free Functor Informally, a **free functor** is left adjoint to a **forgetful functor**.
Examples:

– $F : \mathbf{Set} \to \mathbf{Mon}$
is the free monoid functor that takes a set A to the free monoid $F(A)$, the Kleene closure A^* with concatenation as the monoid operation. F takes set functions to the Kleene closure induced homormophism.
Note: The Kleene closure itself is a functor from **Set** to **Set**, which is the composition of $U \circ F$, the underlying functor $U : \mathbf{Mon} \to \mathbf{Set}$ and the free monoid functor.

– $F : \mathbf{Grf} \to \mathbf{Cat}$
takes graphs to the path category of the graph as objects in **Cat** and graph homomorphisms to arrows (functors between path categories). For a graph homomorphism $\phi : \mathscr{G} \to \mathscr{H}$, $F(\phi)(f_n, f_{n-1}, \ldots, f_1) = (\phi(f_n), \phi(f_{n-1}), \ldots, \phi(f_1))$, which is clearly a path in $\mathscr{H}$ (and thus an arrow in the path category of $\mathscr{H}$).

Full A functor $F : \mathscr{C} \to \mathscr{D}$ is **full** if the set mapping:

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

for each pair of objects A and B in $\mathscr{C}$ induced by F is *surjective*.
Note: A full functor need not be surjective on objects or arrows.
Full Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ such that $\forall A, B \in \mathscr{D}_0$, $\mathrm{Hom}_{\mathscr{D}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$.
Example: **Fin** is a full subcategory of **Set**.
Functor is a "structure preserving" map between categories, similar to homomorphisms between groups, graphs and etc... A **functor** $F : \mathscr{C} \to \mathscr{D}$ from the category $\mathscr{C}$ to the category $\mathscr{D}$ defined as follows:

- $\forall A \in \mathscr{C}$, $F(A) \in \mathscr{D}$.
- If $f : A \to B$ in $\mathscr{C}$, then $F(f) : F(A) \to F(B)$ is an arrow in $\mathscr{D}$ such that:
  – $F(id_A) = id_{F(A)}$ for every object $A \in \mathscr{C}$.
  – $F(g \circ f) = F(g) \circ F(f)$ for all arrows $f : X \to Y, g : Y \to Z$ in $\mathscr{C}$.

Note: All functors preserve isomorphisms.

Godement Calculus Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, $E, F_1, F_2, F_3, G_1, G_2, G_3, H$ be functors and $\alpha, \beta, \gamma, \delta$ be natrual morphisms as shown in the following diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "E"] & \mathscr{B} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{C} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathscr{D} \arrow[r, "H"] & \mathscr{E}
\end{tikzcd}$$

There are 5 laws/equational rules between composition of functors and natural transformations:

1. $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma \circ \alpha)$
2. $(H \circ G_1)\alpha = H(G_1\alpha)$
3. $\gamma(F_1 \circ E) = (\gamma F_1)E$
4. $G_1(\beta \circ \alpha)E = (G_1\beta E) \circ (G_1\alpha E)$
5. $\gamma * \alpha = (\gamma F_2) \circ (G_1\alpha) = (G_2\alpha) \circ (\gamma F_1)$

Note: The first rule is also known as the *Interchange Law*.

Groupoid is a category in which every arrow is an *isomorphism*.

Hom Function Let S be an object and $f : T \to V$ an arrow in a category, a *covariant* **hom function** $\mathrm{Hom}(S, f) : \mathrm{Hom}(S, T) \to \mathrm{Hom}(S, V)$, if defined by:

$$\mathrm{Hom}(S, f)(g) = f \circ g$$

for some $g \in \mathrm{Hom}(S, T)$.
Similarily, a *contravariant* **hom function** $\mathrm{Hom}(f, S) : \mathrm{Hom}(V, S) \to \mathrm{Hom}(T, S)$, if defined by:

$$\mathrm{Hom}(f, S)(g) = g \circ f$$

for some $g \in \mathrm{Hom}(V, S)$.
Hom Functor is a functor that takes a category to a hom-set. For any category $\mathscr{C}$ with an object C, there are three different hom functors:

Covariant Hom Functor $\mathrm{Hom}(C, -) : \mathscr{C} \to \mathbf{Set}$

- $\mathrm{Hom}(C, -)(A) = \mathrm{Hom}(C, A)$ for each object $A \in \mathscr{C}$.
- $\mathrm{Hom}(C, -)(f) = \mathrm{Hom}(C, f) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$ for an arrow $f : A \to B$ in $\mathscr{C}$.

