

Computing the limit of the Nonlocal Curvature of Curves in \mathbb{R}^n

Dan Zimmerman

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Origins of Nonlocal Curvature of Curves

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Consider $u = \chi_E$ for bounded E : $\int_E \int_{E^c} \frac{1}{|x - y|^{n+\sigma}} dx dy$

$$\text{Per}_\sigma(E) := \frac{1}{\alpha_{n-1}} \int_E \int_{E^c} \frac{1}{|x - y|^{n+\sigma}} dx dy$$

For unbounded E , fix Ω bounded, and set

$$\text{Per}_\sigma(E, \Omega) := \frac{1}{\alpha_{n-1}} \left(\int_{E \cap \Omega} \int_{E^c} + \int_{E \cap \Omega^c} \int_{E^c \cap \Omega} \right) \frac{1}{|x - y|^{n+\sigma}} dx dy$$

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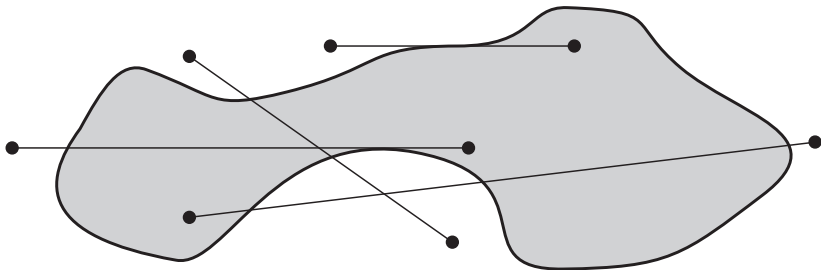
Nonlocal mean curvature at $z \in \partial E$

$$H_\sigma(z) := \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\chi_E(x) - \chi_{E^c}(x)}{|z - x|^{n+\sigma}} dx$$

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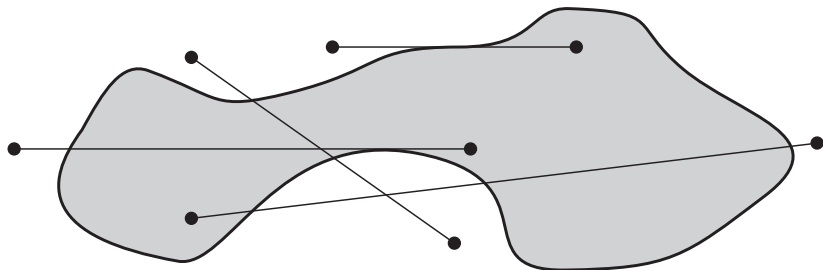
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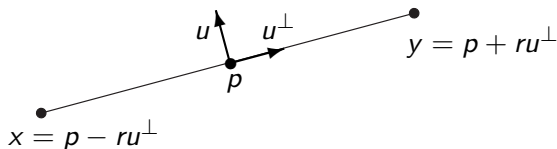
$$\text{Area}_\sigma(S, \Omega) := \frac{1}{2\alpha_{n-1}} \int_{\mathcal{X}(S)} \frac{\max\{\chi_\Omega(x), \chi_\Omega(y)\}}{|x-y|^{n+\sigma}} dx dy$$

$$\text{Area}_\sigma^{(n=2)}(\mathcal{S}, \Omega) = \frac{1}{4} \int_{\mathcal{X}(\mathcal{S})} \frac{\max\{\chi_\Omega(x), \chi_\Omega(y)\}}{|x - y|^{2+\sigma}} dx dy$$

A fractional notion of length and an associated nonlocal curvature - Seguin (2020)

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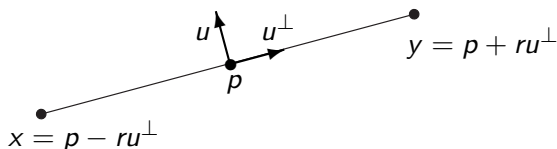
Disks! (intersecting \mathcal{S} odd times)



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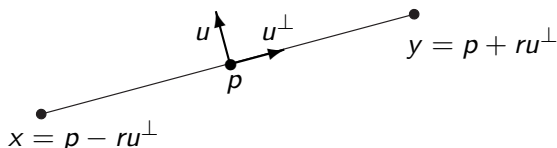


$$= \frac{1}{2} \int_{\mathcal{D}(\mathcal{S})} \frac{\max\{\chi_\Omega(p - ru^\perp), \chi_\Omega(p + ru^\perp)\}}{(2r)^{1+\sigma}} d\mathcal{H}^4(p, u, r)$$

