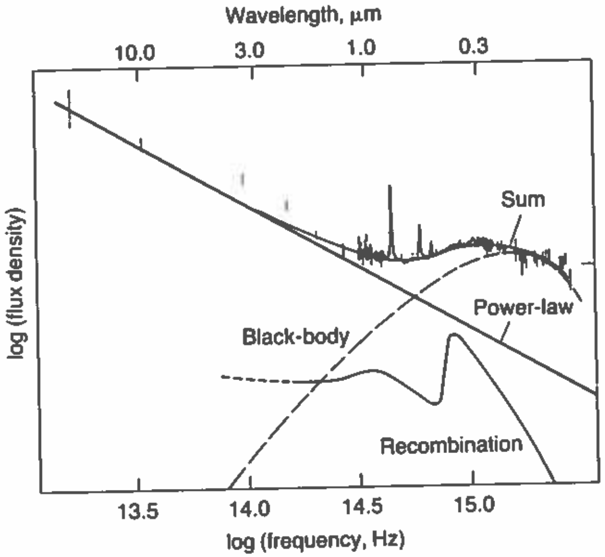

Figure 16.5. The optical–ultraviolet spectrum of the quasar 3C 273. The continuum has been decomposed into a 'power-law' component, a component associated with recombination radiation and a 'blue-bump' component, which has been represented by a black-body curve. The prominent Balmer series in the optical waveband which led to the discovery of the large redshift of 3C 273 can be seen. (From M. Malkan and W. L. W. Sargent (1982). *Astrophys. J.*, **254**, 33.)

This suggests that, for supermassive black holes with, say, $M = 10^8 M_\odot$, the thermal emission would have temperature $T \sim 2 \times 10^5$ K. It is intriguing that many active galactic nuclei have strong ultraviolet continua, 3C273 being a good example of a galaxy with a 'blue bump' (Fig. 16.5). This is, however, a very rough argument, and it certainly cannot be the whole story since these nuclei are just as powerful emitters in the X-ray waveband.

16.3 Thin accretion discs

The simplest cases of disc accretion are thin accretion discs. Many of their essential features are described by Pringle (1981) and in the monograph with the same title as this chapter, *Accretion power in astrophysics* by Frank, King and Raine (1992). In the simplest picture, we consider steady-state discs with a constant rate of mass accretion $\dot{m}$ into the disc. The matter in the disc would take up Keplerian orbits if there were no viscous forces present. We know, however, that these forces are essential in order to transfer the angular momentum of the accreted material outwards and so allow the gas to move inwards to more tightly

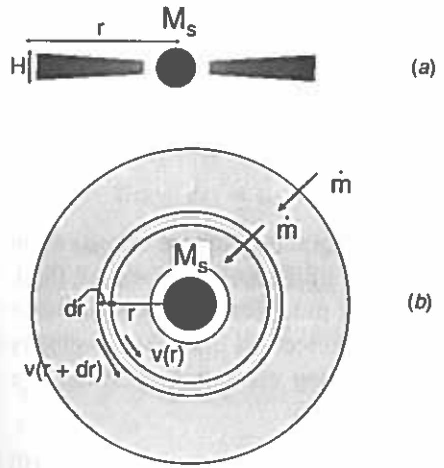

Figure 16.6. A schematic diagram illustrating the geometry of a thin accretion disc. (*a*) A side view; (*b*) a view from above the disc.

16.3.1 *Conditions for thin accretion discs*

The geometry of the thin accretion disc is shown in Fig. 16.6. It is assumed that the mass of material in the disc is much less than the mass of the central star, $M_{disc} \ll M_s$. First, we write down the condition for hydrostatic support in the direction perpendicular to the plane of the disc, that is, in the z direction:

$$\frac{\partial p}{\partial z} = -\frac{G M_s \rho \sin\theta}{r^2}$$

But $\sin\theta \approx z/r$, and, to order of magnitude, we can write $\partial p/\partial z \approx p/H$, where H is the half-thickness of the disc. Therefore, the condition for hydrostatic support in the z direction is

$$\frac{p}{H} = \frac{G M_s \rho H}{r^3} \tag{16.7}$$

Now, although the gas is slowly drifting into the centre, the gas moves in roughly Keplerian orbits about the star and so is in centrifugal equilibrium, that is, at any radius,

$$\frac{v_\phi^2}{r} = \frac{GM}{r^2} \qquad \frac{GM}{r} = v_\phi^2 \tag{16.8}$$