Contravariant Hom Functor $\mathrm{Hom}(-, C) : \mathscr{C}^{op} \to \mathbf{Set}$

- $\mathrm{Hom}(-, C)(A) = \mathrm{Hom}(A, C)$ for each object $A \in \mathscr{C}$.
- $\mathrm{Hom}(-, C)(f) = \mathrm{Hom}(f, C) : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ for an arrow $f : A \to B$ in $\mathscr{C}$.

Two-Variable Hom Functor $\mathrm{Hom}(-, -) : \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set}$

- $\mathrm{Hom}(-, -)((A, B)) = \mathrm{Hom}(A, B)$ for each object $(A, B) \in \mathscr{C}^{op} \times \mathscr{C}$.
- $\mathrm{Hom}(-, -)((f, g)) = \mathrm{Hom}(f, g) : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, D)$ for an arrow $(f : A \to B, g : C \to D)$ in $\mathscr{C}^{op} \times \mathscr{C}$.

Hom-Set A **hom-set** $\mathrm{Hom}(S, T)$ is the set of all morphisms from S to T for any objects S and T in a category.
Horizontal Composition Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ be categories, and $F, F' : \mathscr{A} \to \mathscr{B}, G, G' : \mathscr{B} \to \mathscr{C}$ be functors. **Horizontal composition** between natural transformation transformations $\alpha : F \to F'$ and $\beta : G \to G'$ is $\beta * \alpha : G \circ F \to G' \circ F'$.

Note: The horizontal composite of natural transformations is also a natural transformation.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd} * \begin{tikzcd} \mathscr{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} \end{tikzcd} \longmapsto \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "G\circ F"] \arrow[r, bend right, "G'\circ F'"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathscr{C} \end{tikzcd}$$

Idempotent An arrow $f : A \to A$ in a category is **idempotent** if $f \circ f = f$.
Indiscrete A category in which there is exactly one arrow between any two objects is called **indiscrete**.
Initial Object (0) is the dual of *terminal object* in which the **initial object** has an unique arrow to each object (including itself). The initial object is usually denoted 0.
Note: Any two initial objects in a category are isomorphic.
Example: The empty set $\emptyset$ in **Set** is the initial object.
Inverse For arrows $f : A \to B$ and $g : B \to A$, g is an **inverse** of f if $g \circ f = id_A$ and $f \circ g = id_B$.
Note: If only $f \circ g = id_B$ is satisfied, then f is a **left inverse** of g.
Similarily, if only $g \circ f = id_A$ is satisfied, then f is a **right inverse** of g.
Invertible is an element of a monoid in which it is an *isomorphism* in the category of the monoid (ie: the element has an inverse).
Isomorphism ($\cong$) is an arrow with an *inverse*. Two objects A and B are *isomorphic* if an isomorphism exists between them, thus $A \cong B$.
Note: It follows that all identity arrows are isomorphisms.
Note: A function in **Set**, homomorphism in **Grf** and **Mon** is an isomorphism if and only if it is *bijective*.

Locally Small Category is a category $\mathscr{C}$ with the property that $\mathrm{Hom}(A, B)$ is a set for all objects $A, B \in \mathscr{C}$.

