1 Introduction

Continuing from previous work, we aim to analyze fractional curvature and understand its asymptotics when $\sigma \uparrow 1$. In particular we hope to determine the exact constants which are required to recover classical curvature in the appropriate limit.

Given a curve $\mathcal{C}$ in n dimensions we want to analyze the following quantity

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} \, d\mathcal{H}^{2n-2}(a, b, r),$$

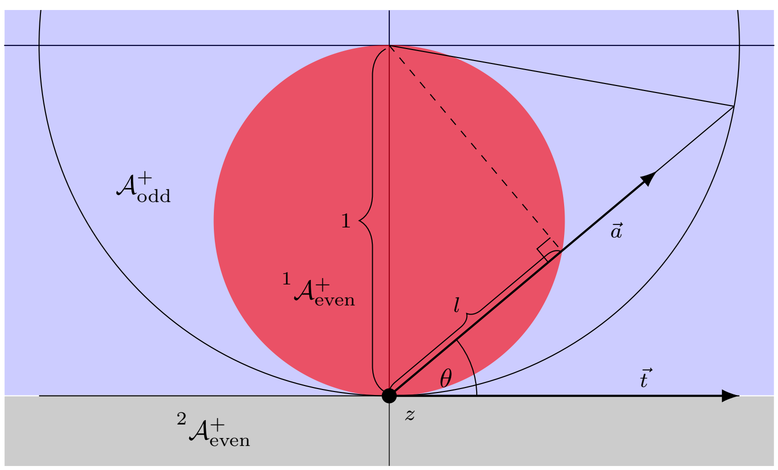
where $t := t(z)$ is the unit tangent of $\mathcal{C}$ at z. Correspondingly, we use $n := n(z)$ to be the unit normal vector of $\mathcal{C}$ at z.

2 Fractional Curvature of Unit Circle in 2D

We wish to compute

$$\kappa_\sigma(z) \cdot n := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}}\right) \frac{(a \cdot t)(b \cdot n) - (b \cdot t)(a \cdot n)}{r^{1+\sigma}} \, d\mathcal{H}^{2}(a, b, r).$$

In order to evaluate this we need to break down $\mathcal{A}^+_{\text{even}}, \mathcal{A}^+_{\text{odd}}$. Consider the following diagram:



Notice a disk only intersects $\mathcal{C}$ if a points in the top half plane & if $r > l$, and notably there is always a single intersection. From the picture we can deduce

$$l = \sin\theta$$

so that[1]:

$$\begin{aligned}
\mathcal{A}^+_{\text{odd}} &= \left\{ \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, r \right) \mid \theta \in [0,\pi], r \in [\sin\theta, \infty) \right\} \\
{}^1\mathcal{A}^+_{\text{even}} &= \left\{ \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, r \right) \mid \theta \in [0,\pi], r \in [0, \sin\theta) \right\} \\
{}^2\mathcal{A}^+_{\text{even}} &= \left\{ \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, r \right) \mid \theta \in [\pi, 2\pi], r \in [0, \infty) \right\} \\
\mathcal{A}^+_{\text{even}} &= {}^1\mathcal{A}^+_{\text{even}} \cup {}^2\mathcal{A}^+_{\text{even}}.
\end{aligned}$$

[1] N.B. b is entirely determined by a, t since $a \cdot b = 0, b \cdot t > 0$

Put $\chi_1 = \chi_{\mathcal{A}^+_{\text{even}}} - \chi_{\mathcal{A}^+_{\text{odd}}}$ so that we can rewrite our calculation as:

$$\kappa_\sigma \cdot n = \int_{\mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}} \frac{\chi_1(r,a)}{r^{1+\sigma}}((a\cdot t)(b \cdot n) - (b\cdot t)(a\cdot n))\, d\mathcal{H}^2(a,b,r)$$

Next consider a change of variables via $\phi : [0,2\pi] \times \mathbb{R}^+ \to \mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}$ given by

$$\phi(\theta, r) = \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, r \right).$$

For $\gamma : \mathbb{R} \to [0,2\pi] \times \mathbb{R}^+$ such that $\gamma(0) = (\theta, r)$ and $\gamma'(0) = (1,0)$ we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma(s))\bigg|_{s=0} = \left(\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, 0 \right) = (-b, a, 0)$$

and similarly when we consider γ such that $\gamma'(0) = (0,1)$ we find:

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma(s))\bigg|_{s=0} = (0,0,1).$$

Since $T_{(a,b,r)}\mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}$ is spanned by $\frac{1}{\sqrt{2}}(b,-a,0), (0,0,1)$ we can write $\nabla\phi$ as follows:

$$\nabla\phi(a,b,r) = \begin{array}{cc} \begin{array}{cc} (1,0) & (0,1) \end{array} & \\ \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{c} \frac{1}{\sqrt{2}}(b,-a,0) \\ (0,0,1) \end{array} \end{array} \implies \sqrt{|\nabla\phi^T \nabla\phi|} = \sqrt{2}.$$

With this change of variables our computation becomes:

$$\kappa_\sigma \cdot n = \sqrt{2} \int_{[0,2\pi]\times\mathbb{R}^+} \frac{\chi_1\left(r, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right)}{r^{1+\sigma}} (-\cos^2\theta - \sin^2\theta)\chi_2\left(r, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right) dr\, d\theta,$$

where $\chi_2 = \chi^1{}_{\mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}} - \chi^2{}_{\mathcal{A}^+_{\text{even}}}$. Fix $\chi = \chi_1\chi_2 = \chi^1{}_{\mathcal{A}^+_{\text{even}}} - \chi^2{}_{\mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}}$ so that

$$\begin{aligned}
\kappa_\sigma \cdot n &= -\sqrt{2} \lim_{\epsilon\downarrow 0} \int_0^{2\pi} \int_\epsilon^\infty \frac{\chi(r,\theta)}{r^{1+\sigma}}\, dr\, d\theta \\
&= -\sqrt{2} \lim_{\epsilon\downarrow 0} \left(\int_0^\pi \int_\epsilon^{\sin\theta} r^{-1-\sigma}\, dr\, d\theta - \int_0^\pi \int_{\sin\theta}^\infty r^{-1-\sigma}\, dr\, d\theta - \int_\pi^{2\pi} \int_\epsilon^\infty r^{-1-\sigma}\, dr\, d\theta \right) \\
&= \frac{\sqrt{2}}{\sigma} \lim_{\epsilon\downarrow 0} \left(\int_0^\pi r^{-\sigma}\big|_\epsilon^{\sin\theta}\, d\theta - \int_0^\pi r^{-\sigma}\big|_{\sin\theta}^\infty\, d\theta - \int_\pi^{2\pi} r^{-\sigma}\big|_\epsilon^\infty \right) \\
&= \frac{\sqrt{2}}{\sigma} \lim_{\epsilon\downarrow 0} \left(\int_0^\pi \sin^{-\sigma}\theta\, d\theta - \pi\epsilon^{-\sigma} + \int_0^\pi \sin^{-\sigma}\theta\, d\theta + \pi\epsilon^{-\sigma} \right) \\
&= \frac{2\sqrt{2}}{\sigma} \int_0^\pi \sin^{-\sigma}\theta\, d\theta = \frac{2\sqrt{2}}{\sigma} \int_{-\pi/2}^{\pi/2} \cos^{-\sigma}\theta\, d\theta = \frac{4\sqrt{2}}{\sigma} \int_0^{\pi/2} \cos^{-\sigma}\theta\, d\theta \\
&= \frac{2\sqrt{2}}{\sigma} B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right)
\end{aligned}$$

