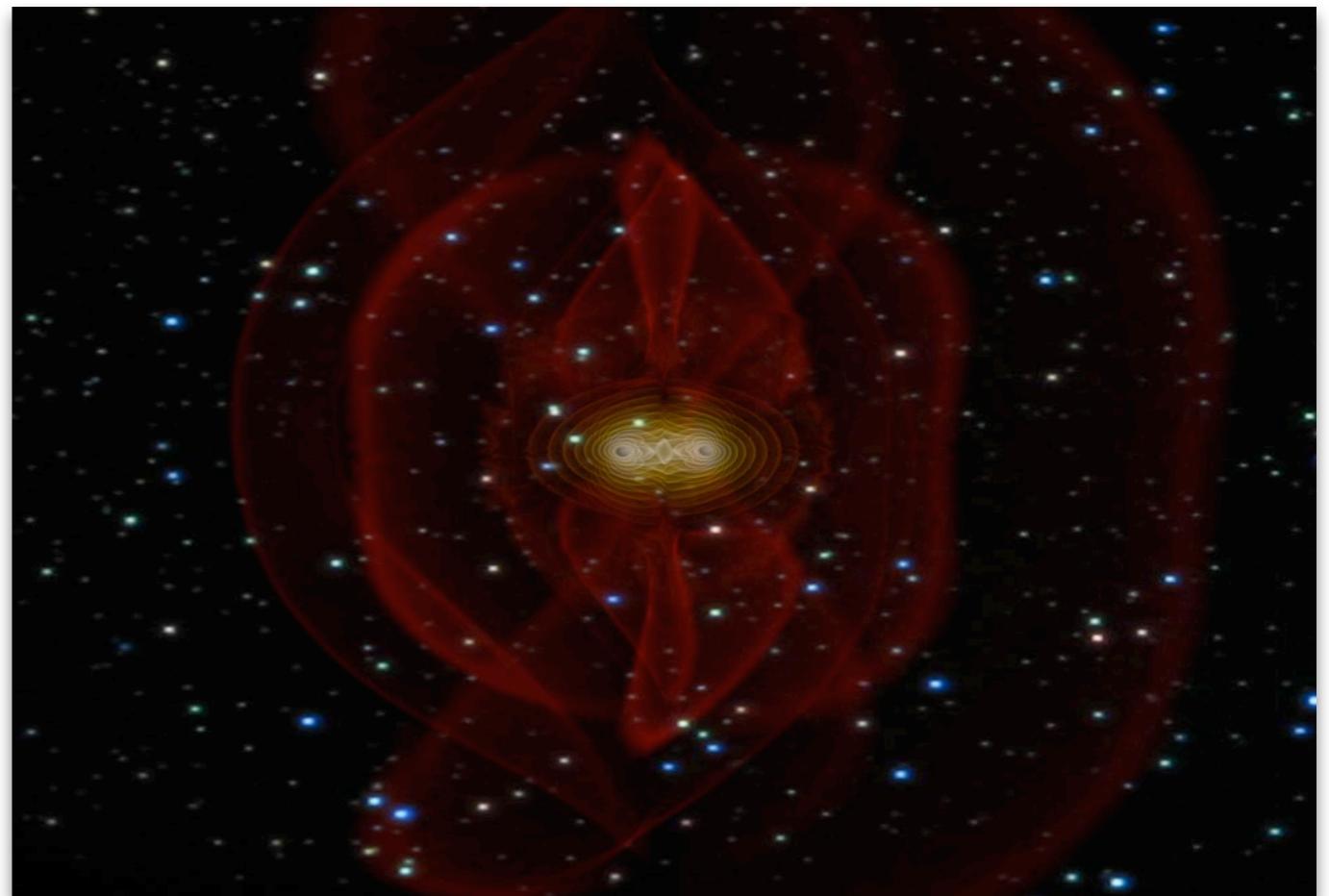


Gravitational Waves

Theory - Sources - Detection

Kostas Glampedakis



Contents

- **Part I: Theory of gravitational waves.**

- ✓ Properties.
- ✓ Wave generation/the quadrupole formula.
- ✓ Basic estimates.

- **Part II: Gravitational wave sources.**

- ✓ Binary systems/the Hulse-Taylor pulsar.

- **Part III: Detection of gravitational waves.**

- ✓ Resonant detectors/Laser interferometers.
- ✓ Ground/Space-based detectors.

Part I

GW theory



Newtonian vs General Relativistic gravity

Newtonian field equations

$$\nabla^2 \Phi = 4 \pi G \rho$$

Source: mass density

Gravitational field: scalar Φ

GR field equations

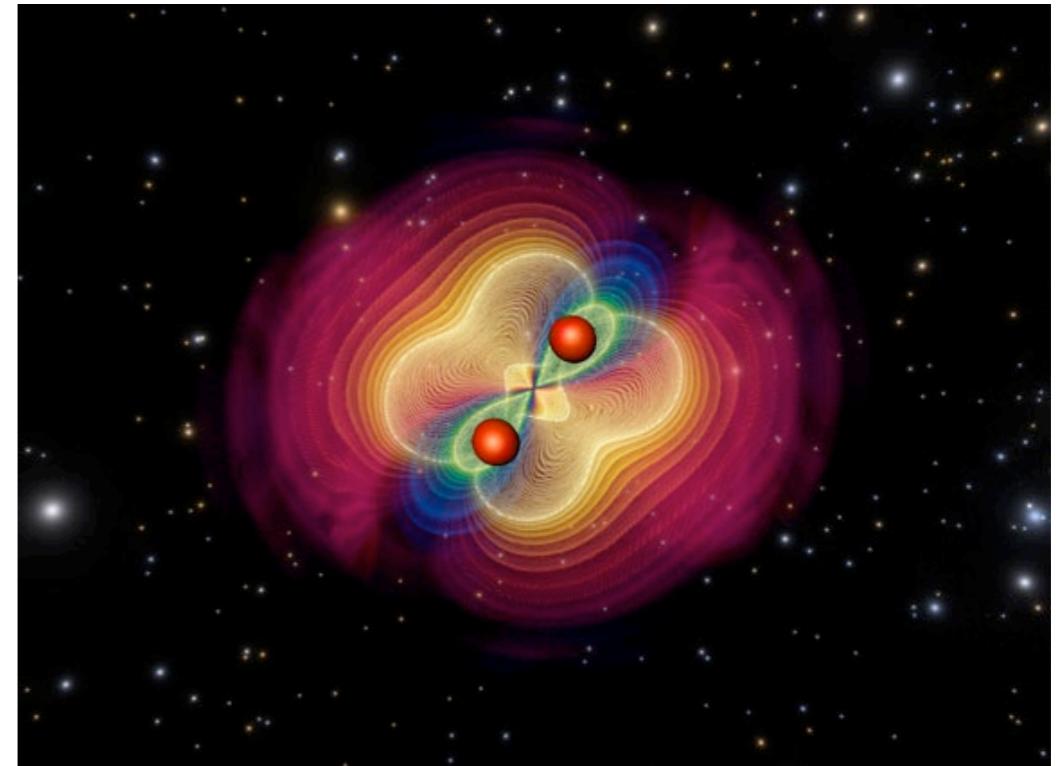
$$G^{ab} = \frac{8\pi G}{c^4} T^{ab}$$

Source: **energy-momentum tensor**
(includes mass densities/currents)

Gravitational field: **metric tensor** g_{ab}

GWs: origins

- Electromagnetism: accelerating charges produce EM radiation.



- Gravitation: accelerating masses produce gravitational radiation.
(another hint: gravity has finite speed.)

GWs in linear gravity

- We consider weak gravitational fields:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h_{\mu\nu}^2)$$

↑
flat Minkowski metric

- The GR field equations in vacuum reduce to the standard wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) h^{\mu\nu} = \square h^{\mu\nu} = 0$$

- Comment: GR gravity like electromagnetism has a “gauge” freedom associated with the choice of coordinate system. The above equation applies in the so-called “transverse-traceless (TT)” gauge where

$$h_{0\mu} = 0, \quad h_\mu^\mu = 0$$

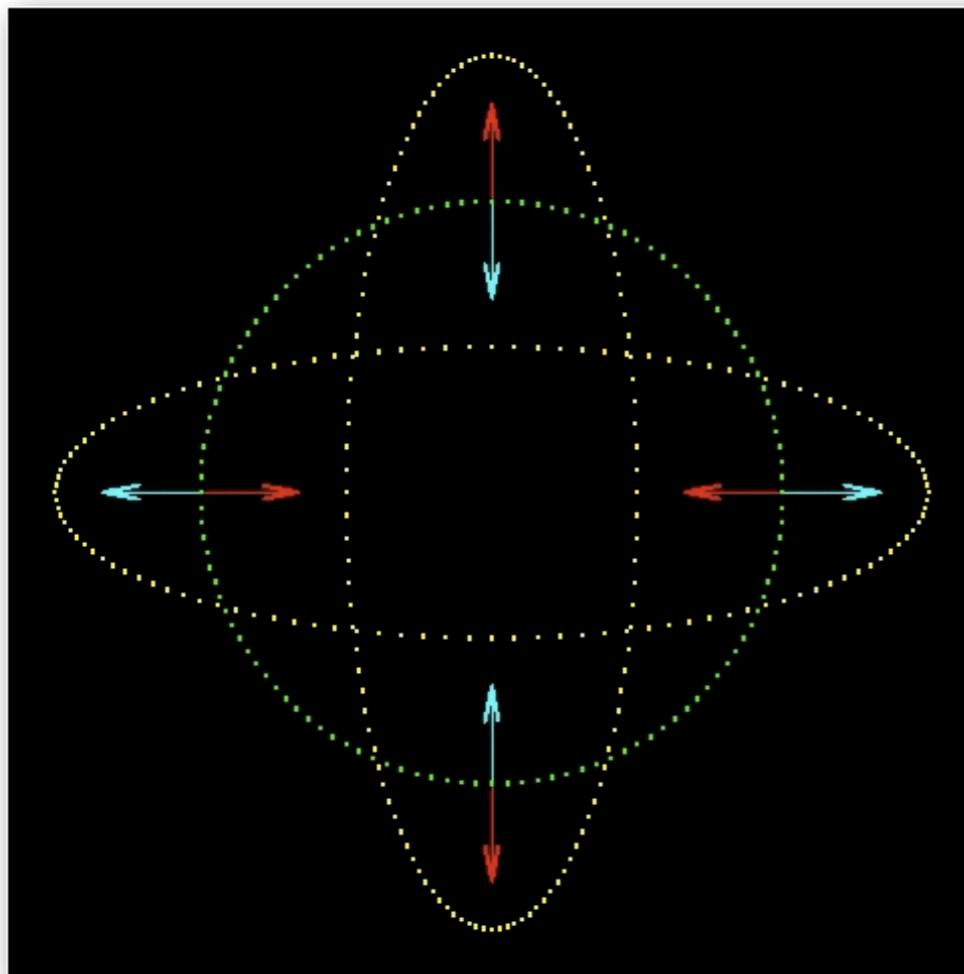
Wave solutions

- Solving the previous wave equation in weak gravity is easy. The solutions represent “plane waves”:

