# Math Behind Earthquake

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# Introduction

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The aim of this study was to develop mathematical models for estimating

- Earthquake casualties such as death, number of injured persons, affected families and total cost of damage and To quantify the direct damages from earthquakes to human beings and properties given the magnitude, intensity, depth of focus, location of epicentre and time duration, the regression models were made.
- Seismic Loads to design earthquake resistant structures using engineering techniques to withstand earthquake agitations.
- Floor displacements of a building. We attempt to model the effect of earthquake on a multistory building and then solve and interpret the mathematics.

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# Mathematical Model for estimation of Death

In this model, we try to find to relation between the independent variables and the dependent variable. Thus to perform this task we use the least square approximation method. To determine the deaths in a particular area we collect the data of a 5-6 zones in the area and then get a generalized equation for deaths.

$$y = X\beta$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} X = \begin{bmatrix} 1 & x_{11} & x_{21} & \dots & x_{k1} \\ 1 & x_{12} & x_{22} & \dots & x_{k2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{1n} & x_{2n} & \dots & x_{kn} \end{bmatrix} \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

$y_i \mapsto$ Number of deaths in $i^{th}$ zone

$X \mapsto$ Parameters

$\beta \mapsto$ Coefficient

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# Mathematical Model for estimation of Death

We find A using least - square approximation method, as follows .

$$A = \bar{X}X$$

$$A = \begin{bmatrix} n & \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{2i} & \dots & \sum_{i=1}^{n} x_{ki} \\ \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{1i}^2 & \sum_{i=1}^{n} x_{1i}x_{2i} & \dots & \sum_{i=1}^{n} x_{1i}x_{ki} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sum_{i=1}^{n} x_{ki} & \sum_{i=1}^{n} x_{1i}x_{ki} & \sum_{i=1}^{n} x_{2i}x_{ki} & \dots & \sum_{i=1}^{n} x_{ki}^2 \end{bmatrix} \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

$$(\bar{X}X)b = \bar{X}y$$

# Mathematical Model for estimation of Death

According to the data collected in different zones of Philippines we get the following matrix equation

$$\begin{bmatrix} 30 & 205 & 199.6 & 710 & 111 & 14436.4 \\ 205 & 1429 & 1374.7 & 4813 & 758 & 101312 \\ 199.6 & 1374.7 & 1341.62 & 4818.8 & 735 & 100163.82 \\ 710 & 4813 & 4818.8 & 23980 & 2399 & 356200 \\ 111 & 758 & 735 & 2399 & 491 & 50919.8 \\ 14436.4 & 101312.2 & 100163.82 & 356200 & 50919.8 & 22596163.36 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$$= \begin{bmatrix} 56.56 \\ 407.78 \\ 399.52 \\ 1355.08 \\ 237.16 \\ 18521.53 \end{bmatrix}$$

# Mathematical Model for estimation of Death

Solving the above equation we get the following data ,

$$b_0 = -13.405 \quad b_1 = -0.001 \quad b_2 = 2.214$$

$$b_3 = -0.013 \quad b_4 = 0.375 \quad b_5 = -0.001$$

The generalized equation for the number of death in Philippines is as follows

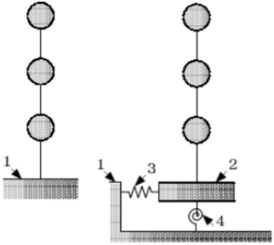
$$\ln \widehat{y_1} = -13.405 - 0.001x_1 + 2.214x_2 - 0.013x_3 + 0.374x_4 - 0.001x_5$$

# Seismic Load

When earthquakes occur, buildings undergoes dynamic motion. This is because the building is subjected to inertia forces that act in opposite direction to the acceleration of earthquake excitations. These inertia forces, called seismic loads, are usually dealt with by assuming forces external to the building. Time histories of earthquake motions are also used to analyze high-rise buildings, and their elements and contents for seismic design. The maximum story shear force for each mode is calculated from the natural frequency and mode that can be calculated through eigenvalue analysis of the model, estimating the damping ratio for each mode.The seismic loads are the maximum story shear forces that can be calculated from the shear force of each mode.

