Non-local Curvature of Curves

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Abstract

Here we study the non-local curvature of a curve and show that there's a natural asymptotic convergence to classical curvature. Concretely, we first study the asymptotics of circles with arbitrary radii, show that, with the right normalization constant, there's an asymptotic relationship to the inverse of the radius, i.e. the classical curvature of a circle. Next we show that the non-curvature of an arbitrary curve can be approximated by the nonlocal curvature of a circle. Finally we show that arbitrary curves' non-local curvature converge to the curve's classical curvature under the right conditions.

Contents

1 Introduction

1 Background

Continuing from [1], we aim to determine the appropriate conditions needed to recover classical curvature κ from non-local curvature κ_σ as $\sigma \to 1$. Concretely, this means, for a given curve $\mathcal{C}$ in n dimensions we want to analyze

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} \, d\mathcal{H}^{2n-2}(a, b, r), \tag{1}$$

where $t(z)$ is the unit tangent of $\mathcal{C}$ at z, and understand the asymptotics as $\sigma \to 1$.

2 Notation

In the following, we always use:

- n to denote the dimension of Euclidean space $\mathbb{R}^n$ with $n \geq 1$
- $\mathcal{C}$ a curve; contextually this will either be a unit circle or arbitrary
- $\lambda : \mathbb{R}^1 \to \mathbb{R}^n$ the parameterization of a curve $\mathcal{C}$
- $\kappa(z)$ the classical curvature of $\mathcal{C}$ at $z \in \mathcal{C}$; $\kappa := \kappa_{\mathcal{C}}(0)$
- $\kappa_\sigma(z)$ the non-local curvature (1) of $\mathcal{C}$ at $z \in \mathcal{C}$; $\kappa_\sigma := \kappa_\sigma(0)$
- $\gamma_v : \mathbb{R} \to \mathcal{M}$ for any manifold $\mathcal{M}$ is a flow such that $\gamma_v'(0) = v$ and $\gamma_(0)$ is set contextually
- $\omega_{k-1} := \frac{2\pi^{k/2}}{\Gamma(k/2)}$ is the surface area of an $k-1$ dimensional unit sphere embedded in k dimensional space
- $\mathcal{H}^k(\cdot)$ the k dimensional Hausdorff measure
- $\mathcal{U}(E) := \{a \in E \mid |a| = 1\}$; $\mathcal{U} := \mathcal{U}(\mathbb{R}^n)$
- $\mathcal{U}_2^\perp(E) := \{(a, b) \in \mathcal{U}(E) \times \mathcal{U}(E) \mid a \cdot b = 0\}$; $\mathcal{U}_2^\perp := \mathcal{U}_2^\perp(\mathbb{R}^n)$
- $E^c := \{x \in \mathbb{R}^n : x \notin E\}$
- $E^\perp := \{x \in \mathbb{R}^n \mid \forall y \in E \, x \cdot y = 0\}$
- $\chi_E(x) := \begin{cases} 1 : x \in E \\ 0 : x \notin E \end{cases}$
- $\overline{\chi}_E := \chi_E - \chi_{E^c}$
- $\mathcal{D}(p, u, r) := \big\{p + \xi v \mid (u, v) \in \mathcal{U}_2^\perp, \xi \in [0, r)\big\}$
- $\mathcal{A}^+(z) := \big\{(a, b, r) \in \mathcal{U}_2^\perp \mid (b \cdot t(z)) > 0\big\}$; $\mathcal{A}^+ = \mathcal{A}^+(0)$
- $\mathcal{A}^+_{\mathsf{Even}}(z) := \big\{(a, b, r) \in \mathcal{A}^+(z) \mid \mathcal{H}^0(\mathcal{D}(z + ra, b, r) \cap \mathcal{C})$ is even$\big\}$; $\mathcal{A}^+_{\mathsf{Even}} := \mathcal{A}^+_{\mathsf{Even}}(0)$
- $\mathcal{A}^+_{\mathsf{Odd}}(z) := \big\{(a, b, r) \in \mathcal{A}^+(z) \mid \mathcal{H}^0(\mathcal{D}(z + ra, b, r) \cap \mathcal{C})$ is odd$\big\}$; $\mathcal{A}^+_{\mathsf{Odd}} := \mathcal{A}^+_{\mathsf{Odd}}(0)$
- $(t, n) := (t(z), n(z))$ are the unit (tangent, normal) vectors of $\mathcal{C}$ at z

- $w_z = (w \cdot z)$, i.e. the $z-th$ component of w
- $v^\perp = ((v \cdot t)t + (v \cdot n)n)^\perp := (v \cdot n)t - (v \cdot t)n$ for any $v \in \mathbb{R}^2$; i.e. a clockwise rotation
- $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt$
- $B(z_1, z_2) := \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt = 2\int_0^{\pi/2} (\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}\, d\theta = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$
- $\operatorname{sgn}(x) := \frac{x}{|x|}$
- $\overline{\mathcal{J}} f := \sqrt{\nabla f^T \nabla f}$ for change of variables
- $\mathcal{J} f := \sqrt{\nabla f \nabla f^T}$ for applying the co-area formula
- $\mathcal{S} := \operatorname{lsp}\{t, n\}$, i.e. the 2D subspace spanned by t, z
- $\mathcal{P}_S := (t \otimes t) + (n \otimes n)$, i.e. the projection onto $\mathcal{S}$
- $\mathcal{T}_p(E)$ is the tangent space of E at p

2 Non-local Curvature of Circle in 2D

$\boxed{\textit{Theorem 2.1}}$ *For a circle of radius R in $\mathbb{R}^2$*

$$\kappa_\sigma(z) = \frac{2\sqrt{2}}{\sigma R^\sigma} B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) n(z)$$

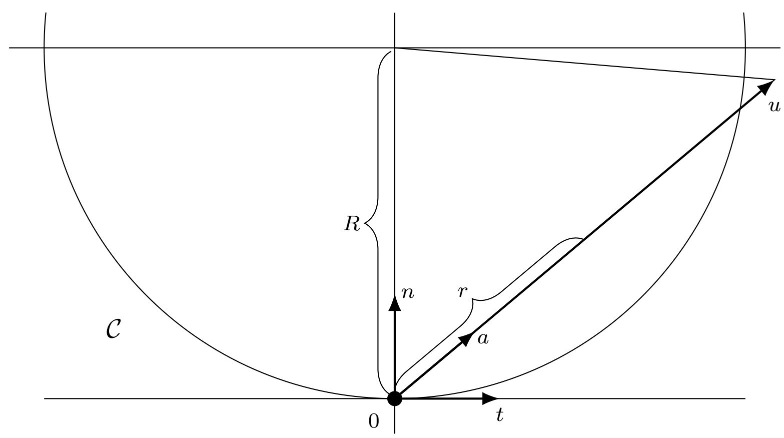
Proof. To begin, notice $\kappa_\sigma(z) = \kappa_\sigma(0) = \kappa_\sigma$ by symmetry and that

$$\kappa_\sigma \cdot t = \left(\int_{\mathcal{A}^+_{\mathsf{Even}}} - \int_{\mathcal{A}^+_{\mathsf{Odd}}} \right) \frac{(a\cdot t)(b\cdot t) - (b\cdot t)(a\cdot t)}{r^{1+\sigma}} \, d\mathcal{H}^2(a,b,r) = 0,$$

thus we only need to worry about $\kappa_\sigma \cdot n$; i.e. we wish to compute

$$\kappa_\sigma \cdot n = \int_{\mathcal{A}^+} \overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}(a,b,r) \frac{(a\cdot t)(b\cdot n) - (b\cdot t)(a\cdot n)}{r^{1+\sigma}} \, d\mathcal{H}^2(a,b,r),$$

where $\mathcal{A}^+ = \mathcal{A}^+_{\mathsf{Even}} \cup \mathcal{A}^+_{\mathsf{Odd}}$ and, for any E $\overline{\chi}_E = \chi_E - \chi_{E^c}$ (i.e. $\overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}} = \chi_{\mathcal{A}^+_{\mathsf{Even}}} - \chi_{\mathcal{A}^+_{\mathsf{Odd}}}$). The picture below shows the geometric relationship between a given a, b, r with t, n and $u := 2ra$:



For any P on the circle the distance between 0 and P is $2R\sin\theta$ where θ is the angle between $\overrightarrow{0P}$ and t (you can see this by e.g. bisecting the triangle formed by $0, P$ and the center of the circle). Consequently, we only have an odd number of intersections when $(a\cdot n) > 0$ and when $|u| > 2R\sin\theta$, where θ is the angle between u and t. Notably $|u| = 2r$ and since a is a unit vector we have

$$(a\cdot n)^2 + (a\cdot t)^2 = 1, \; (a\cdot t) = |a||t|\cos\theta = \cos\theta \implies (a\cdot n)^2 = 1 - \cos^2\theta = \sin^2\theta \implies (a\cdot n) = |\sin\theta|.$$

Since $(a\cdot n) > 0$ for any (a,b,r) giving odd intersections, we know $\theta \in [0,\pi]$ and thus $\sin\theta = |\sin\theta|$, so $|u| > 2R\sin\theta \iff r > R(a\cdot n)$. Finally putting all this together we can explicitly write

$$\overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}(a,b,r) = \begin{cases} 1 & (a\cdot n) > 0, \, r < R(a\cdot n), \\ 1 & (a\cdot n) < 0, \\ -1 & \text{otherwise} \end{cases} . \tag{2}$$

The picture also gives us helps us characterize b in terms of a. Since we only care about the b such that $(b \cdot t) > 0$, $(a \cdot n) > 0 \implies b = a^{\perp}$, i.e. b is a rotated clockwise. Otherwise, b is a counterclockwise rotation so that $b = -a^{\perp}$, thus

$$b = \operatorname{sgn}(a \cdot n) a^{\perp}. \tag{3}$$

We're able to use (2), (3) to simplify our integrand i.e.

$$\begin{aligned}
\kappa_\sigma \cdot n &= \int_{\mathcal{A}^+} \overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}(a,b,r)\operatorname{sgn}(a \cdot n) \frac{(a \cdot t)\big(a^{\perp} \cdot n\big) - \big(a^{\perp} \cdot t\big)(a \cdot n)}{r^{1+\sigma}}\, d\mathcal{H}^2(a,b,r) \\
&= \int_{\mathcal{A}^+} \overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}(a,b,r)\operatorname{sgn}(a \cdot n) \frac{-(a \cdot t)(a \cdot t) - (a \cdot n)(a \cdot n)}{r^{1+\sigma}}\, d\mathcal{H}^2(a,b,r) \\
&= -\int_{\mathcal{A}^+} \frac{\overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}(a,b,r)\operatorname{sgn}(a \cdot n)}{r^{1+\sigma}}\, d\mathcal{H}^2(a,b,r).
\end{aligned}$$

Now, motivated by the picture above we do a change of variables $u := \phi(a,b,r) = 2ra$. We have

$$\nabla\phi(a,b,r) = \begin{array}{cc}
\begin{array}{cc} \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) \end{array} & \\
\begin{pmatrix} \sqrt{2}r(b \cdot t) & 2(a \cdot t) \\ \sqrt{2}r(b \cdot n) & 2(a \cdot n) \end{pmatrix} & \begin{array}{c} t \\ n \end{array}
\end{array}$$

so

$$\begin{aligned}
\big(\overline{\mathcal{J}}\phi(a,b,r)\big)^2 = \big|\nabla\phi^T(a,b,r)\nabla\phi(a,b,r)\big| &= \left|\begin{pmatrix} \sqrt{2}r(b \cdot t) & \sqrt{2}r(b \cdot n) \\ 2(a \cdot t) & 2(a \cdot n) \end{pmatrix}\begin{pmatrix} \sqrt{2}r(b \cdot t) & 2(a \cdot t) \\ \sqrt{2}r(b \cdot n) & 2(a \cdot n) \end{pmatrix}\right| \\
&= \left|\begin{pmatrix} 2r^2 & 2\sqrt{2}r((b \cdot t)(a \cdot t) + (b \cdot n)(a \cdot n)) \\ 2\sqrt{2}r((b \cdot t)(a \cdot t) + (b \cdot n)(a \cdot n)) & 4 \end{pmatrix}\right| \\
&= 8r^2 - 8((a \cdot t))^2(b \cdot t)^2 + (a \cdot n)^2(b \cdot n)^2 + 2(a \cdot n)(a \cdot t)(b \cdot n)(b \cdot t) \\
&= 8r^2 - 8\Big((a \cdot t)^2(a \cdot n)^2 + (a \cdot n)^2(a \cdot t)^2 - 2(a \cdot n)^2(a \cdot t)^2\Big) \\
\implies \overline{\mathcal{J}}\phi = 2\sqrt{2}r.
\end{aligned}$$

Combining this with the fact that

$$\phi\big(\mathcal{A}^+\big) = \mathbb{R}^2,\ \phi^{-1}(u) = \left(\frac{u}{|u|}, \operatorname{sgn}(u \cdot n)\frac{u^{\perp}}{|u|}, \frac{|u|}{2}\right),$$

we now have

$$\kappa_\sigma \cdot n = -\int_{\mathbb{R}^2} \frac{\overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}\Big(\frac{u}{|u|}, \operatorname{sgn}(u \cdot n)\frac{u^{\perp}}{|u|}, \frac{|u|}{2}\Big)\operatorname{sgn}(u \cdot n)}{\Big(\frac{|u|}{2}\Big)^{1+\sigma}} \frac{1}{\sqrt{2}|u|}\, d\mathcal{H}^2(u).$$

To further simplify, put $\Pi^+ := \Big\{u \in \mathbb{R}^2 \mid (u \cdot n) < 0 \text{ or } \frac{|u|}{2} < R\Big(\frac{u}{|u|} \cdot n\Big)\Big\}$ and notice

$$\overline{\chi}_{\mathcal{A}^+_{\mathsf{Even}}}\left(\frac{u}{|u|}, \operatorname{sgn}(u \cdot n)\frac{u^{\perp}}{|u|}\right) = \begin{cases} 1 & (u \cdot n) > 0,\ \frac{|u|}{2} < R\Big(\frac{u}{|u|} \cdot n\Big), \\ 1 & (u \cdot n) < 0, \\ -1 & \text{otherwise} \end{cases} = \overline{\chi}_{\Pi^+}(u).$$

So, further simplifying and leveraging 4.1 we have

$$\kappa_\sigma \cdot n = -2^{\sigma+1/2} \int_{\mathbb{R}^2} \frac{\overline{\chi}_{\Pi^+}(u)\mathrm{sgn}(u \cdot n)}{|u|^{2+\sigma}}\, d\mathcal{H}^2(u) = -2^{\sigma+1/2} \frac{-2^{1-\sigma}}{\sigma R^\sigma} B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) = \frac{2^{3/2}}{\sigma R^\sigma} B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right).$$

□

3 Non-local Curvature of a Circle in N-D

$\boxed{\textit{Theorem 3.1}}$ *For a circle of radius R in $\mathbb{R}^n$*

$$\kappa_\sigma(z) = \frac{2^{1/2}\pi^{n-1}}{\sigma R^\sigma} \frac{\Gamma(1+\sigma/2)\Gamma((1-\sigma)/2)}{\Gamma((n+1)/2)\Gamma((n+\sigma)/2)\Gamma(1-\sigma/2)} n(z)$$

Proof. Our strategy for this proof is as follows

1 Recognize the similarity with the 2D case and slice the domain along each u in (4.1).

2 Show $\kappa_\sigma \cdot m = 0\ \forall\, m \in \{n\}^\perp$ so that we can focus on $\kappa_\sigma \cdot n$.

3 Slice along each radius r to further simplify the computation.

4 Show the domain of the inner-most integral with u, r fixed is a pair of spheres.

5 Evaluate the final integrals.

As with the 2D case, due to symmetry we know $\kappa_\sigma(z) = \kappa_\sigma$, and

$$\begin{aligned} &\frac{\omega_{n-3}^2}{\sigma R^\sigma \sqrt{2}} B\left(\frac{3}{2}, \frac{n-2}{2}\right) B\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right) B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) \\ &\quad= \frac{1}{\sigma R^\sigma \sqrt{2}} \left(\frac{2\pi^{n/2-1}}{\Gamma(n/2-1)}\right)^2 \frac{\Gamma(3/2)\Gamma(n/2-1)}{\Gamma((n+1)/2)} \frac{\Gamma(\sigma/2+1)\Gamma(n/2-1)}{\Gamma((n+\sigma)/2)} \frac{\Gamma(1/2)\Gamma((1-\sigma)/2)}{\Gamma(1-\sigma/2)} \\ &\quad= \frac{2^{3/2}\pi^{n-2}}{\sigma R^\sigma} \frac{\Gamma(3/2)\Gamma(1/2)\Gamma(1+\sigma/2)\Gamma((1-\sigma)/2)}{\Gamma((n+1)/2)\Gamma((n+\sigma)/2)\Gamma(1-\sigma/2)} \\ &\quad= \frac{2^{1/2}\pi^{n-1}}{\sigma R^\sigma} \frac{\Gamma(1+\sigma/2)\Gamma((1-\sigma)/2)}{\Gamma((n+1)/2)\Gamma((n+\sigma)/2)\Gamma(1-\sigma/2)}, \end{aligned} \tag{4}$$

so that our task is to show

$$\kappa_\sigma = \frac{\omega_{n-3}^2}{\sigma R^\sigma \sqrt{2}} B\left(\frac{3}{2}, \frac{n-2}{2}\right) B\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right) B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) n(z).$$

