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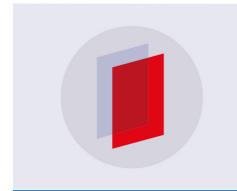
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Gravitational waves as exact solutions of Einstein field equations

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Abstract. Exact solutions of Einstein field equations invariant for a non-Abelian 2-dimensional Lie algebra of Killing fields are described. A sub-class of these gravitational fields have a wave-like character; it is shown that they have spin-1.

1. Introduction

Some decades ago, by using a generalization of the *Inverse Scattering Transform*, Belinsky and Sakharov were able to find [7] solitary wave solutions of Einstein field equations for special metrics belonging to the larger class of metrics invariant for an Abelian 2-dimensional Lie algebra of Killing vector fields.

Lorentzian Ricci-flat metrics represent gravitational fields and deserve special attention when they have a wave-like character. Indeed, presently there are, worldwide, many efforts to detect gravitational radiation, not only because a direct confirmation of their existence is interesting *per se* but also because new insights on the nature of gravity and of the Universe itself could be gained.

Since a 2$-$dimensional Lie algebra is either Abelian ($\mathcal{A}_2$) or non-Abelian ($\mathcal{G}_2$), it has been natural to consider [31, 32, 33] the problem of characterizing all gravitational fields g admitting a Lie algebra $\mathcal{G}$ of Killing fields such that:

I the distribution $\mathcal{D}$, generated by vector fields of $\mathcal{G}$, is 2-dimensional;

II the distribution $\mathcal{D}^\perp$, orthogonal to $\mathcal{D}$ is integrable [1] and transversal to $\mathcal{D}$.

The condition of transversality can be relaxed [12, 13, 2]. This case, when the metric g restricted to any integral (2-dimensional) submanifold (*Killing leaf*) of the distribution $\mathcal{D}$ is degenerate, splits naturally into two sub-cases according to whether the rank of g restricted to Killing leaves is 1 or 0. In order to distinguish various cases occurring in the sequel, the notation $(\mathcal{G}, r)$ has been introduced: metrics satisfying the conditions *I* and *II* are called *of* $(\mathcal{G}, 2)-type$; metrics satisfying conditions *I* and *II*, except the transversality condition, are called *of* $(\mathcal{G}, 0)-type$ or *of* $(\mathcal{G}, 1)-type$ according to the rank of their restriction to Killing leaves.

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[1] A 2$-$dimensional distribution is called *integrable* if the Faraday force lines (*integral curves* in differential geometric language) of two generating vector fields are *surfaces forming*, that is they mesh one another as cotton threads in a web. Such surfaces are called leaves of the distribution. A non integrable 2$-$dimensional distribution is called *semi-integrable* if it is part (i.e., a suitable restriction) of a 3$-$dimensional integrable distribution.

The study of $\mathcal{A}_2$-integrable Einstein metrics goes back to Einstein and Rosen [16], Kompaneyets [19], Geroch [17], Belinsky, Khalatnikov, Zakharov [6, 7], Verdaguer [37] . Recent results can be found in [13].

All the possible situations, corresponding to a $2-$dimensional Lie algebra of isometries, are described by the following table

$$
\begin{array}{|c c c c|}
 & \mathcal{D}^{\perp},\ r=0 & \mathcal{D}^{\perp},\ r=1 & \mathcal{D}^{\perp},\ r=2 \\
\mathcal{G}_2 & integrable & integrable & integrable \\
\mathcal{G}_2 & semi-integrable & semi-integrable & semi-integrable \\
\mathcal{G}_2 & non-integrable & non-integrable & non-integrable \\
\mathcal{A}_2 & integrable & integrable & integrable \\
\mathcal{A}_2 & semi-integrable & semi-integrable & semi-integrable \\
\mathcal{A}_2 & non-integrable & non-integrable & non-integrable \\
\end{array}, \tag{1}
$$

where a non integrable 2-dimensional distribution which is part of a 3-dimensional integrable distribution has been called *semi-integrable* and in which the cases indicated with bold letters have been essentially solved [31, 32, 33, 12, 13, 2].

In section 1, metrics of $(\mathcal{G}_2, 2)-type$ invariant for a non Abelian $2-$dimensional Lie algebra are characterized from a geometric point of view and the solutions of corresponding Einstein field equations are explicitly written. In section 2, the case in which the commutator of generators of the Lie algebra is of *light-type* is analyzed from a physical point of view. Harmonic coordinates are also introduced. Moreover, the wave-like character of the solutions is checked through the Zel'manov and the Pirani criterion. In section 3, the canonical, the Landau-Lifchitz and the Bel energy-momentum pseudo-tensors are introduced and a comparison with the linearised theory is performed. Realistic sources for such gravitational waves are also described. Eventually, the analysis of the polarization leads to the conclusion that these fields are spin-1 gravitational waves.

