Category Theory

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A glossary reference for Category Theory and relevant terms which I have came across. Most of the definitions are from *Category Theory for Computing Science* [1].

Glossary

2-Category is a category $\mathbf{C}$ with the following additional structure:

- A set of objects of $\mathbf{C}$ (called *0-cells*).
- $\forall A, B \in \mathbf{C}_0$ (for every pair of 0-cells), a small category $\mathbf{C}(A, B)$ whose objects are arrows from A to B in $\mathbf{C}$ (objects and arrows in $\mathbf{C}(A, B)$ are called *1-cells* and *2-cells* respectively).
- $\forall A, B, C \in \mathbf{C}_0$ (for every triple of 0-cells), there is a composition functor:

  $$comp : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \longrightarrow \mathbf{C}(A, C)$$

  and that the following diagram commutes:

  $$\begin{tikzcd}
  \mathbf{C}(C, D) \times \mathbf{C}(B, C) \times \mathbf{C}(A, B) \arrow[r, "comp \times id"] \arrow[d, "id \times comp"'] & \mathbf{C}(B, D) \times \mathbf{C}(A, B) \arrow[d, "comp"] \\
  \mathbf{C}(C, D) \times \mathbf{C}(A, C) \arrow[r, "comp"'] & \mathbf{C}(A, D)
  \end{tikzcd}$$

  ie: *comp* is associative

- For each object $A \in \mathbf{C}$, there is the identity functor:

  $$unit_A : 1 \to \mathbf{C}(A, A)$$

  where 1 is the terminal object in **Cat** (ie: the category with one object denoted as 0 and its identity arrow) defined as $unit_A(0) = id_A$ and that the following diagram commutes:

  $$\begin{tikzcd}
  & \mathbf{C}(A, B) \arrow[dl, "unit_B \times id"'] \arrow[d, "id"] \arrow[dr, "id \times unit_A"] & \\
  \mathbf{C}(B, B) \times \mathbf{C}(A, B) \arrow[r, "comp"'] & \mathbf{C}(A, B) & \mathbf{C}(A, B) \times \mathbf{C}(A, A) \arrow[l, "comp"']
  \end{tikzcd}$$

  ie: *unit* is the identity functor with respect to *comp*

The structure of a 2-category can be summarized as follows:

- A *horizontal category* whose objects are 0-cells, arrows are 2-cells.
- A *vertical category* whose objects are 1-cells, arrows are 2-cells.
- A *base category* (that can be derived from the horizontal and vertical categories) whose objects are 0-cells, arrows are 1-cells.

Visual Depictions:

0-cells x, y

1-cells $x \to y$

2-cells $\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$

Cat as a 2-category:

- 0-cells are small categories.
- 1-cells are functors.
- 2-cells are natural transformations.
- *comp* on 1-cells is composition of functors:

$$comp(G, F) = G \circ F$$

  for arrows $F : A \to B, G : B \to C$ in **Cat**.

- *comp* on 2-cells is horizontal composition of natural transformations:

$$comp(\beta, \alpha) = \beta * \alpha$$

  where $F, F' : A \to B$ and $G, G' : B \to C$ are functors $\alpha : F \to F', \beta : G \to G'$ are natural transformations between them.

- By the *Interchange Law*

$$comp((\beta', \alpha') \circ (\beta, \alpha)) = comp(\beta', \alpha') \circ comp(\beta, \alpha)$$

2-Functor If **C** and **D** are 2-categories, a **2-functor** $F : \mathbf{C} \to \mathbf{D}$ consists of three maps of the form $F_i : \mathscr{C}_i \to \mathscr{D}_i$ where $i = b, v, h$ and $\mathscr{C}_i, \mathscr{D}_i$ are base, vertical or horizontal categories respectively.

Automorphism is an *endomorphism* that is *invertible*.

Binary Operation A **binary operation** on an object S is a function $S \times S \to S$ given that the product $S \times S$ exists.

Canonical Product A category has **canonical products** if a specific diagram/representation of the product can be given to each product.
Cartesian Closed Category (CCC) is a cartesian monoidal category $\mathscr{C}$ such that for every pair of object A and B, there is an *exponential object* B^A and an arrow $eval : B^A \times A \to B$ with the property that for any arrow $f : C \times A \to B$, there is an unique arrow $g : C \to B^A$ such that

$$f = eval \circ (g \times id_A)$$

Cartesian Monoidal Category If the products exists for a finite number of objects, a category is said to be a **cartesian monoidal category** or has *finite products*. An alternative definition is that a **cartesian monoidal category** is a monoidal category whose monoidal structure is the categorical product and unit is the terminal object.

Note: If a category has a terminal object and binary products, then it has finite products. Similarily, if a functor preserves terminal objects and binary products, it preserves all finite products.

Note: A lower semilattice is exactly a category that has all finite products. Dually, a upper semilattice is a category that has all finite coproducts.

Categorical Dual (^{op}) Given any category $\mathscr{C}$, $\mathscr{C}^{op}$ is the **dual category** of $\mathscr{C}$ where all the arrows are reversed. $\mathscr{C}^{op}$ is defined as follows:

- The objects and arrows of $\mathscr{C}^{op}$ are the objects and arrows of $\mathscr{C}$.
- If $f : A \to B$ in $\mathscr{C}$, then $f : B \to A$ in $\mathscr{C}^{op}$.
- If $h = g \circ f$ in $\mathscr{C}$, then $h = f \circ g$ in $\mathscr{C}^{op}$.

Since identity arrows have the same source and target objects, identity arrows in $\mathscr{C}$ are the same as in $\mathscr{C}^{op}$ and thus $\mathscr{C}^{op}$ is also a category.

Category A **category** $\mathscr{C}$ contains a collection of *objects* (denoted as $\mathscr{C}_0$) and a collection of *morphisms* (denoted as $\mathscr{C}_1$) where every morphism has $source, target : \mathscr{C}_1 \to \mathscr{C}_0$ functions mapping each arrow to its *source* and *target* objects respectively into the collection of objects.

For any two arrows f, g where the $target(f) = source(g)$, the *composition* of f and g are written as $g \circ f : source(f) \to target(g) \in \mathscr{C}_1$. Composition is associative, $(h \circ g) \circ f = h \circ (g \circ f)$ when either side is defined.

$\forall A \in \mathscr{C}_0$, there exists the *identity* arrow, $id_A : A \to A$ and for any $f : A \to B$, $f \circ id_A = id_B \circ f = f$.

A category is *small* if its objects and arrows constitute sets; otherwise it is *large*.

Category Equivalence A functor $F : \mathscr{C} \to \mathscr{D}$ is an **equivalence of categories** if there is a functor $G : \mathscr{D} \to \mathscr{C}$ such that:

- A family $u_C : C \to G(F(C))$ of isomorphisms of $\mathscr{C}$ indexed by the objects of $\mathscr{C}$ with the property that for every arrow $f : C \to C'$ in $\mathscr{C}$, $G(F(f)) = u_{C'} \circ f \circ u_C^{-1}$.
- A family $v_D : D \to F(G(D))$ of isomorphisms of $\mathscr{D}$ indexed by the objects of $\mathscr{D}$ with the property that for every arrow $g : D \to D'$ in $\mathscr{D}$, $F(G(g)) = v_{D'} \circ g \circ v_D^{-1}$.

Note: In this case, the functor G is called a *pseudo-inverse* of F and u, v are *natural isomorphisms*.

Note: F and G are both full and faithful.

Cocone is the dual of a cone. Thus, a cocone in a category $\mathscr{C}$ is a cone in $\mathscr{C}^{op}$.

Coequalizer is the dual of *equalizer*. For a parallel pair of arrows $f, g : A \to B$, an arrow $h : B \to C$ is the coequalizer of f and g provided $h \circ f = h \circ g$ and for any arrow $k : B \to D$ where $k \circ f = k \circ g$, there is an unique arrow $l : C \to D$ such that $k = l \circ h$.

