Computing the limit of the Nonlocal Curvature of Curves in $\mathbb{R}^n$

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Origins of Nonlocal Curvature of Curves

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Consider $u = \chi_E$ for bounded E: $\int_E \int_{E^c} \frac{1}{|x-y|^{n+\sigma}} dxdy$

$$\mathrm{Per}_\sigma(E) := \frac{1}{\alpha_{n-1}} \int_E \int_{E^c} \frac{1}{|x - y|^{n+\sigma}} dxdy$$

For unbounded E, fix Ω bounded, and set

$$\mathrm{Per}_\sigma(E,\Omega) := \frac{1}{\alpha_{n-1}}\left(\int_{E\cap\Omega}\int_{E^c} + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega}\right)\frac{1}{|x-y|^{n+\sigma}}\,dxdy$$

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Nonlocal mean curvature at $z\in\partial E$

$$H_\sigma(z) := \frac{1}{\omega_{n-1}}\int_{\mathbb{R}^n}\frac{\chi_E(x)-\chi_{E^c}(x)}{|z-x|^{n+\sigma}}dx$$

On the nonlocal curvatures of open surfaces - Paroni, Podio-Guidugli, Seguin (2018)

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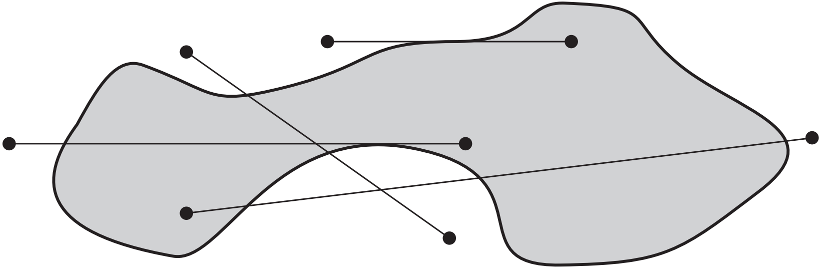

$$\mathcal{X}(\partial E) = \left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n \mid \mathcal{H}^0([x,y]\cap\partial E) \text{ is odd}\right\}$$

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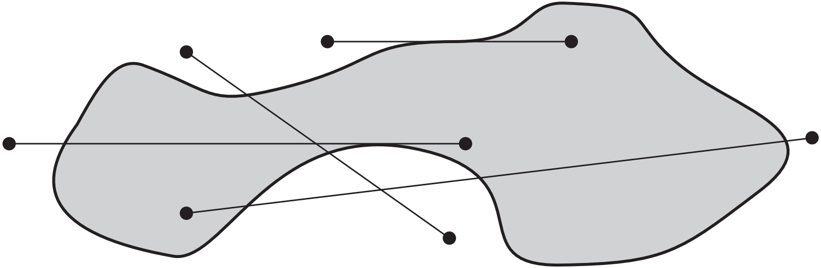

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$$\mathrm{Area}_\sigma(\mathcal{S},\Omega) := \frac{1}{2\alpha_{n-1}} \int_{\mathcal{X}(\mathcal{S})} \frac{\max\{\chi_\Omega(x), \chi_\Omega(y)\}}{|x-y|^{n+\sigma}} dxdy$$

A fractional notion of length and an associated nonlocal curvature - Seguin (2020)

$$\mathrm{Area}_\sigma^{(n=2)}(\mathcal{S},\Omega) = \frac{1}{4}\int_{\mathcal{X}(\mathcal{S})} \frac{\max\{\chi_\Omega(x),\chi_\Omega(y)\}}{|x-y|^{2+\sigma}}dxdy$$

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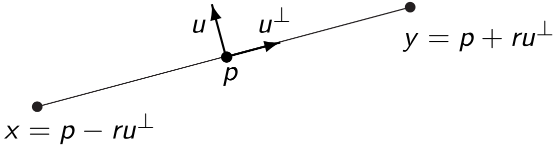