3 Fractional Curvature of Unit Circle in 3D

To begin, fix $\chi(a,b,r)$ to indicate the sign corresponding to a,b,r (i.e. whether a,b,r belong to $\mathcal{A}^+_{\text{even}}$ or $\mathcal{A}^+_{\text{odd}}$, and whether b or $-b$ is needed by the $t \cdot b > 0$ restriction). Put $\mathcal{A}^+ = \mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}$ and

$$g(a,b,r) = \chi(a,b,r)\frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}},$$

so that our desired computation is:

$$\kappa_\sigma(z) = \int_{\mathcal{A}^+} g(a,b,r)\, d\mathcal{H}^4(\mathbf{a},\mathbf{b},r),$$

3.1 Slicing out the 2D plane

To simplify our domain group the disks by their intersection with the $S := \mathrm{lsp}\{t,n\}$ plane, i.e. put $\psi : \mathcal{A}^+ \to \mathbb{R}^2$ so that ψ maps a disk to a vector representing $\mathcal{D}(a,b,r) \cap S$. To determine ψ, put $u := \psi(a,b,r)$, then we must have the following:

- The component of ra in the direction of u must be half of u's length (i.e. an isoceles triangle is formed between the center point of the circle sitting at ra and the chord at the intersection of this circle and S). In other words we must have:
$$ra \cdot \frac{u}{|u|} = \frac{|u|}{2} \implies 2ra \cdot u = |u|^2 = u \cdot u, \tag{1}$$

- Since we're interested in when these circles intersect S, we know
$$u = u_t t + u_n n,\ u_t, u_n \in \mathbb{R}. \tag{2}$$

- Finally, because $z + u$ is the chord of intersection between the disk formed by a,b,r and S we must also have
$$u \cdot b = 0, \tag{3}$$

Due to the combination of (3), (2) we have

$$b_n u_n + b_t u_t = 0.$$

By the construction of the integral we have $b \cdot t > 0 \implies b_t \neq 0$ so that

$$u_t = \frac{-b_n u_n}{b_t}. \tag{4}$$

Plugging this back into (1) (and using (2) to characterize u) we find

$$\begin{aligned} 2ra_n u_n - 2ra_t\frac{b_n u_n}{b_t} &= u_n^2 + \frac{b_n^2 u_n^2}{b_t^2} \\ \implies 0 &= u_n^2\left(1 + \frac{b_n^2}{b_t^2}\right) + 2ru_n\left(\frac{a_t b_n}{b_t} - a_n\right) \\ \implies u_n = 0 \vee u_n\frac{b_t^2 + b_n^2}{b_t^2} &+ 2r\frac{a_t b_n - a_n b_t}{b_t} = 0. \end{aligned}$$

Notably $u_n \neq 0$ since otherwise, by (4), that would force $u_t = 0$, contradicting the assumption that $u_t > 0$. Solving the above equation for u_n we find

$$u_n = -2r\frac{a_t b_n - a_n b_t}{b_t}\frac{b_t^2}{b_t^2 + b_n^2} = 2r\frac{a_n b_t - a_t b_n}{b_t^2 + b_n^2} b_t.$$

Consider $\mathcal{P}_S = (t \otimes t) + (n \otimes n)$ the projection operator onto S so that we can rewrite the above as follows:

$$u_n = 2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2} b_t$$

Plugging this back into (4) we find

$$u_t = -2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2} b_n,$$

so that together, using the n, t coordinate system, we can write

$$\psi(a, b, r) = 2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b|^2} (b_t, -b_n).$$

Consider the $\cdot^\perp$ operator to rotate clockwise in S, so that

$$\mathcal{P}_S b^\perp = ((n \otimes n)b + (t \otimes t)b)^\perp = (b_n n + b_t t)^\perp = b_t n - b_n t \implies \mathcal{P}_S b^\perp \cdot a = a_n b_t - a_t b_n \tag{5}$$

and $\left|\mathcal{P}_S b^\perp\right| = |\mathcal{P}_S b|$. Put $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$, so that, combined with the above, we're able to simplify ψ:

$$\psi(a, b, r) = 2r \frac{\mathcal{P}_S b^\perp \cdot a}{|P_S b^\perp|^2} \mathcal{P}_S b^\perp = 2r(p(b) \cdot a)p(b).$$

Finally, rewriting using a tensor product we come to our final simplified definition:

$$\psi(a, b, r) = 2r(p(b) \otimes p(b))a.$$

3.2 2D Co-Area Calculation

In order to use ψ to slice our domain we must determine $\left|\nabla\psi\nabla\psi^T\right|$. To that end, notice that $T_{(a,b,r)}(\mathcal{U}^2_\perp \times \mathbb{R}^+)$ is spanned by $(c, 0, 0), (0, c, 0), \frac{1}{\sqrt{2}}(b, -a, 0), (0, 0, 1)$ (where c is the orthonormal completion of a, b in $\mathbb{R}^3$), and put $p = p(b)$ so that $p, p^\perp$ spans $T_{\psi(a,b,r)}(\mathbb{R}^2)$. We start with a quick calculation; suppose $\beta : \mathbb{R} \to \mathcal{U}$ such that $\beta(0) = b$, then

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s} p(\beta(s))\right|_{s=0} &= \frac{\mathcal{P}_S \beta'(0)^\perp}{|\mathcal{P}_S b^\perp|} - \frac{1}{|\mathcal{P}_S b^\perp|^3} \mathcal{P}_S b^\perp \otimes \left(\mathcal{P}_S \beta'(0)^\perp\right)^T \mathcal{P}_S b^\perp \\
&= \frac{1}{|\mathcal{P}_S b^\perp|} \left(\mathcal{P}_S \beta'(0)^\perp - \frac{1}{|\mathcal{P}_S b^\perp|^2} \left(\mathcal{P}_S \beta'(0)^\perp \cdot \mathcal{P}_S b^\perp\right) \mathcal{P}_S b^\perp \right) \\
&= \frac{1}{|\mathcal{P}_S b^\perp|} (1 - (p \otimes p)) \mathcal{P}_S \beta'(0)^\perp.
\end{aligned}$$

Since we'll be working in the $p, p^\perp$ coordinate system, it makes sense to expand this result as follows:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} p(\beta(s))\right|_{s=0} = \frac{1}{|\mathcal{P}_S b^\perp|} \left(p \otimes p + p^\perp \otimes p^\perp - p \otimes p\right) \mathcal{P}_S \beta'(0)^\perp = \frac{\left(p^\perp \otimes p^\perp\right)}{|\mathcal{P}_S b^\perp|} \mathcal{P}_S \beta'(0)^\perp \tag{6}$$

Now, put $\gamma_v : \mathbb{R} \to \mathcal{U}^2_\perp \times \mathbb{R}^+$ to be such that $\gamma_v(0) = (a, b, r)$ and $\gamma'_v(0) = v$, then we begin by computing the derivative along the $\gamma_{(c,0,0)}$ flow:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} \psi(\gamma_{(c,0,0)}(s))\right|_{s=0} = 2r(p \otimes p)c = 2r(p \cdot c)p. \tag{7}$$

