# Topology

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A glossary reference for Topology and related terms. Definitions are from various texts which I have read [1].

# Glossary

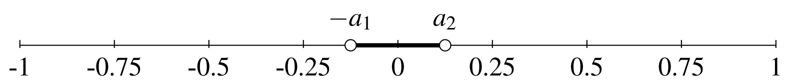
**Accumulation (Limit) Point** Let $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is an **accumulation point** or **limit point** of $A$ iff every open set $G$ containing $p$ contains a point of $A$ different from $p$, i.e.:

$$G \text{ open, } p \in G \text{ implies } A \cap (G \setminus \{p\}) \neq \emptyset$$

The set of accumulation points of $A$, denoted by $A'$, is called the *derived set* of $A$.

*Examples*:

- Every real number $p \in \mathbb{R}$ is a limit point of $\mathbb{Q}$ since every open set contains rational numbers.
- The set of integers $\mathbb{Z}$ does not have any accumulation points, i.e. derived set of $\mathbb{Z}$ is $\emptyset$ (as open sets in $\mathbb{R}$ can span between integers).
- Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, the point 0 is an accumulation point of $A$ since any open set $G$ with $0 \in G$ contains an open interval $(-a_1, a_2) \subseteq G$ with $-a_1 < 0 < a_2$, which contains points in $A$. Note that the limit point 0 of $A$ does not belong to $A$ and there are no other limit points, i.e. $A' = \{0\}$.




*Note*: *"Limit point of a set"* is not to be confused with the concept *"limit of a sequence"*.

*Bolzano-Weierstrass Theorem*: Let $A$ be a bounded, infinite set of real numbers. Then $A$ has at least one accumulation point. However, do note that not every set, even if it is infinite, has a limit point.

**Closed Set** is a subset $A \subseteq \mathbb{R}$, iff its complement, $A^c$, is an open set. Alternatively, $A \subseteq \mathbb{R}$ is closed iff $A$ contains each of its points of accumulation.

*Examples*:

- The closed interval $[a,b]$ is a closed set since its complement $(-\infty, a) \cup (b, \infty)$, the union of two open infinite intervals, is open.
- The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed since 0 is a limit point of $A$ but does not belong to $A$.
- The empty set $\emptyset$ and the entire line $\mathbb{R}$ are closed sets since their complements $\mathbb{R}$ and $\emptyset$, respectively, are open sets.
- Consider the open-closed interval $A = (a,b]$. Note that $A$ is not open since $b \in A$ is not an interior point of $A$, and is not closed since $a \notin A$ but is a limit point of $A$.

*Note*: *Open* and *Closed* sets are not inverses of each other as sets can be neither open nor closed.

**Interior Point** Let $A$ be a set of real numbers. A point $p \in A$ is an **interior point** of $A$ iff $p$ belongs to some open interval $S_p$ which is contained in $A$:

$$p \in S_p \subseteq A$$

**Open Set** A set $A$ is **open** (or $\mathcal{U}$-open) iff each of its points is an interior point. Observer that a set is not open iff there exists a point in the set that is not an interior point.

*Examples*:

- An open interval $A = (a,b)$ is an open set, for we may choose $S_p = A$ for each $p \in A$.
- The real line $\mathbb{R}$ is oepn since any open interval $S_p$ is a subset of $\mathbb{R}$, i.e. $p \in S_p \subseteq \mathbb{R}$
- The empty set $\emptyset$ is open since there is no point in $\emptyset$ which is not an interior point.
- The closed interval $B = [a,b]$ is not an open set, for any open interval containing $a$ or $b$ must contain points outside of $B$, i.e. the end points $a$ and $b$ are not interior points of $B$.
- Infinite open intervals $(a,\infty)$, $(-\infty,a)$ and $(-\infty,\infty)$ are open. On the other hand, infinite closed intervals $[a,\infty)$, $(-\infty,a]$ are not open sets since $a$ is not an interior point.

*Note*: The union of any number of open sets in $\mathbb{R}$ is open and the intersection of any finite number of open sets in $\mathbb{R}$ is open. For consider the class of open intervals:

$$\{A_n = (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\} \text{ i.e. } \{(-1,1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{1}{3}), \dots\}$$

and the intersection, $\bigcap_{n=1}^{\infty} A_n = \{0\}$, is a single point which is not open.

**Topological Spaces (Topology)** Let $X$ be a non-empty set. A class $\mathcal{T}$ of subsets of $X$ is a **topology** on $X$ iff $\mathcal{T}$ satisfies the following axioms:

$(O_1)$ $X$ and $\emptyset$ belong to $\mathcal{T}$.

$(O_2)$ The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

$(O_3)$ The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are then called *$\mathcal{T}$-open sets*, of simply *open sets*, and $X$ together with $\mathcal{T}$, i.e. the pair $(X,\mathcal{T})$ is called a **topological space**.

*Examples*:

- Let $\mathcal{U} = \{\cup_i I_i | I_i \in I\}$ denote the class of all open sets of real numbers where $I = \{(a,b) | a,b \in \mathbb{R}\}$. Then $\mathcal{U}$ is a topology in $\mathbb{R}$ denoted ($\mathbb{R}$,$\mathcal{U}$), and is also called the *usual topology* on $\mathbb{R}$.
- Similarily, the class $\mathcal{U}$ of all open sets in the plane $\mathbb{R}^2$ is a topology and also called the *usual topology* on $\mathbb{R}^2$.

# References


[1] Lipschutz S. *Theory and applications of general topology*. Schaum's outlines. 1965.