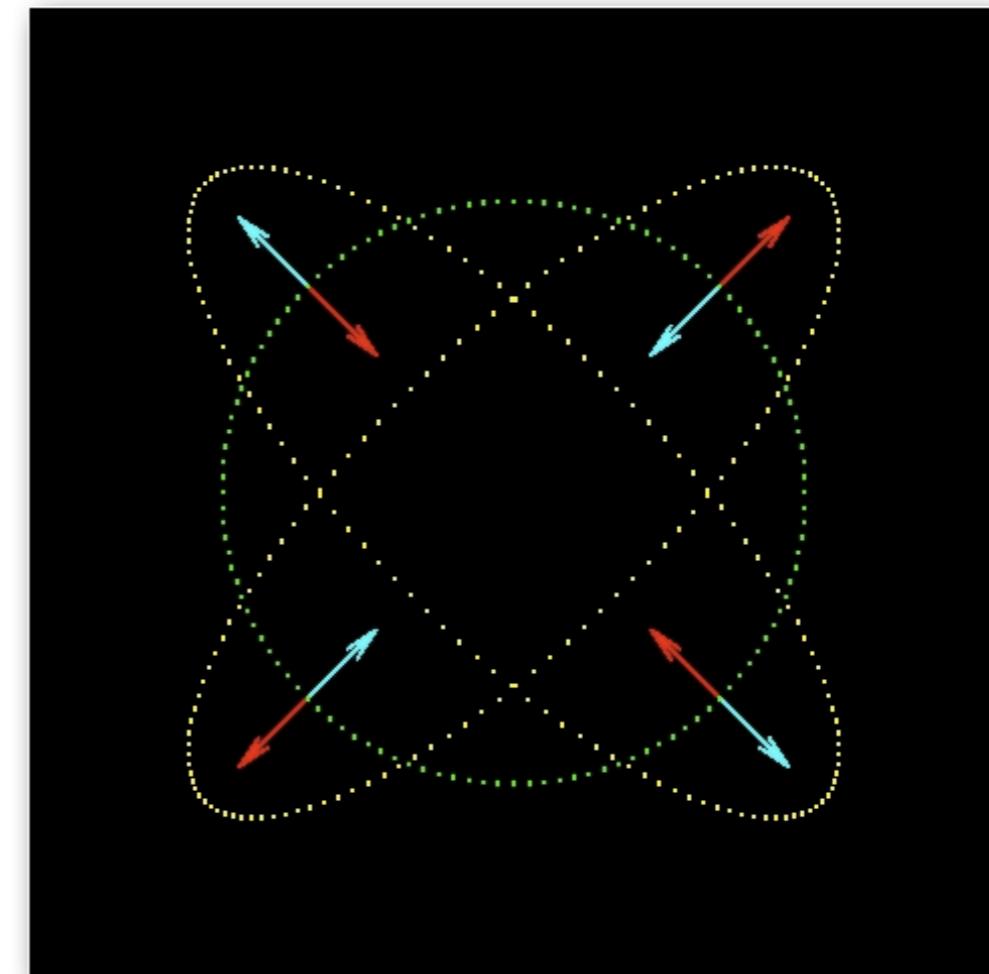
$$h_{\mu\nu} = A_{\mu\nu} e^{ik_a x^a}$$

GWs: polarization

- GWs come in two polarizations:



“+” polarization



“x” polarization

GWs: more properties

- EM waves: at lowest order the radiation can be emitted by a dipole source ($l=1$). Monopolar radiation is forbidden as a result of charge conservation.
- GWs: the lowest allowed multipole is the **quadrupole** ($l=2$). The monopole is forbidden as a result of mass conservation. Similarly, dipole radiation is absent as a result of momentum conservation.
- GWs represents propagating “ripples in spacetime” or, more accurately, a **propagating curvature perturbation**. The perturbed curvature (Riemann tensor) is given by (in the TT gauge):

$$R_{j0k0}^{\text{TT}} = -\frac{1}{2} \partial_t^2 h_{jk}^{\text{TT}}, \quad j, k = 1, 2, 3$$

GWs and curvature

- As we mentioned, GWs represent a fluctuating curvature field.
- Their effect on test particles is of tidal nature.
- Equation of **geodetic deviation** (in weak gravity):

$$\frac{d^2 \xi^k}{dt^2} = -R_{0j0}^{k\text{ TT}} \xi^j$$



distance between geodesics (test particles)

- Newtonian limit: $R_{k0j0}^{\text{TT}} \approx \frac{\partial^2 \Phi}{\partial x^k \partial x^j}$

Newtonian grav. potential

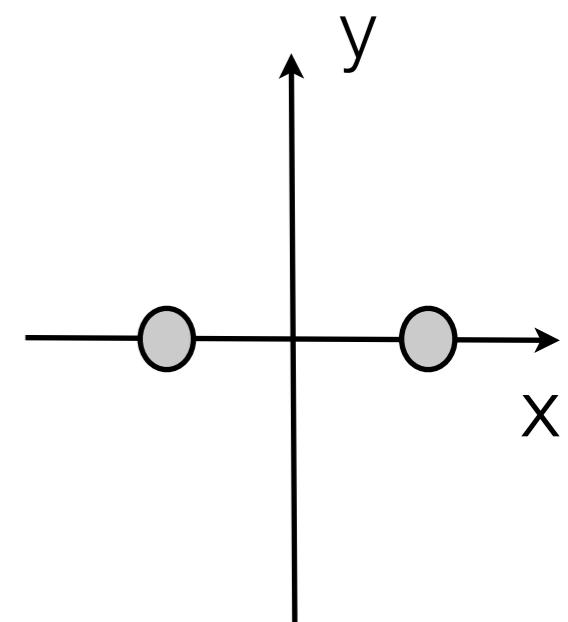
GWs vs EM waves

- Similarities:
 - ✓ Propagation with the speed of light.
 - ✓ Amplitude decreases as $\sim 1/r$.
 - ✓ Frequency redshift (Doppler, gravitational, cosmological).
- Differences:
 - ✓ GWs propagate through matter with little interaction. Hard to detect, but they carry uncontaminated information about their sources.
 - ✓ Strong GWs are generated by bulk (coherent) motion. They require strong gravity/high velocities (compact objects like black holes and neutron star).
 - ✓ EM waves originate from small-scale, incoherent motion of charged particles. They are subject to “environmental” contamination (interstellar absorption etc.).

Effect on test particles (I)

- We consider a pair of test particles on the cartesian axis **Ox** at distances $\pm x_0$ from the origin and we assume a GW traveling in the **z**-direction.
- Their distance will be given by the relation:

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu = \dots = -g_{11} dx^2 = \\ = (1 - h_{11})(2x_0)^2 = [1 - h_+ \cos(\omega t)] (2x_0)^2$$



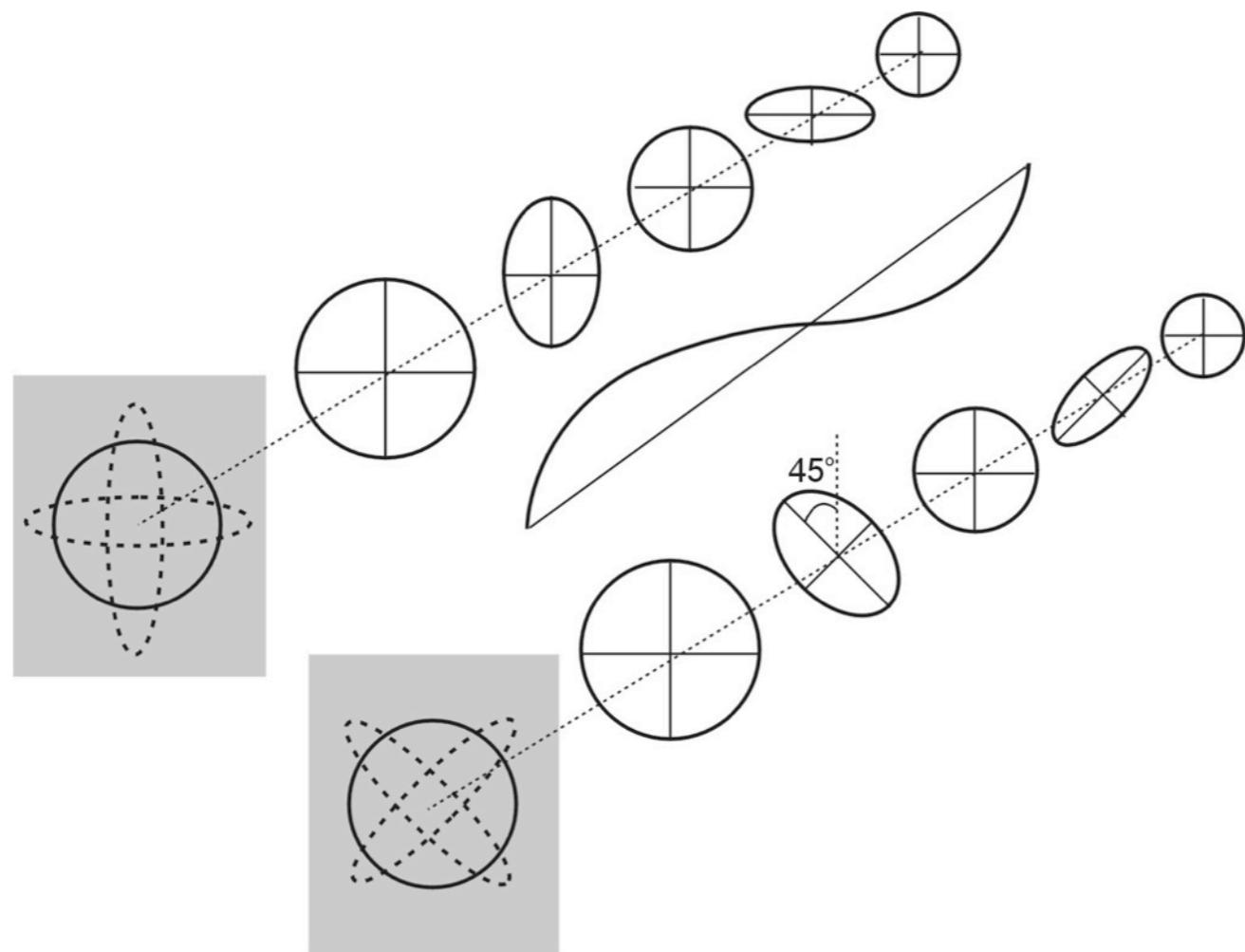
$$dl \approx \left[1 - \frac{1}{2} h_+ \cos(\omega t) \right] (2x_0)$$

Effect on test particles (II)

- Similarly for a pair of particles placed on the y-axis:

- Comment: the same result can be derived using the geodetic deviation equation.

$$dl \approx \left[1 + \frac{1}{2} h_+ \cos(\omega t) \right] (2y_0)$$



The quadrupole formula

- Einstein (1918) derived the quadrupole formula for gravitational radiation by solving the linearized field equations with a source term:

$$\square h^{\mu\nu}(t, \vec{x}) = -\kappa T^{\mu\nu}(t, \vec{x}) \longrightarrow h^{\mu\nu} = -\frac{\kappa}{4\pi} \int_V d^3x' \frac{T^{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}$$

- This solution suggests that the wave amplitude is proportional to the **second time derivative of the quadrupole** moment of the source:

$$h^{\mu\nu} = \frac{2}{r} \frac{G}{c^4} \ddot{Q}_{\text{TT}}^{\mu\nu}(t - r/c) \qquad Q_{\text{TT}}^{\mu\nu} = \int d^3x \rho \left(x^\mu x^\nu - \frac{1}{3} \delta^{\mu\nu} r^2 \right)$$


(quadrupole moment in the “TT gauge” and at the retarded time $t-r/c$)

- This result is quite accurate for all sources, as long as the wavelength is much longer than the source size R .

