Exercise 10.8 Let $x, z \in O$. By construction $\underline{\omega}(x), \underline{\omega}(z)$ are finite. Let $y_k \in O$ s.t. $\omega(y_k) + \|x - y_k\| \to \underline{\omega}(x)$. Then by definition of inf and the triangle inequality

$$\underline{\omega}(z) \leq \omega(y_k) + \|z - y_k\| \leq \omega(y_k) + \|x - y_k\| + \|z - x\|$$

Since this holds for every k we know

$$\underline{\omega}(z) \leq \lim_k \omega(y_k) + \|x - y_k\| + \|z - x\| = \underline{\omega}(x) + \|z - x\| \implies \underline{\omega}(z) - \underline{\omega}(x) \leq \|z - x\|$$

By symmetry we get $\underline{\omega}(x) - \underline{\omega}(z) \leq \|z - x\|$ so that $\underline{\omega}$ is Lipschitz with constant 1.

Notably $\underline{\omega}(x) = \inf_y\{\omega(y) + \|x - y\|\} \leq \omega(x) + \|x - x\| = \omega(x)$. Lastly, we know $\underline{\omega}(x) \geq 0$ since both summed elements are non-negative. For contradiction suppose $\underline{\omega}(x) = 0$ for some x, then there's y_k so that $\omega(y_k) + \|x - y_k\| \to 0$. Because both terms summed are positive we must have $\liminf$ of each term vanish, too. In particular this means $y_k \to x$ and

$$0 = \liminf_k \omega(y_k) \geq \omega(x) > 0$$

a constradiction so that $\underline{\omega}(x) > 0$.

Exercise 10.14 I'm somewhat confused here, and am thinking maybe I should skip this one so I can get onto more convex analysis?

My confusion: I'm not sure how to determine whether A being asymptotically stable for F means A is asymptotically stable for F_K (naively I think this implication shouldn't hold, since there are potentially more solutions to the regularization than there were to the original inclusion). If A is asymptotically stable, then I think we can use Fact 10.13 above to get our desired functions for F_K, but these functions also apply for F since $F(x) \subset F_K(x)$ for every x.

I don't understand how the hint plays into any of this either, but I'm guessing it addresses my above confusion, somehow?

Exercise 10.16 Put $A = \{v \mid v \cdot \nabla f(\overline{x}) \leq 0\}$. Let $v \in T_C(\overline{x})$ then $\exists\, \lambda_i \searrow 0$ and $x_i \in C$ s.t. $x_i \to x$ where $\frac{x_i - x}{\lambda_i} \to v$. First I'll work to show

$$\frac{f(x_i) - f(\overline{x})}{\lambda_i} \to \nabla f(\overline{x}) \cdot v$$

Let $\epsilon > 0$ be given, we know from continuous differentiability for any j $\exists\, N$ so that $i > N$ gives us

$$\left| \frac{f\left(\overline{x} + \lambda_i \frac{x_j - \overline{x}}{\lambda_j}\right) - f(\overline{x})}{\lambda_i} - \nabla f(\overline{x}) \cdot \frac{x_j - \overline{x}}{\lambda_j} \right| < \epsilon/2$$

Similarly, since the dot product is continuous, $\exists\, M$ so that for any $j > M$

$$\left| \nabla f(\overline{x}) \cdot \frac{x_j - \overline{x}}{\lambda_j} - \nabla f(x) \cdot v \right| < \epsilon/2$$

This means for any $i > \max\{M, N\}$

$$\begin{aligned} \left| \frac{f(x_i) - f(\overline{x})}{\lambda_i} - \nabla f(\overline{x}) \cdot v \right| &= \left| \frac{f\left(\overline{x} + \lambda_i \frac{x_i - \overline{x}}{\lambda_i}\right) - f(\overline{x})}{\lambda_i} - \nabla f(\overline{x}) \right| \\ &\leq \left| \frac{f\left(\overline{x} + \lambda_i \frac{x_i - \overline{x}}{\lambda_i}\right) - f(\overline{x})}{\lambda_i} - \nabla f(\overline{x}) \cdot \frac{x_i - \overline{x}}{\lambda_i} \right| + \left| \nabla f(\overline{x}) \cdot \frac{x_i - \overline{x}}{\lambda_i} - \nabla f(x) \cdot v \right| < \epsilon \end{aligned}$$

Since $x_i \in C$ $f(x_i) \leq f(\overline{x})$ for every i so that $f(x_i) - f(\overline{x}) \leq 0$ and thus $\nabla f(\overline{x}) \cdot v \leq 0$ so that $v \in A$.

