Exercise 5.1 First we want to show a lexographically minimal point exists. By compactness of K it's bounded and closed. Projecting onto the first coordinate we get another bounded closed $1D$ set so that there's a minimal element. Call it u_1. Now consider $K_1 := K \cap \{x_1 = u_1\}$, i.e. the intersection with the hyperplane specified by $x_1 = u_1$. This is closed since its the intersection of closed sets & bounded because a subset of a bounded set, hence compact. Project K_1 onto its second coordinate to get another $1D$ compact set, hence with a minimal element u_2. We can iteratively do this m times to come up with $u = (u_1, u_2, ..., u_m)$. By construction this is lexographically minimal.

Now suppose there's a second point w that's also lexographically minimal. By the definition of lexographic minimality of u for some i one of $u_i < w_i$. This means w is not lexographically smaller than u, hence it's not lexographically minimal.

Exercise 5.3 The proof of this is very similar to Lemma 5.2. Put $A \subset [0, T]$ where $\dot{\phi}(t) \in F(\phi(t))$ for $t \in A$. Define $v(t) = \dot{\phi}(t)$ for $t \in A$ and for every $t \notin A$ pick arbitrary $u^*(t) \in U(\phi(t))$ and put $v(t) = f(\phi(t), u(t))$ then $v : [0, T] \to \mathbb{R}^n$ where $v(t) = \dot{\phi}(t)$ a.e. t and hence v is measurable. Define $M : [0, T] \rightrightarrows U(\phi([0, T]))$ by

$$M(t) = \begin{cases} \{u \in U(\phi(t)) \mid v(t) = f(\phi(t), u)\} & t \in A \\ \{u^*(t)\} & t \in [0, T] \setminus A \end{cases}$$

We have the following properties on M at each $t \in A$:

- $M(t)$ is non-empty by construction of v
- Since $\phi([0, T])$ is bounded, U is locally bounded, Exercise 2.12 tells us $U(\phi([0, T]))$ is bounded, hence $M(t)$ is bounded.
- We want to show each $M(t)$ is also closed so that $M(t)$ is compact. Let $u_i \in M(t)$ where $u_i \to u$. By definition of $M(t)$ we know $u_i \in U(\phi(t))$, and so by osc we know $u \in U(\phi(t))$ (our sequence in the domain is the constant sequence $\phi(t) \to \phi(t)$). Continuity of f gives us $v(t) = f(\phi(t), u)$, so that $u \in M(t)$ and thus $M(t)$ is compact.

Trivially all the above conclusions follow for $t \notin [0, T] \setminus A$, so that M takes compact non-empty values. Just as in Lemma 5.2 define $u(t)$ as the lexographic minimal value in $M(t)$. We have $u : [0, t] \to U(\phi([0, T]))$ so that

$$v(t) = f(\phi(t), u(t)) \;\; \forall t \in [0, T]$$

All that's left to show is that u is measurable. We do this just like in Lemma 5.2 inductively with Lusin's Theorem. We start again with $u_1, u_2, ...u_{k-1}$ coordinate functions measurable and will show u_k is measurable. Pick $\epsilon > 0$. Lusin's Theorem gives us a compact $C \subset [0, T]$ so that $t \mapsto (u_1(t), u_2(t), ..., u_{k-1}(t), v(t))$ is continuous and $\mu([0, T] \setminus C) < \epsilon/2$. Notably each $u_1, u_2, ..., u_{k-1}, v$ are continuous on C.

Now the goal is to show for arbitrary r the set

$$S := \{t \in C \mid u_k(t) \leq r\}$$

is closed. If this is the case then we can show our claim the same way the end of Lemma 5.2 does after showing this set is closed.