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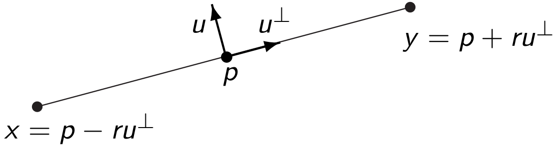

$$=\frac{1}{2}\int_{\mathcal{D}(\mathcal{S})}\frac{\max\left\{\chi_\Omega(p-ru^\perp),\chi_\Omega(p+ru^\perp)\right\}}{(2r)^{1+\sigma}}d\mathcal{H}^4(p,u,r)$$

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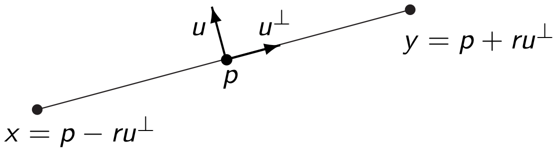

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Generalizes nicely

$$\mathrm{Len}_\sigma(\mathcal{C},\Omega) := \frac{\Gamma\left(\frac{n+1}{2}\right)^2}{2\pi^{n-1}}\int_{\mathcal{D}(\mathcal{C})} \frac{\sup_{v\in\mathcal{U}_n\cap u^\perp}\chi_\Omega(p+rv)}{r^{n-1+\sigma}}d\mathcal{H}^{2n}(p,u,r)$$

Minimizing over $\mathcal{C}$ with fixed boundary, $z \in \mathcal{C}$

$$\left(\int_{\mathcal{A}^+_{\text{odd}}(z)} - \int_{\mathcal{A}^+_{\text{even}}(z)}\right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} d\mathcal{H}^{2n-2}(a, b, r) = 0$$

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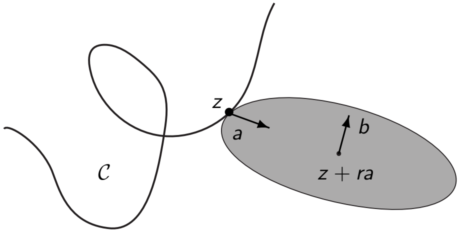

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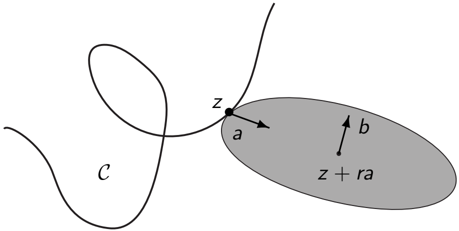

Nonlocal *vector* curvature

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{odd}}(z)} - \int_{\mathcal{A}^+_{\text{even}}(z)}\right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} d\mathcal{H}^{2n-2}(a, b, r)$$

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Argument Outline

1. For $\mathcal{C}_R$ circle with radius R, in $\mathbb{R}^2$

$$\lim_{\sigma \uparrow 1} (1 - \sigma) \quad \kappa_\sigma(z) = \kappa(z) \quad z \in \mathcal{C}_R$$

Argument Outline

1. For $\mathcal{C}_R$ circle with radius R, in $\mathbb{R}^2$

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3. For arbitrary $\mathcal{C}$ and $\mathcal{C}_R$ $(R = 1/\kappa(z;\mathcal{C}))$ at $z \in \mathcal{C}$

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4. Combine 2 & 3, show for arbitrary $\mathcal{C}$, $z \in \mathcal{C}$

$$\boxed{\lim_{\sigma \uparrow 1} (1-\sigma) \frac{1}{K(n)} \kappa_\sigma(z) = \kappa(z)}$$

$z \in \mathcal{C}_R$, $n = 2$

$$\kappa_\sigma = \left(\int_{\mathcal{A}^+_{\text{odd}}} - \int_{\mathcal{A}^+_{\text{even}}} \right) \frac{(a \cdot t)b - (b \cdot t)a}{r^{1+\sigma}} \, d\mathcal{H}^2(a, b, r)$$

