Category Theory

Adam Yin

A glossary reference for Category Theory and relevant terms which I have came across. Most of the definitions are from *Category Theory for Computing Science* [1].

Glossary

2-Category is a category $\mathbf{C}$ with the following additional structure:

- A set of objects of $\mathbf{C}$ (called *0-cells*).
- $\forall A, B \in \mathbf{C}_0$ (for every pair of 0-cells), a small category $\mathbf{C}(A,B)$ whose objects are arrows from A to B in $\mathbf{C}$ (objects and arrows in $\mathbf{C}(A,B)$ are called *1-cells* and *2-cells* respectively).
- $\forall A, B, C \in \mathbf{C}_0$ (for every triple of 0-cells), there is a composition functor:

  $$comp : \mathbf{C}(B,C) \times \mathbf{C}(A,B) \longrightarrow \mathbf{C}(A,C)$$

  and that the following diagram commutes:

  $$\begin{tikzcd} \mathbf{C}(C,D) \times \mathbf{C}(B,C) \times \mathbf{C}(A,B) \arrow[r, "comp \times id"] \arrow[d, "id \times comp"'] & \mathbf{C}(B,D) \times \mathbf{C}(A,B) \arrow[d, "comp"] \\ \mathbf{C}(C,D) \times \mathbf{C}(A,C) \arrow[r, "comp"'] & \mathbf{C}(A,D) \end{tikzcd}$$

  ie: *comp* is associative

- For each object $A \in \mathbf{C}$, there is the identity functor:

  $$unit_A : 1 \to \mathbf{C}(A,A)$$

  where 1 is the terminal object in **Cat** (ie: the category with one object denoted as 0 and its identity arrow) defined as $unit_A(0) = id_A$ and that the following diagram commutes:

  $$\begin{tikzcd} & \mathbf{C}(A,B) \arrow[dl, "unit_B \times id"'] \arrow[d, "id"] \arrow[dr, "id \times unit_A"] & \\ \mathbf{C}(B,B) \times \mathbf{C}(A,B) \arrow[r, "comp"'] & \mathbf{C}(A,B) & \mathbf{C}(A,B) \times \mathbf{C}(A,A) \arrow[l, "comp"'] \end{tikzcd}$$

  ie: *unit* is the identity functor with respect to *comp*

The structure of a 2-category can be summarized as follows:

- A *horizontal category* whose objects are 0-cells, arrows are 2-cells.
- A *vertical category* whose objects are 1-cells, arrows are 2-cells.
- A *base category* (that can be derived from the horizontal and vertical categories) whose objects are 0-cells, arrows are 1-cells.

Visual Depictions:

0-cells x, y

1-cells $x \to y$

2-cells $\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$

Cat as a 2-category:

- 0-cells are small categories.
- 1-cells are functors.
- 2-cells are natural transformations.
- *comp* on 1-cells is composition of functors:

  $$comp(G,F) = G \circ F$$

  for arrows $F : A \to B, G : B \to C$ in **Cat**.
- *comp* on 2-cells is horizontal composition of natural transformations:

  $$comp(\beta, \alpha) = \beta * \alpha$$

  where $F, F' : A \to B$ and $G, G' : B \to C$ are functors $\alpha : F \to F', \beta : G \to G'$ are natural transformations between them.
- By the *Interchange Law*

  $$comp((\beta', \alpha') \circ (\beta, \alpha)) = comp(\beta', \alpha') \circ comp(\beta, \alpha)$$

2-Functor If **C** and **D** are 2-categories, a **2-functor** $F : \mathbf{C} \to \mathbf{D}$ consists of three maps of the form $F_i : \mathscr{C}_i \to \mathscr{D}_i$ where $i = b, v, h$ and $\mathscr{C}_i, \mathscr{D}_i$ are base, vertical or horizontal categories respectively.

Action Induced by Wreath Product Let $\mathscr{A}$, $\mathscr{B}$ be small categories, $G : \mathscr{A} \to$ **Cat**, $H : \mathscr{B} \to$ **Cat** be functors. There is an **induced functor** $G \text{ wr } H : \mathscr{A} \text{ wr}^G \mathscr{B} \to$ **Cat** defined by:

- For an object (A,P) in $\mathscr{A} \text{ wr}^G \mathscr{B}$, $(G \text{ wr } H)(A,P)$ is the split opfibration induced by $H \circ P : G(A) \to$ **Cat**. Thus, objects in this category are (t,x) where x is an object in $G(A)$ and t is an object in $(H \circ P)(x)$. Arrows are $(u, f) : (t,x) \to (t',x')$ where $f : x \to x'$ is an arrow in $G(A)$ and $u : (H \circ P)(f)(x) \to x'$ is an arrow in $G(A')$.
- For an arrow $(h, \lambda) : (G \text{ wr } H)(A,P) \to (G \text{ wr } H)(A',P')$ in $\mathscr{A} \text{ wr}^G \mathscr{B}$ is a functor where $h : A \to A'$ is an arrow in $\mathscr{A}$ and $\lambda : P \to (P' \circ G(h))$ (λ is a natural transformation). For objects (t,x) in $(G \text{ wr } H)(A,P)$, $(G \text{ wr } H)(h,\lambda)(t,x) = (H(\lambda x)(t), G(h)(x))$ where λx is a the natural transformation λ indexed by x in $G(A)$, $\lambda x : P(x) \to P'[G(h)(x)]$ and $H(\lambda x) : H[P(x)] \to H(P'[G(h)(x)])$ is a natural transformation $H\lambda$ indexed at x. For arrows $(u, f) : (t,x) \to (t',x')$ in $(G \text{ wr } H)(A,P)$, $(G \text{ wr } H)(h,\lambda)(h, f) = (H(\lambda x')(u), G(h)(f))$.

Adjunction ($\dashv$) Let GF denote $G \circ F$ and GFA denote $(G \circ F)(A)$. Between functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, F and G form an **adjunction** (denoted as $F \dashv G$) provided that there is a natural transformation $\eta : id \to GF$ such that for any objects $A \in \mathscr{A}$, $B \in \mathscr{B}$ and arrow $f : A \to UB$, there is an unique arrow $g : FA \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[dr, "f"'] & UFA \arrow[d, "Ug"] \\ & UB \end{tikzcd}$$

Note: Dually, for any adjunction $F \dashv G$, there is a natural transformation $\varepsilon : FG \to id$ such that for any $g : FA \to B$, there exists an unique arrow $f : A \to UB$ such that the following diagram commutes:

$$\begin{tikzcd} & FUB \arrow[d, "\varepsilon B"] \\ FA \arrow[ur, "Ff"] \arrow[r, "g"'] & B \end{tikzcd}$$

Note: If $F \dashv G$, then F is *left adjoint* to G and G is *right adjoint* to F. G preserves limits and F preserves colimits.

Note: η and ε are called the *unit* and *counit* of the adjunction.

Theorem: Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors, then $F \dashv G$ if and only if $\mathrm{Hom}(F-,-)$ and $\mathrm{Hom}(-,U-)$ are naturally isomorphic as functors of $\mathscr{A}^{op} \times \mathscr{B} \to$ **Set**.

Note: Adjoint functors are unique up to natural isomorphisms.

Algebra for an Endofunctor For an endofunctor $F : \mathscr{C} \to \mathscr{C}$, a F-**algebra** is a pair (A, α) where A is an object in $\mathscr{C}$ and $\alpha : F(A) \to A$ (A is called the *carrier* of the algebra).

A homomorphism between F-algebras (A, α) and (B, β) is an arrow $f : A \to B$ such that

$$\begin{tikzcd} F(A) \arrow[r, "\alpha"] \arrow[d, "F(f)"'] & A \arrow[d, "f"] \\ F(B) \arrow[r, "\beta"'] & B \end{tikzcd}$$

commutes.

Note: An object (A, α) in $(F : \mathscr{C})$ is a *fixed point* for F if α is an isomorphism.

Note: An initial object in $(F : \mathscr{C})$ is *least fixed point* for F.

Automorphism is an *endomorphism* that is *invertible*.

$*$**-Autonomous Category** is a symmetric monoidal closed category $\mathscr{C}$ with a duality functor $(-)^* : \mathscr{C}^{op} \to \mathscr{C}$ such that there is an natural isomorphism $\mathrm{Hom}(A \otimes B, C^*) \cong \mathrm{Hom}(A, (B \otimes A)^*)$.

Note: $(-)^*$ is related to the functors $\otimes$ and $\multimap$ by the isomorphisms $A \otimes B \cong (A \multimap B^*)^*$ and $A \multimap B \cong (A \otimes B^*)^*$

Note: There is a second monoidal structure in a $*$-autonomous category with $\oplus : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and $\perp = \top^*$ as unit where $A \oplus B = (A^* \otimes B^*)^*$.

Note: There are isomorphisms:

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, A \multimap C) \cong \mathrm{Hom}(B, (A \otimes C^*)^*) \cong \mathrm{Hom}(B, A^* \oplus C)$$

$$A^* \cong \top \multimap A^* \cong A \multimap \top^* \cong A \multimap \bot$$

By the second isomorphism, the duality functor is completely determined by $\multimap$ and $\bot$ and thus $\bot$ is also known as the *dualizing object*.

Binary Operation A **binary operation** on an object S is a function $S \times S \to S$ given that the product $S \times S$ exists.

Canonical Product A category has **canonical products** if a specific diagram/representation of the product can be given to each product.

Cartesian Closed Category (CCC) is a cartesian monoidal category $\mathscr{C}$ such that for every pair of object A and B, there is an *exponential object* B^A and an arrow $eval : B^A \times A \to B$ with the property that for any arrow $f : C \times A \to B$, there is an unique arrow $g : C \to B^A$ such that

$$f = eval \circ (g \times id_A)$$

Cartesian Monoidal Category If the products exists for a finite number of objects, a category is said to be a **cartesian monoidal category** or has *finite products*. An alternative definition is that a **cartesian monoidal category** is a monoidal category whose monoidal structure is the categorical product and unit is the terminal object.

Note: If a category has a terminal object and binary products, then it has finite products. Similarly, if a functor preserves terminal objects and binary products, it preserves all finite products.

Note: A lower semilattice is exactly a category that has all finite products. Dually, a upper semilattice is a category that has all finite coproducts.

Cartesian Morphism Let $P : \mathscr{E} \to \mathscr{C}$ be a functor, $f : C \to D$ is an arrow in $\mathscr{C}$ and $P(Y) = D$ for some object $Y \in \mathscr{E}$. A morphism/arrow $u : X \to Y$ in $\mathscr{E}$ is **cartesian** for f and Y if:

- $P(u) = f$.
- $\forall v : Z \to Y$ in $\mathscr{E}$, $\forall h : P(Z) \to C$ in $\mathscr{C}$ where $f \circ h = P(v)$, $\exists w : Z \to X$ in $\mathscr{E}$ such that $u \circ w = v$ and $P(w) = h$.

Note: A cartesian arrow is universal for f and Y as for any other arrow $v \in \mathscr{E}$, $f \circ h \in \mathscr{C}$ where $target(v) = Y$, $P(v) = f \circ h$, there is an unique factorization w, v factors through u.

Note: Dually, $u : X \to Y$ is **opcatesian/cocartesian** for $f : C \to D$ and $P(X) = C$ if $P(u) = f$ and $\forall v : X \to Z$, $\forall g : D \to P(Z)$ where $P(v) = g \circ f$, there exists an unique arrow $w : Y \to Z$ such that $P(w) = g$.

Categorical Dual (^{op}) Given any category $\mathscr{C}$, $\mathscr{C}^{op}$ is the **dual category** of $\mathscr{C}$ where all the arrows are reversed. $\mathscr{C}^{op}$ is defined as follows:

- The objects and arrows of $\mathscr{C}^{op}$ are the objects and arrows of $\mathscr{C}$.
- If $f : A \to B$ in $\mathscr{C}$, then $f : B \to A$ in $\mathscr{C}^{op}$.
- If $h = g \circ f$ in $\mathscr{C}$, then $h = f \circ g$ in $\mathscr{C}^{op}$.

