1 Introduction

Here I will collect calculations done while exploring fractional curvature.

2 κ_σ of unit circle, $n = 2$

We wish to compute

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r)$$

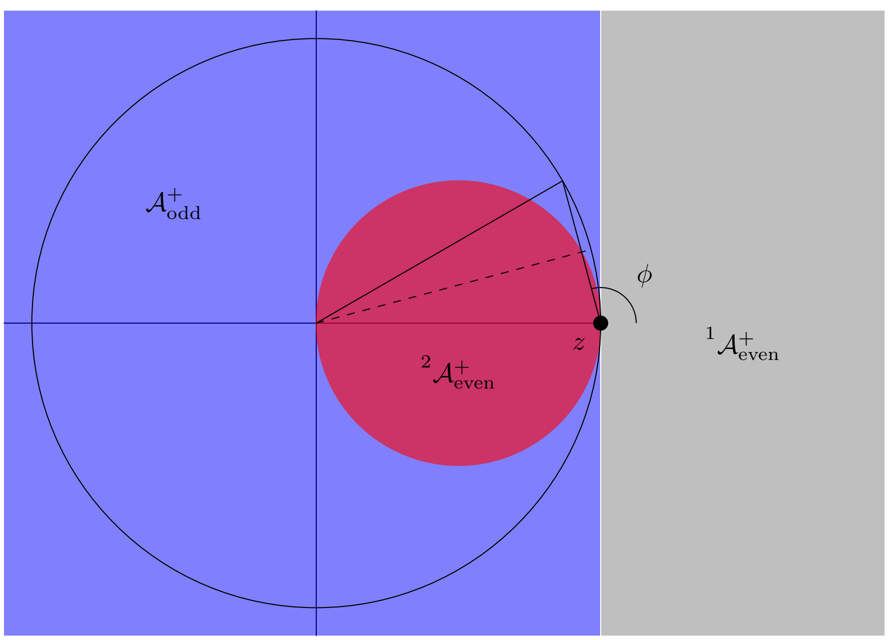
for C given by

$$z(\phi) = (\cos\phi, \sin\phi), \phi \in [0, 2\pi].$$

Due to symmetry $\kappa_\sigma(z(0)) = \kappa_\sigma(z(\phi))\ \forall\, \phi \in (0, 2\pi]$, so we can focus on the case when $z = (1, 0)$. We have $\mathbf{t}(z) = (0, 1)$. in order to help us characterize $\mathcal{A}^+_{\text{even}}, \mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
{}^1\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{3\pi}{2}, 2\pi\right] \cup \left[0, \frac{\pi}{2}\right], r \in [0, \infty)\right\}\\
{}^2\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r \in [0, \cos(\pi-\phi))\right\}\\
\mathcal{A}^+_{\text{even}} &= {}^1\mathcal{A}^+_{\text{even}} \cup {}^2\mathcal{A}^+_{\text{even}}\\
\mathcal{A}^+_{\text{odd}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r \in [\cos(\pi-\phi), \infty)\right\}
\end{aligned}$$

These subsets are motivated by the following picture:



Before jumping into calculations observe that we can parameterize our subset of $\mathbb{R}^5$ via (θ, r), as shown in the definition of the subsets above and put

$$s(\theta) = \begin{cases} -1 & \theta \in [\pi/2, 3\pi/2] \\ 1 & \text{otherwise} \end{cases}.$$

We can simplify our integrand as follows:

$$\begin{aligned}
J(r,\theta) &= \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}} \\
&= \frac{\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \left(s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}}{r^{1+\sigma}} \\
&= \frac{s(\theta)\left(\sin\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \cos\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)}{r^{1+\sigma}} = \frac{-s(\theta)\begin{pmatrix}\sin^2\theta+\cos^2\theta\\ -\sin\theta\cos\theta+\cos\theta\sin\theta\end{pmatrix}}{r^{1+\sigma}} \\
&= \begin{pmatrix}-1\\0\end{pmatrix}\frac{s(\theta)}{r^{1+\sigma}}
\end{aligned}$$

Next we can start computing integrals, we begin by integrating over $\mathcal{A}^+_{\text{even}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{even}}} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2(r,\theta) &= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta + \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{1}{r^{1+\sigma}}\, dr\, d\theta - \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{1}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\frac{1}{\sigma}\left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\left(0 - \frac{1}{\epsilon^\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(\frac{1}{(\cos(\pi-\theta))^\sigma} - \frac{1}{\epsilon^\sigma}\right) d\theta \\
&= \frac{\pi}{\sigma\epsilon^\sigma} - \frac{\pi}{\sigma\epsilon^\sigma} - \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

Now for $\mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{odd}}} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2(r,\theta) &= \int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{1}{r^{1+\sigma}}\, dr\, d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(0 - \frac{1}{(\cos(\pi-\theta))^\sigma}\right) d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= -\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

Putting these computations together we have:

$$\left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r)$$

$$\begin{aligned}
&= \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \begin{pmatrix}-1\\0\end{pmatrix} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2 \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \left(\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta + \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta\right) \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{2}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \text{ by (5.2).}
\end{aligned}$$