Next we compute the derivative along the $(0,c,0)$ flow, and simplify using (6)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,c,0)}(s))\Big|_{s=0} &= 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{(0,c,0),2}(s))\Big|_{s=0}\right)\otimes p + p\otimes\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{(0,c,0),2}(s))\Big|_{s=0}\right)\right)a \\
&= 2r\left(\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S c^\perp\right)\otimes p + p\otimes\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S c^\perp\right)\right)a \\
&= 2r\frac{p^\perp\cdot\mathcal{P}_S c^\perp}{|\mathcal{P}_S b^\perp|}\left(p^\perp\otimes p + p\otimes p^\perp\right)a.
\end{aligned}$$

Taking into account the fact that $p^\perp\cdot\mathcal{P}_S c^\perp = p\cdot\mathcal{P}_S c = p\cdot c$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,c,0)}(s))\Big|_{s=0} = 2r\frac{(p\cdot c)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|}p + 2r\frac{(p\cdot c)(p\cdot a)}{|\mathcal{P}_S b^\perp|}p^\perp. \tag{8}$$

The next derivative we must compute is along the $\frac{1}{\sqrt{2}}(b,-a,0)$ flow:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\Big|_{s=0} &= \sqrt{2}r(p\otimes p)b + 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0),2}(s))\Big|_{s=0}\right)\odot p\right)a \\
&= 2r\left(\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\frac{-a^\perp}{\sqrt{2}}\right)\otimes p + p\otimes\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\frac{-a^\perp}{\sqrt{2}}\right)\right)a \\
&= -\sqrt{2}r\frac{p^\perp\cdot\mathcal{P}_S a^\perp}{|\mathcal{P}_S b^\perp|}\left(p^\perp\otimes p + p\otimes p^\perp\right)a.
\end{aligned}$$

Note that the first term vanishes because $p\cdot b = 0$. Again taking into account the fact that $p^\perp\cdot\mathcal{P}_S a^\perp = p\cdot\mathcal{P}_S a = p\cdot a$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\Big|_{s=0} = -\sqrt{2}r\frac{(p\cdot a)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|}p - \sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^\perp|}p^\perp. \tag{9}$$

Lastly computing the derivative through the $(0,0,1)$ flow we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,0,1)}(s))\Big|_{s=0} = 2(p\otimes p)a = 2(p\cdot a)p \tag{10}$$

Combining (7), (8), (9), (10) we have

$$\begin{array}{rcccccc}
 & & (c,0,0) & (0,c,0) & \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) & \\
\nabla\psi(a,b,r) = & \Bigg(& 2r(p\cdot c) & 2r\frac{(p\cdot c)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|} & -\sqrt{2}r\frac{(p\cdot a)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|} & 2(p\cdot a) & \Bigg) \; p \\
 & & 0 & 2r\frac{(p\cdot c)(p\cdot a)}{|\mathcal{P}_S b^\perp|} & -\sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^\perp|} & 0 & \; p^\perp
\end{array}$$

Put $M = \nabla\psi(a,b,r)\nabla\psi(a,b,r)^T$ so that we desire to compute $\sqrt{|M|}$. Put $g = 2(p\cdot c)^2 + (p\cdot a)^2$. We compute the following:

$$\begin{aligned}
M_{11} &= 4r^2(p\cdot c)^2 + \frac{4r^2(p\cdot c)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^2} + \frac{2r^2(p\cdot a)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^2} + 4(p\cdot a)^2\\
&= 2r^2\frac{\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^2}g + 4\Big(r^2(p\cdot c)^2 + (p\cdot a)^2\Big)\\
M_{22} &= 4r^2\frac{(p\cdot c)^2(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2} + 2r^2\frac{(p\cdot a)^4}{\left|\mathcal{P}_S b^\perp\right|^2}\\
&= 2r^2\frac{(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2}g\\
M_{12} = M_{21} &= 4r^2\frac{(p\cdot c)^2(p\cdot a)\left(p^\perp\cdot a\right)}{\left|\mathcal{P}_S b^\perp\right|^2} + 2r^2\frac{(p\cdot a)^3\left(p^\perp\cdot a\right)}{\left|\mathcal{P}_S b^\perp\right|^2}\\
&= 2r^2\frac{(p\cdot a)\left(p^\perp\cdot a\right)}{\left|\mathcal{P}_S b^\perp\right|^2}g.
\end{aligned}$$

With these we can calculate the determinant as follows:

$$\begin{aligned}
|M| = M_{11}M_{22} - M_{12}M_{21} &= 4r^4\frac{(p\cdot a)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^4}g^2 + 8gr^2\frac{(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2}\Big(r^2(p\cdot c)^2 + (p\cdot a)^2\Big) - 4r^4\frac{(p\cdot a)^2\left(p^\perp\cdot a\right)^2}{\left|\mathcal{P}_S b^\perp\right|^4}g^2\\
&= 8gr^2\frac{(p\cdot a)^2}{\left|\mathcal{P}_S b^\perp\right|^2}\Big(r^2(p\cdot c)^2 + (p\cdot a)^2\Big)
\end{aligned}$$

Thus, altogether we have

$$J\psi(a,b,r) = \sqrt{\left|\nabla\psi(a,b,r)^T\nabla\psi(a,b,r)\right|} = 2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}.$$

Put $\mathcal{D}(u) := \psi^{-1}(\{u\}), p = p(b)$ so that

$$\begin{aligned}
\int_{\mathcal{A}^+} g(a,b,r)\,d\mathcal{H}^4(\mathbf{a},\mathbf{b},r) &= \int_{\mathbb{R}^2}\int_{\mathcal{D}(u)}\frac{g(a,b,r)}{J\psi(a,b,r)}\,d\mathcal{H}^2(a,b,r)d\mathcal{H}^2(u)\\
&= \int_{\mathbb{R}^2}\int_{\mathcal{D}(u)}\frac{g(a,b,r)}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}}\,d\mathcal{H}^2(a,b,r)d\mathcal{H}^2(u).
\end{aligned}$$

3.3 Slicing out Radii

Now, to simplify $\mathcal{D}(u)$ lets group sets of $\mathbf{a},\mathbf{b}$ that correspond to a given r, i.e. put $\phi : \mathcal{D}(u) \to \mathbb{R}^+$ given by

$$\phi(a,b,r) = r.$$

To begin our calculation of $\nabla\phi$ we must characterize $T_{(a,b,r)}(\mathcal{D}(u))$, i.e. via finding an orthonormal basis. Suppose $\gamma : \mathbb{R} \to \mathcal{D}(u)$ is so that $\gamma(0) = (a,b,r)$ and put $(\alpha,\beta,\tau) := \gamma'(0)$. By the definition of $\mathcal{D}(u)$ we know

$$2r(p(b)\otimes p(b))a = u,$$

so that

$$2\gamma_3(s)(p(\gamma_2(s))\otimes p(\gamma_2(s)))\gamma_1(s) = u.$$

Differentiating[2] and evaluating at $s = 0$ we see

$$0 = \tau(p(b) \otimes p(b))a + r\left(\frac{(p^\perp \otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S \beta^\perp \odot p(b)\right)a + r(p(b) \otimes p(b))\alpha.$$

Expanding tensor products, simplifying and using the $p, p^\perp$ coordinate system we get

$$\begin{aligned}
0 &= \tau(p \cdot a)p + r\left(\frac{p \cdot \beta}{|\mathcal{P}_S b^\perp|}p^\perp \odot p\right)a + r(p \cdot \alpha)p \\
&= (\tau(p \cdot a) + r(p \cdot \alpha))p + r\frac{(p \cdot \beta)}{|\mathcal{P}_S b^\perp|}(p^\perp \otimes p + p \otimes p^\perp)a \\
&= \left(\tau(p \cdot a) + r(p \cdot \alpha) + r\frac{(p \cdot \beta)}{|\mathcal{P}_S b^\perp|}(p^\perp \cdot a)\right)p + r\frac{(p \cdot \beta)}{|\mathcal{P}_S b^\perp|}(p \cdot a)p^\perp.
\end{aligned}$$