Since identity arrows have the same source and target objects, identity arrows in $\mathscr{C}$ are the same as in $\mathscr{C}^{op}$ and thus $\mathscr{C}^{op}$ is also a category.

Category A **category** $\mathscr{C}$ contains a collection of *objects* (denoted as $\mathscr{C}_0$) and a collection of *morphisms* (denoted as $\mathscr{C}_1$) where every morphism has $source, target : \mathscr{C}_1 \to \mathscr{C}_0$ functions mapping each arrow to its *source* and *target* objects respectively into the collection of objects.

For any two arrows f, g where the $target(f) = source(g)$, the *composition* of f and g are written as $g \circ f : source(f) \to target(g) \in \mathscr{C}_1$. Composition is associative, $(h \circ g) \circ f = h \circ (g \circ f)$ when either side is defined.

$\forall A \in \mathscr{C}_0$, there exists the *identity* arrow, $id_A : A \to A$ and for any $f : A \to B$, $f \circ id_A = id_B \circ f = f$.

A category is *small* if its objects and arrows constitute sets; otherwise it is *large*.

Category Equivalence A functor $F : \mathscr{C} \to \mathscr{D}$ is an **equivalence of categories** if there is a functor $G : \mathscr{D} \to \mathscr{C}$ such that:

- A family $u_C : C \to G(F(C))$ of isomorphisms of $\mathscr{C}$ indexed by the objects of $\mathscr{C}$ with the property that for every arrow $f : C \to C'$ in $\mathscr{C}$, $G(F(f)) = u_{C'} \circ f \circ u_C^{-1}$.
- A family $v_D : D \to F(G(D))$ of isomorphisms of $\mathscr{D}$ indexed by the objects of $\mathscr{D}$ with the property that for every arrow $g : D \to D'$ in $\mathscr{D}$, $F(G(g)) = v_{D'} \circ g \circ v_D^{-1}$.

Note: In this case, the functor G is called a *pseudo-inverse* of F and u, v are *natural isomorphisms*.

Note: F and G are both full and faithful.

Cleavage A **cleavage** for a fibration $P : \mathscr{E} \to \mathscr{C}$ is a function γ that takes an arrow $f : C \to D$ in $\mathscr{C}$ and each object $Y \in \mathscr{E}$ where $P(Y) = D$ to an arrow $\gamma(f, Y) \in \mathscr{E}$ that is cartesian for f and Y. Dually, an **opcleavage** κ takes f and each object $X \in \mathscr{E}$ where $P(X) = C$ to the arrow $\kappa(f, X) \in \mathscr{E}$ that is opcartesian for f and X.

Note: A cleavage is *normal* if $\gamma(id_D, Y) = id_Y$.

Note: A cleavage is *splitting* if it is normal and the composite of cartesian arrows is a cartesian arrow of the composite, thus $\gamma(g, Z) \circ \gamma(f, Y) = \gamma(g \circ f, Z)$ where $g : D \to E$ is in $\mathscr{C}$ and $P(Z) = E$. Note that this is well defined as as the domain of $\gamma(g, Z)$ is D, and $P(Y) = D$.

Note: A fibration is *split* if it has a splitting. The same can be said for opfibrations and opcleavages that are splitting.

Cocone is the dual of a cone. Thus, a cocone in a category $\mathscr{C}$ is a cone in $\mathscr{C}^{op}$.

Coequalizer is the dual of *equalizer*. For a parallel pair of arrows $f, g : A \to B$, an arrow $h : B \to C$ is the coequalizer of f and g provided $h \circ f = h \circ g$ and for any arrow $k : B \to D$ where $k \circ f = k \circ g$, there is an unique arrow $l : C \to D$ such that $k = l \circ h$.

Note: Coequalizers can be thought of as the least destructive identification necessary (by quotienting into equivalence classes) to force an equation or set of equations to be true.

Example: The *unification* of expressions by finding a common substitution is an example of a coequalizer.

Colimit ($\varinjlim$) is the dual of a limit, which is an universal commutative cocone. A **colimit** is the initial object of the category of cocones.

Note: A category that has all colimits is *cocomplete*, if it has all finite colimits, it is *finitely cocomplete*.

Comma Category Given functors $f : \mathscr{C} \to \mathscr{E}$ and $g : \mathscr{D} \to \mathscr{E}$, their **comma category** (f/g) is defined as:

- An object is a triple (c, d, a) where c, d are objects in $\mathscr{C}, \mathscr{D}$ respectively and $a : f(c) \to g(d)$ is an arrow in $\mathscr{E}$
- An arrow is a pair $(\beta, \gamma) : (c_1, d_1, a_1) \to (c_2, d_2, a_2)$ where $\beta : c_1 \to c_2$, $\gamma : d_1 \to d_2$ are arrows in $\mathscr{C}, \mathscr{D}$ such that $a_2 \circ f(\beta) = g(\gamma) \circ a_1$:

$$\begin{tikzcd}
f(c_1) \arrow[r, "f(\beta)"] \arrow[d, "a_1"'] & f(c_2) \arrow[d, "a_2"] \\
g(d_1) \arrow[r, "g(\gamma)"'] & g(d_2)
\end{tikzcd}$$

- Composition of arrows $(\beta, \gamma) : (c_1, d_1, a_1) \to (c_2, d_2, a_2)$ and $(\beta', \gamma') : (c_2, d_2, a_2) \to (c_3, d_3, a_3)$ is $(\beta' \circ \beta, \gamma' \circ \gamma) : (c_1, d_1, a_1) \to (c_3, d_3, a_3)$

Complete Category is a category $\mathscr{C}$ in which every diagram $D : \mathbb{I} \to \mathscr{C}$ has a limit in $\mathscr{C}$, where $\mathbb{I}$ is a small category. $\mathscr{C}$ has all finite limits or is **finitely complete** if $\mathbb{I}$ is a finite category.

Cone A **cone** over a diagram $D : \mathbb{I} \to \mathscr{C}$ (also known as the base) with an object $U \in \mathscr{C}$ (also known as the vertex) is a family of arrows of in $\mathscr{C}$ from indexed by $D(a)$ for each object $a \in \mathbb{I}$, where each arrow has the form $p_a : U \to D(a)$.

A cone is *commutative* if for any arrow $s : a \to b$ in $\mathbb{I}$, $D(s) \circ p_a = p_b$, ie: the following diagram commutes:

$$\begin{tikzcd}
& U \arrow[dl, "p_a"'] \arrow[dr, "p_b"] & \\
D(a) \arrow[rr, "D(s)"'] & & D(b)
\end{tikzcd}$$

Note: A commutative cone over D with vertex U is precisely a natural transformation from the constant functor Δ_U (the functor that takes every object to U and morphism to id_U) to D, where the commutativity is given by naturality.

Note: A cone over a discrete diagram is vacuously commutative.

Note: A *morphism* between two cones $\alpha : \Delta_U \to D$ and $\alpha' : \Delta_{U'} \to D$ is an arrow $f : U \to U'$ such that for every object $a \in \mathbb{I}$, the following diagram commutes:

$$\begin{tikzcd}
U \arrow[rr, "f"] \arrow[dr, "\alpha_a"'] & & U' \arrow[dl, "\alpha'_a"] \\
& D(a) &
\end{tikzcd}$$

Congruence Relation $(\sim)$ is an equivalence relation $\sim$ on the arrows of a category $\mathscr{C}$ if:

- Whenever $f \sim g$, then f and g have the same domain and the same codomain.

- In the diagram below:

$$A \xrightarrow{h} B \overset{f}{\underset{g}{\rightrightarrows}} C \xrightarrow{k} D$$

If $f \sim g$, then $f \circ h \sim g \circ h$ and $k \circ f \sim k \circ g$ for arrows h, k where $target(h) = source(f)$ and $source(k) = target(f)$.

Note: The congruence class containing the arrow f is denoted by $[f]$.

Constant/Global Element is an arrow from a terminal object. Thus, the **constant** A is an arrow $1 \to A$ for some object A. In **Set**, such an arrow is precisely a function from a singleton set to the set A, where each element $x \in A$ can be uniquely determined by a constant arrow $x : 1 \to A$.

ω-Continuous For any two ω-CPO objects D and D', $f : D \to D'$ is an **ω-continuous** arrow if for any object A and $g_0 \leq g_1 \leq \ldots$ where $g_0, g_1, \ldots$ are arrows in $\mathrm{Hom}(A, D)$ with supremum g, $f \circ g$ is the supremum of $f \circ g_0 \leq f \circ g_1 \leq \ldots$. It is easy to see that f is monotone (order preserving).

Contravariant Functor is a functor $F : \mathscr{C}^{op} \to \mathscr{D}$ for categories $\mathscr{C}$ and $\mathscr{D}$. In contrast to *covariant* functors, for an arrow f in $\mathscr{C}$

$$F(f : B \to A) = F(A) \to F(B)$$

is an arrow in $\mathscr{D}$. This follows from the fact that f is an arrow from A to B in $\mathscr{C}^{op}$.

Coproduct/Sum $(+)$ is the categorial dual of *product*. A **coproduct** $A + B$ of two objects A and B in a category consists of an object $A + B$ together with arrows/*coprojections* $i_1 : A \to A + B$ and $i_2 : B \to A + B$ such that given any arrows $f : A \to C$ and $g : B \to C$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "i_1"'] & C & B \arrow[l, "g"'] \arrow[dl, "i_2"] \\ & A+B \arrow[u, "\langle f|g \rangle"] & \end{tikzcd}$$

ie, $\langle f|g \rangle \circ i_1 = f$ and $\langle f|g \rangle \circ i_2 = g$.

Note: The arrows i_1, i_2 are also known as *canonical injections* or *inclusions*.
Note: The functor $\mathrm{Hom}(A, -) \times \mathrm{Hom}(B, -)$ is represented by $A + B$ and the universal element is the pair of canonical injections.

Diagram A **diagram** in a category $\mathscr{C}$ of shape $\mathbb{I}$ is a functor $D : \mathbb{I} \to \mathscr{C}$ where $\mathbb{I}$ is also a category (usually small or finite).

Note: The purpose of $\mathbb{I}$ is to give shape to the diagram functor.

Discrete A category in which all arrows are identity arrows is called **discrete**.

Disjoint Coproduct A coproduct $A + B$ in a category is **disjoint** if the injections $A \to A + B$ and $B \to A + B$ are monic and the pullback of the injections along each other is the initial object (since the injections are monic, these injections are subobjects and their intersection is the initial object). ie:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A+B \end{tikzcd}$$

is a pullback diagram (this is also a pushout diagram by the definition of coproducts).

Note: In **Set**, this precisely captures the idea of a disjoint union between sets A and B where $A \cap B = \emptyset$.

Note: If $A + B$ is disjoint, the subobjects A and B are known as *complemneted subobjects* and A is the complement of B in C and conversely.

Distributive Category is a category with both finite products and coproducts. Thus for all objects A, B and C, the following diagram commutes

$$\begin{tikzcd} A \times B \arrow[r, "i'_1"] \arrow[dr, "A \times i_1"'] & A \times B + A \times C \arrow[d, "d"] & A \times C \arrow[l, "i'_2"'] \arrow[dl, "A \times i_2"] \\ & A \times (B + C) & \end{tikzcd}$$

and d is an isomorphism.

In a distributive category $\mathscr{C}$, all canonical projections are monic and for any object $A \in \mathscr{C}$, the arrow $\langle id_0, ! \rangle : 0 \to 0 \times A$ is an ismorphism (ie: $0 \cong 0 \times A$ and thus all distributive categories has a strict initial object).

Example: A *Boolean Algebra* is a distributive category with $\wedge$ as product and $\vee$ as coproduct.

Dual Adjunction is an adjunction $F \dashv G$ where $F : C \to D^{op}$ and $G : D \to C^{op}$.

Example: Let A and B be posets. A *Galois Connection* is a dual adjunction between order-reversing maps $f : A \to B$ and $g : B \to A$ such that $a \leq g(f(a))$ and $b \leq f(g(b))$ for all $a \in A$, $b \in B$ (these are the unit and counit of the dual adjunction).

