Exercise 3.1 Consider $e_{i}{}_{i=1}^{n}$ the eigenbasis of $\mathbb{R}^n$ and λ_i the corresponding eigenvalue.

$(a) \implies (b)$ If $|\lambda_i| < 1$ then for any x_0 any solution to the difference equation $x^+ = Ax$ must look like

$$\phi(j) = A^j x_0 = A^j \sum_i (x_0 \cdot e_i) e_i = \sum_i (x_0 \cdot e_i) A^j e_i = \sum_i (x_0 \cdot e_i) \lambda_i^j e_i = \lambda_i^j x_0 \to 0 \text{ as } j \to \infty$$

$(a) \impliedby (b)$ Now suppose all solutions to the difference equation converge to 0, then in particular for each $x_0 = e_i$ we must have

$$\phi(j) = A^j e_i = \lambda_i^j e_i \to 0 \iff |\lambda_i| < 1$$

As for the examples, put

$$A_1 = \begin{pmatrix} 0 & -1/2 \\ 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & -1 \\ 1/2 & 0 \end{pmatrix}$$

then each matrix rotates a point 90 deg counterclockwise, with an additional contraction in either the x or y coordinate to ensure repeated application of either A_1 or A_2 spins towards 0. However

$$\begin{pmatrix} 0 & -1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1/2 & 0 \end{pmatrix} = \begin{pmatrix} -1/4 & 0 \\ 0 & -1 \end{pmatrix}$$

so that, when $y_0 \neq 0$ a solution to the difference inclusion exists where we alternate between A_1 and A_2, however the y coordinate never converges, i.e. it flips between $y_0, -y_0$. If $y_0 = 0$ but $x_0 \neq 0$ then applying A_2 first then A_1 and alternating again will lead to a similar issue with the x coordinate. Together this shows for non 0 initial conditions there are solutions to the difference inclusion which diverge.

Exercise 3.5 Stuck after two hours-ish, so skipping for now.

Exercise 3.6 If $(x, y) \in \operatorname{gph} G$ then $y \in \bigcup_{u \in K(x)} h(x, u)$. Fix some $u \in K(x)$- we'll show for any such u $(x, h(x, u)) \in \overline{\operatorname{gph} h}$. By definition of K, exercise 2.4 and the definition of lim inf of sets we know for any $u \in K(x)$ there's a sequence $u_i \in k(x + i^{-1}\mathbb{B})$ s.t. $u_i \to u$. By construction there's a corresponding $x_i \in x + i^{-1}\mathbb{B}$ where $u_i = k(x_i)$ and $x_i \to x$. Fixing $y_i := h(x_i, u_i) = h(x_i, k(x_i))$ then $(x_i, y_i) \in \operatorname{gph} g$. By continuity of h we know $y_i \to h(x, u)$ and so $(x, h(x, u)) \in \overline{\operatorname{gph} g}$.

Exercise 3.8 Fix $j \in I$ and put $x_i := \phi_i(j)$, then $x_i \to x := \phi(j)$ by hypothesis. Similarly with $y_i := \phi_i(j+1)$, then $y_i \to y := \phi(j+1)$. Since each ϕ_i solves the difference inclusion we know $y_i \in G(x_i)$. By osc we know $y \in G(x)$, hence $\phi(j+1) \in G(\phi(j))$ so that ϕ is also a solution to the difference inclusion.

Exercise 3.10 For both of the cases below consider $\xi_i \to \xi$.

(osc) Consider $(x_{0,i}, x_{1,i}, ..., x_{J,i}) \to (x_0, x_1, ..., x_J)$ where $(x_{0,i}, x_{1,i}, ..., x_{J,i}) \in S(\xi_i)$. By construction we have $x_{k+1,i} \in G(x_{k,i})$ for any fixed k. Since $x_{k+1,i} \to x_{k+1}, x_{k,i} \to x_k$, and since G is osc, we know $x_{k+1} \in G(x_k)$. Since additionally $x_{0,i} = \xi_i \to \xi = x_0$ it's true that $(x_0, x_1, ..., x_J) \in S(\xi)$, so that S is osc.