GW luminosity

- **GWs carry energy.** The stress-energy carried by GWs cannot be localized within a wavelength. Instead, one can say that a certain amount of stress-energy is contained in a region of the space which extends over several wavelengths. The **stress-energy tensor** can be written as:

$$T_{\mu\nu}^{\text{GW}} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{ij}^{\text{TT}} \partial_\nu h_{\text{TT}}^{ij} \rangle$$

- Using the previous quadrupole formula we obtain the **GW luminosity**:

$$L_{\text{GW}} = \frac{dE_{\text{GW}}}{dt} = \int dA T_{0j}^{\text{GW}} \hat{n}^j \quad \longrightarrow \quad L_{\text{GW}} = \frac{1}{5} \frac{G}{c^5} \langle \ddot{Q}_{\mu\nu}^{\text{TT}} \ddot{Q}_{\text{TT}}^{\mu\nu} \rangle$$

Basic estimates (I)

- The quadrupole moment of a system is approximately equal to the **mass M** of the part of the system that moves, times the square of the **size R** of the system. This means that the 3rd-order time derivative of the quadrupole moment is:

$$\ddot{Q} \sim \frac{MR^2}{T^3} \sim \frac{Mv^2}{T} \sim \frac{E_{\text{ns}}}{T}$$

v = mean velocity of source's non-spherical motion,

E_{ns} = kinetic energy of non-spherical motion

T = timescale for a mass to move from one side of the system to the other.

- For a self gravitating system: $T \sim \sqrt{R^3/GM}$
- This relation provides a rough estimate of the characteristic frequency of the system $f \sim 2\pi/T$.

Basic estimates (II)

- The luminosity of GWs from a given source is approximately:

$$L_{\text{GW}} \sim \frac{G^4}{c^5} \left(\frac{M}{R} \right)^5 \sim \frac{G}{c^5} \left(\frac{M}{R} \right)^2 v^6 \sim \frac{c^5}{G} \left(\frac{R_{\text{Sch}}}{R} \right)^2 \left(\frac{v}{c} \right)^6$$

where $R_{\text{Sch}} = 2GM/c^2$ is the Schwarzschild radius of the source. It is obvious that the maximum GW luminosity can be achieved if $R \sim R_{\text{Sch}}$ and $v \sim c$. That is, the source needs to be compact and relativistic.

- Using the above order-of-magnitude estimates, we can get a rough estimate of the amplitude of GWs at a **distance r** from the source:

$$h \sim \frac{G}{c^4} \frac{E_{\text{ns}}}{r} \sim \frac{G}{c^4} \frac{\epsilon E_{\text{kin}}}{r}$$

ϵ = the kinetic energy fraction that is able to produce GWs.

Basic estimates (III)

- Another estimate for the GW amplitude can be derived from the flux formula

$$F_{\text{GW}} = \frac{L_{\text{GW}}}{4\pi r^2} = \frac{c^3}{16\pi G} |\partial_t h|^2$$

- We obtain:

$$h \approx 10^{-22} \left(\frac{E_{\text{GW}}}{10^{-4} M_{\odot}} \right)^{1/2} \left(\frac{1 \text{ kHz}}{f_{\text{GW}}} \right) \left(\frac{\tau}{1 \text{ ms}} \right)^{-1/2} \left(\frac{15 \text{ Mpc}}{r} \right)$$

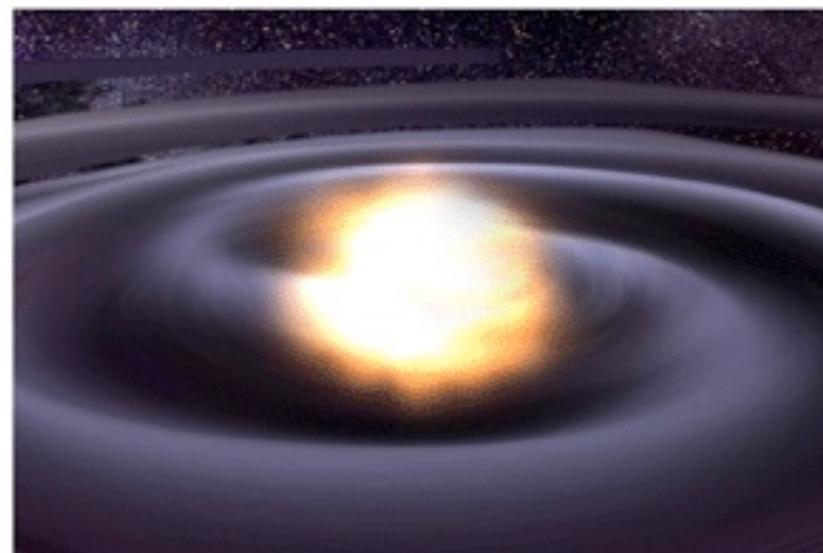
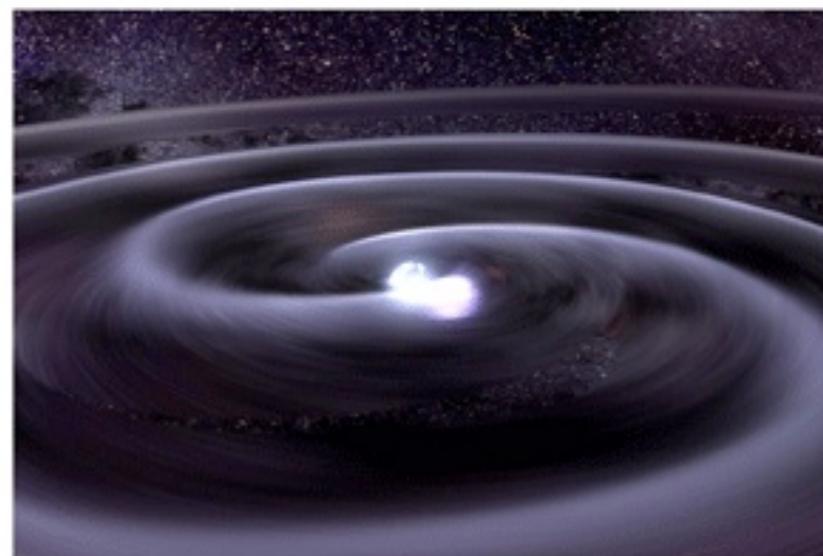
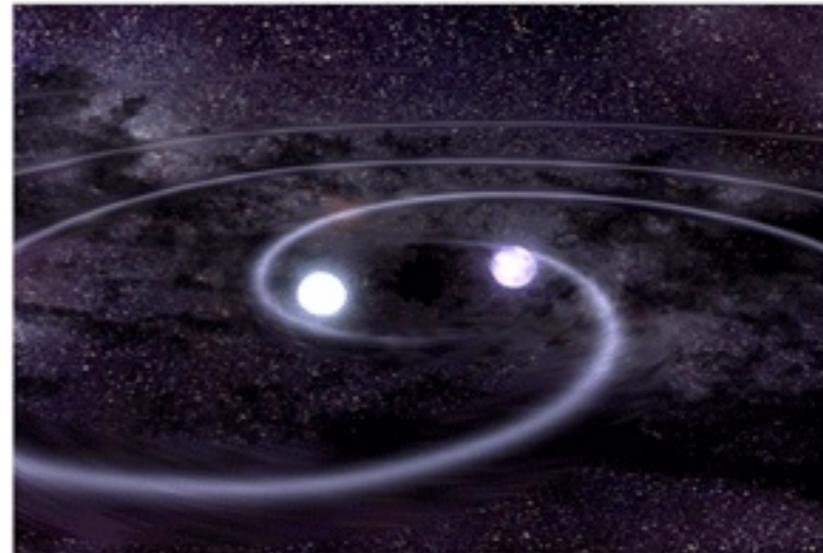
for example, this formula could describe the GW strain from a supernova explosion at the Virgo cluster during which the energy E_{GW} is released in GWs at a frequency of **1 kHz**, and with signal duration of the order of **1 ms**.

- This is why **GWs are hard to detect**: for a GW detector with arm length of $l = 4 \text{ km}$ we are looking for changes in the arm-length of the order of

$$\Delta l = hl = 4 \times 10^{-17} \text{ cm} \quad !!$$

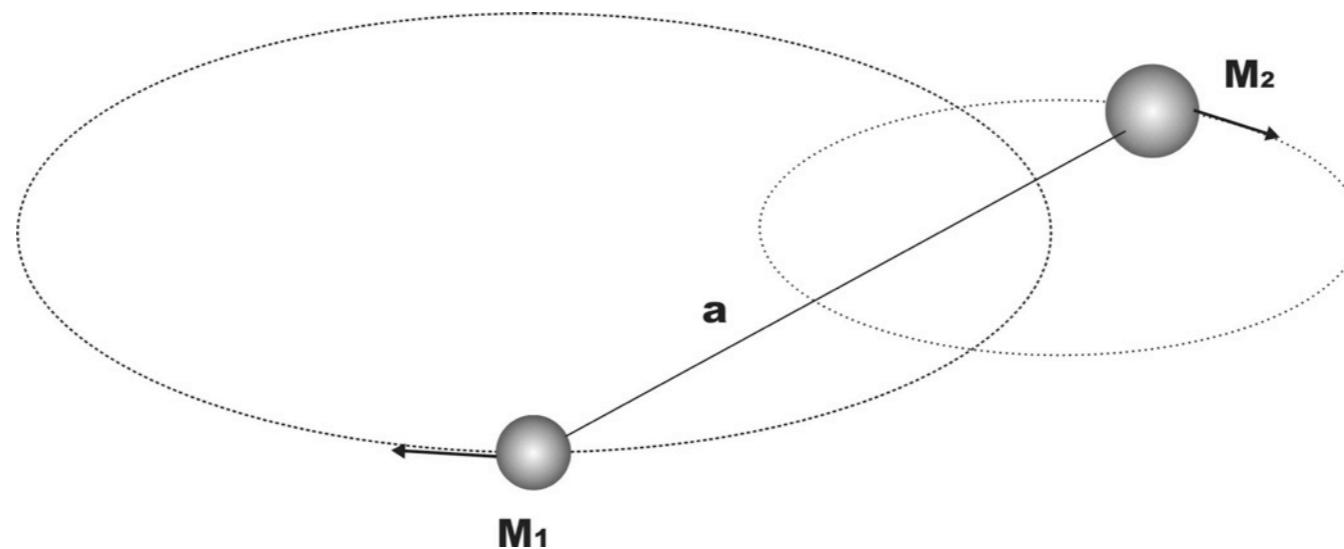
Part II

GW sources



GW emission from a binary system (I)

- The binary consists of the two bodies M_1 and M_2 at distances a_1 and a_2 from the center of mass. The orbits are circular and lie on the x-y plane. The orbital angular frequency is Ω .



- We also define: $a = a_1 + a_2$, $\mu = M_1 M_2 / M$, $M = M_1 + M_2$

GW emission from a binary system (II)

- The only non-vanishing components of the quadrupole tensor are :

$$Q_{xx} = -Q_{yy} = (a_1^2 M_1 + a_2^2 M_2) \cos^2(\Omega t) = \frac{1}{2} \mu a^2 \cos(2\Omega t)$$

$$Q_{xy} = Q_{yx} = \frac{1}{2} \mu a^2 \sin(2\Omega t) \quad (\text{GW frequency} = 2\Omega)$$

- And the **GW luminosity** of the system is (we use Kepler's 3rd law $\Omega^2 = GM/a^3$)

$$L_{\text{GW}} = -\frac{dE}{dt} = \frac{G}{5c^5} (\mu \Omega a^2)^2 \langle 2 \sin^2(2\Omega t) + 2 \cos^2(2\Omega t) \rangle$$

$$= \frac{32}{5} \frac{G}{c^5} \mu^2 a^4 \Omega^6 = \frac{32}{5} \frac{G^4}{c^5} \frac{M^3 \mu^2}{a^5}$$

GW emission from a binary system (III)

- The **total energy** of the binary system can be written as :

$$E = \frac{1}{2} \Omega^2 (M_1 a_1^2 + M_2 a_2^2) - \frac{GM_1 M_2}{a} = -\frac{1}{2} \frac{G\mu M}{a}$$

- As the gravitating system loses energy by emitting radiation, the distance between the two bodies **shrinks at a rate**:

$$\frac{dE}{dt} = \frac{G\mu M}{2a^2} \frac{da}{dt} \rightarrow \frac{da}{dt} = -\frac{64}{5} \frac{G^3}{c^5} \frac{\mu M}{a^3}$$

- The orbital frequency increases accordingly $\dot{T}/T = (3/2)\dot{a}/a$.