Substituting for GM/r from equation (16.8) into equation (16.7), we find

$$\frac{p}{\rho} \approx v_\phi^2 \frac{H^2}{r^2}$$

But the speed of sound in the material comprising the disc is $c_s^2 \approx p/\rho$, and therefore to the local sound speed in the disc. Thus, the condition that the thin disc approximation can be adopted is that the rotation velocity of the disc be very much greater than the sound speed; in other words, internal pressure gradients should not inflate the disc. This is exactly the same condition which is found for the confinement of the cool gas in the plane of the disc of a spiral galaxy such as our own.

16.3.2 *The role of viscosity – the α parameter*

Viscosity plays a central role in determining the structure of accretion discs. We need expressions for the viscous forces in a differentially rotating fluid. Let us recall the definition of viscosity in the case of fluid flow. The simplest case is unidirectional flow of the fluid in the positive x direction but with a velocity gradient in, say, the y direction. Then, the force acting on unit area in the x–z plane, that is the shear stress, is given by the expression

$$f_x(y) = \eta \frac{\partial v_x(y)}{\partial y} \tag{16.10}$$

where η is defined to be the *dynamic* or *shear viscosity*. In the case of two-dimensional flow, the expression for the shear stress acting at the point (x, y) is given by the more general form

$$f_{xy} = \eta \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right] \tag{16.11}$$

because of the symmetry of the stress tensor, $f_{xy} = f_{yx}$ (see, for example, Landau and Lifshiftz, (1959), section 15). For the case of an accretion disc in which the gas moves in circular orbits, it is simplest to convert this expression into polar coordinates, in which $x = r\cos\phi$ and $y = r\sin\phi$. Then, we write the components of the velocity in cylindrical coordinates as

$$v_x = v_r \cos\phi - v_\phi \sin\phi \qquad v_y = v_r \sin\phi + v_\phi \cos\phi$$

and convert the differentials into polar coordinates through the relations

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \qquad \frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}$$

Because of the cylindrical symmetry of the problem, it is convenient to evaluate the differentials at $\phi = 0$, in which case we find

$$f_{xy} = \eta \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right] = \frac{1}{r}\left(\frac{\partial v_x}{\partial \phi}\right)_{\phi=0} + \left(\frac{\partial v_y}{\partial r}\right)_{\phi=0}$$

$$\frac{1}{r}\left(\frac{\partial v_x}{\partial \phi}\right)_{\phi=0} = -\frac{v_\phi}{r} \qquad \left(\frac{\partial v_y}{\partial r}\right)_{\phi=0} = \frac{\partial v_\phi}{\partial r}$$

Hence,

$$f = \eta\left(-\frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r}\right) = \eta r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) = \eta r \frac{\partial \Omega}{\partial r} \tag{16.12}$$

Notice that this result shows correctly that there is no shear stress if the fluid

assume the disc has thickness H, the torque acting on the inner edge of th annulus at r is

$$G = \eta r (2\pi r^2 H)\frac{\partial \Omega}{\partial r}$$

The torque on the outer edge is

$$G(r + \mathrm{d}r) = G(r) + \frac{\partial G}{\partial r}\mathrm{d}r$$

and so the net torque acting on the annulus is the difference of these torque $(\partial G/\partial r)\mathrm{d}r$. The equation of motion of the annulus is therefore

$$\frac{\partial \mathscr{L}}{\partial t} = \frac{\partial}{\partial r}\left[\eta r (2\pi r^2 H)\frac{\partial \Omega}{\partial r}\right]\mathrm{d}r$$

where $\mathscr{L}$ is the angular momentum of the annulus. Since $\mathscr{L} = 2\pi r^2 H \rho v_\phi \mathrm{d}r$, we find

$$r^2 \frac{\partial v_\phi}{\partial t} = \frac{\eta}{\rho}\frac{\partial}{\partial r}\left[r^3 \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