Finally we can recover the classical curvature $\kappa = z''(0) = (-1,0)$ as follows:

$$\begin{aligned}
\lim_{\sigma\uparrow 1} \frac{(1-\sigma)}{4}\kappa_\sigma &= \lim_{\sigma\uparrow 1} \begin{pmatrix}-1\\0\end{pmatrix} \frac{2(1-\sigma)\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{4\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{\sqrt{\pi}}{2} \lim_{\sigma\uparrow 1} \frac{(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{1}{2} \underbrace{\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right)}_{(5.3)} = \begin{pmatrix}-1\\0\end{pmatrix} = \kappa.
\end{aligned}$$

3 κ_σ of unit circle, $n = 3$

To begin, fix $\chi(a,b,r)$ to indicate whether a,b,r belongs to $\mathcal{A}^+_{\text{even}}$ or $\mathcal{A}^+_{\text{odd}}$, put $\mathcal{A}^+ = \mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}$ and

$$g(a,b,r) = \chi(a,b,r)\frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}},$$

so that our desired computation is:

$$\kappa_\sigma(z) = \int_{\mathcal{A}^+} g(a,b,r)\, d\mathcal{H}^4(\mathbf{a},\mathbf{b},r),$$

To simplify our domain group the disks by their intersection with the $S := \operatorname{lsp}\{t,n\}$ plane, i.e. put $\psi : \mathcal{A}^+ \to \mathbb{R}^2$ so that ψ maps a disk to a vector representing $\mathcal{D}(a,b,r)\cap S$. In 3.1 we see that ψ is given by

$$\psi(a,b,r) = 2r(p(b)\otimes p(b))a \text{ where } p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}.$$

Put $\mathcal{D}(u) := \psi^{-1}(\{u\}), p = p(b)$ so that[1]:

$$\begin{aligned}
\int_{\mathcal{A}^+} g(a,b,r)\, d\mathcal{H}^4(\mathbf{a},\mathbf{b},r) &= \int_{\mathbb{R}^2}\int_{\mathcal{D}(u)} \frac{g(a,b,r)}{J\psi(a,b,r)}\, d\mathcal{H}^2(a,b,r)d\mathcal{H}^2(u) \\
&= \int_{\mathbb{R}^2}\int_{\mathcal{D}(u)} \frac{g(a,b,r)}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}}\, d\mathcal{H}^2(a,b,r)d\mathcal{H}^2(u).
\end{aligned}$$

[1] See 3.2 for the calculation of $J\psi$

Now, to simplify $\mathcal{D}(u)$ lets group sets of $\mathbf{a}, \mathbf{b}$ that correspond to a given r, i.e. put $\phi : \mathcal{D}(u) \to \mathbb{R}^+$ given by

$$\phi(a, b, r) = r.$$

Put $\mathcal{D}(u, r) = \phi^{-1}(\{r\})$, $\mathcal{R}(u) = \{r \mid \exists (\mathbf{a}, \mathbf{b}) \in \mathcal{U}_2^\perp \; s.t. (\mathbf{a}, \mathbf{b}, r) \in \mathcal{D}(u)\}$ so that[2]

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathcal{D}(u)} \frac{g(a,b,r)}{2\sqrt{2} r \left| \frac{p\cdot a}{\mathcal{P}_S b^\perp} \right| \sqrt{2(p\cdot c)^2 + (p\cdot a)^2} \sqrt{r^2 (p\cdot c)^2 + (p\cdot a)^2}} \, d\mathcal{H}^2(a,b,r) d\mathcal{H}^2(u) \\
&= \int_{\mathbb{R}^2} \int_{\mathcal{R}(u)} \int_{\mathcal{D}(u,r)} \frac{g(a,b,r)}{2\sqrt{2} r \left| \frac{p\cdot a}{\mathcal{P}_S b^\perp} \right| \sqrt{2(p\cdot c)^2 + (p\cdot a)^2} \sqrt{r^2 (p\cdot c)^2 + (p\cdot a)^2}} \frac{\sqrt{(p\cdot a)^2 + r^2 (p\cdot c)^2}}{r |p\cdot c|} \, d\mathcal{H}^1(a,b) dr d\mathcal{H}^2(u) \\
&= \int_{\mathbb{R}^2} \int_{\mathcal{R}(u)} \int_{\mathcal{D}(u,r)} \frac{g(a,b,r) \left| \mathcal{P}_S b^\perp \right|}{2\sqrt{2} |(p\cdot a)(p\cdot c)| r^2 \sqrt{2(p\cdot c)^2 + (p\cdot a)^2}} \, d\mathcal{H}^1(a,b) dr d\mathcal{H}^2(u)
\end{aligned}$$

After simplifying the integrand[3] we must compute

$$\begin{aligned}
\kappa_\sigma &= \int_{\mathbb{R}^2} \int_{\mathcal{R}(u)} \int_{\mathcal{D}(u,r)} \frac{-\chi(u)}{2\sqrt{2}} \frac{1}{r^{1+\sigma}} \frac{\left| \mathcal{P}_S b^\perp \right|^2}{\sqrt{r^2 - \frac{|u|^2}{4}} \sqrt{2r^2 - \frac{|u|^2}{4}}} \, d\mathcal{H}^1(a,b) dr d\mathcal{H}^2(u) \\
&= \frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \chi(u) \int_{\mathcal{R}(u)} \frac{1}{r^{1+\sigma} \sqrt{r^2 - \frac{|u|^2}{4}} \sqrt{2r^2 - \frac{|u|^2}{4}}} \int_{\mathcal{D}(u,r)} \left| \mathcal{P}_S b^\perp \right|^2 d\mathcal{H}^1(a,b) dr d\mathcal{H}^2(u)
\end{aligned}$$