Eilenberg-Moore Category is the category of M-Algebras denoted as $\mathscr{C}^M$ for a monad $M : \mathscr{C} \to \mathscr{C}$. Thus, objects are (A, v) where A is an object in $\mathscr{C}$ and $v : T(A) \to A$. Arrows are homomorphims between such M-algebras such that the following diagrams commute:

$$\begin{tikzcd} T^2(A) \arrow[r, "\mu A"] \arrow[d, "T(v)"'] & T(A) \arrow[d, "v"] \\ T(A) \arrow[r, "v"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta A"] \arrow[dr, "id"'] & T(A) \arrow[d, "v"] \\ & A \end{tikzcd}$$

Note: It is apparent that there is a forgetful functor $U : \mathscr{C}^M \to \mathscr{C}$ and also a free functor $F : \mathscr{C} \to \mathscr{C}^M$ that is left adjoint to U. It can be shown that the monad induced by $U \circ F$ is isomorphic to M.

Note: The Kleisli category over M is a full subcategory of $\mathscr{C}^M$.

Endomorphism is an arrow $f : A \to A$ in a category where the source and target objects are the same.

Epimorphism ($\twoheadrightarrow$) is the dual to monomorphisms. An arrow $f : S \to T$ is an **epimorphism** if for any arrows $g, h : T \to X$, $g \circ f = h \circ f$ implies $g = h$. Epimorphisms can be denoted by $f : S \twoheadrightarrow T$.

Properties: Let $f : A \to B$ and $g : B \to C$ be arrows:

– If f and g are epimorphisms, so is $g \circ f$.

– If $g \circ f$ is an epimorphism, so is g.

Note: In **Set**, a function is surjective if and only if it is an epimorphism.

Note: In the category determined by a Monoid, an epic element is right cancellable.

Equalizer is the limit over a parallel pair of arrows. Let $f, g : A \to B$ be a parallel pair of arrows. An **equalizer** of f and g is an object E together with an arrow $j : E \to A$ such that:

- $f \circ j = g \circ j$
- If $h : C \to A$ and $f \circ h = g \circ h$, there is an unique arrow $k : C \to E$ such that $j \circ k = h$ (ie: h factors through j).

Note: An arrow that is an equalizer of some parallel pair of arrows is a monomorphism. In addition, any two equalizers for the same pair of arrows belong to the same subobject class of monomorphisms.

Note: Equalizers can be thought of as the largest subobject on which an equation or set of equations is true.

Equivalence Relation For $d, e : E \to A$ a parallel pair of arrows, for an object B, the function $\langle \mathrm{Hom}(B,d), \mathrm{Hom}(B,e) \rangle : \mathrm{Hom}(B,E) \to \mathrm{Hom}(B,A) \times \mathrm{Hom}(B,A)$ sends a function $f : B \to E$ to $(d \circ f, e \circ f)$. If for each object B, $\langle \mathrm{Hom}(B,d), \mathrm{Hom}(B,e) \rangle$ is an isomorphism and there is an equivalence relation on the set $\mathrm{Hom}(B,A)$, then E is an **equivalence relation** on A (note that the image of $\langle \mathrm{Hom}(B,d), \mathrm{Hom}(B,e) \rangle$ is a subset of $\mathrm{Hom}(B,A) \times \mathrm{Hom}(B,A)$ and is an equivalence relation).

Note: An equivalence relation of E on A is *effective* if it is the kernel pair of some arrow from A.

Note: A topos has effective equivalence relations as every equivalence relation is effective.

Examples of Cartesian Closed Categories:

Set For any objects A and B, the exponential object B^A is the set of functions from A to B, effectively the same as $\mathrm{Hom}(A,B)$. $eval : B^A \times A \to B$ is the application of f to a in $(f,a) \in B^A \times A$, ie: $eval(f,a) = f(a)$.

Boolean Algebra In addition to being a distributive category, the exponential object q^p is $\neg p \vee q$. $eval : q^p \times p \to q$ follows from the distributive law, $(\neg p \vee q) \wedge p \Rightarrow q$.

Heyting Algebra The meet operation is product, implication is the exponential object and the arrow *eval* is *modus ponens*.

Grf Let No be the graph with 1 node and no arrows and Ar be the graph with 2 nodes and 1 arrow between them. The exponential object between two graphs $\mathscr{G}$ and $\mathscr{H}$, $\mathscr{H}^{\mathscr{G}}$, is a graph which has a set of nodes $\mathrm{Hom}(\mathscr{G} \times \mathrm{No}, \mathscr{H})$ and a set of arrows $\mathrm{Hom}(\mathscr{G} \times \mathrm{Ar}, \mathscr{H})$. Both sets of nodes and arrows are sets of graph homorphisms.

Func($\mathscr{C}$,Set) For any category $\mathscr{C}$, **Func($\mathscr{C}$,Set)** is a cartesian closed category. For any two functors $F, G : \mathscr{C} \to$ **Set**, the exponential object G^F is a functor defined as follows:

– For any object C, $G^F(C) = (\mathrm{Hom}(C,-) \times F) \to G$ is a set of natural transformations.

– For any arrow $f : C \to D$, $G^F(f)$ takes a natural transformation α : $\mathrm{Hom}(C,-) \times F \to G$ to a natural tranformation (with component at an object A) that is $G^F(f)(\alpha) : \mathrm{Hom}(D,A) \times F(A) \to G(A)$, defined as $G^F(f)(\alpha)(g,a) = \alpha A(v \circ f, x)$ for an arrow $g : D \to A$ and an object $a \in F(A)$.

Cat The exponential object of any two categories $\mathscr{C}$ and $\mathscr{D}$ is $\mathscr{D}^{\mathscr{C}} = \mathbf{Func}(\mathscr{C}, \mathscr{D})$.

ω-CPOs and Scott-continuous Functions

Functional Programming Language The exponential object is the function type and the *eval* arrow is simply the function application function.

Typed Lambda Calculus Objects are types and an arrow from an object A to and object B is an equivalence class of terms of type B with one free variable of type A. The exponential object B^A is the lambda abstraction with a bound variable of type A and a body of body B.

Examples of Categories:

Set is the category whose objects are sets and arrows are functions between sets. Composition is function composition and the identity arrow is the identity function id_S for a set S.

Fin is the category whose objects are finite sets and arrows are all the functions between finite sets.

Pfn is the category whose objects are sets but all arrows are all partial functions. If $f : S \to T, g : T \to V$ are partial functions with f defined on $S_0 \subseteq S$ and g defined on $T_0 \subseteq T$, then the composite $g \circ f : S \to V$ is the partial function from S to V defined on the subset $\{x \in S_0 | f(x) \in T_0\}$ of S by the requirement $(g \circ f)(x) = g(f(x))$.

Rel is the category whose objects are sets and arrows are relations relations between sets. The composite $\beta \circ \alpha$ of relations α from sets S to T and β from T to U is a relation from S to U defined as follows:

If $x \in S$ and $z \in U$, $(x,z) \in \beta \circ \alpha$ if and only if there is an element $y \in T$ for which $(x,y) \in \alpha$ and $(y,z) \in \beta$.

The identity arrow for a set S is the *diagonal relation* $\Delta_S = \{(x,x) | x \in S\}$

Poset is the category whose objects are elements of a poset P and arrows are the partially ordered relation between those elements. The reflexive and transitive properties provides the identity and composition of arrows.

Note: Every hom-set in this category has at most ONE element. Equivalently, any category where every hom-set has at most one element is a poset.

Monoid is the category whose only object is the Monoid M itself and arrows are the elements of the monoid (this category is denoted as $C(M)$). The identity arrow is the identity element and composition being the monoid binary operation.

Note: Any category with one object is a monoid.

Group is the category whose only object is the Group itself and arrows are elements of the group, same as the category determined by a Monoid.

Note: All arrows in this category are isomorphisms.

Grf is the category whose objects are graphs and arrows are graph homomorphisms. Identity arrow being the identity homormophism and composition being composition of homomorphisms.

Posets and Monotone Functions is the category whose objects are posets and arrows are monotone functions between posets.

Sem is the category whose objects are semigroups and arrows are semigroup homomorphisms.

Note: This can be extended to monoids, denoted as **Mon**.

ω-CPOs and Scott-continuous Functions is the category whose objects are ω-CPOs and arrows are Scott-continuous functions.

Note: A category is also formed with *strict* ω-CPOs and *strict* Scott-continous functions.

Path Category is the free category $F(\mathscr{G})$ generated by a graph $\mathscr{G}$ whose objects are the nodes of $\mathscr{G}$ and arrows are paths in $\mathscr{G}$. For each node/object A, the identity arrow is the empty path from A to A. Composition is defined by $(f_1,\dots,f_k)\circ(f_{k+1},\dots,f_n) = (f_1,\dots,f_n)$

Cat is the category whose objects are small categories and arrows are functors between small categories.

M-**Act** is the category whose objects are monoid actions and arrows are equivariant maps. Since equivariant maps are functions, the identity arrow is the identity equivariant map and composition is function composition.

FBool is the category whose objects are finite Boolean algebras and arrows are homormophisms between them.

Pts is the category whose objects are pointed sets and arrows are functions that preserves the pointed object.

Func$(\mathscr{C},\mathscr{D})$ is the category whose objects are functors from categories $\mathscr{C}$ to $\mathscr{D}$ and arrows are natural transformations.

$cone(D)$ is the category whose objects are cones with base D and arrows are morphisms between cones.

SO$(\mathscr{C})$ is the category whose objects are split opfibrations of the category $\mathscr{C}$ and arrows are homomorphisms of split opfibrations.

End(M) is the category of endomorphisms for the monoid M.

$(F:\mathscr{C})$ is the category of F-algebras and homomorphisms between F-algebras for the endofunctor $F:\mathscr{C}\to\mathscr{C}$.

$act(A)$ is the category of algebras for the functor $A\times -$ where A is an object in $\mathscr{C}$. Thus, objects of $act(A)$ are $x : A\times X \to X$ for each object X in $\mathscr{C}$ and arrows are homomorphisms between these algebras.

$\mathscr{K}(M)$ is the *Kleisli* category for a monad (M,η,μ) in a category $\mathscr{C}$. $\mathscr{K}(M)$ has the same objects as $\mathscr{C}$ but an arrow from A to B is an arrow of the form $f : A\to M(B)$ where A and B are objects in $\mathscr{C}$. Let $*$ denote composition in $\mathscr{K}(M)$, for another arrow $g : B\to M(C)$, composition is defined as $g * f = \mu C\circ M(g)\circ f : A\to M(C)$. For any object A, the identity arrow is $\eta A : A\to M(A)$.

Note: There are functors $U : \mathcal{K}(M) \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{K}(M)$ defined by $U(A) = M(A)$, $U(f : A \to B) = \mu B \circ M(f)$ and $F(A) = A$, $F(g : A \to B) = M(g) \circ \eta A$. Then $F \dashv U$ and $M = U \circ F$.

Psh$(\mathcal{C})$ is the category where objects are presheaves on $\mathcal{C}$ and arrows are natural transformations between presheaves.

Note: The category of presheaves on a category $\mathcal{C}$ forms a topos.

Examples of Functors:

Monoid Homomorphism $f : M \to N$ where M and N are monoids is a functor from the category $C(M)$ to the category $C(N)$.

Monotone Map between two posets is a functor between the category determined by the posets.

Projection The first and second projections from the product category to its first or second parts, $P_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $P_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ for categories $\mathcal{C}$ and $\mathcal{D}$ are functors.

Inclusion Map of a subcategory is a functor.

Note: By the definition of *subcategory*, objects and arrows of the subcategory need not be objects and arrows in the bigger category.

Example: **Set** is a subcategory of **Rel** and the functor is defined by taking sets to the sets as objects in **Rel** and set functions $f : S \to T$ to $\{(s,t) | t = f(s)\}$.

Monoid Actions For an action α of a monoid M acting on a set S, let $C(M)$ be the category determined by M and the action α determines a functor $F_\alpha : C(M) \to$ **Set** defined as:

- $F_\alpha(*) = S$ where $*$ denotes the single object in $C(M)$.
- $F_\alpha(m) = s \mapsto \alpha(m,s)$ for $m \in M$, $s \in S$.