It is conventional to work in terms of the *kinematic viscosity* $\nu = \eta/\rho$, and so the equation of motion for the circumferential velocity is

$$\frac{\partial v_\phi}{\partial t} = \frac{\nu}{r^2}\frac{\partial}{\partial r}\left[r^3 \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \tag{16.13}$$

This is the basic relation we have been seeking because it indicates the role of viscosity in determining the structure of the accretion disc. There is, however, a basic problem: the viscosity is very low and the *Reynolds number* is very large. For any flow, the Reynolds number, $\mathscr{R}$, is defined to be the quantity

$$\mathscr{R} \approx \frac{L^2}{\nu T} = \frac{VL}{\nu} \tag{16.14}$$

where L, T and V are typical dimensions of length, time and velocity for the flow. In the typical situations in which viscous forces play an important role, $\mathscr{R} \sim 1$, as, for example, in the flow of a viscous fluid about a cylinder. At large Reynolds numbers, $\mathscr{R} \geq 10^3$, the flow becomes turbulent (see, for example, Landau and Lifshitz (1959), Feynman (1962), Batchelor (1967)).

Let us evaluate the Reynolds number for the material of the disc. In this case, V is the velocity of rotation of the material of the disc and L is the characteristic scale over which the velocity changes, which is just the typical radial distance from the centre of the disc. According to classical kinetic theory, the dynamic viscosity is given by $\eta = \rho\nu = \frac{1}{3}\rho v_s \lambda$, where ρ is the mass density of the gas, v_s is the internal sound speed, which is roughly the same as the typical speed of the particles in the gas, and λ is the mean free path of the particles. To order of magnitude, we therefore find that

$$\mathscr{R} \sim \frac{VL}{v_s \lambda} \tag{16.15}$$

the source is 1.3×10^{31} W, and we can equate this to the accretion luminosity $L = \frac{1}{2}\dot{m}c^2(r_g/r)$, where r is the radius of the region from which most of the luminosity is generated. For illustrative purposes, let us adopt $r = 10r_g$, corresponding to roughly three times the radius of the neutron star. We can then find the accretion rate $\dot{m} \sim 10^{15}$ kg s^{-1}. Now, for the thin disc, the accretion rate $\dot{m}$ is

$$\dot{m} = (2\pi r) \times (2H) \times \rho v_r \tag{16.16}$$

where H is the half-width of the disc at radius r and v_r is the radial inward drift velocity. We can make a rough estimate for H from the expression (16.9) so that $H = r/\mathcal{M}$, where $\mathcal{M}$ is the Mach number of the rotation velocity of the disc relative to the sound speed in the disc. We can find $\mathcal{M}$ from the Keplerian velocity of the disc at $R = 10r_g$ and the sound speed from the fact that the temperature of the disc must be about 10^7 K from the arguments presented in Section 16.2.2. From these, we find $\mathcal{M} \sim 200$. The only remaining uncertainty is the radial drift velocity v_r. This velocity is certainly sub-sonic with respect to the speed of sound in the disc, and so let us write $v_r = \beta v_s$, where $\beta \ll 1$. Inserting these values into the expression (16.16), we find that the mass density in the disc $\rho \sim 0.1\beta^{-1}$ kg m^{-3}. Thus, if, say, $\beta = 0.01$, a typical value which comes out of more detailed models of accretion discs, we find that the mass densities are large, $\rho \sim 10$ kg m^{-3}. This mass density is in good agreement with more detailed models of the structure of accretion discs responsible for the X-ray emission from neutron stars in close binary systems.