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Generalizes nicely

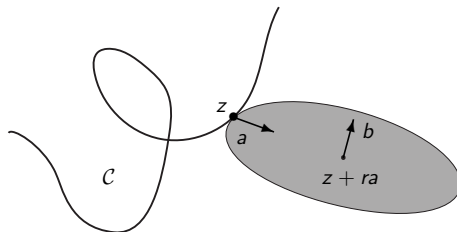
$$\text{Len}_\sigma(\mathcal{C}, \Omega) := \frac{\Gamma\left(\frac{n+1}{2}\right)^2}{2\pi^{n-1}} \int_{\mathcal{D}(\mathcal{C})} \frac{\sup_{v \in \mathcal{U}_n \cap u^\perp} \chi_\Omega(p + rv)}{r^{n-1+\sigma}} d\mathcal{H}^{2n}(p, u, r)$$

Minimizing over \mathcal{C} with fixed boundary, $z \in \mathcal{C}$

$$\left(\int_{\mathcal{A}_{\text{odd}}^+(z)} - \int_{\mathcal{A}_{\text{even}}^+(z)} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} d\mathcal{H}^{2n-2}(a, b, r) = 0$$

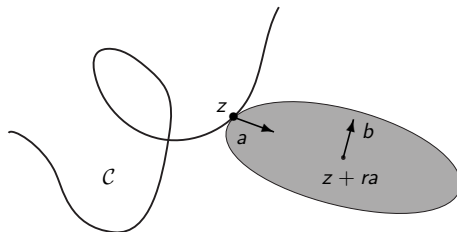
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Nonlocal vector curvature

$$\kappa_{\sigma}(z) := \left(\int_{\mathcal{A}_{\text{odd}}^+(z)} - \int_{\mathcal{A}_{\text{even}}^+(z)} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} d\mathcal{H}^{2n-2}(a, b, r)$$

Asymptotics of fractional & Nonlocal Quantities

$$\lim_{\sigma \uparrow 1} (1 - \sigma) \operatorname{Per}_\sigma(E, B_r) = \operatorname{Per}(E, B_r)$$

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Argument Outline

1. For \mathcal{C}_R circle with radius R , in \mathbb{R}^2

$$\lim_{\sigma \uparrow 1} (1 - \sigma) \quad \kappa_\sigma(z) = \kappa(z) \quad z \in \mathcal{C}_R$$

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3. For arbitrary \mathcal{C} and \mathcal{C}_R ($R = 1/\kappa(z; \mathcal{C})$) at $z \in \mathcal{C}$

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4. Combine 2 & 3, show for arbitrary \mathcal{C} , $z \in \mathcal{C}$

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$$z \in \mathcal{C}_R, \ n = 2$$

$$\kappa_\sigma = \left(\int_{\mathcal{A}_{\text{odd}}^+} - \int_{\mathcal{A}_{\text{even}}^+} \right) \frac{(a \cdot t)b - (b \cdot t)a}{r^{1+\sigma}} d\mathcal{H}^2(a, b, r)$$

