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1 Introduction

Here I will collect calculations done while exploring fractional curvature.

2 κ_σ of the unit circle

We wish to compute

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r)$$

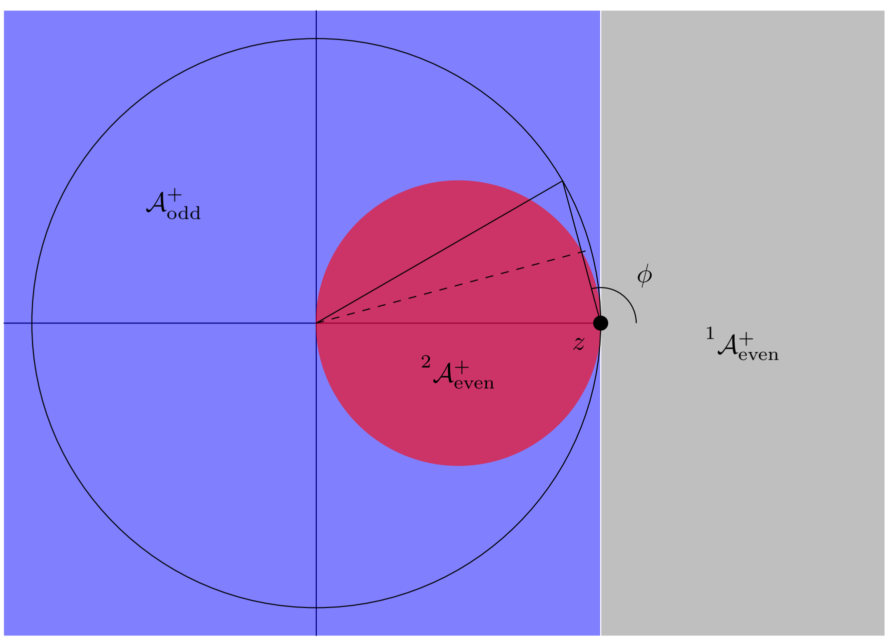
for C given by

$$z(\phi) = (\cos\phi, \sin\phi), \phi \in [0, 2\pi].$$

Due to symmetry $\kappa_\sigma(z(0)) = \kappa_\sigma(z(\phi))\ \forall\, \phi \in (0, 2\pi]$, so we can focus on the case when $z = (1, 0)$. We have $\mathbf{t}(z) = (0, 1)$. in order to help us characterize $\mathcal{A}^+_{\text{even}}, \mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
{}^1\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{3\pi}{2}, 2\pi\right] \cup \left[0, \frac{\pi}{2}\right], r \in [0,\infty)\right\}\\
{}^2\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r \in [0, \cos(\pi-\phi))\right\}\\
\mathcal{A}^+_{\text{even}} &= {}^1\mathcal{A}^+_{\text{even}} \cup {}^2\mathcal{A}^+_{\text{even}}\\
\mathcal{A}^+_{\text{odd}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r \in [\cos(\pi-\phi), \infty)\right\}
\end{aligned}$$

These subsets are motivated by the following picture:



Before jumping into calculations observe that we can parameterize our subset of $\mathbb{R}^5$ via (θ, r), as shown in the definition of the subsets above and put

$$s(\theta) = \begin{cases}-1 & \theta \in [\pi/2, 3\pi/2]\\ 1 & \text{otherwise}\end{cases}.$$

We can simplify our integrand as follows:

$$\begin{aligned}
J(r,\theta) &= \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}} \\
&= \frac{\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \left(s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}}{r^{1+\sigma}} \\
&= \frac{s(\theta)\left(\sin\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \cos\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)}{r^{1+\sigma}} = \frac{-s(\theta)\begin{pmatrix}\sin^2\theta+\cos^2\theta\\ -\sin\theta\cos\theta+\cos\theta\sin\theta\end{pmatrix}}{r^{1+\sigma}} \\
&= \begin{pmatrix}-1\\ 0\end{pmatrix}\frac{s(\theta)}{r^{1+\sigma}}
\end{aligned}$$

Next we can start computing integrals, we begin by integrating over $\mathcal{A}^+_{\text{even}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{even}}} \frac{s(\theta)}{r^{1+\sigma}}\,d\mathcal{H}^2(r,\theta) &= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{s(\theta)}{r^{1+\sigma}}\,dr\,d\theta + \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{s(\theta)}{r^{1+\sigma}}\,dr\,d\theta \\
&= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{1}{r^{1+\sigma}}\,dr\,d\theta - \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{1}{r^{1+\sigma}}\,dr\,d\theta \\
&= -\frac{1}{\sigma}\left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\left(0 - \frac{1}{\epsilon^\sigma}\right)d\theta + \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(\frac{1}{(\cos(\pi-\theta))^\sigma} - \frac{1}{\epsilon^\sigma}\right)d\theta \\
&= \frac{\pi}{\sigma\epsilon^\sigma} - \frac{\pi}{\sigma\epsilon^\sigma} - \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2}(\sec\theta)^\sigma\,d\theta \\
&= \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2}(\sec\theta)^\sigma\,d\theta.
\end{aligned}$$

