1 Introduction

Here I will collect calculations done while exploring fractional curvature.

2 Preliminary Computations for $n = 3$

For the below computations we put t, n the unit tangent, normal vectors of $\mathcal{C}$ at z. Consider e a vector orthonormal to both t, n so that $\operatorname{span}\{t, n, e\} = \mathbb{R}^3$.

In order to simplify the computation in 3D we aim to reuse our calculations in 2D by slicing the domain of the 3D disks appropriately. That is, we will define $\psi : \mathcal{U}^2_\perp \times \mathbb{R}+ \to \mathbb{R}^2$ such that $\psi(a, b, r)$ is the intersection of the disk formed by a, b, r and $\operatorname{span}\{n, t\}$, so that, with the help of the smooth co-area formula, we're able to split our integral into interated integrals: the outer an integral over each 2D disk and the inner an integral over an appropriate subset of 3D disks.

2.1 Defining ψ

To aid in our construction of ψ, put $u = \psi(a, b, r)$, $S = \operatorname{span}\{n, t\}$, then:

- The component of ra in the direction of u must be half of u's length (i.e. an isoceles triangle is formed between the center point of the circle sitting at ra and the chord at the intersection of this circle and S). In other words we must have:
$$ra \cdot \frac{u}{|u|} = \frac{|u|}{2} \implies 2ra \cdot u = |u|^2 = u \cdot u, \tag{1}$$
- Since we're interested in when these circles intersect S, we know
$$u = u_t t + u_n n, \; u_t, u_n \in \mathbb{R}. \tag{2}$$
- Finally, because $z + u$ is the chord of intersection between the disk formed by a, b, r and S we must also have
$$u \cdot b = 0, \tag{3}$$

Due to the combination of (3), (2) we have
$$b_n u_n + b_t u_t = 0.$$
By the construction of the integral we have $b \cdot t > 0 \implies b_t \neq 0$ so that
$$u_t = \frac{-b_n u_n}{b_t}. \tag{4}$$
Plugging this back into (1) (and using (2) to characterize u) we find
$$\begin{aligned} 2ra_n u_n - 2ra_t \frac{b_n u_n}{b_t} &= u_n^2 + \frac{b_n^2 u_n^2}{b_t^2} \\ \implies 0 &= u_n^2 \left(1 + \frac{b_n^2}{b_t^2}\right) + 2ru_n \left(\frac{a_t b_n}{b_t} - a_n\right) \\ \implies u_n = 0 \vee u_n \frac{b_t^2 + b_n^2}{b_t^2} &+ 2r \frac{a_t b_n - a_n b_t}{b_t} = 0. \end{aligned}$$

Notably $u_n \neq 0$ since otherwise, by (4), that would force $u_t = 0$, contradicting the assumption that $u_t > 0$. Solving the above equation for u_n we find

$$u_n = -2r\frac{a_t b_n - a_n b_t}{b_t}\frac{b_t^2}{b_t^2 + b_n^2} = 2r\frac{a_n b_t - a_t b_n}{b_t^2 + b_n^2}b_t.$$

Consider $\mathcal{P}_S = (t \otimes t) + (n \otimes n)$ the projection operator onto S so that we can rewrite the above as follows:

$$u_n = 2r\frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2}b_t$$

Plugging this back into (4) we find

$$u_t = -2r\frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2}b_n,$$

so that together, using the n, t coordinate system, we can write

$$\psi(a, b, r) = 2r\frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2}(b_t, -b_n).$$

Consider the $\cdot^\perp$ operator to rotate clockwise in S, so that

$$\mathcal{P}_S b^\perp = ((n \otimes n)b + (t \otimes t)b)^\perp = (b_n n + b_t t)^\perp = b_t n - b_n t \implies \mathcal{P}_S b^\perp \cdot a = a_n b_t - a_t b_n$$

and $\left|\mathcal{P}_S b^\perp\right| = |\mathcal{P}_S b|$. Put $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$, so that, combined with the above, we're able to simplify ψ:

$$\psi(a, b, r) = 2r\frac{\mathcal{P}_S b^\perp \cdot a}{|P_S b^\perp|^2}\mathcal{P}_S b^\perp = 2r(p(b) \cdot a)p(b).$$