The inner most integral is intergrating over all a, b such that the disk of a fixed radius r intersects the $t - n$ plane along the cord from z to $z + u$. The admissable b form a circle with radius $\sqrt{r^2 - |u|^2/4}$ in the plane which goes through z with normal u. We can parameterize these b via

$$\theta \to \sqrt{r^2 - \frac{|u|^2}{4}} \left((\cos\theta) m + (\sin\theta) p^\perp \right) + z + \frac{u}{2},$$

where $m = t \times n$ and θ ranges over $[\vartheta, \vartheta + \pi]$ for some $\vartheta \in [0, 2\pi)$ (due to the restriction of $t \cdot a > 0$)[4]. With this parameterization we're able to compute the inner-most integral, i.e. we have

$$\int_{\mathcal{D}(u,r)} \left| \mathcal{P}_S b^\perp \right|^2 d\mathcal{H}^1(a,b) = \int_\vartheta^{\vartheta+\pi} \sin^2\theta \sqrt{r^2 - \frac{|u|^2}{4}} \, d\theta = \frac{\pi}{2} \sqrt{r^2 - \frac{|u|^2}{4}}.$$

Plugging this back into our above expression and simplifying we have

$$\kappa_\sigma = \frac{-\pi}{4\sqrt{2}} \int_{\mathbb{R}^2} \chi(u) \int_{\mathcal{R}(u)} \frac{1}{r^{1+\sigma} \sqrt{2r^2 - \frac{|u|^2}{4}}} \, dr \, d\mathcal{H}^2(u)$$

After expressing the inner integral as an incomplete beta function (see 3.5) we see

$$\begin{aligned}
\kappa_\sigma &= \frac{-\pi}{4\sqrt{2}} \int_{\mathbb{R}^2} \chi(u) \frac{\left(2\sqrt{2}\right)^\sigma}{|u|^{\sigma+1}} B_{\frac{1}{2}} \left(\frac{\sigma+1}{2}, \frac{1}{2} \right) d\mathcal{H}^2(u) \\
&= -\pi\sqrt{2} \left(2\sqrt{2}\right)^{\sigma-2} B_{\frac{1}{2}} \left(\frac{\sigma+1}{2}, \frac{1}{2} \right) \int_{\mathbb{R}^2} \frac{\chi(u)}{|u|^{\sigma+1}} \, d\mathcal{H}^2(u)
\end{aligned}$$

[2] See 3.3 for the calculation of $J\phi$

[3] see 3.4

[4] TODO: add picture to help show this geometry

3.1 Defining ψ

To begin, put $u := \psi(a, b, r)$, then we have the following constraints:

- The component of ra in the direction of u must be half of u's length (i.e. an isoceles triangle is formed between the center point of the circle sitting at ra and the chord at the intersection of this circle and S). In other words we must have:

$$ra \cdot \frac{u}{|u|} = \frac{|u|}{2} \implies 2ra \cdot u = |u|^2 = u \cdot u, \tag{1}$$

- Since we're interested in when these circles intersect S, we know

$$u = u_t t + u_n n, \ u_t, u_n \in \mathbb{R}. \tag{2}$$

- Finally, because $z + u$ is the chord of intersection between the disk formed by a, b, r and S we must also have

$$u \cdot b = 0, \tag{3}$$

Due to the combination of (3), (2) we have

$$b_n u_n + b_t u_t = 0.$$

By the construction of the integral we have $b \cdot t > 0 \implies b_t \neq 0$ so that

$$u_t = \frac{-b_n u_n}{b_t}. \tag{4}$$

Plugging this back into (1) (and using (2) to characterize u) we find

$$\begin{aligned} 2ra_n u_n - 2ra_t \frac{b_n u_n}{b_t} &= u_n^2 + \frac{b_n^2 u_n^2}{b_t^2} \\ \implies 0 &= u_n^2 \left(1 + \frac{b_n^2}{b_t^2}\right) + 2ru_n \left(\frac{a_t b_n}{b_t} - a_n\right) \\ \implies u_n = 0 \vee u_n \frac{b_t^2 + b_n^2}{b_t^2} &+ 2r \frac{a_t b_n - a_n b_t}{b_t} = 0. \end{aligned}$$

Notably $u_n \neq 0$ since otherwise, by (4), that would force $u_t = 0$, contradicting the assumption that $u_t > 0$. Solving the above equation for u_n we find

$$u_n = -2r \frac{a_t b_n - a_n b_t}{b_t} \frac{b_t^2}{b_t^2 + b_n^2} = 2r \frac{a_n b_t - a_t b_n}{b_t^2 + b_n^2} b_t.$$

Consider $\mathcal{P}_S = (t \otimes t) + (n \otimes n)$ the projection operator onto S so that we can rewrite the above as follows:

$$u_n = 2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2} b_t$$

Plugging this back into (4) we find

$$u_t = -2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2} b_n,$$

so that together, using the n, t coordinate system, we can write

$$\psi(a, b, r) = 2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2} (b_t, -b_n).$$