Now for $\mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{odd}}} \frac{s(\theta)}{r^{1+\sigma}}\,d\mathcal{H}^2(r,\theta) &= \int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{s(\theta)}{r^{1+\sigma}}\,dr\,d\theta \\
&= -\int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{1}{r^{1+\sigma}}\,dr\,d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(0 - \frac{1}{(\cos(\pi-\theta))^\sigma}\right)d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2}(\sec\theta)^\sigma\,d\theta \\
&= -\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2}(\sec\theta)^\sigma\,d\theta.
\end{aligned}$$

Putting these computations together we have:

$$\begin{aligned}
\left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) &\frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r) \\
&= \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \begin{pmatrix}-1\\0\end{pmatrix} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2 \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \left(\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta + \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta\right) \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{2}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \text{ by (4.2).}
\end{aligned}$$

Finally we can recover the classical curvature $\kappa = z''(0) = (-1,0)$ as follows:

$$\begin{aligned}
\lim_{\sigma\uparrow 1} \frac{(1-\sigma)}{4}\kappa_\sigma &= \lim_{\sigma\uparrow 1} \begin{pmatrix}-1\\0\end{pmatrix} \frac{2(1-\sigma)\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{4\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{\sqrt{\pi}}{2} \lim_{\sigma\uparrow 1} \frac{(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \\
&= \begin{pmatrix}-1\\0\end{pmatrix} \frac{1}{2} \underbrace{\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right)}_{(4.3)} = \begin{pmatrix}-1\\0\end{pmatrix} = \kappa.
\end{aligned}$$

3 Definitions & Properties

For the sake of completeness we use the following definitions are used:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt \text{ where } \Re(z) > 0, \tag{1}$$

$$\mathcal{B}(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt \text{ where } \Re(z_1), \Re(z_2) > 0. \tag{2}$$

And we will assume the following properties:

$$\mathcal{B}(z_1,z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_2)}, \tag{3}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{4}$$

The former can be shown via a direct computation of the product $\Gamma(z_1)\,\Gamma(z_2)$ and change of variables & the latter via Weierstrass products.

4 Calculations

Lemma 4.1

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof. From (4) we have:

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi.$$

□

Lemma 4.2

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{-\sigma}\, d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(1-\frac{\sigma}{2}\right)} \text{ for } \sigma \in (0,1)$$

Proof. Beginning with (2) and using a change of variables $t \to \sin^2\theta$ so that $1-t = \cos^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, thus

$$\mathcal{B}(z_1, z_2) = \int_0^{\pi/2} (\sin\theta)^{2z_1-2}(\cos\theta)^{2z_2-2} \cdot 2\sin\theta\cos\theta\, d\theta = 2\int_0^{\pi/2} (\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}\, d\theta.$$

Now, since $\frac{1-\sigma}{2} > 0$ when $\sigma < 1$ we have:

$$\mathcal{B}\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = 2\int_0^{\pi/2} (\sin\theta)^0(\cos\theta)^{1-\sigma-1}\, d\theta = 2\int_0^{\pi/2} (\cos\theta)^{-\sigma}\, d\theta = \int_{-\pi/2}^{\pi/2} (\cos\theta)^{-\sigma}\, d\theta.$$

Notice the final equality comes from the fact that $\cos\theta$ is even. On the other hand, by (3) we know

$$\mathcal{B}\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = \frac{\Gamma(1/2)\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1-\sigma}{2}\right)}.$$

Leveraging (4.1) we find our desired equality.

□

Lemma 4.3

$$\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right) = 2.$$

Proof. By (4) we know

$$\Gamma\left(\frac{1-\sigma}{2}\right) = \frac{\pi}{\sin\left(\pi\frac{1-\sigma}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}.$$

Thus we have

$$\begin{aligned}\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right) &= \lim_{\sigma\uparrow 1}\frac{(1-\sigma)\pi}{\sin\left(\pi\frac{1-\sigma}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)} = \left(\lim_{\sigma\uparrow 1}\frac{(1-\sigma)\pi}{\sin\left(\pi\frac{1-\sigma}{2}\right)}\right) \cdot \left(\lim_{\sigma\uparrow 1}\frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}\right)\\ &= \pi\lim_{\sigma\uparrow 1}\frac{(1-\sigma)}{\sin\left(\pi\frac{1-\sigma}{2}\right)} = \underbrace{\pi\lim_{\sigma\uparrow 1}\frac{-1}{\cos\left(\pi\frac{1-\sigma}{2}\right)\cdot\frac{-\pi}{2}}}_{\text{L'Hôpital's rule}} = 2\end{aligned}$$

□