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$$= \int_{\mathcal{A}^+} \overline{\chi}_{\mathcal{A}^+_{\text{even}}}(a, b, r) \frac{(a \cdot t)(b \cdot n) - (b \cdot t)(a \cdot n)}{r^{1+\sigma}} d\mathcal{H}^2(a, b, r)$$

$$\overline{\chi}_{\mathcal{A}^+_{\text{even}}} = \chi_{\mathcal{A}^+_{\text{even}}} - \chi_{\left(\mathcal{A}^+_{\text{even}}\right)^c} = \chi_{\mathcal{A}^+_{\text{even}}} - \chi_{\mathcal{A}^+_{\text{odd}}}$$

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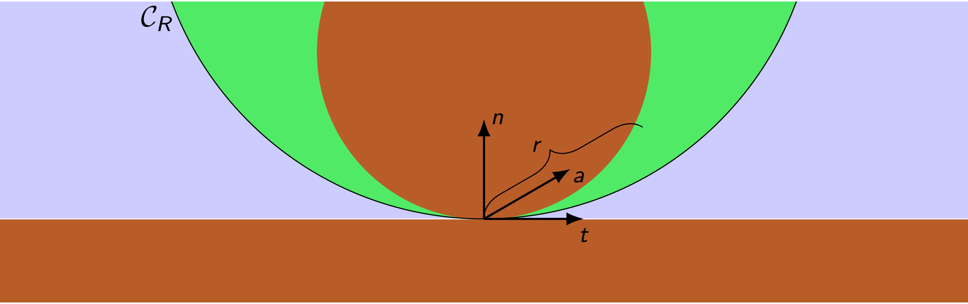

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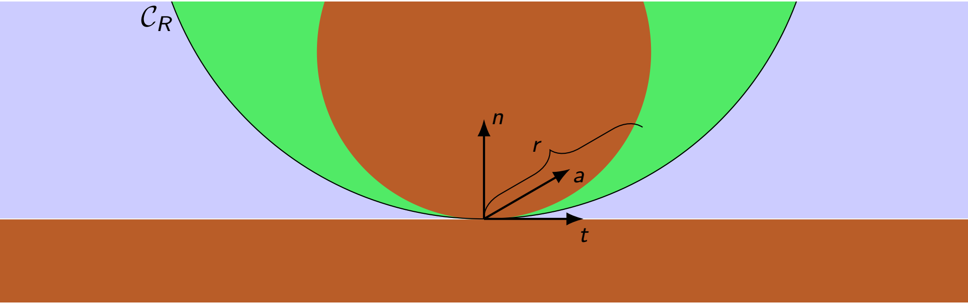

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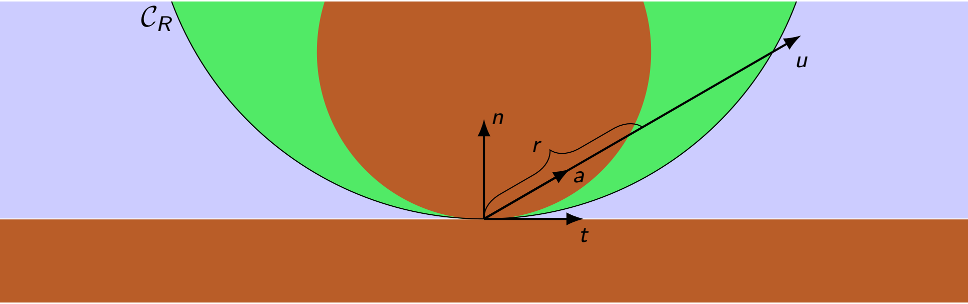