In the following, $\mathcal{K}il\,(g)$ will denote the Lie algebra of all Killing fields of a metric g while *Killing algebra* will denote a sub-algebra of $\mathcal{K}il\,(g)$. Moreover, an integral ($2-$dimensional) submanifold of $\mathcal{D}$ will be called a *Killing leaf*, and an integral (2-dimensional) submanifold of $\mathcal{D}^{\perp}$ *orthogonal leaf* .

2. Geometric aspects of metrics of $(\mathcal{G}_2, 2)-type$

Let g be a metric on the space-time $\mathcal{M}$ and $\mathcal{G}_2$ one of its Killing algebras whose generators X, Y satisfy $[X, Y] = sY$, $s = 0, 1$. The Frobenius distribution $\mathcal{D}$ generated by $\mathcal{G}_2$ is $2-$dimensional and in the neighborhood of a non singular point *adapted coordinates* (x, y, p, q) exist ([31, 32, 33, 12, 13, 34]) such that

$$
X = \frac{\partial}{\partial p}, \qquad Y = \exp\,(sp)\,\frac{\partial}{\partial q}\,.
$$

In the following, we will distinguish two cases according to Y is of *light-type* ($g(Y, Y) = 0$) or not ($g(Y, Y) \neq 0$).

2.1. *Einstein metrics when $g(Y, Y) \neq 0$.*

If the Killing field Y is not of *light type*, i.e. $g(Y, Y) \neq 0$, then in the adapted coordinates (x, y, p, q) the general solution is [31]:

$$
g = f(dx^2 \pm dy^2) + \beta^2[(s^2 k^2 q^2 - 2slq + m)dp^2 + 2(l - skq)dpdq + kdq^2] \tag{2}
$$

with $f = - \triangle_\pm \beta^2/2s^2k$, and $\beta(x,y)$ a solution of the *tortoise equation*

$$\beta + A \ln |\beta - A| = u(x,y), \tag{3}$$

where A is a constant and the function u is a solution either of Laplace or d'Alembert equation, $\triangle_\pm u = 0$, $\triangle_\pm = \partial^2_{xx} \pm \partial^2_{yy}$, such that $(\partial_x u)^2 \pm (\partial_y u)^2 \neq 0$. The constants k, l, m are constrained by $km - l^2 = \mp 1, k \neq 0$ for Lorentzian metrics or by $km - l^2 = \pm 1, k \neq 0$ for Kleinian metrics.

Ricci flat manifolds of Kleinian signature possess a number of interesting geometrical properties and undoubtedly deserve attention in their own right. In recent years the physics of these manifolds has seen a revival of interest [20].

The gauge freedom of the above solution, allowed by the function u, can be locally eliminated by introducing the coordinates (u, v, p, q), the function $v(x,y)$ being conjugate to $u(x,y)$, *i.e.* $\triangle_\pm v = 0$ and $u_x = v_y, u_y = \mp v_x$. In these coordinates the metric g takes the form

$$g = \exp \frac{u - \beta}{A}(du^2 \pm dv^2)/2s^2k\beta + \beta^2[(s^2k^2q^2 - 2slq + m)dp^2 + 2(l - skq)dpdq + kdq^2] \tag{4}$$

with $\beta(u)$ a solution of $\beta + A \ln |\beta - A| = u$.

In *geographic coordinates* (ϑ, φ) along Killing leaves one has

$$g|_S = \beta^2 \left[d\vartheta^2 + \mathcal{F}(\vartheta) d\varphi^2 \right], \tag{5}$$

where $\mathcal{F}(\vartheta)$ is equal either to $\sinh^2 \vartheta$ or $-\cosh^2 \vartheta$, depending on the signature of the metric. Thus, in the *normal coordinates*, $(r = 2s^2k\beta, \tau = v, \vartheta, \varphi)$, the metric takes the form (local "Birkhoff's theorem")

$$g = \varepsilon_1 \left(\left[1 - \frac{A}{r} \right] d\tau^2 \pm \left[1 - \frac{A}{r} \right]^{-1} dr^2 \right) + \varepsilon_2 r^2 \left[d\vartheta^2 + \mathcal{F}(\vartheta) d\varphi^2 \right] \tag{6}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$.

The geometric reason for this form is that, when $g(Y,Y) \neq 0$, a third Killing field exists which together with X and Y constitute a basis of $so(2,1)$. The larger symmetry implies that the geodesic equations describe a *non-commutatively integrable system* [30], and the corresponding geodesic flow projects on the geodesic flow of the metric restricted to the Killing leaves.

