Topology

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A glossary reference for Topology and related terms. Definitions are from various texts which I have read [1].

Glossary

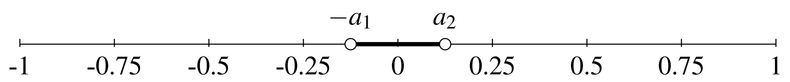
Accumulation (Limit) Point Let $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is an **accumulation point** or **limit point** of A iff every open set G containing p contains a point of A different from p, i.e.:

$$G \text{ open, } p \in G \text{ implies } A \cap (G \setminus \{p\}) \neq \emptyset$$

The set of accumulation points of A, denoted by A', is called the *derived set* of A.

Examples:

- Every real number $p \in \mathbb{R}$ is a limit point of $\mathbb{Q}$ since every open set contains rational numbers.
- The set of integers $\mathbb{Z}$ does not have any accumulation points, i.e. derived set of $\mathbb{Z}$ is $\emptyset$ (as open sets in $\mathbb{R}$ can span between integers).
- Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, the point 0 is an accumulation point of A since any open set G with $0 \in G$ contains an open interval $(-a_1, a_2) \subseteq G$ with $-a_1 < 0 < a_2$, which contains points in A. Note that the limit point 0 of A does not belong to A and there are no other limit points, i.e. $A' = \{0\}$.



Note: *"Limit point of a set"* is not to be confused with the concept *"limit of a sequence"*.

Bolzano-Weierstrass Theorem: Let A be a bounded, infinite set of real numbers. Then A has at least one accumulation point. However, do note that not every set, even if it is infinite, has a limit point.

Closed Set is a subset $A \subseteq \mathbb{R}$, iff its complement, A^c, is an open set. Alternatively, $A \subseteq \mathbb{R}$ is closed iff A contains each of its points of accumulation.

Examples:

- The closed interval $[a,b]$ is a closed set since its complement $(-\infty, a) \cup (b, \infty)$, the union of two open infinite intervals, is open.
- The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed since 0 is a limit point of A but does not belong to A.
- The empty set $\emptyset$ and the entire line $\mathbb{R}$ are closed sets since their complements $\mathbb{R}$ and $\emptyset$, respectively, are open sets.
- Consider the open-closed interval $A = (a,b]$. Note that A is not open since $b \in A$ is not an interior point of A, and is not closed since $a \notin A$ but is a limit point of A.

Note: *Open* and *Closed* sets are not inverses of each other as sets can be neither open nor closed.

Convergent Sequence A sequence $\langle a_1, a_2, \dots \rangle$ of points in a topological space X **converges** to a point $b \in X$, or b is the *limit* of the sequence $\langle a_n \rangle$, denoted by

$$\lim_{n \to \infty} a_n = b, \ \lim a_n = b \text{ or } a_n \to b$$

iff for each open set G containing b there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$n > n_0 \text{ implies } a_n \in G$$

that is, if G contains almost all, i.e. all except a finite number, of the terms of the sequence.

Examples:

- Let $\langle a_1, a_2, \dots \rangle$ be a sequence of points in an indiscrete topological space $(X, \mathcal{G})$. Note that: (i) X is the only open set containing any point $b \in X$; and (ii) X contains every term of the sequence $\langle a_n \rangle$. Accordingly, the seqeuence $\langle a_1, a_2, \dots \rangle$ contverges to every point $b \in X$.
- Let $\langle a_1, a_2, \dots \rangle$ be a sequence of points in a discrete topological space $(X, \mathcal{D})$. Now for every point $b \in X$, the singleton set $\{b\}$ is an open set containing b. So, if $a_n \to b$, then the set $\{b\}$ must contain almost all of the terms of the sequence. In other wrods, the sequence $\langle a_n \rangle$ converges to a point $b \in X$ iff the sequence is of the form $\langle a_1, a_2, \dots, a_{n_0}, b, b, b, \dots \rangle$
- Let $\mathcal{T}$ be a cocountable topology. A sequence $\langle a_1, a_2, \dots \rangle$ converges to $b \in X$, $(a_n \to b)$, iff the sequence is eventually constant, i.e. $\langle a_1, a_2, \dots, a_{n_0}, b, b, b, \dots \rangle$. Prove $\Rightarrow$, we define A^c to be the set consisting of the terms in the sequence $\langle a_n \rangle$ not equal to b, and this set is finite. Thus A is an open set in $\mathcal{T}$ also containing b. Since $a_n \to b$, there exists $n > n_0$ such that $a_n = b$ for all $n > n_0$, hence enventually constant. [2]