Note: Coequalizers can be thought of as the least destructive identification necessary (by quotienting into equivalence classes) to force an equation or set of equations to be true.

Example: The *unification* of expressions by finding a common substitution is an example of a coequalizer.

Colimit ($\varinjlim$) is the dual of a limit, which is an universal commutative cocone.
A **colimit** is the initial object of the category of cocones.

Note: A category that has all colimits is *cocomplete*, if it has all finite colimits, it is *finitely cocomplete*.

Complete Category is a category $\mathscr{C}$ in which every diagram $D : \mathbb{I} \to \mathscr{C}$ has a limit in $\mathscr{C}$, where $\mathbb{I}$ is a small category. $\mathscr{C}$ has all finite limits or is **finitely complete** if $\mathbb{I}$ is a finite category.

Cone A **cone** over a diagram $D : \mathbb{I} \to \mathscr{C}$ (also known as the base) with an object $U \in \mathscr{C}$ (also known as the vertex) is a family of arrows of in $\mathscr{C}$ from indexed by $D(a)$ for each object $a \in \mathbb{I}$, where each arrow has the form $p_a : U \to D(a)$.
A cone is *commutative* if for any arrow $s : a \to b$ in $\mathbb{I}$, $D(s) \circ p_a = p_b$, ie: the following diagram commutes:

$$\begin{tikzcd} & U \arrow[dl, "p_a"'] \arrow[dr, "p_b"] & \\ D(a) \arrow[rr, "D(s)"'] & & D(b) \end{tikzcd}$$

Note: A commutative cone over D with vertex U is precisely a natural transformation from the constant functor Δ_U (the functor that takes every object to U and morphism to id_U) to D, where the commutativity is given by naturality.

Note: A cone over a discrete diagram is vacuously commutative.

Note: A *morphism* between two cones $\alpha : \Delta_U \to D$ and $\alpha' : \Delta_{U'} \to D$ is an arrow $f : U \to U'$ such that for every object $a \in \mathbb{I}$, the following diagram commutes:

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "\alpha_a"'] & & U' \arrow[dl, "\alpha'_a"] \\ & D(a) & \end{tikzcd}$$

Congruence Relation ($\sim$) is an equivalence relation $\sim$ on the arrows of a category $\mathscr{C}$ if:

- Whenever $f \sim g$, then f and g have the same domain and the same codomain.
- In the diagram below:

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "k"] & D \end{tikzcd}$$

  If $f \sim g$, then $f \circ h \sim g \circ h$ and $k \circ f \sim k \circ g$ for arrows h, k where $target(h) = source(f)$ and $source(k) = target(f)$.

Note: The congruence class containing the arrow f is denoted by $[f]$.

Constant/Global Element is an arrow from a terminal object. Thus, the **constant** A is an arrow $1 \to A$ for some object A. In **Set**, such an arrow is precisely a function from a singleton set to the set A, where each element $x \in A$ can be uniquely determined by a constant arrow $x : 1 \to A$.

ω-Continuous For any two ω-CPO objects D and D', $f : D \to D'$ is an **ω-continuous** arrow if for any object A and $g_0 \leq g_1 \leq \ldots$ where $g_0, g_1, \ldots$ are arrows in $\mathrm{Hom}(A, D)$ with supremum g, $f \circ g$ is the supremum of $f \circ g_0 \leq f \circ g_1 \leq \ldots$. It is easy to see that f is monotone (order preserving).
Contravariant Functor is a functor $F : \mathscr{C}^{op} \to \mathscr{D}$ for categories $\mathscr{C}$ and $\mathscr{D}$. In contrast to *covariant* functors, for an arrow f in $\mathscr{C}$

$$F(f : B \to A) = F(A) \to F(B)$$

is an arrow in $\mathscr{D}$. This follows from the fact that f is an arrow from A to B in $\mathscr{C}^{op}$.
Coproduct/Sum $(+)$ is the categorial dual of *product*. A **coproduct** $A + B$ of two objects A and B in a category consists of an object $A + B$ together with arrows/*coprojections* $i_1 : A \to A + B$ and $i_2 : B \to A + B$ such that given any arrows $f : A \to C$ and $g : B \to C$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "i_1"'] & C & B \arrow[l, "g"'] \arrow[dl, "i_2"] \\ & A + B \arrow[u, "\langle f|g \rangle"] & \end{tikzcd}$$

ie, $\langle f|g \rangle \circ i_1 = f$ and $\langle f|g \rangle \circ i_2 = g$.

Note: The arrows i_1, i_2 are also known as *canonical injections* or *inclusions*.
Note: The functor $\mathrm{Hom}(A, -) \times \mathrm{Hom}(B, -)$ is represented by $A + B$ and the universal element is the pair of canonical injections.

Diagram A **diagram** in a category $\mathscr{C}$ of shape $\mathbb{I}$ is a functor $D : \mathbb{I} \to \mathscr{C}$ where $\mathbb{I}$ is also a category (usually small or finite).
Note: The purpose of $\mathbb{I}$ is to give shape to the diagram functor.
Discrete A category in which all arrows are identity arrows is called **discrete**.
Disjoint Coproduct A coproduct $A + B$ in a category is **disjoint** if the injections $A \to A + B$ and $B \to A + B$ are monic and the pullback of the injections along each other is the initial object (since the injections are monic, these injections are subobjects and their intersection is the initial object). ie:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A + B \end{tikzcd}$$

is a pullback diagram (this is also a pushout diagram by the definition of coproducts).
Note: In **Set**, this precisely captures the idea of a disjoint union between sets A and B where $A \cap B = \emptyset$.
Note: If $A + B$ is disjoint, the subobjects A and B are known as *complemneted subobjects* and A is the complement of B in C and conversely.

Distributive Category is a category with both finite products and coproducts. Thus for all objects A, B and C, the following diagram commutes

$$\begin{tikzcd}
A \times B \arrow[r, "i_1'"] \arrow[dr, "A\times i_1"'] & A \times B + A \times C \arrow[d, "d"] & A \times C \arrow[l, "i_2'"'] \arrow[dl, "A\times i_2"] \\
& A \times (B + C) &
\end{tikzcd}$$

and d is an isomorphism.

In a distributive category $\mathscr{C}$, all canonical projections are monic and for any object $A \in \mathscr{C}$, the arrow $\langle id_0, !\rangle : 0 \to 0 \times A$ is an ismorphism (ie: $0 \cong 0 \times A$ and thus all distributive categories has a strict initial object).

Example: A *Boolean Algebra* is a distributive category with $\wedge$ as product and $\vee$ as coproduct.

Endomorphism is an arrow $f : A \to A$ in a category where the source and target objects are the same.

Epimorphism ($\twoheadrightarrow$) is the dual to monomorphisms. An arrow $f : S \to T$ is an **epimorphism** if for any arrows $g, h : T \to X$, $g \circ f = h \circ f$ implies $g = h$. Epimorphisms can be denoted by $f : S \twoheadrightarrow T$.

Properties: Let $f : A \to B$ and $g : B \to C$ be arrows:

- If f and g are epimorphisms, so is $g \circ f$.
- If $g \circ f$ is an epimorphism, so is g.

Note: In **Set**, a function is surjective if and only if it is an epimorphism.

Note: In the category determined by a Monoid, an epic element is right cancellable.

Equalizer is the limit over a parallel pair of arrows. Let $f, g : A \to B$ be a parallel pair of arrows. An **equalizer** of f and g is an object E together with an arrow $j : E \to A$ such that:

- $f \circ j = g \circ j$
- If $h : C \to A$ and $f \circ h = g \circ h$, there is an unique arrow $k : C \to E$ such that $j \circ k = h$ (ie: h factors through j).