1 Similarity to 2D

Since $\mathcal{C} \subset \mathrm{lsp}\{t, n\}$, we can simplify our calculation by slicing at each $u \in \mathbb{R}^2$ representing $\mathcal{D}(ra, b, r) \cap \mathcal{S}$. To that end, put $\Psi : \mathcal{A}^+ \to \mathbb{R}^2$ so that

$$\mathcal{D}(ra, b, r) \cap S = \{t\Psi(a, b, r) \mid t \in [0, 1]\}.$$

In order to to determine an exact formula for Ψ, note the following, where $u := \Psi(a, b, r)$:

- There is an isoceles triangle formed by the center of $\mathcal{D}(ra,b,r)$ and the endpoints of $\mathcal{D}(ra,b,r)\cap S$, and thus the component of ra in the direction of u must be half of u's length, i.e.

$$ra\cdot\frac{u}{|u|}=\frac{|u|}{2}\implies 2(ra\cdot u)=|u|^2=u\cdot u, \tag{5}$$

- Since $\mathcal{D}(ra,b,r)\cap\mathcal{S}\subset\mathcal{D}(ra,b,r),\ \forall x\in\mathcal{D}(ra,b,r)\cap\mathcal{S}$

$$b\cdot x=0\implies b\cdot u=0. \tag{6}$$

Since $u\in\mathcal{S}$, we can expand via the $\{t,n\}$ basis, take into account (6), and the fact that $b\cdot t>0$ to see

$$0=b\cdot u=b_tu_t+b_nu_n\implies u_t=\frac{-b_nu_n}{b_t}. \tag{7}$$

Substituting this back into (5) we have

$$\begin{aligned}2ra_nu_n-2ra_t\frac{b_nu_n}{b_t}=u_n^2+\frac{b_n^2u_n^2}{b_t^2}&\implies u_n^2\left(1+\frac{b_n^2}{b_t^2}\right)+2ru_n\left(\frac{a_tb_n}{b_t}-a_n\right)=0\\&\implies u_n=0\vee u_n\frac{b_t^2+b_n^2}{b_t^2}+2r\frac{a_tb_n-a_nb_t}{b_t}=0.\end{aligned}$$

Notably $u_n\neq 0$ since otherwise (7) would force $u_t=0$, i.e. $u=0$, which only occurs for a measure-zero set of $(a,b,r)\in\mathcal{A}^+$. Solving the above equation for u_n we find

$$u_n=-2r\frac{a_tb_n-a_nb_t}{b_t}\frac{b_t^2}{b_t^2+b_n^2}=2r\frac{a_nb_t-a_tb_n}{b_t^2+b_n^2}b_t.$$

Putting $\mathcal{P}_S:=(t\otimes t)+(n\otimes n)$ and plugging this back into (7) we find

$$\Psi(a,b,r)=2r\frac{a_nb_t-a_tb_n}{|\mathcal{P}_Sb^\perp|^2}(b_tn-b_nt)=2r\frac{a_tb_n-a_nb_t}{|\mathcal{P}_Sb^\perp|^2}\mathcal{P}_Sb^\perp$$

We can further rewrite this by noticing

$$\mathcal{P}_Sb^\perp\cdot a=(b_tt+b_nn)^\perp\cdot a=(b_nt-b_tn)\cdot a=b_na_t-b_ta_n,$$

and putting

$$p(b)=\frac{\mathcal{P}_Sb^\perp}{|\mathcal{P}_Sb^\perp|}, \tag{8}$$

so that

$$\Psi(a,b,r)=2r\frac{\mathcal{P}_Sb^\perp\cdot a}{|\mathcal{P}_Sb^\perp|^2}\mathcal{P}_Sb^\perp=2r(p(b)\cdot a)p(b)=2r(p(b)\otimes p(b))a. \tag{9}$$

Put $\mathcal{E}(u):=\Psi^{-1}(\{u\})$, with 4.2 we see

$$\kappa_\sigma=\frac{1}{\sqrt{2}}\int_{\mathbb{R}^2}\frac{1}{|u|}\int_{\mathcal{E}(u)}\overline{\chi}_{\mathcal{A}^+}(a,b,r)\frac{(a\cdot t)b-(b\cdot t)a}{r^{1+\sigma}}\frac{\left|\mathcal{P}_Sb^\perp\right|}{\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}\,d\mathcal{H}^{2n-4}(a,b,r)\,d\mathcal{H}^2(u).$$

As noticed above $\mathcal{C} \in \mathcal{S}$ so $\overline{\chi}_{\mathcal{A}^+}(a,b,r)$ must only depend on where $\mathcal{D}(a,b,r) \cap \mathcal{S}$, i.e. u. Putting

$$\Pi^+ := \left\{ u \in \mathcal{S} \mid (u \cdot n) < 0 \text{ or } \frac{|u|}{2} < R\left(\frac{u}{|u|} \cdot n\right) \right\}$$

we must have $\overline{\chi}_{\mathcal{A}^+}(a,b,r) = \overline{\chi}_{\Pi^+}(u)$[1], and so

$$\kappa_\sigma = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^2} \frac{\overline{\chi}_{\Pi^+}(u)}{|u|} \int_{\mathcal{E}(u)} \frac{(a \cdot t)b - (b \cdot t)a}{r^{1+\sigma}} \frac{\left|\mathcal{P}_S b^\perp\right|}{\sqrt{2 - \frac{|u|^2}{4r^2}} \sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}} \, d\mathcal{H}^{2n-4}(a,b,r) \, d\mathcal{H}^2(u). \tag{10}$$

2 Normal Component Projection

We want to show $\kappa_\sigma = (n \otimes n)\kappa_\sigma$. Just as with the 2D case it's trivial to see $(\kappa_\sigma \cdot t) = 0$. In order to show $(\kappa_\sigma \cdot m) = 0$ for $m \in \mathcal{U}\Big(\{n,t\}^\perp\Big)$, we'll show that the integrand of (10) dotted with m

$$\frac{\overline{\chi}_{\Pi^+}(u)}{|u|} \frac{(a \cdot t)(b \cdot m) - (b \cdot t)(a \cdot m)}{r^{1+\sigma}} \frac{\left|\mathcal{P}_S b^\perp\right|}{\sqrt{2 - \frac{|u|^2}{4r^2}} \sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}}, \tag{11}$$

is odd over a symmetric domain.

Specifically, we hope to show, for $R_m := I - 2(m \otimes m)$, $(u,a,b,r) \to (u, R_m a, R_m b, r)$ is an isomorphism that the integrand is odd under. From (8) $p(R_m b) = p(b)$ and thus (9) says

$$\Psi(R_m a, R_m b, r) = 2r(p(R_m b) \cdot R_m a)p(R_m b) = 2r(p(b) \cdot R_m a)p(b) = 2r(p(b) \cdot a)p(b) = \Psi(a,b,r).$$

This tells us that u is invariant under $(a,b) \to (R_m a, R_m b)$ transformation. By a similar observation $\left|\mathcal{P}_S b^\perp\right|$ is invariant under a $b \to R_m b$ transformation, and thus the task of showing (11) is odd under $(u,a,b,r) \to (u, R_m a, R_m b, r)$ reduces to showing

$$J(a,b) := (a \cdot t)(b \cdot m) - (b \cdot t)(a \cdot m)$$

is odd under the $(a,b) \to (R_m a, R_m b)$ transformation. Indeed we can see

$$\begin{aligned}
J(R_m a, R_m b) &= (R_m a \cdot t)(R_m b \cdot m) - (R_m b \cdot t)(R_m a \cdot m) \\
&= ((a - 2(a \cdot m)m) \cdot t)((b - 2(b \cdot m)m) \cdot m) - ((b - 2(b \cdot m)m) \cdot t)((a - 2(a \cdot m)m) \cdot m) \\
&= ((a \cdot t) - 2(a \cdot m)(m \cdot t))((b \cdot m) - 2(b \cdot m)) - ((b \cdot t) - 2(b \cdot m)(m \cdot t))((a \cdot m) - 2(a \cdot m)) \\
&= ((a \cdot t) - 0)(-(b \cdot m)) - ((b \cdot t) - 0)(-(a \cdot m)) \\
&= -(a \cdot t)(b \cdot m) + (b \cdot t)(a \cdot m) = -J(a,b,r).
\end{aligned}$$