# Seismic Load

The seismic load used for the equivalent static analysis shall be calculated with the following procedure.It is called a response spectrum method in which an eigenvalue analysis of a building model is first implemented and then superposition of each modal response based on the response spectrum is made to determine the seismic load.




a. Fixed base model b. Sway-rocking model

# Seismic Loads

The equation of motion of a multi-degree-of-freedom (MDOF) system :

$$[m]\ddot{x} + [c]\dot{x} + [k]x = -[m]\ddot{x_g}$$

where $[m], [c], [k]$ are the mass, damping and stiffness matrces, respectively, $\ddot{x}, \dot{x}, x$ are the acceleration vector, velocity vector, displacement vector

The equation of motion for free vibration becomes as follows :

$$[m]\ddot{x} + [c]\dot{x} + [k]x = 0$$

The solution of the above equation can be assumed to be as follows :

$$x = \phi e^{\lambda t}$$

The characteristic equation is given as follows :

$$\left|\lambda^2[m] + \lambda[c] + [k]\right| = 0$$

# Seismic Load

That is, the undamped equation of motion of free vibration is given as follows :

$$[m]\ddot{x} + [k]x = 0$$

The displacement vector $x$ is assumed to be the product of $\phi$ and the function of time $e^{i\omega t}$

$$x = \phi e^{\omega t}$$

$$\left|[k] - \omega^2[m]\right| = 0$$

Since this equation (called as eigenvalue equation or frequency equation) is an n-degree equation, n real eigenvalues are obtained and they give n natural frequencies $\omega_j(j = 1, 2, , n)$.Then, natural mode $\phi_j(j = 1, 2, , n)$ can be calculated for each natural frequency.

# Seismic Load

$$[\phi] = [\{\phi_1\}\{\phi_2\}\cdots\{\phi_n\}] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$

$$\{x\} = [\phi]\{x^*\} = [\{\phi_1\}\ \{\phi_2\}\ \cdots\ \{\phi_n\}] \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix}$$

Then the equation of motion becomes as follows:

$$[m][\phi]\ddot{x} + [c][\phi]\dot{x} + [k][\phi]x = -[m]\ddot{x_g}$$

$$[\phi]^T[m][\phi]\ddot{x} + [\phi]^T[c][\phi]\dot{x} + [\phi]^T[k][\phi]x = -[\phi]^T[m]\ddot{x_g}$$

# Seismic Load

We have Orthogonality of natural mode,that is
${\phi_j}^T[m][\phi_k]$ ,${\phi_j}^T[m][\phi_k] = 0$ for $j! = k$, the next relationships are obtained.

$$[\phi]^T[m][\phi] = \begin{bmatrix} m_1^* & 0 & \dots & 0 \\ 0 & m_2^* & \dots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & m_n^* \end{bmatrix}$$

$$[\phi]^T[k][\phi] = \begin{bmatrix} k_1^* & 0 & \dots & 0 \\ 0 & k_2^* & \dots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k_n^* \end{bmatrix}$$

# Seismic Load

The natural modes are not always orthogonal with respect to damping matrix, therefore the damping matrix generally becomes as follows:

$$[\phi]^T[c][\phi] = \begin{bmatrix} c_{11}^* & c_{12}^* & \dots & c_{1n}^* \\ c_{21}^* & c_{23}^* & \dots & c_{2n}^* \\ \vdots & \ddots & \ddots & \vdots \\ c_{n1}^* & c_{n2}^* & \dots & c_{nn}^* \end{bmatrix}$$

In the analysis, it is assumed that orthogonality exists as Rayleigh damping.

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# Seismic Load

$$\begin{bmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n^* \end{bmatrix} \begin{Bmatrix} \ddot{x}_1^* \\ \ddot{x}_2^* \\ \vdots \\ \ddot{x}_n^* \end{Bmatrix} + \begin{bmatrix} c_1^* & 0 & \cdots & 0 \\ 0 & c_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_n^* \end{bmatrix} \begin{Bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \\ \vdots \\ \dot{x}_n^* \end{Bmatrix} + \begin{bmatrix} k_1^* & 0 & \cdots & 0 \\ 0 & k_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & k_n^* \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{Bmatrix} = - \begin{Bmatrix} \sum_{i=1}^n m_i\phi_{i1} \\ \sum_{i=1}^n m_i\phi_{i2} \\ \vdots \\ \sum_{i=1}^n m_i\phi_{in} \end{Bmatrix} \ddot{x}_g$$

