

5. When Gravity is Weak

The elegance of the Einstein field equations ensures that they hold a special place in the hearts of many physicists. However, any fondness you may feel for these equations will be severely tested if you ever try to solve them. The Einstein equations comprise ten, coupled partial differential equations. While a number of important solutions which exhibit large amounts symmetry are known, the general solution remains a formidable challenge.

We can make progress by considering situations in which the metric is almost flat. We work with $\Lambda = 0$ and consider metrics which, in so-called *almost-inertial coordinates* x^μ , takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (5.1)$$

Here $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the Minkowski metric. The components $h_{\mu\nu}$ are assumed to be small perturbation of this metric: $h_{\mu\nu} \ll 1$.

Our strategy is to expand the Einstein equations to linear order in the small perturbation $h_{\mu\nu}$. At this order, we can think of gravity as a symmetric “spin 2” field $h_{\mu\nu}$ propagating in flat Minkowski space $\eta_{\mu\nu}$. To this end, all indices will now be raised and lowered with $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$. For example, we have

$$h^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}$$

Our theory will exhibit a Lorentz invariance, under which $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$ and the gravitational field transforms as

$$h^{\mu\nu}(x) \rightarrow \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma h^{\rho\sigma}(\Lambda^{-1}x)$$

In this way, we construct a theory around flat space that starts to look very much like the other field theories that we meet in physics.

5.1 Linearised Theory

To proceed, we need to construct the various curvature tensors from the metric (5.1). For each, we work at linear order in h . To leading order, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

The Christoffel symbols are then

$$\Gamma_{\nu\rho}^\sigma = \frac{1}{2} \eta^{\sigma\lambda} (\partial_\nu h_{\lambda\rho} + \partial_\rho h_{\nu\lambda} - \partial_\lambda h_{\nu\rho}) \quad (5.2)$$

The Riemann tensor is

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda}$$

The Γ terms are second order in h , so to linear order we have

$$\begin{aligned} R^\sigma{}_{\rho\mu\nu} &= \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} \\ &= \frac{1}{2} \eta^{\sigma\lambda} (\partial_\mu \partial_\rho h_{\nu\lambda} - \partial_\mu \partial_\lambda h_{\nu\rho} - \partial_\nu \partial_\rho h_{\mu\lambda} + \partial_\nu \partial_\lambda h_{\mu\rho}) \end{aligned} \quad (5.3)$$

The Ricci tensor is then

$$R_{\mu\nu} = \frac{1}{2} (\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h)$$

with $h = h^\mu{}_\mu$ the trace of $h_{\mu\nu}$ and $\square = \partial^\mu \partial_\mu$. The Ricci scalar is

$$R = \partial^\mu \partial^\nu h_{\mu\nu} - \square h \quad (5.4)$$

By the time we get to the Einstein tensor, we've amassed quite a collection of terms

$$G_{\mu\nu} = \frac{1}{2} \left[\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h - (\partial^\rho \partial^\sigma h_{\rho\sigma} - \square h) \eta_{\mu\nu} \right] \quad (5.5)$$

The Bianchi identity for the full Einstein tensor is $\nabla^\mu G_{\mu\nu} = 0$. For the linearised Einstein tensor, this reduces to

$$\partial^\mu G_{\mu\nu} = 0 \quad (5.6)$$

It's simple to check explicitly that this is indeed obeyed by (5.5).

The Einstein equations then become the linear, but somewhat complicated, set of partial differential equations

$$\partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\nu h_{\mu\rho} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h - (\partial^\rho \partial^\sigma h_{\rho\sigma} - \square h) \eta_{\mu\nu} = 16\pi G T_{\mu\nu} \quad (5.7)$$

where, for consistency, the source $T_{\mu\nu}$ must also be suitably small. The left-hand side of this equation should be viewed as a second order, linear differential operator acting on $h_{\mu\nu}$. This is known as the *Lichnerowicz operator*.

The Fierz-Pauli Action

The linearised equations of motion can be derived from an action principle, first written down by Fierz and Pauli,

$$S_{FP} = \frac{1}{8\pi G} \int d^4x \left[-\frac{1}{4} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\nu h^{\rho\mu} + \frac{1}{4} \partial_\mu h \partial^\mu h - \frac{1}{2} \partial_\nu h^{\mu\nu} \partial_\mu h \right] \quad (5.8)$$

This is the expansion of the Einstein-Hilbert action to quadratic order in h (after some integration by parts). (At linear order, the expansion of the Lagrangian is equal to the linearised Ricci scalar (5.4) which is a total derivative.)

Varying the Fierz-Pauli action, and performing some integration by parts, we have

$$\begin{aligned}\delta S_{FP} &= \frac{1}{8\pi G} \int d^4x \left[\frac{1}{2} \partial_\rho \partial^\rho h_{\mu\nu} - \partial^\rho \partial_\nu h_{\rho\mu} - \frac{1}{2} \partial^\rho \partial_\rho h \eta_{\mu\nu} + \frac{1}{2} \partial_\nu \partial_\mu h + \frac{1}{2} \partial_\rho \partial_\sigma h^{\rho\sigma} \eta_{\mu\nu} \right] \delta h^{\mu\nu} \\ &= \frac{1}{8\pi G} \int d^4x \left[-G_{\mu\nu} \delta h^{\mu\nu} \right]\end{aligned}\tag{5.9}$$

We see that the Fierz-Pauli action does indeed give the vacuum Einstein equations $G_{\mu\nu} = 0$. We can then couple matter by adding $T_{\mu\nu} h^{\mu\nu}$ to the action.

5.1.1 Gauge Symmetry

Linearised gravity has a rather pretty gauge symmetry. This is inherited from the diffeomorphisms of the full theory. To see this, we repeat our consideration of infinitesimal diffeomorphisms from Section 4.1.3. Under an infinitesimal change of coordinates

$$x^\mu \rightarrow x^\mu - \xi^\mu(x)$$

with ξ assumed to be small. The metric changes by (4.6)

$$\delta g_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

When the metric takes the form (5.1), this can be viewed as a transformation of the linearised field $h_{\mu\nu}$. Because both ξ and h are small, the covariant derivative should be taken using the vanishing connection of Minkowski space. We then have

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + (\mathcal{L}_\xi h)_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\tag{5.10}$$

This looks very similar to the gauge transformation of Maxwell theory, where the gauge potential shifts as $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$. Just as the electromagnetic field strength $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ is gauge invariant, so is the linearised Riemann tensor $R^\sigma{}_{\rho\mu\nu}$.

We can quickly check that the Fierz-Pauli action is invariant under the gauge symmetry (5.10). From (5.9), we have

$$\delta S_{FP} = -\frac{1}{8\pi G} \int d^4x \, 2G_{\mu\nu} \partial^\mu \xi^\nu = +\frac{1}{8\pi G} \int d^4x \, 2(\partial^\mu G_{\mu\nu}) \xi^\nu = 0$$

where, in the second equality, we've integrated by parts (and discarded the boundary term) and in the third equality we've invoked the linearised Bianchi identity (5.6). In fact, this is just the same argument that we used to derive the Bianchi identity in Section 4.1.3, now played backwards.

When doing calculations in electromagnetism, it's often useful to pick a gauge. One of the most commonly used is Lorentz gauge,

$$\partial^\mu A_\mu = 0$$

Once we impose this condition, the Maxwell equations $\partial^\mu F_{\mu\nu} = j_\nu$ reduce to the wave equations

$$\square A_\nu = j_\nu$$

We solved these equations in detail in the lectures on [Electromagnetism](#).

We can impose a similar gauge fixing condition in linearised gravity. In this case, the analog of Lorentz gauge is called *de Donder gauge*

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0 \quad (5.11)$$

To see that this is always possible, suppose that you are handed a metric that doesn't obey the de Donder condition but instead satisfies $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = f_\nu$ for some functions f_ν . Then do a gauge transformation (5.10). Your new gauge potential will satisfy $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h + \square \xi_\nu = f_\nu$. So if you pick a gauge transformation ξ_μ that obeys $\square \xi_\mu = f_\mu$ then your new metric will be in de Donder gauge.