Lastly due to the invariance of u, the fact that R_m is an isomorphism, $(R_m b \cdot t) = (b \cdot t)$ and

$$(R_m a \cdot R_m b) = ((a - 2(m \cdot a)m) \cdot (b - 2(m \cdot b)m)) = (a \cdot b)$$

we have $(a,b,r) \in \mathcal{E}(u) \iff (R_m a, R_m b, r) \in \mathcal{E}(u)$. Thus, for the rest of the proof we may focus on $\kappa_\sigma \cdot n$. Before expanding our integral, notice

$$(a \cdot t)(b \cdot n) - (b \cdot t)(a \cdot n) = ((b \cdot n)t - (b \cdot t)n) \cdot a = -\mathcal{P}_S b^\perp \cdot a$$

[1] Π^+ is notably the same as in the 2D case. This comes from the inherent geometry of the circle, i.e. this set includes all u who point below the circle (i.e. $(u \cdot n) < 0$), or u that lie within the interior of the circle, i.e. $|u| < 2R\sin\theta$ where θ is the angle between t and u.

and since $u = \frac{2r}{|\mathcal{P}_S b^\perp|^2}\big(\mathcal{P}_S b^\perp \cdot a\big)\mathcal{P}_S b^\perp \implies \big|\mathcal{P}_S b^\perp \cdot a\big| = \frac{|u|\left|\mathcal{P}_S b^\perp\right|}{2r}$, $\operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot a\big) = \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)$ we have

$$\kappa_\sigma \cdot n = \frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\mathcal{E}(u)} \frac{1}{r^{2+\sigma}} \frac{\operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}}\, d\mathcal{H}^{2n-4}(a,b,r)\, d\mathcal{H}^2(u). \tag{12}$$

3 Slicing Along Radii

Next, to further simplify (12), let's slice along each r, i.e. put $\Phi : \mathcal{E}(u) \to \mathbb{R}^+$ given by

$$\Phi(a,b,r) = r. \tag{13}$$

Put $\mathcal{E}(u,r) = \Phi^{-1}(\{r\})$, so that with the co-area calculation in 4.3 we find (12) becomes

$$\begin{aligned}
&\frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\mathcal{E}(u)} \frac{1}{r^{2+\sigma}} \frac{\operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}}\, d\mathcal{H}^{2n-4}(a,b,r)\, d\mathcal{H}^2(u) \\
&\quad = \frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\Phi(\mathcal{E}(u))} \int_{\mathcal{E}(u,r)} \frac{1}{r^{3+\sigma}} \frac{\operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{1 - \frac{|u|^2}{4r^2}}}\, d\mathcal{H}^{2n-5}(a,b)\, dr\, d\mathcal{H}^2(u) \\
&\quad = \frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\Phi(\mathcal{E}(u))} \frac{1}{r^{3+\sigma}\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{1 - \frac{|u|^2}{4r^2}}} \int_{\mathcal{E}(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^{2n-5}(a,b)\, dr\, d\mathcal{H}^2(u)
\end{aligned} \tag{14}$$

4 Splitting Spheres

Since the inner most integrand in (14) only depends on b, and by the uniform nature of

$$\mathcal{E}(u,r) = \big\{(a,b) \in \mathcal{U}_2^\perp \mid b \cdot t > 0,\ u \cdot b = 0 \text{ and } 2r(u \cdot a) = (u \cdot u)\big\},$$

it's natural to slice on each b, so put $\Xi : \mathcal{E}(u,r) \to \mathcal{E}_2(u,r)$ where $\Xi(a,b) = b$. With the calculation done in 4.4 we can see, after putting $\mathcal{E}(u,r,b) = \Xi^{-1}(b)$ and including 4.4, 4.5, (14) becomes

$$\begin{aligned}
&\frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\Phi(\mathcal{E}(u))} \frac{}{r^{3+\sigma}\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{1 - \frac{|u|^2}{4r^2}}} \int_{\mathcal{E}_2(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2 \frac{\mathcal{H}^{n-3}(\mathcal{E}(u,r,b))}{\left(2 - \frac{|u|^2}{4r^2}\right)^{-1/2}}\, d\mathcal{H}^{n-2}(b)\, dr\, d\mathcal{H}^2(u) \\
&= \frac{-\omega_{n-3}}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\Phi(\mathcal{E}(u))} \frac{\left(1 - \frac{|u|^2}{4r^2}\right)^{(n-4)/2}}{r^{3+\sigma}} \int_{\mathcal{E}_2(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^{n-2}(b)\, dr\, d\mathcal{H}^2(u)
\end{aligned} \tag{15}$$

To evaluate this second integral we'll need one more application of the co-area formula; we'll split using

$$\zeta : \mathcal{E}_2(u,r) \to [-1,1] \text{ given by } \zeta(b) = b \cdot \frac{u^\perp}{|u^\perp|}.$$

This is a natural transformation since

$$\mathcal{P}_S b^\perp = \left(\left(b \cdot \frac{u}{|u|}\right)\frac{u}{|u|} + \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\frac{u^\perp}{|u^\perp|}\right)^\perp = -\left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\frac{u}{|u^\perp|} = -\zeta(b)\frac{u}{|u^\perp|} \tag{16}$$

Further, since $\mathbb{R}^n = \operatorname{lsp}\left\{\frac{u}{|u|}, \frac{u^\perp}{|u^\perp|}, f_1, f_2, \dots f_{n-2}\right\}$ for some orthonormal f_i, it's true that

$$\zeta(b)^2 + (b\cdot f_1)^2 + (b\cdot f_2)^2 + \cdots + (b\cdot f_{n-2})^2 = 1 \implies (b\cdot f_1)^2 + (b\cdot f_2)^2 + \cdots + (b\cdot f_{n-2})^2 = 1 - \zeta(b)^2,$$

and so

$$\begin{aligned}\zeta^{-1}(v) &= \left\{b \in \mathcal{U}\Big(\{u\}^\perp\Big) \mid v = \left(b\cdot \frac{u^\perp}{|u^\perp|}\right)\right\}\\ &= \left\{b \in \mathcal{U}\Big(\{u\}^\perp\Big) \mid (b\cdot f_1)^2 + (b\cdot f_2)^2 + \cdots + (b\cdot f_{n-2})^2 = 1 - v^2\right\}\\ &= S_{n-3}\Big(\sqrt{1-v^2}\Big) \qquad (17)\end{aligned}$$

Combining (16), (17) and 4.6, the inner most integral of (15) becomes

$$\begin{aligned}\int_{\mathcal{E}_2(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot u\big)\big|\mathcal{P}_S b^\perp\big|^2\, d\mathcal{H}^{n-2}(b) &= \int_{\zeta(\mathcal{E}_2(u,r))} \operatorname{sgn}\left(-v\frac{u}{|u^\perp|}\cdot u\right)v^2\,\frac{\mathcal{H}^{n-3}\big(\zeta^{-1}(v)\big)}{\sqrt{1-v^2}}\,dv\\ &= -\omega_{n-3}\int_{\zeta(\mathcal{E}_2(u,r))} \operatorname{sgn}(v)v^2\big(1-v^2\big)^{(n-4)/2}\,dv. \qquad (18)\end{aligned}$$