Monomorphism ($\rightarrowtail$) is an arrow $f : A \to B$ such that for any object T and arrows $x, y : T \to A$, if $x \neq y$, then $f \circ x \neq f \circ y$. Monomorphisms can be denoted by $f : A \rightarrowtail B$.
Note: In this context, $x, y : T \to A$ may be regarded as *variable elements* and can be written as if $x \neq y$, then $f(x) \neq f(y)$.
Properties: Let $f : A \to B$ and $g : B \to C$ be arrows:

- If f and g are monomorphisms, so is $g \circ f$.
- If $g \circ f$ is a monomorphism, so is f.

Note: In **Set**, a function is injective if and only if it is a monomorphism.
Note: In the category determined by a Monoid, a monic element is left cancellable.
Morphism/Arrow is a directed connective between two objects (*source* and *target* objects) in a category, synonymous with *map* or a *directed edge* in a *directed graph*. A morphism f with source x and target y objects is denoted as:

$$f : x \to y$$

Natural Isomorphism is a natural transformation $\alpha : F \to G$ in which there is a natural transformation $\beta : G \to F$ is an inverse to α in $\mathbf{Func}(\mathscr{C}, \mathscr{D})$.

Note: Natural isomorphisms are also known as *natural equivalences*.

Theorem: Suppose $F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C}$ are functors and $\alpha : F \to G$ is a natural transformation between them. Then α is a *natural isomorphism* if and only if for each object $C \in \mathscr{C}$, $\alpha C : F(C) \to G(C)$ is an isomorphism in $\mathscr{D}$.

Natural Transformation Let $F, G : \mathscr{C} \to \mathscr{D}$ be 2 functors with the same domain and codomain. A **natural transformation** $\alpha : F \to G$ is given by a family of arrows αC for each object $C \in \mathscr{C}$ such that:

- $\alpha C : F(C) \to G(C)$ for each object $C \in \mathscr{C}$, also say that α is natural in C.
- For any arrow $f : C \to D$ in $\mathscr{C}$, the following diagram commutes.

$$\begin{tikzcd} F(C) \arrow[r, "\alpha C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(D) \arrow[r, "\alpha D"'] & G(D) \end{tikzcd}$$

Composition with Functors: Let $H : \mathscr{B} \to \mathscr{C}$ be a functor and $\alpha : F \to G$ where $F, G : \mathscr{A} \to \mathscr{B}$ be are functors. Then, postcomposing α with the functor H is a natural transformation $H\alpha : H \circ F \to H \circ G$ defined by $(H\alpha)A = H(\alpha A)$ for any object $A \in \mathscr{A}$.
A similar result can be shown when precomposing a natural transformation with a functor.

Order-Enriched Category is a poset-enriched category in which each poset $\mathrm{Hom}(A, B)$ is a strict ω-CPO where every pair of elements has a supremum (least upper bound) and an infimum (greatest lower bound).

Poset-Enriched Category is a category $\mathscr{C}$ with a partial ordering on every $\mathrm{Hom}_{\mathscr{C}}(A, B)$ such that for every triple of objects $A, B, C \in \mathscr{C}$, the composition of hom-sets:

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$

is monotone. Thus if $f \leq f' : A \to B, g \leq g' : B \to C$ then $g \circ f \leq g' \circ f' : A \to C$.

As a 2-category: Exactly ONE 2-cell can be defined from f to g for arrows $f, g : A \to B$ if and only if $f \leq g$. Otherwise, no 2-cells from f to g. For each pair of objects A and B, the category $C(\mathrm{Hom}(A, B))$ can be constructed as a poset.

Powerset Functor is a functor that takes a set S to its powerset $\mathcal{P}(S)$. There are three different powerset functors:

Inverse Image $\mathscr{P} : \mathbf{Set}^{op} \to \mathbf{Set}$
is a contravariant functor that takes sets S to its powerset $\mathcal{P}(S)$ and functions $f : A \to B$ to its inverse image $f : \mathcal{P}(B) \to \mathcal{P}(A)$.

Direct/Existential Image is a covariant functor that takes sets S to its powerset $\mathcal{P}(S)$ and functions $f : A \to B$ to the function $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ where $f_*(A_0) = \{f(x) | x \in A_0\}$.

