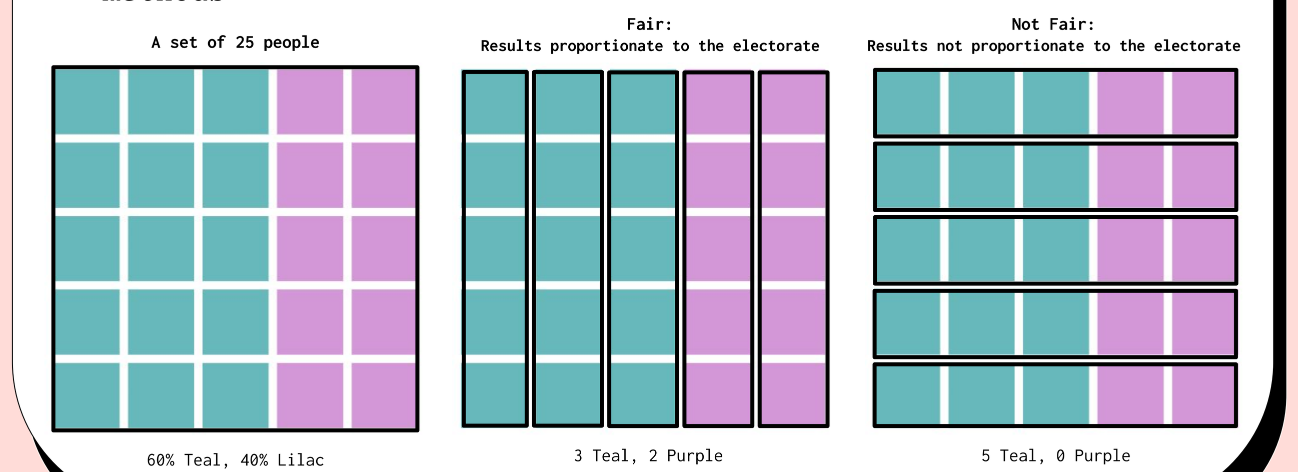
CONNECTING THE DOTS

An Exploration of Dual Graphs in Political Redistricting

Sarah Cannon, Mehrin Khan, Claire Vlases, Xinran (Joy) Zhang (Claremont McKenna College); Andy Emerson, Sara Anderson (Claremont Graduate University)

Motivation

- Gerrymandering is when political districts are drawn in an **unfair way**
- Many mathematical tools exist for studying gerrymandering
- Nearly all **use *dual graphs* in their analysis**
- Little is known about the structure of dual graphs
- Learning more about dual graphs can **help improve methods** for studying redistricting and guide the development of new future methods



Dual Graphs Background

- A way to **represent geographic** structure
- In Iowa: Make a vertex for each county, connect neighboring counties
- Can build redistricting plans out of this graph

For **more detailed** redistricting plans, use **finer levels** of geography

Counties Census Tracts Census Block Groups Census Blocks

Bigger regions Smaller regions

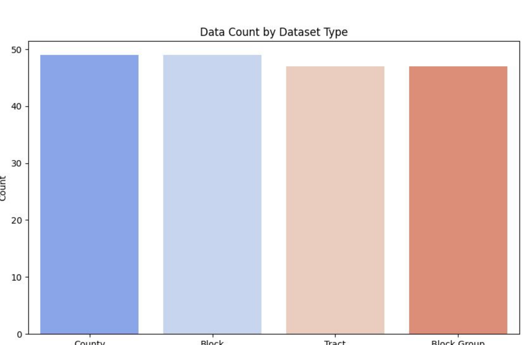
Ethical Considerations

Dual Graphs Considered:

- From Redistricting Data Hub, prepared by Daryl DeFord (Washington State), which uses Census data
- Our data covers four levels of census geographies: counties, census tracts, census block groups, and census blocks. For each, we have data available from all states for which there were not data formatting issues:
  - Counties: All states except Nevada
  - Census tracts: All states except Nebraska, Nevada, and Wisconsin
  - Census block groups: All states except Nebraska, Nevada, Virginia, and Wisconsin
  - Census blocks: All states except Nevada