There is a version of de Donder gauge condition (5.11) that we can write down in the full non-linear theory. We won't need it in this course, but it's useful to know it exists. It is

$$g^{\mu\nu}\Gamma_{\mu\nu}^\rho = 0 \quad (5.12)$$

This isn't a tensor equation because the connection $\Gamma_{\mu\nu}^\rho$ is not a tensor. Indeed, if a tensor vanishes in one choice of coordinates then it vanishes for all choices while the whole point of a gauge fixing condition is to pick out a preferred choice of coordinates. If we substitute in the linearised Christoffel symbols (5.2), this reduces to the de Donder gauge condition.

The non-linear gauge condition (5.12) has a number of nice features. For example, in general the wave operator (or, on a Riemannian manifold, the Laplacian Δ) is $\square = \nabla^\mu \nabla_\mu = g^{\mu\nu}(\partial_\nu \partial_\mu - \Gamma_{\nu\mu}^\rho \partial_\rho)$. If we fix the gauge (5.12), the annoying connection term vanishes and we simply have $\square = g^{\mu\nu} \partial_\mu \partial_\nu$. A similar simplification happens if we compute the covariant divergence of a one-form in this gauge: $\nabla^\mu \omega_\mu = g^{\mu\nu} \nabla_\mu \omega_\nu = g^{\mu\nu}(\partial_\mu \omega_\nu - \Gamma_{\mu\nu}^\rho \omega_\rho) = \partial^\mu \omega_\mu$.

Back in our linearised world, de Donder gauge greatly simplifies the Einstein equation (5.7), which now become

$$\square h_{\mu\nu} - \frac{1}{2}\square h\eta_{\mu\nu} = -16\pi GT_{\mu\nu} \quad (5.13)$$

It is useful to define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$$

Taking the trace of both sides gives $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$ so, given $\bar{h}_{\mu\nu}$ we can trivially reconstruct $h_{\mu\nu}$ as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu} \quad (5.14)$$

Written in terms of $\bar{h}_{\mu\nu}$, the linearised Einstein equations in de Donder gauge (5.13) then reduce once again to a bunch of wave equations

$$\square \bar{h}_{\mu\nu} = -16\pi GT_{\mu\nu} \quad (5.15)$$

and we can simply import the solutions from electromagnetism to learn something about gravity. We'll look at some examples shortly.

5.1.2 The Newtonian Limit

Under certain circumstances, the linearised equations of general relativity reduce to the familiar Newtonian theory of gravity. These circumstances occur when we have a low-density, slowly moving distribution of matter.

For simplicity, we'll look at a stationary matter configuration. This means that we take

$$T_{00} = \rho(\mathbf{x})$$

with the other components vanishing. Since nothing depends on time, we can replace the wave operator is just the Laplacian in \mathbf{R}^3 : $\square = -\partial_t^2 + \nabla^2 = \nabla^2$. The Einstein equations are then simply

$$\nabla^2 \bar{h}_{00} = -16\pi G\rho(\mathbf{x}) \quad \text{and} \quad \nabla^2 \bar{h}_{0i} = \nabla^2 \bar{h}_{ij} = 0$$

With suitable boundary conditions, the solutions to these equations are

$$\bar{h}_{00} = -4\Phi(\mathbf{x}) \quad \text{and} \quad \bar{h}_{0i} = \bar{h}_{ij} = 0 \quad (5.16)$$

where the field Φ is identified with the Newtonian gravitational potential, obeying (0.1)

$$\nabla^2 \Phi = 4\pi G\rho$$

Translating this back to $h_{\mu\nu}$ using (5.14), we use $\bar{h} = +4\Phi$ to find

$$h_{00} = -2\Phi, \quad h_{ij} = -2\Phi\delta_{ij}, \quad h_{0i} = 0$$

Putting this back into the full metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)d\mathbf{x} \cdot d\mathbf{x}$$

If we take a $\Phi = -GM/r$ as expected for a point mass, we find that this coincides with the leading expansion of the Schwarzschild metric (4.8). (The g_{00} term turns out to be exact; the g_{ij} term is the leading order Taylor expansion of $(1 + 2\Phi)^{-1}$.)

Way back in Section 1.2, we gave a naive, intuitive discussion of curved spacetime. There we already anticipated that the Newtonian potential Φ would appear in the g_{00} component of the metric (1.26). However, in solving the Einstein equations, we learn that this is necessarily accompanied by an appearance of Φ in the g_{ij} component. Ultimately, this is the reason for the factor of 2 discrepancy between the Newtonian and relativistic predictions for light bending that we met in Section 1.3

5.2 Gravitational Waves

A long time ago, in a galaxy far far away, two black holes collided. Here a “long time ago” means 1.3 billion years ago. And “far far away” means a distance of about 1.3 billion light years.

To say that this was a violent event is something of an understatement. One of the black holes was roughly 35 times heavier than the Sun, the other about 30 times heavier. When they collided they merged to form a black hole whose mass was about 62 times heavier than the Sun. Now $30 + 35 \neq 62$. This means that some mass, or equivalently energy, went missing during the collision. In a tiny fraction of a second, this pair of black holes emitted an energy equivalent to three times the mass of the Sun.

That, it turns out, is quite a lot of energy. For example, nuclear bombs convert the mass of a handful of atoms into energy. But here we’re talking about solar masses, not atomic masses. In fact, for that tiny fraction of a second, these colliding black holes released more energy than all the stars in all the galaxies in the visible universe put together.

But the most astonishing part of the story is how we know this collision happened. It's because, on September 14th, 2015, at 9.30 in the morning UK time, we felt it. The collision of the black holes was so violent that it caused an enormous perturbation of spacetime. Like dropping a stone in a pond, these ripples propagated outwards as gravitational waves. These ripples started 1.3 billion years ago, roughly at the time that multi-cellular life was forming here on Earth. They then travelled through the cosmos at the speed of light. The ripples hit the outer edge of our galaxy about 50,000 years ago, at a time when humans were hanging out with neanderthals. The intervening 50,000 years gave us just enough time to band together into hunter-gatherer tribes, develop cohesive societies bound by false religions, invent sophisticated language and writing, discover mathematics, understand the theory that governs the spacetime continuum and, finally, build a machine that is capable of detecting the ripples, turning it on just in time for the gravitational wave to hit the south pole and pass, up through the Earth, triggering the detector.

The purpose of this section is to tell the story above in equations.

5.2.1 Solving the Wave Equation

Gravitational waves propagate in vacuum, in the absence of any sources. This means that we need to solve the linearised equation

$$\square \bar{h}_{\mu\nu} = 0 \quad (5.17)$$

One solution is provided by the gravitational wave

$$\bar{h}_{\mu\nu} = \text{Re} \left(H_{\mu\nu} e^{ik_\rho x^\rho} \right) \quad (5.18)$$

Here $H_{\mu\nu}$ a complex, symmetric polarisation matrix and the wavevector k^μ is a real 4-vector. Usually when writing these solutions we are lazy and drop the Re on the right-hand side, leaving it implicit that one takes the real part. This plane wave ansatz solves the linearised Einstein equation (5.17) provided that the wavevector is null,

$$k_\mu k^\mu = 0$$

This tells us that gravitational waves, like light waves, travel at the speed of light. If we write the wavevector as $k^\mu = (\omega, \mathbf{k})$, with ω the frequency, then this condition becomes $\omega = \pm |\mathbf{k}|$.

Because the wave equation is linear, we may superpose as many different waves of the form (5.18) as we wish. In this way, we build up the most general solution to the wave equation.

Naively, the polarisation matrix $H_{\mu\nu}$ has 10 components. But we still have to worry about gauge issues. The ansatz (5.18) satisfies the de Donder gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ only if

$$k^\mu H_{\mu\nu} = 0 \quad (5.19)$$

This tells us that the polarisation is transverse to the direction of propagation. Furthermore, the choice of de Donder gauge does not exhaust our ability to make gauge transformations. If we make a further gauge transformation $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, then

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial^\rho \xi_\rho \eta_{\mu\nu}$$

This transformation leaves the solution in de Donder gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ provided that

$$\square \xi_\nu = 0$$

In particular, we can take

$$\xi_\mu = \lambda_\mu e^{ik_\rho x^\rho}$$

which obeys $\square \xi_\mu = 0$ because $k_\rho k^\rho = 0$. A gauge transformation of this type shifts the polarisation matrix to

$$H_{\mu\nu} \rightarrow H_{\mu\nu} + i(k_\mu \lambda_\nu + k_\nu \lambda_\mu - k^\rho \lambda_\rho \eta_{\mu\nu}) \quad (5.20)$$

Polarisation matrices that differ in this way describe the same gravitational wave. We now choose the gauge transformation λ_μ in order to further set

$$H_{0\mu} = 0 \quad \text{and} \quad H^\mu{}_\mu = 0 \quad (5.21)$$

These conditions, in conjunction with (5.19), are known as *transverse traceless gauge*. Because H is traceless, this choice of gauge has the advantage that $h_{\mu\nu} = \bar{h}_{\mu\nu}$.