The reason for carrying out this calculation is that we need the mass density, ρ, to work out the mean free path, λ, of the particles in the disc. We have already performed this calculation for the case of protons in Solar Wind, and exactly the same physics applies in this case. According to the expression (10.4), the mean free path of a proton is

$$\lambda \approx v_s t_c = 11.4 \times 10^6 \frac{T^{3/2} A^{1/2} v_s}{N Z^4 \ln\Lambda} \quad \mathrm{m} \tag{16.17}$$

Inserting the values we have derived into the expression for the Reynolds number, we find $\mathcal{R} \sim 10^{12}$.

This is a key result in the astrophysics of accretion discs. A similar answer is found for the accretion discs about white dwarfs. The result has two implications. First, ordinary viscosity associated with the deflection of charged particles in the plasma (Section 10.4) cannot play a major role in determining the structure of the disc. According to this analysis, the flow would be strongly turbulent. The result also provides a possible solution to the problem. The generation of turbulence within the disc results in a *turbulent viscosity*, which can perform all the functions of normal molecular viscosity but now the transport of momentum is associated with the motion of turbulent eddies in the plasma. There is also likely to be a magnetic field in the disc, and this provides a further means of transporting momentum on a large scale. Probably we should be talking

can write down an expression for the turbulent viscosity as $\nu_{turb} \sim \lambda_{turb} v_{turb}$, wh λ_{turb} is the scale of the eddies and v_{turb} is their rotational or 'turn-over' velocit

To overcome this problem, Sunyaev and Shakura (1972) introduced the f lowing prescription for the turbulent viscosity, $\nu = \alpha v_s H$, where v_s is the spe of sound in the disc and H is its scale height in the z direction. The turb lent eddies must have dimension less than the scale height of the disc, and t turn-over velocities must be less than the speed of sound. Therefore, we exp α to be less than or equal to one. There is, therefore, little physics involved this prescription of the viscosity to be used in the study of accretion discs, a there is little theoretical understanding of how α should vary through the disc a function of density and temperature. The advantage of this formalism is th analytic solutions can be found for the structure of thin accretion discs in terr of the single parameter α – these solutions are often referred to as *α discs*. On thin disc models have been found which account for the observed properties X-ray sources, empirical values for the parameter α can be found.

16.3.3 *The structure of thin discs*

We can now derive further properties of thin discs by estimating the ra at which mass drifts in through the disc, that is, the radial drift velocity, v_r. L us consider only steady-state discs in which the inflow velocity v_r is determine by the viscosity ν. We have already used the equation of conservation of mas which simply states that, in the steady state, the mass flow through any radius i a constant:

$$\dot{m} = 2\pi r v_r \int \rho\, \mathrm{d}z = \text{constant}$$

where the integral takes account of the inflow through the full thickness of th disc in the z direction. It is convenient to work in terms of the *surface density* o the disc, which is defined to be $\Sigma = \int \rho \mathrm{d}z$. Therefore,

$$\dot{m} = 2\pi r v_r \Sigma = \text{constant} \tag{16.18}$$

First of all, we need an expression for the torque G acting on the cylindrica surface at radius r from the centre of the disc. According to equation (16.12),

$$f = \eta r \frac{\partial \Omega}{\partial r}$$

The total shear force acting on the cylindrical surface at radius r is therefore fA, and the torque $G = fAr$. Setting $v_\phi = \Omega r$, we find

$$G = 2\pi r^3 \nu \frac{\mathrm{d}\Omega}{\mathrm{d}r} \int \rho\, \mathrm{d}z = 2\pi r^3 \nu \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}r} \tag{16.19}$$

Now we consider the transport of angular momentum through an annular region of the disc between radii r and $r + \Delta r$. In the steady state ...

r and $r + \Delta r$, that is,

$$G(r) - G(r + \Delta r) = -\frac{\partial G}{\partial r}\Delta r \tag{16.20}$$

where G is given by the expression (16.19).