Consider the $\cdot^{\perp}$ operator to rotate clockwise in S, so that

$$\mathcal{P}_S b^{\perp} = ((n \otimes n)b + (t \otimes t)b)^{\perp} = (b_n n + b_t t)^{\perp} = b_t n - b_n t \implies \mathcal{P}_S b^{\perp} \cdot a = a_n b_t - a_t b_n \tag{5}$$

and $\left|\mathcal{P}_S b^{\perp}\right| = |\mathcal{P}_S b|$. Put $p(b) = \frac{\mathcal{P}_S b^{\perp}}{|\mathcal{P}_S b^{\perp}|}$, so that, combined with the above, we're able to simplify ψ:

$$\psi(a,b,r) = 2r \frac{\mathcal{P}_S b^{\perp} \cdot a}{|P_S b^{\perp}|^2} \mathcal{P}_S b^{\perp} = 2r(p(b) \cdot a)p(b).$$

Finally, rewriting using a tensor product we come to our final simplified definition:

$$\psi(a,b,r) = 2r(p(b) \otimes p(b))a$$

3.2 Computing $J\psi$

With our above definition, we're now able to compute the smooth gradient of ψ to use in the co-area formula. To begin, notice that $T_{(a,b,r)}(\mathcal{U}_{\perp}^2 \times \mathbb{R}^+)$ is spanned by $(c,0,0), (0,c,0), \frac{1}{\sqrt{2}}(b,-a,0), (0,0,1)$ (where c is the orthonormal completion of a, b in $\mathbb{R}^3$), and put $p = p(b)$ so that $p, p^{\perp}$ spans $T_{\psi(a,b,r)}(\mathbb{R}^2)$. We start with a quick calculation; suppose $\beta : \mathbb{R} \to \mathcal{U}$ such that $\beta(0) = b$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} p(\beta(s))\bigg|_{s=0} &= \frac{\mathcal{P}_S \beta'(0)^{\perp}}{|\mathcal{P}_S b^{\perp}|} - \frac{1}{|\mathcal{P}_S b^{\perp}|^3} \mathcal{P}_S b^{\perp} \otimes \left(\mathcal{P}_S \beta'(0)^{\perp}\right)^T \mathcal{P}_S b^{\perp} \\
&= \frac{1}{|\mathcal{P}_S b^{\perp}|} \left(\mathcal{P}_S \beta'(0)^{\perp} - \frac{1}{|\mathcal{P}_S b^{\perp}|^2} \left(\mathcal{P}_S \beta'(0)^{\perp} \cdot \mathcal{P}_S b^{\perp}\right) \mathcal{P}_S b^{\perp} \right) \\
&= \frac{1}{|\mathcal{P}_S b^{\perp}|} (1 - (p \otimes p)) \mathcal{P}_S \beta'(0)^{\perp}.
\end{aligned}$$

Since we'll be working in the $p, p^{\perp}$ coordinate system, it makes sense to expand this result as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s} p(\beta(s))\bigg|_{s=0} = \frac{1}{|\mathcal{P}_S b^{\perp}|} \left(p \otimes p + p^{\perp} \otimes p^{\perp} - p \otimes p\right) \mathcal{P}_S \beta'(0)^{\perp} = \frac{\left(p^{\perp} \otimes p^{\perp}\right)}{|\mathcal{P}_S b^{\perp}|} \mathcal{P}_S \beta'(0)^{\perp} \tag{6}$$

Now, put $\gamma_v : \mathbb{R} \to \mathcal{U}_{\perp}^2 \times \mathbb{R}^+$ to be such that $\gamma_v(0) = (a,b,r)$ and $\gamma_v'(0) = v$, then we begin by computing the derivative along the $\gamma_{(c,0,0)}$ flow:

$$\frac{\mathrm{d}}{\mathrm{d}s} \psi(\gamma_{(c,0,0)}(s))\bigg|_{s=0} = 2r(p \otimes p)c = 2r(p \cdot c)p. \tag{7}$$

Next we compute the derivative along the $(0,c,0)$ flow, and simplify using (6)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \psi(\gamma_{(0,c,0)}(s))\bigg|_{s=0} &= 2r \left(\left(\frac{\mathrm{d}}{\mathrm{d}s} p(\gamma_{(0,c,0),2}(s))\bigg|_{s=0} \right) \otimes p + p \otimes \left(\frac{\mathrm{d}}{\mathrm{d}s} p(\gamma_{(0,c,0),2}(s))\bigg|_{s=0} \right) \right) a \\
&= 2r \left(\left(\frac{\left(p^{\perp} \otimes p^{\perp}\right)}{|\mathcal{P}_S b^{\perp}|} \mathcal{P}_S c^{\perp} \right) \otimes p + p \otimes \left(\frac{\left(p^{\perp} \otimes p^{\perp}\right)}{|\mathcal{P}_S b^{\perp}|} \mathcal{P}_S c^{\perp} \right) \right) a \\
&= 2r \frac{p^{\perp} \cdot \mathcal{P}_S c^{\perp}}{|\mathcal{P}_S b^{\perp}|} \left(p^{\perp} \otimes p + p \otimes p^{\perp}\right) a.
\end{aligned}$$