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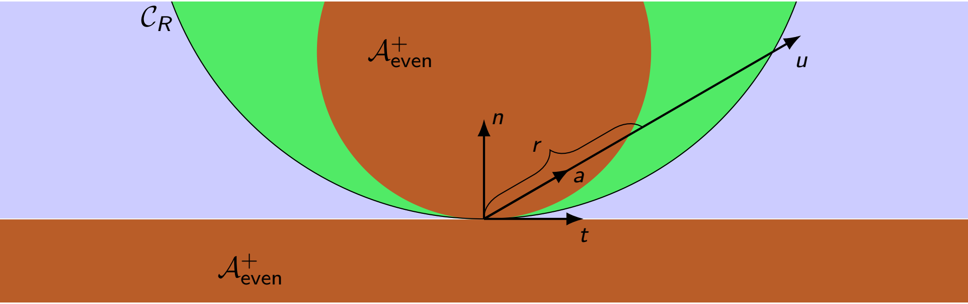

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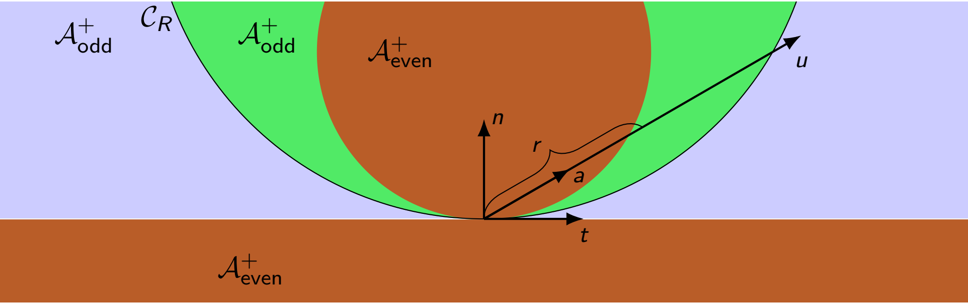

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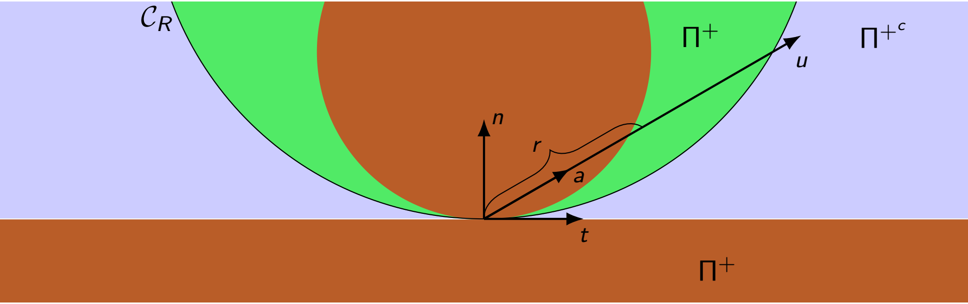

$$\Pi^+ := \left\{ u \in \mathbb{R}^2 \mid (u \cdot n) < 0 \text{ or } \frac{|u|}{2} < R\left(\frac{u}{|u|} \cdot n\right) \right\}$$

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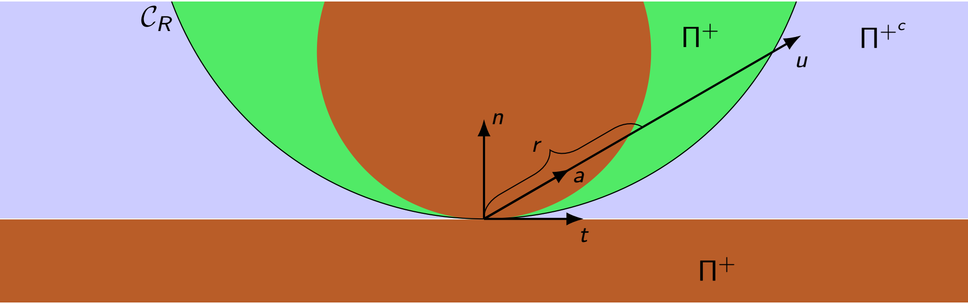
$$\text{Disks: } u := 2ra$$



$$\Pi^+ := \left\{ u \in \mathbb{R}^2 \mid (u \cdot n) < 0 \text{ or } \frac{|u|}{2} < R\left(\frac{u}{|u|} \cdot n\right) \right\}$$

$$\kappa_\sigma \cdot n = -\, 2^{\sigma + 1/2} \int_{\mathbb{R}^2} \frac{\overline{\chi}_{\Pi^+}(u) \operatorname{sgn}(u \cdot n)}{|u|^{2+\sigma}} \, d\mathcal{H}^2(u)$$