Universal Image is a covariant functor that takes sets S to its powerset $\mathcal{P}(S)$ and functions $f : A \to B$ to the function $f_! : \mathcal{P}(A) \to \mathcal{P}(B)$ where $f_!(A_0) = \{y \in B | (f(x) = y) \Rightarrow (x \in A_0)\}$.

Preserve An arrow $f : A \to B$ **preserves** a property P if whenever A has P, then so does B.

Note: A functor $F : \mathscr{C} \to \mathscr{D}$ preserves P of arrows if whenever $f \in \mathscr{C}_1$ has P, so does $F(f) \in \mathscr{D}_1$.

Note: A property is preserved by isomorphisms if for any object A with the property, any object isomorphic to A must also have the property.

Product of Categories If $\mathscr{C}$ and $\mathscr{D}$ are categories their **product** $\mathscr{C} \times \mathscr{D}$ is the category whose objects are all ordered pairs (C, D) where $C \in \mathscr{C}_0, D \in \mathscr{D}_0$ and arrows are $(f, g) : (C, D) \to (C', D')$ where $f : C \to C' \in \mathscr{C}_1, g : D \to D' \in \mathscr{D}_1$. The identity arrow of (C, D) is (id_C, id_D) and composition of arrows is defined component wise.

Quotient Category $(/ \sim)$ For a congruence relation $\sim$ on the arrows of a category $\mathscr{C}$, the **quotient category** $\mathscr{C}/\sim$ is defined as follows:

- The objects of $\mathscr{C}/\sim$ are the objects of $\mathscr{C}$.
- The arrows of $\mathscr{C}/\sim$ are the congruence classes of arrows of $\mathscr{C}$.
- If $f : A \to B$ in $\mathscr{C}$, then $[f] : A \to B$ in $\mathscr{C}/\sim$.
- If $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, then $[g] \circ [f] = [g \circ f] : A \to C$ in $\mathscr{C}$.

Note: For any functor $F : \mathscr{C} \to \mathscr{D}$ with the property that if $f \sim g$ then $F(f) = F(g)$, then there is an unique functor $F_0 : \mathscr{C}/\sim \to \mathscr{D}$ for which $F_0 \circ Q = F$. Here, Q is the quotient functor, $Q : \mathscr{C} \to \mathscr{C}/\sim$.

Reflect A functor $F : \mathscr{C} \to \mathscr{D}$ **reflects** a property P of arrows if whenever $F(f)$ has property P then so does f (where f is any arrow for which F takes to $F(f)$).

Theorem: Consider a full and faithful functor $F : \mathscr{C} \to \mathscr{D}$. Let A and B be objects in C and $u : F(A) \to F(B)$ is an isomorphism in $\mathscr{D}$, then there is an unique isomorphism $f : A \to B$ in $\mathscr{C}$ for which $F(f) = u$.

Corollary: A full and faithful functor reflects isomorphisms.

Corollary: For a full an faithful functor F, if $F(A) = F(B)$ for objects A and B in the domain of F, then $A \cong B$.

Representable Functor A *set-valued functor* is representable if it is naturally isomorphic to a *hom functor*.

Note: If a covariant functor $F : \mathscr{C} \to \mathbf{Set}$ is isomorphic to $\mathrm{Hom}(C, -)$ for some object $C \in \mathscr{C}$, we say that C represents F. The same can be said for contravariant functors and contravariant hom functors.

Representative Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ in which every object of $\mathscr{C}$ is isomorphic to some object in $\mathscr{D}$.

Set-Valued Functor is any functor from any category $\mathscr{C}$ to **Set**.

Note: For small categories $\mathscr{C}$, a set valued functor $F : \mathscr{C} \to \mathbf{Set}$ generalizes the concept of monoid actions as functors.