Note: An arrow that is an equalizer of some parallel pair of arrows is a monomorphism. In addition, any two equalizers for the same pair of arrows belong to the same subobject class of monomorphisms.

Note: Equalizers can be thought of as the largest subobject on which an equation or set of equations is true.

Examples of Cartesian Closed Categories:

Set For any objects A and B, the exponential object B^A is the set of functions from A to B, effectively the same as $\mathrm{Hom}(A, B)$. $eval : B^A \times A \to B$ is the application of f to a in $(f, a) \in B^A \times A$, ie: $eval(f, a) = f(a)$.

Boolean Algebra In addition to being a distributive category, the exponential object q^p is $\neg p \vee q$. $eval : q^p \times p \to q$ follows from the distributive law, $(\neg p \vee q) \wedge p \Rightarrow q$.

Heyting Algebra The meet operation is product, implication is the exponential object and the arrow *eval* is *modus ponens*.

Grf Let No be the graph with 1 node and no arrows and Ar be the graph with 2 nodes and 1 arrow between them. The exponential object between two graphs $\mathscr{G}$ and $\mathscr{H}$, $\mathscr{H}^{\mathscr{G}}$, is a graph which has a set of nodes $\mathrm{Hom}(\mathscr{G} \times \mathrm{No}, \mathscr{H})$ and a set of arrows $\mathrm{Hom}(\mathscr{G} \times \mathrm{Ar}, \mathscr{H})$. Both sets of nodes and arrows are sets of graph homorphisms.

Func($\mathscr{C}$,Set) For any category $\mathscr{C}$, **Func($\mathscr{C}$,Set)** is a cartesian closed category. For any two functors $F, G : \mathscr{C} \to \mathbf{Set}$, the exponential object G^F is a functor defined as follows:

- For any object C, $G^F(C) = (\mathrm{Hom}(C, -) \times F) \to G$ is a set of natural transformations.
- For any arrow $f : C \to D$, $G^F(f)$ takes a natural transformation $\alpha : \mathrm{Hom}(C, -) \times F \to G$ to a natural tranformation (with component at an object A) that is $G^F(f)(\alpha) : \mathrm{Hom}(D, A) \times F(A) \to G(A)$, defined as $G^F(f)(\alpha)(g, a) = \alpha A(v \circ f, x)$ for an arrow $g : D \to A$ and an object $a \in F(A)$.

Cat The exponential object of any two categories $\mathscr{C}$ and $\mathscr{D}$ is $\mathscr{D}^{\mathscr{C}} = \mathbf{Func}(\mathscr{C}, \mathscr{D})$.

ω-CPOs and Scott-continuous Functions

Functional Programming Language The exponential object is the function type and the *eval* arrow is simply the function application function.

Typed Lambda Calculus Objects are types and an arrow from an object A to and object B is an equivalence class of terms of type B with one free variable of type A. The exponential object B^A is the lambda abstraction with a bound variable of type A and a body of body B.

Examples of Categories:

Set is the category whose objects are sets and arrows are functions between sets. Composition is function composition and the identity arrow is the identity function id_S for a set S.

Fin is the category whose objects are finite sets and arrows are all the functions between finite sets.

Pfn is the category whose objects are sets but all arrows are all partial functions. If $f : S \to T, g : T \to V$ are partial functions with f defined on $S_0 \subseteq S$ and g defined on $T_0 \subseteq T$, then the composite $g \circ f : S \to V$ is the partial function from S to V defined on the subset $\{x \in S_0 | f(x) \in T_0\}$ of S by the requirement $(g \circ f)(x) = g(f(x))$.

Rel is the category whose objects are sets and arrows are relations relations between sets. The composite $\beta \circ \alpha$ of relations α from sets S to T and β from T to U is a relation from S to U defined as follows:

If $x \in S$ and $z \in U$, $(x, z) \in \beta \circ \alpha$ if and only if there is an element $y \in T$ for which $(x, y) \in \alpha$ and $(y, z) \in \beta$.

The identity arrow for a set S is the *diagonal relation* $\Delta_S = \{(x, x)|x \in S\}$

Poset is the category whose objects are elements of a poset P and arrows are the partially ordered relation between those elements. The reflexive and transitive properties provides the identity and composition of arrows.

Note: Every hom-set in this category has at most ONE element. Equivalently, any category where every hom-set has at most one element is a poset.

Monoid is the category whose only object is the Monoid M itself and arrows are the elements of the monoid (this category is denoted as $C(M)$). The identity arrow is the identity element and composition being the monoid binary operation.

Note: Any category with one object is a monoid.

Group is the category whose only object is the Group itself and arrows are elements of the group, same as the category determined by a Monoid.

Note: All arrows in this category are isomorphisms.

Grf is the category whose objects are graphs and arrows are graph homomorphisms. Identity arrow being the identity hormormophism and composition being composition of homomorphisms.

Posets and Monotone Functions is the category whose objects are posets and arrows are monotone functions between posets.

Sem is the category whose objects are semigroups and arrows are semigroup homomorphisms.

Note: This can be extended to monoids, denoted as **Mon**.

ω-CPOs and Scott-continuous Functions is the category whose objects are ω-CPOs and arrows are Scott-continuous functions.

Note: A category is also formed with *strict* ω-CPOs and *strict* Scott-continous functions.

Path Category is the free category $F(\mathscr{G})$ generated by a graph $\mathscr{G}$ whose objects are the nodes of $\mathscr{G}$ and arrows are paths in $\mathscr{G}$. For each node/object A, the identity arrow is the empty path from A to A. Composition is defined by $(f_1, \ldots, f_k) \circ (f_{k+1}, \ldots, f_n) = (f_1, \ldots, f_n)$

Cat is the category whose objects are small categories and arrows are functors between small categories.

M-**Act** is the category whose objects are monoid actions and arrows are equivariant maps. Since equivariant maps are functions, the identity arrow is the identity equivariant map and composition is function composition.

FBool is the category whose objects are finite Boolean algebras and arrows are hormormophisms between them.

Pts is the category whose objects are pointed sets and arrows are functions that preserves the pointed object.

Func$(\mathscr{C}, \mathscr{D})$ is the category whose objects are functors from categories $\mathscr{C}$ to $\mathscr{D}$ and arrows are natural transformations.

cone(D) is the category whose objects are cones with base D and arrows are morphisms between cones.

Examples of Functors:

Monoid Homomorphism $f : M \to N$ where M and N are monoids is a functor from the category $C(M)$ to the category $C(N)$.

Monotone Map between two posets is a functor between the category determined by the posets.

Projection The first and second projections from the product category to its first or second parts, $P_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $P_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ for categories $\mathscr{C}$ and $\mathscr{D}$ are functors.

Inclusion Map of a subcategory is a functor.

Note: By the definition of *subcategory*, objects and arrows of the subcategory need not be objects and arrows in the bigger category.

Example: **Set** is a subcategory of **Rel** and the functor is defined by taking sets to the sets as objects in **Rel** and set functions $f : S \to T$ to $\{(s,t)|t = f(s)\}$.

Monoid Actions For an action α of a monoid M acting on a set S, let $C(M)$ be the category determined by M and the action α determines a functor $F_\alpha : C(M) \to \mathbf{Set}$ defined as:

- $F_\alpha(*) = S$ where $*$ denotes the single object in $C(M)$.
- $F_\alpha(m) = s \mapsto \alpha(m,s)$ for $m \in M$, $s \in S$.

Quotient Functor $Q : \mathscr{C} \to \mathscr{C}/\sim$

is the functor thats takes a category to its quotient category by where $Q(A) = A$ for any object A and $Q(f) = [f]$ for arrows f in the category $\mathscr{C}$.