To determine an initial basis vector suppose $\beta = 0, \alpha \neq 0, \tau \neq 0$, then we must have

$$0 = \tau(p \cdot a) + r(p \cdot \alpha),$$

but since $\mathcal{D}(u) \subset \mathcal{U}_2^\perp \times \mathbb{R}^+$ we must have $\alpha = c$ so that one of our basis vectors is

$$\mu = \frac{1}{\sqrt{1 + r^2\frac{(p \cdot c)^2}{(p \cdot a)^2}}}\left(c, 0, -r\frac{(p \cdot c)}{(p \cdot a)}\right).$$

Next suppose $\tau = 0, \alpha \neq 0, \beta \neq 0$ so that we must have

$$(p \cdot \alpha) + \frac{(p \cdot \beta)}{|\mathcal{P}_S b^\perp|}(p^\perp \cdot a) = 0 \wedge (p \cdot \beta)(p \cdot a) = 0.$$

Notice $u = 2r(p \cdot a)p \implies u = |u|p$ and by construction we have $2ra \cdot u = u \cdot u$ so that $p \cdot a \neq 0$. Together with the second statment this leads us to see

$$(p \cdot \beta) = 0.$$

Again, since $\mathcal{D}(u) \subset \mathcal{U}_2^\perp \times \mathbb{R}^+$ we must have $\beta \cdot b = 0$, so that β is perpendicular to both p and b, i.e. we have

$$\beta = p \times b.$$

Revisiting the first equality above we also find

$$(p \cdot \alpha) = 0,$$

and for similar reasoning we have $\alpha \cdot a = 0$, thus[3]

$$\alpha = p \times a.$$

Altogether this gives us a second basis vector:

$$\nu = \frac{1}{\sqrt{|p \times a|^2 + |p \times b|^2}}(p \times a, p \times b, 0)$$

Note that[4]

$$c \cdot (p \times a) = -c \cdot (a \times p) = -(c \times a) \cdot p = -b \cdot p = 0,$$

[2] N.B. we use (6)

[3] TODO: Talk about signs of α, β?

[4] TODO: formalize c so that we know whether $c \times a = \pm b$.

i.e. $\mu \cdot \nu = 0$ so that μ, ν are an orthonormal basis of $T_{(a,b,r)}\mathcal{D}(u)$.

Now, to compute $J\phi$, for $v \in \{\mu, \nu\}$, put $\gamma_v : \mathbb{R} \to \mathcal{D}(u)$ so that $\gamma_v(0) = (a, b, r)$ and $\gamma'(0) = v$. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_\mu(s))\bigg|_{s=0} &= \frac{-r(p\cdot c)}{(p\cdot a)\sqrt{1 + r^2\frac{(p\cdot c)^2}{(p\cdot a)^2}}} \\ &= -r\frac{(p\cdot c)}{\sqrt{(p\cdot a)^2 + r^2(p\cdot c)^2}} \\ \frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_\nu(s))\bigg|_{s=0} &= 0.\end{aligned}$$

so that

$$J\phi = \frac{r|p\cdot c|}{\sqrt{(p\cdot a)^2 + r^2(p\cdot c)^2}}.$$

3.4 Simplifying Computation

Put $\mathcal{D}(u, r) = \phi^{-1}(\{r\})$, $\mathcal{R}(u) = \big\{r \mid \exists\,(\mathbf{a}, \mathbf{b}) \in \mathcal{U}_2^\perp \; s.t.\, (\mathbf{a}, \mathbf{b}, r) \in \mathcal{D}(u)\big\}$ so that

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathcal{D}(u)} \frac{g(a,b,r)}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}}\, d\mathcal{H}^2(a,b,r)d\mathcal{H}^2(u) \\ &= \int_{\mathbb{R}^2}\int_{\mathcal{R}(u)}\int_{\mathcal{D}(u,r)} \frac{g(a,b,r)}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}} \frac{\sqrt{(p\cdot a)^2 + r^2(p\cdot c)^2}}{r|p\cdot c|}\, d\mathcal{H}^1(a,b)drd\mathcal{H}^2(u) \\ &= \int_{\mathbb{R}^2}\int_{\mathcal{R}(u)}\int_{\mathcal{D}(u,r)} \frac{g(a,b,r)\left|\mathcal{P}_S b^\perp\right|}{2\sqrt{2}|(p\cdot a)(p\cdot c)|r^2\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}}\, d\mathcal{H}^1(a,b)drd\mathcal{H}^2(u)\end{aligned}$$

In order to simplify this, recall

$$g(a,b,r) = \chi(a,b,r)\frac{(a\cdot t)b - (b\cdot t)a}{r^{1+\sigma}},$$

where χ is a signing function. Notice 5 we see

$$(a\cdot t)(b\cdot n) - (b\cdot t)(a\cdot n) = -\mathcal{P}_S b^\perp \cdot a = -\left|\mathcal{P}_S b^\perp\right|(p\cdot a),$$

so

$$g(a,b,r)\cdot n = \chi(a,b,r)\frac{(a\cdot t)(b\cdot n) - (b\cdot t)(a\cdot n)}{r^{1+\sigma}} = -\chi(a,b,r)\frac{\left|\mathcal{P}_S b^\perp\right|(p\cdot a)}{r^{1+\sigma}}.$$

Additionally, since a, b, c span $\mathbb{R}^3$ & $p \cdot b = 0$ we know

$$(p\cdot a)^2 + (p\cdot b)^2 + (p\cdot c)^2 = |p| = 1 \implies |p\cdot c| = \sqrt{1 - (p\cdot a)^2}.$$

And finally, before simplifying our integrand let's note that

$$2r(p\cdot a)p = u \implies |u| = 2r(p\cdot a) \implies (p\cdot a) = \frac{|u|}{2r},$$

and in particular this means $p \cdot a > 0$. Altogether, substituting this into our integrand, and acknowledging χ only depends on u we find

$$\frac{1}{2\sqrt{2}}\frac{1}{r^2}\left|\frac{\mathcal{P}_S b^\perp}{(p\cdot a)(p\cdot c)}\right|\frac{g(a,b,r)\cdot n}{\sqrt{2(p\cdot c)^2+(p\cdot a)^2}} = \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{3+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2(p\cdot a)}{(p\cdot a)|p\cdot c|}\frac{1}{\sqrt{2(p\cdot c)^2+(p\cdot a)^2}}$$
$$= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{3+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{1-(p\cdot a)^2}\sqrt{2-(p\cdot a)^2}}$$
$$= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{3+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{1-\frac{|u|^2}{4r^2}}\sqrt{2-\frac{|u|^2}{4r^2}}}$$
$$= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{1+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{r^2-\frac{|u|^2}{4}}\sqrt{2r^2-\frac{|u|^2}{4}}}.$$

Thus, putting this back into our integral we find

$$\kappa_\sigma \cdot n = \int_{\mathbb{R}^2}\int_{\mathcal{R}(u)}\int_{\mathcal{D}(u,r)}\frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{1+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{r^2-\frac{|u|^2}{4}}\sqrt{2r^2-\frac{|u|^2}{4}}}\,d\mathcal{H}^1(a,b)drd\mathcal{H}^2(u)$$
$$= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\mathcal{R}(u)}\frac{1}{r^{1+\sigma}\sqrt{r^2-\frac{|u|^2}{4}}\sqrt{2r^2-\frac{|u|^2}{4}}}\int_{\mathcal{D}(u,r)}\left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^1(a,b)drd\mathcal{H}^2(u)$$