$$\left.\begin{array}{c} m_1^*\ddot{x}_1^* + c_1^*\dot{x}_1^* + k_1^*x_1^* = -\ddot{x}_g \sum_{i=1}^n m_i\phi_{i1} \\ m_2^*\ddot{x}_2^* + c_2^*\dot{x}_2^* + k_2^*x_2^* = -\ddot{x}_g \sum_{i=1}^n m_i\phi_{i2} \\ \vdots \\ m_n^*\ddot{x}_n^* + c_n^*\dot{x}_n^* + k_n^*x_n^* = -\ddot{x}_g \sum_{i=1}^n m_i\phi_{in} \end{array}\right\}$$

$$\left.\begin{array}{c} \ddot{x}_1^* + 2\zeta_1\omega_1\dot{x}_1^* + \omega_1{}^2x_1^* = -\beta_1\ddot{x}_g \\ \ddot{x}_2^* + 2\zeta_2\omega_2\dot{x}_2^* + \omega_2{}^2x_2^* = -\beta_2\ddot{x}_g \\ \vdots \\ \ddot{x}_n^* + 2\zeta_n\omega_n\dot{x}_n^* + \omega_n{}^2x_n^* = -\beta_n\ddot{x}_g \end{array}\right\}$$

$$\omega_j = \sqrt{\frac{k_j^*}{m_j^*}} \qquad 2\zeta_j\omega_j = \frac{c_j^*}{m_j^*} \qquad \beta_j = \frac{\sum_{i=1}^n m_i\phi_{ij}}{\sum_{i=1}^n m_i\phi_{ij}^2}$$

# Seismic Load

Each equation of previous slide is the equation of motion of a SDOF system with natural circular frequency $\omega_j$ and damping ratio $\zeta_j$ for j = 1, 2,. . . , n subjected to the input motion multiplied by the participation factor $\beta_j$ .

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_3^* \end{Bmatrix} = \begin{Bmatrix} \phi_{11}x_1^* + \phi_{12}x_2^* + \cdots + \phi_{1n}x_n^* \\ \phi_{21}x_1^* + \phi_{22}x_2^* + \cdots + \phi_{2n}x_n^* \\ \vdots \\ \phi_{n1}x_1^* + \phi_{n2}x_2^* + \cdots + \phi_{nn}x_n^* \end{Bmatrix}$$

In this manner, the response for each natural mode is calculated independently, and the total response is obtained as the sum of these responses. This method of analysis is called modal analysis or mode superposition method. This method gives exact solution for linear systems if all modes are superimposed. It is also used as an approximate method when superimposing the important modes only.

# Spring Model

We assume that the $i^{th}$ floor of a building has mass mi and that successive floors are connected by an elastic connector whose effect resembles a spring.Typically, the structural elements in large building are made of steel, a highly elastic material. Each connector supplies a restoring force when the floors are displaced with respect to each other. We assume that the Hookes law holds, with proportionality constant $k_i$ between the $i^{th}$ and the $(i+1)^{th}$ floor.

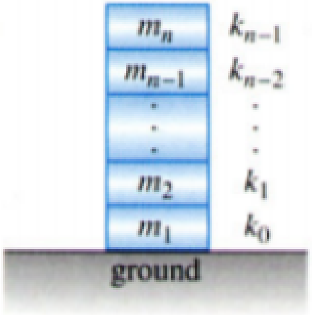


Figure 1

# Spring Method

That is, the restoring force between the two floors is:

$$F = k_i(x_{i+1} - x_i)$$

Where $x_i$ represents the horizontal displacement from the equilibrium and $x_i + 1 - x_i$ is the displacement(shift) of the $(i+1)^{th}$ floor relative to the $i^{th}$ floor.