Finally, rewriting using a tensor product we come to our final simplified definition:

$$\psi(a, b, r) = 2r(p(b) \otimes p(b))a$$

2.2 Computing $\nabla\psi$

With our above definition, we're now able to compute the smooth gradient of ψ to use in the co-area formula. To begin, notice that $T_{(a,b,r)}(\mathcal{U}_\perp^2 \times \mathbb{R}^+)$ is spanned by $(c, 0, 0), (0, c, 0), \frac{1}{\sqrt{2}}(b, -a, 0), (0, 0, 1)$ (where c is the orthonormal completion of a, b in $\mathbb{R}^3$), and put $p = p(b)$ so that $p, p^\perp$ spans $T_{\psi(a,b,r)}(\mathbb{R}^2)$. We start with a quick calculation; suppose $\beta : \mathbb{R} \to \mathcal{U}$ such that $\beta(0) = b$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}p(\beta(s))\bigg|_{s=0} &= \frac{\mathcal{P}_S \beta'(0)^\perp}{|\mathcal{P}_S b^\perp|} - \frac{1}{|\mathcal{P}_S b^\perp|^3}\mathcal{P}_S b^\perp \otimes \left(\mathcal{P}_S \beta'(0)^\perp\right)^T \mathcal{P}_S b^\perp \\
&= \frac{1}{|\mathcal{P}_S b^\perp|}\left(\mathcal{P}_S \beta'(0)^\perp - \frac{1}{|\mathcal{P}_S b^\perp|^2}\left(\mathcal{P}_S \beta'(0)^\perp \cdot \mathcal{P}_S b^\perp\right)\mathcal{P}_S b^\perp\right) \\
&= \frac{1}{|\mathcal{P}_S b^\perp|}(1 - (p \otimes p))\mathcal{P}_S \beta'(0)^\perp.
\end{aligned}$$

Since we'll be working in the $p, p^\perp$ coordinate system, it makes sense to expand this result as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}p(\beta(s))\bigg|_{s=0} = \frac{1}{|\mathcal{P}_S b^\perp|}\left(p \otimes p + p^\perp \otimes p^\perp - p \otimes p\right)\mathcal{P}_S \beta'(0)^\perp = \frac{\left(p^\perp \otimes p^\perp\right)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S \beta'(0)^\perp \tag{5}$$

Now, put $\gamma_v : \mathbb{R} \to \mathcal{U}_\perp^2 \times \mathbb{R}^+$ to be such that $\gamma_v(0) = (a, b, r)$ and $\gamma_v'(0) = v$, then we begin by computing the derivative along the $\gamma_{(c,0,0)}$ flow:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(c,0,0)}(s))\bigg|_{s=0} = 2r(p \otimes p)c = 2r(p \cdot c)p. \tag{6}$$

Next we compute the derivative along the $(0, c, 0)$ flow, and simplify using (5)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,c,0)}(s))\bigg|_{s=0} &= 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{(0,c,0),2}(s))\bigg|_{s=0}\right) \otimes p + p \otimes \left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{(0,c,0),2}(s))\bigg|_{s=0}\right)\right)a \\ &= 2r\left(\left(\frac{(p^\perp \otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S c^\perp\right) \otimes p + p \otimes \left(\frac{(p^\perp \otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S c^\perp\right)\right)a \\ &= 2r\frac{p^\perp \cdot \mathcal{P}_S c^\perp}{|\mathcal{P}_S b^\perp|}\big(p^\perp \otimes p + p \otimes p^\perp\big)a.\end{aligned}$$

Taking into account the fact that $p^\perp \cdot \mathcal{P}_S c^\perp = p \cdot \mathcal{P}_S c = p \cdot c$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,c,0)}(s))\bigg|_{s=0} = 2r\frac{(p \cdot c)\big(p^\perp \cdot a\big)}{|\mathcal{P}_S b^\perp|}p + 2r\frac{(p \cdot c)(p \cdot a)}{|\mathcal{P}_S b^\perp|}p^\perp. \tag{7}$$