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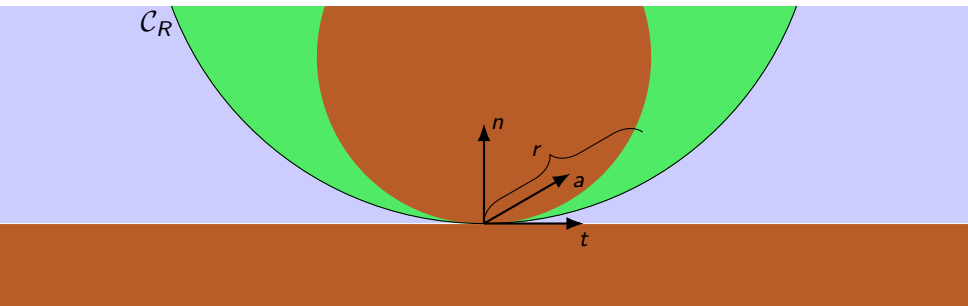
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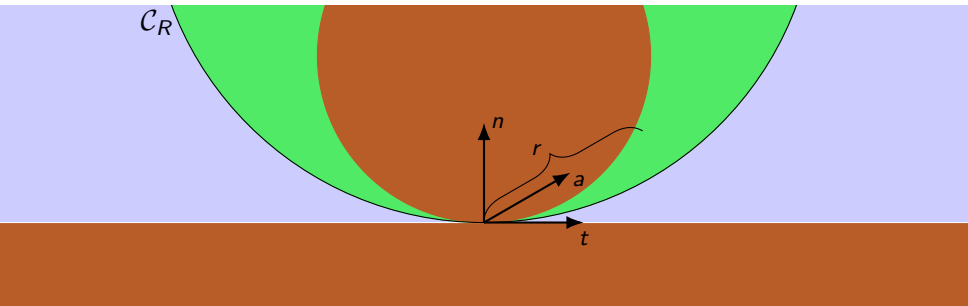
$$\bar{\chi}_{\mathcal{A}_{\text{even}}^+} = \chi_{\mathcal{A}_{\text{even}}^+} - \chi_{(\mathcal{A}_{\text{even}}^+)^c} = \chi_{\mathcal{A}_{\text{even}}^+} - \chi_{\mathcal{A}_{\text{odd}}^+}$$

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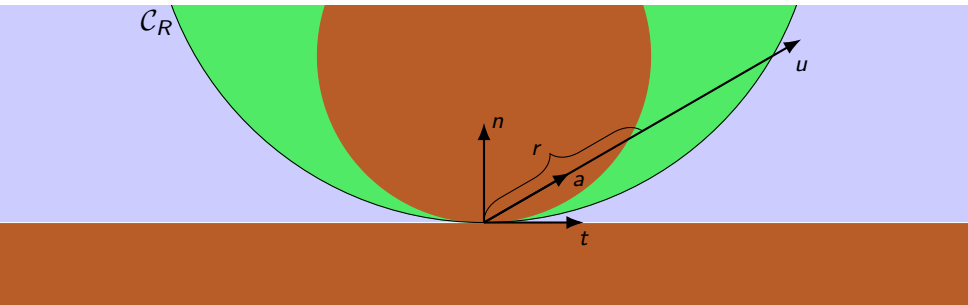
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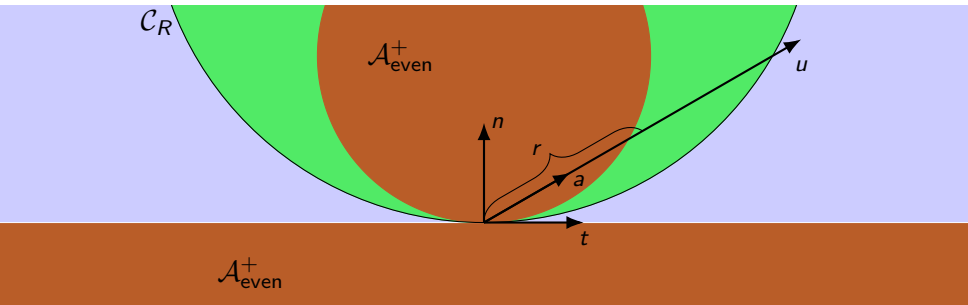
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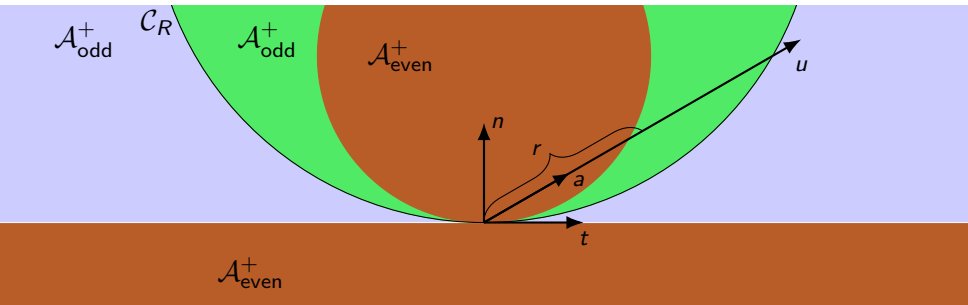
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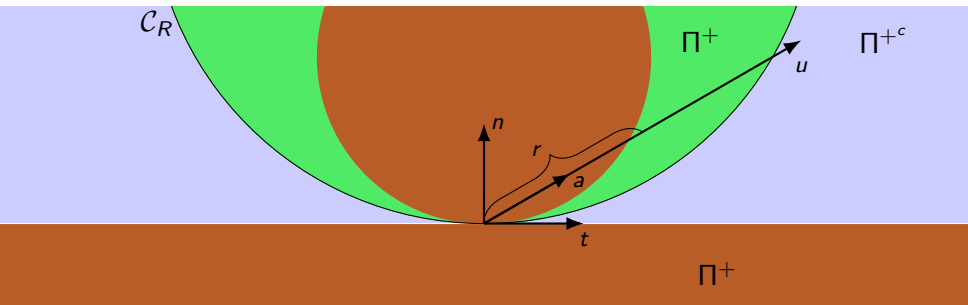
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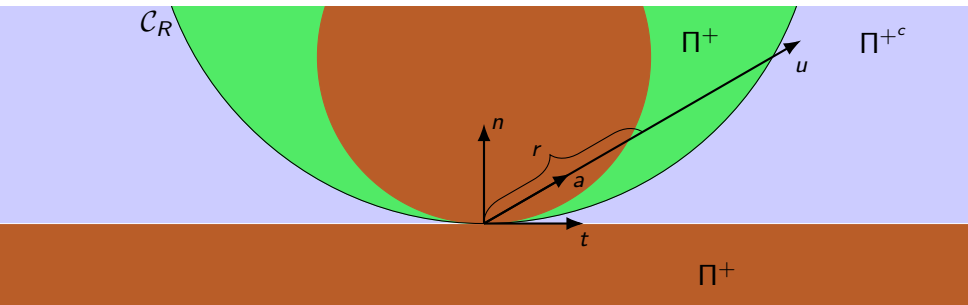