Exponential Object An **exponential object** B^A is an *internal hom*$[A,B]$ in a *cartesian closed category*. B^A is equipped with the arrow $eval : B^A \times A \to B$ and is the universal element of the functor $\mathrm{Hom}(- \times A, B)$. This means for any other object C and the $eval' : C \times A \to B$, there is an unique arrow from $f : C \to B^A$ such that

$$eval' = eval \circ (f \times id_A)$$

An exponential object is also known as *internal hom*.

Note: Since *eval* is an universal element of the functor $\mathrm{Hom}(- \times A, B)$, the process of *currying*, which takes the function $f : C \times A \to B$ to $f' : C \to B^A$ forms a natural isomorphism of functors

$$\mathrm{Hom}(- \times A, B) \cong \mathrm{Hom}(-, B^A)$$

Note: Consider the sets A, B where $|A|$ is the cardinality of A. The exponential object B^A is a set of functions from A to B. $\forall f \in B^A$, f maps each element of A to an element of B, which can be described as $|A|$-tuple of elements of B. Hence, the notation for B^A which is an $|A|$ product of B.

Note: Any cartesian closed category with finite coproducts is a distributive category.

Extremal Morphisms An **extremal epimorphism** in a category $\mathscr{C}$ is a morphism e such that if $e = m \circ g$ where m is monic, then m is an isomorphism. Dually, an **extremal monomoprhism** is a a morphism m such that if $m = g \circ e$ where e is epic, then e is an isomorphism.

Factor Let $f : A \to B$ and $g : C \to B$ be arrows. If there is an arrow $h : A \to C$ such that $f = g \circ h$, then f **factors** through g.

Factorization System A **factorization system** in a category $\mathscr{C}$ consists of two subclasses $\mathscr{E}$ and $\mathscr{M}$ of the arrows of $\mathscr{C}$ such that:

- If $\mathscr{I}$ is the class of isomorphisms, then $\mathscr{M} \circ \mathscr{I} \subseteq \mathscr{M}$ and $\mathscr{I} \circ \mathscr{E} \subseteq \mathscr{E}$.
- Every arrow $f \in \mathscr{C}$ factors as $f = m \circ e$ where $m \in \mathscr{M}$ and $e \in \mathscr{E}$.
- In any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

  with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, there is a unique $h : B \to C$ such that $h \circ e = f$ and $m \circ h = g$.

Faithful A functor $F : \mathscr{C} \to \mathscr{D}$ is **faithful** if the set mapping:

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

for each pair of objects A and B in $\mathscr{C}$ induced by F is *injective*.

Note: If $f : A \to B$ and $g : C \to D$, as long as either $A \neq C$ or $C \neq D$, $F(f) = F(g)$ may hold and F can still be faithful.

Note: A faithful functor need not be injective on objects or arrows.

Finite Category is a category which consists of a finite set of objects and a finite set of arrows.

Forgetful/Underlying Functor is the functor which "forgets" some of the structure in a category of structures and structure-preserving functions.

Examples:

- U : **Mon** $\to$ **Sem**
- U : **Sem** $\to$ **Set**
- U : **Grf** $\to$ **Set** $\times$ **Set**
  that takes a graph $\mathscr{G}$ to the set of nodes and set of arrows, $U(\mathscr{G}) = (\mathscr{G}_0, \mathscr{G}_1)$. It follows that there is also an arrowset functor A : **Grf** $\to$ **Set** and nodeset functor N : **Grf** $\to$ **Set**.
- U : **Cat** $\to$ **Grf**
  by forgetting the identity arrows and composition, the remains of a category becomes a graph.
  Note: If F is a functor, then F is a graph homomorphism. Note that the converse is not true.

Simiarily, there is the functor $A : \mathbf{Cat} \to \mathbf{Set}$ and $O : \mathbf{Cat} \to \mathbf{Set}$ which takes a category to its set of arrows and objects respectively.

– $U : \mathscr{C}/A \to \mathscr{C}$
for an object A in the category $\mathscr{C}$.
Note: If $\mathscr{C}$ is **Set**, the underlying functor $U : \mathbf{Set}/S \to \mathbf{Set}$ forgets the indexing of S-indexed sets.

Free Functor Informally, a **free functor** is left adjoint to a **forgetful functor**.
Examples:

– $F : \mathbf{Set} \to \mathbf{Mon}$
is the free monoid functor that takes a set A to the free monoid $F(A)$, the Kleene closure A^* with concatenation as the monoid operation. F takes set functions to the Kleene closure induced homormophism.
Note: The Kleene closure itself is a functor from **Set** to **Set**, which is the composition of $U \circ F$, the underlying functor $U : \mathbf{Mon} \to \mathbf{Set}$ and the free monoid functor.

– $F : \mathbf{Grf} \to \mathbf{Cat}$
takes graphs to the path category of the graph as objects in **Cat** and graph homomorphisms to arrows (functors between path categories). For a graph homomorphism $\phi : \mathscr{G} \to \mathscr{H}$, $F(\phi)(f_n, f_{n-1}, \ldots, f_1) = (\phi(f_n), \phi(f_{n-1}), \ldots, \phi(f_1))$, which is clearly a path in $\mathscr{H}$ (and thus an arrow in the path category of $\mathscr{H}$).

Full A functor $F : \mathscr{C} \to \mathscr{D}$ is **full** if the set mapping:

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

for each pair of objects A and B in $\mathscr{C}$ induced by F is *surjective*.
Note: A full functor need not be surjective on objects or arrows.
Full Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ such that $\forall A, B \in \mathscr{D}_0$, $\mathrm{Hom}_{\mathscr{D}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$.
Example: **Fin** is a full subcategory of **Set**.
Functor is a "structure preserving" map between categories, similar to homomorphisms between groups, graphs and etc… A **functor** $F : \mathscr{C} \to \mathscr{D}$ from the category $\mathscr{C}$ to the category $\mathscr{D}$ defined as follows:

- $\forall A \in \mathscr{C}$, $F(A) \in \mathscr{D}$.
- If $f : A \to B$ in $\mathscr{C}$, then $F(f) : F(A) \to F(B)$ is an arrow in $\mathscr{D}$ such that:
  – $F(id_A) = id_{F(A)}$ for every object $A \in \mathscr{C}$.
  – $F(g \circ f) = F(g) \circ F(f)$ for all arrows $f : X \to Y, g : Y \to Z$ in $\mathscr{C}$.

Note: All functors preserve isomorphisms.

Godement Calculus Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, $E, F_1, F_2, F_3, G_1, G_2, G_3, H$ be functors and $\alpha, \beta, \gamma, \delta$ be natrual morphisms as shown in the following diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "E"] & \mathscr{B} \arrow[r, bend left=50, "F_1"] \arrow[r, dashed, "F_2" description] \arrow[r, bend right=50, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{C} \arrow[r, bend left=50, "G_1"] \arrow[r, dashed, "G_2" description] \arrow[r, bend right=50, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathscr{D} \arrow[r, "H"] & \mathscr{E}
\end{tikzcd}$$

There are 5 laws/equational rules between composition of functors and natural transformations:

1. $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma \circ \alpha)$
2. $(H \circ G_1)\alpha = H(G_1\alpha)$
3. $\gamma(F_1 \circ E) = (\gamma F_1)E$
4. $G_1(\beta \circ \alpha)E = (G_1\beta E) \circ (G_1\alpha E)$
5. $\gamma * \alpha = (\gamma F_2) \circ (G_1\alpha) = (G_2\alpha) \circ (\gamma F_1)$

Note: The first rule is also known as the *Interchange Law*.

Groupoid is a category in which every arrow is an *isomorphism*.

