Four Converse-Optimal Systems (Quadratic–Gaussian): Math Summary

Shared QG Converse Identities

In the Quadratic–Gaussian (QG) case with $q(s) = s^\top Q s$, $\rho(a) = a^\top R a$, and $V^*(s) = s^\top P s + b$ with $Q \succeq 0$, $R \succ 0$, $P \succ 0$, the discounted Schur metric and the analytic policy are

$$H_\gamma = \gamma\big(P^{-1} + \gamma\, g R^{-1} g^\top\big)^{-1} \succ 0, \qquad a^*(s) = -\gamma\big(R + \gamma g^\top P g\big)^{-1} g^\top P\, f(s).$$

The metric-normalized drift and Bellman energy identity are

$$f(s) = H_\gamma^{-1/2}\, S(s)\, (P - Q)^{1/2} s, \quad S(s)^\top S(s) = I, \qquad s^\top (P - Q) s = f(s)^\top H_\gamma f(s),$$

and the noise offset is $b = \frac{\gamma}{1-\gamma} \mathrm{tr}(P\Sigma)$.

S1 Unicycle

$\Delta t = 0.10$, $\gamma = 0.980$, $\rho(A_0) = 1.153$, $b = 0.041514$.

Matrices. $P = \mathrm{diag}(0.1111, 0.1111, 1.0000, 1.0000, 1.0000)$, $R = \mathrm{diag}(0.0036, 0.0036)$, $\Sigma = \mathrm{diag}(1.0000e-04, 1.0000e-04, 2.5000e-05, 4.0000e-04, 4.0000e-04)$, and

$$G = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix}.$$

Validation. Bellman residual stats on a random box: mean=5.794e-16, max-abs=7.105e-15, std=1.265e-15.

Figure. See the overlay figure (optimal vs no-control) provided alongside this report.

S2 2-DOF Manipulator

$\Delta t = 0.05$, $\gamma = 0.990$, $\rho(A_0) = 1.139$, $b = 0.039600$.

Matrices. $P = \mathrm{diag}(0.1013, 0.1013, 1.0000, 1.0000)$, $R = \mathrm{diag}(0.0016, 0.0016)$, $\Sigma = \mathrm{diag}(0.0000e+00, 0.0000e+00, 2.0000e-04, 2.0000e-04)$, and

$$G = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0500 & 0.0000 \\ 0.0000 & 0.0500 \end{bmatrix}.$$

Validation. Bellman residual stats on a random box: mean=-5.616e-17, max-abs=1.776e-15, std=5.249e-16.

Figure. See the overlay figure (optimal vs no-control) provided alongside this report.

S3 Point Mass

$\Delta t = 0.10$, $\gamma = 0.985$, $\rho(A_0) = 1.238$, $b = 0.053059$.

Matrices. $P = \mathrm{diag}(0.0400, 0.0400, 1.0000, 1.0000)$, $R = \mathrm{diag}(0.0043, 0.0043)$, $\Sigma = \mathrm{diag}(1.0000e-04, 1.0000e-04, 4.0000e-04, 4.0000e-04)$, and

$$G = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix}.$$

Validation. Bellman residual stats on a random box: mean=7.763e-16, max-abs=3.553e-15, std=1.025e-15.

Figure. See the overlay figure (optimal vs no-control) provided alongside this report.

S4 Planar Rocket

$\Delta t = 0.08$, $\gamma = 0.985$, $\rho(A_0) = 1.083$, $b = 0.054540$.

Matrices. $P = \mathrm{diag}(0.0278, 0.0278, 1.0000, 1.0000, 1.0000)$, $R = \mathrm{diag}(0.0036, 0.0036)$, $\Sigma = \mathrm{diag}(1.0000e-04, 1.0000e-04, 2.5000e-05, 4.0000e-04, 4.0000e-04)$, and

$$G = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0800 & 0.0000 \\ 0.0000 & 0.0800 \end{bmatrix}.$$

Validation. Bellman residual stats on a random box: mean=5.487e-16, max-abs=5.329e-15, std=8.893e-16.

Figure. See the overlay figure (optimal vs no-control) provided alongside this report.