The above local metric may be interpreted as the gravitational field generated by a star "outside" the universe. To be specific, we recall that the Schwarzschild solution shows a "star" generating a space "around" itself and it is an $so(3)$-invariant solution of the vacuum Einstein equations. On the contrary, its $so(2,1)$-analogue shows *a "star" generating the space only on "one side of itself". More exactly, the fact that the space in the Schwarzschild universe is formed by a 1-parametric family of "concentric" spheres allows one to give a sense to the adverb "around". In the $so(2,1)$-case the space is formed by a 1-parameter family of "concentric"* hyperboloids. The adjective "concentric" means that the curves orthogonal to hyperboloids are geodesics and metrically converge to a singular point. This explains in what sense this singular point generates the space only on "one side of itself".

2.2. *Einstein metrics when* $g(Y,Y) = 0$

If the Killing field Y is of *light type*, then the general solution [31] of vacuum Einstein equations, in the adapted coordinates (x, y, p, q), is given by

$$g = 2f(dx^2 \pm dy^2) + \mu[(w(x,y) - 2sq)dp^2 + 2dpdq], \tag{7}$$

where $\mu = A\Phi + B$ with $A, B \in R$, Φ is a non constant harmonic function of x and y, $f = (\nabla\Phi)^2 \sqrt{|\mu|}/\mu$, and $w(x,y)$ is solution of the μ-*deformed Laplace equation*:

$$\Delta_{\pm} w + (\partial_x \ln|\mu|) \partial_x w \pm (\partial_y \ln|\mu|) \partial_y w = 0, \qquad (8)$$

where Δ_+ (Δ_-) is the Laplace (d'Alembert) operator in the $(x,y)-$plane. Metrics (7) are Lorentzian if the orthogonal leaves are conformally Euclidean, *i.e.* the positive sign is chosen, and Kleinian if not. Only the Lorentzian case will be analyzed and these metrics will be called of $(\mathcal{G}_2, 2)-isotropic\ type$.

In the particular case $s = 1$, $f = 1/2$ and $\mu = 1$, the above (Lorentzian) metrics are locally diffeomorphic to a subclass of the vacuum Peres solutions [26], that for later purpose we rewrite in the form

$$g = dx^2 \pm dy^2 + 2dudv + 2(\varphi_{,x}dx + \varphi_{,y}dy)du. \qquad (9)$$

The correspondence between (7) and (9) depends on the special choice of the function $\varphi(x,y,u)$ (which, in general, is harmonic in x and y arbitrarily dependent on u); in our case $x \rightarrow x$, $y \rightarrow y$, $u \rightarrow u$, $v \rightarrow v + \varphi(x,y,u)$, $h = \varphi_{,u}$. In the case $\mu = const$, the μ-deformed Laplace equation reduces to the Laplace equation; for $\mu = 1$, in the harmonic coordinates system (x,y,z,t) defined in [10], the Einstein metrics (7) take the particularly simple form

$$g = 2f(dx^2 \pm dy^2) + dz^2 - dt^2 + d(w)\,d(\ln|z-t|)\,. \qquad (10)$$

This shows that, when w is constant, the Einstein metrics given by Eq. (10) are static and, under the further assumption $\Phi = x\sqrt{2}$, they reduce to the Minkowski one. Moreover, when w is not constant, gravitational fields (10) look like a *disturbance* propagating at light velocity along the z direction on the Killing leaves.

3. Physical properties of $(\mathcal{G}_2, 2)-type$ metrics

From a physical point of view, only Lorentzian metrics will be analyzed in the following, even if Ricci flat manifolds of Kleinian signature appear in the 'no boundary' proposal of Hartle and Hawking [18] in which the idea is suggested that the signature of the space-time metric may have changed in the early universe. So, assuming that particles are free to move between Lorentzian and Kleinian regions some surprising physical phenomena, like time travelling, would be observable (see [1] and [29]). Some other examples of Kleinian geometry in physics occur in the theory of heterotic $N = 2$ string (see [24] and [4]) for which the target space is four dimensional. The analysis will be devoted to metrics of $(\mathcal{G}_2, 2) - type$, when the vector field Y, *i.e.* the commutator $[X, Y]$, is of light-type: $g(Y, Y) = 0$. The wave character of gravitational fields (7) has been checked by using covariant criteria. In the following we will shortly review the most important properties of these waves which will turn out to have spin-1.

In the first part of the section the standard theory of linearized gravitational waves will be shortly described. In the second part, the theoretical reality of spin-1 gravitational waves will be discussed.

3.1. The standard linearized theory

The usual transverse-traceless gauge in the linearised vacuum Einstein equations and the (usually implicit) assumptions needed to reduce to this gauge play an important role: the generality of the usual claim "the graviton has spin-2" is strictly related to these assumptions. Thus, it is quite useful to discuss the physical and mathematical hypothesis leading to this result to check if there are physical interesting situations in which they are not fulfilled. Here, only the case of gravitational waves propagating on flat space-time will be considered since the principal ingredients needed in the following are already present in this case.