Taking into account the fact that $p^{\perp} \cdot \mathcal{P}_S c^{\perp} = p \cdot \mathcal{P}_S c = p \cdot c$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s} \psi(\gamma_{(0,c,0)}(s))\bigg|_{s=0} = 2r \frac{(p \cdot c)\left(p^{\perp} \cdot a\right)}{|\mathcal{P}_S b^{\perp}|} p + 2r \frac{(p \cdot c)(p \cdot a)}{|\mathcal{P}_S b^{\perp}|} p^{\perp}. \tag{8}$$

The next derivative we must compute is along the $\frac{1}{\sqrt{2}}(b,-a,0)$ flow:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\bigg|_{s=0} &= \sqrt{2}r(p\otimes p)b + 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0),2}(s))\bigg|_{s=0}\right)\odot p\right)a \\ &= 2r\left(\left(\frac{\left(p^{\perp}\otimes p^{\perp}\right)}{|\mathcal{P}_S b^{\perp}|}\mathcal{P}_S\frac{-a}{\sqrt{2}}^{\perp}\right)\otimes p + p\otimes\left(\frac{\left(p^{\perp}\otimes p^{\perp}\right)}{|\mathcal{P}_S b^{\perp}|}\mathcal{P}_S\frac{-a}{\sqrt{2}}^{\perp}\right)\right)a \\ &= -\sqrt{2}r\frac{p^{\perp}\cdot\mathcal{P}_S a^{\perp}}{|\mathcal{P}_S b^{\perp}|}\left(p^{\perp}\otimes p + p\otimes p^{\perp}\right)a.\end{aligned}$$

Note that the first term vanishes because $p\cdot b = 0$. Again taking into account the fact that $p^{\perp}\cdot\mathcal{P}_S a^{\perp} = p\cdot\mathcal{P}_S a = p\cdot a$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\bigg|_{s=0} = -\sqrt{2}r\frac{(p\cdot a)\left(p^{\perp}\cdot a\right)}{|\mathcal{P}_S b^{\perp}|}p - \sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^{\perp}|}p^{\perp}. \tag{9}$$

Lastly computing the derivative through the $(0,0,1)$ flow we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,0,1)}(s))\bigg|_{s=0} = 2(p\otimes p)a = 2(p\cdot a)p \tag{10}$$

Combining (7), (8), (9), (10) we have

$$\begin{array}{ccccccc} & & (c,0,0) & (0,c,0) & \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) & \\ \nabla\psi(a,b,r) = & \Bigg(& 2r(p\cdot c) & 2r\frac{(p\cdot c)\left(p^{\perp}\cdot a\right)}{|\mathcal{P}_S b^{\perp}|} & -\sqrt{2}r\frac{(p\cdot a)\left(p^{\perp}\cdot a\right)}{|\mathcal{P}_S b^{\perp}|} & 2(p\cdot a) & \Bigg) \; p \\ & & 0 & 2r\frac{(p\cdot c)(p\cdot a)}{|\mathcal{P}_S b^{\perp}|} & -\sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^{\perp}|} & 0 & \; p^{\perp}\end{array}$$

Put $M = \nabla\psi(a,b,r)\nabla\psi(a,b,r)^T$ so that we desire to compute $\sqrt{|M|}$. Put $g = 2(p\cdot c)^2 + (p\cdot a)^2$. We compute the following:

$$\begin{aligned}M_{11} &= 4r^2(p\cdot c)^2 + \frac{4r^2(p\cdot c)^2\left(p^{\perp}\cdot a\right)^2}{|\mathcal{P}_S b^{\perp}|^2} + \frac{2r^2(p\cdot a)^2\left(p^{\perp}\cdot a\right)^2}{|\mathcal{P}_S b^{\perp}|^2} + 4(p\cdot a)^2 \\ &= 2r^2\frac{\left(p^{\perp}\cdot a\right)^2}{|\mathcal{P}_S b^{\perp}|^2}g + 4\left(r^2(p\cdot c)^2 + (p\cdot a)^2\right) \\ M_{22} &= 4r^2\frac{(p\cdot c)^2(p\cdot a)^2}{|\mathcal{P}_S b^{\perp}|^2} + 2r^2\frac{(p\cdot a)^4}{|\mathcal{P}_S b^{\perp}|^2} \\ &= 2r^2\frac{(p\cdot a)^2}{|\mathcal{P}_S b^{\perp}|^2}g \\ M_{12} = M_{21} &= 4r^2\frac{(p\cdot c)^2(p\cdot a)\left(p^{\perp}\cdot a\right)}{|\mathcal{P}_S b^{\perp}|^2} + 2r^2\frac{(p\cdot a)^3\left(p^{\perp}\cdot a\right)}{|\mathcal{P}_S b^{\perp}|^2} \\ &= 2r^2\frac{(p\cdot a)\left(p^{\perp}\cdot a\right)}{|\mathcal{P}_S b^{\perp}|^2}g.\end{aligned}$$