To better understand $\zeta(\mathcal{E}_2(u,r))$ notice $b\in\mathcal{E}_2(u,r) \implies b\cdot t > 0$, so

$$0 < b\cdot t = \left(b\cdot\frac{u}{|u|}\right)\left(t\cdot\frac{u}{|u|}\right) + \left(b\cdot\frac{u^\perp}{|u^\perp|}\right)\left(t\cdot\frac{u^\perp}{|u^\perp|}\right) = \left(b\cdot\frac{u^\perp}{|u^\perp|}\right)\left(t\cdot\frac{u^\perp}{|u^\perp|}\right),$$

i.e. $\zeta(\mathcal{E}_2(u,r)) = [-1,0]$ if $\operatorname{sgn}\big(t\cdot u^\perp\big) = -1$ and $[0,1]$ otherwise. Since the integrand in (18) is odd, we can integrate over $[0,1]$ and multiply by $\operatorname{sgn}\big(t\cdot u^\perp\big)$ to take this into account. As we expand (18), note $\operatorname{sgn}\big(t\cdot u^\perp\big) = -\operatorname{sgn}(n\cdot u)$, so that

$$\begin{aligned}\omega_{n-3}\int_{\zeta(\mathcal{E}_2(u,r))} \operatorname{sgn}(v)v^2\big(1-v^2\big)^{(n-4)/2}\,dv &= \operatorname{sgn}(n\cdot u)\omega_{n-3}\int_0^1 v^2\big(1-v^2\big)^{(n-4)/2}\,dv\\ &= \frac{\operatorname{sgn}(n\cdot u)\omega_{n-3}}{2}\int_0^1 v^{1/2}(1-v)^{(n-4)/2}\,dv\\ &= \frac{\operatorname{sgn}(n\cdot u)\omega_{n-3}}{2}B\left(\frac{3}{2},\frac{n-2}{2}\right) \qquad (19)\end{aligned}$$

Putting (19) back into (15) our calculation becomes

$$\kappa_\sigma\cdot n = \frac{-\omega_{n-3}^2}{4\sqrt{2}}B\left(\frac{3}{2},\frac{n-2}{2}\right)\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(n\cdot u)\int_{\Phi(\mathcal{E}(u))}\frac{\left(1-\frac{|u|^2}{4r^2}\right)^{(n-4)/2}}{r^{3+\sigma}}\,dr\,d\mathcal{H}^2(u). \qquad (20)$$

5 Evaluating Integrals

To evaluate the final integrals we recall $\mathcal{E}(u) = \Psi^{-1}(\{u\})$ where Ψ is given in (9). By definition of Φ in (13) we have

$$\Phi(\mathcal{E}(u)) = \Phi\big(\Psi^{-1}(\{u\})\big) = \big\{r \mid 2r(p(b)\cdot a)p(b) = u \text{ for some } (a,b,r)\in\mathcal{A}^+\big\} = \{r \mid r \geq |u|2\},$$

so that the inner integral in (20), after applying the transformation $r \to \frac{|u|}{2}s^{-1/2}$, is

$$\begin{aligned}
\int_{\Phi(\mathcal{E}(u))} \frac{\left(1-\frac{|u|^2}{4r^2}\right)^{(n-4)/2}}{r^{3+\sigma}}\,dr &= \int_{|u|/2}^{\infty}\left(r^2-\frac{|u|^2}{4}\right)^{(n-4)/2} r^{1-\sigma-n}\,dr \\
&= \int_1^0 \left(\frac{|u|^2}{4}\frac{1}{s}-\frac{|u|^2}{4}\right)^{(n-4)/2}\left(\frac{|u|}{2}\right)^{1-\sigma-n} s^{(n+\sigma-1)/2}\frac{-|u|}{4}s^{-3/2}\,ds \\
&= \left(\frac{|u|}{2}\right)^{(n-4)+1-\sigma-n}\frac{|u|}{4}\int_0^1\left(\frac{1-s}{s}\right)^{(n-4)/2} s^{(n+\sigma-4)/2}\,ds \\
&= \frac{|u|^{-2-\sigma}}{2^{-1-\sigma}}\int_0^1 (1-s)^{(n-4)/2}s^{\sigma/2}\,ds \\
&= \frac{2^{1+\sigma}}{|u|^{2+\sigma}}B\left(\frac{\sigma+2}{2},\frac{n-2}{2}\right). \qquad (21)
\end{aligned}$$

Plugging (21) back into (20) our calculation simplifies to

$$\begin{aligned}
\kappa_\sigma \cdot n &= \frac{-\omega_{n-3}^2}{4\sqrt{2}}B\left(\frac{3}{2},\frac{n-2}{2}\right)\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\mathrm{sgn}(n\cdot u)\int_{\Phi(\mathcal{E}(u))}\frac{\left(1-\frac{|u|^2}{4r^2}\right)^{(n-4)/2}}{r^{3+\sigma}}\,dr\,d\mathcal{H}^2(u) \\
&= -2^{1+\sigma-5/2}\omega_{n-3}^2 B\left(\frac{3}{2},\frac{n-2}{2}\right)B\left(\frac{\sigma+2}{2},\frac{n-2}{2}\right)\int_{\mathbb{R}^2}\frac{\overline{\chi}_{\Pi^+}(u)\mathrm{sgn}(n\cdot u)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u) \\
&= -2^{\sigma-3/2}\omega_{n-3}^2 B\left(\frac{3}{2},\frac{n-2}{2}\right)B\left(\frac{\sigma+2}{2},\frac{n-2}{2}\right)\frac{-2^{1-\sigma}}{\sigma R^\sigma}B\left(\frac{1}{2},\frac{1-\sigma}{2}\right) \qquad (22) \\
&= \frac{\omega_{n-3}^2}{\sigma R^\sigma\sqrt{2}}B\left(\frac{3}{2},\frac{n-2}{2}\right)B\left(\frac{\sigma+2}{2},\frac{n-2}{2}\right)B\left(\frac{1}{2},\frac{1-\sigma}{2}\right), \qquad (23)
\end{aligned}$$

where in (22) we leveraged 4.1. Finally our desired equality follows after combining (23) with (4). ☐

4 Appendix

1 Supplementary Calculations

Lemma 4.1 *For* $\Pi^+ := \left\{u \in \mathbb{R}^2 \mid (u\cdot n) < 0 \text{ or } \frac{|u|}{2} < R\left(\frac{u}{|u|}\cdot n\right)\right\}$ *we have*

$$\int_{\mathbb{R}^2}\frac{\overline{\chi}_{\Pi^+}(u)\mathrm{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u) = \frac{-2^{1-\sigma}}{\sigma R^\sigma}B\left(\frac{1}{2},\frac{1-\sigma}{2}\right)$$

Proof. Begin by putting

$$I = \int_{\mathbb{R}^2}\frac{\overline{\chi}_{\Pi^+}(u)\mathrm{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u).$$

Notably, this integral needs to be taken in a principal value sense, i.e.

$$I = \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^2\setminus B_\epsilon}\frac{\overline{\chi}_{\Pi^+}(u)\mathrm{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u),$$

where $B_\epsilon = \{x \in \mathbb{R}^2 \mid |x| = \epsilon\}$. Next, substituting $u = r\cos\theta t + r\sin\theta n$ shows that

$$I = \lim_{\epsilon \downarrow 0} \int_0^{2\pi} \int_\epsilon^\infty \frac{\overline{\chi}_{\Pi^+}(r\cos\theta t + r\sin\theta n)\operatorname{sgn}(\sin\theta)}{r^{2+\sigma}} r\,dr\,d\theta,$$

and further noting

$$\overline{\chi}_{\Pi^+}(r\cos\theta t + r\sin\theta n) = \begin{cases} 1 & \theta \in [0,\pi], \frac{r}{2} < R\sin\theta \\ 1 & \theta \in [\pi, 2\pi] \\ 0 & \text{otherwise} \end{cases}$$

$$\operatorname{sgn}(\sin\theta) = \begin{cases} 1 & \theta \in [0,\pi] \\ -1 & \theta \in [\pi, 2\pi] \end{cases},$$

means

$$\begin{aligned}
I &= \lim_{\epsilon \downarrow 0} \int_0^\pi \left(\int_\epsilon^{2R\sin\theta} \frac{1 \cdot 1}{r^{1+\sigma}}\,dr + \int_{2R\sin\theta}^\infty \frac{-1 \cdot 1}{r^{1+\sigma}}\,dr \right) + \int_\pi^{2\pi} \left(\int_\epsilon^\infty \frac{1 \cdot (-1)}{r^{1+\sigma}}\,dr \right) d\theta \\
&= \lim_{\epsilon \downarrow 0} \frac{1}{-\sigma} \int_0^\pi \left(r^{-\sigma}\big|_\epsilon^{2R\sin\theta} - r^{-\sigma}\big|_{2R\sin\theta}^\infty \right) d\theta - \frac{1}{-\sigma} \int_\pi^{2\pi} r^{-\sigma}\big|_\epsilon^\infty\,d\theta \\
&= \frac{-1}{\sigma} \lim_{\epsilon \downarrow 0} \int_0^\pi (2R\sin\theta)^{-\sigma} - \epsilon^{-\sigma} + (2R\sin\theta)^{-\sigma}\,d\theta + \int_\pi^{2\pi} \epsilon^{-\sigma}\,d\theta \\
&= \frac{-1}{\sigma} \lim_{\epsilon \downarrow 0} 2 \int_0^\pi (2R\sin\theta)^{-\sigma}\,d\theta - \pi\epsilon^{-\sigma} + \pi\epsilon^{-\sigma} \\
&= \frac{-2^{1-\sigma}}{\sigma R^\sigma} \int_0^\pi (\sin\theta)^{-\sigma}\,d\theta \\
&= \frac{-2^{1-\sigma}}{\sigma R^\sigma} \int_{\pi/2}^{-\pi/2} (\cos\theta)^{-\sigma}\,(-d\theta) \text{ via } \theta \to \frac{\pi}{2} - \theta \\
&= \frac{-2^{1-\sigma}}{\sigma R^\sigma} 2 \int_0^{\pi/2} (\cos\theta)^{-\sigma}\,d\theta = \frac{-2^{1-\sigma}}{\sigma R^\sigma} B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right)
\end{aligned}$$

