Exercise 11.2

- For fixed y we know $-g(x) := e^x - y \cdot x$ is differentiable with $-g'(x) = e^x - y$, which is trivially non-decreasing, hence convex. This says $g(x) := y \cdot x - e^x$ is concave and so its maximum is achieved and occurs when $g'(x) = 0$, i.e. when $x = \ln y$. This tells us for $f(x) = e^x$

$$f^*(y) := \sup_{x \in \mathbb{R}} \{y \cdot x - e^x\} = y \ln y - 1$$

- For fixed y

$$g(x) := y \cdot x - |x| = \begin{cases} x(y-1) & x \geq 0 \\ x(y+1) & x < 0 \end{cases}$$

When $y > 1$ so $y - 1 > 0$ and thus $\sup_x g(x) = +\infty$ by the first case. When $y < -1$ $y + 1 < 0$ so that $\sup_x g(x) = +\infty$ by the second case. When $y \in [-1, 0]$ $y - 1 \leq 0$ so that sup over the first case is 0. On the other hand, $y + 1 \geq 0$ so that sup of the second case is also 0. A similar argument is made for $y \in [0, 1]$ so that for $f(x) = |x|$

$$f^*(y) = \sup_x g(x) = \begin{cases} 0 & y \in [-1, 1] \\ +\infty & \text{otherwise} \end{cases}$$

- Using the alternate characterization of the dual, for $f(x) = x^3$ we have

$$-f^*(y) = \sup\big(b \in \mathbb{R} \mid x^3 \geq y \cdot x + b \ \forall x \in \mathbb{R}\big)$$

Notably $x^3 \to -\infty$ as $x \to -\infty$ so that for every finite fixed b it is not true that $x^3 \geq y \cdot x + b$, i.e. $-f^*(y) = -\infty \implies f^*(y) = +\infty$.

Exercise 11.7 Taking the hint, we can leverage proposition 7.36. Take the V, γ that it gives you and square both quantities so that, because both are non-negative we have

$$V^2(g) \leq \gamma^2 V^2(x)$$

Notably $\gamma \in (0, 1)$ so $\gamma^2 \in (0, 1)$ and it's very easy to see that V^2 is positively homogeneous of degree 2 (since positive scalars can be pulled out of both the norm and the sup). Finally if $x = 0$ it's clear $V^2(x) = 0$. On the other hand if $V^2(x) = 0$ then, since $e^k j > 0$ we must have the contents of the norm vanish, which only occurs if $x = 0$.

I thought this was the end of the problem, but now that I'm typing it up I'm realizing there might be a flaw with the last statement: what if product of the matrices result in the 0 matrix? As far as I can tell there aren't any restrictions on the matrices.. but maybe (probably) I'm missing why this is a "trivial" case where V can be taken as the 0 function, or something?

Exercise 11.8

(a) Let $s \geq 0$ then

$$\begin{aligned} f^*(sy) &= \sup_x \{sy \cdot x - f(x)\} = \sup_{sx} \{sy \cdot x - f(x)\} \\ &= \sup_x \{s^2 y \cdot x - f(sx)\} = \sup_x \{s^2(y \cdot x - f(x))\} \\ &= s^2 \sup_x \{y \cdot x - f(x)\} = s^2 f^*(y) \end{aligned}$$

(b) Suppose f is positive definite then by 11.4 f^* is proper. If $f^*(y) = +\infty$ then $\exists\, x_k$ so that $y \cdot x_k - f(x_k) \nearrow +\infty$. Because f is positive definite this means $y \cdot x_k \nearrow +\infty$ so that $\|x_k\| \to \infty$. Because f is positively homogeneous of degree 2 we know

$$f(x_k) = f\left(\|x_k\| \frac{x_k}{\|x_k\|}\right) = \|x_k\|^2 f\left(\frac{x_k}{\|x_k\|}\right)$$

and so

$$y \cdot x_k - f(x_k) = y \cdot x_k - \|x_k\|^2 f\left(\frac{x_k}{\|x_k\|}\right) \leq \|y\|\|x_k\| - \|x_k\|^2 f\left(\frac{x_k}{\|x_k\|}\right) = \|x_k\|\left(\|y\| - \|x_k\| f\left(\frac{x_k}{\|x_k\|}\right)\right)$$