3.5 Evaluating Inner 1D Integral

The inner most integral is intergrating over all a, b such that the disk of a fixed radius r intersects the $t-n$ plane along the cord from z to $z+u$. We can parameterize a, b via θ as follows[5]:

$$\theta \to \left(\frac{1}{r}\left(\sqrt{r^2-\frac{|u|^2}{4}}\left(\cos\theta p^\perp + \sin\theta m\right)+\frac{|u|}{2}p\right), -\sin\theta p^\perp + \cos\theta m\right)$$

where $m = t \times n$ and θ ranges over $[\vartheta, \vartheta+\pi]$ for some $\vartheta \in [0, 2\pi)$ (due to the restriction of $t \cdot a > 0$)[6]. With this parameterization we're able to compute the inner-most integral, i.e. we have

$$\int_{\mathcal{D}(u,r)}\left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^1(a,b) = \int_{\vartheta}^{\vartheta+\pi}\sin^2\theta\frac{\sqrt{2r^2-\frac{|u|^2}{4}}}{r}\,d\theta = \frac{\pi}{2}\frac{\sqrt{2r^2-\frac{|u|^2}{4}}}{r}$$

3.6 Evaluating Integral over Radii

Our computation simplifies to

$$\kappa_\sigma \cdot n = \frac{-\pi}{4\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\mathcal{R}(u)}\frac{1}{r^{2+\sigma}\sqrt{r^2-\frac{|u|^2}{4}}}\,dr\,d\mathcal{H}^2(u) = \frac{-\pi}{4\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{|u|/2}^{\infty}\frac{1}{r^{2+\sigma}\sqrt{r^2-\frac{|u|^2}{4}}}\,dr\,d\mathcal{H}^2(u).$$

[5]TODO add picture, formalize sign of b
[6]TODO: add picture to help show this geometry

Substitute $r = \frac{|u|}{2}s^{-1/2}$ so that $r \to |u|/2 \implies s \to 1,\ r \to \infty \implies s \to 0,\ dr = \frac{-|u|}{4}s^{-3/2}ds$ and

$$\begin{aligned}
\kappa_\sigma \cdot n &= \frac{\pi}{4\sqrt{2}} \int_{\mathbb{R}^2} \chi(u) \int_1^0 \frac{|u|}{4} \frac{s^{-3/2}}{\frac{|u|^{2+\sigma}}{2^{2+\sigma}} s^{-(2+\sigma)/2} \sqrt{\frac{|u|^2}{4}s^{-1} - \frac{|u|^2}{4}}}\, ds\, d\mathcal{H}^2(u) \\
&= -\pi 2^{\sigma-1/2} \int_{\mathbb{R}^2} \frac{\chi(u)}{|u|^{2+\sigma}} \int_0^1 \frac{|u|}{4} \frac{s^{(2+\sigma-3)/2}}{\frac{|u|}{2}\sqrt{\frac{1}{s}-1}}\, ds\, d\mathcal{H}^2(u) \\
&= -\pi 2^{\sigma-3/2} \int_{\mathbb{R}^2} \frac{\chi(u)}{|u|^{2+\sigma}} \int_0^1 \frac{s^{(\sigma-1)/2}}{\sqrt{\frac{1-s}{s}}}\, ds\, d\mathcal{H}^2(u) \\
&= -\pi 2^{\sigma-3/2} \int_{\mathbb{R}^2} \frac{\chi(u)}{|u|^{2+\sigma}} \int_0^1 s^{\sigma/2}(1-s)^{-1/2}\, ds\, d\mathcal{H}^2(u) \\
&= -\pi 2^{\sigma-3/2} B\left(\frac{\sigma+2}{2}, \frac{1}{2}\right) \int_{\mathbb{R}^2} \frac{\chi(u)}{|u|^{2+\sigma}}\, d\mathcal{H}^2(u).
\end{aligned}$$

After converting to polar coordinates we're able to reuse our 2D calculation to evaluate the final integral, thus

$$\begin{aligned}
\kappa_\sigma \cdot n &= \frac{\pi 2^{\sigma-1/2}}{\sigma} B\left(\frac{\sigma+2}{2}, \frac{1}{2}\right) B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) \\
&= \frac{\pi 2^{\sigma-1/2}}{\sigma} \frac{\Gamma\left(\frac{\sigma+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\sigma+3}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{2-\sigma}{2}\right)} \\
&= \frac{\pi^2 2^{\sigma-1/2}}{\sigma} \frac{\Gamma\left(\frac{\sigma+2}{2}\right)\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{\sigma+3}{2}\right)\Gamma\left(\frac{2-\sigma}{2}\right)}.
\end{aligned}$$

4 Fractional Curvature of Unit Circle in N-D

To begin, fix $\chi(a,b,r)$ to indicate the sign corresponding to a,b,r (i.e. whether a,b,r belong to $\mathcal{A}^+_{\text{even}}$ or $\mathcal{A}^+_{\text{odd}}$, and whether b or $-b$ is needed by the $t \cdot b > 0$ restriction). Put $\mathcal{A}^+ = \mathcal{A}^+_{\text{even}} \cup \mathcal{A}^+_{\text{odd}}$ and

$$g(a,b,r) = \chi(a,b,r) \frac{(\mathbf{a}\cdot\mathbf{t}(z))\mathbf{b} - (\mathbf{b}\cdot\mathbf{t}(z))\mathbf{a}}{r^{1+\sigma}},$$

so that our desired computation is:

$$\kappa_\sigma(z) \cdot n = \int_{\mathcal{A}^+} g(a,b,r) \cdot n\, d\mathcal{H}^{2n-2}(\mathbf{a},\mathbf{b},r),$$

4.1 Slicing out the 2D plane

To simplify our domain group the disks by their intersection with the $S := \operatorname{lsp}\{t,n\}$ plane, i.e. put $\psi : \mathcal{A}^+ \to \mathbb{R}^2$ so that ψ maps a disk to a vector representing $\mathcal{D}(a,b,r) \cap S$. The same constraints that applied in 3D above apply once again here, so that, with $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$ we have

$$\psi(a,b,r) = 2r(p(b) \otimes p(b))a.$$

4.2 2D Co-Area Calculation

To begin our calculation put $c = \frac{u-(u\cdot a)a}{\sqrt{u\cdot u-(u\cdot a)^2}}$ and $\{e_i\}_{i=1}^{n-3}$ so that $\operatorname{lsp}\{a, b, c, e_1, ..., e_{n-3}\} = \mathbb{R}^n$. Notice that

$$\operatorname{lsp}\{a, b, p\} = \operatorname{lsp}\left\{a, b, \frac{u}{|u|}\right\} = \operatorname{lsp}\{a, b, u\} \subset \operatorname{lsp}\{a, b, c\},$$

and that a, b, c are orthonormal. Further, we have

$$T_{(a,b,r)}(\mathcal{U}_\perp^2 \times \mathbb{R}^+) = \operatorname{lsp}\left\{(e_i, 0, 0), (0, e_i, 0), (c, 0, 0), (0, c, 0), \frac{1}{\sqrt{2}}(b, -a, 0), (0, 0, 1) \mid i = 1, ...(n-3)\right\}.$$