(isc) Consider $(x_0, x_1, ..., x_J) \in S(\xi)$. Since $x_1 \in G(\xi)$ with $\xi_i \to \xi$, isc of G says for $i > N_1$ $\exists x_{1,i} \in G(\xi_i)$ with $x_{1,i} \to x_1$. Similarly $x_2 \in G(x_1)$, so for $i > N_2$ $\exists x_{2,i} \in G(x_{1,i})$ where $x_{2,i} \to x_2$. We can do this for all x_k so that, fixing $x_{0,i} = \xi_i$, we find $(x_{0,i}, x_{1,i}, ..., x_{J,i})$ for $i > \max\{N_1, N_2, ..., N_J\}$, with the property that each $x_{k,i} \to x_k$ and $x_{k+1,i} \in G(x_{k,i})$. This means $(x_{0,i}, x_{1,i}, ..., x_{J,i}) \in G(\xi_i)$ with $(x_{0,i}, x_{1,i}, ..., x_{J,i}) \to (x_0, x_1, ..., x_J)$ and so G is isc.

$\boxed{\textit{Exercise 3.13}}$ Since a finite union of compact sets is compact, we only need to show $\mathcal{R}_{j=J}(K)$ is compact. Additionally, since $\mathcal{R}_{j=J}(K) = G^J(K)$, if we can show $G(K)$ is compact, then we can apply our argument $J-1$ more times to show the desired result. To that end we strive to show $G(K)$ is bounded and closed. In order to see it's bounded, notice we can build up an open covering of K by considering $\bigcup_{x\in K} N(x)$ where $N(x)$ is the (open) neighborhood of each x so that $S(N(x))$ is bounded. By compactness there are $N(x_k)$ for $k = 1, 2, ...N$ so that $K \subset \bigcup_{k=1}^{N} N(x_k)$. Notably, then $G(K) \subset \bigcup_{k=1}^{N} G(N(x_k))$, and since a finite union of bounded sets is bounded that means $G(K)$ is bounded.
Now to show $G(K)$ is closed take a sequence $x_i \in G(K)$ such that $x_i \to x$. Since $x_i \in G(K)$ that means $G^{-1}(x_i) \neq \emptyset$. Put $k_i \in G^{-1}(x_i)$ for each i. Since $k_i \in K$, and K is compact there's a subsequence k_{i_j} such that $k_{i_j} \to k \in K$. Now by osc of G, since $k_{i_j} \to k$ and $x_{i_j} \to x$ with $x_{i_j} \in G(k_{i_j})$ we must have $x \in G(k) \subset G(K)$, hence $G(K)$ is closed, and thus compact.

$\boxed{\textit{Exercise 3.15}}$ Put $G(x) = \frac{1}{x^2}$ when $x \neq 0$ and 1 when $x = 0$, then ϕ is a forward complete solution to $x^+ \in G(x)$ when defined by

$$\phi(j) = \begin{cases} 2 & j = 0, \\ 2^{(-2)^j} & \text{otherwise} \end{cases}$$

Notably $0 \in \omega(\phi)$ since $\lim_{i\to\infty} \phi(2i+1) = \lim_{i\to\infty} 2^{(-2)^{2i+1}} = \lim_{i\to\infty} 2^{-(2)^{2i+1}} = 0$. However $G(0) = 1$ is not in $\omega(\phi)$ as for large i $\phi(i)$ is either a really big number or a number very close to 0, hence it doesn't stay in any close neighborhood of 1. Together this shows $\omega(\phi)$ is not weakly forward invariant.

N.B. I feel like I'm missing something here. Maybe the above solution is good enough, but I'm not sure I understand the 'spirit' of the problem.

$\boxed{\textit{Exercise 3.17}}$ It's clear that $\inf_\phi |\phi(0)| + |\phi(2)| \geq 0$. We aim to find a sequence of ϕ_i that are solutions whereas the limit of the objective of these solutions converges to 0. To that end put

$$\phi_i(0) = i^{-1},\ \phi_i(1) = i,\ \phi_i(2) = i^{-1}$$

then ϕ_i is a solution to $x^+ = H(x)$ and $\lim_{i\to\infty} |\phi(0)| + |\phi(1)| = \lim_{i\to\infty} 2i^{-1} = 0$, so $\inf_\phi |\phi(0)| + |\phi(2)| = 0$. Now suppose there is an optimal solution ϕ where this inf is achieved. Then $\phi(0) = 0$. In order for ϕ to be a solution then $\phi(1) = 1$, which in turn forces $\phi(2) = 1$, but then the cost is not 0, hence such a solution doesn't exist.

