Exercise 2.3 Fix a point $x \in \overline{\limsup_i C_i}$, then there are $x_k \in \limsup C_i$ such that $x_k \to x$. By definition, for each x_k there are C_{i_j} and $x_k^{(i_j)} \in C_{i_j}$ so that $x_k^{(i_j)} \to x_k$. Now define y_j by $x_j^(q)$ where q is such that $|x_j^q - x_j| < \frac{1}{2j}$. I claim $y_j \to x$, thus showing $x \in \limsup_i C_i$. Let $\epsilon > 0$ then fix N_1 so that $\frac{1}{N_1} < \epsilon$ and put N_2 so that

$$|x_k - x| < \frac{1}{2N_1} \ \forall\, k > M.$$

Finally put $M = \max\{N_1, N_2\}$ so that $\forall\, j > M$ we know

$$|y_j - x| \leq |y_j - x_j| + |x_j - x| < \frac{1}{2N_1} + \frac{1}{2N_1} = \frac{1}{N_1} < \epsilon.$$

The proof for $\liminf_i C_i$ is identical except y_j's definition changes according to the definition of $\liminf$. If $\lim_i C_i$ exists, then due to the previous two facts, it must also be closed.

Exercise 2.4 I'll show this by showing

$$\limsup_i C_i \subset \bigcap_i \overline{C_i} \subset \liminf_i C_i$$

and then equality comes from the fact that $\liminf_i C_i \subset \limsup_i C_i$.
Starting with the first inclusion, let $x \in \limsup_i C_i$ then by definition $\exists\, C_{i_k}$ and $x_{i_k} \in C_{i_k}$ where $x_{i_k} \to x$. By the fact that the sets are non-increasing we know, for any i the tail of the sequence $x_{i_k} \in C_i$ (tail meaning with at most finite terms removed, dependent on i). Since the tail exists in C_i then $\lim_i x_{i_k} \in \overline{C_i}$ and thus $x \in \overline{C_i}$, showing the claim.
For the second inclusion take $x \in \bigcap_i \overline{C_i}$. For every i put $x_i \in \overline{C_i}$ such that $|x_i - x| < 1/i$– this is possible, since otherwise $x \notin \overline{C_i}$. These $x_i \to x$, thus showing the claim.

Exercise 2.5 Let $x \in \limsup_i C_i$ then $\exists\, C_{i_k}, x_{i_k} \in C_{i_k}$ s.t. $x_{i_k} \to x$. By uniform convergence for any $\epsilon > 0$ $\exists\, N$ s.t. $i_k > N$ gives us

$$|f_{i_k}(x_{i_k}) - f(x_{i_k})| < \epsilon/2$$

By continuity we also know, for $i_k > M$

$$|f(x_{i_k}) - f(x)| < \epsilon/2$$

so that for $i_k > \max\{N, M\}$

$$|f_{i_k}(x_{i_k}) - f(x)| = |f_{i_k}(x_{i_k}) - f(x_{i_k}) + f(x_{i_k}) - f(x)| \leq |f_{i_k}(x_{i_k}) - f(x_{i_k})| + |f(x_{i_k}) - f(x)| < \epsilon$$

so that $f_{i_k}(x_{i_k}) \to f(x)$ as $k \to \infty$. By uniform convergence we also know for any $z \in [a, b]$ $f_{i_k}(z) \to f(z)$, so combined with the previous gives us $f_{i_k}(z) - f_{i_k}(x_{i_k}) \to f(z) - f(x)$. By construction

$$0 \leq f_{i_k}(z) - f_{i_k}(x_{i_k}) \implies 0 \leq f(z) - f(x) \implies f(x) \leq f(z)$$

so that $x \in C$.

Exercise 2.6

(a) Take $x \in C$, then, by continuity $\exists\, N(x)$ so that $\rho \geq M > 0$ on $N(x)$. For any $y \in \operatorname{Int}((x + M\mathbb{B}) \cap N(x))$

$$|y - x| \leq M \leq \rho(y) \implies x \in y + \rho(y)\mathbb{B} \implies y \in C_\rho \implies \operatorname{Int}((x + M\mathbb{B}) \cap N(x)) \subset C_\rho.$$

Since $x \in \operatorname{Int}((x + M\mathbb{B}) \cap N(x))$ $x \in \operatorname{Int} C_\rho$.