Now let us work out the rate of transport of angular momentum through the surface at r. The mass transfer per second is given by the expression (16.18), and therefore the angular momentum transport is

$$\dot{m}v_\phi r = 2\pi r^3 \Sigma v_r \Omega \tag{16.21}$$

In exactly the same way, the angular momentum transport at radius $r + \Delta r$ is

$$(\dot{m}v_\phi r)_{r+\Delta r} = (2\pi r^3 \Sigma v_r \Omega)_{r+\Delta r}$$

We can now make a Taylor expansion and subtract from the expression (16.21). We find

$$\Delta\mathscr{L} = 2\pi \frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 \Sigma v_r \Omega\right) \Delta r \tag{16.22}$$

per unit time. Equating equations (16.20) and (16.22), we obtain

$$\frac{\mathrm{d}G}{\mathrm{d}r} = 2\pi \frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 \Sigma v_r \Omega\right) \tag{16.23}$$

We can immediately take the integral of this equation to find

$$G = 2\pi r^3 \Sigma v_r \Omega + C$$

or, using the expression (16.19) for G,

$$\nu\Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}r} = \Sigma v_r \Omega + \frac{C}{2\pi r^3} \tag{16.24}$$

where C is a constant. This constant is to be found from the boundary conditions, in particular, by the matching of the rotational velocity at the surface of the star to the velocity of rotation at the inner edge of the accretion disc. The matching of these velocities takes place through a boundary layer of thickness b, and this slightly complicates the analysis. We will, however, press on with the relevant approximations for thin discs in which the boundary layer is assumed to be thin compared with the radius of the star, $r_\bullet$. These complications are clearly described by Frank, King and Raine (1992). Within the boundary layer, the matter is dragged round the star in solid-body rotation and so out to radius $r_\bullet + b$, $\mathrm{d}\Omega/\mathrm{d}r = 0$. Hence, at the radius $r_\bullet + b$,

$$C = -2\pi(r^3 \Sigma v_r \Omega)_{r_\bullet + b} \tag{16.25}$$

Now, at this radius, the velocity of rotation must also be the rotation velocity of material in a Keplerian orbit about the star, $\Omega^2 = GM_\bullet/r^3$. Hence,

$$C = -2\pi[r^{3/2}\Sigma v_r (GM_\bullet)^{1/2}]_{r_\bullet + b}$$

But $\dot{m} = 2\pi r v_r \Sigma =$ constant at any radius, and so

[illegible]

Notice that the constant C is just the rate of transfer of angular momentum int the boundary layer.

In the disc itself, the velocities are close to Keplerian, and so, substitutin equations (16.26) and (16.18) into equation (16.24), we find the pleasant result

$$\nu\Sigma = \frac{\dot{m}}{3\pi}\left[1 - \left(\frac{r_\bullet}{r}\right)^{1/2}\right] \tag{16.27}$$

The next result we need is the rate of dissipation of energy by the viscous force acting in the disc. The expression found in the standard text-books for the hea generated per unit volume (for example, Landau and Lifshitz (1959), section 16 is given in Cartesian coordinates as:

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{1}{2}\eta \sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 \tag{16.28}$$

where the vector $\mathbf{v} = (v_x, v_y, v_z)$ is the fluid velocity. It is convenient to convert this expression into cylindrical polar coordinates using the usual relations $x = r\cos\phi; y = r\sin\phi; z = z$ and $v_x = v_r\cos\phi - v_\phi\sin\phi; v_y = v_r\sin\phi + v_\phi\cos\phi; v_z = 0$. In the case of axial symmetry, the dissipation rate takes the simple form

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \eta r^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 \tag{16.29}$$

As usual, we integrate over the z coordinate, and then the dissipation rate is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \int \eta r^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 \mathrm{d}z = \nu\Sigma r^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 \tag{16.30}$$

We can now substitute for $\nu\Sigma$ from the expression (16.27), and hence, assuming the orbits of the matter in the disc are closely Keplerian,

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{3G\dot{m}M_\bullet}{4\pi r^3}\left[1 - \left(\frac{r_\bullet}{r}\right)^{1/2}\right] \tag{16.31}$$