Let us consider a generic metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where η is the Minkowski metric and h can be thought as a perturbation. We formally write the inverse metric as $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$ with $k = \sum_{n\in N} k(n)$ and $k(n)$ is of order h^n and $k(1) = -\eta h \eta$. Then we have for the Christoffel symbols

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}) \tag{11}$$

and

$$\Gamma^\lambda_{\mu\nu} = \sum_{n\in N} \Gamma^{(n)\lambda}_{\mu\nu} \tag{12}$$

where $\Gamma^{(n)\lambda}_{\mu\nu} = k^{(n-1)\lambda\rho}\Gamma_{\rho\mu\nu}$ and $k^{(0)} = \eta^2$.

Thus the Ricci tensor may be written as $R_{\mu\nu} = \sum_n R^{(n)}_{\mu\nu}$, with

$$R^{(n)}_{\mu\nu} = \Gamma^{(n)\lambda}_{\mu\lambda,\nu} - \Gamma^{(n)\lambda}_{\mu\nu,\lambda} + \sum_{m+m'=n}\left(\Gamma^{(m)\lambda}_{\nu\beta}\Gamma^{(m')\beta}_{\mu\lambda} - \Gamma^{(m)\lambda}_{\lambda\beta}\Gamma^{(m')\beta}_{\mu\nu}\right). \tag{13}$$

The harmonicity condition reads

$$0 = \Gamma^\lambda = g^{\lambda\rho}g^{\mu\nu}\Gamma_{\rho\mu\nu} = \sum_n \Gamma^{(n)\lambda}. \tag{14}$$

Up to now, we made no hypothesis on $h_{\mu\nu}$. The gravitational field is said to be *weak* (in M') if there exists a (harmonic) coordinates system and a region $M' \subset M$ of space-time in which the following conditions hold:

$$|h_{\mu\nu}| << 1, |h_{\mu\nu,\alpha}| << 1. \tag{15}$$

In the space-time regions where the linearized theory can be applied, one can take into account only terms which are of first order in h. In particular only the term

$$\Gamma^{(1)\lambda} = \eta^{\mu\nu}[(\eta^{\lambda\rho}h_{\mu\rho})_{,\nu} + (\eta^{\lambda\rho}h_{\nu\rho})_{,\mu}] - \eta^{\lambda\rho}(\eta^{\mu\nu}h_{\mu\nu})_{,\rho} \tag{16}$$

contributes to the sum (14). In this approximation the components of Ricci tensor read

$$R^{(1)}_{\mu\nu} \equiv \frac{1}{2}[\eta^{\rho\sigma}h_{\rho\sigma,\mu\nu} - \eta^{\rho\sigma}(h_{\mu\sigma,\rho\nu} + h_{\nu\rho,\mu\sigma})] \tag{17}$$

and, because of the harmonicity condition $\Gamma^{(1)\lambda} = 0$, the Einstein equations reduce to the well known *wave-equation*

$$h_{\mu\nu} = 0. \tag{18}$$

Thus, we see that the harmonicity condition has a key role in derive equation (18). Up to now, apart from the harmonicity condition, no special assumptions either on the form or on the analytic properties of the perturbation h has been done. Then, a natural question arises: is the original gauge freedom of the theory completely fixed or is there a residual gauge that can simplify the form of $h_{\mu\nu}$?

The answer is, of course, well known: the residual gauge transformations are nothing but the so called *gauge transformations of the linearized theory*, that is, the coordinates transformations with the following two properties:

1) they preserve the harmonicity of the coordinates system and, then, the form of equations (18);

2) they preserve the "weak character" of the gravitational field, namely, conditions (15).

---

[2] In fact, $\Gamma_{\rho\mu\nu}$ is intrinsically of the first order in h

3.2. The spin

It is worth to stress here that the definition of *spin* or *polarization* for a theory, such as general relativity, which is non-linear and possesses a much bigger invariance than just the Poincaré one, deserves a careful analysis. It is well known that the concept of particle together with its degrees of freedom like the spin may be only introduced for linear theories (for example for the Yang-Mills theories, which are non linear, it is necessary to perform a perturbative expansion around the linearized theory). In these theories, when Poincaré invariant, the particles are classified in terms of the eigenvalues of two Casimir operators of the Poincaré group, P^2 and W^2 where P_μ are the translation generators and $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ is the *Pauli-Ljubanski polarization vector* with $M^{\mu\nu}$ Lorentz generators. Then, the total angular momentum $J = L + S$ is defined in terms of the generators $M_{\mu\nu}$ as $J^i = \frac{1}{2}\epsilon^{0ijk}M_{jk}$.