- The system will **coalesce** after a time: $\tau = \frac{5}{256} \frac{c^5}{G^3} \frac{a_0^4}{\mu M^4}$

(initial separation)

GW emission from a binary system (IV)

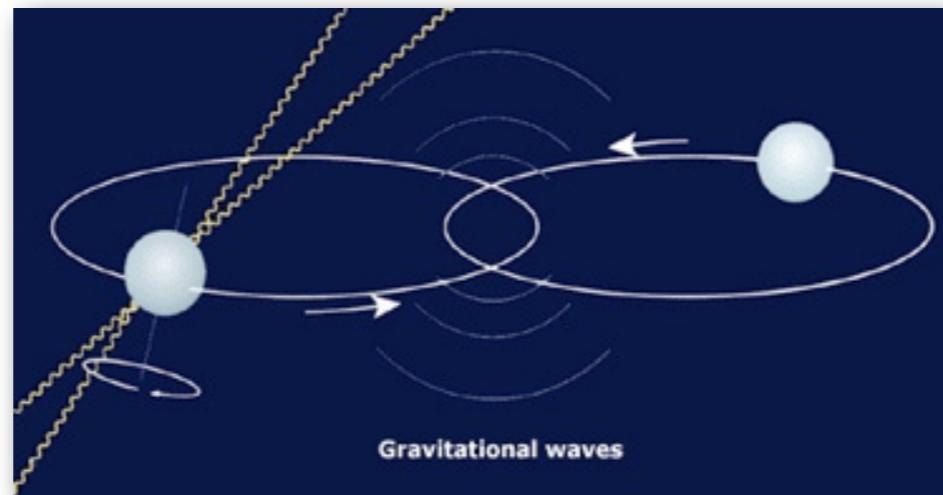
- In this analysis we have assumed circular orbits. In general the orbits can be elliptical, but it has been shown that GW emission **circularizes** them faster than the coalescence timescale.
- The GW amplitude is (ignoring geometrical factors):

$$h \approx 5 \times 10^{-22} \left(\frac{M}{2.8M_{\odot}} \right)^{2/3} \left(\frac{\mu}{0.7M_{\odot}} \right) \left(\frac{f}{100 \text{ Hz}} \right)^{2/3} \left(\frac{15 \text{ Mpc}}{r} \right)$$

↑
(set distance to the Virgo cluster, why?)

PSR 1913+16: a Nobel-prize GW source

- The now famous [Hulse & Taylor](#) binary neutron star system provided the first astrophysical evidence of the existence of GWs !

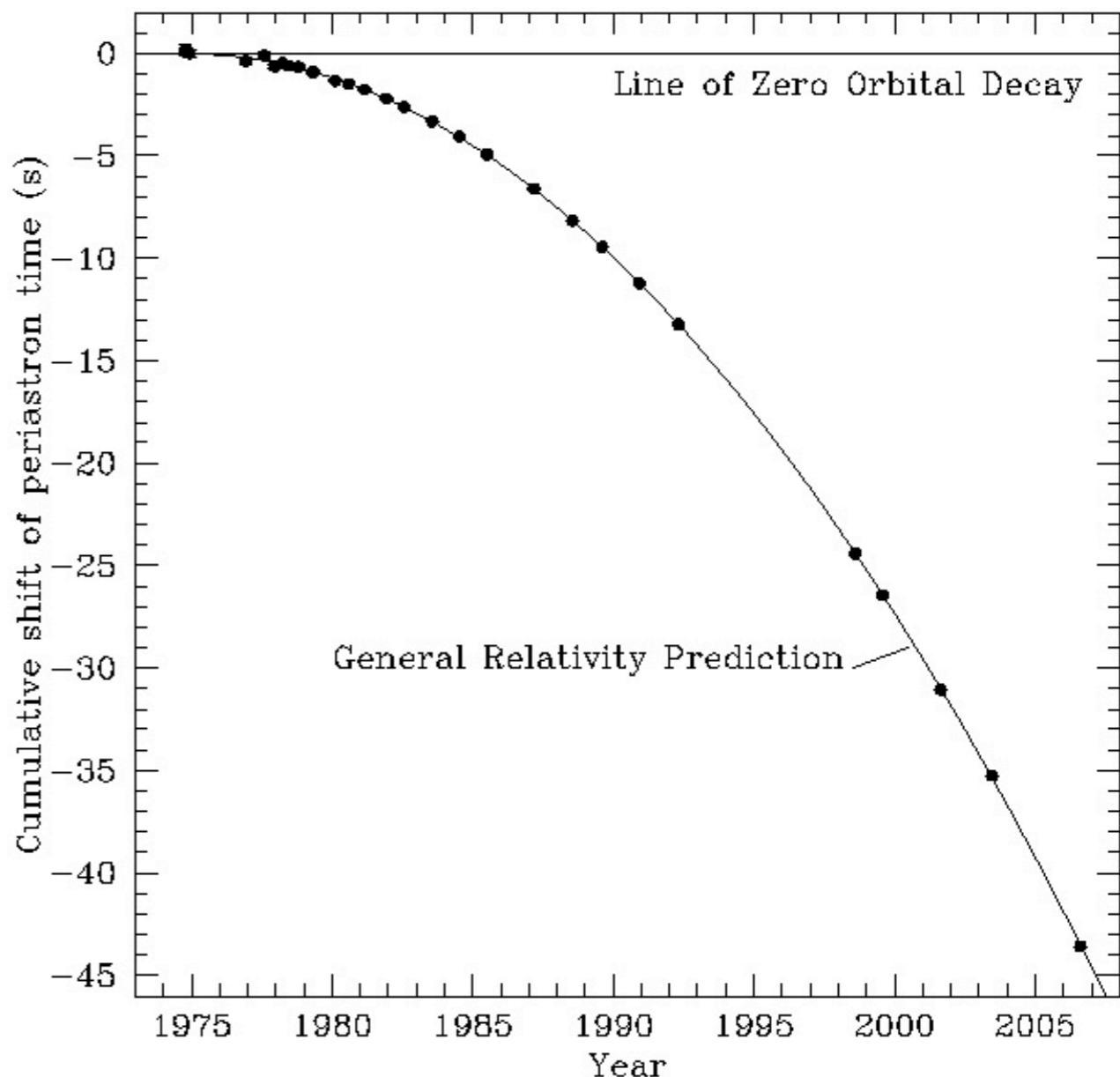


- The system's parameters: $r = 5 \text{ Kpc}$, $M_1 \approx M_2 \approx 1.4 M_\odot$, $T = 7 \text{ h } 45 \text{ min}$
- Using the previous equations we can predict:

$$\dot{T} = -2.4 \times 10^{-12} \text{ sec/sec}, \quad f_{\text{GW}} = 7 \times 10^{-5} \text{ Hz}, \quad h \sim 10^{-23}, \quad \tau \approx 3.5 \times 10^8 \text{ yr}$$

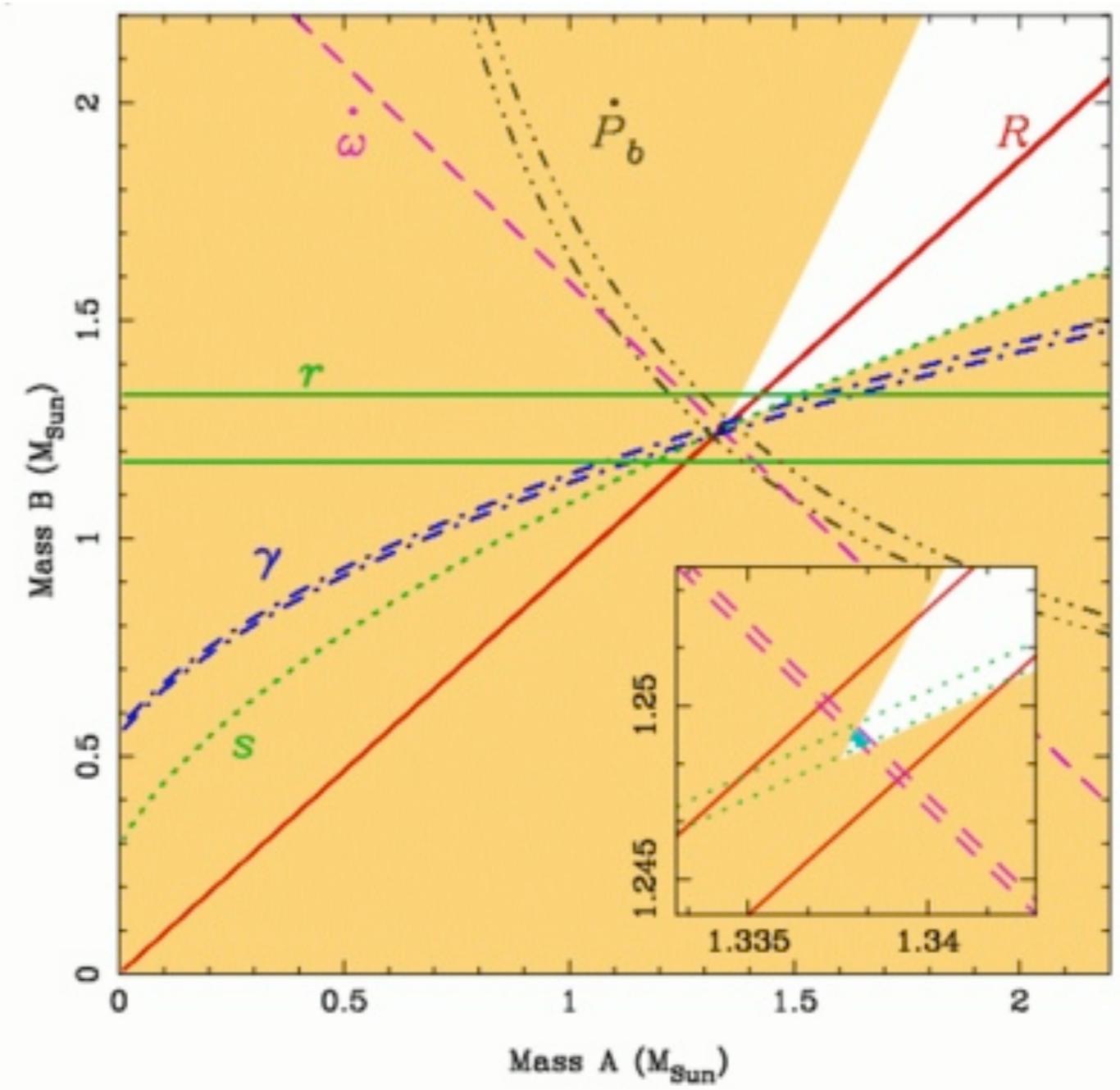
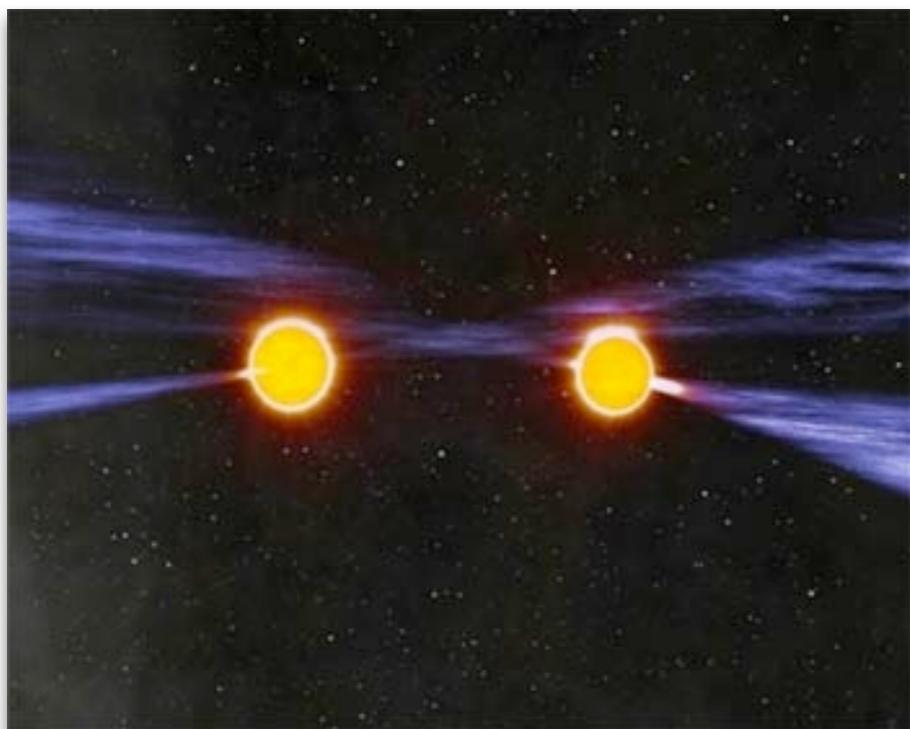
Theory vs observations