Quotient Functor $Q : \mathcal{C} \to \mathcal{C}/\sim$
is the functor thats takes a category to its quotient category by where $Q(A) = A$ for any object A and $Q(f) = [f]$ for arrows f in the category $\mathcal{C}$.

Pullback Functor Let $\mathcal{C}$ be a category with pullbacks and $f : A \to B$ is an arrow in $\mathcal{C}$. The **pullback functor** $f^* : \mathcal{C}/B \to \mathcal{C}/A$ is defined as:

* For an object $x : X \to B$, $f^*(x) = p_1 : P \to A$ of the following pullback diagram:

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "x"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

* For an arrow $t : x \to x'$ and objects $x : X \to B$, $x' : X' \to B$, $x' \circ t = x$ and $f^*(t) = s$ of the following pullback diagram:

$$\begin{tikzcd}
P \arrow[dr, "s"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "p_1"'] & & \\
& P' \arrow[r, "p_2'"] \arrow[d, "p_1'"'] & X' \arrow[d, "x'"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

where $s : P \to P'$ is the unique arrow by the definition of a pullback.

Exponential Object An **exponential object** B^A is an *internal hom*$[A,B]$ in a *cartesian closed category*. B^A is equipped with the arrow $eval : B^A \times A \to B$ and is the universal element of the functor $\mathrm{Hom}(- \times A, B)$. This means for any other object C and the $eval' : C \times A \to B$, there is an unique arrow from $f : C \to B^A$ such that

$$eval' = eval \circ (f \times id_A)$$

An exponential object is also known as *internal hom*.

Note: Since *eval* is an universal element of the functor $\mathrm{Hom}(- \times A, B)$, the process of *currying*, which takes the function $f : C \times A \to B$ to $f' : C \to B^A$ forms a natural isomorphism of functors

$$\mathrm{Hom}(- \times A, B) \cong \mathrm{Hom}(-, B^A)$$

Note: Consider the sets A, B where $|A|$ is the cardinality of A. The exponential object B^A is a set of functions from A to B. $\forall f \in B^A$, f maps each element of A to an element of B, which can be described as $|A|$-tuple of elements of B. Hence, the notation for B^A which is an $|A|$ product of B.

Note: Any cartesian closed category with finite coproducts is a distributive category.

Extremal Morphisms An **extremal epimorphism** in a category $\mathscr{C}$ is a morphism e such that if $e = m \circ g$ where m is monic, then m is an isomorphism. Dually, an **extremal monomoprhism** is a a morphism m such that if $m = g \circ e$ where e is epic, then e is an isomorphism.

Factor Let $f : A \to B$ and $g : C \to B$ be arrows. If there is an arrow $h : A \to C$ such that $f = g \circ h$, then f **factors** through g.

Factorization System A **factorization system** in a category $\mathscr{C}$ consists of two subclasses $\mathscr{E}$ and $\mathscr{M}$ of the arrows of $\mathscr{C}$ such that:

- If $\mathscr{I}$ is the class of isomorphisms, then $\mathscr{M} \circ \mathscr{I} \subseteq \mathscr{M}$ and $\mathscr{I} \circ \mathscr{E} \subseteq \mathscr{E}$.
- Every arrow $f \in \mathscr{C}$ factors as $f = m \circ e$ where $m \in \mathscr{M}$ and $e \in \mathscr{E}$.
- In any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

  with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there is a unique $h : B \to C$ such that $h \circ e = f$ and $m \circ h = g$.

Faithful A functor $F : \mathscr{C} \to \mathscr{D}$ is **faithful** if the set mapping:

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

for each pair of objects A and B in $\mathscr{C}$ induced by F is *injective*.

Note: If $f : A \to B$ and $g : C \to D$, as long as either $A \neq C$ or $C \neq D$, $F(f) = F(g)$ may hold and F can still be faithful.

Note: A faithful functor need not be injective on objects or arrows.

Fiber For any functor $P : \mathscr{E} \to \mathscr{C}$, the **fiber** over an object $C \in \mathscr{C}$ is the collection of objects X and arrows f where $P(X) = C$ and $P(f) = id_C$. It can be shown that this fiber is a subcategory of $\mathscr{E}$.

Note: The fibers of a functor forms an "indexed category". A functor $F : \mathscr{C} \to \mathbf{Cat}$ can be defined for an opfibration $P : \mathscr{E} \to \mathscr{C}$ with opcleavage κ, where $\forall C \in \mathscr{C}, F(C)$ is the fiber over C and for any arrow $f : C \to D$ in $\mathscr{C}$, $F(f)$ is the functor between the fiber over C and D. Dually, a contravariant functor $F : \mathscr{C}^{op} \to \mathbf{Cat}$ can be defined for a fibration $P : \mathscr{E} \to \mathscr{C}$ with cleavage γ.

Fibration A functor $P : \mathscr{E} \to \mathscr{C}$ is a **fibration** if there is a cartesian arrow for every $f : C \to D$ in $\mathscr{C}$ and object $Y \in \mathscr{E}$ such that $P(Y) = D$. Dually, P is an **opfibration** if there is an opcartesian arrow for every f and object $X \in \mathscr{E}$ such that $P(X) = C$.

Note: P is a fibration if and only if $P^{op} : \mathscr{E}^{op} \to \mathscr{C}^{op}$ is an opfibration.

Note: If $P : \mathscr{E} \to \mathscr{C}$ is a functor, then $\mathscr{E}$ is *fibered over* $\mathscr{C}$. $\mathscr{C}$ is also known as the *base category*, $\mathscr{E}$ as the *total category*.

Finitary Functor A set-valued endofunctor R is **finitary** if for each set S and element $x \in R(S)$, there is a finite subset $S_0 \subseteq S$ and an element $x_0 \in R(S_0)$ such that $x = R(i_0)(x_0)$ where $i_0 : S_0 \to S$ is the inclusion.

Finite Category is a category which consists of a finite set of objects and a finite set of arrows.

Forgetful/Underlying Functor is the functor which "forgets" some of the structure in a category of structures and structure-preserving functions.

Examples:

- $U : \mathbf{Mon} \to \mathbf{Sem}$
- $U : \mathbf{Sem} \to \mathbf{Set}$
- $U : \mathbf{Grf} \to \mathbf{Set} \times \mathbf{Set}$
  that takes a graph $\mathscr{G}$ to the set of nodes and set of arrows, $U(\mathscr{G}) = (\mathscr{G}_0, \mathscr{G}_1)$. It follows that there is also an arrowset functor $A : \mathbf{Grf} \to \mathbf{Set}$ and nodeset functor $N : \mathbf{Grf} \to \mathbf{Set}$.
- $U : \mathbf{Cat} \to \mathbf{Grf}$
  by forgetting the identity arrows and composition, the remains of a category becomes a graph.

  Note: If F is a functor, then F is a graph homomorphism. Note that the converse is not true.

  Simiarily, there is the functor $A : \mathbf{Cat} \to \mathbf{Set}$ and $O : \mathbf{Cat} \to \mathbf{Set}$ which takes a category to its set of arrows and objects respectively.
- $U : \mathscr{C}/A \to \mathscr{C}$
  for an object A in the category $\mathscr{C}$.

  Note: If $\mathscr{C}$ is $\mathbf{Set}$, the underlying functor $U : \mathbf{Set}/S \to \mathbf{Set}$ forgets the indexing of S-indexed sets.

Free Functor Informally, a **free functor** is left adjoint to a **forgetful functor**.

Examples:

- $F : \mathbf{Set} \to \mathbf{Mon}$
  is the free monoid functor that takes a set A to the free monoid $F(A)$, the Kleene closure A^* with concatenation as the monoid operation. F takes set functions to the Kleene closure induced homormophism.

Note: The Kleene closure itself is a functor from **Set** to **Set**, which is the composition of $U \circ F$, the underlying functor $U : \textbf{Mon} \to \textbf{Set}$ and the free monoid functor.

- $F : \textbf{Grf} \to \textbf{Cat}$
  takes graphs to the path category of the graph as objects in **Cat** and graph homomorphisms to arrows (functors between path categories). For a graph homomorphism $\phi : \mathscr{G} \to \mathscr{H}$, $F(\phi)(f_n, f_{n-1}, \ldots, f_1) = (\phi(f_n), \phi(f_{n-1}), \ldots, \phi(f_1))$, which is clearly a path in $\mathscr{H}$ (and thus an arrow in the path category of $\mathscr{H}$).
- $F : \textbf{Set} \to \mathscr{L}$
  takes a set S to the category $\mathscr{L}$ of join-semilattice and homomorphisms where semilattices are non-empty subsets of S with inclusion as ordering. If a join-semilattice has bottom, the empty $\emptyset$ is the bottom element of the semi-lattice.

  Note: Meet-semilattices can be constructed by dualizing the inclusion for ordering.

Full A functor $F : \mathscr{C} \to \mathscr{D}$ is **full** if the set mapping:

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(F(A),F(B))$$

for each pair of objects A and B in $\mathscr{C}$ induced by F is *surjective*.
Note: A full functor need not be surjective on objects or arrows.
Full Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ such that $\forall A, B \in \mathscr{D}_0$, $\mathrm{Hom}_{\mathscr{D}}(A,B) = \mathrm{Hom}_{\mathscr{C}}(A,B)$.
Example: **Fin** is a full subcategory of **Set**.
Functor is a "structure preserving" map between categories, similar to homomorphisms between groups, graphs and etc... A **functor** $F : \mathscr{C} \to \mathscr{D}$ from the category $\mathscr{C}$ to the category $\mathscr{D}$ defined as follows:

- $\forall A \in \mathscr{C}$, $F(A) \in \mathscr{D}$.
- If $f : A \to B$ in $\mathscr{C}$, then $F(f) : F(A) \to F(B)$ is an arrow in $\mathscr{D}$ such that:
  - $F(id_A) = id_{F(A)}$ for every object $A \in \mathscr{C}$.
  - $F(g \circ f) = F(g) \circ F(f)$ for all arrows $f : X \to Y, g : Y \to Z$ in $\mathscr{C}$.

Note: All functors preserve isomorphisms.

Godement Calculus Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, $E, F_1, F_2, F_3, G_1, G_2, G_3, H$ be functors and $\alpha, \beta, \gamma, \delta$ be natrual morphisms as shown in the following diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "E"] & \mathscr{B} \arrow[r, bend left=60, "F_1"] \arrow[r, dashed, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \mathscr{C} \arrow[r, bend left=60, "G_1"] \arrow[r, dashed, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta"] & \mathscr{D} \arrow[r, "H"] & \mathscr{E}
\end{tikzcd}$$

There are 5 laws/equational rules between composition of functors and natural transformations:

1. $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma \circ \alpha)$

2. $(H \circ G_1)\alpha = H(G_1\alpha)$

3. $\gamma(F_1 \circ E) = (\gamma F_1)E$

4. $G_1(\beta \circ \alpha)E = (G_1\beta E) \circ (G_1\alpha E)$

5. $\gamma * \alpha = (\gamma F_2) \circ (G_1\alpha) = (G_2\alpha) \circ (\gamma F_1)$

Note: The first rule is also known as the *Interchange Law*.

Grothendieck Construction For any category valued functor, $F : \mathscr{C} \to \mathbf{Cat}$, the **Grothendieck Construction** constructs a split opfibration from $\mathbf{G}(\mathscr{C}, F)$ to $\mathscr{C}$ which is induced by F. The category $\mathbf{G}(\mathscr{C}, F)$ is defined as:

- Objects are pairs (x, C), where C is an object in $\mathscr{C}$ and x is an object in $F(C)$.
- Arrows are pairs $(u, f) : (x, C) \to (x', C')$ where $f : C \to C'$ is an arrow in $\mathscr{C}$ and $u : F(f)(x) \to x'$ is an arrow in $F(C')$.
- Composition of $(u, f) : (x, C) \to (x', C')$ and $(v, g) : (x', C') \to (x'', C'')$ is $(v, g) : (x, C) \to (x'', C'')$ defined by $(v, g) \circ (u, f) = (v \circ F(g)(u), g \circ f)$

The corresponding opfibration is the second projection functor of the pairs and has a splitting $\kappa(f, X) = (id_{F(f)(x)}, f) : (x, C) \to (F(f)(x), C')$ where $f : C \to C'$ is an arrow in $\mathscr{C}$ and (x, C) is an object in $\mathbf{G}(\mathscr{C}, F)$.