With these we can calculate the determinant as follows:

$$\begin{aligned}|M| = M_{11}M_{22} - M_{12}M_{21} &= 4r^4\frac{(p\cdot a)^2\left(p^{\perp}\cdot a\right)^2}{|\mathcal{P}_S b^{\perp}|^4}g^2 + 8gr^2\frac{(p\cdot a)^2}{|\mathcal{P}_S b^{\perp}|^2}\left(r^2(p\cdot c)^2 + (p\cdot a)^2\right) - 4r^4\frac{(p\cdot a)^2\left(p^{\perp}\cdot a\right)^2}{|\mathcal{P}_S b^{\perp}|^4}g^2 \\ &= 8gr^2\frac{(p\cdot a)^2}{|\mathcal{P}_S b^{\perp}|^2}\left(r^2(p\cdot c)^2 + (p\cdot a)^2\right)\end{aligned}$$

Thus, altogether we have

$$J\psi(a,b,r) = \sqrt{|\nabla\psi(a,b,r)^T\nabla\psi(a,b,r)|} = 2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2+(p\cdot a)^2}\sqrt{r^2(p\cdot c)^2+(p\cdot a)^2}.$$

3.3 Computing $J\phi$

For $u \in \mathbb{R}^2$ we have $\phi : \mathcal{D}(u) \to \mathbb{R}^+$ given by

$$\phi(a,b,r) = r.$$

To begin our calculation we must characterize $T_{(a,b,r)}(\mathcal{D}(u))$, i.e. via finding an orthonormal basis. Suppose $\gamma : \mathbb{R} \to \mathcal{D}(u)$ is so that $\gamma(0) = (a,b,r)$ and put $(\alpha,\beta,\tau) := \gamma'(0)$. By the definition of $\mathcal{D}(u)$ we know

$$2r(p(b)\otimes p(b))a = u,$$

so that

$$2\gamma_3(s)(p(\gamma_2(s))\otimes p(\gamma_2(s)))\gamma_1(s) = u.$$

Differentiating[5] and evaluating at $s=0$ we see

$$0 = \tau(p(b)\otimes p(b))a + r\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\beta^\perp \odot p(b)\right)a + r(p(b)\otimes p(b))\alpha.$$

Expanding tensor products, simplifying and using the $p, p^\perp$ coordinate system as we do in 3.2 we get

$$\begin{aligned}
0 &= \tau(p\cdot a)p + r\left(\frac{p\cdot\beta}{|\mathcal{P}_S b^\perp|}p^\perp\odot p\right)a + r(p\cdot\alpha)p \\
&= (\tau(p\cdot a)+r(p\cdot\alpha))p + r\frac{(p\cdot\beta)}{|\mathcal{P}_S b^\perp|}(p^\perp\otimes p + p\otimes p^\perp)a \\
&= \left(\tau(p\cdot a)+r(p\cdot\alpha)+r\frac{(p\cdot\beta)}{|\mathcal{P}_S b^\perp|}(p^\perp\cdot a)\right)p + r\frac{(p\cdot\beta)}{|\mathcal{P}_S b^\perp|}(p\cdot a)p^\perp.
\end{aligned}$$

To determine an initial basis vector suppose $\beta = 0, \alpha \neq 0, \tau \neq 0$, then we must have

$$0 = \tau(p\cdot a) + r(p\cdot\alpha),$$

but since $\mathcal{D}(u) \subset \mathcal{U}_2^\perp \times \mathbb{R}^+$ we must have $\alpha = c$ so that one of our basis vectors is

$$\mu = \frac{1}{\sqrt{1+r^2\frac{(p\cdot c)^2}{(p\cdot a)^2}}}\left(c, 0, -r\frac{(p\cdot c)}{(p\cdot a)}\right).$$

Next suppose $\tau = 0, \alpha \neq 0, \beta \neq 0$ so that we must have

$$(p\cdot\alpha) + \frac{(p\cdot\beta)}{|\mathcal{P}_S b^\perp|}(p^\perp\cdot a) = 0 \wedge (p\cdot\beta)(p\cdot a) = 0.$$

Notice $u = 2r(p\cdot a)p \implies u = |u|p$ and by construction we have $2ra\cdot u = u\cdot u$ so that $p\cdot a \neq 0$. Together with the second statment this leads us to see

$$(p\cdot\beta) = 0.$$

[5] N.B. we use (6)

Again, since $\mathcal{D}(u) \subset \mathcal{U}_2^\perp \times \mathbb{R}^+$ we must have $\beta \cdot b = 0$, so that β is perpendicular to both p and b, i.e. we have

$$\beta = p \times b.$$

Revisiting the first equality above we also find

$$(p \cdot \alpha) = 0,$$

and for similar reasoning we have $\alpha \cdot a = 0$, thus[6]

$$\alpha = p \times a.$$

Altogether this gives us a second basis vector:

$$\nu = \frac{1}{\sqrt{|p \times a|^2 + |p \times b|^2}} (p \times a, p \times b, 0)$$

Note that[7]

$$c \cdot (p \times a) = -c \cdot (a \times p) = -(c \times a) \cdot p = -b \cdot p = 0,$$

i.e. $\mu \cdot \nu = 0$ so that μ, ν are an orthonormal basis of $T_{(a,b,r)}\mathcal{D}(u)$.