(b) Let $x_i \to x$ with $x_i \in C_\rho$. Since $x_i \in C_\rho$ $\exists\, y_i \in (x_i + \rho(x_i)\mathbb{B}) \cap C$. Due to convergence and continuity of ρ y_i are in a compact space for large i and hence $\exists\, y_{i_k} \to y$ for some $y \in C$ (the last inclusion due to closedness of C). If $y \notin x + \rho(x)\mathbb{B}$ then you could form a neighborhood N about $x + \rho(x)\mathbb{B}$ such that $y \notin N$ (since $x + \rho(x)\mathbb{B}$ is closed). Since $x + \rho(x)$ is continuous $x_i + \rho(x_i)\mathbb{B} \cap N \neq \emptyset$ for large i, however, since $y \notin N$, $y_{i_k} \notin N$ for large i_k, a contradiction.

(c) Since $C \subset C_{\rho_i}$ $\forall\, i$ we know $C \subset \limsup_i C_{\rho_i}, C \subset \liminf_i C_{\rho_i}$. We know $\liminf_i C_i \subset \limsup_i C_i$, so we must show $\limsup_i \subset C$ to complete the proof.
Take $x \in \limsup_i C_{\rho_i}, x \notin C$ for contradiction. Then $\exists\, C_{i_k}, x_{i_k} \in C_{i_k}$ s.t. $x_{i_k} \to x$. Because C is closed we know $D := d_H(x, C) > 0$. Now fix M_1 s.t. $\forall\, i_k > M_1$, $x_{i_k} \in N(x) := x + \frac{D}{4}\mathbb{B}$. Also fix M_2 s.t. $\forall\, i_k > M_2$, $|\rho_{i_k}| < \frac{D}{4}$ $\forall\, y \in N(x)$ (this is using uniform local convergence since $N(x)$ is compact). Together, for $i_k > \max M_1, M_2$

$$|x_{i_k} + \rho_{i_k} - x| < \frac{D}{2}$$

so that $x_{i_k} + \rho_{i_k}\mathbb{B} \cap C = \emptyset$, a contradiction with the construction of x_{i_k}.

Exercise 2.10

(osc) Let $(a_i, b_i, c_i) \to \overline{(a, b, c)}$, $x_i \in S((a_i, b_i, c_i))$ with $x_i \to \overline{x}$. We must show $\overline{x} \in S(\overline{(a, b, c)})$. Since $(a, b, c, x) \to ax^2 + bx + c$ is a continuous function we know

$$a_i x_i^2 + b_i x_i + c_i \to ax^2 + bx + c \implies ax^2 + bx + c = 0,$$

hence the claim is proven.

(not isc) Put $(a, b, c) = (1, 0, 0)$ and $(a_i, b_i, c_i) = (1, 0, 1/i)$, then $0 \in S(x)$, however $S((a_i, b_i, c_i)) = \emptyset$ $\forall\, i$.

Exercise 2.11

($\implies$) Let S be osc. Take $(x_i, y_i) \in \operatorname{gph} S$ s.t. $(x_i, y_i) \to (x, y)$. In particular $x_i \to x$, $y_i \to y$ and $y_i \in S(x_i)$ by definition of $\operatorname{gph} S$, so osc tells us $y \in S(x) \implies (x, y) \in \operatorname{gph} S$.

($\impliedby$) Consider $\operatorname{gph} S$ closed, and let $x_i \to x$, $y_i \to y$ with $y_i \in S(x_i)$. By the convergence of x_i, y_i we know $(x_i, y_i) \to (x, y)$, and so by closedness of $\operatorname{gph} S$, and the fact that $y_i \in S(x_i) \iff (x_i, y_i) \in \operatorname{gph} S$, $(x, y) \in \operatorname{gph} S$ hence $y \in S(x)$, thus S is osc

Exercise 2.12

($\implies$) Let S be locally bounded and let $U \subset \mathbb{R}^m$ be bounded. For the sake of contradiction suppose $S(U)$ is unbounded so that $\exists\, u_i \in S(U)$ s.t. $|u_i| \to \infty$ and $|u_i|$ is monotonically increasing (i.e. we want $\forall\, M \in \mathbb{R}^+$ $\exists\, N \in \mathbb{N}$ s.t. $\forall\, i > N\, |u_i| > M$).

By boundedness of U there's a convergent subsequence $u_{i_k} \to u \in \overline{U}$. By local boundedness of S there's a neighborhood $N(u)$ of u such that $S(N(u))$ is bounded. For large enough k $u_{i_k} \in N(u)$, hence $S(u_{i_k})$ is bounded, but this contradicts the construction of u_i

($\impliedby$) Let $x \in \mathbb{R}^n$ and fix U a bounded neighborhood of x then by hypothesis $S(U)$ is bounded, completing the proof.