It is commonly believed that, with a suitable gauge transformation with the above properties **1** and **2,** one can always kill the "spin-1" components of the gravitational waves. However, even if not explicitly declared, the standard textbook analysis of the polarization is performed for *square integrable* solutions of the wave-equation (18) but, as we will see in the following, some very interesting solutions do not belong to this class. To make this point clear, now we will briefly describe the standard analysis to kill the spin-1 components stressing the role played by the *square integrability* assumption.

Indeed, *square integrable* solutions of equations (18) can be always Fourier expanded in terms of plane-wave functions with a (*real*) light-like vector wave k_μ. The standard plane wave solutions of Eq. (18) are $h_{\mu\nu} = e_{\mu\nu}e^{i\rho} + e_{\mu\nu}^*e^{-i\rho}$ with $\rho = k_\mu x^\mu$, k_μ being the propagation direction vector fulfilling $k_\mu k^\mu = 0$, the harmonicity condition reduces to

$$\frac{1}{2}k_\lambda \eta^{\mu\nu}e_{\mu\nu} = \eta^{\mu\nu}k_\nu e_{\mu\lambda}, \tag{19}$$

while the *gauge transformations of the linearized theory* are in this case:

$$e_{\mu\nu} \rightarrow e'_{\mu\nu} = e_{\mu\nu} + k_\mu l_\nu + k_\nu l_\mu; \quad k^\mu l_\mu = 0, \tag{20}$$

l_μ being a real vector too.

It is easy to see that the symmetry group of this equation, which encodes the harmonic nature of the coordinate system, reduces to linear transformations and more precisely to Poincaré transformations [38]. This characteristic, is, of course, essential for a meaningful definition of polarization of gravitational waves. In particular, it allows to show the spin contents of a gravitational perturbation. Namely, a *real* propagation direction vector k_μ can always be chosen in the following form $k_\mu = (1, 0, 0, -1)$, so that the gravitational perturbation is propagating along the $z-$axis. Then, it is trivial to show (see, for example, [38]) that the spin-1 components of $e_{\mu\nu}$ are e_{it} (where $i, j = x, y$) while the spin-2 are e_{ij} (that is, the spin-2 components are the ones with both index in the plane orthogonal to the propagation direction). Moreover, it can be easily shown that with a suitable transformation (20) the e_{it} components can be killed. From the formal point of view this means that the system of equations for l_μ

$$e_{it} + k_i l_t + k_t l_i = 0 \tag{21}$$

with $k_\mu = (1, 0, 0, 1)$ has always a solution [15].

However, for complex k_μ such that $k_\mu k^\mu = 0$ (for example $(1, a, \pm ia, 1)$ with $a \in R$), this is not true anymore, that is, the above system of equations has in general no solution as can be easily checked. Thus, in this case, the spin-1 components are not *pure gauge*. A complex k_μ could look quite strange; however this simply corresponds to split the four-dimensional d'Alembert operator $^{(4)}\Box$ in a two-dimensional d'Alembert operator in the $z-t$ plane plus a Laplace operator $^{(2)}\Delta$ in the $x-y$ plane. In other words, we are thinking solutions of the four-dimensional d'Alembert

equation as products of solutions of the two-dimensional d'Alembert equation and of the two-dimensional Laplace equation. What is lacking in this case is, obviously, the *square integrability* of such solutions due to the presence of the harmonic function solution of the two-dimensional Laplace equation. To be more precise, the global *square integrability* is lacking, but there exist singular solutions of this form which far away the singularities are perfectly well-behaved. Now the question is, can be these solutions declared unphysical? Put in another way, can reasonable sources be found to smooth out the singularities? The answer is positive as we will see in more details in the next sections where a simple and interesting example will be given.

A different but equivalent procedure makes use of the so-called *energy-momentum pseudo-tensors.*

4. The energy-momentum pseudo-tensors

The definition of *momentum* and *energy* associated with a gravitational field is an intrinsically controversial problem because these quantities are connected to the space-time translation invariance, whereas the group of invariance of general relativity is much bigger. With this cautionary remark in mind, various definitions are available which attain to different physical situations (for a recent discussion see [3]).

4.1. The canonical energy-momentum pseudo-tensor

A commonly accepted definition is based on the *canonical* energy-momentum pseudo-tensor [15]:

$$\tau_\mu^\nu = \frac{\partial L}{\partial \left(\partial_\nu g_{\alpha\beta}\right)} \partial_\mu g_{\alpha\beta} - g_\mu^\nu L, \tag{22}$$

where

$$L = g^{\mu\nu} \left[\Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\sigma}^\sigma - \Gamma_{\mu\rho}^\sigma \Gamma_{\nu\sigma}^\rho \right] \tag{23}$$

is the *Ricci scalar* deprived of terms containing the second derivative of the metric.