Note: On set-valued functors $F : \mathscr{C} \to \mathbf{Set}$, the Grothendieck construction constructs a *split discrete opfibration* where the fibers are discrete categories (sets can be viewed as discrete categories where elements are objects and the only arrows are the identity arrows).

Note: The monoid homomorphism for a semidirect product of M and T is $\phi : M \to \mathbf{End}(T)$, which is also a category valued functor. By the Grothendieck construction, the category $\mathbf{G}(M, \phi)$ is exactly the semidirect product of M and T.

Grothendieck Functor is a functor $\mathbf{G} : \mathbf{Func}(\mathscr{C}, \mathbf{Cat}) \to \mathbf{SO}(\mathscr{C})$ defined as:

On <u>objects</u>, $\mathbf{G}(F : \mathscr{C} \to \mathbf{Cat}) = \mathbf{G}(\mathscr{C}, F)$.

On <u>arrows</u>, $\mathbf{G}(\alpha : F \to G : \mathscr{C} \to \mathbf{Cat}) : \mathbf{G}(F) \to \mathbf{G}(G)$ is a functor. Let $\mathbf{G}\alpha = \mathbf{G}(\alpha)$, on objects, $\mathbf{G}\alpha(x, C) = (\alpha C(x), C)$. On arrows, $\mathbf{G}\alpha((u, f) : (x, C) \to (x', C')) = (\alpha C'(u), f)$.

Note: $\alpha C : F(C) \to G(C)$ is a natural transformation indexed at C but is also functor because $F(C)$ and $G(C)$ are categories.

Groupoid is a category in which every arrow is an *isomorphism*.

Hom Function Let S be an object and $f : T \to V$ an arrow in a category, a *covariant* **hom function** $\mathrm{Hom}(S, f) : \mathrm{Hom}(S, T) \to \mathrm{Hom}(S, V)$, if defined by:

$$\mathrm{Hom}(S, f)(g) = f \circ g$$

for some $g \in \mathrm{Hom}(S, T)$.
Similarly, a *contravariant* **hom function** $\mathrm{Hom}(f, S) : \mathrm{Hom}(V, S) \to \mathrm{Hom}(T, S)$, if defined by:

$$\mathrm{Hom}(f,S)(g) = g \circ f$$

for some $g \in \mathrm{Hom}(V,S)$.
Hom Functor is a functor that takes a category to a hom-set. For any category $\mathscr{C}$ with an object C, there are three different hom functors:

Covariant Hom Functor $\mathrm{Hom}(C,-) : \mathscr{C} \to \mathbf{Set}$

* $\mathrm{Hom}(C,-)(A) = \mathrm{Hom}(C,A)$ for each object $A \in \mathscr{C}$.
* $\mathrm{Hom}(C,-)(f) = \mathrm{Hom}(C,f) : \mathrm{Hom}(C,A) \to \mathrm{Hom}(C,B)$ for an arrow $f : A \to B$ in $\mathscr{C}$.

Contravariant Hom Functor $\mathrm{Hom}(-,C) : \mathscr{C}^{op} \to \mathbf{Set}$

* $\mathrm{Hom}(-,C)(A) = \mathrm{Hom}(A,C)$ for each object $A \in \mathscr{C}$.
* $\mathrm{Hom}(-,C)(f) = \mathrm{Hom}(f,C) : \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$ for an arrow $f : A \to B$ in $\mathscr{C}$.

Two-Variable Hom Functor $\mathrm{Hom}(-,-) : \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set}$

* $\mathrm{Hom}(-,-)((A,B)) = \mathrm{Hom}(A,B)$ for each object $(A,B) \in \mathscr{C}^{op} \times \mathscr{C}$.
* $\mathrm{Hom}(-,-)((f,g)) = \mathrm{Hom}(f,g) : \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,D)$ for an arrow $(f : A \to B, g : C \to D)$ in $\mathscr{C}^{op} \times \mathscr{C}$.

Note: All covariant hom functors preserves products.

Homomorphism of Split Opfibrations For split opfibrations $P : \mathscr{E} \to \mathscr{C}$ and $P : \mathscr{E}' \to \mathscr{C}$ with splitting κ and κ' respectively, a **homomorphism of split opfibrations** $\zeta : \mathscr{E} \to \mathscr{E}'$ is a functor such that for any arrow $f : C \to D$ in $\mathscr{C}$ and object X in $\mathscr{E}$ where $P(X) = C$, $\zeta(\kappa(f,X)) = \kappa'(f,\zeta(X))$ and the following diagram commutes:

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\zeta"] \arrow[dr, "P"'] & & \mathscr{E}' \arrow[dl, "P'"] \\ & \mathscr{C} & \end{tikzcd}$$

Hom-Set A **hom-set** $\mathrm{Hom}(S,T)$ is the set of all morphisms from S to T for any objects S and T in a category.
Horizontal Composition Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ be categories, and $F, F' : \mathscr{A} \to \mathscr{B}, G, G' : \mathscr{B} \to \mathscr{C}$ be functors. **Horizontal composition** between natural transformation transformations $\alpha : F \to F'$ and $\beta : G \to G'$ is $\beta * \alpha : G \circ F \to G' \circ F'$.
Note: The horizontal composite of natural transformations is also a natural transformation.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} & * & \mathscr{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} \arrow[r, mapsto] & \mathscr{A} \arrow[r, bend left, "G\circ F"] \arrow[r, bend right, "G'\circ F'"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathscr{C} \end{tikzcd}$$

Idempotent An arrow $f : A \to A$ in a category is **idempotent** if $f \circ f = f$.
Indiscrete A category in which there is exactly one arrow between any two objects is called **indiscrete**.
Infinite Product is a product of infinitely many objects. Let I be a set and $\{A_i | i \in I\}$ be an indexed set of objects in the category $\mathscr{C}$. An **infinite product** $\Pi_{i=1} A_i$ of an indexed set is an object P with an indexed set of arrows $p_i : P \to A_i$ such that for any object $A \in \mathscr{C}$ with arrows $q_i : A \to A_i$, there is an unique arrow $q = \langle q_i \rangle : A \to P$ and $p_i \circ q = q_i$ for all $i \in I$.

Initial Object (0) is the dual of *terminal object* in which the **initial object** has an unique arrow to each object (including itself). This object is usually denoted 0 and the unique arrow $! : 0 \to A$ for every object A.
Note: Any two initial objects in a category are isomorphic.
Example: The empty set $\emptyset$ in **Set** is the initial object.
Internal Hom Functors In any Cartesian Closed Category $\mathscr{C}$, for an object A, there are two **internal hom functors** $F : \mathscr{C} \to \mathscr{C}$ and $G : \mathscr{C}^{op} \to \mathscr{C}$ defined as:

- For any object B, $F(B) = B^A$ and $G(B) = A^B$
- For an arrow $f : B \to C$, $F(f) = F(B) \to F(C)$ and $G(f) = G(C) \to G(B)$

Intersection If C is an object of $\mathscr{C}$ with subobjects C_1 and C_2, the **intersection** of C_1 and C_2, if it exists, is the pullback:

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C_1 \arrow[d] \\ C_2 \arrow[r] & C \end{tikzcd}$$

Note: A category has *wide intersections* if every object C and every class of subobjects $\{C_i\}$ of C, there is an intersection.
Note: If exists, the intersection is the meet of the poset of subobjects of C.

Inverse For arrows $f : A \to B$ and $g : B \to A$, g is an **inverse** of f if $g \circ f = id_A$ and $f \circ g = id_B$.
Note: If only $f \circ g = id_B$ is satisfied, then f is a **left inverse** of g.
Similarly, if only $g \circ f = id_A$ is satisfied, then f is a **right inverse** of g.
Invertible is an element of a monoid in which it is an *isomorphism* in the category of the monoid (ie: the element has an inverse).
Isomorphism ($\cong$) is an arrow with an *inverse*. Two objects A and B are *isomorphic* if an isomorphism exists between them, thus $A \cong B$.
Note: It follows that all identity arrows are isomorphisms.
Note: A function in **Set**, homomorphism in **Grf** and **Mon** is an isomorphism if and only if it is *bijective*.

Kernel Pair A **kernel pair** of an arrow $f : X \to Y$ is a parallel pair of arrows d, e from any object R to X that forms a limit:

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "e"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

This is also the fiber product $X \times_Y X$, or the pullback of f along itself.
Note: Dually, a *cokernel pair* of f is the pushout of f along itself.

Limit ($\varprojlim$) A **limit** is an universal commutative cone $\alpha : \Delta_U \to D$ such that for every other cone over the same diagram $D : \mathbb{I} \to \mathscr{C}$, there exists an unique arrow to it. Thus:

- For every arrow $s : a \to b \in \mathbb{I}$, $D(s) \circ \alpha_a = \alpha_b$.
- If $\alpha' : \Delta'_U \to D$ is another commutative cone, there exists an unique arrow $f : U' \to U$ such that for each object $a \in \mathbb{I}$, $\alpha_a \circ f = \alpha'_a$.

Note: A limit can also be defined as a terminal object in the category $cone(D)$.

Note: A limit is also an universal element of the functor $cone(-,D) : \mathscr{C}^{op} \to \mathbf{Set}$, where $cone(U,D)$ is the set of commutative cones with vertex U over the diagram D, and $cone(f : U' \to U, D) : cone(U,D) \to cone(U',D)$ defined as $cone(f,D)(\alpha_a : \Delta_{Ua} \to D_a) = \alpha_a \circ f$ where a is some object in the domain of D. A universal element of such a functor is precisely a cone with which there is an unique arrow to it from any other cone with the same base.

Locally Cartesian Closed Category is a category $\mathscr{C}$ if for every object $A \in \mathscr{C}$, the slice category $\mathscr{C}/A$ is cartesian closed. In other words, $\mathscr{C}$ has pullbacks and for each arrow $f : A \to B$, the pullback functor $f^* : \mathscr{C}/B \to \mathscr{C}/A$ has a right adjoint.

Note: Since $\mathscr{C}/1$ is isomorphic to $\mathscr{C}$, a locally cartesian closed category which has a terminal object 1 is cartesian closed.

Locally Small Category is a category $\mathscr{C}$ with the property that $\mathrm{Hom}(A,B)$ is a set for all objects $A, B \in \mathscr{C}$.

Monad is a triple (M, η, μ) in a category $\mathscr{C}$, where $M : \mathscr{C} \to \mathscr{C}$ is an endofunctor, $\eta : id \to M$ and $\mu : M^2 \to M$ are natural transformations such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\eta M"] \arrow[dr, "id"'] & M^2 \arrow[d, "\mu"] & M \arrow[l, "M\eta"'] \arrow[dl, "id"] \\ & M & \end{tikzcd}$$

$$\begin{tikzcd} M^3 \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd}$$

η is called *unit* and μ is *multiplication*, the diagrams depict unitary identities and associativity of multiplication.

Note: Dually, a *comonad* $(M', \varepsilon, \delta)$ in a category $\mathscr{C}$ is a monad in $\mathscr{C}^{op}$. Thus, M' is an endofunctor in $\mathscr{C}$ and $\varepsilon : M' \to id$, $\delta : M' \to M'^2$ are natural transformations where the corresponding dual diagrams commute.

Note: Given functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ where $F \dashv G$, $(GF, \eta, G\varepsilon F)$ is a monad in $\mathscr{A}$ where η and ε are the unit and counit of the adjunction. Dually, $(FG, \varepsilon, F\eta G)$ is a comonad.