- How can the orbital parameters be measured with such high precision?
- One of the neutron stars is a **pulsar**, emitting extremely stable periodic radio pulses. The emission is modulated by the orbital motion.
- Since the discovery of the H-T system in 1974 more such binaries were found by astronomers.

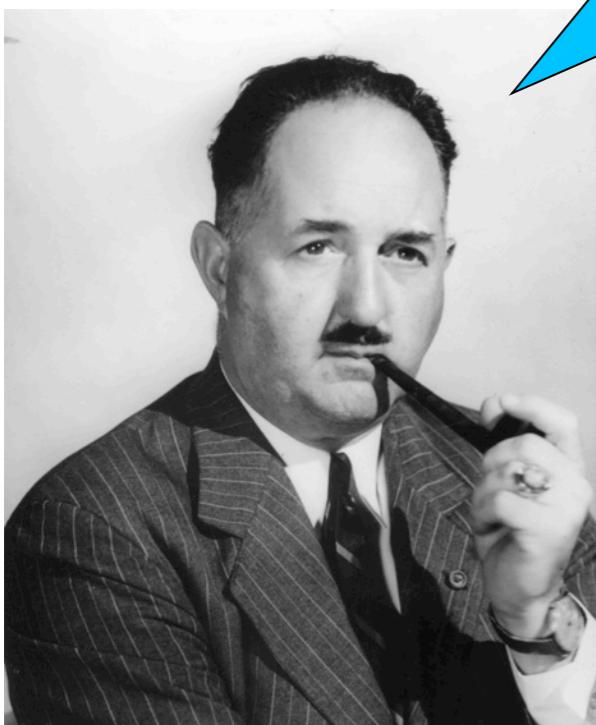


The double pulsar system

- Discovered in 2003, this binary system consists of two pulsars: PSR J0737–3039A & B.
- This rare system allows for high-precision tests of GR.

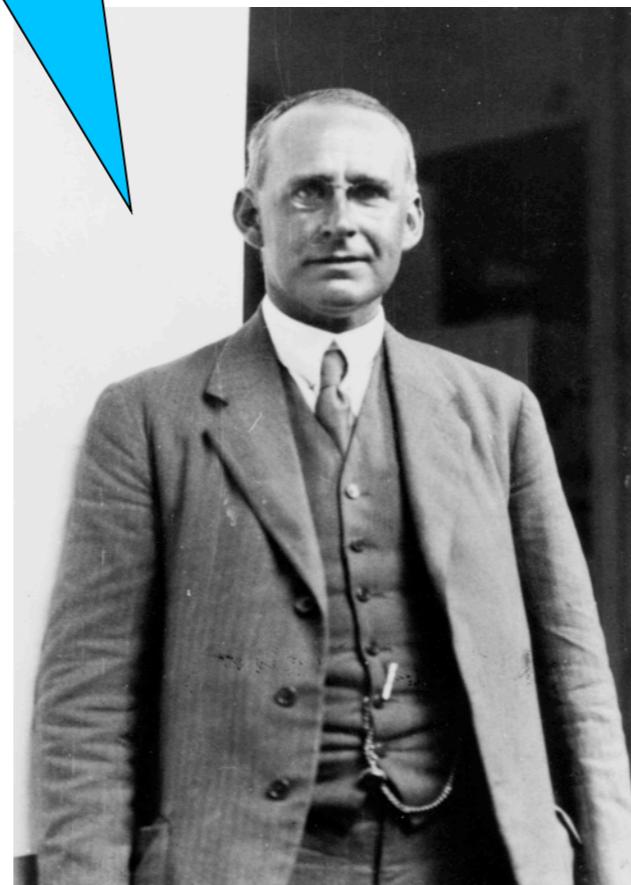


Do GWs exist? a historical footnote



They do Exist

I'm skeptical

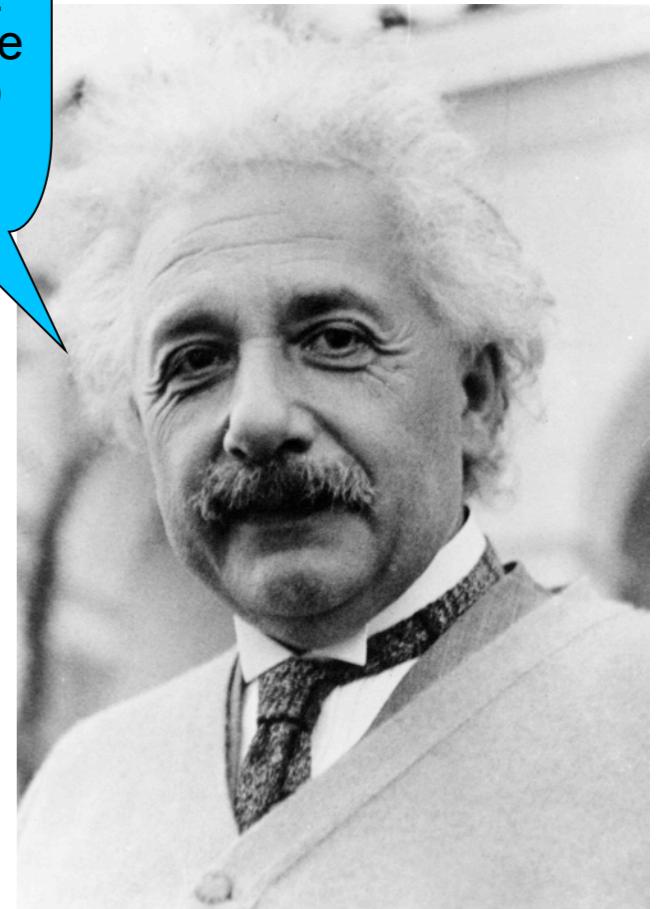


Howard P. Robertson

Arthur S. Eddington

They
don't ...
They do!

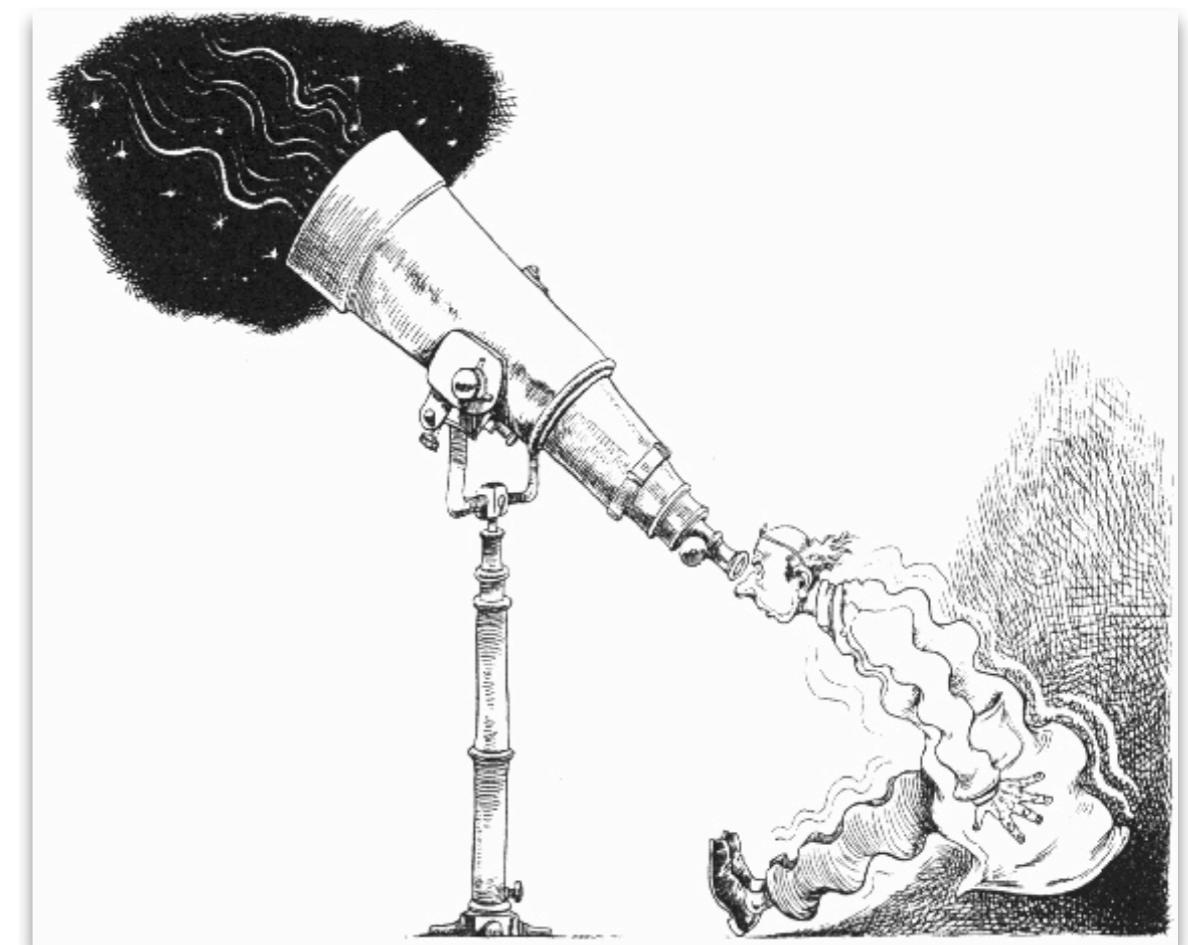
No, wait,
they
don't ...
or maybe
they do