$$\Pi^+ := \left\{ u \in \mathbb{R}^2 \mid (u \cdot n) < 0 \text{ or } \frac{|u|}{2} < R \left(\frac{u}{|u|} \cdot n \right) \right\}$$

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$$\kappa_\sigma \cdot n = -2^{\sigma+1/2} \int_{\mathbb{R}^2} \frac{\bar{\chi}_{\Pi^+}(u) \operatorname{sgn}(u \cdot n)}{|u|^{2+\sigma}} d\mathcal{H}^2(u)$$

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$$z \in \mathcal{C}_R, \ n \geq 2$$

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}_{\text{odd}}^+(z)} - \int_{\mathcal{A}_{\text{even}}^+(z)} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} d\mathcal{H}^{2n-2}(a, b, r)$$

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Conjecture Any curve in k dimensional subspace only depends on k -dimensional curvature

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 &= \omega_{n-3}^2 \sqrt{2\pi} \mathrm{B}\left(\frac{3}{2}, \frac{n-2}{2}\right) \mathrm{B}\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right) \frac{1}{R}
 \end{aligned}$$

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 &= \overbrace{\omega_{n-3}^2 \sqrt{2\pi} \mathrm{B}\left(\frac{3}{2}, \frac{n-2}{2}\right) \mathrm{B}\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right)}^{K(n)} \overbrace{\frac{1}{R}}^{\kappa}
 \end{aligned}$$

2nd order approximation

Want $\lim_{\sigma \uparrow 1} (1 - \sigma) |\kappa_\sigma(z; \mathcal{C}) - \kappa_\sigma(z; \mathcal{C}_R)| = 0$ for arbitrary \mathcal{C} , $\mathcal{C}_{R=1/\kappa_\sigma}$