Put $p = p(b)$ so that $p, p^\perp$ spans $T_{\psi(a,b,r)}(\mathbb{R}^2)$. We start with a quick calculation; suppose $\beta : \mathbb{R} \to \mathcal{U}(\mathbb{R}^n)$ such that $\beta(0) = b$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} p(\beta(s))\bigg|_{s=0} &= \frac{\mathcal{P}_S \beta'(0)^\perp}{|\mathcal{P}_S b^\perp|} - \frac{1}{|\mathcal{P}_S b^\perp|^3} \mathcal{P}_S b^\perp \otimes \left(\mathcal{P}_S \beta'(0)^\perp\right)^T \mathcal{P}_S b^\perp \\
&= \frac{1}{|\mathcal{P}_S b^\perp|}\left(\mathcal{P}_S \beta'(0)^\perp - \frac{1}{|\mathcal{P}_S b^\perp|^2}\left(\mathcal{P}_S \beta'(0)^\perp \cdot \mathcal{P}_S b^\perp\right)\mathcal{P}_S b^\perp\right) \\
&= \frac{1}{|\mathcal{P}_S b^\perp|}(1 - (p \otimes p))\mathcal{P}_S \beta'(0)^\perp.
\end{aligned}$$

Since we'll be working in the $p, p^\perp$ coordinate system, it makes sense to expand this result as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s} p(\beta(s))\bigg|_{s=0} = \frac{1}{|\mathcal{P}_S b^\perp|}\left(p \otimes p + p^\perp \otimes p^\perp - p \otimes p\right)\mathcal{P}_S \beta'(0)^\perp = \frac{\left(p^\perp \otimes p^\perp\right)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S \beta'(0)^\perp \tag{11}$$

Now, put $\gamma_v : \mathbb{R} \to \mathcal{U}_\perp^2 \times \mathbb{R}^+$ to be such that $\gamma_v(0) = (a, b, r)$ and $\gamma_v'(0) = v$, then we begin by computing the derivative along the $\gamma_{(e_i,0,0)}$ flow:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(e_i,0,0)}(s))\bigg|_{s=0} = 2r(p \otimes p)e_i = 2r(p \cdot e_i)p = 0. \tag{12}$$

Similarly we can compute along the $(c, 0, 0)$ flow:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(c,0,0)}(s))\bigg|_{s=0} = 2r(p \otimes p)e_i = 2r(p \cdot c)p. \tag{13}$$

Next we compute along the $\{(0, v, 0) \mid v = e, e_1, e_2, ..., e_{n-3}\}$ flows, and simplify using (11)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,v,0)}(s))\bigg|_{s=0} &= 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{(0,v,0),2}(s))\bigg|_{s=0}\right) \otimes p + p \otimes \left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{(0,v,0),2}(s))\bigg|_{s=0}\right)\right)a \\
&= 2r\left(\left(\frac{\left(p^\perp \otimes p^\perp\right)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S v^\perp\right) \otimes p + p \otimes \left(\frac{\left(p^\perp \otimes p^\perp\right)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S v^\perp\right)\right)a \\
&= 2r\frac{p^\perp \cdot \mathcal{P}_S v^\perp}{|\mathcal{P}_S b^\perp|}\left(p^\perp \otimes p + p \otimes p^\perp\right)a.
\end{aligned}$$

Taking into account the fact that $p^\perp \cdot \mathcal{P}_S v^\perp = p \cdot \mathcal{P}_S v = p \cdot v$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,v,0)}(s))\bigg|_{s=0} = 2r\frac{(p \cdot v)\left(p^\perp \cdot a\right)}{|\mathcal{P}_S b^\perp|}p + 2r\frac{(p \cdot v)(p \cdot a)}{|\mathcal{P}_S b^\perp|}p^\perp. \tag{14}$$

Notably, when $v = e_i$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,e_i,0)}(s))\bigg|_{s=0} = 0.$$

The next derivative we must compute is along the $\frac{1}{\sqrt{2}}(b,-a,0)$ flow:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\bigg|_{s=0} &= \sqrt{2}r(p\otimes p)b + 2r\left(\left(\frac{\mathrm{d}}{\mathrm{d}s}p(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0),2}(s))\bigg|_{s=0}\right)\odot p\right)a \\
&= 2r\left(\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\frac{-a^\perp}{\sqrt{2}}\right)\otimes p + p\otimes\left(\frac{(p^\perp\otimes p^\perp)}{|\mathcal{P}_S b^\perp|}\mathcal{P}_S\frac{-a^\perp}{\sqrt{2}}\right)\right)a \\
&= -\sqrt{2}r\frac{p^\perp\cdot\mathcal{P}_S a^\perp}{|\mathcal{P}_S b^\perp|}(p^\perp\otimes p + p\otimes p^\perp)a.
\end{aligned}$$

Note that the first term vanishes because $p\cdot b = 0$. Again taking into account the fact that $p^\perp\cdot\mathcal{P}_S a^\perp = p\cdot\mathcal{P}_S a = p\cdot a$, and expanding the tensor products we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{\frac{1}{\sqrt{2}}(b,-a,0)}(s))\bigg|_{s=0} = -\sqrt{2}r\frac{(p\cdot a)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|}p - \sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^\perp|}p^\perp. \tag{15}$$

Lastly computing the derivative through the $(0,0,1)$ flow we have:

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi(\gamma_{(0,0,1)}(s))\bigg|_{s=0} = 2(p\otimes p)a = 2(p\cdot a)p \tag{16}$$

Combining (13), (14), (15), (16) we have

$$\nabla\psi(a,b,r) = \begin{array}{cc} \begin{array}{cccccc} (e_i,0,0) & (0,e_i,0) & (c,0,0) & (0,c,0) & \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) \end{array} & \\ \begin{pmatrix} 0 & 0 & 2r(p\cdot c) & 2r\frac{(p\cdot c)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|} & -\sqrt{2}r\frac{(p\cdot a)(p^\perp\cdot a)}{|\mathcal{P}_S b^\perp|} & 2(p\cdot a) \\ 0 & 0 & 0 & 2r\frac{(p\cdot c)(p\cdot a)}{|\mathcal{P}_S b^\perp|} & -\sqrt{2}r\frac{(p\cdot a)^2}{|\mathcal{P}_S b^\perp|} & 0 \end{pmatrix} & \begin{array}{c} p \\ p^\perp \end{array} \end{array}$$

Notably the jacobian factor is identical to the jacobian factor in 3D, so we have

$$J\psi(a,b,r) = \sqrt{|\nabla\psi(a,b,r)^T\nabla\psi(a,b,r)|} = 2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}.$$

Put $\mathcal{D}(u) := \psi^{-1}(\{u\}), p = p(b)$ so that

$$\begin{aligned}
\kappa_\sigma\cdot n &= \int_{\mathbb{R}^2}\int_{\mathcal{D}(u)}\frac{g(a,b,r)\cdot n}{J\psi(a,b,r)}\,d\mathcal{H}^{2n-4}(a,b,r)d\mathcal{H}^2(u) \\
&= \int_{\mathbb{R}^2}\int_{\mathcal{D}(u)}\frac{g(a,b,r)\cdot n}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2 + (p\cdot a)^2}\sqrt{r^2(p\cdot c)^2 + (p\cdot a)^2}}\,d\mathcal{H}^{2n-4}(a,b,r)d\mathcal{H}^2(u).
\end{aligned}$$