Albert Einstein

Part III

Detection of GWs



GW detectors: prehistory

- For decades after the formulation of Einstein's GR the notion of GWs was a topic for speculations and remote from real astrophysics.
- Joe Weber pioneered the construction of the first “primitive” bar detector. However, his claims of a GW detection were never verified ...
- Theoretical work in the 1970s-1990s (and the discovery of the Hulse-Taylor pulsar) advanced the popularity of GWs.
- **GW astronomy is expected to become reality in the present decade.**



SCIENCE PHOTO LIBRARY

A toy model GW detector

- Consider a GW propagating along the z-axis (with a “+” polarization and frequency ω), impinging on an idealized detector consisting of two masses joined by a spring (of length L) along the x-axis



- The resulting motion is that of a forced oscillator (with friction τ , natural frequency ω_0):

$$\ddot{\xi} + \dot{\xi}/\tau + \omega_0^2 \xi = -\frac{1}{2} \omega^2 L h_+ e^{i\omega t}$$

- The solution is:

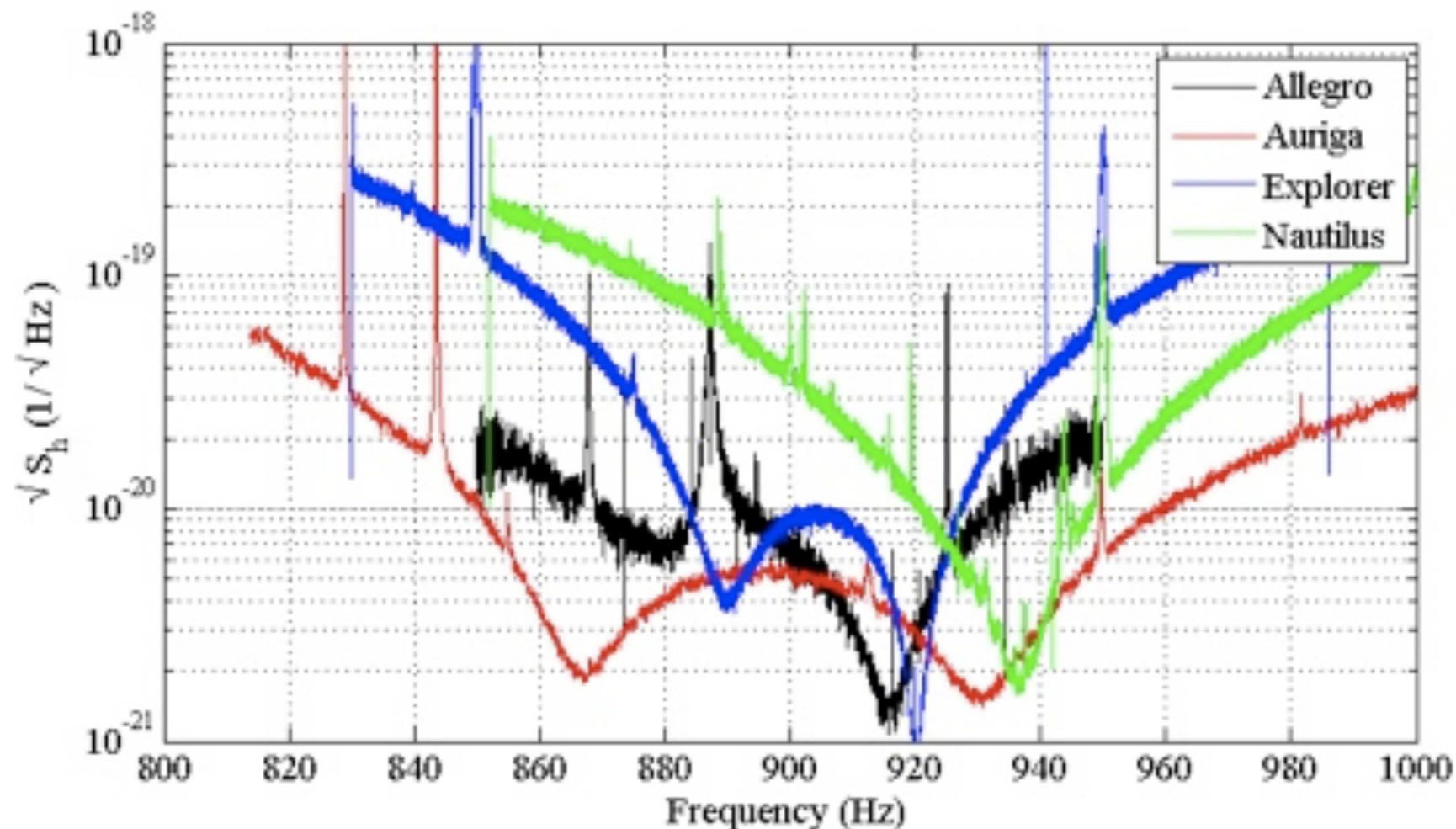
$$\xi = \frac{\omega^2 L h_+}{2(\omega_0^2 - \omega^2 + i\omega/\tau)} e^{i\omega t}$$

- The **maximum amplitude** is achieved at $\omega \approx \omega_0$ and has a size: $\xi_{\max} = \frac{1}{2} \omega_0 \tau L h_+$

- The detector can be optimized by increasing $\omega_0 \tau L$.

Bar detectors

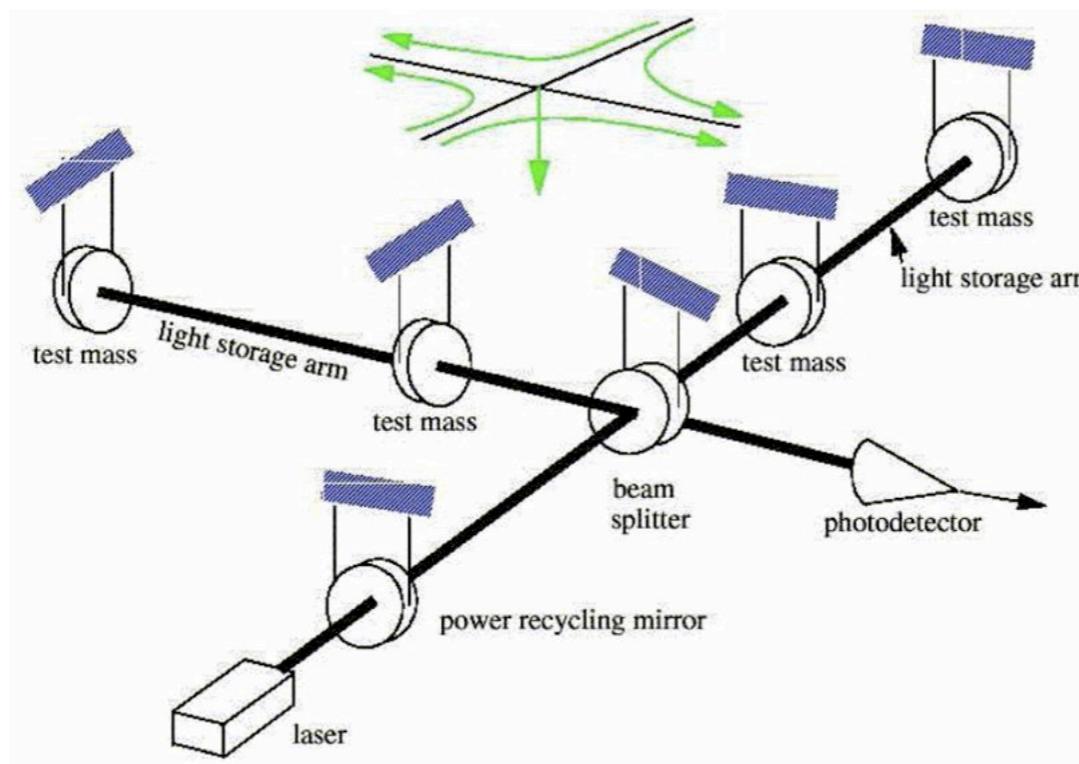
- Bar detectors are narrow bandwidth instruments (like the previous toy-model)



Sensitivity curves of various bar detectors

Detectors: laser interferometry

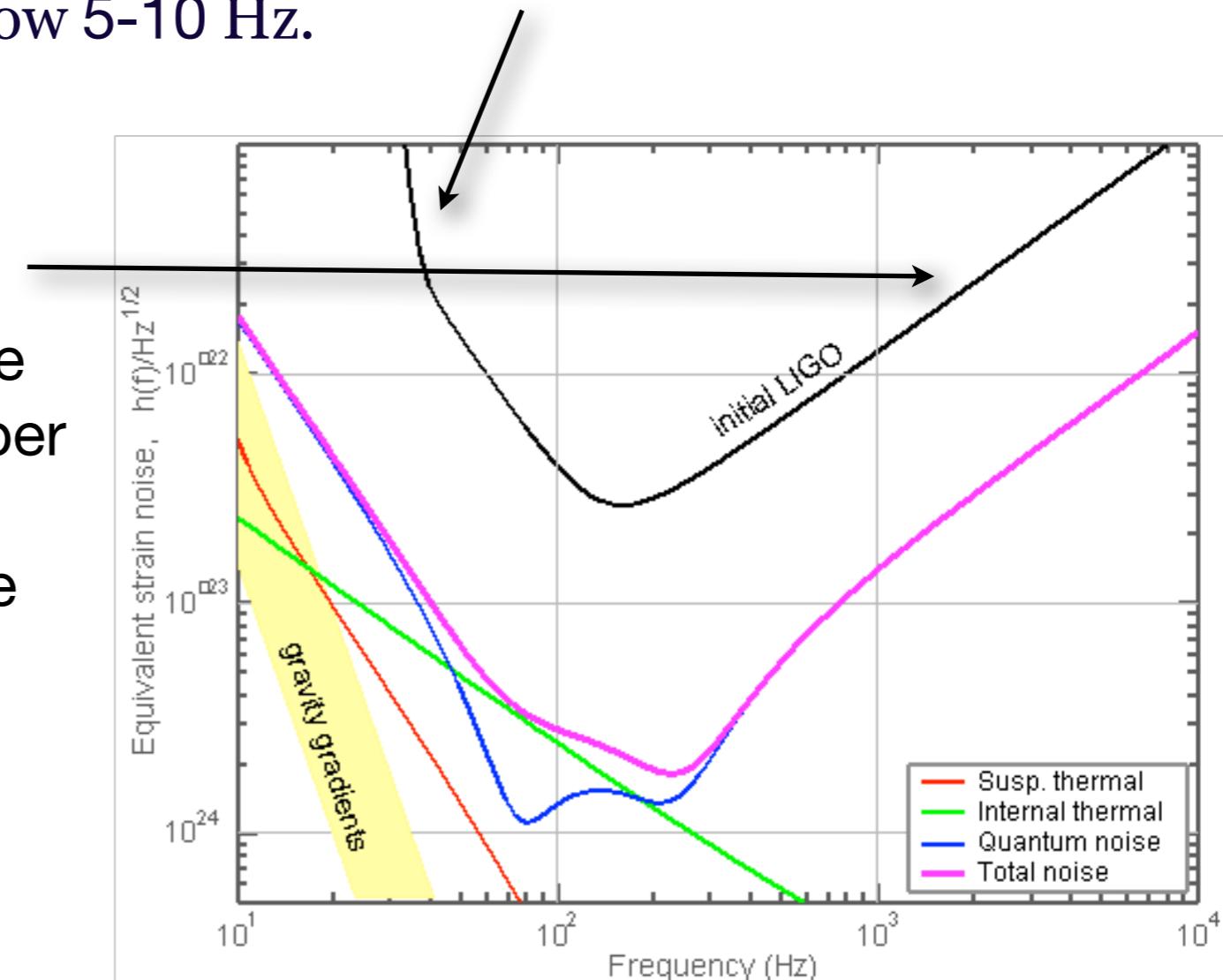
- A laser interferometer is an alternative choice for GW detection, offering a combination of **very high sensitivities over a broad frequency band**.
- **Suspended mirrors** play the role of “test-particles”, placed in perpendicular directions. The light is reflected on the mirrors and returns back to the beam splitter and then to a photodetector where the fringe pattern is monitored.



