1 Introduction

Here I will collect calculations done while exploring fractional curvature.

2 κ_σ of the unit circle

We wish to compute

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r)$$

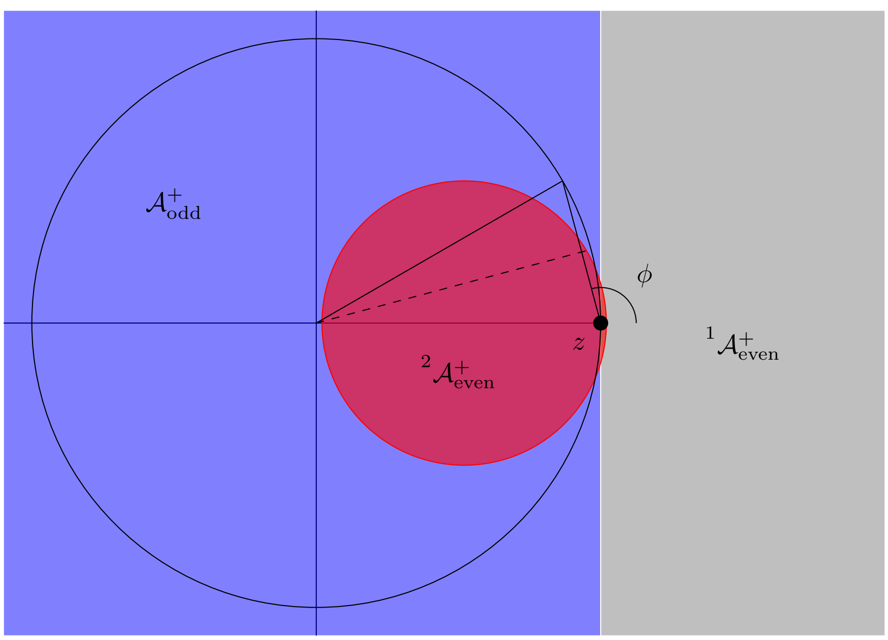
for C given by

$$z(\phi) = (\cos\phi, \sin\phi), \phi \in [0, 2\pi].$$

Due to symmetry $\kappa_\sigma(z(0)) = \kappa_\sigma(z(\phi))\ \ \forall\, \phi \in (0, 2\pi]$, so we can focus on the case when $z = (1, 0)$. We have $\mathbf{t}(z) = (0, 1)$. in order to help us characterize $\mathcal{A}^+_{\text{even}}, \mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
{}^1\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{3\pi}{2}, 2\pi\right] \cup \left[0, \frac{\pi}{2}\right], r \in [0,\infty)\right\}\\
{}^2\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r \in [0, \cos(\pi-\phi))\right\}\\
\mathcal{A}^+_{\text{even}} &= {}^1\mathcal{A}^+_{\text{even}} \cup {}^2\mathcal{A}^+_{\text{even}}\\
\mathcal{A}^+_{\text{odd}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r \in [\cos(\pi-\phi), \infty)\right\}
\end{aligned}$$

These subsets are motivated by the following picture:



Before jumping into calculations observe that we can parameterize our subset of $\mathbb{R}^5$ via (θ, r), as shown in the definition of the subsets above and put

$$s(\theta) = \begin{cases} -1 & \theta \in [\pi/2, 3\pi/2] \\ 1 & \text{otherwise} \end{cases}.$$

We can simplify our integrand as follows:

$$\begin{aligned}
J(r,\theta) &= \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}} \\
&= \frac{\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \left(s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}}{r^{1+\sigma}} \\
&= \frac{s(\theta)\left(\sin\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \cos\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)}{r^{1+\sigma}} = \frac{-s(\theta)\begin{pmatrix}\sin^2\theta+\cos^2\theta\\ -\sin\theta\cos\theta+\cos\theta\sin\theta\end{pmatrix}}{r^{1+\sigma}} \\
&= \begin{pmatrix}-1\\0\end{pmatrix}\frac{s(\theta)}{r^{1+\sigma}}
\end{aligned}$$

Next we can start computing integrals, we begin by integrating over $\mathcal{A}^+_{\text{even}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{even}}} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2(r,\theta) &= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta + \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{1}{r^{1+\sigma}}\, dr\, d\theta - \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{1}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\frac{1}{\sigma}\left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\left(0 - \frac{1}{\epsilon^\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(\frac{1}{(\cos(\pi-\theta))^\sigma} - \frac{1}{\epsilon^\sigma}\right) d\theta \\
&= \frac{\pi}{\sigma\epsilon^\sigma} - \frac{\pi}{\sigma\epsilon^\sigma} - \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

Now for $\mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{odd}}} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2(r,\theta) &= \int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{1}{r^{1+\sigma}}\, dr\, d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(0 - \frac{1}{(\cos(\pi-\theta))^\sigma}\right) d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= -\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

Putting these computations together we have:

$$\begin{aligned}
\left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r) &= \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2 \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \left(\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta + \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta \right) \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{2}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

For now I don't have the know-how to evaluate this integral myself, but Wolfram Alpha tells me the following:

$$\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta = \frac{\sqrt{\pi}\,\Gamma\big(\tfrac{1}{2} - \tfrac{\sigma}{2}\big)}{\Gamma\big(1 - \tfrac{\sigma}{2}\big)}$$

Thus we find:

$$\begin{aligned}
\lim_{\sigma\uparrow 1} \frac{(1-\sigma)}{4}\kappa_\sigma &= \lim_{\sigma\uparrow 1} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{2(1-\sigma)\sqrt{\pi}\,\Gamma\big(\tfrac{1}{2} - \tfrac{\sigma}{2}\big)}{4\sigma\,\Gamma\big(1 - \tfrac{\sigma}{2}\big)} \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{\sqrt{\pi}}{2} \lim_{\sigma\uparrow 1} \frac{(1-\sigma)\,\Gamma\big(\tfrac{1}{2} - \tfrac{\sigma}{2}\big)}{\sigma\,\Gamma\big(1 - \tfrac{\sigma}{2}\big)} \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{1}{2} \lim_{\sigma\uparrow 1} (1-\sigma)\,\Gamma\left(\frac{1}{2} - \frac{\sigma}{2}\right) \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \kappa,
\end{aligned}$$

i.e. κ_σ converges to κ with the factor of $(1-\sigma)/4$ in front of it.