4.3 Slicing out Radii

Now, to simplify $\mathcal{D}(u)$ lets group sets of $\mathbf{a},\mathbf{b}$ that correspond to a given r, i.e. put $\phi:\mathcal{D}(u)\to\mathbb{R}^+$ given by

$$\phi(a,b,r) = r.$$

To begin our calculation of $\nabla\phi$ we must characterize $T_{(a,b,r)}(\mathcal{D}(u))$, i.e. via finding an orthonormal basis. Suppose $\gamma : \mathbb{R} \to \mathcal{D}(u)$ is so that $\gamma(0) = (a, b, r)$ and put $(\alpha, \beta, \tau) := \gamma'(0)$. Note $\alpha \cdot a = \beta \cdot b = 0$ and $\alpha \cdot b + a \cdot \beta = 0$, since $(a, b) \in \mathcal{U}_2^{\perp}$. By the definition of $\mathcal{D}(u)$ we know

$$2r(p(b) \otimes p(b))a = u,$$

so that

$$2\gamma_3(s)(p(\gamma_2(s)) \otimes p(\gamma_2(s)))\gamma_1(s) = u.$$

Differentiating[7] and evaluating at $s = 0$ we see

$$0 = \tau(p(b) \otimes p(b))a + r\left(\frac{(p^{\perp} \otimes p^{\perp})}{|\mathcal{P}_S b^{\perp}|}\mathcal{P}_S \beta^{\perp} \odot p(b)\right)a + r(p(b) \otimes p(b))\alpha.$$

Expanding tensor products, simplifying and using the $p, p^{\perp}$ coordinate system we get

$$\begin{aligned}
0 &= \tau(p \cdot a)p + r\left(\frac{p \cdot \beta}{|\mathcal{P}_S b^{\perp}|}p^{\perp} \odot p\right)a + r(p \cdot \alpha)p \\
&= (\tau(p \cdot a) + r(p \cdot \alpha))p + r\frac{(p \cdot \beta)}{|\mathcal{P}_S b^{\perp}|}\left(p^{\perp} \otimes p + p \otimes p^{\perp}\right)a \\
&= \left(\tau(p \cdot a) + r(p \cdot \alpha) + r\frac{(p \cdot \beta)}{|\mathcal{P}_S b^{\perp}|}(p^{\perp} \cdot a)\right)p + r\frac{(p \cdot \beta)}{|\mathcal{P}_S b^{\perp}|}(p \cdot a)p^{\perp}.
\end{aligned}$$

Notably, we must have $p \cdot \beta = 0$, so our constraint simplifies to

$$0 = \tau(p \cdot a) + r(p \cdot \alpha).$$

Observe $(e_i, 0, 0), (0, e_i, 0)$ are $2n - 6$ valid basis vectors abiding by our constraints. Additionally, fixing $\beta = 0$ and considering $\alpha \neq e_i$ we must have $\alpha = c$, and thus

$$\mu = \frac{1}{\sqrt{1 + r^2\frac{(p\cdot c)^2}{(p\cdot a)^2}}}\left(c, 0, -r\frac{(p \cdot c)}{(p \cdot a)}\right),$$

is an additional basis vector. Put $\mathcal{R}(u, r) = \sqrt{r^2 - \frac{|u|^2}{4}}$ and consider the vector

$$\nu = \left(\frac{\mathcal{R}(u, r)}{\mathcal{R}(u, \sqrt{2}r)}b, \frac{r}{2\mathcal{R}(u, r)\mathcal{R}(u, \sqrt{2}r)}(u - 2ra), 0\right).$$

Observe

$$\begin{aligned}
|\nu|^2 &= \frac{\mathcal{R}^2(u, r)}{\mathcal{R}^2(u, \sqrt{2}r)} + \frac{r^2}{4\mathcal{R}^2(u, r)\mathcal{R}^2(u, \sqrt{2}r)}\left(|u|^2 + 4r^2 - 4r(u \cdot a)\right) \\
&= \frac{\mathcal{R}^2(u, r)}{\mathcal{R}^2(u, \sqrt{2}r)} + \frac{r^2}{4\mathcal{R}^2(u, r)\mathcal{R}^2(u, \sqrt{2}r)}\left(|u|^2 + 4r^2 - 4r\frac{|u|^2}{2r}\right) \\
&= \frac{\mathcal{R}^2(u, r)}{\mathcal{R}^2(u, \sqrt{2}r)} + \frac{r^2}{4\mathcal{R}^2(u, r)\mathcal{R}^2(u, \sqrt{2}r)}\left(4r^2 - |u|^2\right) \\
&= \frac{\mathcal{R}^2(u, r)}{\mathcal{R}^2(u, \sqrt{2}r)} + \frac{r^2}{\mathcal{R}^2(u, \sqrt{2}r)} = 1,
\end{aligned}$$

[7] N.B. we use (11)

so that ν is the final basis vector spanning $T_{(a,b)}\mathcal{D}(u)$. Now, to compute $J\phi$, for $v \in \{\mu, \nu\}$, put $\gamma_v : \mathbb{R} \to \mathcal{D}(u)$ so that $\gamma_v(0) = (a, b, r)$ and $\gamma'(0) = v$. Then we have a few trivial derivatives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_{(e_i,0,0)}(s))\bigg|_{s=0} &= 0.\\ \frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_{(0,e_i,0)}(s))\bigg|_{s=0} &= 0.\\ \frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_{\nu}(s))\bigg|_{s=0} &= 0,\end{aligned}$$

Now we're able to compute the derivative along the non-trivial flow

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi(\gamma_{\mu}(s))\bigg|_{s=0} &= \frac{-r(p\cdot c)}{(p\cdot a)\sqrt{1+r^2\frac{(p\cdot c)^2}{(p\cdot a)^2}}}\\ &= -r\frac{(p\cdot c)}{\sqrt{(p\cdot a)^2+r^2(p\cdot c)^2}}\end{aligned}$$

so that

$$J\phi = \frac{r|p\cdot c|}{\sqrt{(p\cdot a)^2+r^2(p\cdot c)^2}}.$$

4.4 Simplifying Computation

Put $\mathcal{D}(u, r) = \phi^{-1}(\{r\})$ so that

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathcal{D}(u)} \frac{g(a,b,r)\cdot n}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2+(p\cdot a)^2}\sqrt{r^2(p\cdot c)^2+(p\cdot a)^2}}\,d\mathcal{H}^{2n-4}(a,b,r)d\mathcal{H}^2(u)\\ &= \int_{\mathbb{R}^2}\int_{\phi(\mathcal{D}(u))}\int_{\mathcal{D}(u,r)} \frac{g(a,b,r)\cdot n}{2\sqrt{2}r\left|\frac{p\cdot a}{\mathcal{P}_S b^\perp}\right|\sqrt{2(p\cdot c)^2+(p\cdot a)^2}\sqrt{r^2(p\cdot c)^2+(p\cdot a)^2}}\frac{\sqrt{(p\cdot a)^2+r^2(p\cdot c)^2}}{r|p\cdot c|}\,d\mathcal{H}^{2n-5}(a,b)drd\mathcal{H}^2(u)\\ &= \int_{\mathbb{R}^2}\int_{\phi(\mathcal{D}(u))}\int_{\mathcal{D}(u,r)} \frac{(g(a,b,r)\cdot n)\left|\mathcal{P}_S b^\perp\right|}{2\sqrt{2}|(p\cdot a)(p\cdot c)|r^2\sqrt{2(p\cdot c)^2+(p\cdot a)^2}}\,d\mathcal{H}^{2n-5}(a,b)drd\mathcal{H}^2(u)\end{aligned}$$