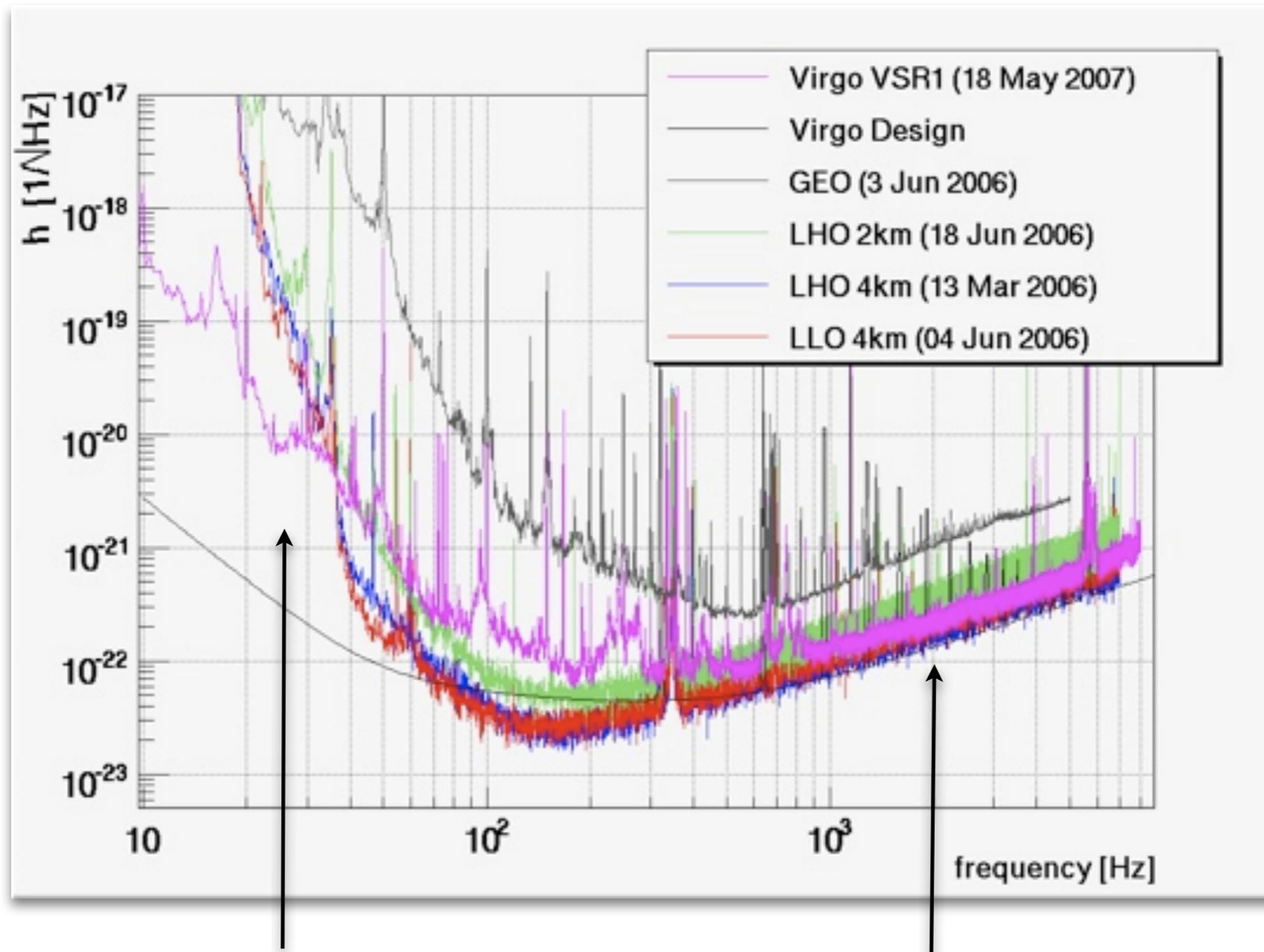
Noise in interferometric detectors

- **Seismic noise (low frequencies).** At frequencies below 60 Hz, the noise in the interferometers is dominated by seismic noise. The vibrations of the ground couple to the mirrors via the wire suspensions which support them. This effect is strongly suppressed by properly designed suspension systems. Still, seismic noise is very difficult to eliminate at frequencies below 5-10 Hz.

- **Photon shot noise (high frequencies).** The precision of the measurements is restricted by fluctuations in the fringe pattern due to fluctuations in the number of detected photons. The number of detected photons is proportional to the intensity of the laser beam. Statistical fluctuations in the number of detected photons imply an uncertainty in the measurement of the arm length.



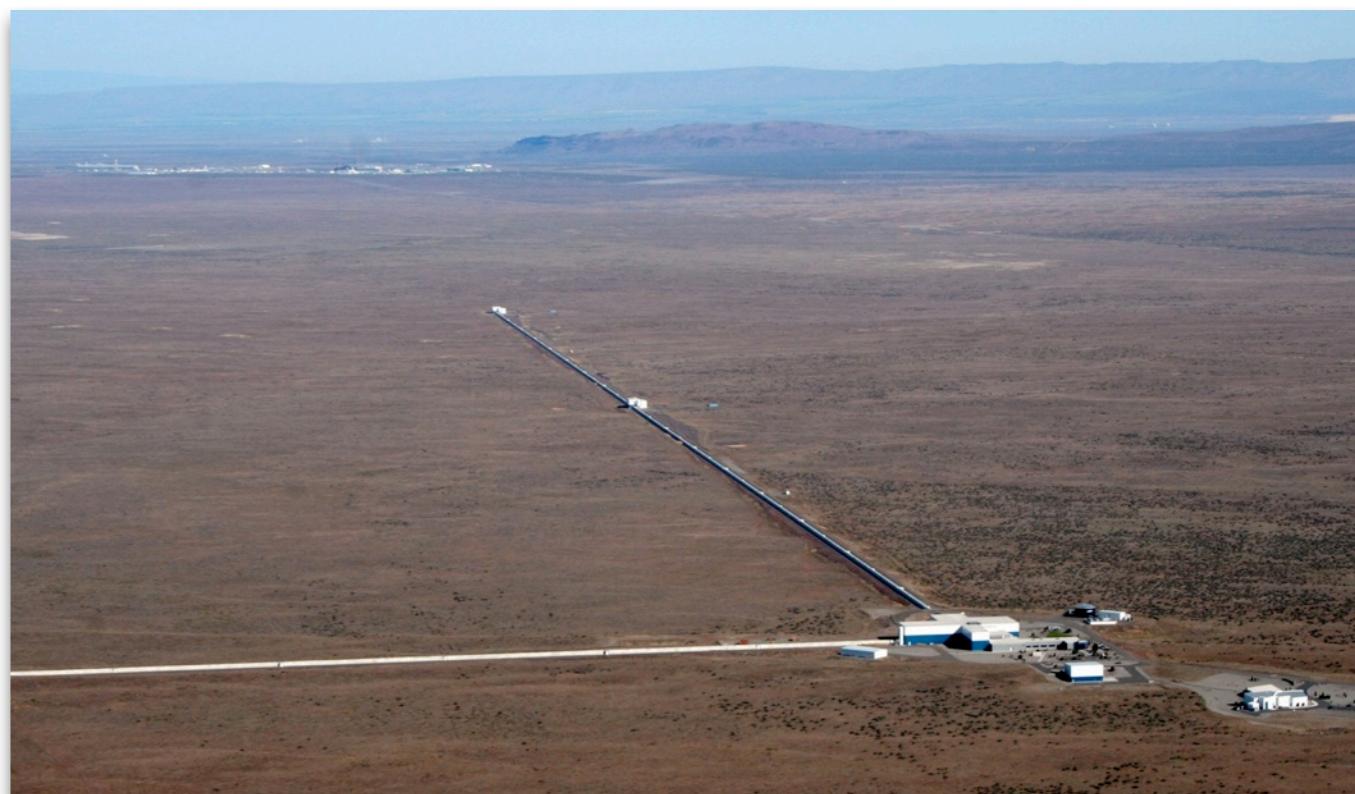
Detectors: real-life sensitivity



Seismic noise

laser photon noise

Detectors: the present (I)



The twin LIGO detectors ($L = 4$ km) at Livingston Louisiana and Hanford Washington (US).