2nd order approximation

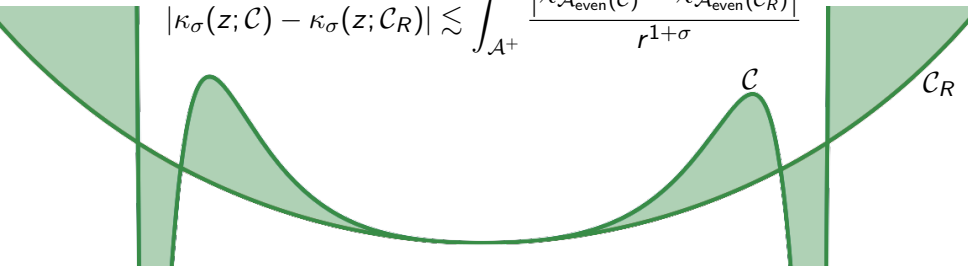
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$$|\kappa_\sigma(z; \mathcal{C}) - \kappa_\sigma(z; \mathcal{C}_R)| \lesssim \int_{\mathcal{A}^+} \frac{|\bar{\chi}_{\mathcal{A}^+_{\text{even}}(\mathcal{C})} - \bar{\chi}_{\mathcal{A}^+_{\text{even}}(\mathcal{C}_R)}|}{r^{1+\sigma}}$$

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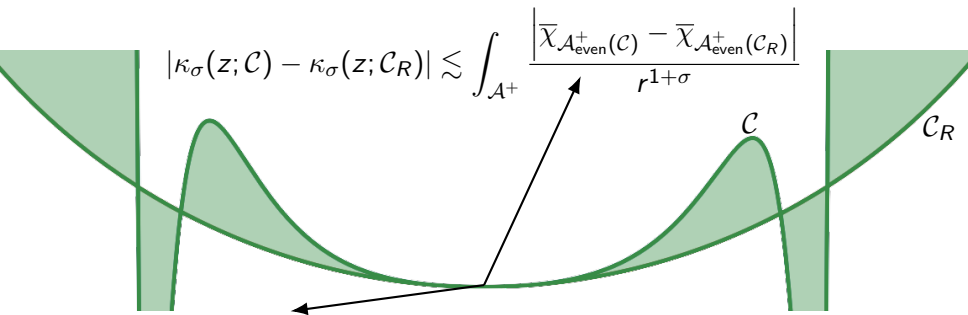
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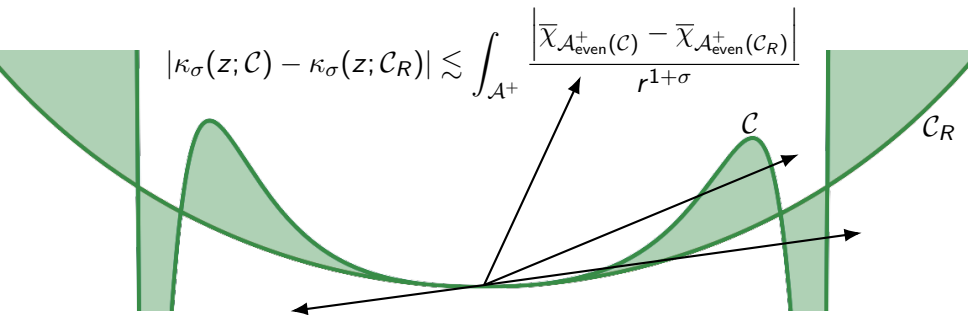
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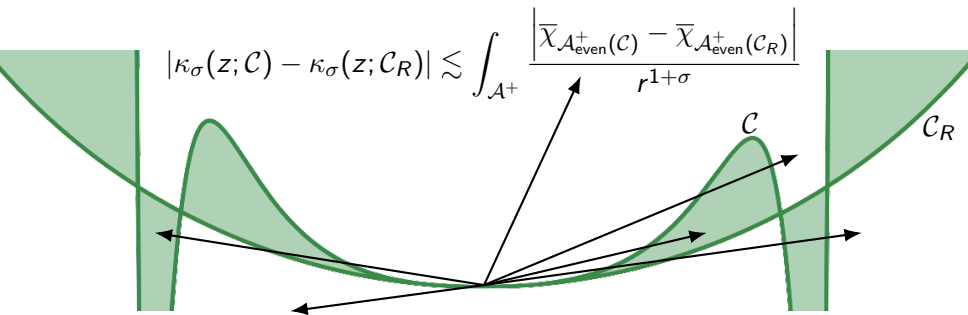
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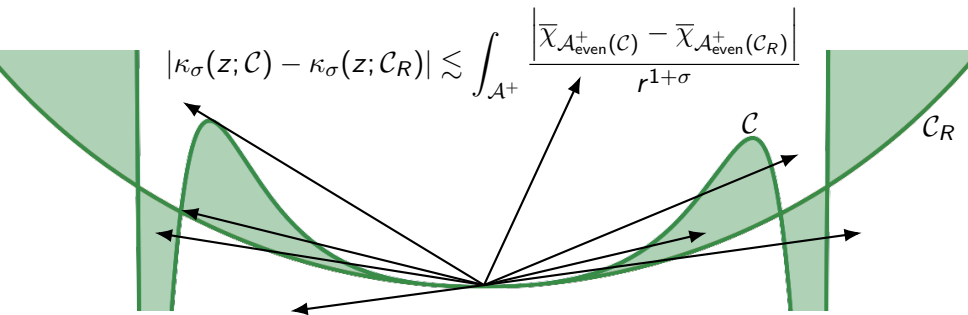
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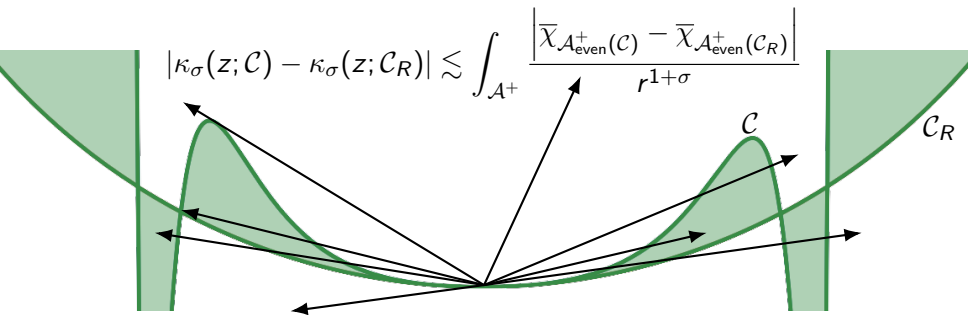
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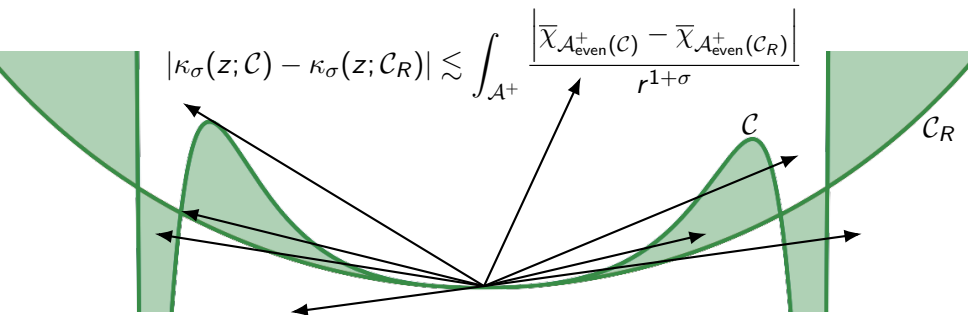
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C1 Disks whose boundary intersects ribbon are all that matter