The next derivative we must compute is along the $\frac{1}{\sqrt{2}}(b, -a, 0)$ flow:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\bigg|_{s=0} &= \sqrt{2}r(p \otimes p)b + 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0),2}(s))\bigg|_{s=0}\right) \odot p\right)a \\ &= 2r\left(\left(\frac{(p^\perp \otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\frac{-a^\perp}{\sqrt{2}}\right) \otimes p + p \otimes \left(\frac{(p^\perp \otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\frac{-a^\perp}{\sqrt{2}}\right)\right)a \\ &= -\sqrt{2}r\frac{p^\perp \cdot \mathcal{P}_S a^\perp}{|\mathcal{P}_S b^\perp|}\big(p^\perp \otimes p + p \otimes p^\perp\big)a.\end{aligned}$$

Note that the first term vanishes because $p \cdot b = 0$. Again taking into account the fact that $p^\perp \cdot \mathcal{P}_S a^\perp = p \cdot \mathcal{P}_S a = p \cdot a$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\bigg|_{s=0} = -\sqrt{2}r\frac{(p \cdot a)\big(p^\perp \cdot a\big)}{|\mathcal{P}_S b^\perp|}p - \sqrt{2}r\frac{(p \cdot a)^2}{|\mathcal{P}_S b^\perp|}p^\perp. \tag{8}$$

Lastly computing the derivative through the $(0, 0, 1)$ flow we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,0,1)}(s))\bigg|_{s=0} = 2(p \otimes p)a = 2(p \cdot a)p \tag{9}$$

Combining (6), (7), (8), (9) we have

$$\nabla\psi(a, b, r) = \begin{array}{c} \begin{array}{cccc} (c,0,0) & (0,c,0) & \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) \end{array} \\ \begin{pmatrix} 2r(p \cdot c) & 2r\frac{(p\cdot c)\left(p^\perp \cdot a\right)}{|\mathcal{P}_S b^\perp|} & -\sqrt{2}r\frac{(p\cdot a)\left(p^\perp\cdot a\right)}{|\mathcal{P}_S b^\perp|} & 2(p \cdot a) \\ 0 & 2r\frac{(p\cdot c)(p\cdot a)}{|\mathcal{P}_S b^\perp|} & -\sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^\perp|} & 0 \end{pmatrix} \end{array} \begin{array}{c} p \\ p^\perp \end{array}$$

Put $M = \nabla\psi(a,b,r)^T\nabla\psi(a,b,r)$ so that we desire to compute $\sqrt{|M|}$. Put $g = 2(p\cdot c)^2 + (p\cdot a)^2$. We compute the following:

$$\begin{aligned}
M_{11} &= 4r^2(p\cdot c)^2 + \frac{4r^2(p\cdot c)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^2} + \frac{2r^2(p\cdot a)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^2} + 4(p\cdot a)^2 \\
&= 2r^2\frac{\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^2}g + 4\left(r^2(p\cdot c)^2 + (p\cdot a)^2\right) \\
M_{22} &= 4r^2\frac{(p\cdot c)^2(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2} + 2r^2\frac{(p\cdot a)^4}{\left|\mathcal{P}_S b^\perp\right|^2} \\
&= 2r^2\frac{(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2}g \\
M_{12} = M_{21} &= 4r^2\frac{(p\cdot c)^2(p\cdot a)\left(p^\perp\cdot a\right)}{\left|\mathcal{P}_S b^\perp\right|^2} + 2r^2\frac{(p\cdot a)^3\left(p^\perp\cdot a\right)}{\left|\mathcal{P}_S b^\perp\right|^2} \\
&= 2r^2\frac{(p\cdot a)\left(p^\perp\cdot a\right)}{\left|\mathcal{P}_S b^\perp\right|^2}g.
\end{aligned}$$