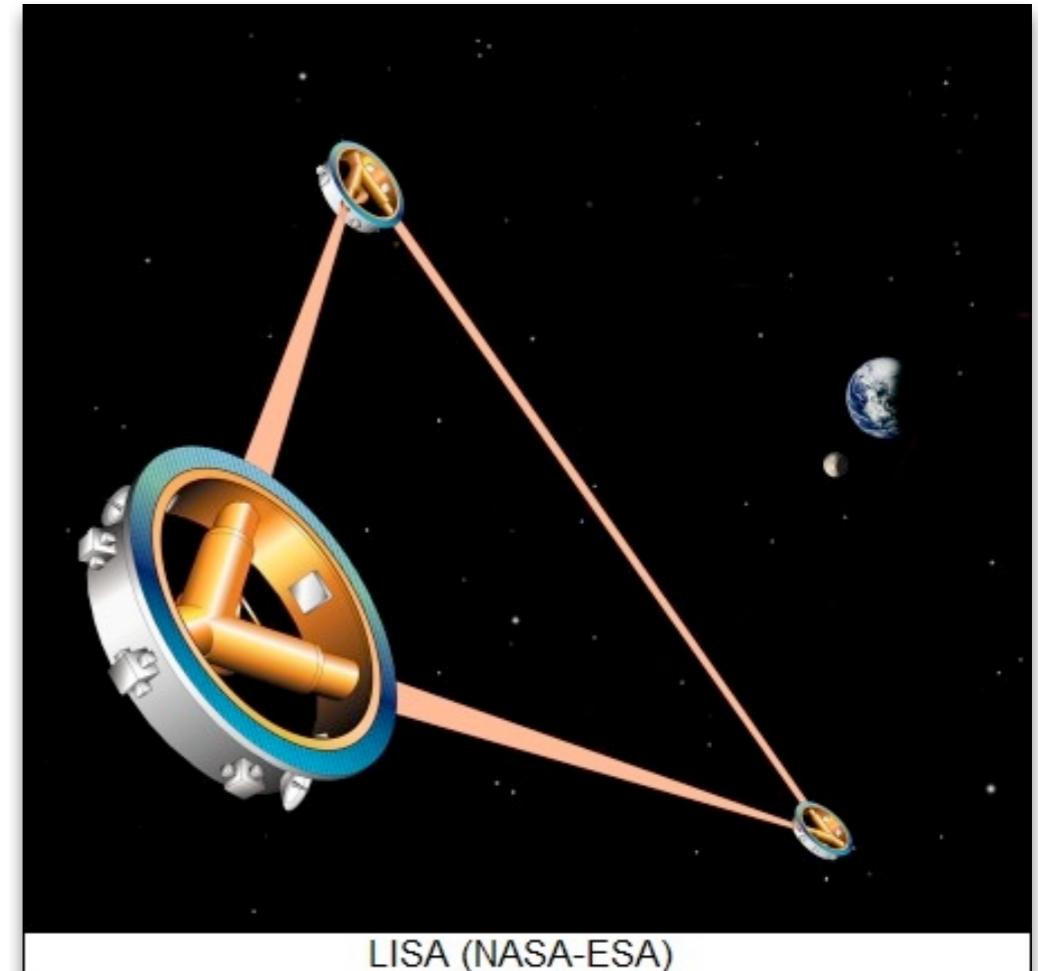
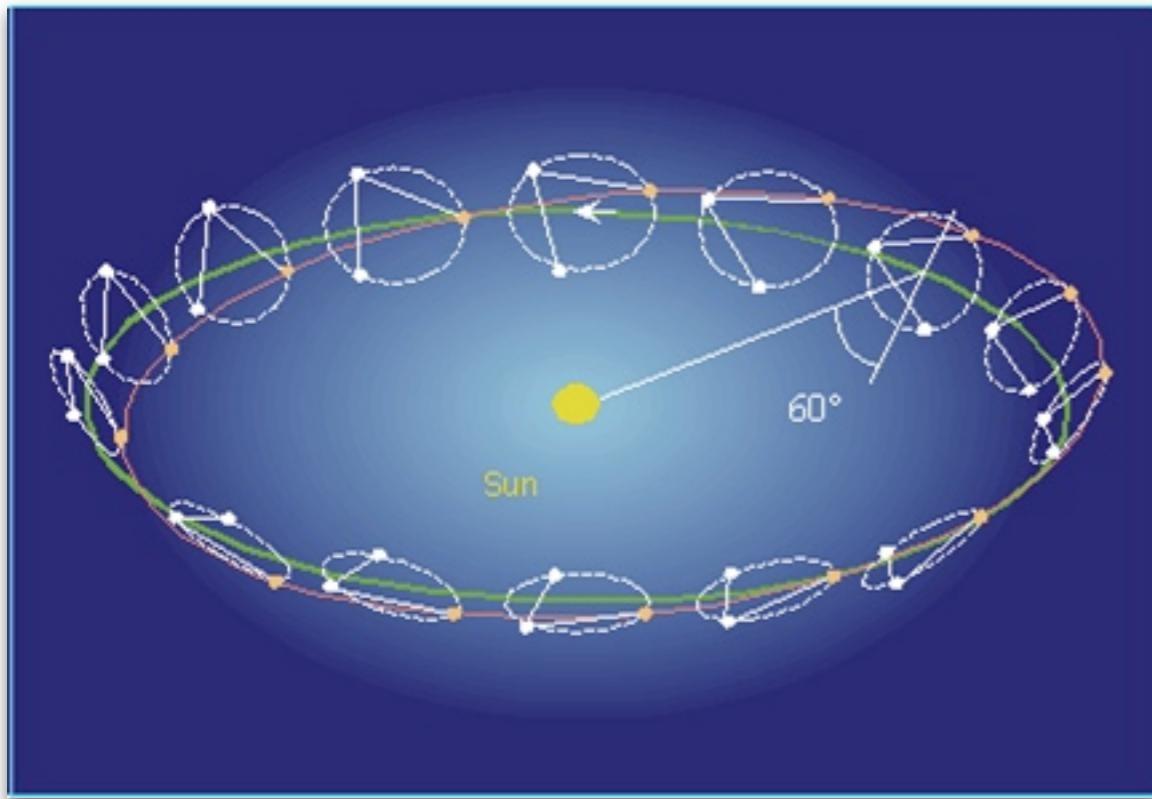
Detectors: the present (II)



The VIRGO detector ($L= 3$ km) near Pisa, Italy

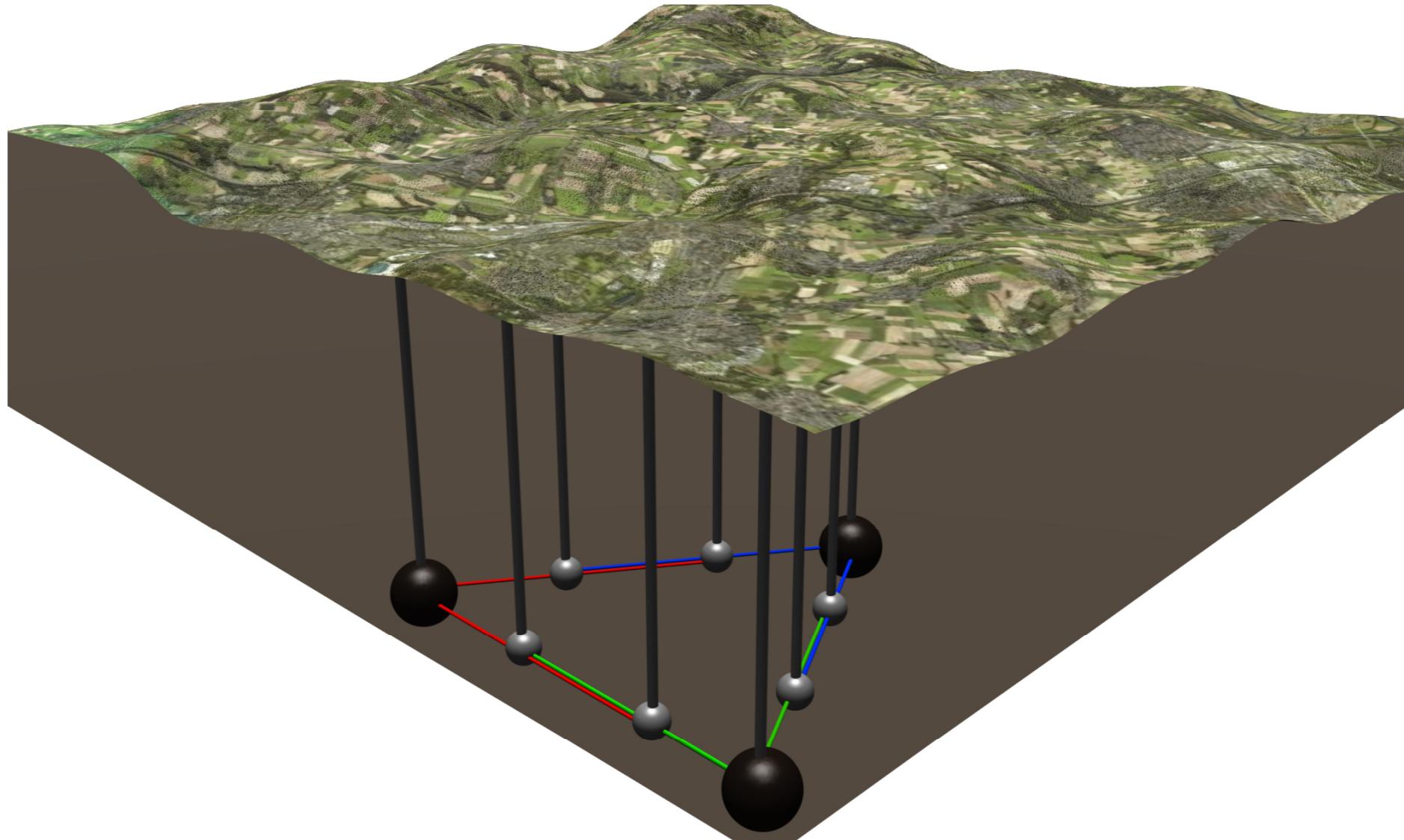
Going to space: the LISA detector

- Space-based detectors: “noise-free” environment, abundance of space!
- Long-arm baseline, **low frequency sensitivity**
- **LISA**: Up until recently a joint NASA/ESA mission, now an ESA mission only.
To be launched around 2020.

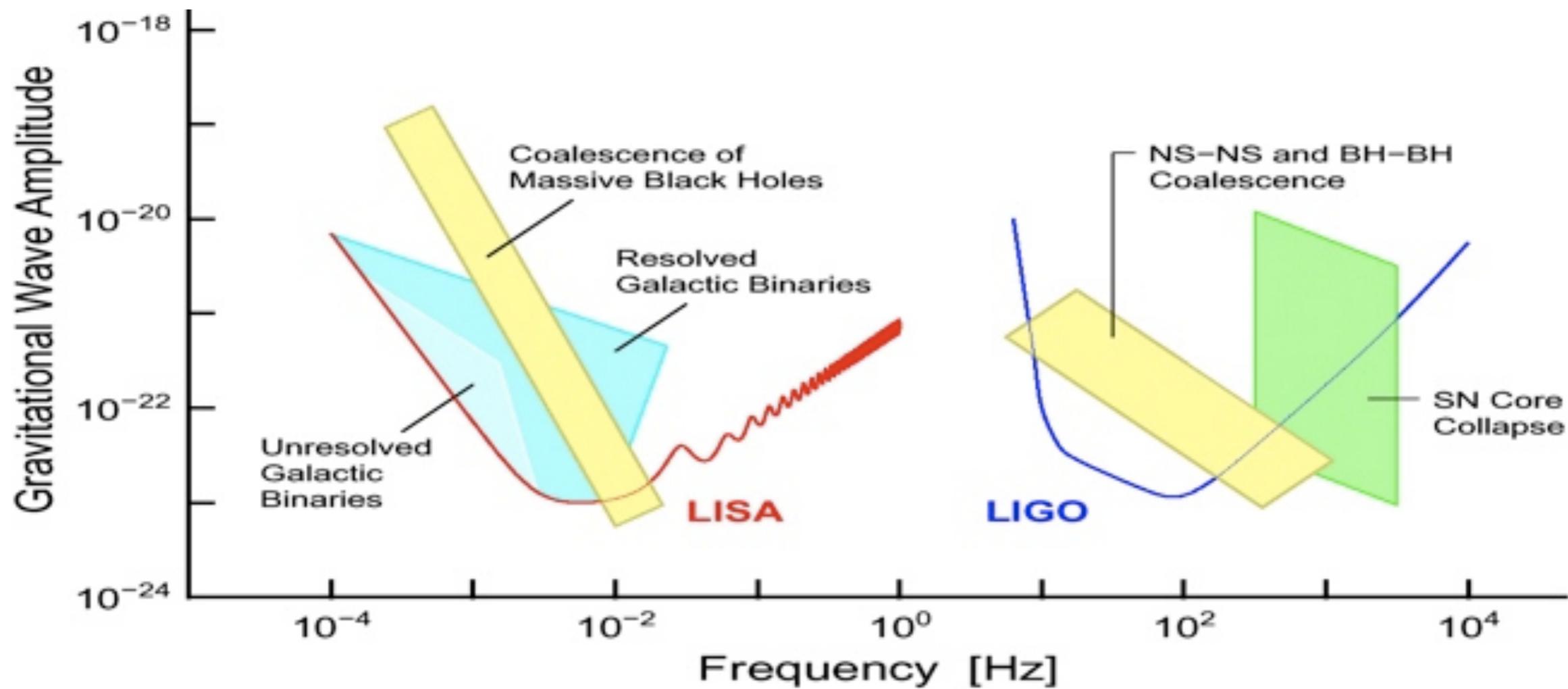


Going underground: the ET

- The Einstein Telescope will be the next generation underground detector.



GWs detectors: ground and space



GW-related research in Spain

- [Barcelona](#): binary BH systems, GW data analysis.
- [Valencia](#): numerical GR/hydrodynamics, neutron star physics.
- [Palma de Mallorca](#): GW data analysis, binary BHs.
- [Alicante](#): neutron star physics.