Hom Function Let S be an object and $f : T \to V$ an arrow in a category, a *covariant* **hom function** $\mathrm{Hom}(S, f) : \mathrm{Hom}(S, T) \to \mathrm{Hom}(S, V)$, if defined by:

$$\mathrm{Hom}(S, f)(g) = f \circ g$$

for some $g \in \mathrm{Hom}(S, T)$.
Similarily, a *contravariant* **hom function** $\mathrm{Hom}(f, S) : \mathrm{Hom}(V, S) \to \mathrm{Hom}(T, S)$, if defined by:

$$\mathrm{Hom}(f, S)(g) = g \circ f$$

for some $g \in \mathrm{Hom}(V, S)$.
Hom Functor is a functor that takes a category to a hom-set. For any category $\mathscr{C}$ with an object C, there are three different hom functors:

Covariant Hom Functor $\mathrm{Hom}(C, -) : \mathscr{C} \to \mathbf{Set}$

- $\mathrm{Hom}(C, -)(A) = \mathrm{Hom}(C, A)$ for each object $A \in \mathscr{C}$.
- $\mathrm{Hom}(C, -)(f) = \mathrm{Hom}(C, f) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$ for an arrow $f : A \to B$ in $\mathscr{C}$.

Contravariant Hom Functor $\mathrm{Hom}(-, C) : \mathscr{C}^{op} \to \mathbf{Set}$

- $\mathrm{Hom}(-, C)(A) = \mathrm{Hom}(A, C)$ for each object $A \in \mathscr{C}$.
- $\mathrm{Hom}(-, C)(f) = \mathrm{Hom}(f, C) : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ for an arrow $f : A \to B$ in $\mathscr{C}$.

Two-Variable Hom Functor $\mathrm{Hom}(-, -) : \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Set}$

- $\mathrm{Hom}(-, -)((A, B)) = \mathrm{Hom}(A, B)$ for each object $(A, B) \in \mathscr{C}^{op} \times \mathscr{C}$.
- $\mathrm{Hom}(-, -)((f, g)) = \mathrm{Hom}(f, g) : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, D)$ for an arrow $(f : A \to B, g : C \to D)$ in $\mathscr{C}^{op} \times \mathscr{C}$.

Note: All covariant hom functors preserves products.

Hom-Set A **hom-set** $\mathrm{Hom}(S, T)$ is the set of all morphisms from S to T for any objects S and T in a category.
Horizontal Composition Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ be categories, and $F, F' : \mathscr{A} \to \mathscr{B}, G, G' : \mathscr{B} \to \mathscr{C}$ be functors. **Horizontal composition** between natural transformation transformations $\alpha : F \to F'$ and $\beta : G \to G'$ is $\beta * \alpha : G \circ F \to G' \circ F'$.
Note: The horizontal composite of natural transformations is also a natural transformation.

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} & * & \mathscr{B} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{C} & \longmapsto & \mathscr{A} \arrow[r, bend left=50, "G\circ F"] \arrow[r, bend right=50, "G'\circ F'"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathscr{C}
\end{tikzcd}
$$

Idempotent An arrow $f : A \to A$ in a category is **idempotent** if $f \circ f = f$.
Indiscrete A category in which there is exactly one arrow between any two objects is called **indiscrete**.
Infinite Product is a product of infinitely many objects. Let I be a set and $\{A_i | i \in I\}$ be an indexed set of objects in the category $\mathscr{C}$. An **infinite product** $\Pi_{i=1} A_i$ of an indexed set is an object P with an indexed set of arrows $p_i : P \to A_i$ such that for any object $A \in \mathscr{C}$ with arrows $q_i : A \to A_i$, there is an unique arrow $q = \langle q_i \rangle : A \to P$ and $p_i \circ q = q_i$ for all $i \in I$.
Initial Object (0) is the dual of *terminal object* in which the **initial object** has an unique arrow to each object (including itself). This object is usually denoted 0 and the unique arrow $! : 0 \to A$ for every object A.
Note: Any two initial objects in a category are isomorphic.
Example: The empty set $\emptyset$ in **Set** is the initial object.
Internal Hom Functors In any Cartesian Closed Category $\mathscr{C}$, for an object A, there are two **internal hom functors** $F : \mathscr{C} \to \mathscr{C}$ and $G : \mathscr{C}^{op} \to \mathscr{C}$ defined as:

- For any object B, $F(B) = B^A$ and $G(B) = A^B$
- For an arrow $f : B \to C$, $F(f) = F(B) \to F(C)$ and $G(f) = G(C) \to G(B)$

Intersection If C is an object of $\mathscr{C}$ with subobjects C_1 and C_2, the **intersection** of C_1 and C_2, if it exists, is the pullback:

$$
\begin{tikzcd}
C_0 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
C_2 \arrow[r] & C
\end{tikzcd}
$$

Note: A category has *wide intersections* if every object C and every class of subobjects $\{C_i\}$ of C, there is an intersection.
Note: If exists, the intersection is the meet of the poset of subobjects of C.

Inverse For arrows $f : A \to B$ and $g : B \to A$, g is an **inverse** of f if $g \circ f = id_A$ and $f \circ g = id_B$.

Note: If only $f \circ g = id_B$ is satisfied, then f is a **left inverse** of g.
Similarily, if only $g \circ f = id_A$ is satisfied, then f is a **right inverse** of g.

Invertible is an element of a monoid in which it is an *isomorphism* in the category of the monoid (ie: the element has an inverse).

Isomorphism ($\cong$) is an arrow with an *inverse*. Two objects A and B are *isomorphic* if an isomorphism exists between them, thus $A \cong B$.

Note: It follows that all identity arrows are isomorphisms.

Note: A function in **Set**, homomorphism in **Grf** and **Mon** is an isomorphism if and only if it is *bijective*.

Kernel Pair A **kernel pair** of an arrow $f : X \to Y$ is a parallel pair of arrows from any object R to X that forms a limit:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

This is also the fiber product $X \times_Y X$, or the pullback of f along itself.

Note: Dually, a *cokernel pair* of f is the pushout of f along itself.

Limit ($\varprojlim$) A **limit** is an universal commutative cone $\alpha : \Delta_U \to D$ such that for every other cone over the same diagram $D : \mathbb{I} \to \mathscr{C}$, there exists an unique arrow to it. Thus:

- For every arrow $s : a \to b \in \mathbb{I}$, $D(s) \circ \alpha_a = \alpha_b$.
- If $\alpha' : \Delta'_U \to D$ is another commutative cone, there exists an unique arrow $f : U' \to U$ such that for each object $a \in \mathbb{I}$, $\alpha_a \circ f = \alpha'_a$.

Note: A limit can also be defined as a terminal object in the category $cone(D)$.

Note: A limit is also an universal element of the functor $cone(-, D) : \mathscr{C}^{op} \to$ **Set**, where $cone(U, D)$ is the set of commutative cones with vertex U over the diagram D, and $cone(f : U' \to U, D) : cone(U, D) \to cone(U', D)$ defined as $cone(f, D)(\alpha_a : \Delta_{Ua} \to D_a) = \alpha_a \circ f$ where a is some object in the domain of D. A universal element of such a functor is precisely a cone with which there is an unique arrow to it from any other cone with the same base.

Locally Small Category is a category $\mathscr{C}$ with the property that $\mathrm{Hom}(A, B)$ is a set for all objects $A, B \in \mathscr{C}$.

Monomorphism ($\rightarrowtail$) is an arrow $f : A \to B$ such that for any object T and arrows $x, y : T \to A$, if $x \neq y$, then $f \circ x \neq f \circ y$. Monomorphisms can be denoted by $f : A \rightarrowtail B$.

Note: In this context, $x, y : T \to A$ may be regarded as *variable elements* and can be written as if $x \neq y$, then $f(x) \neq f(y)$.

Properties: Let $f : A \to B$ and $g : B \to C$ be arrows:

- If f and g are monomorphisms, so is $g \circ f$.
- If $g \circ f$ is a monomorphism, so is f.