$\boxed{\textit{Exercise 3.18}}$ Put $c(a, b) = |a| + |b|$ and consider

$$g(x) = \begin{cases} e^{\sqrt{2}} & x = 0, \\ e^{2i-1} & x = e^{-i}, i \in \mathbb{N}^+ \\ e^{2i} & x = e^{2i-1}, i \in \mathbb{N}^+, \\ e^{-i} & x = e^{2i}, i \in \mathbb{N}^+, \\ 0 & \text{otherwise} \end{cases}$$

g is indeed locally bounded and discontinuous. In order to minimize $c(\phi(0), \phi(2))$ consider $\phi(0) = 0, \phi(1) = e^{\sqrt{2}}, \phi(2) = 0$ then ϕ is a minimizer since c is bounded below by 0, and $c(\phi(0), \phi(2)) = 0$. Consider ϕ_i defined by

$$\phi_i(0) = e^{-i}, \phi_i(1) = e^{2i+1}, \phi_i(2) = e^{2i}, \phi_i(3) = e^{-i},$$

then ϕ_i are solutions to $x^+ = g(x)$, and $\lim_{i\to\infty} c(\phi(0), \phi(1)) = \lim_{i\to\infty} 2e^{-i} = 0$, hence since also $c(\phi(0), \phi(3))$, $\inf_\phi c(\phi(0), \phi(3)) = 0$. Now suppose we can find ϕ optimial so that $c(\phi(0), \phi(3)) = 0$, then $\phi(1) = e^{\sqrt{2}}$, $\phi(2) = 0$ and $\phi(3) = e^{\sqrt{2}}$ hence $c(\phi(0), \phi(3)) = e^{\sqrt{2}}$, a contradiction so no such ϕ can exist.

$\boxed{\textit{Exercise 3.20}}$ Our plan is to use the 'direct' method to show an optimal solution exists. Since $\mathcal{U}$ is non-empty $\exists\, u \in \mathcal{U}$ and so we can form at least one solution to $x^+ \in G(x)$ by putting $x(1) = g(\xi, u), x(2) = g(x(1), u)$ all the way up to $x(J)$. Next, the cost function is bounded below by 0 since g, l (and thus L) only map to non-negative numbers. Together these points tell us

$$\alpha := \inf\{c(x) \mid x(0) = \xi, x(j+1) = x(j)\ \forall\, j \in J_-\} \text{ is finite, where } c(x) := \sum_{j=0}^{J-1} L(x(j), x(j+1)) + d(x(J))$$

By definition of inf there's a sequence of solutions x_i such that $c(x_i) \to \alpha$. By local boundedness of G $x_i(1) \in G(x(0) + i^{-1}\mathbb{B})$, which is bounded, hence theres a convergent subsequence $x_{i_k}(1)$ and corresponding. Since $G^{-1}(x_{i_k}) \neq \emptyset$ we can find a subsequence of $x^{(i_k)}$ (without relabeling) s.t. $x^{(i_k)}(0) \to x(0)$. If we fix $x(1) := \lim_k x_{i_k}(1)$, then by osc of G we know $x(1) \in G(x(0))$. We can proceed this way for $x(2), x(3), ...x(J)$ to show that $x := \lim_i x_i$ is a solution to $x^+ \in G$. Since L is lsc & d is continuous, we must have

$$c(x) = \sum_{j=0}^{J-1} L(x(j), x(j+1)) + d(x(J)) \leq \lim_{i\to\infty} \sum_{j=0}^{J-1} L(x_i(j), x_i(j+1)) + d(x_i(J)) = \lim_{i\to\infty} c(x_i) = \alpha,$$

so x must be an optimal solution.

N.B. The sequences $x^{(i_k)}(j)$ are not necessarily the same as the sequences $x_{i_k}(j)$ for any fixed j, hence the change in notation.