Data validation:

- Check total number of vertices is
- correct
- Check total population (added across all vertices) is correct



Key Questions

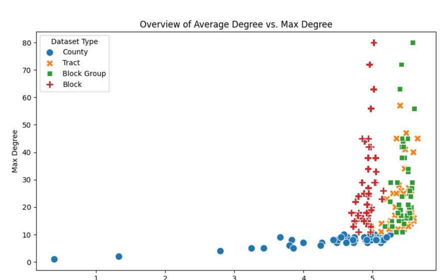
- How similar are **dual graphs to grids**?
- What **properties** do dual graphs have that are **similar** across all levels of geography?
- Are there any **key differences** between the different levels of geography?
- These graphs are sometimes described as "nearly triangulated and nearly planar." Is this true?

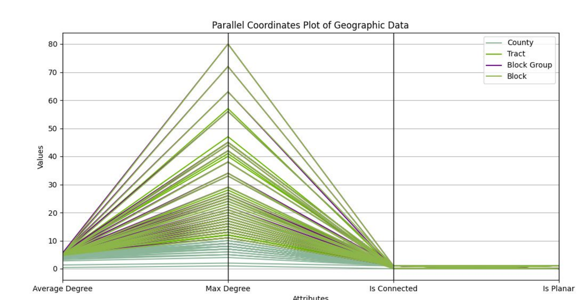
Sequencing

A degree sequence is a list or sequence that represents the degrees of all nodes in a graph.

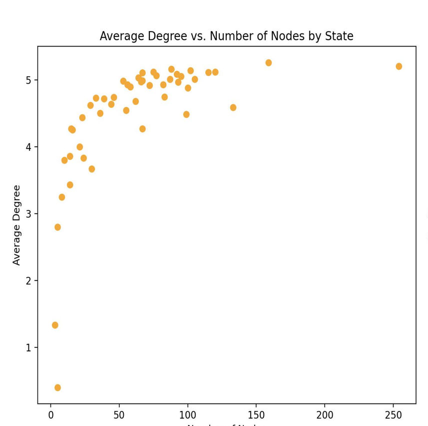
In a degree sequence, each element of the sequence corresponds to the degree of a node in the graph. The sequence is usually sorted in non-increasing order, meaning that the largest degree is listed first, followed by the second largest, and so on, until the smallest degree.

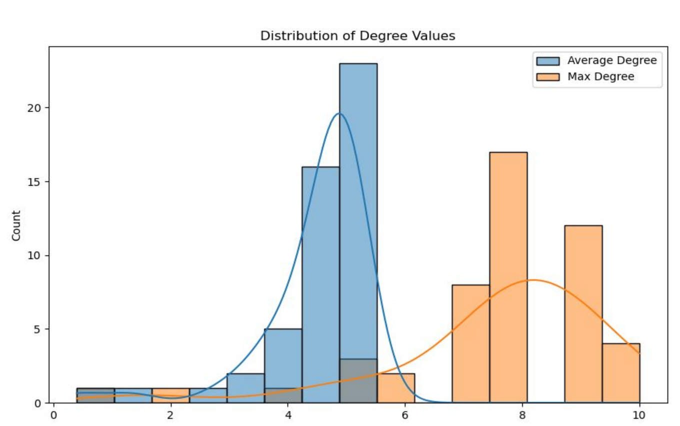
Looking at the entire data set: there are trends apparent in all 4 data sets, including a correlation between average and max degree.





Zooming in on county data: there are noticeable trends within the specific level too. For example, in county data, there is a correlation between average degree, max degree, and a distribution of degree values.