Since $y \cdot x_k - f(x_k) \nearrow +\infty$ and $\|x_k\| \to +\infty$ it must be the case that

$$f\left(\frac{x_k}{\|x_k\|}\right) \to 0$$

Notably there must be a convergent subsequence of $\frac{x_k}{\|x_k\|}$, which we'll use without relabeling going forward, and whose limit is $\bar{x}$. By lsc of f we know

$$0 = \liminf_k f\left(\frac{x_k}{\|x_k\|}\right) \geq f(\bar{x}) \geq 0$$

where the right most inequality comes from the positive-definiteness assumption. This means $\bar{x} = 0$, but by its construction we must have $\|\bar{x}\| = 1$, a contradiction so that f^* must be finite.

Now suppose f^* is finite so that there's some M where $|f^*(y)| \leq M$. Suppose $\exists\, x$ s.t. $f(x) < 0$. This means for any $s > 0$ and fixed y where $y \cdot x \geq 0$

$$sy \cdot x - f(sx) = sy \cdot x - s^2 f(x) \nearrow +\infty \text{ as } s \nearrow +\infty \implies f^*(y) = +\infty$$

which is a contradiction with our initial assumption.

Now if $f(x) = 0$ then $y \cdot x - f(x) = y \cdot x$ and so for any $y \neq 0$ we must have $x = 0$, otherwise $f^*(y) \nearrow +\infty$. On the other hand if $x = 0$ but $f(x) \neq 0$ then $y \cdot x - f(0) = -s^2 f(0)$ for any positive s. This means $f^*(y) = \pm\infty$ depending on the sign of $f(0)$, so that we get a contradiction with our assumption hence $f(0) = 0$.

Exercise 11.11

- We'll use the fact that $x \in \partial g(y) \iff y$ is a minimizer of a lsc convex function. For this situation we're analyzing, for a fixed x

$$g(y) := |y| + \frac{1}{2\alpha}\|x - y\|^2,$$

which is easily seen as convex from properties from chapter 7. For $y \neq 0$ g is differentiable and so

$$\partial g(y) = g'(y) = \operatorname{sgn} y + \frac{1}{\alpha}(y - x)$$

Next we want to set this expression equal to 0 and solve for y (if such a y exists). First for $y > 0$ $y = x - \alpha$, so that a minimum occurs there as long as $x > \alpha$. Similarly for $y < 0$ we get $y = x + \alpha$, so that a minimum occurs there as long as $x < -\alpha$.

For $x \in [-\alpha, \alpha]$ and $y < 0$ $g'(y) = -1 + \frac{y}{\alpha} - \frac{x}{\alpha} < 0$ so that g is decreasing. A similar calculation can be shown for $y > 0$, but that $g'(y) > 0$ so that it's increasing. Because g is lsc this tells us $g(0)$ is a minimum, for $x \in [-\alpha, \alpha]$.

Putting this all together we get

$$e_\alpha f(x) = \inf_y \left\{ |y| + \frac{1}{2\alpha}|x-y|^2 \right\} = \begin{cases} \frac{x^2}{2\alpha} & x \in [-\alpha, \alpha] \\ x - \alpha + \frac{\alpha}{2} & x > \alpha \\ -x - \alpha + \frac{\alpha}{2} & x < \alpha \end{cases}$$

- Since $f(x)$ is smooth and convex then $g(y) := f(y) + \frac{1}{2\alpha}\|x-y\|^2$ is smooth and convex so that the minimum occurs when $\nabla g(y) = 0$, i.e. when

$$0 = \nabla g(y) = y + \frac{1}{\alpha}(y - x) \implies y = \frac{x}{\alpha + 1}$$

so that

$$e_\alpha f(x) = \inf_y \left\{ \frac{1}{2}\|y\|^2 + \frac{1}{2\alpha}\|x-y\|^2 \right\} = \frac{1}{2(\alpha+1)^2}\|x\|^2 + \frac{\alpha}{2(\alpha+1)^2}\|x\|^2 = \frac{1}{2(\alpha+1)}\|x\|^2$$

