Topology

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A glossary reference for Topology and related terms. Definitions are from various texts which I have read [1].

Glossary

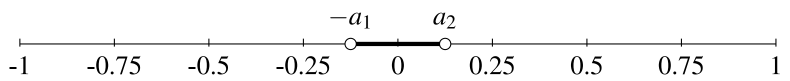
Accumulation (Limit) Point Let $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is an **accumulation point** or **limit point** of A iff every open set G containing p contains a point of A different from p, i.e.:

$$G \text{ open}, p \in G \text{ implies } A \cap (G \setminus \{p\}) \neq \emptyset$$

The set of accumulation points of A, denoted by A', is called the *derived set* of A.

Examples:

- Every real number $p \in \mathbb{R}$ is a limit point of $\mathbb{Q}$ since every open set contains rational numbers.
- The set of integers $\mathbb{Z}$ does not have any accumulation points, i.e. derived set of $\mathbb{Z}$ is $\emptyset$ (as open sets in $\mathbb{R}$ can span between integers).
- Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, the point 0 is an accumulation point of A since any open set G with $0 \in G$ contains an open interval $(-a_1, a_2) \subseteq G$ with $-a_1 < 0 < a_2$, which contains points in A. Note that the limit point 0 of A does not belong to A and there are no other limit points, i.e. $A' = \{0\}$.



Note: *"Limit point of a set"* is not to be confused with the concept *"limit of a sequence"*.

Bolzano-Weierstrass Theorem: Let A be a bounded, infinite set of real numbers. Then A has at least one accumulation point. However, do note that not every set, even if it is infinite, has a limit point.

Closed Set is a subset $A \subseteq \mathbb{R}$, iff its complement, A^c, is an open set. Alternatively, $A \subseteq \mathbb{R}$ is closed iff A contains each of its points of accumulation.

Examples:

- The closed interval $[a,b]$ is a closed set since its complement $(-\infty, a) \cup (b, \infty)$, the union of two open infinite intervals, is open.
- The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed since 0 is a limit point of A but does not belong to A.
- The empty set $\emptyset$ and the entire line $\mathbb{R}$ are closed sets since their complements $\mathbb{R}$ and $\emptyset$, respectively, are open sets.
- Consider the open-closed interval $A = (a,b]$. Note that A is not open since $b \in A$ is not an interior point of A, and is not closed since $a \notin A$ but is a limit point of A.

Note: *Open* and *Closed* sets are not inverses of each other as sets can be neither open nor closed.

Interior Point Let A be a set of real numbers. A point $p \in A$ is an **interior point** of A iff p belongs to some open interval S_p which is contained in A:

$$p \in S_p \subseteq A$$

Open Set A set A is **open** (or $\mathscr{U}$-open) iff each of its points is an interior point. Observer that a set is not open iff there exists a point in the set that is not an interior point.

Examples:

- An open interval $A = (a, b)$ is an open set, for we may choose $S_p = A$ for each $p \in A$.
- The real line $\mathbb{R}$ is oepn since any open interval S_p is a subset of $\mathbb{R}$, i.e. $p \in S_p \subseteq \mathbb{R}$
- The empty set $\emptyset$ is open since there is no point in $\emptyset$ which is not an interior point.
- The closed interval $B = [a, b]$ is not an open set, for any open interval containing a or b must contain points outside of B, i.e. the end points a and b are not interior points of B.
- Infinite open intervals (a, ∞), $(-\infty, a)$ and $(-\infty, \infty)$ are open. On the other hand, infinite closed intervals $[a, \infty)$, $(-\infty, a]$ are not open sets since a is not an interior point.

Note: The union of any number of open sets in $\mathbb{R}$ is open and the intersection of any finite number of open sets in $\mathbb{R}$ is open. For consider the class of open intervals:

$$\{A_n = (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\} \text{ i.e. } \{(-1,1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{1}{3}), \ldots\}$$

and the intersection, $\bigcap_{n=1}^{\infty} A_n = \{0\}$, is a single point which is not open.

References

[1] Lipschutz S. *Theory and applications of general topology*. Schaum's outlines. 1965.