Exercise 2.13 W.l.o.g. we operate as if S is fully single valued, since we can look locally enough (i.e. large enough i) so that it is, and so consider $S(x) = s(x)$ for some single valued s.

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(a) $\implies$ (c) by definition. To show (b) $\implies$ (c) let $x_i \to \overline{x}$, $\overline{y} \in S(\overline{x}) \iff \overline{y} = s(\overline{x})$. By local boundedness $\exists\, N(x)$ bounded such that $s(N(x))$ is bounded. For large i $x_i \in N(x)$ and so $y_i := s(x_i) \in s(N(x))$. Since $s(N(x))$ is bounded there's a convergent subsequence $y_{i_k} \to y^*$. By osc we know $y^* \in S(x) \iff y^* = s(x) = \overline{y}$, hence isc.

—

So, I think there's an error above, but I can't figure out at the moment how to get around it: we need to show that y_i converges, not that y_{i_k} converges, since osc operates on convergent $y_i := S(x_i)$.

—

Finally to show (c) $\implies$ (a), since continuity of S at $\overline{x}$ is the same as standard continuity of s at $\overline{x}$, it suffices to show $\lim_i s(x_i) = s(\overline{x})$ for any $x_i \to \overline{x}$. Due to isc, for $\overline{y} = s(\overline{x})$, and any $x_i \to \overline{x}$ we have $\lim_i(y_i := s(x_i)) = y = s(x)$, hence s is continuous and so S is as well.

Exercise 2.15

(osc) Let $x_i \to \overline{x}$ and $y_i \to \overline{y}$ where $y_i \in F(x_i)$. By construction $y_i \in \bigcup_{u \in U(x_i)} f(x, u)$, so that $y_i = f(x, u_i)$ for some $u_i \in U(x_i)$ (since U never takes on empty values). Because U is locally bounded we know $\exists\, N(x)$ s.t. $U(N(x))$ is bounded. Since $x_i \in N(x)$ for large i we have $u_i \in U(N(x))$ for large i, so we can find a subsequence $u_{i_k} \to \overline{u}$ for some $\overline{u}$. Since U is assumed to be osc, $\overline{u} \in U(x)$. Together with the fact that f is continuous we know $f(x_i, u_{i_k}) \to f(\overline{x}, \overline{u}) \in F(\overline{x})$.

(isc) Let $x_i \to \overline{x}$ and $\overline{y} \in F(\overline{x})$. By construction $\overline{y} \in \bigcup_{u \in U(\overline{x})} f(\overline{x}, u)$ so that $\overline{y} = f(\overline{x}, \overline{u})$ for some $\overline{u} \in U(\overline{x})$. By isc of U $\exists\, u_i \in U(x_i)$ (which we can assume exist for all i since U never takes empty values) s.t. $u_i \to \overline{u}$. Fix $y_i = f(x_i, u_i)$ then $y_i \to \overline{y}$ by continuity of f.

(continuous) The two results above together show if U is continuous then F is as well.

Exercise 2.18

(isc) Let $x_i \to \overline{x}$ and $\overline{y} \in S(\overline{x})$. Suppose there doesn't exist a sequence $y_i \in S(x_i)$ for large i such that $y_i \to \overline{y}$. Then there's some $\delta > 0$ so that $y_i \notin \overline{y} + k\delta\mathbb{B}$ for infinitely many i. However, eventually $x_i \in x + \delta\mathbb{B}$ for all $i > N \in \mathbb{N}$, so by Lipschitz, $\forall\, i > N$

$$S(x_i) \subset S(\overline{x}) + k\delta\mathbb{B}$$

but $y_i \in S(x_i)$, a contradiction.

(usc) Let O be open containing $S(\overline{x})$, then $\exists\, \delta > 0$ so that $S(\overline{x}) + k\delta\mathbb{B} \subset O$. Put $N(x) = x + \delta\mathbb{B}$ then $\forall\, x' \in N(x)$

$$S(x') \subset S(\overline{x}) + k\|x - x'\|\mathbb{B}$$

by construction of S and so

$$S(N(\overline{x})) = \bigcup_{x' \in N(\overline{x})} S(x') \subset \bigcup_{x' \in N(\overline{x})} S(\overline{x}) + k\|x - x'\|\mathbb{B} = S(\overline{x}) + \bigcup_{x' \in N(x)} k\|x' - x\|\mathbb{B} \subset S(\overline{x}) + k\delta\mathbb{B} \subset 0.$$

(osc) When $S(\overline{x})$ is closed, with the fact that S is usc at $\overline{x}$, Prop 2.16 tells us S is osc at $\overline{x}$.