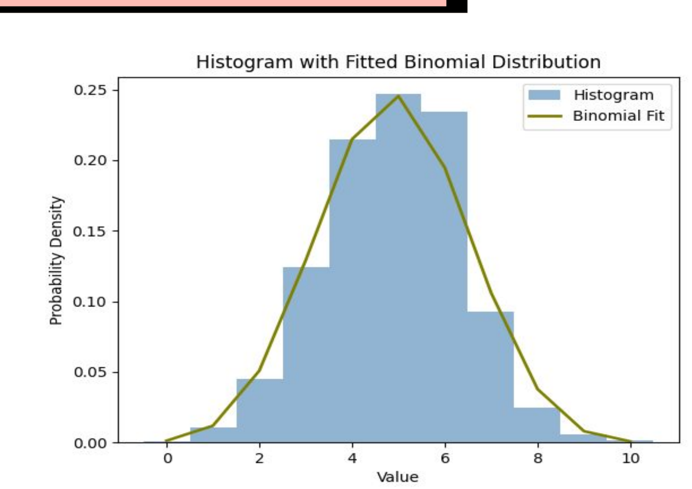


Degree Distributions

What is the distribution of degrees in a dual graph?

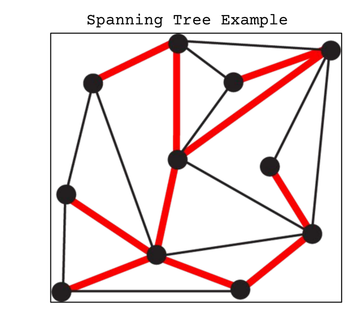
For county data, degrees appear to be **normally distributed**!

Notably, this pattern doesn't hold for other geographies due to the presence of large degree nodes.



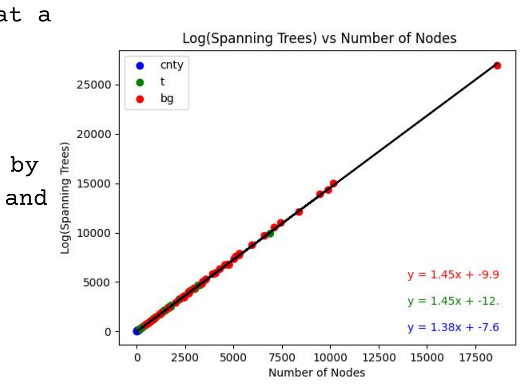
Spanning Trees

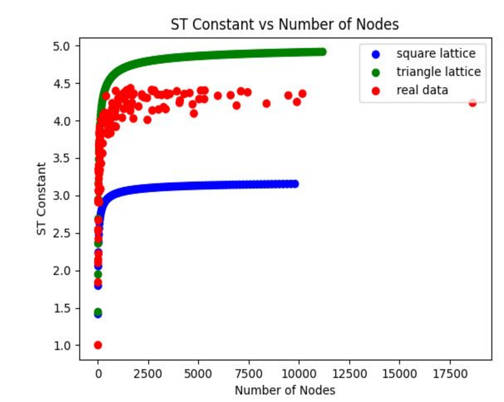
Many algorithms for studying redistricting rely heavily on spanning trees. **A spanning tree of a connected graph is a subgraph which uses all nodes and contains no cycles.** A disconnected graph will have no spanning trees, and a graph that is itself a tree will have exactly one spanning tree. **The more edges a graph has, compared to the number of nodes, the more spanning trees it will have.**



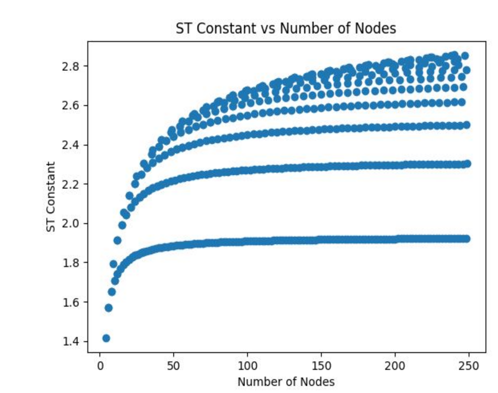
A large number of spanning trees indicates that a graph has many different paths between its vertices, and is well-connected.

We can use the number of spanning trees to calculate the **"spanning tree constant,"** given by $S^{1/n}$ where S is the number of spanning trees and n is the number of nodes in the graph. For a square lattice, as we increase the size, our spanning tree constant converges somewhere between 3 and 3.25. For a triangular lattice, this number converges to just under 5.