Slice Category (/) If $\mathscr{C}$ is a category, for any object $A \in \mathscr{C}_0$, the **slice category** $\mathscr{C}/A$ is defined as follows:

- An object of $\mathscr{C}/A$ is an arrow $f : C \to A$ in $\mathscr{C}$ for some $C \in \mathscr{C}_0$.
- An arrow of $\mathscr{C}/A$ from $f : C \to A$ to $f' : C' \to A$ is an arrow $h : C \to C'$ such that $f = f' \circ h$.
- The composite of $h : f \circ f'$ and $h' : f' \circ f''$ is $h' \circ h$.

Note: Since the same h can satisfy both $f = f' \circ h$ and $g = g' \circ h$ where $f \neq g$ or $f' \neq g'$, $h : f \to f'$ and $h : g \to g'$ are regarded as different arrows in $\mathscr{C}/A$.

Note: The *indexed function* of an *indexed set* indexed by S, is precisely an arrow in the slice category $\mathbf{Set}/S$.

Split Epimorphism is an arrow that has a *right inverse* (this arrow can be shown to be an *epimorphism*, by composing its right inverse to its right).
Split Monomorphism is an arrow that has a *left inverse* (this arrow can be shown to be a *monomorphism*, by composing its left inverse to its left).
Subcategory A **subcategory** $\mathscr{D}$ of a category $\mathscr{C}$ is a category for which:

- All objects and arrows in $\mathscr{D}$ are objects and arrows in $\mathscr{C}$ (ie: $\mathscr{D}_0 \subseteq \mathscr{C}_0$ and $\mathscr{D}_1 \subseteq \mathscr{C}_1$).
- The source and target of an arrow in $\mathscr{D}$ are the same as its souce and target in $\mathscr{C}$ (ie: source and target maps in $\mathscr{D}$ are restrictions of those in $\mathscr{C}$). Thus, $\forall A, B \in \mathscr{C}_0$, $\mathrm{Hom}_{\mathscr{D}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A, B)$.
- If A is an object in $\mathscr{C}$, its identity arrow id_A in $\mathscr{C}$ is in $\mathscr{D}$.
- If $f : A \to B$ and $g : B \to C$ in $\mathscr{D}$, then the composite $g \circ f \in \mathscr{C}_1$ is also the composite in $\mathscr{D}$.

Subfunctor A **subfunctor** of a functor $F : \mathscr{C} \to \mathbf{Set}$ is a functor $G : \mathscr{C} \to \mathbf{Set}$ with the property that for each object $C \in \mathscr{C}$, $G(C) \subseteq F(C)$ and for each arrow $f : C \to C'$ and element $x \in G(C)$, $G(f)(x) = F(f)(x)$.

Note: The inclusion function $i_C : G(C) \to F(C)$ that takes the the subfunctor G to the functor F is a natural transformation.

Subobject In a category $\mathscr{C}$, a **subobject** of an object $C \in \mathscr{C}$ is an equivalence class of monomorphisms under the relation $\sim$. The subobject is a **proper subobject** if it does not contain id_C. The equivalence relation $\sim$ is defined as follows:

For monomorphisms $f_0 : C_0 \rightarrowtail C$ and $f_1 : C_1 \rightarrowtail C$, $f_0 \sim f_1$ if both f_0 and f_1 factors through each other.

Terminal Object (1) is an object T in $\mathscr{C}$ if there is exactly ONE arrow $A \to T$ for each object $A \in \mathscr{C}$ (including itself). This object is usually denoted 1 and the unique arrow $\langle\rangle : A \to 1$.

Note: Any two terminal objects in a category are isomorphic.

Example: Any singleton set in **Set** is a terminal object.

Universal Element By the *Yoneda Lemma*, an element $c \in F(C)$ is given by $c = h^C(id_C)$. c is an **universal element** of a functor $F : \mathscr{C} \to$ **Set** if and only if the induced $h^C : Hom(C, -) \to F$ is a *natural isomorphism*. Consequently, for any object $C' \in \mathscr{C}$ and any element $x \in F(C')$, there is an unique arrow $f : C \to C'$ in $\mathscr{C}$ for which $x = F(f)(c)$.

Note: There is a one-to-one correspondance between representations and universal elements of F.