(continuous) If S takes on closed values then S is osc $\forall\overline{x}$ and so, combined with being isc, this tells us S is continuous.

Exercise 2.20

- Finding an open loop control so that $u^2(0) + u^2(1)$ is minimized:
  If x_0 is outside $[-2, 2]$ then no such u exists. If $x_0 = 0$ then $u = 0$. If $x_0 > 0$ then we must have $u(0)+u(1) = -x_0$. To minimize $u^2(0)+u^2(1)$ it must be that $u(0), u(1) < 0$, otherwise either $x_0 > 1$ and u cannot be constructed, or $u(0) = -x_0, u(1)$ has smaller objective than one with e.g. $u(1) > 0$. Now we must decide how much of x_0 goes into either $u(0) \vee u(1)$, i.e. for what λ is

$$(-x_0\lambda)^2 + (-x_0(1-\lambda))^2 = x_0^2\big(1 - 2\lambda + 2\lambda^2\big)$$

  minimized (since indeed $-x_0\lambda + -x_0(1-\lambda)$ generically describes the set of possible $u(0), u(1)$. Taking the derivative and finding critical points in terms of λ we see $\lambda = \frac{1}{2}$ minmizes the above expression (as the objective is 1 at endpoints and 1/2 at $\lambda = 1/2$), so that $u(0) = u(1) = \frac{-x_0}{2}$ is a control minimizing the objective. For $x_0 < 0$ the analysis is identical except with $u(0) = u(1) = \frac{x_0}{2}$ appropriately.

- Finding $M(x_0)$: The same analysis from above works to show

$$M(x_0) = -(\operatorname{sgn} x_0)\Big(\frac{x_0}{2}, \frac{x_0}{2}\Big)$$

- Showing $m(x_0) = \frac{1}{2}x_0^2$: By our characterization of $M(x_0)$ we know int inf in $m(x_0)$ is achieved and so

$$m(x_0) = p(x_0, M(x_0)) = \Big(-(\operatorname{sgn} x_0)\frac{x_0}{2}\Big)^2 + \Big(-(\operatorname{sgn} x_0)\frac{x_0}{2}\Big)^2 = \frac{1}{4}x_0^2$$

  This disagrees with what the book says $m(x_0)$ should be, I'm not sure how to close the gap.

- When $p(x_0, u) = |u_0| + |u_1|$: Comparing with above, the same logic applies to show $u_0, u_1 < 0$, but now our minmization objective in terms of λ becomes, w.l.o.g. with $x_0 > 0$

$$|-x_0\lambda| + |-x_0(1-\lambda)| = 1$$

  so that any λ can minimize the objective, hence $M(x_0) = \{-(\lambda, 1-\lambda) \mid \lambda \in [0,1]\}$. When $x_0 < 0$ then appropriately $M(x_0) = \{(\lambda, 1-\lambda) \mid \lambda \in [0,1]\}$ and finally with $x_0 = 0$ $M(x_0) = 0$. We can summarize

$$M(x_0) = \begin{cases} -\operatorname{sgn} x_0\{(\lambda, 1-\lambda) \mid \lambda \in [0,1]\} & x_0 \neq 0 \\ 0 & x_0 = 0 \end{cases}$$

  Consequently, since the objective is 1 for $x_0 \neq 0$ and 0 for $x_0 = 0$ we have

$$m(x_0) = \begin{cases} 1 & x_0 \neq 0 \\ 0 & x_0 = 0 \end{cases}$$

Exercise 2.21

$(\implies)$ Suppose f is lsc and let $r \in \mathbb{R}$. Take a sequence $x_i \to \overline{x}$ where $x_i \in f^{-1}((-\infty, r]) = \{x \in \mathbb{R}^n \mid f(x) \leq r\}$. By lsc and the definition of $f^{-1}((-\infty, r])$ we know

$$f(\overline{x}) \leq \liminf_i f(x_i) \leq r \implies f(\overline{x}) \leq r \implies \overline{x} \in f^{-1}((-\infty, r])$$

so that $f^{-1}((-\infty,])$ is closed.