Note: In **Set**, a function is injective if and only if it is a monomorphism.

Note: In the category determined by a Monoid, a monic element is left cancellable.

Morphism/Arrow is a directed connective between two objects (*source* and *target* objects) in a category, synonymous with *map* or a *directed edge* in a *directed graph*. A morphism f with source x and target y objects is denoted as:

$$f : x \to y$$

n-Ary Product is the product of a list of n objects $A_1, \ldots, A_n$ with projectiions $p_i : \Pi_{i=1}^{n} A_i \to A_i$ and the property that given any object B and arrows $f_i : B \to A_i$, there is an unique arrow $\langle f_1, \ldots, f_n \rangle : B \to \Pi_{i=1}^{n} A_i$ for which $p_i \circ \langle f_1, \ldots, f_n \rangle = f_i$.

Let the product of n number of the object A, $A \times \cdots \times A$ be denoted as A_n:

Nullary Product denoted as A_0, is the terminal object.

Unary Product denoted as A_1, is effectively a product "wrapper" around A, where the only projection arrow is id_A. (Analogous to characters vs strings of length 1).

Note: *Binary products* (2-ary product) can be used to construct *Ternary products* (3-ary products) as

$$A \times B \times C \cong (A \times B) \times C \cong A \times (B \times C)$$

Natural Isomorphism is a natural transformation $\alpha : F \to G$ in which there is a natural transformation $\beta : G \to F$ is an inverse to α in $\mathbf{Func}(\mathscr{C}, \mathscr{D})$.

Note: Natural isomorphisms are also known as *natural equivalences*.

Theorem: Suppose $F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C}$ are functors and $\alpha : F \to G$ is a natural transformation between them. Then α is a *natural isomorphism* if and only if for each object $C \in \mathscr{C}$, $\alpha C : F(C) \to G(C)$ is an isomorphism in $\mathscr{D}$.

Natural Number Object (NNO) An object $\mathbf{N}$ in a category $\mathscr{C}$ together with arrows $zero : 1 \to \mathbf{N}$ and $succ : \mathbf{N} \to \mathbf{N}$ is a parametrized **natural numbers object** if for all objects A, B and arrows $f_0 : A \to X, t : X \to X$, there is an unique arrow $f : A \times \mathbf{N} \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
& A \times \mathbf{N} \arrow[r, "id_A \times succ"] \arrow[dd, "f"] & A \times \mathbf{N} \arrow[dd, "f"] \\
A \arrow[ur, "\langle id_A, zero \rangle"] \arrow[dr, "f_0"'] & & \\
& X \arrow[r, "t"'] & X
\end{tikzcd}$$

Natural Transformation Let $F, G : \mathscr{C} \to \mathscr{D}$ be 2 functors with the same domain and codomain. A **natural transformation** $\alpha : F \to G$ is given by a family of arrows αC for each object $C \in \mathscr{C}$ such that:

- $\alpha C : F(C) \to G(C)$ for each object $C \in \mathscr{C}$, also say that α is natural in C.
- For any arrow $f : C \to D$ in $\mathscr{C}$, the following diagram commutes.

$$\begin{tikzcd} F(C) \arrow[r, "\alpha C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(D) \arrow[r, "\alpha D"'] & G(D) \end{tikzcd}$$

Composition with Functors: Let $H : \mathscr{B} \to \mathscr{C}$ be a functor and $\alpha : F \to G$ where $F, G : \mathscr{A} \to \mathscr{B}$ be are functors. Then, postcomposing α with the functor H is a natural transformation $H\alpha : H \circ F \to H \circ G$ defined by $(H\alpha)A = H(\alpha A)$ for any object $A \in \mathscr{A}$.
A similar result can be shown when precomposing a natural transformation with a functor.

Order-Enriched Category is a poset-enriched category in which each poset $\mathrm{Hom}(A, B)$ is a strict ω-CPO where every pair of elements has a supremum (least upper bound) and an infimum (greatest lower bound).

Parallel Morphisms Morphisms which have the same *source* and *target* are **parallel**.
Note: The limit of a *parallel pair* of arrows is their equalizer. Dually, the colimit of a parallel pair of arrows is their coequalizer.
Partially Ordered Object D is a **partially ordered object** in a cartesian closed category if for every $\mathrm{Hom}(A, D)$ where A is any object, there is a partial order relation such that for any $f : B \to A$ and $g \leq h$ in $\mathrm{Hom}(A, D)$, $g \circ f \leq h \circ f$ in $\mathrm{Hom}(B, D)$.
Note: D is an ω-CPO object if for every $\mathrm{Hom}(A, D)$ for any object A, $\mathrm{Hom}(A, D)$ is an ω-CPO (ie: $\mathrm{Hom}(A, D)$ is ω-CPO).
Note: D is *strict* if there is an arrow $\perp : 1 \to D$ such that for any object A and any $f \in \mathrm{Hom}(A, D)$, $\perp \circ \langle\rangle \leq f$.
Note: If D is a strict ω-CPO and $f : D \to D$ is an ω-continuous arrow, then $fix(f) : 1 \to D$ is the least element with the property $f \circ fix(f) = fix(f)$.
Pointwise Products For categories $\mathscr{C}, \mathscr{D}$, if $\mathscr{D}$ has products then $\mathbf{Func}(\mathscr{C}, \mathscr{D})$ has **pointwise products** defined as:

- $(F \times G)(C) = F(C) \times G(C)$ for any object $C \in \mathscr{C}$
- $(F \times G)(f) = F(f) \times G(f)$ for any arrow $f \in \mathscr{C}$

where $F, G : \mathscr{C} \to \mathscr{D}$ are functors.
Poset-Enriched Category is a category $\mathscr{C}$ with a partial ordering on every $\mathrm{Hom}_{\mathscr{C}}(A, B)$ such that for every triple of objects $A, B, C \in \mathscr{C}$, the composition of hom-sets:

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$

is monotone. Thus if $f \leq f' : A \to B, g \leq g' : B \to C$ then $g \circ f \leq g' \circ f' : A \to C$.

As a 2-category: Exactly ONE 2-cell can be defined from f to g for arrows $f, g : A \to B$ if and only if $f \leq g$. Otherwise, no 2-cells from f to g. For each pair of objects A and B, the category $C(\text{Hom}(A, B))$ can be constructed as a poset.

Powerset Functor is a functor that takes a set S to its powerset $\mathcal{P}(S)$. There are three different powerset functors:

Inverse Image $\mathscr{P} : \textbf{Set}^{op} \to \textbf{Set}$
is a contravariant functor that takes sets S to its powerset $\mathcal{P}(S)$ and functions $f : A \to B$ to its inverse image $f : \mathcal{P}(B) \to \mathcal{P}(A)$.

Direct/Existential Image is a covariant functor that takes sets S to its powerset $\mathcal{P}(S)$ and functions $f : A \to B$ to the function $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ where $f_*(A_0) = \{f(x) | x \in A_0\}$.

Universal Image is a covariant functor that takes sets S to its powerset $\mathcal{P}(S)$ and functions $f : A \to B$ to the function $f_! : \mathcal{P}(A) \to \mathcal{P}(B)$ where $f_!(A_0) = \{y \in B | (f(x) = y) \Rightarrow (x \in A_0)\}$.

Preserve An arrow $f : A \to B$ **preserves** a property P if whenever A has P, then so does B.

Note: A functor $F : \mathscr{C} \to \mathscr{D}$ preserves P of arrows if whenever $f \in \mathscr{C}_1$ has P, so does $F(f) \in \mathscr{D}_1$.

Note: A property is preserved by isomorphisms if for any object A with the property, any object isomorphic to A must also have the property.