Now, to compute $J\phi$, for $v \in \{\mu, \nu\}$, put $\gamma_v : \mathbb{R} \to \mathcal{D}(u)$ so that $\gamma_v(0) = (a, b, r)$ and $\gamma'(0) = v$. Then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_\mu(s))\bigg|_{s=0} &= \frac{-r(p \cdot c)}{(p \cdot a)\sqrt{1 + r^2 \frac{(p \cdot c)^2}{(p \cdot a)^2}}} \\
&= -r \frac{(p \cdot c)}{\sqrt{(p \cdot a)^2 + r^2 (p \cdot c)^2}} \\
\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_\nu(s))\bigg|_{s=0} &= 0.
\end{aligned}$$

so that

$$J\phi = \frac{r|p \cdot c|}{\sqrt{(p \cdot a)^2 + r^2 (p \cdot c)^2}}.$$

3.4 Simplifying Integrand of κ_σ

By definition we have

$$g(a, b, r) = \chi(a, b, r) \frac{(a \cdot t)b - (b \cdot t)a}{r^{1+\sigma}},$$

where χ is a signing function indicating whether we're integrating over $\mathcal{A}^+_{\text{even}}$ or $\mathcal{A}^+_{\text{odd}}$. Notice that[8]

$$(a \cdot t)(b \cdot n) - (b \cdot t)(a \cdot n) = ((b \cdot n)t - (b \cdot t)n) \cdot a = -\mathcal{P}_S b^\perp \cdot a = -\left|\mathcal{P}_S b^\perp\right|(p \cdot a),$$

so

$$g(a, b, r) \cdot n = \chi(a, b, r) \frac{(a \cdot t)(b \cdot n) - (b \cdot t)(a \cdot n)}{r^{1+\sigma}} = -\chi(a, b, r) \frac{-\left|\mathcal{P}_S b^\perp\right|(p \cdot a)}{r^{1+\sigma}}.$$

[6] TODO: Talk about signs of α, β?

[7] TODO: formalize c so that we know whether $c \times a = \pm b$.

[8] see the definition of $\cdot^\perp$ at 5.

Additionally, since a, b, c span $\mathbb{R}^3$ & $p \cdot b = 0$ we know

$$(p \cdot a)^2 + (p \cdot b)^2 + (p \cdot c)^2 = |p| = 1 \implies (p \cdot c)^2 = 1 - (p \cdot a)^2.$$

And finally, before simplifying our integrand let's note that

$$2r(p \cdot a)p = u \implies |u| = 2r(p \cdot a) \implies (p \cdot a) = \frac{|u|}{2r},$$

and in particular this means $p \cdot a > 0$. Altogether, substituting this into our integrand, and acknowledging χ only depends on u we find

$$\begin{aligned}
\frac{1}{2\sqrt{2}} \frac{1}{r^2} \left| \frac{\mathcal{P}_S b^{\perp}}{(p \cdot a)(p \cdot c)} \right| \frac{g(a,b,r) \cdot n}{\sqrt{2(p \cdot c)^2 + (p \cdot a)^2}} &= \frac{-\chi(u)}{2\sqrt{2}} \frac{1}{r^{3+\sigma}} \frac{\left|\mathcal{P}_S b^{\perp}\right|^2 |p \cdot a|}{(p \cdot a)(p \cdot c)} \frac{1}{\sqrt{2(p \cdot c)^2 + (p \cdot a)^2}} \\
&= \frac{-\chi(u)}{2\sqrt{2}} \frac{1}{r^{3+\sigma}} \frac{\left|\mathcal{P}_S b^{\perp}\right|^2}{\sqrt{1 - (p \cdot a)^2}\sqrt{2 - (p \cdot a)^2}} \\
&= \frac{-\chi(u)}{2\sqrt{2}} \frac{1}{r^{3+\sigma}} \frac{\left|\mathcal{P}_S b^{\perp}\right|^2}{\sqrt{1 - \frac{|u|^2}{4r^2}}\sqrt{2 - \frac{|u|^2}{4r^2}}} \\
&= \frac{-\chi(u)}{2\sqrt{2}} \frac{1}{r^{1+\sigma}} \frac{\left|\mathcal{P}_S b^{\perp}\right|^2}{\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}},
\end{aligned}$$

3.5 Simplifying integrand of r-integral in terms of $B_x(a, b)$

Recall

$$\mathcal{R}(u) = \left\{ r \mid \exists\, (a,b) \in \mathcal{U}_2^{\perp} \; s.t. \; (a,b,r) \in \mathcal{D}(u) \right\}.$$

Notice this set is unbounded towards $+\infty$ and that it's bounded below by $|u|/2$ (i.e. since $|u|$ is the minimal diameter of a disk). Additionally, it's easy to see that all values between $|u|/2, +\infty$ are also in $\mathcal{R}(u)$ so that $\mathcal{R}(u) = [|u|/2, \infty)$. Thus, we hope to compute the following definite integral

$$\int_{|u|/2}^{\infty} \frac{1}{r^{1+\sigma}\sqrt{2r^2 - \frac{|u|^2}{4}}}\, dr = \frac{1}{\sqrt{2}} \int_{|u|/2}^{\infty} \frac{1}{r^{1+\sigma}\sqrt{r^2 - \left(\frac{|u|}{2\sqrt{2}}\right)^2}}\, dr,$$

