Exercise 6.1 We have

$$\nabla V(x) = \begin{pmatrix} 4x_1 \\ x_2 \end{pmatrix} \implies \nabla V(x) \cdot f(x) = -4x_1^2 + 4x_1^2 x_2 - 2x_2^2 - 4x_1^2 x_2 = -4x_1^2 - 2x_2^2 \leq 0$$

We also know $2x_1^2 + \frac{1}{2}x_2^2 = 0 \implies x_1 = x_2 = 0$, so that $V(x) = 0 \iff x = 0$. Lastly for $x \notin \mathbb{B}$, $V(x) \notin \frac{1}{2}\|x\|\mathbb{B}$ so that $V(x) \to \infty$ as $\|x\| \to \infty$, showing that V is Lyapunov for f.

Considering $V(x) = x_1^2 + x_2^2$ we have

$$\nabla V(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 2x \implies \nabla V(x) \cdot f(x) = 2\big(-x_1^2 + x_1^2 x_2 - 2x_2^2 - 4x_1^2 x_2\big) = 2\big(-x_1^2 - 2x_2^2 - 3x_1^2 x_2\big)$$

But when $x_1 = x_2 = -2$

$$\nabla V(x) \cdot f(x) = 2(-4 - 8 + 3 \cdot 4 \cdot 2) = 24 > 0$$

so that V is not Lyapunov for f.