Our data falls between a square and a triangular grid

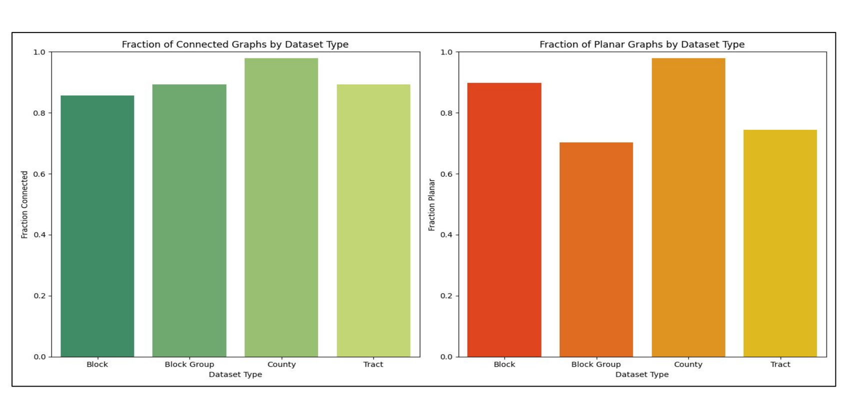


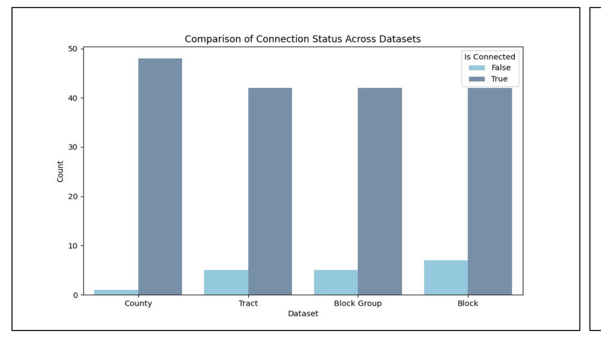
The shape of the grid has a large effect on the spanning tree constant

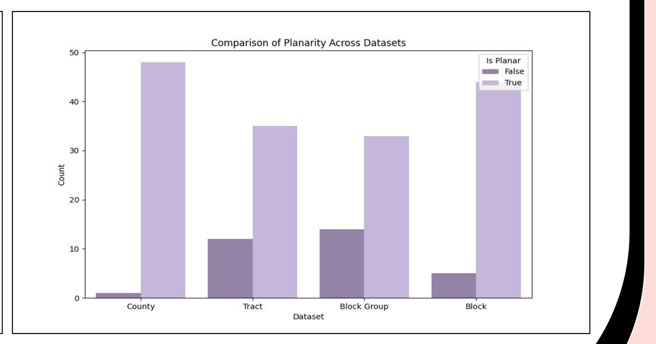
Connectivity and Planarity

A graph **is connected if for every pair of nodes there exists a path between them.** A graph **is planar if it can be drawn in the plane such that no edges cross.** While one might expect dual graphs to be both connected and planar because they arise from real-world geographies, there are many reasons why they might not be.

For example, a graph might not be connected because a state has islands; a graph might not be planar because five geographic regions meet at a single point, resulting in a complete graph K_5 as a subgraph of the dual graph.



