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1 Intro

Below I present a proof for a well known relationship between $\inf \sup$ and $\sup \inf$.

2 Helper Lemmas

To begin let's prove a few lemmas which will simplify our proof below

Lemma 2.1 *Given $X \subset \mathbb{R}$ bounded below, $y \in \mathbb{R}$ if $x \geq y \, \forall x \in X$ then*

$$\inf_{x \in X} x \geq y$$

Proof. Since X is bounded below we know $\alpha := \inf_{x \in X} x$ is finite. Suppose, for the sake of contradiction, $\alpha < y$. We know $\forall \boldsymbol{\gamma} > 0 \, \exists x \in X$ such that

$$\alpha + \boldsymbol{\gamma} > x$$

since otherwise $\alpha + \boldsymbol{\gamma}$ would be a lower bound of X, but $\alpha + \boldsymbol{\gamma} > \alpha$, contradicting the definition of α. In particular $y - \alpha > 0 \implies \exists x \in X$ such that

$$y = \alpha + y - \alpha > x \Rightarrow\Leftarrow x \geq y$$

$\square$

Lemma 2.2 *Let $X \subset \mathbb{R}$ be bounded below then*

$$-\inf_{x \in X} x = \sup_{x \in X} -x$$

Proof. Since X is bounded below we know $\alpha := \inf_{x \in X}$ and $\beta := \sup_{x \in X} -x$ are finite. We have $\forall x \in X$

$$\alpha \leq x \implies -\alpha \geq -x$$

thus $-\alpha$ is an upper bound on $\{-x \mid x \in X\}$ so that

$$-\alpha \geq \beta \tag{1}$$

Additionally $\forall x \in X$

$$-x \leq \beta \implies x \geq -\beta$$

thus $-\beta$ is a lower bound of X so that $\alpha \geq -\beta$ or equivalently

$$-\alpha \leq \beta \tag{2}$$

Combining (1) & (2) gives us our desired equality. $\square$

Lemma 2.3 *Given $X \subset \mathbb{R}$ bounded above, $y \in \mathbb{R}$ if $x \leq y \, \forall x \in X$ then*

$$\sup_{x \in X} x \leq y$$

Proof. By Lemmas 2.1 and 2.2 we have

$$-\sup_{x \in X} x = \inf_{x \in X} -x \geq -y \implies \sup_{x \in X} x \leq y$$

$\square$

3 Main Results

Theorem 3.1 *Let X, Y be two non-empty sets, $Z \subset \mathbb{R}$ and $f : X \times Y \to Z$, then*

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \geq \sup_{y \in Y} \inf_{x \in X} f(x, y)$$

Proof. Let $x' \in X, y' \in Y$ (as X, Y are non-empty we know these exist). We have

$$\sup_{y \in Y} f(x', y) \geq f(x', y') \geq \inf_{x \in X} f(x, y')$$

By Lemma 2.1 we have

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \geq \inf_{x \in X} f(x, y')$$

and by Lemma 2.3 we have

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \geq \sup_{y \in Y} \inf x \in X f(x, y)$$

$\square$