With these we can calculate the determinant as follows:

$$\begin{aligned}
|M| = M_{11}M_{22} - M_{12}M_{21} &= 4r^4\frac{(p\cdot a)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^4}g^2 + 8r^2\frac{(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2}\left(r^2(p\cdot c)^2 + (p\cdot a)^2\right) - 4r^4\frac{(p\cdot a)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^4}g^2 \\
&= 8r^2\frac{(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2}\left(r^2(p\cdot c)^2 + (p\cdot a)^2\right)
\end{aligned}$$

Thus, altogether we have

$$\sqrt{\left|\nabla\psi(a,b,r)^T\nabla\psi(a,b,r)\right|} = 2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}.$$

3 κ_σ of the unit circle

We wish to compute

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right)\frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\,d\mathcal{H}^2(\mathbf{a},\mathbf{b},r)$$

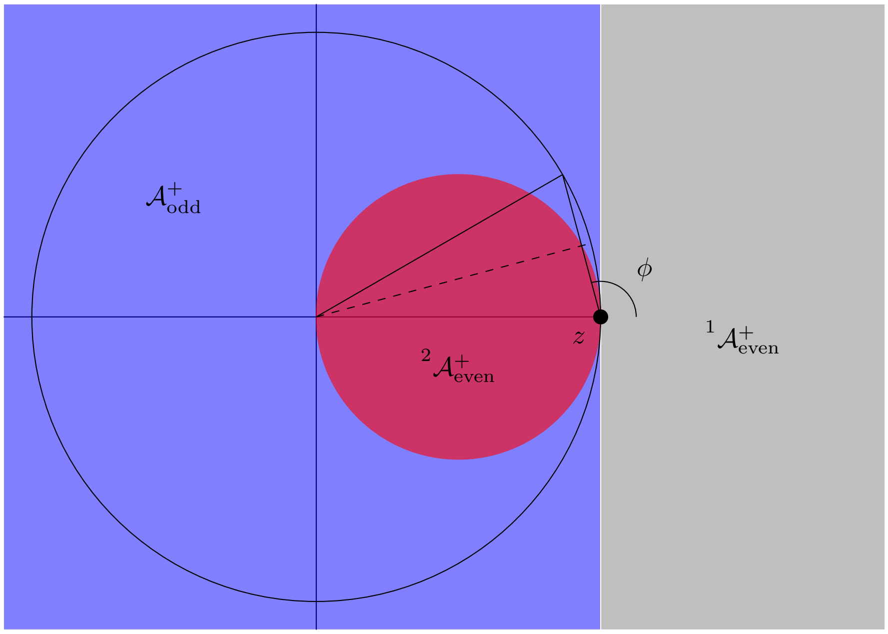
for C given by

$$z(\phi) = (\cos\phi, \sin\phi), \phi\in[0,2\pi].$$

Due to symmetry $\kappa_\sigma(z(0)) = \kappa_\sigma(z(\phi))\ \forall\,\phi\in(0,2\pi]$, so we can focus on the case when $z=(1,0)$. We have $\mathbf{t}(z) = (0,1)$. in order to help us characterize $\mathcal{A}^+_{\text{even}}, \mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
{}^1\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}, r\right) \mid \phi\in\left[\frac{3\pi}{2}, 2\pi\right]\cup\left[0,\frac{\pi}{2}\right], r\in[0,\infty)\right\} \\
{}^2\mathcal{A}^+_{\text{even}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi\in\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r\in[0,\cos(\pi-\phi))\right\} \\
\mathcal{A}^+_{\text{even}} &= {}^1\mathcal{A}^+_{\text{even}}\cup{}^2\mathcal{A}^+_{\text{even}} \\
\mathcal{A}^+_{\text{odd}} &= \left\{\left(\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\sin\phi\\ -\cos\phi\end{pmatrix}, r\right) \mid \phi\in\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], r\in[\cos(\pi-\phi),\infty)\right\}
\end{aligned}$$

These subsets are motivated by the following picture:



Before jumping into calculations observe that we can parameterize our subset of $\mathbb{R}^5$ via (θ, r), as shown in the definition of the subsets above and put

$$s(\theta) = \begin{cases} -1 & \theta \in [\pi/2, 3\pi/2] \\ 1 & \text{otherwise} \end{cases}.$$

We can simplify our integrand as follows:

$$\begin{aligned} J(r,\theta) &= \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}} \\ &= \frac{\left(\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \left(s(\theta)\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right)s(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}}{r^{1+\sigma}} \\ &= \frac{s(\theta)\left(\sin\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} - \cos\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right)}{r^{1+\sigma}} = \frac{-s(\theta)\begin{pmatrix}\sin^2\theta+\cos^2\theta\\ -\sin\theta\cos\theta+\cos\theta\sin\theta\end{pmatrix}}{r^{1+\sigma}} \\ &= \begin{pmatrix}-1\\0\end{pmatrix}\frac{s(\theta)}{r^{1+\sigma}} \end{aligned}$$

Next we can start computing integrals, we begin by integrating over $\mathcal{A}^+_{\text{even}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{even}}} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2(r,\theta) &= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta + \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= \left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\int_\epsilon^\infty \frac{1}{r^{1+\sigma}}\, dr\, d\theta - \int_{\pi/2}^{3\pi/2}\int_\epsilon^{\cos(\pi-\theta)} \frac{1}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\frac{1}{\sigma}\left(\int_{3\pi/2}^{2\pi} + \int_0^{\pi/2}\right)\left(0 - \frac{1}{\epsilon^\sigma}\right) d\theta + \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(\frac{1}{(\cos(\pi-\theta))^\sigma} - \frac{1}{\epsilon^\sigma}\right) d\theta \\
&= \frac{\pi}{\sigma\epsilon^\sigma} - \frac{\pi}{\sigma\epsilon^\sigma} - \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

Now for $\mathcal{A}^+_{\text{odd}}$:

$$\begin{aligned}
\int_{\mathcal{A}^+_{\text{odd}}} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2(r,\theta) &= \int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{s(\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\int_{\pi/2}^{3\pi/2}\int_{\cos(\pi-\theta)}^\infty \frac{1}{r^{1+\sigma}}\, dr\, d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{3\pi/2}\left(0 - \frac{1}{(\cos(\pi-\theta))^\sigma}\right) d\theta \\
&= \frac{1}{\sigma}\int_{\pi/2}^{-\pi/2} (\sec\theta)^\sigma\, d\theta \\
&= -\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2} (\sec\theta)^\sigma\, d\theta.
\end{aligned}$$

Putting these computations together we have:

$$\begin{aligned}
&\left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}}\, d\mathcal{H}^2(\mathbf{a},\mathbf{b},r) \\
&\qquad = \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right)\begin{pmatrix}-1\\0\end{pmatrix} \frac{s(\theta)}{r^{1+\sigma}}\, d\mathcal{H}^2 \\
&\qquad = \begin{pmatrix}-1\\0\end{pmatrix}\left(\frac{1}{\sigma}\int_{-\pi/2}^{\pi/2}(\sec\theta)^\sigma\, d\theta + \frac{1}{\sigma}\int_{-\pi/2}^{\pi/2}(\sec\theta)^\sigma\, d\theta\right) \\
&\qquad = \begin{pmatrix}-1\\0\end{pmatrix}\frac{2}{\sigma}\int_{-\pi/2}^{\pi/2}(\sec\theta)^\sigma\, d\theta \\
&\qquad = \begin{pmatrix}-1\\0\end{pmatrix}\frac{2\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\sigma\,\Gamma\left(1-\frac{\sigma}{2}\right)} \text{ by (5.2).}
\end{aligned}$$