Now let $v \in A$. If $v \cdot \nabla f(\overline{x}) < 0$ then
$$\frac{f(\overline{x} + hv) - f(\overline{x})}{h} < 0$$
for small enough h. Form a sequence $h_i \to 0$ of these small enough h and put $x_i = \overline{x} + h_i v$ then $f(x_i) < f(\overline{x})$ so that $x_i \in C$, $h_i \searrow 0$ and
$$\frac{x_i - \overline{x}}{h_i} = \frac{\overline{x} + h_i v - \overline{x}}{h_i} = v$$
so that $v \in T_C(\overline{x})$. If $v \cdot \nabla f(\overline{x}) = 0$ then, because $\nabla f(\overline{x}) \neq 0$ there's a direction w so that $w \cdot \nabla f(\overline{x}) < 0$. Put $\psi : [0,1] \to \mathbb{R}$ given by
$$\psi(\lambda) = ((1-\lambda)w + \lambda v) \cdot \nabla f(\overline{x})$$
then ψ is continuous and $\psi(0) < 0$ and $\psi(1) = 0$. By IVT ψ achieves each value in between $\psi(0)$ and $\psi(1)$ so we can find $\lambda_i \to 1$ so that $\psi(\lambda_i)$ is negative, increasing and converges to 0.Because $((1-\lambda_i)w + \lambda_i v) \cdot \nabla f(\overline{x}) < 0$ we know $(1-\lambda_i)w + \lambda_i v \in T_C(\overline{x})$. Since $T_C(\overline{x})$ is closed we know $v = \lim_i (1-\lambda_i)w + \lambda_i v \in C$, completing the proof.

If $\nabla f(\overline{x}) = 0$ then the above equality breaks at saddle points. For example $f(x) = x^3$, $f'(0) = 0$ and $T_C(0) = \mathbb{R}^-$, but $1 \cdot f'(0) = 0 \implies 1 \in A$, so that $T_C(0) \neq A$.

Exercise 10.17 Trivially if $x \notin C$ then neither side of the $\iff$ can ever be true, so that vacuously the $\iff$ holds. For $x \in \operatorname{int} C$ $T_C(x) = \mathbb{R}^n$, so we only need to show the right hand implication holds for any v. For small enough h $x + hv \in C$ so that $d_C(x + hv) = 0$, thus the right hand lim inf vanishes, so the claim holds. Lastly consider $x \in \partial C$. If $v \in T_C(x)$ then $\exists\, x_i \in C \to x$ and $\lambda_i \searrow 0$ so that
$$0 = \lim_{i\to\infty} \left\| \frac{x_i - x}{\lambda_i} - v \right\| = \lim_{i\to\infty} \frac{\|x_i - x - \lambda_i v\|}{\lambda_i} \geq \lim_{i\to\infty} \frac{d_C(x + \lambda_i v)}{\geq} \liminf_{h\searrow 0} \frac{d_C(x + hv)}{h} \geq 0$$
thus the right hand side holds. For the reverse implication we know $\exists\, \lambda_i \searrow 0$ so that
$$0 = \liminf_{h\searrow 0} \frac{d_C(x + hv)}{h} = \lim_{i\to\infty} \frac{d_C(x + \lambda_i v)}{\lambda_i}$$
And for each i by definition $\exists\, x_{i,k} \in C$ so that $\lim_{k\to\infty} \|x + \lambda_i v - x_{i,k}\| \to d_C(x + \lambda_i v)$. Notably
$$\|x - x_{i,i}\| \leq \|x + \lambda_i v - x_{i,i}\| + \lambda_i \|v\| \to 0 \text{ as } i \to \infty$$
so that $x_{i,i} \to x$ and
$$0 = \lim_{i\to\infty} \frac{d_C(x + \lambda_i v)}{\lambda_i} = \lim_{i\to\infty} \frac{\|x + \lambda_i v - x_{i,i}\|}{\lambda_i} = \lim_{i\to\infty} \left\| \frac{x_{i,i} - x}{\lambda_i} - v \right\|$$
so that $\frac{x_{i,i} - x}{\lambda_i} \to v$, thus $v \in T_C(x)$.