Convergent Sequence In Real Numbers is defined as:

The sequence $\langle a_1, a_2, \dots \rangle$ of real numbers converges to $b \in \mathbb{R}$ or, equivalently, the real number b is the limit of the sequence $\langle a_n : n \in \mathbb{N} \rangle$, denoted by

$$\lim_{n \to \infty} a_n = b, \ \lim a_n = b \text{ or } a_n \to b$$

if for every $\varepsilon > 0$ there exists a positive integer n_0 such that

$$n > n_0 \text{ implies } |a_n - b| < \varepsilon$$

Observe that $|a_n - b| < \varepsilon$ means that $b - \varepsilon < a_n < b + \varepsilon$ and a_n belongs to the open interval $(b - \varepsilon, b + \varepsilon)$ containing b. Thus, since each term after the n_0th lies inside the interval $(b - \varepsilon, b + \varepsilon)$, only the terms before a_{n_0}, and there are only a finite number of them, can lie outside the interval $(b - \varepsilon, b + \varepsilon)$.

Alternative Definition: The sequence $\langle a_n : n \in \mathbb{N} \rangle$ converges to b if every open set containing b contains almost all, i.e. all but a finite number, of the terms of the sequence.

Examples:

- A constant sequence $\langle a_0, a_0, \dots \rangle$ such as $\langle -2, -2, \dots \rangle$ converges to a_0 since each open set containing a_0 contains every term of the sequence.
- Each of the sequence

  $$\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle, \langle 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \dots \rangle, \langle 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots \rangle$$

  converges to 0 since any open interval containing 0 contains almost all of the terms of each of the sequences.

- Consider the sequence $\langle \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \ldots \rangle$, i.e. the sequence

$$a_n = \begin{cases} \frac{1}{2^{(n+2)/2}} & \text{if } n \text{ is even} \\ 1 - \frac{1}{2^{(n+2)/2}} & \text{if } n \text{ is odd} \end{cases}$$

  Observe that any open interval containing either 0 or 1 contains an infinite number of the terms of the sequence. Neither 0 nor 1, however, is a limit of the sequence. Observe, though, that 0 and 1 are accumulation points of the *range* of the sequence, that is, of the set $\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \ldots\}$.

Cover A class of sets, $\mathscr{A} = \{A_i\}$, is said to **cover** a set A if A is contained in the union of the members of $\mathscr{A}$, i.e. $A \subseteq \cup_i A_i$.

Heine-Borel Theorem: Let $A = [c,d]$ be a closed and bounded interval, and $\mathcal{G} = \{G_i : i \in I\}$ be a class of open intervals which covers A, i.e. $A \subseteq \cup_i G_i$. Then $\mathcal{G}$ contains a finite subclass, say $\{Gi_1, Gi_2, \ldots, Gi_m\}$, which also covers A, i.e.,

$$A \subseteq G_{i_1} \subseteq G_{i_2} \subseteq \cdots \subseteq G_{i_m}$$

Both conditions, closed and bounded, must be satisfied by A or else this theorem is not true.

Examples of Heine-Borel Theorem:

- Consider the open, bounded unit interval $A = (0,1)$. The class

$$\mathbb{G} = \{G_n = (\frac{1}{n+2}, \frac{1}{n}) : n \in \mathbb{N}\}$$

  of open intervals covers A, i.e.,

$$A \subseteq (\frac{1}{3}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{5}, \frac{1}{3}) \cup \ldots$$

  But the union of no finite subclass of $\mathbb{G}$ contains A.

- Consider the closed infinite interval $A = [1, \infty)$. The class

$$\mathbb{G} = \{(0,2), (1,3), (2,4), \ldots\}$$

  of open intervals covers A, but no infinite subclass does.

Interior Point Let A be a set of real numbers. A point $p \in A$ is an **interior point** of A iff p belongs to some open interval S_p which is contained in A:

$$p \in S_p \subseteq A$$

Open Set A set A is **open** (or $\mathcal{U}$-open) iff each of its points is an interior point. Observer that a set is not open iff there exists a point in the set that is not an interior point.

Examples:

- An open interval $A = (a,b)$ is an open set, for we may choose $S_p = A$ for each $p \in A$.