This is the elegant result we have been seeking and which makes the study of thin discs so attractive – the energy dissipation rate does not depend explicitly upon the viscosity ν. The viscosity has been absorbed into the requirement of steady-state accretion at a constant rate $\dot{m}$. On the other hand, more detailed properties of the disc, such as the surface density, do depend upon ν. We assume that, in the steady state, this heat energy is carried away by radiation, and so the luminosity of the disc is found by integrating the heat dissipation rate (16.31) from $r_\bullet$ to infinity:

$$L = \int_{r_\bullet}^{\infty}\left(-\frac{\mathrm{d}E}{\mathrm{d}t}\right) 2\pi r \mathrm{d}r = \frac{G\dot{m}M_\bullet}{2r_\bullet} \tag{16.32}$$

This is a very sensible result. The matter falling in from infinity passes through a series of bound Keplerian orbits for which the kinetic energy is equal to half of the gravitational potential energy. Thus, the matter has to dissipate half of the total potential energy which it would acquire in falling from infinity to that radius. This gravitational potential energy can be dissipated. This calculation indicates that the boundary layer can be just as important a source of luminosity as the disc itself.

It is interesting to look at this result in terms of the rate of dissipation of energy, or luminosity $L(r)$, in the annulus between radii r and $r+\Delta r$. Multiplying equation (16.31) by $2\pi r\Delta r$, the luminosity of the annulus is

$$L(r) = -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{3G\dot{m}M_*}{2r^2}\left[1-\left(\frac{r_*}{r}\right)^{1/2}\right]\Delta r \tag{16.33}$$

This is an interesting answer because it is apparent that this expression is more than simply the release of gravitational binding energy when matter moves from a Keplerian orbit at $r+\Delta r$ to one at r, which is simply $(G\dot{m}M_*/2r^2)\Delta r$. The difference between this expression and equation (16.33) represents the net flow of energy into the annulus Δr associated with the transport of angular momentum outwards. Thus, although the total energy released in reaching the surface of the star is simply half the gravitational potential energy, at any radius, the energy dissipation rate consists of both the energy loss due to angular momentum transport as well as the release of gravitational binding energy. From the expression (16.33), it can be seen that, at distances $r \gg r_*$, the energy dissipation rate is

$$L(r) = -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{3G\dot{m}M_*}{2r^2}\Delta r \tag{16.34}$$

which is three times the rate of release of gravitational binding energy.

16.3.4 *Accretion discs about black holes*

Now let us extend the analysis to the case of black holes rather than considering objects with a solid surface. The problem is to determine what boundary condition should replace the expression (16.26). In the case of black holes, the matter drifts inwards through the accretion disc until it reaches the last stable orbit, from which it rapidly spirals into the black hole. As indicated in Section 15.6.1, the angular velocity now helps, rather than hinders, the collapse of matter into the black hole, since, crudely speaking, the rotational energy now contributes to the inertial mass of the infalling matter. Let us suppose that the last stable orbit has radius r_I. We recall that, in the case of a non-rotating, spherically symmetric black hole, $r_\mathrm{I} = 3r_\mathrm{g} = 6GM/c^2$. Then, according to an entirely classical calculation, the condition that the matter can spiral into the hole is simply that the rotational energy of the matter should be less than or equal to half the gravitational potential energy. In other words, $\frac{1}{2}\mathscr{L}^2/I \le \frac{1}{2}GMm/r_\mathrm{I}$, where $\mathscr{L}$ is the angular momentum of the element of mass m and $I = mr_\mathrm{I}^2$ is its moment of inertia about the hole in the last stable orbit. It is convenient to work in terms of the *specific angular momentum* $J = \mathscr{L}/m$. Therefore, the condition that the matter fall into the hole is

$$J \le (GMr_\mathrm{I})^{1/2}$$

the black hole. It is conventional to write the angular momentum of the mater as it arrives at r_I as $\mathscr{L} = \beta\dot{m}(GMr_\mathrm{I})^{1/2}$, where $\beta \le 1$. This, then, becomes t boundary condition which replaces the expression (16.26). Therefore, for the ca of black holes, we can write the expression for the luminosity of the disc betwee r and $r+\Delta r$ as follows:

$$L(r) = -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{3G\dot{m}M_*}{2r^2}\left[1-\beta\left(\frac{r_\mathrm{I}}{r}\right)^{1/2}\right]\Delta r \tag{16.3}$$