2nd order approximation

Want $\lim_{\sigma \uparrow 1} (1 - \sigma) |\kappa_\sigma(z; \mathcal{C}) - \kappa_\sigma(z; \mathcal{C}_R)| = 0$ for arbitrary \mathcal{C} , $\mathcal{C}_R = 1/\kappa_\sigma$



- C1 Disks whose boundary intersects ribbon are all that matter
- C2 With the right area & coarea applications the integral over relevant disks vanishes

Final Result

Arbitrary \mathcal{C} , $K(n) = \omega_{n-3}^2 \sqrt{2\pi} B\left(\frac{3}{2}, \frac{n-2}{2}\right) B\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right)$, $z \in \mathcal{C}$

$$\lim_{\sigma \uparrow 1} (1 - \sigma) \frac{1}{K(n)} \kappa_{\sigma}(z) = \kappa(z)$$

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$$\lim_{\sigma \uparrow 1} (1 - \sigma) \frac{1}{K(n)} \kappa_\sigma(z) = \kappa(z)$$

New definition?

$$\kappa_\sigma(z) := \frac{1}{K(n)} \left(\int_{\mathcal{A}_{\text{odd}}^+(z)} - \int_{\mathcal{A}_{\text{even}}^+(z)} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} d\mathcal{H}^{2n-2}(a, b, r)$$

Fin