To this end, suppose S is not closed, so that $\exists\, t_i \to \tau$ where $t_i \in S$. Since C is closed and $t_i \in C$ we know $\tau \in C$. Because U is locally bounded $u(t_i)$ is bounded and hence there's a convergent subsequence. Without relabeling we have $u(t_i) \to \overline{u}$, and by osc of U we know $\overline{u} \in U(\tau)$. Continuity of f, ϕ tells us $f(\phi(t_i), u(t_i)) \to f(\phi(\tau), \overline{u})$ so that $\overline{u} \in M(\tau)$.

Continuity of $u_1, u_2, ..., u_{k-1}$ on C gives us $\overline{u}_i = u_i(\tau)$ for $i = 1, 2, ..., k-1$. However, by our initial assumption $u_k(\tau) > r$, and since $u(t_i)_k \leq r \implies \overline{u}_k \leq r$ so that

$$u_k(\tau) > \overline{u}_k,$$

but this contradicts the fact that $u(\tau)$ is lexographically minimal in $M(\tau)$ (since $\overline{u} \in M(\tau)$), hence indeed S is closed.

Exercise 5.4 (a) The forward implication comes from the definition of sup. To prove the backward one, we start with v so that $v \cdot p \leq \sup_{w \in C} w \cdot p$ for every $p \in \mathbb{R}^n$. Suppose for contradiction $v \notin C$. By Theorem 4.16 that $\exists\, p \in \mathbb{R}^n, \epsilon > 0$ so that

$$w \cdot p + \epsilon \leq v \cdot p \;\; \forall\, w \in C \implies \sup_{w \in C} w \cdot p + \epsilon \leq v \cdot p$$

which contradicts the initial assumption, so $v \in C$.

(b)
- Let $(x_i, p_i) \to (x, p) \in C \times \mathbb{R}^n$. Because F is locally bounded $H(x_n, p_n)$ is finite. Thus, for each n, by definition of sup $\exists\, v_n \in F(x_n)$ where

$$\sup_{v \in F(x_n)} v \cdot p_n - \frac{1}{n} \leq v_n \cdot p_n \implies \limsup_{n \to \infty} H(x_n, p_n) \leq \limsup_{n \to \infty} v_n \cdot p_n$$

  Next, by definition of lim sup $\exists\, v_{n_k} \cdot p_{n_k} \to \limsup_{n \to \infty} v_n \cdot p_n$. Since F is locally bounded $\exists\, v_{n_{k_i}} \to \nu$, and by osc $\nu \in F(x)$. Using the fact that $p_n \to p$ we see $v_{n_k} \cdot p_{n_k} \to \nu \cdot p$ and so $\nu \cdot p = \limsup_{n \to \infty} v_n \cdot p_n$. Combined with the definition of sup, the fact that $\nu \in F(x)$ means

$$\limsup_{n \to \infty} v_n \cdot p_n = \nu \cdot p \leq \sup_{v \in F(x)} v \cdot p = H(x, p)$$

  Pulling in the initial major expression above we find our desired result

$$\limsup_{n \to \infty} H(x_n, p_n) \leq H(x, p)$$

- Let $p_n \to p \in C$. By the definition of sup we can construct, $v_n \in F(x)$ so that $v_n \cdot p \to \sup_{v \in F(x)} v \cdot p$. By osc and local boundedness of F there's a convergent subsequence so that

$$v_{n_i} \to \nu \in F(x) \implies v_n \cdot p \to \nu \cdot p$$

  We also have

$$\nu \cdot p_n \leq \sup_{v \in F(x)} v \cdot p_n \implies \liminf_n \nu \cdot p_n \leq \liminf_n \sup_{v \in F(x)} v \cdot p_n = \liminf_n H(x, p_n)$$

  So that combining $\liminf_n \nu \cdot p_n = \nu \cdot p = H(x, p)$, and usc of H our claim is shown since

$$\limsup_n H(x, p_n) \leq H(x, p) \leq \liminf_n H(x, p_n) \implies H(x, p_n) \to H(x, p)$$

Exercise 5.12

Exercise 5.13

Exercise 5.14

Exercise 5.24

Exercise 5.27