Monoidal Category is a category $\mathscr{C}$ equipped with an object $\top$ and a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ with natural isomorphisms:

- $a(A,B,C) : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$
- $lA : \top \otimes A \to A$
- $rA : A \otimes \top \to A$

such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes (\top \otimes B) \arrow[rr, "a(A,\top,B)"] \arrow[dr, "A\otimes l(B)"'] & & (A \otimes \top) \otimes B \arrow[dl, "r(A)\otimes B"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes (B \otimes (C \otimes D)) \arrow[dl, "A\otimes a(B,C,D)"'] \arrow[dr, "a(A,B,C\otimes D)"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "a(A,B\otimes C,D)"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "a(A\otimes B,C,D)"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a(A,B,C)\otimes D"'] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

illustrating the unit $\top$ with respect to $\otimes$ and the associativity of $\otimes$.

Note: A monoidal category is *symmetric* if it has an additional natural isomorphism $s(A,B) : A \otimes B \to B \otimes A$ such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes B \arrow[rr, "s(A,B)"] \arrow[dr, "id"'] & & B \otimes A \arrow[dl, "s(B,A)"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes \top \arrow[rr, "s(A,\top)"] \arrow[dr, "r(A)"'] & & \top \otimes A \arrow[dl, "l(A)"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "a(A,B,C)"] \arrow[d, "A\otimes s(B,C)"'] & (A \otimes B) \otimes C \arrow[d, "s(A\otimes B,C)"] \\
A \otimes (C \otimes B) \arrow[d, "a(A,C,B)"'] & C \otimes (A \otimes B) \arrow[d, "a(C,A,B)"] \\
(A \otimes C) \otimes B \arrow[r, "s(A,C)\otimes B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

Note: $\otimes$ is often known as the tensor product and $A \otimes - : \mathscr{C} \to \mathscr{C}$ forms an endofunctor.

Note: A monoidal category $\mathscr{C}$ is *closed* if for each object A, the functor $A \otimes -$ has a right adjoint $A \multimap - : \mathscr{C} \to \mathscr{C}$ where for any object C in $\mathscr{C}$, $A \multimap C$ is the internal hom from A to C.

Thus by the definition of adjoints,

$$\mathrm{Hom}(A \otimes B, C) \xrightarrow{\cong} \mathrm{Hom}(B, A \multimap C)$$

is a natural isomorphism that is natural in B and C. It follows that in a symmetric monoidal closed category,

$$\mathrm{Hom}(A \otimes B, C) \xrightarrow{\cong} \mathrm{Hom}(A, B \multimap C)$$

is also a natural isomorphism and

$$(A \otimes B) \multimap C \cong A \multimap (B \multimap C)$$

The counit of this adjunction is the evaluation map of the internal hom object, $e : A \otimes (A \multimap B) \to B$.

Monomorphism ($\rightarrowtail$) is an arrow $f : A \to B$ such that for any object T and arrows $x, y : T \to A$, if $x \neq y$, then $f \circ x \neq f \circ y$. Monomorphisms can be denoted by $f : A \rightarrowtail B$.

Note: In this context, $x, y : T \to A$ may be regarded as *variable elements* and can be written as if $x \neq y$, then $f(x) \neq f(y)$.

Properties: Let $f : A \to B$ and $g : B \to C$ be arrows:

- If f and g are monomorphisms, so is $g \circ f$.
- If $g \circ f$ is a monomorphism, so is f.

Note: In **Set**, a function is injective if and only if it is a monomorphism.

Note: In the category determined by a Monoid, a monic element is left cancellable.

Morphism/Arrow is a directed connective between two objects (*source* and *target* objects) in a category, synonymous with *map* or a *directed edge* in a *directed graph*. A morphism f with source x and target y objects is denoted as:

$$f : x \to y$$

n-Ary Product is the product of a list of n objects $A_1, \ldots, A_n$ with projectiions $p_i : \Pi_{i=1}^{n} A_i \to A_i$ and the property that given any object B and arrows $f_i : B \to A_i$, there is an unique arrow $\langle f_1, \ldots, f_n \rangle : B \to \Pi_{i=1}^{n} A_i$ for which $p_i \circ \langle f_1, \ldots, f_n \rangle = f_i$.

Let the product of n number of the object A, $A \times \cdots \times A$ be denoted as A_n:

Nullary Product denoted as A_0, is the terminal object.

Unary Product denoted as A_1, is effectively a product "wrapper" around A, where the only projection arrow is id_A. (Analogous to characters vs strings of length 1).

Note: *Binary products* (2-ary product) can be used to construct *Ternary products* (3-ary products) as

$$A \times B \times C \cong (A \times B) \times C \cong A \times (B \times C)$$

Natural Isomorphism is a natural transformation $\alpha : F \to G$ in which there is a natural transformation $\beta : G \to F$ is an inverse to α in $\mathbf{Func}(\mathscr{C}, \mathscr{D})$.

Note: Natural isomorphisms are also known as *natural equivalences*.

Theorem: Suppose $F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C}$ are functors and $\alpha : F \to G$ is a natural transformation between them. Then α is a *natural isomorphism* if and only if for each object $C \in \mathscr{C}$, $\alpha C : F(C) \to G(C)$ is an isomorphism in $\mathscr{D}$.

Natural Number Object (NNO) An object $\mathbf{N}$ in a category $\mathscr{C}$ together with arrows $zero : 1 \to \mathbf{N}$ and $succ : \mathbf{N} \to \mathbf{N}$ is a parametrized **natural numbers object** if for all objects A, B and arrows $f_0 : A \to X, t : X \to X$, there is an unique arrow $f : A \times \mathbf{N} \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
 & A \times \mathbf{N} \arrow[r, "id_A \times succ"] \arrow[d, "f"] & A \times \mathbf{N} \arrow[d, "f"] \\
A \arrow[ur, "\langle id_A, zero \rangle"] \arrow[dr, "f_0"'] & & \\
 & X \arrow[r, "t"'] & X
\end{tikzcd}$$

Natural Transformation Let $F, G : \mathscr{C} \to \mathscr{D}$ be 2 functors with the same domain and codomain. A **natural transformation** $\alpha : F \to G$ is given by a family of arrows αC for each object $C \in \mathscr{C}$ such that:

- $\alpha C : F(C) \to G(C)$ for each object $C \in \mathscr{C}$, also say that α is natural in C.
- For any arrow $f : C \to D$ in $\mathscr{C}$, the following diagram commutes.

$$\begin{tikzcd}
F(C) \arrow[r, "\alpha C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\
F(D) \arrow[r, "\alpha D"'] & G(D)
\end{tikzcd}$$

Composition with Functors: Let $H : \mathscr{B} \to \mathscr{C}$ be a functor and $\alpha : F \to G$ where $F, G : \mathscr{A} \to \mathscr{B}$ be are functors. Then, postcomposing α with the functor H is a natural transformation $H\alpha : H \circ F \to H \circ G$ defined by $(H\alpha)A = H(\alpha A)$ for any object $A \in \mathscr{A}$.
A similar result can be shown when precomposing a natural transformation with a functor.

Order-Enriched Category is a poset-enriched category in which each poset $\mathrm{Hom}(A, B)$ is a strict ω-CPO where every pair of elements has a supremum (least upper bound) and an infimum (greatest lower bound).

Parallel Morphisms Morphisms which have the same *source* and *target* are **parallel**.
Note: The limit of a *parallel pair* of arrows is their equalizer. Dually, the colimit of a parallel pair of arrows is their coequalizer.
Partial Morphism A **partial morphism** from objects A to B is a subobject $A_0 \subseteq A$ and an arrow from $f : A_0 \to B$. If $g : A_1 \to B$ is another partial morphism, $f \leq g$ if $A_0 \subseteq A_1$ and f is the restriction to A_0 of g, ie: $f = g \circ i$ where $i : A_0 \to A_1$ is the inclusion.
Note: If both $f \leq g$ and $g \leq f$, then f and g have the same subobject as their domain.
Note: Partial arrows from A to B are *representable* if $\widetilde{B}$ is an object where $B \rightarrowtail \widetilde{B}$ and $\mathrm{Hom}(A, \widetilde{B})$ is isomorphic to the partial arrows from A to B. This one to one correspondence is given by the pullback:

$$\begin{tikzcd}
A_0 \arrow[r, tail] \arrow[d, "f"'] & A \arrow[d, "g"] \\
B \arrow[r, tail] & \widetilde{B}
\end{tikzcd}$$

where there is a one to one correspondence between $f \in \text{Hom}(A, \widetilde{B})$ and g, which is an arrow from A to B.

Partially Ordered Object D is a **partially ordered object** in a cartesian closed category if for every $\text{Hom}(A, D)$ where A is any object, there is a partial order relation such that for any $f : B \to A$ and $g \leq h$ in $\text{Hom}(A, D)$, $g \circ f \leq h \circ f$ in $\text{Hom}(B, D)$.

Note: D is an ω-CPO object if for every $\text{Hom}(A, D)$ for any object A, $\text{Hom}(A, D)$ is an ω-CPO (ie: $\text{Hom}(A, D)$ is ω-CPO).

Note: D is *strict* if there is an arrow $\perp : 1 \to D$ such that for any object A and any $f \in \text{Hom}(A, D)$, $\perp \circ \langle\rangle \leq f$.

Note: If D is a strict ω-CPO and $f : D \to D$ is an ω-continuous arrow, then $fix(f) : 1 \to D$ is the least element with the property $f \circ fix(f) = fix(f)$.

Pointwise Products For categories $\mathscr{C}, \mathscr{D}$, if $\mathscr{D}$ has products then $\textbf{Func}(\mathscr{C}, \mathscr{D})$ has **pointwise products** defined as:

- $(F \times G)(C) = F(C) \times G(C)$ for any object $C \in \mathscr{C}$
- $(F \times G)(f) = F(f) \times G(f)$ for any arrow $f \in \mathscr{C}$

where $F, G : \mathscr{C} \to \mathscr{D}$ are functors.

Polynomial Endofunctors on a category with finite products and coproducts is the least class that contains the constant endofunctors, the identity functor, and is closed under the operations of finite products and coproducts.

Poset-Enriched Category is a category $\mathscr{C}$ with a partial ordering on every $\text{Hom}_{\mathscr{C}}(A, B)$ such that for every triple of objects $A, B, C \in \mathscr{C}$, the composition of hom-sets:

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \longrightarrow \text{Hom}(A, C)$$

is monotone. Thus if $f \leq f' : A \to B, g \leq g' : B \to C$ then $g \circ f \leq g' \circ f' : A \to C$.

As a 2-category: Exactly ONE 2-cell can be defined from f to g for arrows $f, g : A \to B$ if and only if $f \leq g$. Otherwise, no 2-cells from f to g. For each pair of objects A and B, the category $C(\text{Hom}(A, B))$ can be constructed as a poset.

Powerset Functor is a functor that takes a set S to its powerset $\mathscr{P}(S)$. There are three different powerset functors:

Inverse Image $\mathscr{P} : \textbf{Set}^{op} \to \textbf{Set}$
is a contravariant functor that takes sets S to its powerset $\mathscr{P}(S)$ and functions $f : A \to B$ to its inverse image $f : \mathscr{P}(B) \to \mathscr{P}(A)$.

Direct/Existential Image is a covariant functor that takes sets S to its powerset $\mathscr{P}(S)$ and functions $f : A \to B$ to the function $f_* : \mathscr{P}(A) \to \mathscr{P}(B)$ where $f_*(A_0) = \{f(x) | x \in A_0\}$.

Universal Image is a covariant functor that takes sets S to its powerset $\mathscr{P}(S)$ and functions $f : A \to B$ to the function $f_! : \mathscr{P}(A) \to \mathscr{P}(B)$ where $f_!(A_0) = \{y \in B | (f(x) = y) \Rightarrow (x \in A_0)\}$.

Preserve An arrow $f : A \to B$ **preserves** a property P if whenever A has P, then so does B.

Note: A functor $F : \mathscr{C} \to \mathscr{D}$ preserves P of arrows if whenever $f \in \mathscr{C}_1$ has P, so does $F(f) \in \mathscr{D}_1$.

Note: A property is preserved by isomorphisms if for any object A with the property, any object isomorphic to A must also have the property.