At this stage we can do some counting. The polarisation matrix $H_{\mu\nu}$ has 10 components. The de Donder condition (5.19) gives 4 constraints, and there are 4 residual gauge transformations (5.20). The upshot is that there are just $10 - 4 - 4 = 2$ independent polarisations in $H_{\mu\nu}$.

(There is a similar counting in Maxwell theory. The polarisation of A_μ seemingly has 4 components. The Lorentz gauge $\partial^\mu A_\mu = 0$ kills one of them, and a residual gauge transformation kills another, leaving the 2 familiar polarisation states of light.)

An Example

Consider a wave propagating in the z direction. The wavevector is

$$k_\mu = (\omega, 0, 0, \omega)$$

The condition (5.19) sets $H_{0\nu} + H_{3\nu} = 0$. The additional constraint (5.21) restricts the polarisation matrix to be

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_X & 0 \\ 0 & H_X & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.22)$$

Both H_+ and H_X can be complex; we take the real part when computing the metric in (5.18). Here we see explicitly the two polarisation states H_+ and H_X . We'll see below how to interpret these two polarisations.

5.2.2 Bobbing on the Waves

What do you feel if a gravitational wave passes you by? Well, if you're happy to be modelled as a pointlike particle, moving along a geodesic, then the answer is simple: you feel nothing at all. This follows from the equivalence principle. Instead, it's all about your standing relative to your neighbours.

This relative physics is captured by the geodesic deviation equation that we met in Section 3.3.4. Consider a family of geodesics $x^\mu(\tau, s)$, with s labelling the different geodesics, and τ the affine parameter along any geodesic. The vector field tangent to these geodesics is the velocity 4-vector

$$u^\mu = \left. \frac{\partial x^\mu}{\partial \tau} \right|_s$$

Meanwhile, the displacement vector S^μ takes us between neighbouring geodesics,

$$S^\mu = \left. \frac{\partial x^\mu}{\partial s} \right|_\tau$$

We previously derived the geodesic deviation equation (3.36)

$$\frac{d^2 S^\mu}{d\tau^2} = R^\mu{}_{\rho\sigma\nu} u^\rho u^\sigma S^\nu$$

We'll consider the situation where, in the absence of the gravitational wave, our family of geodesics are sitting happily in a rest frame, with $u^\mu = (1, 0, 0, 0)$. As the gravitational wave passes, the geodesics will change as

$$u^\mu = (1, 0, 0, 0) + \mathcal{O}(h)$$

Fortunately, we won't need to compute the details of this. We will compute the deviation to leading order in the metric perturbation h , but the Riemann tensor is already $\mathcal{O}(h)$, which means that we can neglect the corrections in the other terms. Similarly, we can replace the proper time τ for the coordinate time t . We then have

$$\frac{d^2 S^\mu}{dt^2} = R^\mu{}_{00\nu} S^\nu$$

The Riemann tensor in the linearised regime was previously computed in (5.3)

$$R^\mu{}_{\rho\sigma\nu} = \frac{1}{2} g^{\mu\lambda} (\partial_\sigma \partial_\rho h_{\nu\lambda} - \partial_\sigma \partial_\lambda h_{\nu\rho} - \partial_\nu \partial_\rho h_{\sigma\lambda} + \partial_\nu \partial_\lambda h_{\sigma\rho})$$

Using $h_{\mu 0} = 0$, the component we need is simply

$$R^\mu{}_{00\nu} = \frac{1}{2} \partial_0^2 h^\mu{}_\nu$$

Our geodesic deviation equation is then

$$\frac{d^2 S^\mu}{dt^2} = \frac{1}{2} \frac{d^2 h^\mu{}_\nu}{dt^2} S^\nu \quad (5.23)$$

We see that the gravitational wave propagating, say, the z direction with polarisation vector (5.22) affects neither S^0 nor S^3 . The only effect on the geodesics is in the (x, y) -plane, transverse to the direction of propagation. For simplicity, we will solve this equation in the $z = 0$ plane.

H_+ Polarisation: If we set $H_X = 0$ in (5.22), then the geodesic deviation equation (5.23) becomes

$$\frac{d^2 S^1}{dt^2} = -\frac{\omega^2}{2} H_+ e^{i\omega t} S^1 \quad \text{and} \quad \frac{d^2 S^2}{dt^2} = +\frac{\omega^2}{2} H_+ e^{i\omega t} S^2$$

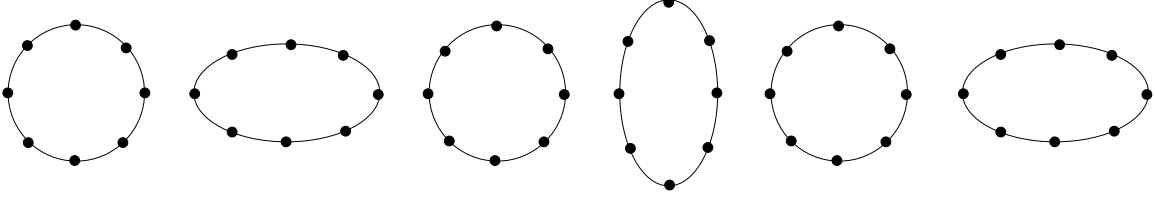
We can solve these perturbatively in H_+ . Keeping terms of order $\mathcal{O}(h)$ only, we have

$$S^1(t) = S^1(0) \left(1 + \frac{1}{2} H_+ e^{i\omega t} + \dots \right) \quad \text{and} \quad S^2(t) = S^2(0) \left(1 - \frac{1}{2} H_+ e^{i\omega t} + \dots \right) \quad (5.24)$$

where, as we mentioned previously, we should take the real part of the right-hand-side. (Recall that H_+ can also be complex.)

From these solutions, we can determine the way in which geodesics are affected by a passing wave. Think of the displacement vector S^μ as the distance from the origin to a neighbouring geodesic. We will consider a family of neighbouring geodesics corresponding to a collection of particles which, at time $t = 0$, are arranged around a circle of radius R . This means that we have initial conditions $S^a(t = 0)$ satisfying $S^1(0)^2 + S^2(0)^2 = R^2$.

The solutions (5.24) tell us how these geodesics evolve. The relative minus sign between the two equations means that when geodesics move outwards in, say, the $x^1 = x$ direction, they move inwards in the $x^2 = y$ direction, and vice-versa. The net result is that, as time goes on, these particles will evolve from a circle to an ellipse and back again, displaced like this:



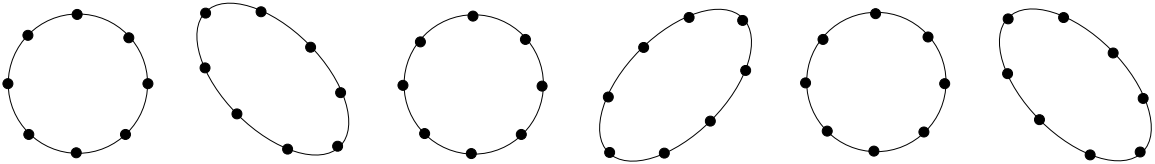
H_X Polarisation: If we set $H_+ = 0$ in (5.22), then the geodesic deviation equation (5.23) becomes

$$\frac{d^2 S^1}{dt^2} = -\frac{\omega^2}{2} H_X e^{i\omega t} S^2 \quad \text{and} \quad \frac{d^2 S^2}{dt^2} = -\frac{\omega^2}{2} H_X e^{i\omega t} S^1$$

Again, we solve these perturbatively in H_X . We have

$$S^1(t) = S^1(0) + \frac{1}{2} S^2(0) H_X e^{i\omega t} + \dots \quad \text{and} \quad S^2(t) = S^2(0) + \frac{1}{2} S^1(0) H_X e^{i\omega t} + \dots$$