$z \in \mathcal{C}_R$, $n = 2$

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$z \in \mathcal{C}_R$, $n \geq 2$

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{odd}}(z)} - \int_{\mathcal{A}^+_{\text{even}}(z)} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} \, d\mathcal{H}^{2n-2}(a, b, r)$$

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2nd order approximation

Want $\lim_{\sigma\uparrow 1}(1-\sigma)|\kappa_\sigma(z;\mathcal{C})-\kappa_\sigma(z;\mathcal{C}_R)|=0$ for arbitrary $\mathcal{C}$, $\mathcal{C}_{R=1/\kappa_\sigma}$

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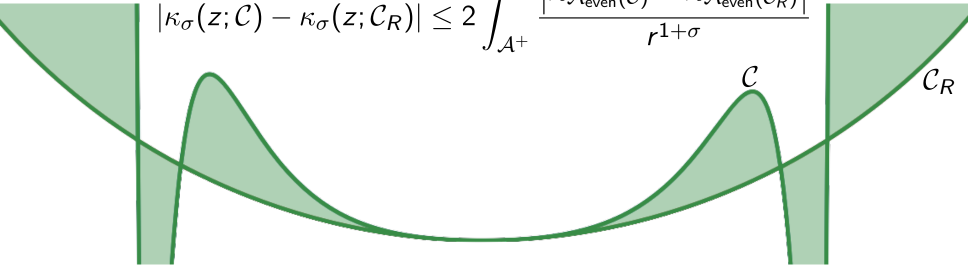

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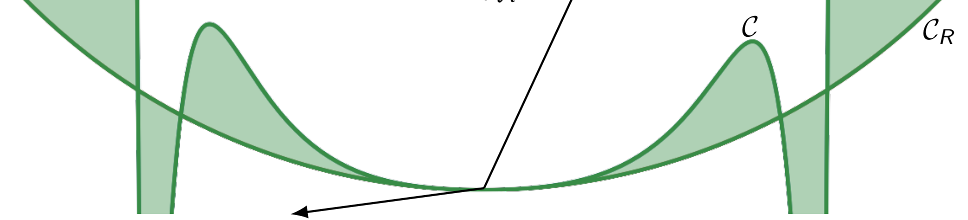

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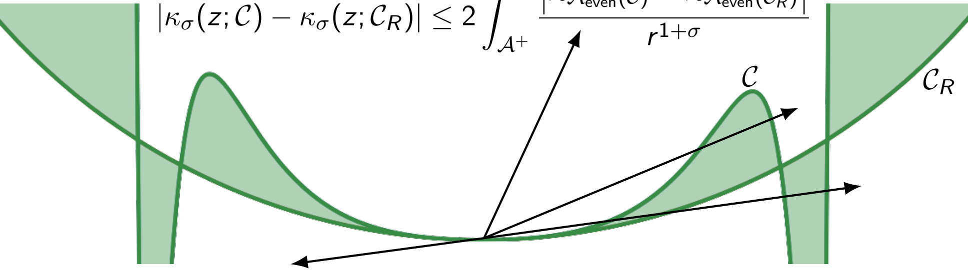

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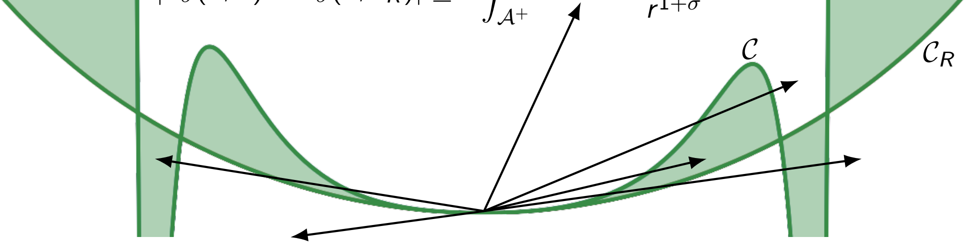

