Gravitational Waves

January 2, 2019

1 Very General Equations

From Wikipedia::

In general relativity, the stress tensor is studied in the context of the Einstein field equations which are often written as

$$R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar (the tensor contraction of the Ricci tensor), $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constant (negligible at the scale of a galaxy or smaller), and G is the universal gravitational constant.

For starters, have $\Lambda = 0$ and $\kappa = \frac{8\pi G}{c^4}$:

$$R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2}$$

then (from [1]) and by assuming that the metric $g_{\mu\nu}$ representing the gravitational field has the form of a slightly perturbed Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \tag{3}$$

Here $0 < \epsilon \ll 1$, and this linearization simply means that we develope the left hand side of (0.1) in powers of ϵ and neglected all terms involving ϵ^k with $k > 1$. As a result of this linearization Einstein found the field equations of linearized general relativity, which can conveniently be written for an unknown

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu}\, h_{\alpha\beta}\, \eta^{\alpha\beta} \tag{4}$$

as

$$\Box \bar{h}_{\mu\nu} = 2\kappa T_{\mu\nu}, \qquad \Box = \eta_{\mu\nu}\, \partial^\mu \partial^\nu. \tag{5}$$

These equations, outside the sources where

$$T_{\mu\nu} = 0 \tag{6}$$

constitute a system of decoupled relativistic wave equations:

$$\Box h_{\mu\nu} = 0 \tag{7}$$

for each component of $h_{\mu\nu}$. This enabled Einstein to conclude that *linearized* general relativity theory admits solutions in which the perturbations of Minkowski spacetime $h_{\mu\nu}$ are plane waves traveling with the speed of light. Because of the *linearity*, by superposing plane wave solutions with different propagation vectors k_μ one can get waves having any desirable wave front. Einstein named these gravitational waves. He also showed that within the linearized theory these waves carry energy, and he found a formula for the energy loss in terms of the third time derivative of the quadrupole moment of the sources.

Since far from the sources the gravitational field is very weak, solutions from the linearized theory should coincide with solutions from the full theory. Actually the wave detected by the LIGO/Virgo team was so weak that it was treated as if it were a gravitational plane wave from the linearized theory. We also mention that essentially all visualizations of gravitational waves presented during popular lectures or in the news are obtained using linearized theory only.

e.g., notes from COTB 2014, Holz.

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x) \tag{8}$$

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z) \tag{9}$$

Quadrupole formula gives the total power radiated in gravitational waves:

$$L_{\mathrm{GW}} = \frac{G}{5c^5} \left\langle \dddot{I}_{ij} \dddot{I}^{ij} \right\rangle \tag{10}$$

Luminosity of GW sources:

$$L_{\mathrm{GW}} \sim \frac{c^5}{G} \sim \frac{2 \times 10^{42}}{7 \times 10^{-11}} \sim 10^{52} \text{ Joules} \tag{11}$$

Notes from THE MATHEMATICS OF GRAVITATIONAL WAVES A Two-Part Feature;
[1]: Part One: How the Green Light Was Given for Gravitational Wave Search by C. Denson Hill and Pawe Nurowski p.686
[2]: Part Two: Gravitational Waves and Their Mathematics, by Lydia Bieri,

David Garfinkle, and Nicolas Yunes, p 693
DOI: http://dx.doi.org/10.1090/noti1551.

Quora

Final paragraph:: In other words, we solve the penultimate wave equation and the solution represents plane gravitational waves traveling at the speed of light: transverse and longitudinal but only the transverse carry energy.

Also from Quora

2 Gravitational waves demystified

Straight from:: Teviet Creighton's Caltech page.

3 The strength of the (gravitational) quadrupole field

The strength of the quadrupole field, i.e. the amplitude of gravitational radiation, scales as:

$$g' \sim \frac{\ddddot{I}}{r} \tag{12}$$

$$g' \sim \frac{GM}{c^4}\frac{f^4 s^2}{r} \tag{13}$$

$$\sim GM\frac{s^2}{\lambda^4 r} \tag{14}$$

$$\tag{15}$$

where s is the mass (charge in EM) separation,

4 Dimensionless amplitude

$$g' = \frac{\Delta g}{d} \tag{16}$$

where Δg is the change in gravity and d is displacement.

The tidal field g' is the physically measurable part of gravitational phenomena: it represents an observable relative acceleration or force between two displaced "test masses". However, when discussing gravitational waves, the most common parameter describing the amplitude is a dimensionless *strain*,

$$h = 2\int\int g' dt^2. \tag{17}$$

What does this quantity mean? Remember that g' is a gravity gradient, so $g'\,d$ gives the difference in gravity, i.e. the differential acceleration, between two objects separated by a small displacement d. Two time integrals of acceleration give us the instantaneous change in this displacement as a function of time. Thus h is twice the fractional change in displacement between two nearby masses due to the gravitational wave. This change in displacement occurs in the plane *transverse to the direction of radiation*, and causes a stretch along one axis and a squeeze along the orthogonal axis. The net distortion is twice as much as a stretching or squeezing alone, which is the reason for the factor of 2 in the equations for h.

h is not itself directly observeable. A constant h, or an h that varies linearly with time, is exactly equivalent to starting the masses at slightly different positions, or with a slight relative velocity. *Only the second and higher derivatives of* h *produce accelerations that would indicate the presence of gravitational radiation.*

From the above scaling for g' we get $h \sim GMs^2/\lambda^2 r$ or:

$$h \sim \frac{GM}{c^2}\frac{1}{r}(\frac{v}{c})^2. \tag{18}$$

The first term is roughly the size of a black hole of mass M, so the distance r to the system must clearly be much greater. Similarly, v/c is the ratio of the speeds of masses in the system to the speed of light, which must be less than (usually much less than) unity. Thus h approaches unity when one is standing in the immediate vicinity of black holes moving about at lightspeed, and is less for any other circumstance.

In particular, the length scale of a "typical" black hole 10× as massive as our Sun is 14km, and such objects achive speeds around c only when they collide, which might occur on a yearly basis within a volume of radius $6{\times}10^{20}$km (20 Megaparsecs). So the strongest waves we expect to observe passing the Earth will have $h \sim 10^{-20}$ or less. This is enough to distort the shape of the Earth by 10^{-13} metres, or about 1% of the size of an atom. By contrast, the (nonradiative) tidal field of the Moon raises a tidal bulge of about 1 metre on the Earth's oceans.

5 Detecting GWs

5.1 Pulsar timing arrays

- 10^{-6} - 10^{-9} Hz.
- Sensitive to supermassive binary black holes with orbital periods of months.