The displacement is the same as previously, but rotated by 45° . (To see this, note that the displacements $S^1(t) \pm S^2(t)$ have the same functional form as (5.24).) This means that this time the displacement of geodesics looks like this:



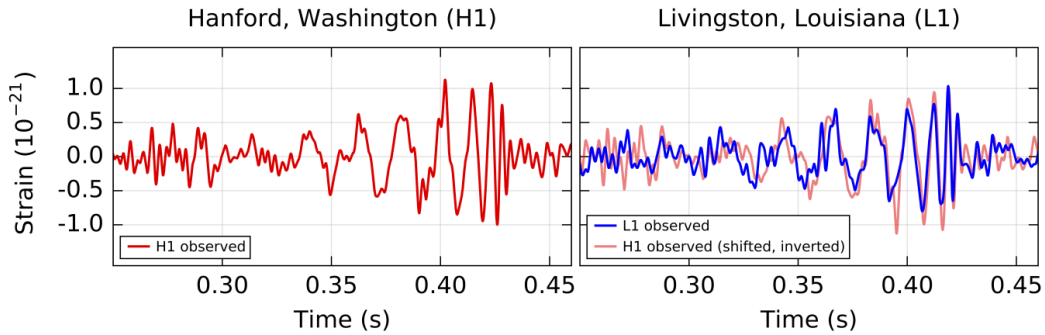


Figure 42: The discovery of gravitational waves by the LIGO detectors.

We can also take linear combinations of the polarisation states. Adding the two polarisations above give an elliptic displacement whose axis rotates. This is analogous to the circular polarisation of light.

The displacements due to gravitational waves are invariant under rotations by π . This contrasts with polarisation of light which is described by a vector, and so is only invariant under 2π rotations. This reflects the fact that graviton has spin 2, while the photon has spin 1.

Gravitational Wave Detectors

Gravitational wave detectors are interferometers. They bounce light back and forth between two arms, with the mirrors at either end playing the role of test masses.

If the gravitational wave travels perpendicular to the plane of the detector, it will shorten one arm and lengthen the other. With the arms aligned along the x and y axes, the maximum change in length can be read from (5.24),

$$L' = L \left(1 \pm \frac{H_+}{2} \right) \quad \Rightarrow \quad \frac{\delta L}{L} = \frac{H_+}{2}$$

To get a ballpark figure for this, we need to understand how large we expect H_+ to be from any plausible astrophysical source. We'll do this in Section 5.3.2. It turns out it's not really very large at all: typical sources have $H_+ \sim 10^{-21}$. The lengths of each arm in the LIGO detectors is around $L \sim 3$ km, meaning that we have to detect a change in length of $\delta L \sim 10^{-18}$ m. This seems like a crazy small number: it's smaller than the radius of a proton, and around 10^{12} times smaller than the wavelength of the light used in the interferometer. Nonetheless, the sensitivity of the detectors is up to the task and the LIGO observatories detected gravitational waves for the first time in 2015. For this, three members of the collaboration were awarded the 2017 Nobel prize. Subsequently,

the LIGO and VIRGO detectors have observed a large number of mergers involving black holes and neutron stars.

5.2.3 Exact Solutions

We have found a wave-like solution to the linearised Einstein equations. The metric for a wave moving in, say, the positive z direction takes the form

$$ds^2 = -dt^2 + (\delta_{ab} + h_{ab}(z-t))dx^a dx^b + dz^2 \quad (5.25)$$

where the $a, b = 1, 2$ indices run over the spatial directions transverse to the direction of the wave. Because the wave equation is linear, any function $h_{ab}(z-t)$ is a solution to the linearised Einstein equations; the form that we gave in (5.18) is simply the Fourier decomposition of the general solution.

Because gravitational waves are so weak, the linearised metric is entirely adequate for any properties of gravitational waves that we wish to calculate. Nonetheless, it's natural to ask if this solution has an extension to the full non-linear Einstein equations. Rather surprisingly, it turns out that there is.

For a wave propagating in the positive z direction, we first introduce lightcone coordinates

$$u = t - z \quad , \quad v = t + z$$

Then we consider the *plane wave* ansatz, sometimes called the *Brinkmann metric*

$$ds^2 = -dudv + dx^a dx^a + H_{ab}(u)x^a x^b du^2$$

Note that our linearised gravitational wave (5.25) is not of this form; there is some (slightly fiddly) change of coordinates that takes us between the two metrics. One can show that the Brinkmann metric is Ricci flat, and hence solves the vacuum Einstein equations, for *any* traceless metric H_{ab}

$$R_{\mu\nu} = 0 \quad \Leftrightarrow \quad H^a_a(u) = 0$$

The general metric again has two independent polarisation states,

$$H_{ab}(u) = \begin{pmatrix} H_{11}(u) & H_{12}(u) \\ H_{12}(u) & -H_{11}(u) \end{pmatrix}$$

It is unusual to find solutions on non-linear PDEs which depend on arbitrary functions, like $H_{11}(u)$ and $H_{12}(u)$. The Brinkmann metrics are a rather special exception.

5.3 Making Waves

The gravitational wave solutions described in the previous section are plane waves. They come in from infinity, and go out to infinity. In reality however, gravitational waves start at some point and radiate out.

As we will see, the story is entirely analogous to what we saw in our earlier course on [Electromagnetism](#). There, you generate electromagnetic waves by shaking electric charges. Similarly, we generate gravitational waves by shaking masses. The purpose of this section is to make this precise.

5.3.1 The Green's Function for the Wave Equation

Our starting point is the linearised Einstein equation (5.15),

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \quad (5.26)$$

which assumes that both the source, in the guise of the energy momentum tensor $T_{\mu\nu}$, and the perturbed metric $\bar{h}_{\mu\nu}$ are small. This is simply a bunch of decoupled wave equations. We already solved these in Section 6 of the lectures on [Electromagnetism](#), and our discussion here will parallel the presentation there.

We will consider a situation in which matter fields are localised to some spatial region Σ . In this region, there is a time-dependent source of energy and momentum $T_{\mu\nu}(\mathbf{x}', t)$, such as two orbiting black holes. Outside of this region, the energy-momentum tensor vanishes: $T_{\mu\nu}(\mathbf{x}', t) = 0$ for $\mathbf{x}' \notin \Sigma$. We want to know what the metric $h_{\mu\nu}$ looks like a long way from the region Σ . The solution to (5.26) outside of Σ can be given using the (retarded) Green's function; it is

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = 4G \int_{\Sigma} d^3x' \frac{T_{\mu\nu}(\mathbf{x}', t_{\text{ret}})}{|\mathbf{x} - \mathbf{x}'|} \quad (5.27)$$

here t_{ret} is the retarded time, given by

$$t_{\text{ret}} = t - |\mathbf{x} - \mathbf{x}'|$$

It's not too hard to show that this solution satisfies the de Donder gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ provided that the energy momentum tensor is conserved, $\partial^\mu T_{\mu\nu} = 0$. The solution does not, however, automatically satisfy the temporal and traceless conditions (5.21). The solution (5.27) captures the causality of the wave equation: the gravitational field $\bar{h}_{\mu\nu}(\mathbf{x}, t)$ is influenced by the matter at position \mathbf{x}' at the earlier time t_{ret} , so that there is time for this influence to propagate from \mathbf{x}' to \mathbf{x} .