Exercise 11.12 First let g, h be functions on $\mathbb{R}^n$ taking $\mathbb{R} \cup \{\infty\}$ values and consider

$$\left(g \underset{\inf}{*} h \right)(x) := \inf_y g(y) + h(x-y)$$

I aim to show $\left(g \underset{\inf}{*} h \right)^*(y) = g^* + h^*$. By definition

$$\begin{aligned}
\left(g \underset{\inf}{*} h \right)^*(y) &= \sup_x \left\{ y \cdot x - \inf_z \{ g(z) + h(x-z) \} \right\} \\
&= \sup_x \sup_z \{ y \cdot x - g(z) - h(x-z) \} \\
&= \sup_x \sup_z \{ y \cdot z - g(z) + y \cdot (x-z) - h(x-z) \} \\
&= \sup_w \sup_z \{ y \cdot z - g(z) + y \cdot w - h(w) \} \\
&= \sup_w \{ g^*(y) + y \cdot w - h(w) \} \\
&= g^*(y) + h^*(y)
\end{aligned}$$

The variable change in the 4th line comes by setting $w := x - z$, which can be done because x is unconstrained.

Now our job is to compute the dual of $\frac{1}{2\alpha}\|x\|^2$. Since this function is smooth and convex we can use the characterization that a the maximum of a concave function occurs when the gradient vanishes. That is, when

$$0 = \nabla_x \left(y \cdot x - \frac{1}{2\alpha}\|x\|^2 \right) = y - \frac{1}{\alpha}x \implies x = \alpha y$$

This gives us, for $h(x) := \frac{1}{2\alpha}\|x\|^2$

$$h^*(y) = \sup_x \left\{ y \cdot x - \frac{1}{2\alpha}\|x\|^2 \right\} = \frac{\alpha}{2}\|y\|^2$$

Combining this result with the above we find

$$(e_\alpha f)^*(y) = \inf_y \left\{ f(y) + \frac{1}{2\alpha}\|x-y\|^2 \right\} = \left(f \underset{\inf}{*} h \right)^*(y) = f^*(y) + \frac{\alpha}{2}\|y\|^2$$

Exercise 11.14

(a) Let $x \in \operatorname{argmin} f$, then $0 \in \partial_x f(x)$. Now we know (by smoothness) that

$$\partial_y \left(f(y) + \frac{\alpha}{2}\|x-y\|^2 \right) = \partial_y f(y) + \nabla_y \left(\frac{\alpha}{2}\|x-y\|^2 \right) = \partial_y f(y) + \frac{1}{\alpha}(y-x)$$

This means when $y = x$ $0 \in \partial_y \left(f(y) + \frac{\alpha}{2}\|x-y\|^2 \right)$ and so $e_\alpha f(x) = f(x)$ (by plugging in $y = x$ into the definition).

Now let $z \neq x$. Because $x \in \operatorname{argmin} f$ we know for any y that

$$f(x) \leq f(y) + \frac{1}{2\alpha}\|y-z\|^2 \implies e_\alpha f(x) = f(x) \leq \inf_y f(y) + \frac{1}{2\alpha}\|y-z\|^2 = e_\alpha f(z)$$

so that $e_\alpha f(x)$ is a minimum, thus $x \in \operatorname{argmin} e_\alpha f$.

Now let $x \in \operatorname{argmin} e_\alpha f$. By lemma 11.13 we know the inf in the definition of $e_\alpha f$ is achieved, fix y where it's achieved so that

$$f(y) + \frac{1}{2\alpha}\|x-y\|^2 = e_\alpha f(x) \leq e_\alpha f(y) \leq f(y) \implies x = y$$

This tells us $e_\alpha f(x) = f(x)$. Now fix an arbitrary z, then

$$f(x) = e_\alpha f(x) \leq e_\alpha f(z) \leq f(z)$$

so that $x \in \operatorname{argmin} f$.

(b)