□

$\boxed{\textit{Lemma 4.2}}$ *For $\Psi : \mathcal{A}^+ \to \mathbb{R}^2$ given by $\Psi(a,b,r) = 2r(p(b) \otimes p(b))a$, where $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$*

$$\mathcal{J}\Psi(a,b,r) = \frac{\sqrt{2}|u|}{|\mathcal{P}_S b^\perp|} \sqrt{2 - \frac{|u|^2}{4r^2}} \sqrt{r^2 - (r^2 - 1)\frac{|u|^2}{4r^2}}.$$

Proof. Put $c = \frac{u - (u \cdot a)a}{\sqrt{(u \cdot u) - (u \cdot a)}}$ and $\{e_i\}_{i=1}^{n-3}$ so that $\{a, b, c, e_1, ..., e_{n-3}\}$ is an orthonormal basis spanning $\mathbb{R}^n$. We can use this basis to characterize

$$\mathcal{T}_{(a,b,r)}(\mathcal{A}^+) = \operatorname{lsp}\{(e_i, 0, 0), (0, e_i, 0), (c, 0, 0), (0, c, 0), \epsilon_{a,b}, (0, 0, 1) \mid i = 1, ...(n-3)\},$$

where $\epsilon_{a,b} := \frac{1}{\sqrt{2}}(b, -a, 0)$. Additionally, we can use the fact that $p(b), p(b)^\perp$ are an orthonormal basis spanning $\mathcal{S}$ to see

$\mathcal{T}_{\Psi(a,b,r)}(\mathcal{S}) = \operatorname{lsp}\{p(b), p(b)^\perp\}$. Next, we compute derivatives along flows as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(e_i,0,0)}(s)\big)\Big|_{s=0} &= 2r(p(b)\otimes p(b))e_i = 2r(p(b)\cdot e_i)p(b) = 0,\\
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(c,0,0)}(s)\big)\Big|_{s=0} &= 2r(p(b)\otimes p(b))c = 2r(p(b)\cdot c)p(b),\\
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(0,0,1)}(s)\big)\Big|_{s=0} &= 2(p(b)\cdot a)p(b).
\end{aligned}$$

For our last calculations we'll need the following, for $\beta : \mathbb{R}^1 \to \mathcal{U}$ such that $\beta(0) = b$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}p(\beta(s))\Big|_{s=0} &= \frac{\mathcal{P}_S\beta'(0)^\perp}{|\mathcal{P}_S b^\perp|} - \frac{1}{|\mathcal{P}_S b^\perp|^3}\mathcal{P}_S b^\perp \otimes \big(\mathcal{P}_S\beta'(0)^\perp\big)^T \mathcal{P}_S b^\perp = \frac{1}{|\mathcal{P}_S b^\perp|}(1-(p(b)\otimes p(b)))\mathcal{P}_S\beta'(0)^\perp\\
&= \frac{1}{|\mathcal{P}_S b^\perp|}\big(p(b)\otimes p(b) + p(b)^\perp\otimes p(b)^\perp - p(b)\otimes p(b)\big)\mathcal{P}_S\beta'(0)^\perp = \frac{(p(b)\cdot\beta'(0))}{|\mathcal{P}_S b^\perp|}p(b)^\perp. \qquad (24)
\end{aligned}$$

For $v \in \{c\} \cup \{e_i\}_1^{(n-3)}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(0,v,0)}(s)\big)\Big|_{s=0} &= 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p\big(\gamma_{(0,v,0),2}(s)\big)\Big|_{s=0}\right)\otimes p + p\otimes\left(\frac{\mathrm{d}}{\mathrm{d}s}p\big(\gamma_{(0,v,0),2}(s)\big)\Big|_{s=0}\right)\right)a\\
&= 2r\left(\left(\frac{(p(b)\cdot v)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\right)\otimes p + p\otimes\left(\frac{(p(b)\cdot v)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\right)\right)a\\
&= 2r\frac{p(b)\cdot v}{|\mathcal{P}_S b^\perp|}\big((p(b)\cdot a)p(b)^\perp + \big(p(b)^\perp\cdot a\big)p(b)\big),
\end{aligned}$$

particularly $\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(0,e_i,0)}(s)\big)\big|_{s=0} = 0$. Finally,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{\epsilon_{a,b}}(s)\big)\Big|_{s=0} &= \sqrt{2}r(p\otimes p)b + 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p\big(\gamma_{\epsilon_{a,b}}(s)\big)\Big|_{s=0}\right)\odot p\right)a\\
&= 2r\left(\left(\frac{\big(p(b)\cdot(-a/\sqrt{2})\big)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\right)\otimes p + p\otimes\left(\frac{\big(p(b)\cdot(-a/\sqrt{2}\big)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\right)\right)a\\
&= -\sqrt{2}r\frac{p(b)\cdot a}{|\mathcal{P}_S b^\perp|}\big((p(b)\cdot a)p(b)^\perp + \big(p(b)^\perp\cdot a\big)p(b)\big)
\end{aligned}$$

Altogether, after putting $p := p(b)$ we get

$$\nabla\Psi(a,b,r) = \frac{2r}{|\mathcal{P}_S b^\perp|}\begin{array}{c} \begin{array}{cccccc} (e_i,0,0) & (0,e_i,0) & (c,0,0) & (0,c,0) & \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) \end{array} \\ \begin{pmatrix} 0 & 0 & (p\cdot c)\big|\mathcal{P}_S b^\perp\big| & (p\cdot c)\big(p^\perp\cdot a\big) & -\frac{(p\cdot a)(p^\perp\cdot a)}{\sqrt{2}} & \frac{(p\cdot a)|\mathcal{P}_S b^\perp|}{r} \\ 0 & 0 & 0 & (p\cdot c)(p\cdot a) & -\frac{(p\cdot a)^2}{\sqrt{2}} & 0 \end{pmatrix} \end{array} \begin{array}{c} p \\ p^\perp \end{array}.$$

To aid in our calculation of $\mathcal{J}\Psi$ put $M := \frac{|\mathcal{P}_S b^\perp|^2}{4r^2}\nabla\Psi(a,b,r)\Psi(a,b,r)^\perp$. We have

$$\begin{aligned}
M_{1,1} &= (p\cdot c)^2|\mathcal{P}_S b^\perp|^2 + (p\cdot c)^2(p^\perp\cdot a)^2 + \frac{(p\cdot a)^2(p^\perp\cdot a)^2}{2} + \frac{(p\cdot a)^2|\mathcal{P}_S b^\perp|^2}{r^2}\\
&= \frac{(p^\perp\cdot a)^2}{2}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big) + \frac{|\mathcal{P}_S b^\perp|^2}{r^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\\
M_{1,2} = M_{2,1} &= (p\cdot c)^2(p\cdot a)(p^\perp\cdot a) + \frac{(p\cdot a)^3(p^\perp\cdot a)}{2} = \frac{(p\cdot a)(p^\perp\cdot a)}{2}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)\\
M_{2,2} &= (p\cdot a)^2(p\cdot c)^2 + \frac{(p\cdot a)^2}{2} = \frac{(p\cdot a)^2}{2}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
M_{1,1}M_{2,2} &= \frac{(p\cdot a)^2(p^\perp\cdot a)^2}{4}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)^2 + \frac{(p\cdot a)^2|\mathcal{P}_S b^\perp|^2}{2r^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)\\
M_{1,2}M_{2,1} &= \frac{(p\cdot a)^2(p^\perp\cdot a)^2}{4}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)^2\\
\implies |M| &= \frac{(p\cdot a)^2|\mathcal{P}_S b^\perp|^2}{2r^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)\\
\implies (\mathcal{J}\Psi(a,b,r))^2 &= \frac{8r^2(p\cdot a)^2}{|\mathcal{P}_S b^\perp|^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big).
\end{aligned}$$