Finally we can recover the classical curvature $\kappa = z''(0) = (-1, 0)$ as follows:

$$\begin{aligned}\lim_{\sigma\uparrow 1}\frac{(1-\sigma)}{4}\kappa_\sigma &= \lim_{\sigma\uparrow 1}\begin{pmatrix}-1\\0\end{pmatrix}\frac{2(1-\sigma)\sqrt{\pi}\,\Gamma\big(\frac{1-\sigma}{2}\big)}{4\sigma\,\Gamma\big(1-\frac{\sigma}{2}\big)}\\ &= \begin{pmatrix}-1\\0\end{pmatrix}\frac{\sqrt{\pi}}{2}\lim_{\sigma\uparrow 1}\frac{(1-\sigma)\,\Gamma\big(\frac{1-\sigma}{2}\big)}{\sigma\,\Gamma\big(1-\frac{\sigma}{2}\big)}\\ &= \begin{pmatrix}-1\\0\end{pmatrix}\frac{1}{2}\underbrace{\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right)}_{(5.3)} = \begin{pmatrix}-1\\0\end{pmatrix} = \kappa.\end{aligned}$$

4 Definitions & Properties

For the sake of completeness we use the following definitions are used:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt \text{ where } \Re(z) > 0, \tag{10}$$

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt \text{ where } \Re(z_1), \Re(z_2) > 0. \tag{11}$$

And we will assume the following properties:

$$\mathcal{B}(z_1, z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_2)}, \tag{12}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{13}$$

The former can be shown via a direct computation of the product $\Gamma(z_1)\,\Gamma(z_2)$ and change of variables & the latter via Weierstrass products.

5 Calculations

Lemma 5.1

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof. From (13) we have:

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi.$$

□

Lemma 5.2

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{-\sigma}\, d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(1-\frac{\sigma}{2}\right)} \text{ for } \sigma \in (0,1)$$

Proof. Beginning with (11) and using a change of variables $t \to \sin^2\theta$ so that $1 - t = \cos^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, thus

$$\mathcal{B}(z_1, z_2) = \int_0^{\pi/2} (\sin\theta)^{2z_1-2}(\cos\theta)^{2z_2-2} \cdot 2\sin\theta\cos\theta\, d\theta = 2\int_0^{\pi/2} (\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}\, d\theta.$$

Now, since $\frac{1-\sigma}{2} > 0$ when $\sigma < 1$ we have:

$$\mathcal{B}\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = 2\int_0^{\pi/2} (\sin\theta)^0(\cos\theta)^{1-\sigma-1}\, d\theta = 2\int_0^{\pi/2} (\cos\theta)^{-\sigma}\, d\theta = \int_{-\pi/2}^{\pi/2} (\cos\theta)^{-\sigma}\, d\theta.$$

Notice the final equality comes from the fact that $\cos\theta$ is even. On the other hand, by (12) we know

$$\mathcal{B}\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = \frac{\Gamma(1/2)\,\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1-\sigma}{2}\right)}.$$

Leveraging (5.1) we find our desired equality.

□

Lemma 5.3

$$\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right) = 2.$$

Proof. By (13) we know

$$\Gamma\left(\frac{1-\sigma}{2}\right) = \frac{\pi}{\sin\left(\pi\frac{1-\sigma}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}.$$

Thus we have

$$\begin{aligned}\lim_{\sigma\uparrow 1}(1-\sigma)\,\Gamma\left(\frac{1-\sigma}{2}\right) &= \lim_{\sigma\uparrow 1}\frac{(1-\sigma)\pi}{\sin\left(\pi\frac{1-\sigma}{2}\right)} \cdot \frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)} = \left(\lim_{\sigma\uparrow 1}\frac{(1-\sigma)\pi}{\sin\left(\pi\frac{1-\sigma}{2}\right)}\right) \cdot \left(\lim_{\sigma\uparrow 1}\frac{1}{\Gamma\left(\frac{1+\sigma}{2}\right)}\right)\\ &= \pi\lim_{\sigma\uparrow 1}\frac{(1-\sigma)}{\sin\left(\pi\frac{1-\sigma}{2}\right)} \underbrace{= \pi\lim_{\sigma\uparrow 1}\frac{-1}{\cos\left(\pi\frac{1-\sigma}{2}\right)\cdot\frac{-\pi}{2}}}_{\text{L'Hôpital's rule}} = 2\end{aligned}$$

□