($\Longleftarrow$) Now suppose $\forall r \in \mathbb{R}$ $f^{-1}((-\infty, r])$ is closed and let $x_i \to \overline{x}$ be a sequence. If $\liminf_i f(x_i) = \infty$ then the inequality is trivially true. If $\liminf_i f(x_i) = -\infty$ then there's a subsequence x_{i_k} so that $f(x_{i_k}) \to -\infty$ and $f(x_{i_k})$ is bounded above by M, so that $f^{-1}((-\infty, M])$ is not closed, hence $\liminf_i f(x_i)$ must be finite. Now

$$f^{-1}((-\infty, \liminf_i f(x_i)]) = \left\{x \in \mathbb{R}^n \mid f(x) \leq \liminf_i f(x_i)\right\}$$

By definition of lim inf

$$\liminf_i f(x_i) = \lim_i \inf\{f(x_j) \mid j \geq i\} \implies f(x_j) \leq \liminf_i f(x_i) \ \forall j,$$

the implication coming from the fact that the set whose inf is taken is shrinking as i increases, causing the inf to only possibly increase, so $x_i \in f^{-1}((-\infty, \liminf_i f(x_i)])$, and thus closedness tells us $\overline{x} \in f^{-1}((-\infty, \liminf_i f(x_i)])$ showing $f(\overline{x}) \leq \liminf_i f(x_i)$, thus f is lsc at $\overline{x}$.

Exercise 2.22 Let f be lsc and C compact and suppose for contradiction f is not bounded below on C, so that $\exists\, x_i \in C$ s.t. $x_i \to \overline{x}$ but $f(x_i) \to -\infty$. By compactness $\overline{x} \in C$, hence

$$f(\overline{x}) \leq \liminf_i f(x_i)$$

If f isn't bounded below then $\liminf_i f(x_i) = -\infty$ but $f(\overline{x}) \neq -\infty$ (by construction), a contradiction, hence f is bounded below. Since f is bounded below $\inf_{y \in C} f(y)$ is finite, and so there's a minimizing sequence y_i s.t. $y_i \in C$ and $f(y_i) \to \inf_{y \in C} f(y)$. By compactness there's a convergent subsequence $y_{i_k} \to \overline{y} \in C$ and so

$$f(\overline{y}) \leq \liminf_i f(y_{i_k}) = \lim_i f(y_i) = \inf_{y \in C} f(y)$$

where the middle equality comes from the fact that $\lim_i f(x_i)$ exists and is finite. Since $f(\overline{y}) \leq \inf_{y \in C} f(y)$ it must be $f(\overline{y}) = \inf_{y \in C} f(y)$ so $\inf_{y \in C} f(y) = \min_{y \in C} f(y)$, hence every sequence in $\operatorname{argmin}_{y \in C} f(y)$ converges in $\operatorname{argmin}_{y \in C} f(y)$ so that $\operatorname{argmin}_{y \in C} f(y)$ is compact.

Exercise 2.25

(a) Suppose S, T are locally bounded at $\overline{x}$, so there's $N_S(\overline{x}), N_T(\overline{x})$ neighborhoods of x so that $S(N_S(\overline{x})), T(N_T(\overline{x}))$ are bounded. $N_S(\overline{x}) \cap N_T(\overline{x})$ is a neighborhood of $\overline{x}$, then, so that

$$(S+T)(N_S(\overline{x}) \cap N_T(\overline{x})) = \{s + t \mid s \in S(N_S(\overline{x}) \cap N_T(\overline{x})), t \in T(N_S(\overline{x}) \cap N_T(\overline{x}))\}$$

is bounded (since each value in the summand is guaranteed bounded).

(b) Let $x_i \to x$ and fix $y \in (S+T)(x)$, then $y = s + t$ for some $s \in S(x), t \in T(x)$. By isc of S, T $\exists\, s_i \in S(x_i), t_i \in T(x_i)$ where $s_i \to s, t_i \to t$. By continuity of standard addition we have

$$s_i + t_i \to s + t$$

and since $s_i + t_i \in (S+T)(x_i)$ it's shown $S + T$ is isc at x.

(c) Continuity follows from showing the osc case, and combining with the above isc case. Let $x_i \to x$ and let $y_i \in (S+T)(x_i)$ be so that $y_i \to y$. By definition for each y_i $\exists\, s_i \in S(x_i), t_i \in T(x_i)$ so that $y_i = s_i + t_i$. W.l.o.g. consider S locally bounded, so that there's a convergent subsequence $s_{i_k} \to s$. By osc of S $s \in S(x)$. Since $s_{i_k} \to s$ $t_{i_k} \to y - s$ and hence $y - s \in T(x)$. Altogether this shows $y \in (S+T)(x)$, thus osc.