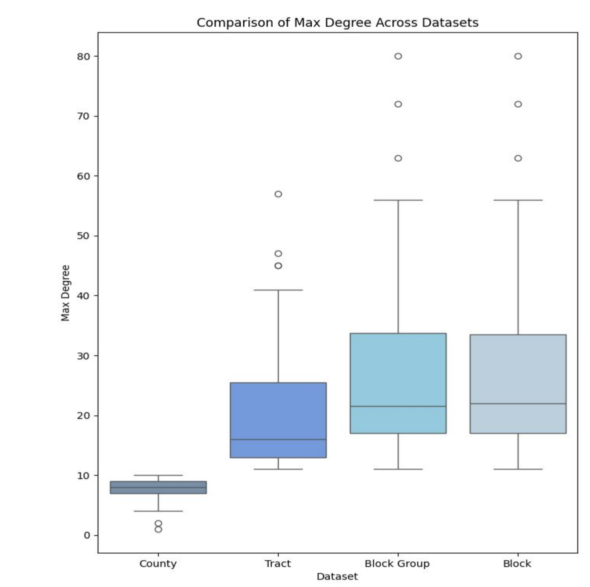


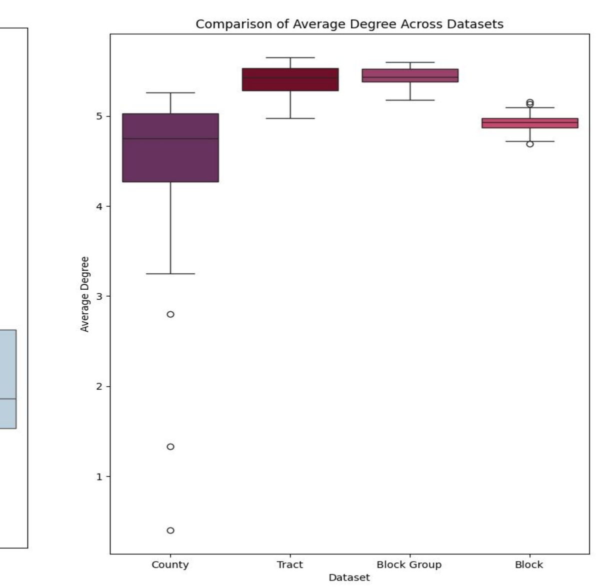


There is a variation of connectivity and planarity across the four datasets

Degrees

The degree of a node is the number of edges incident to it, which is equivalent to the number of connections or neighbors that the node has. The degree of nodes is one of the most fundamental measures in the study of a graph's underlying structure.

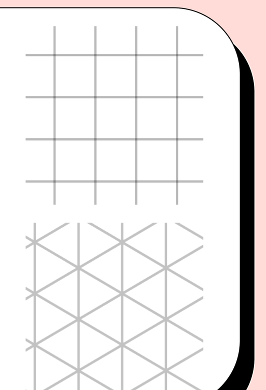




- We expected average degree to increase as graphs got larger, so were **surprised to see block graphs had smaller average degree.**
- Square grids have max degree 4 and average degree just less than 4; triangular grids have max degree 6 and average degree just less than 6
- In terms of degrees, **our graphs are closer to triangular grids, but have significantly larger max degrees**

Key Takeaways

- We understand the **typical statistics** of dual graphs
- County data has different features than the other levels of geography
- Block data has some different features than the other levels of geography
- Dual graphs fall somewhere **between square grids and triangular grids** in terms of number of spanning trees
- The term **"nearly triangulated"** may be apt; the term **"nearly planar"** is clearly not: a significant **majority of dual graphs are planar!**



Next Steps

Splitability of spanning trees: A new polynomial-time algorithm for sampling districting plans on grids relies on the key fact that a polynomial fraction of spanning trees on grids can be split exactly in half. For our real-world graphs, how likely is it that a random spanning tree can be split exactly in half?

Clustering coefficients:

- Local: The local clustering coefficient measures the probability that neighbors of a given node are also connected to each other, forming a triangle. For a node i, the local clustering coefficient C_i is defined as:

$$C_i = \frac{2e_i}{k_i(k_i - 1)}$$

  where e_i is the number of edges between the neighbors of node i, and k_i is the degree of node i (the number of neighbors). High local clustering coefficient for a city would indicate that its neighboring cities are also well connected among themselves, forming a tightly knit regional network. This could be important for logistics planning, for example.
- Global: This could be indicative of strong regional cohesion or the presence of natural barriers that limit inter-regional connections.

Centrality measures are used to identify the most important vertices within a graph with respect to different utilities.
These measures provide a way to quantify the significance of individual nodes based on their position in the network. We want to continue by exploring the four most common types of centrality: degree centrality, betweenness centrality, closeness centrality, and eigenvector centrality.