Product ($\times$) Let A, B be objects in a category $\mathscr{C}$. A **product** of A and B is an object $A \times B$ together with projections/arrows $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ such that for any object D and arrows $q_1 : D \to A$, $q_2 : D \to B$, there is an unique arrow $q : D \to A \times B$ and that the following diagram commutes:

$$\begin{tikzcd} & D \arrow[dl, "q_1"'] \arrow[d, "q"] \arrow[dr, "q_2"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

ie, $p_1 \circ q = q_1$ and $p_2 \circ q = q_2$.

The concept of a product is defined up to an unique isomorphism and is defined by a *universal mapping property*. Thus, all products of the objects A and B are isomorphic and any object isomorphic to $A \times B$ is a product of A and B.

Since for each pair of arrows $(q1, q2)$ produces an unique arrow q, there is a natural isomorphism:

$$\pi : \text{Hom}_{\mathscr{C}}(-, A) \times \text{Hom}_{\mathscr{C}}(-, B) \to \text{Hom}_{\mathscr{C}}(-, A \times B)$$

such that $\pi D(q_1, q_2) = q$ and is denoted as $\langle q_1, q_2 \rangle$.

Note: $(p1, p2)$ is an *universal element* of the functor $\text{Hom}(-, A) \times \text{Hom}(-, B)$.

Product of Arrows For arrows $f : S \to S', t : T \to T'$, the **product of**

arrows, $f \times g$ is defined such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[d, "f"'] & S \times T \arrow[l, "p1"'] \arrow[r, "p2"] \arrow[d, "f \times g"] & T \arrow[d, "g"] \\ S' & S' \times T' \arrow[l, "p1'"] \arrow[r, "p2'"'] & T' \end{tikzcd}$$

where $p1, p2$ and $p1', p2'$ are the projections of $S \times T$ to S, T and $S' \times T'$ to S', T' respectively.

Note: Products are distributed through composition, for arrows $f_i : A_i \to B_i, g_i : B_i \to C_i$ for $i = 1, 2$:

$$(g_1 \circ f_1) \times (g_2 \circ f_2) = (g_1 \times g_2) \circ (f_1 \times f_2)$$

Product of Categories If $\mathscr{C}$ and $\mathscr{D}$ are categories their **product** $\mathscr{C} \times \mathscr{D}$ is the category whose objects are all ordered pairs (C, D) where $C \in \mathscr{C}_0, D \in \mathscr{D}_0$ and arrows are $(f, g) : (C, D) \to (C', D')$ where $f : C \to C' \in \mathscr{C}_1, g : D \to D' \in \mathscr{D}_1$. The identity arrow of (C, D) is (id_C, id_D) and composition of arrows is defined component wise.

Pullback For the diagram:

$$B \xrightarrow{g} C \xleftarrow{f} A$$

a **pullback** is its limit:

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

This is known as the pullback diagram and the composite arrow $f \circ p_1 = g \circ p_2 : P \to C$ is omitted by convention.

Other Names: P together with p_1 and p_2 is a pullback or *fiber product* of f and g. p_2 is the pullback of f along g, p_1 is the pullback of g along f. P is sometimes be denoted as $A \times_C B$.

Note: In **Set**, $P = \{(a, b) | \forall a \in A, \forall b \in B, f(a) = g(b)\}$. p_1 and p_2 are just the coordinate projections of P.

Pushout is the dual of pullback, such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "q_1"] \\ B \arrow[r, "q_2"'] & Q \end{tikzcd}$$

is a commutative square for any object R and arrows $r_1 : A \to R, r_2 : B \to R, r_1 \circ f = r_2 \circ g$, there is an unique arrow $r : Q \to R$ where $r \circ q_1 = r_1$ and $r \circ q_2 = r_2$.

Quotient Category $(/ \sim)$ For a congruence relation $\sim$ on the arrows of a category $\mathscr{C}$, the **quotient category** $\mathscr{C} / \sim$ is defined as follows:

- The objects of $\mathscr{C}/\sim$ are the objects of $\mathscr{C}$.
- The arrows of $\mathscr{C}/\sim$ are the congruence classes of arrows of $\mathscr{C}$.
- If $f : A \to B$ in $\mathscr{C}$, then $[f] : A \to B$ in $\mathscr{C}/\sim$.
- If $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, then $[g] \circ [f] = [g \circ f] : A \to C$ in $\mathscr{C}$.

Note: For any functor $F : \mathscr{C} \to \mathscr{D}$ with the property that if $f \sim g$ then $F(f) = F(g)$, then there is an unique functor $F_0 : \mathscr{C}/\sim \to \mathscr{D}$ for which $F_0 \circ Q = F$. Here, Q is the quotient functor, $Q : \mathscr{C} \to \mathscr{C}/\sim$.

Reflect A functor $F : \mathscr{C} \to \mathscr{D}$ **reflects** a property P of arrows if whenever $F(f)$ has property P then so does f (where f is any arrow for which F takes to $F(f)$).

Theorem: Consider a full and faithful functor $F : \mathscr{C} \to \mathscr{D}$. Let A and B be objects in C and $u : F(A) \to F(B)$ is an isomorphism in $\mathscr{D}$, then there is an unique isomorphism $f : A \to B$ in $\mathscr{C}$ for which $F(f) = u$.

Corollary: A full and faithful functor reflects isomorphisms.

Corollary: For a full an faithful functor F, if $F(A) = F(B)$ for objects A and B in the domain of F, then $A \cong B$.

Regular Category is a finitely complete category which has finite limits, all arrows are coequalizers of its kernel pair and the pullback of any regular epimorphism along any other arrow is an regular epimorphism.

Note: A functor between regular categories is called a *regular functor* if it preserves finite limits and regular epimorphisms.

Regular Epimorphism is an epimorphism that is also a coequalizer of a parallel pair of arrows.

Regular Monomorphism is a monomorphism that is also an equalizer of a parallel pair of arrows.

Note: An arrow that is both a regular monomorphism and an epimorphism is an isomorphism.

Representable Functor A *set-valued functor* is representable if it is naturally isomorphic to a *hom functor*.

Note: If a covariant functor $F : \mathscr{C} \to \mathbf{Set}$ is isomorphic to $\mathrm{Hom}(C, -)$ for some object $C \in \mathscr{C}$, we say that C represents F. The same can be said for contravariant functors and contravariant hom functors.

Representative Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ in which every object of $\mathscr{C}$ is isomorphic to some object in $\mathscr{D}$.

Set-Valued Functor is any functor from any category $\mathscr{C}$ to **Set**.

Note: For small categories $\mathscr{C}$, a set valued functor $F : \mathscr{C} \to \mathbf{Set}$ generalizes the concept of monoid actions as functors.

Slice Category (/) If $\mathscr{C}$ is a category, for any object $A \in \mathscr{C}_0$, the **slice category** $\mathscr{C}/A$ is defined as follows:

- An object of $\mathscr{C}/A$ is an arrow $f : C \to A$ in $\mathscr{C}$ for some $C \in \mathscr{C}_0$.
- An arrow of $\mathscr{C}/A$ from $f : C \to A$ to $f' : C' \to A$ is an arrow $h : C \to C'$ such that $f = f' \circ h$.
- The composite of $h : f \circ f'$ and $h' : f' \circ f''$ is $h' \circ h$.

Note: Since the same h can satisfy both $f = f' \circ h$ and $g = g' \circ h$ where $f \neq g$ or $f' \neq g'$, $h : f \to f'$ and $h : g \to g'$ are regarded as different arrows in $\mathscr{C}/A$.

Note: The *indexed function* of an *indexed set* indexed by S, is precisely an arrow in the slice category $\mathbf{Set}/S$.

Split Epimorphism is an arrow that has a *right inverse* (this arrow can be shown to be an *epimorphism*, by composing its right inverse to its right).

Split Monomorphism is an arrow that has a *left inverse* (this arrow can be shown to be a *monomorphism*, by composing its left inverse to its left).