Exactly as before, we find the total luminosity of the disc by integrating equatic (16.35) from r_I to infinity. The result is

$$L = \left(\frac{3}{2}-\beta\right)\frac{G\dot{m}M_*}{r_\mathrm{I}} \tag{16.3}$$

16.3.5 *The temperature distribution and emission spectra of thin discs*

Let us now make some simple estimates of the temperature distributio and emission spectrum of the disc. Suppose the disc is optically thick to radiatio and that there is sufficient scattering to ensure that we can approximate th emission as black-body radiation at each point in the disc. The disc can radiat from its top and bottom surfaces, and so we can equate the heat dissipate between r and $r+\Delta r$, expression (16.34) for simplicity, to $2\sigma T^4 \times 2\pi r\Delta r$, wher σ is the Stefan–Boltzmann constant:

$$\sigma T^4 = \frac{3G\dot{m}M_*}{8\pi r^3} \qquad T = \left(\frac{3G\dot{m}M_*}{8\pi r^3\sigma}\right)^{1/4} \tag{16.37}$$

Thus, in the outer regions of the disc, $r \gg r_*$, the temperature of the matter is expected to increase towards the centre as $T \propto r^{-3/4}$. A consequence of this result is that the sound speed in the disc increases towards the centre, and we need more detailed calculations to ensure that all the approximations we have made are indeed appropriate.

If we assume that each annulus of the disc radiates like a black body at the appropriate temperature given by the expression (16.37), we can derive the form of the integrated spectrum of the disc. The total intensity of the disc is proportional to the surface area at temperature T times the black-body intensity at that temperature. Thus,

$$I(\nu) \propto \int_{r_\mathrm{I}}^{r_{\max}} 2\pi r B[T(r),\nu]\,\mathrm{d}r \tag{16.38}$$

where $B(T,\nu)$ is the Planck function $B(\nu) \propto \nu^3[\exp(h\nu/kT)-1]^{-1}$. But, we know that $T \propto r^{-3/4}$ and so $\mathrm{d}r \propto (1/T)^{1/3}\mathrm{d}(1/T)$. Therefore, we can convert the integral over $\mathrm{d}r$ into an integral over $(1/T)$. Carrying this out and preserving the dependence upon frequency, we find

$$I(\nu) \propto \int_{r_\mathrm{I}}^{r_{\max}} \left(\frac{1}{T}\right)^{4/3}\nu^3\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}\left(\frac{1}{T}\right)^{1/3}\mathrm{d}\left(\frac{1}{T}\right) \tag{16.39}$$

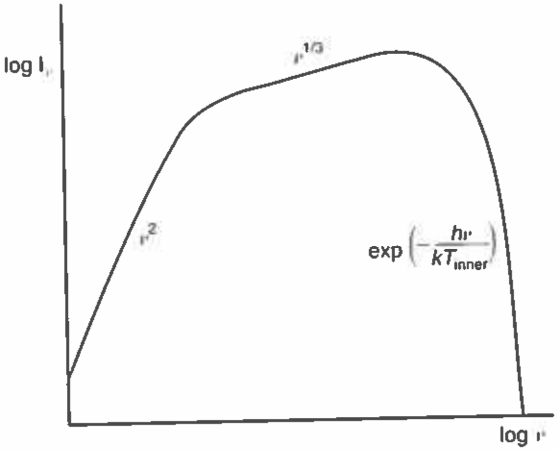

Figure 16.7. A schematic representation of the emission spectrum of an optically thick accretion disc. The exponential cut-off at high frequencies occurs at frequency $\nu = kT_{\rm inner}/h$, where $T_{\rm inner}$ is the temperature of the innermost layers of the thin accretion disc.