- The real line $\mathbb{R}$ is oepn since any open interval S_p is a subset of $\mathbb{R}$, i.e. $p \in S_p \subseteq \mathbb{R}$
- The empty set $\emptyset$ is open since there is no point in $\emptyset$ which is not an interior point.
- The closed interval $B = [a,b]$ is not an open set, for any open interval containing a or b must contain points outside of B, i.e. the end points a and b are not interior points of B.
- Infinite open intervals (a,∞), $(-\infty,a)$ and $(-\infty,\infty)$ are open. On the other hand, infinite closed intervals $[a,\infty)$, $(-\infty,a]$ are not open sets since a is not an interior point.

Note: The union of any number of open sets in $\mathbb{R}$ is open and the intersection of any finite number of open sets in $\mathbb{R}$ is open. For consider the class of open intervals:

$$\{A_n = (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\} \text{ i.e. } \{(-1,1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{1}{3}), \dots\}$$

and the intersection, $\bigcap_{n=1}^{\infty} A_n = \{0\}$, is a single point which is not open.

Sequence , denoted by

$$\langle s_1, s_2, \dots \rangle,\ \langle s_n : n \in \mathbb{N} \rangle \text{ or } \langle s_n \rangle$$

is a function whose domain is $\mathbb{N} = \{1,2,3,\dots\}$, i.e. a sequence assigns a point s_n to each positive integer $n \in \mathbb{N}$. The image s_n or $s(n)$ of $n \in \mathbb{N}$ is called the nth *term* of the sequence.

Bounded: A sequence $\langle s_n : n \in \mathbb{N} \rangle$ is said to be *bounded* if its range $\{s_n : n \in \mathbb{N}\}$ is a bounded set.

Notation: Observe that $\langle s_n : n \in \mathbb{N} \rangle$ denotes a sequence and is a function. Whereas $\{s_n : n \in \mathbb{N}\}$ denotes the range of the sequence and is a set.

Subsequence Consider a sequence $\langle a_1, a_2, a_3, \dots \rangle$. If $\langle i_n \rangle$ is a sequence of positive integers such that $i_1 < i_2 < \dots$, then

$$\langle a_{i_1}, a_{i_2}, a_{i_3}, \dots \rangle$$

is called a **subsequence** of $\langle a_n : n \in \mathbb{N} \rangle$.

Note: Every bounded sequence of real numbers contains a convergent subsequence.

Examples:

- Consider the sequence $\langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$. Observe that $\langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \rangle$ is a subsequence of $\langle a_n \rangle$ but that $\langle a_n \rangle = \langle \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \dots \rangle$ is not a subsequence of $\langle a_n \rangle$ since 1 appears before $\frac{1}{2}$ in the original sequence.
- Although the sequence $\langle \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \dots \rangle$ does not converge, it does have a convergent subsequence such as $\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \rangle$ and $\langle \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots \rangle$. On the other hand, the sequence $\langle 1, 3, 5, \dots \rangle$ does not have any convergent subsequences.

Topological Spaces (Topology) Let X be a non-empty set. A class $\mathcal{T}$ of subsets of X is a **topology** on X iff $\mathcal{T}$ satisfies the following axioms:

(O_1) X and $\emptyset$ belong to $\mathcal{T}$.

(O_2) The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

(O_3) The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are then called $\mathcal{T}$*-open sets*, of simply *open sets*, and X together with $\mathcal{T}$, i.e. the pair $(X,\mathcal{T})$ is called a **topological space**.
Examples:

- Let $\mathcal{U} = \{\cup_i I_i | I_i \in I\}$ denote the class of all open sets of real numbers where $I = \{(a,b) | a,b \in \mathbb{R}\}$. Then $\mathcal{U}$ is a topology in $\mathbb{R}$ denoted $(\mathbb{R},\mathcal{U})$, and is also called the *usual topology* on $\mathbb{R}$.
- Similarily, the class $\mathcal{U}$ of all open sets in the plane $\mathbb{R}^2$ is a topology and also called the *usual topology* on $\mathbb{R}^2$.
- Let X be an infinite set. We can define a *Cocountable Topology* (also known as *Countable Complement Topology*) by declaring the empty set to be open, and a non-empty subset $U \subseteq X$ to be open if $X \setminus U$ is countable. In this case, if X is countable then the cocountable topology is just the discrete topology, as the complement of any set is countable, and thus open.

References

[1] Lipschutz S. *Theory and applications of general topology*. Schaum's outlines. 1965.

[2] Severin Schraven (https://math.stackexchange.com/users/331816/severin schraven). Convergent sequence in co-countable topology iff sequence is eventually constant. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1768567 (version: 2016-05-02).