where $\sigma \in [0, 1]$. Substitute $r \to \frac{|u|}{2\sqrt{2}} t^{-1/2}$ so that $dr \to -\frac{|u|}{4\sqrt{2}} t^{-3/2} dt$ and the limits transform as $0 \to \frac{1}{2}, \infty \to 0$ so that our integral becomes

$$\begin{aligned}
\frac{1}{\sqrt{2}} \int_{|u|/2}^{\infty} \frac{1}{r^{1+\sigma}\sqrt{r^2 - \left(\frac{|u|}{2\sqrt{2}}\right)^2}}\, dr &= -\frac{|u|}{8} \int_{1/2}^{0} \frac{t^{-3/2}}{\left(\frac{|u|}{2\sqrt{2}}\right)^{1+\sigma} \left(t^{-1/2}\right)^{1+\sigma} \sqrt{\left(\frac{|u|}{2\sqrt{2}}\right)^2 t^{-1} - \left(\frac{|u|}{2\sqrt{2}}\right)^2}}\, dt \\
&= \frac{|u|}{8} \left(\frac{2\sqrt{2}}{|u|}\right)^{1+\sigma} \int_0^{1/2} \frac{t^{(1+\sigma-3)/2}}{\frac{|u|}{2\sqrt{2}}\sqrt{\frac{1-t}{t}}}\, dt = \frac{\left(2\sqrt{2}\right)^{1+\sigma}}{8|u|^{\sigma}} \frac{2\sqrt{2}}{|u|} \int_0^{1/2} \frac{t^{(\sigma-2+1)/2}}{(1-t)^{1/2}}\, dt \\
&= \frac{\left(2\sqrt{2}\right)^{\sigma}}{|u|^{\sigma+1}} \int_0^{1/2} t^{(\sigma-1)/2}(1-t)^{-1/2}\, dt = \frac{\left(2\sqrt{2}\right)^{\sigma}}{|u|^{\sigma+1}} B_{\frac{1}{2}}\left(\frac{\sigma+1}{2}, \frac{1}{2}\right).
\end{aligned}$$

4 Definitions & Properties

For the sake of completeness we use the following definitions are used:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt \text{ where } \Re(z) > 0, \tag{11}$$

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt \text{ where } \Re(z_1), \Re(z_2) > 0. \tag{12}$$

And we will assume the following properties:

$$\mathcal{B}(z_1, z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_2)}, \tag{13}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{14}$$

The former can be shown via a direct computation of the product $\Gamma(z_1)\,\Gamma(z_2)$ and change of variables & the latter via Weierstrass products.

5 Calculations

Lemma 5.1

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof. From (14) we have:

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi.$$

□

Lemma 5.2

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{-\sigma}\,d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(1-\frac{\sigma}{2}\right)} \text{ for } \sigma \in (0,1)$$

Proof. Beginning with (12) and using a change of variables $t \to \sin^2\theta$ so that $1-t = \cos^2\theta$ and $dt = 2\sin\theta\cos\theta\,d\theta$, thus

$$\mathcal{B}(z_1, z_2) = \int_0^{\pi/2} (\sin\theta)^{2z_1-2}(\cos\theta)^{2z_2-2}\cdot 2\sin\theta\cos\theta\,d\theta = 2\int_0^{\pi/2}(\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}\,d\theta.$$

Now, since $\frac{1-\sigma}{2} > 0$ when $\sigma < 1$ we have:

$$\mathcal{B}\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = 2\int_0^{\pi/2}(\sin\theta)^0(\cos\theta)^{1-\sigma-1}\,d\theta = 2\int_0^{\pi/2}(\cos\theta)^{-\sigma}\,d\theta = \int_{-\pi/2}^{\pi/2}(\cos\theta)^{-\sigma}\,d\theta.$$

Notice the final equality comes from the fact that $\cos\theta$ is even. On the other hand, by (13) we know

$$\mathcal{B}\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = \frac{\Gamma(1/2)\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1-\sigma}{2}\right)}.$$

Leveraging (5.1) we find our desired equality.

□

Lemma 5.3

$$\lim_{\sigma \uparrow 1} (1-\sigma)\, \Gamma\left(\frac{1-\sigma}{2}\right) = 2.$$

Proof. By (14) we know

$$\Gamma\left(\frac{1-\sigma}{2}\right) = \frac{\pi}{\sin\left(\pi \frac{1-\sigma}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}.$$

Thus we have

$$\begin{aligned}
\lim_{\sigma \uparrow 1} (1-\sigma)\, \Gamma\left(\frac{1-\sigma}{2}\right) &= \lim_{\sigma \uparrow 1} \frac{(1-\sigma)\pi}{\sin\left(\pi \frac{1-\sigma}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)} = \left(\lim_{\sigma \uparrow 1} \frac{(1-\sigma)\pi}{\sin\left(\pi \frac{1-\sigma}{2}\right)}\right) \cdot \left(\lim_{\sigma \uparrow 1} \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}\right) \\
&= \pi \lim_{\sigma \uparrow 1} \frac{(1-\sigma)}{\sin\left(\pi \frac{1-\sigma}{2}\right)} \underbrace{= \pi \lim_{\sigma \uparrow 1} \frac{-1}{\cos\left(\pi \frac{1-\sigma}{2}\right) \cdot \frac{-\pi}{2}}}_{\text{L'Hôpital's rule}} = 2
\end{aligned}$$

$\square$