But, we now notice that the integral is a definite integral, which is just a constant, and all the dependences upon ν are outside the integral. Therefore,

$$I(\nu) \propto \nu^{1/3} \tag{16.41}$$

Thus, the predicted spectrum of a thin, optically thick accretion disc in the black-body approximation is that the spectrum should have the form $I(\nu) \propto \nu^{1/3}$ between the frequencies corresponding to $r_{\rm I}$ and $r_{\rm max}$, as illustrated in Fig. 16.7. At frequencies less than that corresponding to the temperature of the disc at $r_{\rm max}$, the spectrum tends towards a Rayleigh–Jeans spectrum, $I_\nu \propto \nu^2$.

16.3.6 *Detailed models of thin discs*

The analysis of Section 16.3.3 is almost as far as we can go without constructing proper models for thin discs. It was shown by Sunyaev and Shakura (1972) that, adopting the α-disc approach to the definition of the turbulent viscosity, it is possible to derive eight equations which can be solved in closed form for the structure of thin accretion discs in terms of eight unknown parameters. This analysis is straightforward, if lengthy, and the results are quoted by Frank, King and Raine (1992). It turns out that many of the properties of thin discs are only weakly dependent upon the viscosity parameter α, which is encouraging from the point of view of the confrontation with observation but disappointing from the point of view of understanding more about the nature of the viscosity in the disc.

An important aspect of these solutions is that they enable the detailed properties

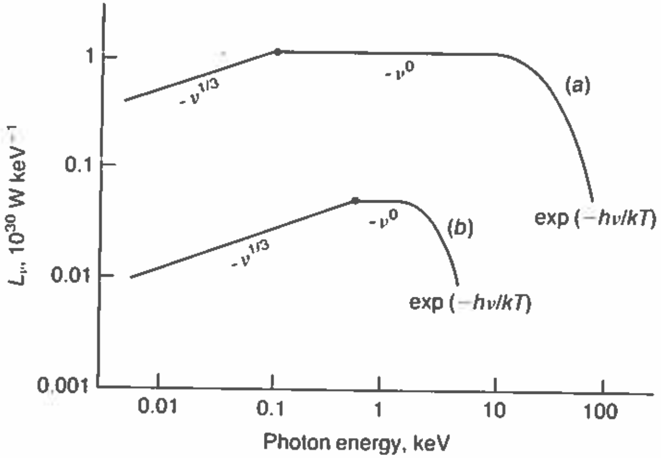

Figure 16.8. The spectra of the radiation emitted by accretion discs about a spherical, non-rotating black hole. These models are due to Sunyaev and Shakura (1972). In mode (*a*), the black-hole mass is assumed to be $1M_\odot$, the accretion rate $3 \times 10^{-8} M_\odot$ year^{-1} and the accretion takes place at the Eddington limiting luminosity. In model (*b*), the mass accretion rate is less, $3 \times 10^{-10} M_\odot$ year^{-1}, and the luminosity is 10^{29} W. The radiation generated in the outer cool regions of the disc result in a power-law spectrum $I(\nu) \propto \nu^{1/3}$. In the inner regions, electron scattering is the dominant source of opacity, and the spectrum is approximately independent of frequency. The temperature of the exponential tail corresponds to the surface temperature of the inner regions of the disc modified by the effects of electron scattering. (After S.L. Shapiro and S.A. Teukolsky (1983). *Black holes, white dwarfs and neutron stars: the physics of compact objects*, p. 446 New York: Wiley Interscience.)

regions: the outer region, in which the gas pressure is much greater than the radiation pressure and the opacity is dominated by free–free (or bremsstrahlung absorption; a middle region, in which the gas pressure is still dominant but the dominant source of opacity is electron scattering; and an inner region, in which the radiation pressure dominates and electron scattering is the most important source of opacity. These results are derived from detailed models of the structure of the accretion discs. As a result, the emitted spectrum of the disc in these regions cannot be approximated by a black-body spectrum. In the innermost regions, the disc may be optically thin to free–free absorption, even taking into account the multiple scattering of the radiation by the electrons, and then the predicted emission continuum spectrum is characteristic of optically thin bremsstrahlung, $I(\nu) \propto \nu^0$. Examples of the predicted spectra of the radiation for various assumptions about the accretion rate and the viscosity parameter α are shown in Fig. 16.8.

16.3.7 *Thick discs*