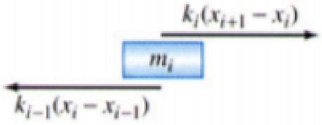


Figure 2

# Spring Model

We can apply Newtons second law of motion, $F = ma$ , to each section of the building to arrive at the following system of linear differential equations:

$$m_1 x_1'' = -k_0 x_1 + k_1(x_2 - x_1)$$

$$m_2 x_2'' = -k_1(x_2 - x_1) + k_2(x_3 - x_2)$$

$$\vdots \qquad \vdots$$

$$m_n x_n'' = k_{n-1}(x_n - x_{n-1})$$

# Spring Model

## Example

Lets take an example, consider a two-story building where each floor has mass m=5,000 Kg and each restoring force constant has the value k = 10,000 Kg/S then the system of differential equations simplifies to

$$x_1'' = -4x_1 + 2x_2$$

$$x_2'' = 2x_1 + 2x_2$$

Solution is $x_1(t) = c_1 \cos \omega_1 t + c_2 \sin \omega_1 t + c_3 \cos \omega_2 t + c_4 \sin \omega_2 t$

$$x_1(t) = \frac{1(4-\omega_1^2)c_1 \cos \omega_1 t}{2} + \frac{1(4-\omega_1^2)c_2 \sin \omega_1 t}{2} +$$

$$\frac{1(4-\omega_1^2)c_3 \cos \omega_2 t}{2} + \frac{1(4-\omega_2^2)c_2 \sin \omega_2 t}{2}$$

# Spring Model

We can represent the system in the matrix form as follows,

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} M = \begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & m_n \end{pmatrix}$$

$$K = \begin{pmatrix} -(k_0 + k_1) & k_1 & \dots & 0 \\ k1 & -(k_1 + k_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -(k_{n-1} + k_n) \end{pmatrix}$$

The n*n matrices M and K are called the mass matrix and the stiffness matrix of the building, respectively.

$$MX'' = KX$$

# Spring Model

We obtain the matrix equation of a homogeneous second order system in normal form

$$X'' = AX$$

The coefficient matrix is $A = M^{-1}K$ The eigenvalues of A reveal the stability of the building during an earthquake. The eigenvalues of A are negative and distinct. The natural frequencies of the building are the square roots of the negatives of the eigenvalues. If i is the $i^{th}$ eigenvalue of A , then $\omega_i = \sqrt{-\lambda_i}$ is the $i^{th}$ frequency for i = 1,2,3,. . . n. During an earthquake, a large horizontal force is applied to the first floor. If this force is oscillatory in nature, say of the form

$$F(t) = G \cos \gamma t$$

where G is a column matrix constants, then large displacements may develop in the building, especially if the frequency $\gamma$ of the forcing term F is close to one of the natural frequencies $\omega_i$ of the building. This is reminiscent of the resonance phenomenon.

# Spring Model (Example)

Suppose we have a ten-story building, where each floor has a mass 10,000kg and each ki has the value $5000 kg/s^2$. Since the sizes of the matrix M and K are both 10*10, the matrix A is also 10*10. With the help of a CAS we find

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{pmatrix}.$$

# Spring Model (Example)

The eigenvalues $\lambda_i$ of the matrix A were obtained with the aid of a CAS. The values of $\lambda_i$ , along with the corresponding frequencies $\omega_i = \sqrt{-\lambda_i}$ and periods $T_i = \frac{2\pi}{\omega_i}$ (in seconds) are summarized in Table 1.

**TABLE 1**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_i$ | −1.956 | −1.826 | −1.623 | −1.365 | −1.075 | −0.777 | −0.500 | −0.267 | −0.099 | −0.011 |
| $\omega_i$ | 1.399 | 1.351 | 1.274 | 1.168 | 1.037 | 0.881 | 0.707 | 0.517 | 0.315 | 0.105 |
| $T_i$ | 4.491 | 4.651 | 4.932 | 5.379 | 6.059 | 7.132 | 8.887 | 12.153 | 19.947 | 59.840 |

From the last row of numbers in the table we see that this building does not seem to be in any danger of developing resonance during a typical earthquake whose period is in the range of 2-3 seconds.

# Conclusion

First, we tried to analyse the damage cost and the number of deaths due to earthquakes. We realised that this is a really huge loss to humanity and to prevent it we need to use engineering skills and design structures which can withstand earthquake agitations. For designing these structures we need to estimate some things in prior. So, we used the response spectrum method of analysis to calculate the seismic loads. We also used the spring model of multi-storey building to calculate the floor displacements of the building. These helped us design structures and also improve them. This is dealt by the disaster management department.

# The End