We denote the size of the region Σ as d . We're interested in what's happening at a point \mathbf{x} which is a distance $r = |\mathbf{x}|$ away. If $|\mathbf{x} - \mathbf{x}'| \gg d$ for all $\mathbf{x}' \in \Sigma$ then we can approximate

$$|\mathbf{x} - \mathbf{x}'| = r - \frac{\mathbf{x} \cdot \mathbf{x}'}{r} + \dots \Rightarrow \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x} \cdot \mathbf{x}'}{r^3} + \dots \quad (5.28)$$

We also have a factor of $|\mathbf{x} - \mathbf{x}'|$ that sits inside $t_{\text{ret}} = t - |\mathbf{x} - \mathbf{x}'|$. This means that we should also Taylor expand the argument of the energy-momentum tensor

$$T_{\mu\nu}(\mathbf{x}', t_{\text{ret}}) = T_{\mu\nu}(\mathbf{x}', t - r + \mathbf{x} \cdot \mathbf{x}'/r + \dots)$$

Now we'd like to further expand out this argument. But, to do that, we need to know something about what the source is doing. We will assume that the motion of matter is *non-relativistic*, so that the energy momentum tensor doesn't change very much over the time $\tau \sim d$ that it takes light to cross the region Σ . For example, if we have a system comprised of two objects (say, neutron stars or black holes) orbiting each other with characteristic frequency ω then $T_{\mu\nu} \sim e^{-i\omega t}$ and the requirement that the motion is non-relativistic becomes $d \ll 1/\omega$. Then we can further Taylor expand the current to write

$$T_{\mu\nu}(\mathbf{x}', t_{\text{ret}}) = T_{\mu\nu}(\mathbf{x}', t - r) + \dot{T}_{\mu\nu}(\mathbf{x}', t - r) \frac{\mathbf{x} \cdot \mathbf{x}'}{r} + \dots \quad (5.29)$$

We have two Taylor expansions, (5.28) and (5.29). At leading order in d/r we take the first term from both these expansions to find

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) \approx \frac{4G}{r} \int_{\Sigma} d^3x' T_{\mu\nu}(\mathbf{x}', t - r)$$

We first look at the expressions for \bar{h}_{00} and \bar{h}_{0i} . The first of these is

$$\bar{h}_{00}(\mathbf{x}, t) \approx \frac{4G}{r} E \quad \text{with} \quad E = \int_{\Sigma} d^3x' T_{00}(\mathbf{x}', t - r) \quad (5.30)$$

This is simply a recapitulation of the Newtonian limit (5.16), with the long distance gravitational potential given by $\Phi = -GE/r$ where E is the total energy inside the region Σ . At the linear order to which we're working, current conservation $\partial^\mu T_{\mu\nu} = 0$ ensures that the energy E inside Σ is constant, so the time dependence drops out.

Similarly, we have

$$\bar{h}_{0i}(\mathbf{x}, t) \approx -\frac{4G}{r} P_i \quad \text{with} \quad P_i = - \int_{\Sigma} d^3x' T_{0i}(\mathbf{x}', t - r) \quad (5.31)$$

Here P_i is the total momentum of the matter inside Σ which, again, is conserved. We can always go to a rest frame where this matter is stationary in which case $P_i = 0$ and hence $\bar{h}_{0i} = 0$. This was the choice we implicitly made in describing the Newtonian limit (5.16).

Neither the expression for \bar{h}_{00} nor \bar{h}_{ij} captures the physics that we are interested in. The results only know about the conserved quantities inside the region Σ , not about how they're moving. However, things become more interesting when we look at the spatial components of the metric,

$$\bar{h}_{ij}(\mathbf{x}, t) \approx \frac{4G}{r} \int_{\Sigma} d^3x' T_{ij}(\mathbf{x}', t - r)$$

with $i, j = 1, 2, 3$. Now the integral on the right-hand side is not a conserved quantity. However, it is possible to relate it to certain properties of the energy distribution inside Σ .

Claim:

$$\int_{\Sigma} d^3x' T_{ij}(\mathbf{x}', t) = \frac{1}{2} \ddot{I}_{ij}(t)$$

where I_{ij} is the quadrupole moment of the energy,

$$I_{ij}(t) = \int_{\Sigma} d^3x T^{00}(\mathbf{x}, t) x_i x_j \quad (5.32)$$

Proof: We start by writing

$$T^{ij} = \partial_k (T^{ik} x^j) - (\partial_k T^{ik}) x^j = \partial_k (T^{ik} x^j) + \partial_0 T^{0i} x^j$$

where, in the second equality, we've used current conservation $\partial_{\mu} T^{\mu\nu} = 0$. (Note that current conservation in the full theory is $\nabla_{\mu} T^{\mu\nu} = 0$, but in our linearised analysis this reduces to $\partial_{\mu} T^{\mu\nu} = 0$.) For the T^{0i} term, we play the same trick again. Symmetrising over (ij) , we have

$$T^{0(i} x^{j)} = \frac{1}{2} \partial_k (T^{0k} x^i x^j) - \frac{1}{2} (\partial_k T^{0k}) x^i x^j = \frac{1}{2} \partial_k (T^{0k} x^i x^j) + \frac{1}{2} \partial_0 T^{00} x^i x^j$$

When we integrate this over Σ , we drop the terms that are total spatial derivatives. We're left with

$$\int_{\Sigma} d^3x' T^{ij}(\mathbf{x}', t) = \frac{1}{2} \partial_0^2 \int_{\Sigma} d^3x' T^{00}(\mathbf{x}', t) x'^i x'^j$$

which is the claimed result. □

We learn that, far from the source, the metric takes the form

$$\bar{h}_{ij}(\mathbf{x}, t) \approx \frac{2G}{r} \ddot{I}_{ij}(t - r) \quad (5.33)$$

This is the physics that we want: if we shake the matter distribution in some way then, once the signal has had time to propagate, this will affect the metric. Because the equations are linear, if the matter shakes at some frequency ω the spacetime will respond by creating waves at parametrically same frequency. (In fact, we'll see a factor of 2 arises in the example of a binary system (5.36).)

In fact, we can now revisit the other components \bar{h}_{00} and \bar{h}_{0i} . The gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ tells us that

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji} \quad \text{and} \quad \partial_0 \bar{h}_{00} = \partial_i \bar{h}_{i0}$$

The first of these equations gives

$$\partial_0 \bar{h}_{0i} \approx \partial_j \left(\frac{2G}{r} \ddot{I}_{ij}(t - r) \right) = -\frac{2G\hat{x}_j}{r^2} \ddot{I}_{ij}(t - r) - \frac{2G\hat{x}_j}{r} \dddot{I}_{ij}(t - r) \quad (5.34)$$

where we've used the fact that $\partial_j r = x_j/r = \hat{x}_j$. Which of these two terms in (5.34) is bigger? As we get further from the source, we would expect the second, $1/r$, term to dominate over the first, $1/r^2$ term. But the second term has an extra time derivative, which means an extra factor of the characteristic frequency of the source, ω . This means that the second term dominates provided that $r \gg 1/\omega$ or, in terms of the wavelength λ of the emitted gravitational wave, $r \gg \lambda$. This is known as the *far-field zone* or, sometimes, the *radiation zone*. In this regime, we have

$$\bar{h}_{0i} \approx -\frac{2G\hat{x}_j}{r} \ddot{I}_{ij}(t - r)$$

where we've integrated (5.34). In general, the integration constant is given by the P_i term that we previously saw in (5.31). In the answer above, we've set this integration constant to zero by choosing coordinates in which $P_i = 0$, meaning that the centre of mass of the source doesn't move. We can now repeat this to determine \bar{h}_{00} . The same argument means that we discard one term, and retain

$$\bar{h}_{00} = \frac{4G}{r} E + \frac{2G\hat{x}_i\hat{x}_j}{r} \ddot{I}_{ij}(t - r)$$

If we tried to compute these \ddot{I} terms in \bar{h}_{00} and \bar{h}_{0i} directly from (5.27), we would have to go to higher order in the expansion. Implementing the gauge condition, as above, saves us this work.

5.3.2 An Example: Binary Systems

As an example, consider two stars (or neutron stars, or black holes) each with mass M , separated by distance R , orbiting in the (x, y) plane. Using Newtonian gravity, the stars orbit with frequency

$$\omega^2 = \frac{2GM}{R^3} \quad (5.35)$$

If we treat these stars as point particles, then the energy density is simply a product of delta-functions

$$T^{00}(\mathbf{x}, t) = M\delta(z) \left[\delta\left(x - \frac{R}{2}\cos\omega t\right) \delta\left(y - \frac{R}{2}\sin\omega t\right) + \delta\left(x + \frac{R}{2}\cos\omega t\right) \delta\left(y + \frac{R}{2}\sin\omega t\right) \right]$$

The quadrupole (5.32) is then easily evaluated

$$\begin{aligned} I_{ij}(t) &= \frac{MR^2}{2} \begin{pmatrix} \cos^2\omega t & \cos\omega t \sin\omega t & 0 \\ \cos\omega t \sin\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{MR^2}{4} \begin{pmatrix} 1 + \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & 1 - \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (5.36)$$

The resulting metric perturbation is then

$$\bar{h}_{ij} \approx -\frac{2GMR^2\omega^2}{r} \begin{pmatrix} \cos 2\omega t_{\text{ret}} & \sin 2\omega t_{\text{ret}} & 0 \\ \sin 2\omega t_{\text{ret}} & -\cos 2\omega t_{\text{ret}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $t_{\text{ret}} = t - r$ is the retarded time.