Note: A similar argument for contravariant set valued functors of the form $F : \mathscr{C}^{op} \to$ **Set** can be made where $c \in F(C)$ is an **universal element** if for any object $C' \in \mathscr{C}$ and any element $x \in F(C')$, there is an unique arrow $f : C' \to C$ for which $x = F(f)(c)$.

Vertical Composition ($\circ$) Composition of natural transformations $\alpha : F \to G$ and $\beta : G \to H$ is $\beta \circ \alpha : F \to H$ and is known as **vertical composition**, where F, G and H are functors from the categories $\mathscr{C}$ to $\mathscr{D}$. This follows from the fact that the outer rectangle of the following diagram is commutative:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\beta A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\
F(B) \arrow[r, "\alpha B"'] & G(B) \arrow[r, "\beta B"'] & H(B)
\end{tikzcd}$$

for each arrow $f : A \to B$.

Note: The vertical composite of natural transformations is also a natural transformation.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"'{name=G}] \arrow[from=F, to=G, phantom, "\Downarrow\alpha"] & \mathscr{D} \circ \mathscr{C} \arrow[r, "G"{name=G2}] \arrow[r, bend right=60, "H"'{name=H}] \arrow[from=G2, to=H, phantom, "\Downarrow\beta"] & \mathscr{D} \longmapsto \mathscr{C} \arrow[r, bend left=60, "F"{name=F3}] \arrow[r, bend right=60, "H"'{name=H3}] \arrow[from=F3, to=H3, Rightarrow, "\beta\circ\alpha"] & \mathscr{D}
\end{tikzcd}$$

Wide Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ such that $\mathscr{D}_0 = \mathscr{C}_0$.

Example: **Set** is a wide subcategory of **Pfn**.

Yoneda Embedding For any category $\mathscr{C}$, there is a contravariant *Yoneda Functor* $Y : \mathscr{C}^{op} \to$ **Func**$(C,$ **Set**$)$ defined as:

- for objects $C \in \mathscr{C}, Y(C) = \mathrm{Hom}(C, -)$
- for an arrow $f : D \to C$ and an object $A \in \mathscr{C}, Y(f) = \mathrm{Hom}(f, -)$ and $Y(f)(A) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(D, A)$

Thus, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(D, A) \arrow[r, "\mathrm{Hom}(D,k)"] & \mathrm{Hom}(D, B) \\
\mathrm{Hom}(C, A) \arrow[r, "\mathrm{Hom}(C,k)"'] \arrow[u, "Y(f)A"] & \mathrm{Hom}(C, B) \arrow[u, "Y(f)B"']
\end{tikzcd}$$

for an arrow $k : A \to B$.

Note: $Y(f) : \mathrm{Hom}(C, -) \to \mathrm{Hom}(D, -)$ is the induced natural transformation between hom functors corresponding to f. Generalizing $\mathrm{Hom}(D, -)$ to an arbitary set valued functor, by the *Yoneda Lemma*, Y is both full and faithful.

Note: By the contravariant *Yoneda Lemma*, there also exists the covariant functor $J : \mathscr{C} \to \mathbf{Func}(\mathscr{C}^{op}, \mathbf{Set})$.

Yoneda Lemma Let $F : \mathscr{C} \to \mathbf{Set}$ be a *set-valued functor* and C is an object in C. There is a one-to-one correspondance between elements of $F(C)$ and $h^C : \mathrm{Hom}(C, -) \to F$, that is:

$$Nat(h^C, F) \cong F(C)$$

Thus, this is an isomorphism that is both an injective and surjective mapping between elements of $F(C)$ and h^C.

Note: $h^C(X) = \mathrm{Hom}(C, X) \to F(X)$ for an object $X \in \mathscr{C}$ is defined as $h^C(X)(f) = F(f)(c)$ where $f : C \to X$ is an arrow and $c \in F(C)$ is an element.

Note: There is also a contravariant version of Yoneda Lemma involving the contravariant hom functor $h_C : \mathrm{Hom}(-, C) \to F$.