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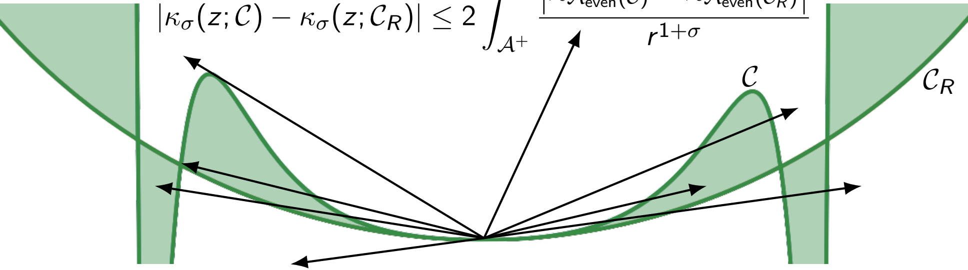

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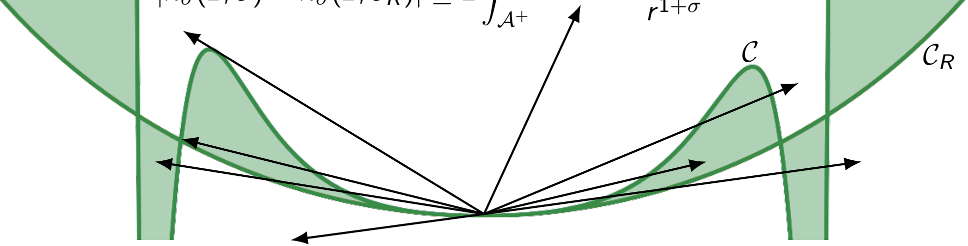

C1 Disks whose boundary intersects ribbon are all that matter

2nd order approximation

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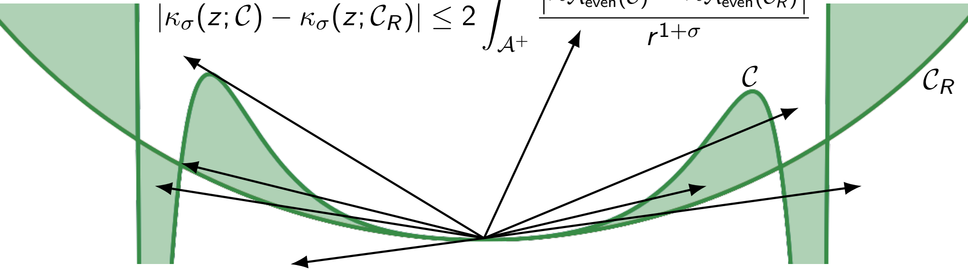

C1 Disks whose boundary intersects ribbon are all that matter

C2 With the right area & coarea applications the integral over relevant disks vanishes

Final Result

Arbitrary $\mathcal{C}$, $K(n) = \omega_{n-3}^2 \sqrt{2\pi}\,\mathrm{B}\left(\frac{3}{2}, \frac{n-2}{2}\right)\mathrm{B}\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right)$, $z \in \mathcal{C}$

$$\boxed{\lim_{\sigma \uparrow 1} (1-\sigma)\frac{1}{K(n)}\kappa_\sigma(z) = \kappa(z)}$$

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New definition?

$$\kappa_\sigma(z) := \frac{1}{K(n)}\left(\int_{\mathcal{A}^+_{\text{odd}}(z)} - \int_{\mathcal{A}^+_{\text{even}}(z)}\right)\frac{(a\cdot t(z))b - (b\cdot t(z))a}{r^{1+\sigma}}\,d\mathcal{H}^{2n-2}(a,b,r)$$

Fin