Strict Initial Object Let $\mathscr{C}$ be a category with products. An initial product, 0, in $\mathscr{C}$ is a **strict initial object** if it has one of the following equivalent properties:

- $\forall A \in \mathscr{C}$, if there is an $u : A \to 0$, then $A \cong 0$.
- $\forall A \in \mathscr{C}, 0 \times A \cong 0$.

Subcategory A **subcategory** $\mathscr{D}$ of a category $\mathscr{C}$ is a category for which:

- All objects and arrows in $\mathscr{D}$ are objects and arrows in $\mathscr{C}$ (ie: $\mathscr{D}_0 \subseteq \mathscr{C}_0$ and $\mathscr{D}_1 \subseteq \mathscr{C}_1$).
- The source and target of an arrow in $\mathscr{D}$ are the same as its souce and target in $\mathscr{C}$ (ie: source and target maps in $\mathscr{D}$ are restrictions of those in $\mathscr{C}$). Thus, $\forall A, B \in \mathscr{C}_0$, $\mathrm{Hom}_{\mathscr{D}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A, B)$.
- If A is an object in $\mathscr{C}$, its identity arrow id_A in $\mathscr{C}$ is in $\mathscr{D}$.
- If $f : A \to B$ and $g : B \to C$ in $\mathscr{D}$, then the composite $g \circ f \in \mathscr{C}_1$ is also the composite in $\mathscr{D}$.

Subfunctor A **subfunctor** of a functor $F : \mathscr{C} \to \mathbf{Set}$ is a functor $G : \mathscr{C} \to \mathbf{Set}$ with the property that for each object $C \in \mathscr{C}$, $G(C) \subseteq F(C)$ and for each arrow $f : C \to C'$ and element $x \in G(C)$, $G(f)(x) = F(f)(x)$.

Note: The inclusion function $i_C : G(C) \to F(C)$ that takes the the subfunctor G to the functor F is a natural transformation.

Subobject In a category $\mathscr{C}$, a **subobject** of an object $C \in \mathscr{C}$ is an equivalence class of monomorphisms under the relation $\sim$. The subobject is a **proper subobject** if it does not contain id_C. The equivalence relation $\sim$ is defined as follows:

- For monomorphisms $f_0 : C_0 \rightarrowtail C$ and $f_1 : C_1 \rightarrowtail C$, $f_0 \sim f_1$ if both f_0 and f_1 factors through each other.

Note: A preorder for subobjects $(i : A \to C) \leq (j : B \to C)$ is defined by the condition that there exists $k : A \to B$ such that $i = j \circ k$. Since j is monic, k is unique. And i is monic implies k is monic. In **Set**, $A \subseteq B \subseteq C$ where i is the inclusion of A to C, j the inclusion of B to C and k the inclusion of A to B.

Switch Map is a function that takes $A \times B$ with projections p_1, p_2 to $B \times A$ denoted as $\langle p_2, p_1 \rangle : A \times B \to B \times A$.

Terminal Object (1) is an object T in $\mathscr{C}$ if there is exactly ONE arrow $A \to T$ for each object $A \in \mathscr{C}$ (including itself). This object is usually denoted 1 and the unique arrow $\langle\rangle : A \to 1$.
Note: Any two terminal objects in a category are isomorphic.
Example: Any singleton set in **Set** is a terminal object.

Universal Element By the *Yoneda Lemma*, an element $c \in F(C)$ is given by $c = h^C(id_C)$. c is an **universal element** of a functor $F : \mathscr{C} \to \mathbf{Set}$ if and only if the induced $h^C : Hom(C, -) \to F$ is a *natural isomorphism*. Consequently, for any object $C' \in \mathscr{C}$ and any element $x \in F(C')$, there is an unique arrow $f : C \to C'$ in $\mathscr{C}$ for which $x = F(f)(c)$.
Note: There is a one-to-one correspondance between representations and universal elements of F.
Note: A similar argument for contravariant set valued functors of the form $F : \mathscr{C}^{op} \to \mathbf{Set}$ can be made where $c \in F(C)$ is an **universal element** if for any object $C' \in \mathscr{C}$ and any element $x \in F(C')$, there is an unique arrow $f : C' \to C$ for which $x = F(f)(c)$.

Vertical Composition ($\circ$) Composition of natural transformations $\alpha : F \to G$ and $\beta : G \to H$ is $\beta \circ \alpha : F \to H$ and is known as **vertical composition**, where F, G and H are functors from the categories $\mathscr{C}$ to $\mathscr{D}$. This follows from the fact that the outer rectangle of the following diagram is commutative:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\beta A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\
F(B) \arrow[r, "\alpha B"'] & G(B) \arrow[r, "\beta B"'] & H(B)
\end{tikzcd}$$

for each arrow $f : A \to B$.
Note: The vertical composite of natural transformations is also a natural transformation.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"'] \arrow[r, phantom, bend left=30, "\Downarrow\alpha"] & \mathscr{D}
\end{tikzcd}
\circ
\begin{tikzcd}
\mathscr{C} \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, bend right=30, "\Downarrow\beta"] & \mathscr{D}
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathscr{D}
\end{tikzcd}$$

Wide Subcategory is a subcategory $\mathscr{D}$ of $\mathscr{C}$ such that $\mathscr{D}_0 = \mathscr{C}_0$.
Example: **Set** is a wide subcategory of **Pfn**.

Yoneda Embedding For any category $\mathscr{C}$, there is a contravariant *Yoneda Functor* $Y : \mathscr{C}^{op} \to \mathbf{Func}(C, \mathbf{Set})$ defined as:

- for objects $C \in \mathscr{C}, Y(C) = \mathrm{Hom}(C, -)$

- for an arrow $f : D \to C$ and an object $A \in \mathscr{C}, Y(f) = \mathrm{Hom}(f, -)$ and $Y(f)(A) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(D, A)$

Thus, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(D, A) \arrow[r, "\mathrm{Hom}(D,k)"] & \mathrm{Hom}(D, B) \\
\mathrm{Hom}(C, A) \arrow[u, "Y(f)A"] \arrow[r, "\mathrm{Hom}(C,k)"'] & \mathrm{Hom}(C, B) \arrow[u, "Y(f)B"']
\end{tikzcd}$$

for an arrow $k : A \to B$.

Note: $Y(f) : \mathrm{Hom}(C, -) \to \mathrm{Hom}(D, -)$ is the induced natural transformation between hom functors corresponding to f. Generalizing $\mathrm{Hom}(D, -)$ to an arbitary set valued functor, by the *Yoneda Lemma*, Y is both full and faithful.

Note: By the contravariant *Yoneda Lemma*, there also exists the covariant functor $J : \mathscr{C} \to \mathbf{Func}(\mathscr{C}^{op}, \mathbf{Set})$.

Yoneda Lemma Let $F : \mathscr{C} \to \mathbf{Set}$ be a *set-valued functor* and C is an object in C. There is a one-to-one correspondance between elements of $F(C)$ and $h^C : \mathrm{Hom}(C, -) \to F$, that is:

$$Nat(h^C, F) \cong F(C)$$

Thus, this is an isomorphism that is both an injective and surjective mapping between elements of $F(C)$ and h^C.

Note: $h^C(X) = \mathrm{Hom}(C, X) \to F(X)$ for an object $X \in \mathscr{C}$ is defined as $h^C(X)(f) = F(f)(c)$ where $f : C \to X$ is an arrow and $c \in F(C)$ is an element.

Note: There is also a contravariant version of Yoneda Lemma involving the contravariant hom functor $h_C : \mathrm{Hom}(-, C) \to F$.

References

[1] Charles Wells Michael Barr. *Category Theory for Computing Science*. Reprints in Theory and Applications of Categories #22. 2013.