This gravitational wave propagates out more or less radially. If we look along the z -axis, then the wave takes the same form as the plane wave (5.22) that we saw previously, now with combination of H_+ and H_X polarisations, $\pi/2$ out of phase, also known as circular polarisation.

We can use this to give us a ballpark figure for the expected strength of gravitational waves. Using (5.35) to replace the frequency, we have

$$|h_{ij}| \sim \frac{G^2 M^2}{Rr}$$

Clearly the signal is largest for large masses M , orbiting as close as possible so R is small. The densest objects are black holes whose size is given by the Schwarzschild radius $R_s = 2GM$. As the black holes come close, we take $R \approx R_s$ to get

$$|h_{ij}| \sim \frac{GM}{r}$$

A black hole weighing a few solar masses has Schwarzschild radius $R_s \sim 10$ km. Now it's a question of how far away these black holes are. If two such black holes were orbiting in, say, the Andromeda galaxy which, at 2.5 million light years, has $r \approx 10^{18}$ km, we would get $h \sim 10^{-17}$. At a distance of a billion light-years, we're looking at $h \sim 10^{-20}$. These are small numbers. Nonetheless, as we mentioned previously, this is the sensitivity that has been achieved by gravitational wave detectors.

5.3.3 Comparison to Electromagnetism

For both electromagnetic and gravitational waves, there is a multipole expansion that determines the long distance wave behaviour in terms of the source. (Full details of the calculations in Maxwell theory can be found in the lectures on [Electromagnetism](#).) In electromagnetism, the multipoles of the charge distribution $\rho(\mathbf{x})$ are the charge

$$Q = \int_{\Sigma} d^3x \rho(\mathbf{x})$$

the dipole

$$\mathbf{p} = \int_{\Sigma} d^3x \rho(\mathbf{x}) \mathbf{x}$$

the quadrupole

$$\mathbb{Q}_{ij} = \int_{\Sigma} d^3x \rho(\mathbf{x}) (3x_i x_j - \delta_{ij} x^2)$$

and so on. Charge conservation tells us that $\dot{Q} = 0$: the total charge cannot change which means that there is no monopole contribution to electromagnetic waves. Instead the leading order contribution comes from the dipole. Indeed, repeating the calculation that we saw above in the context of Maxwell theory shows that the leading order contribution to electromagnetic waves

$$\mathbf{A}(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi r} \dot{\mathbf{p}}(t - r) \quad (5.37)$$

We can compare this to the situation in gravity. The multipoles of the energy distribution $T_{00}(\mathbf{x})$ are the total energy

$$E = \int_{\Sigma} d^3x T_{00}(\mathbf{x})$$

the dipole which, for energy, is related to the centre of mass of the distribution

$$\mathbf{X} = \frac{1}{E} \int_{\Sigma} d^3x T_{00}(\mathbf{x}) \mathbf{x}$$

the quadrupole

$$I_{ij}(t) = \int d^3x T_{00}(\mathbf{x}, t) x_i x_j$$

The conservation of energy, $\dot{E} = 0$, is responsible for the lack of a monopole contribution to gravitational radiation. But, as we saw above, in contrast to electromagnetism, the dipole contribution also vanishes. This too can be traced to a conservation law: we have

$$E \dot{X}_i = \int_{\Sigma} d^3x (\partial_0 T_{00}) x_i = \int_{\Sigma} d^3x (\partial_j T_{j0}) x_i = - \int_{\Sigma} d^3x T_{i0} = P_i$$

where, in the penultimate equality, we have integrated by parts and, in the final equality, we have used the definition of the total momentum P_i defined in (5.31). But conservation of momentum \mathbf{P} means that the second time derivative of the dipole vanishes

$$E \ddot{\mathbf{X}} = \dot{\mathbf{P}} = 0$$

This is the physical reason that there's no gravitational dipole: it would violate the conservation of momentum.

In electromagnetism, there is another dipole contribution to the gauge potential: this is

$$\mathbf{A}^{\text{MD}}(\mathbf{x}, t) = -\frac{\mu_0}{4\pi r} \hat{\mathbf{x}} \times \mathbf{m}(t - r)$$

where the magnetic dipole \mathbf{m} is defined by

$$\mathbf{m} = \frac{1}{2} \int_{\Sigma} d^3x \mathbf{x} \times \mathbf{J}(\mathbf{x})$$

In our gravity, the analogous term comes from the \dot{T}_{ij} in the expansion (5.29). The analog of the magnetic dipole in gravity is

$$J_i = \int_{\Sigma} d^3x \epsilon_{ijk} x_j T_{0k}$$

But this is again something familiar: it is the angular momentum of the system. This too is conserved, $\dot{\mathbf{J}} = 0$, which means that, again, the dipole contribution vanishes in gravity. The leading order effect is the quadrupole.

5.3.4 Power Radiated: The Quadrupole Formula

A source which emits gravitational waves will lose energy. We'd like to know how much energy is emitted. In other words, we'd like to understand how much energy is carried by the gravitational waves.

In the context of electromagnetism, it is fairly easy to calculate the analogous quantity. The energy current in electromagnetic waves is described by the T^{0i} components of the energy-momentum tensor, better known as the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

To compute the power \mathcal{P} emitted by an electromagnetic source, we simply integrate this energy flux over a sphere \mathbf{S}^2 that surrounds the source,

$$\mathcal{P} = \int_{\mathbf{S}^2} d^2\mathbf{r} \cdot \mathbf{S}$$

Evaluating this using the dipole approximation for electromagnetic waves (5.37), and doing a suitable average, we find the Larmor formula

$$\mathcal{P} = \frac{\mu_0}{6\pi c} |\ddot{\mathbf{p}}|^2$$

Our task in this section is to perform the same calculations for gravitational waves.

This is not as easy as it sounds. The problem is the one we addressed in Section 4.5.5: there is no local energy-momentum tensor for gravitational fields. This means that there is no analog of the Poynting vector for gravitational waves. It looks like we're scuppered.

There is, however, a way forward. The idea is that we will attempt to define an energy-momentum tensor $t_{\mu\nu}$ for gravitational waves which, in the linearised theory, obeys

$$\partial^\mu t_{\mu\nu} = 0$$

The problem is that, as we mentioned in Section 4.5.5, there is no way to achieve this in a diffeomorphism invariant way. In the full non-linear theory, this means that $t_{\mu\nu}$ is not actually a tensor. In our linearised theory, it means that $t_{\mu\nu}$ will not be invariant under the gauge transformations (5.10). Nonetheless, we'll first define an appropriate $t_{\mu\nu}$, and then worry about the lack of gauge invariance later.

A Quick and Dirty Approach: the Fierz-Pauli Action

When asked to construct an energy-momentum tensor for the metric perturbations, the first thing that springs to mind is to return to the Fierz-Pauli action (5.8). Viewed as an action describing a spin 2 field propagating in Minkowski space, we can then treat it as any other classical field theory and compute the energy-momentum tensor in the usual ways.