Since $p \in \operatorname{lsp}\{a,b,c\}$ we know $1 = (p\cdot p) = (p\cdot a)^2 + (p\cdot b)^2 + (p\cdot c)^2 = (p\cdot a)^2 + (p\cdot c)^2$, and $u = 2r(p\cdot a)p \implies (p\cdot a) = \frac{|u|}{2r}$ so

$$(p\cdot c)^2 = 1 - (p\cdot a)^2 = 1 - \frac{|u|^2}{4r^2}.$$

Plugging this back in we find

$$\mathcal{J}\Psi(a,b,r) = \frac{\sqrt{2}|u|}{|\mathcal{P}_S b^\perp|^2}\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}$$

$\square$

$\boxed{\textit{Lemma 4.3}}$ *Fixing $u \in \mathbb{R}^2$, for Ψ as defined in 4.2, if $\Phi : \Psi^{-1}(u) \to \mathbb{R}^+$ is given by $\Phi(a,b,r) = r$ then*

$$\mathcal{J}\Phi(a,b,r) = \frac{r\sqrt{1 - \frac{|u|^2}{4r^2}}}{\sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}}.$$

Proof. In order to compute $\mathcal{J}\Phi$ we need to characterize $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$, i.e. by finding $2n-4$ orthonormal vectors forming a basis. We can leverage the constraints imposed by $\mathcal{E}(u)$, i.e.

$$u = 2r(p(b)\otimes p(b))a, \tag{25}$$
$$a\cdot a = 1, \tag{26}$$
$$b\cdot b = 1, \tag{27}$$
$$a\cdot b = 0, \tag{28}$$

where $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$. Put $\gamma : \mathbb{R}^1 \to \mathcal{E}(u)$ such that $\gamma(0) = (a, b, r)$ and $\gamma'(0) = (\delta_a, \delta_b, \delta_r)$ be an arbitrary flow, then (25) combined with (24) shows us

$$0 = \delta_r(p(b) \otimes p(b))a + r\left(\frac{p(b) \cdot \delta_b}{|\mathcal{P}_S b^\perp|} p(b)^\perp \odot p(b)\right)a + r(p(b) \otimes p(b))\delta_a. \tag{29}$$

Next, to simplify notation put $p = p(b)$ so that $\operatorname{lsp}\{p, p^\perp\} = \mathbb{R}^2$ so that (29) shows us

$$\begin{aligned} 0 &= \delta_r(p \cdot a)p + r\left(\frac{p \cdot \delta_b}{|\mathcal{P}_S b^\perp|} p^\perp \odot p\right)a + r(p \cdot \delta_a)p \\ &= (\delta_r(p \cdot a) + r(p \cdot \delta_a))p + r\frac{(p \cdot \delta_b)}{|\mathcal{P}_S b^\perp|}(p^\perp \otimes p + p \otimes p^\perp)a \\ &= \left(\delta_r(p \cdot a) + r(p \cdot \delta_a) + r\frac{(p \cdot \delta_b)}{|\mathcal{P}_S b^\perp|}(p^\perp \cdot a)\right)p + r\frac{(p \cdot \delta_b)}{|\mathcal{P}_S b^\perp|}(p \cdot a)p^\perp. \end{aligned} \tag{30}$$

From the $p^\perp$ component of (30) we see

$$p \cdot \delta_b = 0, \tag{31}$$

since $p \cdot a = 0$ for only a measure zero set of (a, b). The p component of (30) then simplifies to

$$0 = \delta_r(p \cdot a) + r(p \cdot \delta_a). \tag{32}$$

Lastly, doing the same flow calculation with (26), (27), (28) tells us

$$a \cdot \delta_a = 0, \tag{33}$$

$$b \cdot \delta_b = 0 \tag{34}$$

$$a \cdot \delta_b + \delta_a \cdot b = 0 \tag{35}$$

Put $c = \frac{u - (u \cdot a)a}{\sqrt{(u \cdot u) - (u \cdot a)}}$ and $\{e_i\}_1^{n-3}$ so that $\operatorname{lsp}\left\{\{a, b, c\} \cup \{e_i\}_1^{n-3}\right\} = \mathbb{R}^n$. Since $p \in \operatorname{lsp}\{a, b, c\}$ we know $(p \cdot e_i) = 0$ and thus $\{(e_i, 0, 0)\}_1^{n-3} \cup \{(0, e_i, 0)\}_i^{n-3}$ are $2n - 6$ orthonormal vectors which span a subspace of $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$.

In order to find two more orthonormal vectors, let's begin by supposing $\delta_b = 0$. In order to find an additional orthoganol basis vector, we want to consider δ_a where $\delta_a \cdot e_i = 0$, and in order to satisfy (33), (35) we must have $\delta_a \cdot a = \delta \cdot b = 0$, so, w.l.o.g. we can consider $\delta_a = c$. Plugging this data into (32) and solving for δ_r gives us

$$\delta_r = \frac{-r(p \cdot \delta_a)}{p \cdot a},$$

so after normalization, we can see an additional orthornomal basis vector spanning $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$ is

$$\begin{aligned} \mu &= \frac{1}{\sqrt{1 + r^2 \frac{(p \cdot c)^2}{(p \cdot a)^2}}}\left(c, 0, -r\frac{p \cdot c}{p \cdot a}\right) \\ &= \frac{1}{\sqrt{1 - r^2 + \frac{4r^4}{|u|^2}}}\left(c, 0, \frac{-2r^2\sqrt{1 - \frac{|u|^2}{4r^2}}}{|u|}\right), \end{aligned} \tag{36}$$

where we got the second line from the first by noticing $1 = (p \cdot p)^2 = (p \cdot a)^2 + (p \cdot b)^2 + (p \cdot c)^2$ and that $u = 2r(p \cdot a)p \implies$

$|u|2r = (p \cdot a)$. Lastly it's easy to check that

$$\nu = \left(\frac{\sqrt{r^2 - \frac{|u|^2}{4}}}{\sqrt{2r^2 - \frac{|u|^2}{4}}} b, \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}(u - 2ra), 0 \right), \tag{37}$$

is a final orthonormal basis vector so that

$$\mathcal{T}_{(a,b,r)}(\mathcal{E}(u)) = \mathrm{lsp}\Big\{ \{(e_i, 0, 0)\}_1^{n-3} \cup \{(0, e_i, 0)\}_1^{n-3} \cup \{\mu, \nu\} \Big\}.$$

Next, to compute $\mathcal{J}\Phi$ put $\gamma_v : \mathbb{R}^1 \to \mathcal{E}(u)$ so that $\gamma_v(0) = (a, b, r)$ and $\gamma_v'(0) = v$, then the only non-vanishing derivative is

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\gamma_\mu(s))\Big|_{s=0} = \frac{-2r^2\sqrt{1 - \frac{|u|^2}{4r^2}}}{|u|\sqrt{1 - r^2 + \frac{4r^4}{|u|^2}}} = \frac{-r\sqrt{1 - \frac{|u|^2}{4r^2}}}{\sqrt{\frac{|u|^2}{4r^2} - \frac{r^2|u|^2}{4r^2} + r^2}} = \frac{-r\sqrt{1 - \frac{|u|^2}{4r^2}}}{\sqrt{r^2 - (r^2 - 1)\frac{|u|^2}{4r^2}}},$$

so

$$\nabla\Phi(a, b, r) = \frac{-r\sqrt{1 - \frac{|u|^2}{4r^2}}}{\sqrt{r^2 - (r^2 - 1)\frac{|u|^2}{4r^2}}} \implies \mathcal{J}\Phi(a, b, r) = \frac{r\sqrt{1 - \frac{|u|^2}{4r^2}}}{\sqrt{r^2 - (r^2 - 1)\frac{|u|^2}{4r^2}}}.$$