Owing to τ_μ^ν not being a real tensor, we do not get, in general, a clear result independent of the coordinate system. But there is one special case in which we do get a clear result; namely when the waves are all moving in the same direction. If the $h_{\mu\nu}$ are assumed to be a *square integrable* solution of the wave equation (18) we may consider the general case in which they are all functions of the single variable $r = k_\mu x^\mu$, the k_μ's being real constants satisfying $k_\mu k^\mu \equiv \eta^{\mu\nu} k_\mu k_\nu = 0$. We then have

$$h_{\mu\nu,\sigma} = u_{\mu\nu} k_\sigma, \tag{24}$$

where $u_{\mu\nu} = \partial_r h_{\mu\nu}$.

The harmonic condition $\Gamma^{(1)\lambda} = 0$ gives

$$\eta^{\mu\nu} u_{\mu\rho} k_\nu = \frac{1}{2} u k_\rho, \tag{25}$$

with $u = \eta^{\mu\nu} u_{\mu\nu} = u_\mu^\mu$.

It is easily seen that the action density L, defined by expression (23), vanishes on account of harmonic coordinates (25) and condition and $k^\mu k_\mu = 0$. There is a corresponding result for the electromagnetic field, for which the action density also vanishes in the case of waves moving only in one direction. It is easy to show that

$$\tau_\mu^\nu = \frac{1}{2} \left(u_{\alpha\beta} u^{\alpha\beta} - \frac{1}{2} u^2 \right) k_\mu k^\nu. \tag{26}$$

We have the result that τ_μ^ν transforms like a tensor under those transformations that preserve the character of the field consisting only of waves moving in the direction k_σ so that $h_{\mu\nu}$ remain functions of the single variable $r = k_\mu x^\mu$.

To understand the physical significance of the above expression, let us go back to the case of waves moving in the direction x^3, so that $k_0 = 1$, $k_1 = 0$, $k_2 = 0$, $k_3 = -1$; by using the harmonic conditions we get

$$16\pi\tau_0^0 = \frac{1}{4}\left(u_{11} - u_{22}\right)^2 + \left(u_{12}\right)2, \quad \tau_0^3 = \tau_0^0. \tag{27}$$

We see that the energy density is positive definite and the energy flows in the direction x^3 with the velocity of light. To discuss the polarization of waves, we introduce the infinitesimal rotation operator $\mathcal{R}$ in the plane (x^1, x^2). Applied to $u_{\alpha\beta}$, it has the effect

$$\mathcal{R}u_{11} = 2u_{12}; \quad \mathcal{R}u_{12} = u_{22} - u_{11}; \quad \mathcal{R}u_{22} = -2u_{12}. \tag{28}$$

Since $u_{11} + u_{22}$ is invariant, while $i\mathcal{R}$ has eigenvalues ± 2 when applied to $u_{11} - u_{22}$ or u_{12}, the components of $u_{\alpha\beta}$ that contribute to the energy correspond to spin-2.

Then, for the square integrable solutions of the linearised Einstein equations the energy density τ_0^0 is expressed [15] as the sum of squares of derivatives of some metric components which do represent the physical degrees of freedom of the gravitational field.

Under a transformation preserving the propagation direction and the harmonic character of the coordinates system, in particular a rotation in the (x, y) plane, the physical components of the metric transform like a spin-2 field.

4.2. Landau–Lifshitz's and Bel's energy-momentum pseudo-tensors

Besides the canonical energy-momentum pseudo-tensor, a deep physical significance can be given to the Landau-Lifshitz energy-momentum pseudo-tensor $\tau^{\rho\kappa}$ [21] $\tau^{\rho\kappa} = \frac{1}{16\pi k}\left\{(2\Gamma_{\lambda\mu}^\nu\Gamma_{\nu\sigma}^\sigma - \Gamma_{\lambda\sigma}^\nu\Gamma_{\mu\nu}^\sigma - \Gamma_{\lambda\nu}^\nu\Gamma_{\mu\sigma}^\sigma)(g^{\rho\lambda}g^{\kappa\mu} - g^{\rho\kappa}g^{\lambda\mu})\right.$
$+ g^{\rho\lambda}g^{\mu\nu}(\Gamma_{\lambda\sigma}^\kappa\Gamma_{\mu\nu}^\sigma + \Gamma_{\mu\nu}^\kappa\Gamma_{\lambda\sigma}^\sigma - \Gamma_{\nu\sigma}^\kappa\Gamma_{\lambda\mu}^\sigma - \Gamma_{\lambda\mu}^\kappa\Gamma_{\nu\sigma}^\sigma)$
$+ g^{\kappa\lambda}g^{\mu\nu}(\Gamma_{\lambda\sigma}^\rho\Gamma_{\mu\nu}^\sigma + \Gamma_{\mu\nu}^\rho\Gamma_{\lambda\sigma}^\sigma - \Gamma_{\nu\sigma}^\rho\Gamma_{\lambda\mu}^\sigma - \Gamma_{\lambda\mu}^\rho\Gamma_{\nu\sigma}^\sigma)$
$\left. + g^{\mu\lambda}g^{\sigma\nu}(\Gamma_{\lambda\nu}^\rho\Gamma_{\mu\sigma}^\kappa - \Gamma_{\lambda\mu}^\rho\Gamma_{\nu\sigma}^\kappa)\right\}.$