For example if we work in transverse traceless gauge, with $h = 0$ and $\partial^\mu h_{\mu\nu} = 0$ then, after an integration by parts, the Fierz-Pauli action becomes

$$S_{FP} = -\frac{1}{8\pi G} \int d^4x \frac{1}{4} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu}$$

which looks like the action for a bunch of massless scalar fields. The energy density then takes the schematic form

$$t^{00} \sim \frac{1}{G} \dot{h}_{\mu\nu} \dot{h}^{\mu\nu}$$

There are also gradient terms but, for wave equations, these contribute in the same way as time derivatives. Strictly speaking, we should be working with the momentum t^{0i} , but this scales in the same way and calculation is somewhat easier if we work with t^{00} . Our previous expression (5.33) for the emitted gravitational wave wasn't in transverse-traceless gauge. If we were to massage it into this form, we have

$$h_{ij}(\mathbf{x}, t) \sim \frac{G}{r} \ddot{\mathcal{Q}}_{ij}(t - r)$$

where \mathcal{Q}_{ij} is the traceless part of the quadrupole moment,

$$\mathcal{Q}_{ij} = I_{ij} - \frac{1}{3} I_{kk} \delta_{ij}$$

Putting this together, suggests that the energy density carried in gravitational waves is schematically of the form

$$t^{00} \sim \frac{G}{r^2} \ddot{\mathcal{Q}}_{ij}^2$$

Integrating over a sphere at a large distance, suggests that the energy lost in gravitational waves should depend on the square of the third derivative of the quadrupole,

$$\mathcal{P} \sim G \ddot{\mathcal{Q}}_{ij}^2$$

It turns out that this is indeed correct. A better treatment gives

$$\mathcal{P} = \frac{G}{5} \ddot{\mathcal{Q}}_{ij} \ddot{\mathcal{Q}}^{ij} \quad (5.38)$$

where, as in all previous formulae, $\ddot{\mathcal{Q}}_{ij}$ should be evaluated in retarded time $t_{\text{ret}} = t - r$. This is the quadrupole formula, the gravitational equivalent of the Larmor formula.

Before the direct detection of gravitational waves, the quadrupole formula gave us the best observational evidence of their existence. The *Hulse-Taylor pulsar* is a binary neutron star system, discovered in 1974. One of these neutron stars is a pulsar, emitting a sharp beam every 59 ms. This can be used to very accurately track the orbit of the stars and show that the period – which is about 7.75 hours – is getting shorter by around $10\ \mu\text{s}$ each year. This is in agreement with the quadrupole formula (5.38). Hulse and Taylor were awarded the 1993 Nobel prize for this discovery.

Looking for a Better Approach

Any attempt to improve on the discussion above opens up a can of worms. The calculation needed to nail the factor of $1/5$ is rather arduous. More importantly, however, there are also a number of conceptual issues that we need to overcome. Rather than explaining the detailed integrals that give the factor of $1/5$, we'll instead focus on some of these conceptual ideas.

Our first task is to do a better job of defining $t_{\mu\nu}$. There are a number of ways to proceed.

- First, we could try to do a less shoddy job of computing the energy-momentum tensor $t_{\mu\nu}$ from the Fierz-Pauli action (5.8). This, it turns out, suffers a number of ambiguities. If, for example, we attempted to compute $t_{\mu\nu}$ as the Noether currents associated to spacetime translations, then we would find that the result is neither symmetric in μ and ν , nor gauge invariant. That's not such a surprise as it's also true for Maxwell theory. We can then try to add an “improvement” term

$$t_{\mu\nu} \rightarrow t_{\mu\nu} + \partial^\rho \Theta_{\rho\mu\nu}$$

where $\Theta_{\rho\mu\nu} = -\Theta_{\mu\rho\nu}$ which ensures that $\partial^\mu \partial^\rho \Theta_{\rho\mu\nu} = 0$ and the extra term doesn't ruin conservation of the current. In Maxwell theory, such a term can be added to make the resulting energy-momentum tensor both symmetric and gauge invariant. For the Fierz-Pauli action, we can make it symmetric but not gauge invariant.

A similar approach is to forget the origin of the Fierz-Pauli action and then attempt to write a generalisation of the action in “curved spacetime” by contracting indices with a metric $g^{\mu\nu}$ and replacing derivatives with ∇_μ . We could then evaluate the energy-momentum tensor using the usual formula (4.44), subsequently restricting to flat space. Here too there are ambiguities which now arise from the possibility of including terms like $R_{\mu\nu} h^{\mu\rho} h^\nu{}_\rho$ or $R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma}$ in the action. These vanish in Minkowski space, but give different energy-momentum tensors. For any choice, the result is again symmetric but not gauge invariant.

- Another approach is to take the lack of energy-conservation of the matter fields seriously, and try to interpret this as energy transferred into the gravitational field. To this end, let's look again at the covariant conservation $\nabla_\mu T^{\mu\nu} = 0$. As we stressed in Section 4.5.5, covariant conservation is not the same thing as actual conservation. In particular, we can rewrite the covariant conservation equation as

$$\begin{aligned}\nabla_\mu T^\mu{}_\nu &= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^\mu{}_\nu) - \Gamma^\rho_{\mu\nu} T^\mu{}_\rho \\ &= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^\mu{}_\nu) - \frac{1}{2} \partial_\nu g_{\mu\rho} T^{\mu\rho} = 0\end{aligned}$$

where, to get the second line, we've invoked the symmetry of $T^{\mu\rho}$. Note that the simplification of the Christoffel symbol to $g_{\mu\rho,\nu}$ only happens when the ν index is down; this reflects the fact we're writing the equations in a non-covariant way. Next, we use the Einstein equation to replace $T^{\mu\rho}$ on the right-hand side by $\frac{1}{8\pi G} G^{\mu\rho}$. This gives

$$\partial_\mu (\sqrt{-g} T^\mu{}_\nu) = \frac{1}{16\pi G} \sqrt{-g} \partial_\nu g_{\mu\rho} \left(R^{\mu\rho} - \frac{1}{2} R g^{\mu\rho} \right) = \frac{1}{16\pi G} \sqrt{-g} \partial_\nu g_{\mu\rho} R^{\mu\rho}$$

The idea is to massage the right-hand side so that this expression becomes

$$\partial_\mu (\sqrt{-g} T^\mu{}_\nu) = -\partial_\mu (\sqrt{-g} t^\mu{}_\nu)$$

for some $t^\mu{}_\nu$ which is referred to as the *Landau-Lifshitz pseudotensor*. This equation suggests that the sum of the matter energy $T^\mu{}_\nu$ and the gravitational energy $t^\mu{}_\nu$ is conserved. However, this statement should be treated with suspicion because it's coordinate dependent: the pseudotensor $t_{\mu\nu}$ is not a real tensor: its expression is long and horrible involving many terms, each of which is quadratic in Γ and quadratic in g . (You can find it in (101.6) of Landau and Lifshitz, volume 2 but it's unlikely to give you a sense of enlightenment.) The expression for the pseudo-tensor is slightly nicer in the linearised theory, albeit only slightly.

- The final approach is perhaps the least intuitive, but has the advantage that it gives a straightforward and unambiguous path to find an appropriate non-tensor $t_{\mu\nu}$. Motivated by the expectation that any putative $t_{\mu\nu}$ will be quadratic in $h_{\mu\nu}$, we expand the Einstein equations to the next order. We keep $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Expanding to second order, the Einstein equations becomes

$$\left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right]^{(1)} + \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right]^{(2)} = 8\pi G T_{\mu\nu}$$

where the subscript (n) means restrict to terms of order h^n . We rewrite this as

$$\left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right]^{(1)} = 8\pi G (T_{\mu\nu} + t_{\mu\nu}) \quad (5.39)$$

with the second order expansion of the Einstein tensor now sitting suggestively on the right-hand side it is interpreted as the gravitational energy-momentum non-tensor

$$\begin{aligned} t_{\mu\nu} &= -\frac{1}{8\pi G} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right]^{(2)} \\ &= -\frac{1}{8\pi G} \left[R_{\mu\nu}^{(2)} - \frac{1}{2} R^{(2)} \eta_{\mu\nu} - \frac{1}{2} R^{(1)} h_{\mu\nu} \right] \end{aligned}$$

If we're far from the source then we can neglect the term $R^{(1)}$ since it vanishes by the equation of motion. (More precisely, it vanishes at linear order and so fails to contribute at the quadratic order that we care about.) We end up with seemingly simple expression

$$t_{\mu\nu} = -\frac{1}{8\pi G} \left[R_{\mu\nu}^{(2)} - \frac{1}{2} R^{(2)} \eta_{\mu\nu} \right] \quad (5.40)$$

The linearised Bianchi identity is $\partial^\mu [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}]^{(1)} = 0$. But this means that if we are far from sources, so $T^{\mu\nu} = 0$, and the equation of motion (5.39) is satisfied, then we necessarily have $\partial^\mu t_{\mu\nu} = 0$ as befits a conserved current. All that's left is to evaluate the Ricci tensor to second order in the perturbation $h_{\mu\nu}$. This is painful. The answer turns out to be

$$\begin{aligned} R_{\mu\nu}^{(2)}[h] &= \frac{1}{2} h^{\rho\sigma} \partial_\mu \partial_\nu h_{\rho\sigma} - h^{\rho\sigma} \partial_\rho \partial_{(\mu} h_{\nu)\sigma} + \frac{1}{4} \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} + \partial^\sigma h^\rho{}_\nu \partial_{[\sigma} h_{\rho]\mu} \\ &\quad + \frac{1}{2} \partial_\sigma (h^{\sigma\rho} \partial_\rho h_{\mu\nu}) - \frac{1}{4} \partial^\rho h \partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2} \partial^\rho h \right) \partial_{(\mu} h_{\nu)\rho} \end{aligned}$$

Pretty huh? Substituting this into the expression (5.40) gives an equally pretty expression for $t_{\mu\nu}$. Once again however, $t_{\mu\nu}$ is not gauge invariant.