□

Lemma 4.4 *Fixing $u \in \mathbb{R}^2$, $r \in \mathbb{R}^+$, for Φ as defined in 4.3, if $\Xi : \Phi^{-1}(r) \to \mathcal{E}_2(u, r)$ is given by $\Xi(a, b) = b$ then*

$$\mathcal{J}\Xi(a, b) = \frac{1}{\sqrt{2 - \frac{|u|^2}{4r^2}}}.$$

Proof. In order to characterize $\mathcal{T}_{(a,b)}(\mathcal{E}(u, r))$ we must find $2n-5$ orthornomal basis vectors spanning the space. Similar to 4.3 we can leverage the constraints imposed by $\mathcal{E}(u)$, since $\mathcal{E}(u, r)$ simply introduces a new constraint that r is unchanging. That is, every basis vector of $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$ such that r vanishes is a a valid basis vector for $\mathcal{T}_{(a,b)}(\mathcal{E}(u, r))$ when projected onto the first two coordinates, so

$$\mathcal{T}_{(a,b)}(\mathcal{E}(u)) = \{(e_i, 0)\}_1^{n-3} \cup \{(0, e_i)\}_1^{n-3} \cup \{\nu\},$$

where ν comes from (37), i.e.

$$\nu = \left(\frac{\sqrt{r^2 - \frac{|u|^2}{4}}}{\sqrt{2r^2 - \frac{|u|^2}{4}}} b, \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}(u - 2ra) \right), \tag{38}$$

and the $\{e_i\}_1^{n-3}$ are defined so that, for $c = \frac{u - (u \cdot a)a}{\sqrt{(u \cdot u) - (u \cdot a)}}$ $\{a, b, c\} \cup \{e_i\}_1^{n-3}$ forms an orthonormal basis spanning $\mathbb{R}^n$.

Next we need to characterize the tangent space of the codomain, i.e. $\mathcal{T}_b(\mathcal{E}_2(u, r))$. Since, for the calculation of $\mathcal{J}\Xi(a, b)$ both a, b are fixed, we can use both a, b for this characterization. Notice

$$\mathcal{E}_2(u, r) = \Big\{ b \in \mathcal{U}\Big(\{u\}^\perp\Big) \mid b \cdot t > 0 \Big\},$$

so that we need $n - 2$ basis vectors respecting the constraints that $b \cdot u = 0, b \cdot b = 1$. Implicitly differentiating these constraints shows us that

$$\delta_b \cdot u = 0, \delta_b \cdot b = 0,$$

and so it becomes clear that every $\delta_b = e_i$ is a valid basis vector for $\mathcal{T}_b(\mathcal{E}_2(u,r))$ for every e_i. It can also be shown that $\frac{\nu_2}{|\nu_2|}$, where ν_2 is the second component of (38), is a valid normalized basis vector orthoganol to every e_i, so that

$$\mathcal{T}_b(\mathcal{E}_2(u,r)) = \mathrm{lsp}\left\{ \{e_i\}_1^{n-3} \cup \left\{ \frac{\nu_2}{|\nu_2|} \right\} \right\}.$$

Now that we've characterized the tangent spaces we can begin to calculate the jacobian factor, i.e. it's easy to see

$$\nabla\xi(a,b) = \begin{array}{c} \begin{array}{ccc} (e_i,0) & (0,e_i) & \nu \end{array} \\ \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & |\nu_2| \end{pmatrix} \end{array} \begin{array}{c} \\ e_i \\ \frac{\nu_2}{|\nu_2|} \end{array} \implies \nabla\xi(a,b)(\nabla\xi(a,b))^T = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{n-3} & \overbrace{\quad}^{1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & |\nu_2|^2 \end{pmatrix} \end{array} \begin{array}{c} \\ n-3 \\ 1 \end{array},$$

and so

$$\begin{aligned} \mathcal{J}\Xi(a,b) = |\nu_2| &= \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}\sqrt{|u|^2 - 4r(a \cdot u) + 4r^2} \\ &= \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}\sqrt{|u|^2 - 2|u|^2 + 4r^2} \\ &= \frac{r}{\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}\sqrt{r^2 - \frac{|u|^2}{4}} = \frac{1}{\sqrt{2 - \frac{|u|^2}{4r^2}}} \end{aligned}$$

□

$\boxed{\textit{Lemma 4.5}}$ *Fixing* $u \in \mathbb{R}^2$, $r \in \mathbb{R}^+$ $b \in \left\{ b \in \mathcal{U}\left(\{u\}^\perp\right) \mid b \cdot t > 0 \right\}$ *then, for* Ξ *as defined in 4.4*

$$\mathcal{H}^{n-3}\big(\Xi^{-1}(b)\big) = \omega_{n-3}\left(1 - \frac{|u|^2}{4r^2}\right)^{(n-3)/2}.$$

Proof. By definition

$$\Xi^{-1}(b) = \left\{ (a,b) \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u \cdot a) = (u \cdot u) \right\} = \left\{ a \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u \cdot a) = (u \cdot u) \right\} \times \{b\},$$

so

$$\mathcal{H}^{n-3}\big(\Xi^{-1}(b)\big) = \mathcal{H}^{n-3}\left(\left\{ a \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u \cdot a) = (u \cdot u) \right\}\right).$$

Now, for $a \in \mathcal{F}$ for $\mathcal{F} = \left\{ a \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u \cdot a) = (u \cdot u) \right\}$

$$\left(a - \frac{u}{2r}\right) \cdot \left(a - \frac{u}{2r}\right) = 1 + \frac{|u|^2}{4r^2} - 2\frac{(a \cdot u)}{2r} = 1 + \frac{|u|^2}{4r^2} - 2\frac{|u|^2}{4r^2} = 1 - \frac{|u|^2}{4r^2},$$

which makes it easy to see

$$\mathcal{F} = \left\{ a \in \mathcal{U}\left(\{b\}^\perp\right) : \left|a - \frac{u}{2r}\right|^2 = 1 - \frac{|u|^2}{4r^2} \right\}. \tag{39}$$

Since (39) is an $n-3$ dimensional sphere located at $\frac{u}{2r}$ with radius $\sqrt{1-\frac{|u|^2}{4r^2}}$ we have

$$\mathcal{H}^{n-3}(\mathcal{F}) = S_{n-3}\left(\sqrt{1-\frac{|u|^2}{4r^2}}\right) = \omega_{n-3}\left(1-\frac{|u|^2}{4r^2}\right)^{(n-3)/2}.$$

$\square$

$\boxed{\textit{Lemma 4.6}}$ *Fix* $u \in \mathbb{R}^2$, $r \in \mathbb{R}^+$, $\mathcal{E}(u,r) = \Phi^{-1}(\{r\})$ *for* Φ *as defined in 4.3, then if* $\zeta : \mathcal{E}_2(u,r) \to [-1,1]$ *is given by* $\zeta(b) = b \cdot \frac{u^\perp}{|u^\perp|}$,

$$\mathcal{J}\zeta(b) = \sqrt{1-\left(b\cdot\frac{u^\perp}{|u^\perp|}\right)^2}.$$

Proof. We begin by characterizing $\mathcal{T}_b(\mathcal{E}_2(u,r))$. Put $\{f_i\}_1^{n-3}$

$$u^* = \frac{\frac{u^\perp}{|u^\perp|} - \left(b\cdot\frac{u^\perp}{|u^\perp|}\right)b}{\sqrt{1-\left(b\cdot\frac{u^\perp}{|u^\perp|}\right)^2}}$$

so that $\left\{\frac{u}{|u|}, b, u^*\right\} \cup \{f_i\}_1^{n-3}$ is an orthonormal basis spanning $\mathbb{R}^n$. We must have $|b| = 1$, $b\cdot u = 1$ since $b \in \mathcal{E}_2(u,r)$, thus if we fix a tangent vector δ_b, we must have $\delta_b \cdot b = 0$, $\delta_b \cdot u = 0$, i.e.

$$\mathcal{T}_b(\mathcal{E}_2(u,r)) = \operatorname{lsp}\{u^*, f_1, f_2, \ldots, f_{n-3}\}.$$

With this, we can compute

$$\nabla\zeta(b) = \begin{array}{cc} \begin{array}{cccc} u^* & g_1 & \cdots & g_{n-3} \end{array} & \\ \begin{pmatrix} \sqrt{1-\left(b\cdot\frac{u^\perp}{|u^\perp|}\right)^2} & 0 & \cdots & 0 \end{pmatrix} & 1 \end{array} \implies \mathcal{J}\zeta(b) = \sqrt{1-\left(b\cdot\frac{u^\perp}{|u^\perp|}\right)^2}.$$

$\square$

References

[1] Brian Seguin. A fractional notion of length and an associated nonlocal curvature. *The Journal of Geometric Analysis*, 30(1):161–181, 2020.