There are strong evidences that, in some cases, it gives the correct definition of energy [28]. In fact, the energy flux radiated at infinity for an asymptotically flat space-time, evaluated with the Landau-Lifshitz energy-momentum pseudo-tensor, has been seen to agree with the Bondi flux [8] that is with the energy flux evaluated in the exact theory.

It is easy to check that for the metric (10) with $f = 1/2$,

$$g = dx^2 + dy^2 + dz^2 - dt^2 + d\left(w\right)d\left(\ln|z - t|\right). \tag{29}$$

the non vanishing components $p^\mu \equiv \tau_0^\mu$ of the 4-momentum density are

$$t_0^0 = 4\left(t - z\right)^{-2}\left[c_1\left(w_{,xx}\right)^2 + c_2\left(w_{,xy}\right)^2\right], \quad t_0^3 = t_0^0 \tag{30}$$

where c_i are some positive numerical constants and the harmonicity condition for w has been used.

The use of the Bel's superenergy tensor [5]

$$T^{\alpha\beta\lambda\mu} = \frac{1}{2}\left(R^{\alpha\rho\lambda\sigma}R^\beta{}_{\rho}{}^\mu{}_\sigma + {}^*R^{\alpha\rho\lambda\sigma}{}^*R^\beta{}_{\rho}{}^\mu{}_\sigma\right), \tag{31}$$

where the symbol $*$ denotes the *volume dual*, leads to the same result. Indeed, the only non vanishing independent components of the covariant Riemann tensor $R_{\alpha\beta\gamma\delta} = g_{\alpha\rho}R^\rho{}_{\beta\gamma\delta}$ are

$$R_{xpxp} = -w_{,xx}; \quad R_{xpyp} = -w_{,xy}; \quad R_{ypyp} = -w_{,yy}. \tag{32}$$

It follows that the density energy represented by the Bel's scalar

$$W = T_{\alpha\beta\lambda\mu} U^\alpha U^\beta U^\lambda U^\mu, \tag{33}$$

the U^α's denoting the components of a time-like unit vector field, depends on the squares of $w_{,ij}$.

Thus, both the Landau-Lifshitz pseudo tensor and the Bel superenergy tensor single out the same physical degrees of freedom.

As it has been said, gravitational fields (29) represent gravitational waves moving at the velocity of light, that is, in the would be quantised theory, particles with zero rest mass. Thus, if a classification in terms of Poincaré group invariants could be performed, these waves would belong to the class of unitary (infinite-dimensional) representations of the Poincaré group characterized by $P^2 = 0$, $W^2 = 0$. But, in order for such a classification to be meaningful, P^2 and W^2 have to be invariants of the theory.

For these reasons, only gravitational fields represented by Eq. (29) will be considered, which, besides being exact solutions, solve the linearised vacuum Einstein equations as well. Following the previous analysis one can see that these two components are eigenvectors of $i\mathcal{R}$ belonging to the eigenvalues ± 1. In other words, gravitational fields (10), which are not pure gauge since the Riemann tensor is not vanishing, represent spin-1 gravitational waves propagating along the $z-$axis at light velocity.

5. A simple example
Thus, more generally, it is natural to consider perturbations of the form $h = dw(x,y) \cdot df(z-t)$ which are not square integrable and cannot be Fourier expanded. Nevertheless, in next section it will be shown that the metric

$$g = \eta + dw(x,y) \cdot df(u), \quad u = z - t, \qquad \left(\partial_x^2 + \partial_y^2\right) w = 0, \tag{34}$$

with f an arbitrary function, is asymptotically flat for a wide choice of harmonic functions w; thus, it represents a physically interesting gravitational field which, besides to be a solution of the linearized Einstein equations on flat background, it is also an exact solution of Einstein equations too.

It is trivial to verify that the above perturbation h is written in harmonic coordinates and moreover has an *off-diagonal* form, that is, this perturbation has only one index in the plane $x - y$ orthogonal to the propagation direction z: for this reason the above gravitational wave has spin equal to 1 and obviously is not a pure gauge [10], that is, its Riemann tensor does not vanish. One could think that with a gauge transformation it is possible to bring the above gravitational wave in the standard *transverse-traceless* form. Indeed, it is possible [35] to find a diffeomorphism which gives to the metric (34) the standard *transverse-traceless* form but one can check that the new coordinates are not harmonic anymore.