We see that there are a number of different ways to construct an energy-momentum tensor $t_{\mu\nu}$ for gravitational waves. But none are gauge invariant. In order to relate this to something physical, we clearly have to construct something which is gauge invariant.

It is possible to extract something gauge invariant from $t_{\mu\nu}$ provided that our space-time is asymptotically Minkowski. We could, for example, integrate t^{00} over an infinite spatial hypersurface. This defines the *ADM energy* which can be shown to be constant in time. We'll revisit this in Section ?? in the full non-linear theory.

Alternatively, we could integrate t^{0i} over a sphere at \mathcal{I}^+ . This too gives a gauge invariant quantity, which is the time dependence of the so-called *Bondi energy*. This too can be defined in the full non-linear theory.

Here we give a less rigorous but slightly simpler construction. The gravitational wave, like any wave, varies over some typical length scale λ . We average over this oscillations by introducing a coarse-grained energy tensor

$$\langle t_{\mu\nu} \rangle = \int_V d^4x W(x-y) t_{\mu\nu}(y)$$

where the integral is over some region V of typical size a . The weighting function $W(x)$ has the property that it varies smoothly over V with $\int_V d^4x W(x) = 1$ and $W(x) = 0$ on ∂V . The coarse graining means that averages of total derivatives scale as $\langle \partial X \rangle \sim 1/a$. For large a , we can neglect such terms. Similarly, we can “integrate by parts” inside averages, so that $\langle X \partial Y \rangle = -\langle (\partial X) Y \rangle + \mathcal{O}(1/a)$. A fairly straightforward calculation shows that, in transverse-traceless gauge, the averaged energy-momentum tensor is simply

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle$$

where we neglect total derivatives. We can check that this is indeed conserved,

$$\partial^\mu \langle t_{\mu\nu} \rangle = \frac{1}{32\pi G} \langle (\Box h_{\rho\sigma}) \partial_\nu h^{\rho\sigma} + \frac{1}{2} \partial_\nu (\partial_\mu h_{\rho\sigma} \partial^\mu h^{\rho\sigma}) \rangle = 0$$

The first term vanishes by the equation of motion, while the second is a total derivative and so can be neglected. More importantly, under a gauge transformation

$$\delta \langle t_{\mu\nu} \rangle = \frac{1}{16\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu (\partial^\rho \xi^\sigma + \partial^\sigma \xi^\rho) \rangle$$

But now we can integrate by parts and use the de Donder gauge condition $\partial^\rho h_{\rho\sigma} = 0$. We see that the averaged $\langle t_{\mu\nu} \rangle$ is gauge invariant, with $\delta \langle t_{\mu\nu} \rangle = 0$ up to total derivative term of order $\mathcal{O}(1/a)$. In other words, $\langle t_{\mu\nu} \rangle$ is almost gauge invariant. A better way of saying “almost gauge invariant” is “not gauge invariant”. If we really want something gauge invariant, which we do, we must take $a \rightarrow \infty$, meaning that we average over all of spacetime.

Finally, we can compute the power emitted by gravitational wave at infinity by

$$\mathcal{P} = \int_{\mathbf{S}^2} d^2x \hat{n}_i \langle t^{0i} \rangle$$

with \hat{n}_i a normal vector to \mathbf{S}_∞^2 . With some tedious integrals, we then find the answer (5.38).

5.3.5 Gravitational Wave Sources on the ☒

We can do some quick, back-of-the-envelope calculations to get a sense for how much energy is emitted by a gravitational wave source. Assuming Newtonian gravity is a good approximation, two masses M , separated by a distance R , will orbit with frequency

$$\omega^2 R \sim \frac{GM}{R^2}$$

The quadrupole is $\mathcal{Q} \sim MR^2$ and so $\ddot{\mathcal{Q}} \sim \omega^3 MR^2$. We learn that the power emitted scales as (5.38)

$$\mathcal{P} \sim G \ddot{\mathcal{Q}}^2 \sim \frac{G^4 M^5}{R^5} \quad (5.41)$$

To get numbers out of this, we need to put the factors of c back in. Recall that the Schwarzschild radius of an object is $R_s = 2GM/c^2$. and the dimensions of Newton's constant are $[G] = M^{-1}L^3T^{-2}$. So we can write this as

$$\mathcal{P} = \left(\frac{R_s}{R}\right)^5 L_{\text{Planck}} \quad (5.42)$$

where the *Planck luminosity* is

$$L_{\text{Planck}} = \frac{c^5}{G} \approx 3.6 \times 10^{52} \text{ J s}^{-1}$$

This is a silly luminosity. The luminosity of the Sun is $L_\odot \approx 10^{-26} L_{\text{Planck}}$. With 10^{11} stars, the luminosity of the galaxy is $L_{\text{galaxy}} \approx 10^{-15} L_{\text{Planck}}$. There are roughly 10^{10} galaxies in the visible universe, which means that all the stars in the all the galaxies shine with a luminosity $\approx 10^{-5} L_{\text{Planck}}$.

Yet, when two black holes orbit and spiral towards each other, at the point where their separation is comparable to their Schwarzschild radius, the formula (5.42) tells us that the power they emit in gravitational waves is approximately L_{Planck} . For that brief moment before they collide, spiralling black holes emit more energy than all the stars in the visible universe.

Since the power emitted by colliding black holes is so ridiculously large, we might harbour some hope that we will still get a significant energy from more mundane systems. We could, for example, look at our solar system. The formula (5.42) assumes that the orbiting objects have the same mass. If two objects with masses $M_1 \gg M_2$ are in orbit, then (5.41) is replaced by

$$\mathcal{P} \sim \frac{G^4 M_1^3 M_2^2}{R^5}$$

(A derivation of this can be found on Examples Sheet 4.) Jupiter has a mass $10^{-3}M_{\odot}$ and orbits at a distance $\approx 10^9$ km from the Sun. Using the fact that the Schwarzschild radius of the Sun is $R_s \approx 3$ km, we find that the power emitted in gravitational waves by Jupiter is

$$\mathcal{P} \approx 10^{-50} L_{\text{Planck}} \approx 10^{-24} L_{\odot}$$

This is completely negligible. We can trace this to the power of 5 in (5.42) which means the fall-off in power is quick: extreme events in the universe emit a ridiculous amount of energy in gravitational waves. Events involving objects that are merely heavy emit essentially zero.

Of course, the question that we all really want to ask is: how much gravitational radiation can we emit by shaking our arms around? Suppose that we go really crazy, doing jumping jacks and generally acting like a loon. For once, SI units are useful. The mass of our arms is few kg, moving a distance of around a metre, with a frequency around a second. So $Q \approx 1 \text{ kg m}^2$ and $\ddot{Q} \approx 1 \text{ kg m}^2 \text{ s}^{-3}$. The power is then

$$\mathcal{P} \sim \frac{G\ddot{Q}^2}{c^5} \approx 10^{-52} \text{ J s}^{-1}$$

To put this in perspective, let's remind ourselves that ultimately the world is quantum and although we have no hope of detecting individual gravitons it is surely the case that gravitational waves come in quanta with energy $E = \hbar\omega$. So we could ask: how long do we have to wave our arms before we emit a single graviton? The energy of a graviton with frequency $\omega \approx 1 \text{ s}^{-1}$ is $E \approx 10^{-34} \text{ J}$. So the calculation above tells us that we can expect to emit a single graviton if we wave our hands around for

$$T = 10^{18} \text{ s}$$

This is more or less the age of the universe. You may be many things, but you are not a factory for making gravitons.