Presheaf is a contravariant set-valued functor $E : \mathscr{C}^{op} \to \textbf{Set}$ on a category $\mathscr{C}$.

Note: A presheaf $E : \mathscr{C}^{op} \to \mathbf{Set}$ is *constant* if E is a constant functor. Thus, for any object x in $\mathscr{C}^{op}$, $E(x)$ is the same set and for any arrow f in $\mathscr{C}^{op}$, $E(f) = id_{E(x)}$.

Product ($\times$) Let A, B be objects in a category $\mathscr{C}$. A **product** of A and B is an object $A \times B$ together with projections/arrows $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ such that for any object D and arrows $q_1 : D \to A$, $q_2 : D \to B$, there is an unique arrow $q : D \to A \times B$ and that the following diagram commutes:

$$\begin{tikzcd} & D \arrow[dl, "q_1"'] \arrow[d, "q"] \arrow[dr, "q_2"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

ie, $p_1 \circ q = q_1$ and $p_2 \circ q = q_2$.

The concept of a product is defined up to an unique isomorphism and is defined by a *universal mapping property*. Thus, all products of the objects A and B are isomorphic and any object isomorphic to $A \times B$ is a product of A and B.

Since for each pair of arrows $(q1, q2)$ produces an unique arrow q, there is a natural isomorphism:

$$\pi : \mathrm{Hom}_{\mathscr{C}}(-, A) \times \mathrm{Hom}_{\mathscr{C}}(-, B) \to \mathrm{Hom}_{\mathscr{C}}(-, A \times B)$$

such that $\pi D(q_1, q_2) = q$ and is denoted as $\langle q_1, q_2 \rangle$.

Note: $(p1, p2)$ is an *universal element* of the functor $\mathrm{Hom}(-, A) \times \mathrm{Hom}(-, B)$.

Product of Arrows For arrows $f : S \to S', t : T \to T'$, the **product of arrows**, $f \times g$ is defined such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[d, "f"'] & S \times T \arrow[l, "p1"'] \arrow[r, "p2"] \arrow[d, "f \times g"] & T \arrow[d, "g"] \\ S' & S' \times T' \arrow[l, "p1'"] \arrow[r, "p2'"'] & T' \end{tikzcd}$$

where $p1, p2$ and $p1', p2'$ are the projections of $S \times T$ to S, T and $S' \times T'$ to S', T' respectively.

Note: Products are distributed through composition, for arrows $f_i : A_i \to B_i, g_i : B_i \to C_i$ for $i = 1, 2$:

$$(g_1 \circ f_1) \times (g_2 \circ f_2) = (g_1 \times g_2) \circ (f_1 \times f_2)$$

Product of Categories If $\mathscr{C}$ and $\mathscr{D}$ are categories their **product** $\mathscr{C} \times \mathscr{D}$ is the category whose objects are all ordered pairs (C, D) where $C \in \mathscr{C}_0, D \in \mathscr{D}_0$ and arrows are $(f, g) : (C, D) \to (C', D')$ where $f : C \to C' \in \mathscr{C}_1, g : D \to D' \in \mathscr{D}_1$. The identity arrow of (C, D) is (id_C, id_D) and composition of arrows is defined component wise.

Pullback For the diagram:

$$B \xrightarrow{g} C \xleftarrow{f} A$$

a **pullback** is its limit:

$$\begin{array}{ccc} P & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

This is known as the pullback diagram and the composite arrow $f \circ p_1 = g \circ p_2 : P \to C$ is omitted by convention.

Other Names: P together with p_1 and p_2 is a pullback or *fiber product* of f and g. p_2 is the pullback of f along g, p_1 is the pullback of g along f. P is sometimes be denoted as $A \times_C B$.

Note: In **Set**, $P = \{(a,b) | \forall a \in A, \forall b \in B, f(a) = g(b)\}$. p_1 and p_2 are just the coordinate projections of P.

Pushout is the dual of pullback, such that

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle q_1} \\ B & \xrightarrow[q_2]{} & Q \end{array}$$

is a commutative square for any object R and arrows $r_1 : A \to R, r_2 : B \to R, r_1 \circ f = r_2 \circ g$, there is an unique arrow $r : Q \to R$ where $r \circ q_1 = r_1$ and $r \circ q_2 = r_2$.

Quotient Category $(/ \sim)$ For a congruence relation $\sim$ on the arrows of a category $\mathscr{C}$, the **quotient category** $\mathscr{C}/\sim$ is defined as follows:

- The objects of $\mathscr{C}/\sim$ are the objects of $\mathscr{C}$.
- The arrows of $\mathscr{C}/\sim$ are the congruence classes of arrows of $\mathscr{C}$.
- If $f : A \to B$ in $\mathscr{C}$, then $[f] : A \to B$ in $\mathscr{C}/\sim$.
- If $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, then $[g] \circ [f] = [g \circ f] : A \to C$ in $\mathscr{C}$.

Note: For any functor $F : \mathscr{C} \to \mathscr{D}$ with the property that if $f \sim g$ then $F(f) = F(g)$, then there is an unique functor $F_0 : \mathscr{C}/\sim \to \mathscr{D}$ for which $F_0 \circ Q = F$. Here, Q is the quotient functor, $Q : \mathscr{C} \to \mathscr{C}/\sim$.

Recursive Category is a category $\mathscr{C}$ if for every object A, the underlying functor $U_A : act(A) \to \mathscr{C}$ has a left adjoint $F_A : \mathscr{C} \to act(A)$.

Note: $U_A : act(A) \to \mathscr{C}$ is defined as $U_A(x : A \times X \to X) = X$ and $U_A(f_A : (x : A \times X \to X) \to y : (A \times Y \to Y)) = f : X \to Y$.

Note: For an object B and arrow $f : B \to C$, let $rec(A,B) = U_A(F_A(B))$ and $rec(A, f) = U_A(F_A(f)) : rec(A,B)) \to rec(A,C)$. F_A is defined as $F_A(B) = A \times rec(A,B) \to rec(A,B)$.

Note: If for all A and B, $\langle rec(A, \langle\rangle), a\rangle : rec(A,B) \to rec(A,1) \times B$ is an ismorphism, then $\mathscr{C}$ has *local recursion*. A recursive category with local recursion for which every slice category $\mathscr{C}/C$ is recursive is known as a *locally recursive category*.

Note: A locally recursive category is a *locos* if it is coherent.

Reflect A functor $F : \mathscr{C} \to \mathscr{D}$ **reflects** a property P of arrows if whenever $F(f)$ has property P then so does f (where f is any arrow for which F takes to $F(f)$).

Theorem: Consider a full and faithful functor $F : \mathscr{C} \to \mathscr{D}$. Let A and B be objects in C and $u : F(A) \to F(B)$ is an isomorphism in $\mathscr{D}$, then there is an unique isomorphism $f : A \to B$ in $\mathscr{C}$ for which $F(f) = u$.

Corollary: A full and faithful functor reflects isomorphisms.

Corollary: For a full an faithful functor F, if $F(A) = F(B)$ for objects A and B in the domain of F, then $A \cong B$.

Regular Category is a finitely complete category which has finite limits, all arrows are coequalizers of its kernel pair and the pullback of any regular epimorphism along any other arrow is an regular epimorphism.

Note: A functor between regular categories is called a *regular functor* if it preserves finite limits and regular epimorphisms.

Regular Epimorphism is an epimorphism that is also a coequalizer of a parallel pair of arrows.

Regular Monomorphism is a monomorphism that is also an equalizer of a parallel pair of arrows.

Note: An arrow that is both a regular monomorphism and an epimorphism is an isomorphism.

Representable Functor A *set-valued functor* is representable if it is naturally isomorphic to a *hom functor*.

Note: If a covariant functor $F : \mathscr{C} \to \mathbf{Set}$ is isomorphic to $\mathrm{Hom}(C, -)$ for some object $C \in \mathscr{C}$, we say that C represents F. The same can be said for contravariant functors and contravariant hom functors.

Representative Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ in which every object of $\mathscr{C}$ is isomorphic to some object in $\mathscr{D}$.

Retract is an object A for an object B when there are arrows $i : A \to B$ and $r : B \to A$ such that $r \circ i = id_A$. r is also called a **retraction** of B onto A and i is a section of r.

Note: Retraction is a split epimorphisms.

Right Regular Representation The **right regular representation** for a category $\mathscr{C}$ is the functor $R_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$ where objects $R_{\mathscr{C}}(C)$ for any object C in $\mathscr{C}$ is the set of arrows in $\mathscr{C}$ with codomain is C. For any arrow $f : C \to C'$ in $\mathscr{C}$, $R_{\mathscr{C}}(f)(g) = f \circ g$ for any $g \in R_{\mathscr{C}}(C)$.

Note: The *standard wreath product* for $\mathscr{A}$ and $\mathscr{B}$ is $\mathscr{A}\ \mathrm{wr}^{R_{\mathscr{A}}}\ \mathscr{B}$.

Section is an arrow that is also a right-inverse, ie: a split monomorphism.

Note: The left inverse of a section is also called a retraction or *cosection*.

Set-Valued Functor is any functor from any category $\mathscr{C}$ to **Set**.

Note: For small categories $\mathscr{C}$, a set valued functor $F : \mathscr{C} \to \mathbf{Set}$ generalizes the concept of monoid actions as functors.

Shape Functor Let $\mathscr{A}, \mathscr{B}$ be categories, $G : \mathscr{A} \to \mathbf{Cat}$ be a functor. $S(G, \mathscr{B}) : \mathscr{A}^{op} \to \mathbf{Cat}$ is a **shape functor** such that:

On objects, $\forall A \in \mathscr{A}$, $S(G, \mathscr{B})(A) = \mathbf{Func}(G(A), \mathscr{B})$.

On arrows, for any arrow $f : A \to A'$ in $\mathscr{A}$, $S(G, \mathscr{B})(f) : \mathbf{Func}(G(A'), \mathscr{B}) \to \mathbf{Func}(G(A), \mathscr{B})$ is a functor. On objects/functors, $S(G, \mathscr{B})(f)(H : G(A') \to \mathscr{B}) = H \circ G(f)$. On arrows/natural transformations, $S(G, \mathscr{B})(f)(\alpha : H \to H' : G(A') \to \mathscr{B}) = \alpha G(f) : H \circ G(f) \to H' \circ G(f)$. The component at an object $X \in G(A)$ for

$S(G,\mathscr{B})(f)(\alpha)$ is the component of α for $G(f)(X)$, ie: $\alpha G(f)X : H[G(f)(X)] \to H'[G(f)(X)]$.

Note: $S(G,\mathscr{B})(A)$ is the category of diagrams (diagram functors) of shape $G(A)$ in $\mathscr{B}$, where arrows are diagram homomorphisms (natural transformations between these diagram functors).

Sheaf is a presheaf E over a complete Heyting algebra such that:

$$E(\bigvee x_i) \xrightarrow{e} \prod_i E(x_i) \underset{d}{\overset{c}{\rightrightarrows}} \prod_{ij} E(x_i \wedge x_j)$$

the arrow e is an equalizer.

Note: Let $x = \bigvee x_i$ for all i. $e : E(x) \xrightarrow{e} \prod_i E(x_i)$ is induced by universal property of the categorical product where since forall $i, E(x \geq x_i) : E(x) \to E(x_i)$, there exists $\prod_i E(x_i)$ such that $E(x \geq x_i) = proj_i \circ e$. The arrows c and d are induced by the arrows $E(x_i \geq x_i \wedge x_j) : E(x_i) \to E(x_i \wedge x_j)$, $E(x_j \geq x_i \wedge x_j) : E(x_j) \to E(x_i \wedge x_j)$, $proj_i : \prod_i E(x_i) \to E(x_i)$, $proj_j : \prod_i E(x_i) \to E(x_i)$ and thus there exists $\prod_{ij} E(x_i \wedge x_j)$ such that $proj_{ij} \circ c = E(x_i \geq x_i \wedge x_j) \circ proj_i$ and $proj_{ij} \circ d = E(x_i \geq x_j \wedge x_j) \circ proj_j$.

Note: The category of sheaves on a Heyting algebra is a topos.