5.1. Asymptotic flatness
From the physical point of view, it is important to understand under which conditions metrics (34) are asymptotically flat. In the vacuum case, the coordinates (x, y, z, t) are harmonic. Being z the propagation direction, the physical effects manifest themselves in the $x - y$ planes orthogonal to the propagation direction. This suggests to call (spatially) *asymptotically flat* a metric approaching, for $x^2 + y^2 \to \infty$, the Minkowski metric. This intuitive definition of *asymptotic flatness* allows to obtain qualitative results by using the standard theory of Partial Differential Equations. In terms of the function w the asymptotic flatness condition reads:

$$\lim_{x^2+y^2\to\infty} (w - c_1 x - c_2 y - c_3) = 0,$$

where c_1, c_2 and c_3 are suitable arbitrary constants and the behaviour of w can be easily recognized by looking at the Riemann tensor of the metrics (34):

$$R_{uiuj} = f_{,u} w_{,ij} \tag{35}$$

which depends on the second derivatives of the two-dimensional harmonic function w.

Therefore, to have an asymptotically flat metric, the function w must be asymptotically close to a linear function. But, due to standard results in the theory of linear Partial Differential Equations, this is impossible unless w is a linear function everywhere and this would imply the flatness of the metrics (34). However, if we admit δ-like singularities in the $x - y$ planes, non trivial spatially asymptotically flat vacuum solutions with $w \neq const$ can exist [9]. Of course, it is not necessary to consider $\delta-$like singularities: it is enough to take into account matter sources. For example, in the presence of an electromagnetic wave propagating along the z axis with energy density equal to ρ, the exact non vacuum Einstein equations for metrics (34 read (see, for example, [9])

$$f_{,u} \left(\partial_x^2 + \partial_y^2 \right) w = \kappa \rho,$$

where κ is the gravitational coupling constant. Thus, one can have non singular spin-1 gravitational waves by considering suitable matter sources which smooth out the singularities: in the above case one can, for example, consider an energy density ρ which vanishes outside a compact region of the $x - y$ planes.

From the phenomenological point of view, it is worth to note that these kind of wave-like gravitational fields, unlike standard spin-2 gravitational waves which can be singularities free even in the vacuum case, have to be coupled to matter sources in order to represent reasonable gravitational fields.

5.2. Wave character of the field

Up to now, it has been assumed that metrics (34) indeed represent wave-like gravitational fields. Even if from a "linearised" perspective this is obvious, being the above metrics solutions of the exact Einstein equations too, one should try to use covariant criteria in order to establish their wave character.

Here the following gravitational fields, a little more general than the ones expressed by Eq. (9), will be considered:

$$g = dx^2 + dy^2 + 2dudv + 2(\varphi_{,x}dx + \varphi_{,y}dy)du \tag{36}$$

where Eqs. (38) vacuum Einstein equations and harmonicity conditions read, respectively,

$$\partial_u \left(\partial_x^2 + \partial_y^2 \right) \varphi = 0, \qquad \left(\partial_x^2 + \partial_y^2 \right) \varphi = 0. \tag{37}$$

The wave character and the polarization of these gravitational fields can be analyzed in many ways. For example, we could use the Zel'manov criterion [39] to show that these are gravitational waves and the Landau-Lifshitz pseudo-tensor to find the propagation direction of the waves [10, 11]. However, the algebraic Pirani's criterion is easier to handle since it determines the wave character of the solutions and the propagation direction both at once. Moreover, it has been shown that, in the vacuum case, the two methods agree [11]. To use this criterion the Weyl scalars must be evaluated according to the Petrov-Penrose classification [27, 25, 23] (for a recent discussion see [22]).

To perform the Petrov-Penrose classification, one has to choose a *tetrad* basis with two real null vector fields and two real spacelike (or two complex null) vector fields. Then, according to the Pirani's criterion, if the metric belongs to type **N** [14, 39] of the Petrov classification, it is a

gravitational wave propagating along one of the two real null vector fields. Since ∂_u and ∂_v are null real vector fields and ∂_x and ∂_y are spacelike real vector fields, the above set of coordinates is the right one to apply for the Pirani's criterion.

Since the only nonvanishing components of the Riemann tensor, corresponding to the metric (37), are

$$R_{iuju} = -\partial_{ij}^2 \partial_u \varphi, \quad i,j = x,y, \tag{38}$$

this gravitational fields belong to Petrov type **N**. Then, according to the Pirani's criterion, the metric (37) does indeed represent a gravitational wave propagating along the null vector field ∂_u.

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