Slice Category $(/)$ If $\mathscr{C}$ is a category, for any object $A \in \mathscr{C}_0$, the **slice category** $\mathscr{C}/A$ is defined as follows:

- An object of $\mathscr{C}/A$ is an arrow $f : C \to A$ in $\mathscr{C}$ for some $C \in \mathscr{C}_0$.
- An arrow of $\mathscr{C}/A$ from $f : C \to A$ to $f' : C' \to A$ is an arrow $h : C \to C'$ such that $f = f' \circ h$.
- The composite of $h : f \circ f'$ and $h' : f' \circ f''$ is $h' \circ h$.

Note: Since the same h can satisfy both $f = f' \circ h$ and $g = g' \circ h$ where $f \neq g$ or $f' \neq g'$, $h : f \to f'$ and $h : g \to g'$ are regarded as different arrows in $\mathscr{C}/A$.

Note: The *indexed function* of an *indexed set* indexed by S, is precisely an arrow in the slice category **Set**$/S$.

Split Epimorphism is an arrow that has a *right inverse* (this arrow can be shown to be an *epimorphism*, by composing its right inverse to its right).

Split Monomorphism is an arrow that has a *left inverse* (this arrow can be shown to be a *monomorphism*, by composing its left inverse to its left).

Strict Initial Object Let $\mathscr{C}$ be a category with products. An initial product, 0, in $\mathscr{C}$ is a **strict initial object** if it has one of the following equivalent properties:

- $\forall A \in \mathscr{C}$, if there is an $u : A \to 0$, then $A \cong 0$.
- $\forall A \in \mathscr{C}, 0 \times A \cong 0$.

Subcategory A **subcategory** $\mathscr{D}$ of a category $\mathscr{C}$ is a category for which:

- All objects and arrows in $\mathscr{D}$ are objects and arrows in $\mathscr{C}$ (ie: $\mathscr{D}_0 \subseteq \mathscr{C}_0$ and $\mathscr{D}_1 \subseteq \mathscr{C}_1$).

- The source and target of an arrow in $\mathscr{D}$ are the same as its souce and target in $\mathscr{C}$ (ie: source and target maps in $\mathscr{D}$ are restrictions of those in $\mathscr{C}$). Thus, $\forall A, B \in \mathscr{C}_0$, $\mathrm{Hom}_{\mathscr{D}}(A,B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A,B)$.
- If A is an object in $\mathscr{C}$, its identity arrow id_A in $\mathscr{C}$ is in $\mathscr{D}$.
- If $f : A \to B$ and $g : B \to C$ in $\mathscr{D}$, then the composite $g \circ f \in \mathscr{C}_1$ is also the composite in $\mathscr{D}$.

Subfunctor A **subfunctor** of a functor $F : \mathscr{C} \to \mathbf{Set}$ is a functor $G : \mathscr{C} \to \mathbf{Set}$ with the property that for each object $C \in \mathscr{C}$, $G(C) \subseteq F(C)$ and for each arrow $f : C \to C'$ and element $x \in G(C)$, $G(f)(x) = F(f)(x)$.

Note: The inclusion function $i_C : G(C) \to F(C)$ that takes the the subfunctor G to the functor F is a natural transformation.

Subobject In a category $\mathscr{C}$, a **subobject** of an object $C \in \mathscr{C}$ is an equivalence class of monomorphisms under the relation $\sim$. The subobject is a **proper subobject** if it does not contain id_C. The equivalence relation $\sim$ is defined as follows:

- For monomorphisms $f_0 : C_0 \rightarrowtail C$ and $f_1 : C_1 \rightarrowtail C$, $f_0 \sim f_1$ if both f_0 and f_1 factors through each other.

Note: A preorder for subobjects $(i : A \to C) \leq (j : B \to C)$ is defined by the condition that there exists $k : A \to B$ such that $i = j \circ k$. Since j is monic, k is unique. And i is monic implies k is monic. In **Set**, $A \subseteq B \subseteq C$ where i is the inclusion of A to C, j the inclusion of B to C and k the inclusion of A to B.

Subobject Classifier is the object representing a subobject functor, usually denoted Ω where an universal element is a subobject $true : \Omega_0 \rightarrowtail \Omega$ and for any other subobject $A_0 \subseteq A$, there is an unique arrow $\chi : A \to \Omega$ such that the following is a pullback diagram:

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, tail] & \Omega_0 \arrow[d, tail, "true"] \\ A \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

It can be proven that Ω_0 is a terminal object and thus the arrow from A_0 to Ω_0 is the unique arrow by the definition of a terminal object.

Note: The arrow χ is called the *characteristic arrow* of the subobject. Since the subobject functor *Sub* is represented by Ω, there is a natural isomorphism from $Sub(-)$ to $\mathrm{Hom}(-, \Omega)$, which takes a subobject to its characteristic function.

Subobject Functor is a functor $Sub : \mathscr{C}^{op} \to \mathbf{Set}$ defined as:

On objects C in $\mathscr{C}$, $Sub(C)$ is the set of subobjects of C.

On arrows $k : C' \to C$, $Sub(k) : Sub(C) \to Sub(C')$ is a set function that takes a subobject $f_0 : C_0 \rightarrowtail C$ to f_0' in the following pullback diagram:

$$\begin{tikzcd} C_0' \arrow[r, "k_0"] \arrow[d, tail, "f_0'"'] & C_0 \arrow[d, tail, "f_0"] \\ C' \arrow[r, "k"'] & C \end{tikzcd}$$

Note: f'_0 is a monic due to being a pullback of a monomorphism.

Switch Map is a function that takes $A \times B$ with projections p_1, p_2 to $B \times A$ denoted as $\langle p_2, p_1 \rangle : A \times B \to B \times A$.

Terminal Object (1) is an object T in $\mathscr{C}$ if there is exactly ONE arrow $A \to T$ for each object $A \in \mathscr{C}$ (including itself). This object is usually denoted 1 and the unique arrow $\langle\rangle : A \to 1$.

Note: Any two terminal objects in a category are isomorphic.

Example: Any singleton set in **Set** is a terminal object.

Topos is a cartesian closed category which has finite limits and has a representable subobject functor.

Note: **Set** is a topos where the subobject classifier is a two-element set, $\Omega = \{True, False\}$. For any set $S_0 \subseteq S$, the characteristic function $\chi : S \to \Omega$ is defined as:

$$\chi(x) = \begin{cases} True & \text{if } x \in S_0 \\ False & \text{if } x \notin S_0 \end{cases}$$

Note: The exponential object Ω^A has the property that:

$$\mathrm{Hom}(B, \Omega^A) \cong \mathrm{Hom}(A \times B, \Omega) \cong Sub(A \times B)$$

In topos theory, Ω^A is often called the *power object* of A and is denoted $\mathscr{P}A$

Note: A topos $\mathscr{E}$ has finite colimits, disjoint sums, universal sums, every epimorphism is regular and $\mathscr{E}$ is a regular category.

Universal Element By the *Yoneda Lemma*, an element $c \in F(C)$ is given by $c = h^C(id_C)$. c is an **universal element** of a functor $F : \mathscr{C} \to$ **Set** if and only if the induced $h^C : Hom(C, -) \to F$ is a *natural isomorphism*. Consequently, for any object $C' \in \mathscr{C}$ and any element $x \in F(C')$, there is an unique arrow $f : C \to C'$ in $\mathscr{C}$ for which $x = F(f)(c)$.

Note: There is a one-to-one correspondance between representations and universal elements of F.

Note: A similar argument for contravariant set valued functors of the form $F : \mathscr{C}^{op} \to$ **Set** can be made where $c \in F(C)$ is an **universal element** if for any object $C' \in \mathscr{C}$ and any element $x \in F(C')$, there is an unique arrow $f : C' \to C$ for which $x = F(f)(c)$.

Vertical Composition ($\circ$) Composition of natural transformations $\alpha : F \to G$ and $\beta : G \to H$ is $\beta \circ \alpha : F \to H$ and is known as **vertical composition**, where F, G and H are functors from the categories $\mathscr{C}$ to $\mathscr{D}$. This follows from the fact that the outer rectangle of the following diagram is commutative:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\beta A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\ F(B) \arrow[r, "\alpha B"'] & G(B) \arrow[r, "\beta B"'] & H(B) \end{tikzcd}$$

for each arrow $f : A \to B$.

Note: The vertical composite of natural transformations is also a natural transformation.

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, "G"'] \arrow[r, phantom, bend left=25, "\Downarrow\alpha"] & \mathscr{D} \end{tikzcd} \circ \begin{tikzcd} \mathscr{C} \arrow[r, "G"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, bend right=25, "\Downarrow\beta"] & \mathscr{D} \end{tikzcd} \longmapsto \begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow \beta\circ\alpha"] & \mathscr{D} \end{tikzcd}$$

Wide Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ such that $\mathscr{D}_0 = \mathscr{C}_0$.
Example: **Set** is a wide subcategory of **Pfn**.
Wreath Product Let $\mathscr{A}$, $\mathscr{B}$ be categories and $G : \mathscr{A} \to$ **Cat** be a functor. The **wreath product** $\mathscr{A}\ \mathrm{wr}^G\ \mathscr{B}$ is category of the split opfibration from the Grothendieck construction $\mathbf{G}(\mathscr{A}, S(G,\mathscr{B}))$.
Note: If G is a set valued functor, then the construction produces a *discrete wreath product*.

Yoneda Embedding For any category $\mathscr{C}$, there is a contravariant *Yoneda Functor* $Y : \mathscr{C}^{op} \to \mathbf{Func}(C,\mathbf{Set})$ defined as:

- for objects $C \in \mathscr{C}, Y(C) = \mathrm{Hom}(C,-)$
- for an arrow $f : D \to C$ and an object $A \in \mathscr{C}, Y(f) = \mathrm{Hom}(f,-)$ and $Y(f)(A) : \mathrm{Hom}(C,A) \to \mathrm{Hom}(D,A)$

Thus, the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}(D,A) \arrow[r, "\mathrm{Hom}(D,k)"] & \mathrm{Hom}(D,B) \\ \mathrm{Hom}(C,A) \arrow[u, "Y(f)A"] \arrow[r, "\mathrm{Hom}(C,k)"'] & \mathrm{Hom}(C,B) \arrow[u, "Y(f)B"'] \end{tikzcd}$$

for an arrow $k : A \to B$.

Note: $Y(f) : \mathrm{Hom}(C,-) \to \mathrm{Hom}(D,-)$ is the induced natural transformation between hom functors corresponding to f. Generalizing $\mathrm{Hom}(D,-)$ to an arbitary set valued functor, by the *Yoneda Lemma*, Y is both full and faithful.
Note: By the contravariant *Yoneda Lemma*, there also exists the covariant functor $J : \mathscr{C} \to \mathbf{Func}(\mathscr{C}^{op},\mathbf{Set})$.

Yoneda Lemma Let $F : \mathscr{C} \to \mathbf{Set}$ be a *set-valued functor* and C is an object in C. There is a one-to-one correspondance between elements of $F(C)$ and $h^C : \mathrm{Hom}(C,-) \to F$, that is:

$$Nat(h^C,F) \cong F(C)$$

Thus, this is an isomorphism that is both an injective and surjective mapping between elements of $F(C)$ and h^C.
Note: $h^C(X) = \mathrm{Hom}(C,X) \to F(X)$ for an object $X \in \mathscr{C}$ is defined as $h^C(X)(f) = F(f)(c)$ where $f : C \to X$ is an arrow and $c \in F(C)$ is an element.
Note: There is also a contravariant version of Yoneda Lemma involving the contravariant hom functor $h_C : \mathrm{Hom}(-,C) \to F$.

Zero Object is an object in a category that is both an initial and a terminal object.
Note: A zero object is also as a *null object*.
Note: A category with a zero object is sometimes called a *pointed category*.
Note: In a category with a zero object 0, between any two objects A and B, there exists a canonical morphism given by the composition $A \to 0 \to B$.

References

[1] Charles Wells Michael Barr. *Category Theory for Computing Science*. Reprints in Theory and Applications of Categories #22. 2013.