In order to simplify this, recall

$$g(a,b,r) = \chi(a,b,r)\frac{(a\cdot t)b-(b\cdot t)a}{r^{1+\sigma}},$$

Notice from 5 we see

$$(a\cdot t)(b\cdot n)-(b\cdot t)(a\cdot n) = -\mathcal{P}_S b^\perp\cdot a = -\left|\mathcal{P}_S b^\perp\right|(p\cdot a),$$

so

$$g(a,b,r)\cdot n = \chi(a,b,r)\frac{(a\cdot t)(b\cdot n)-(b\cdot t)(a\cdot n)}{r^{1+\sigma}} = -\chi(a,b,r)\frac{\left|\mathcal{P}_S b^\perp\right|(p\cdot a)}{r^{1+\sigma}}.$$

We also know $p \in \operatorname{lsp}\{a, b, c\}$ so that

$$(p\cdot a)^2+(p\cdot b)^2+(p\cdot c)^2 = |p| = 1 \implies |p\cdot c| = \sqrt{1-(p\cdot a)^2}.$$

And finally, before simplifying our integrand let's note that

$$2r(p\cdot a)p = u \implies |u| = 2r(p\cdot a) \implies (p\cdot a) = \frac{|u|}{2r},$$

and in particular this means $p\cdot a > 0$. Altogether, substituting this into our integrand, and acknowledging χ only depends on u we find

$$\begin{aligned}
\frac{1}{2\sqrt{2}}\frac{1}{r^2}\left|\frac{\mathcal{P}_S b^\perp}{(p\cdot a)(p\cdot c)}\right|\frac{g(a,b,r)\cdot n}{\sqrt{2(p\cdot c)^2+(p\cdot a)^2}} &= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{3+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2(p\cdot a)}{(p\cdot a)|p\cdot c|}\frac{1}{\sqrt{2(p\cdot c)^2+(p\cdot a)^2}}\\
&= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{3+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{1-(p\cdot a)^2}\sqrt{2-(p\cdot a)^2}}\\
&= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{3+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{1-\frac{|u|^2}{4r^2}}\sqrt{2-\frac{|u|^2}{4r^2}}}\\
&= \frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{1+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\mathcal{R}(u,r)\mathcal{R}(u,\sqrt{2}r)}.
\end{aligned}$$

Thus, putting this back into our integral we find

$$\begin{aligned}
\kappa_\sigma\cdot n &= \int_{\mathbb{R}^2}\int_{\phi(\mathcal{D}(u))}\int_{\mathcal{D}(u,r)}\frac{-\chi(u)}{2\sqrt{2}}\frac{1}{r^{1+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|^2}{\mathcal{R}(u,r)\mathcal{R}(u,\sqrt{2}r)}\,d\mathcal{H}^{2n-5}(a,b)drd\mathcal{H}^2(u)\\
&= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{1}{r^{1+\sigma}\mathcal{R}(u,r)\mathcal{R}(u,\sqrt{2}r)}\int_{\mathcal{D}(u,r)}\left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^{2n-5}(a,b)drd\mathcal{H}^2(u)
\end{aligned}$$

4.5 Splitting Spheres

In order to compute the inner most integral we will apply the co-area formula once again to slice out all $\mathcal{B}(u) = \mathcal{U}\Big(\{u\}^\perp\Big)$, i.e. the projection of $\mathcal{D}(u,r)$ onto the b coordinates. Concretely, our slicing function $\xi : \mathcal{D}(u,r) \to \mathcal{B}(u)$ is given by $\xi(a,b) = b$. From our above calculations we know

$$T_{(a,b)}\mathcal{D}(u,r) = \operatorname{lsp}\{(e_i,0),(0,e_i),\nu\},$$

where, for $\mathcal{R}(u,r) = \sqrt{r^2 - \frac{|u|^2}{4}}$, we put

$$\nu = \Bigg(\frac{\mathcal{R}(u,r)}{\mathcal{R}(u,\sqrt{2}r)}b, \frac{r}{2\mathcal{R}(u,r)\mathcal{R}(u,\sqrt{2}r)}(u-2ra)\Bigg).$$

Similarly, we know $T_b\mathcal{B}(u,r) = \operatorname{lsp}\Big\{e_i, \frac{\nu_2}{|\nu_2|}\Big\}$ and so we can compute

$$\nabla\xi(a,b) = \begin{array}{cc} \begin{array}{ccc} (e_i,0) & (0,e_i) & \nu \end{array} & \\ \begin{pmatrix} 0 & I & 0\\ 0 & 0 & |\nu_2| \end{pmatrix} & \begin{array}{c} e_i \\ \frac{\nu_2}{|\nu_2|}\end{array} \end{array} \implies \nabla\xi(a,b)(\nabla\xi(a,b))^T = \begin{array}{cc} \begin{array}{cc} \overbrace{\quad}^{n-3} & \overbrace{\quad}^{1} \end{array} & \\ \begin{pmatrix} I & 0\\ 0 & |\nu_2|^2 \end{pmatrix} & \begin{array}{c} n-3 \\ 1 \end{array} \end{array}.$$

This tells us

$$\begin{aligned}
J\xi(a,b) = |\nu_2| &= \sqrt{\frac{r^2}{4\mathcal{R}^2(u,r)\mathcal{R}^2(u,\sqrt{2}r)}\Big(|u|^2 + 4r^2 - 4r(u\cdot a)\Big)} \\
&= \sqrt{\frac{r^2}{4\mathcal{R}^2(u,r)\mathcal{R}^2(u,\sqrt{2}r)}\left(|u|^2 + 4r^2 - 4r\frac{|u|^2}{2r}\right)} \\
&= \sqrt{\frac{r^2}{4\mathcal{R}^2(u,r)\mathcal{R}^2(u,\sqrt{2}r)}4\mathcal{R}^2(u,r)} \\
&= \sqrt{\frac{r^2}{\mathcal{R}^2(u,\sqrt{2}r)}} = \frac{r}{\mathcal{R}(u,\sqrt{2}r)}.
\end{aligned}$$

Putting $\mathcal{A}(u,r,b) = \xi^{-1}(\{b\})$ and plugging this back into our desired calculation we find

$$\begin{aligned}
\kappa_\sigma \cdot n &= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{1}{r^{1+\sigma}\mathcal{R}(u,r)\mathcal{R}(u,\sqrt{2}r)}\int_{\mathcal{B}(u)}\left|\mathcal{P}_S b^\perp\right|^2\int_{\mathcal{A}(u,r,b)}\frac{1}{J\xi(a,b)}\,d\mathcal{H}^{n-3}(a)\,d\mathcal{H}^{n-2}(b)\,dr\,d\mathcal{H}^2(u) \\
&= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{\mathcal{R}(u,\sqrt{2}r)}{r^{2+\sigma}\mathcal{R}(u,r)\mathcal{R}(u,\sqrt{2}r)}\int_{\mathcal{B}(u)}\left|\mathcal{P}_S b^\perp\right|^2\int_{\mathcal{A}(u,r,b)}d\mathcal{H}^{n-3}(a)\,d\mathcal{H}^{n-2}(b)\,dr\,d\mathcal{H}^2(u) \\
&= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{1}{r^{2+\sigma}\mathcal{R}(u,r)}\int_{\mathcal{B}(u)}\left|\mathcal{P}_S b^\perp\right|^2\int_{\mathcal{A}(u,r,b)}d\mathcal{H}^{n-3}(a)\,d\mathcal{H}^{n-2}(b)\,dr\,d\mathcal{H}^2(u)
\end{aligned}$$

4.6 Inner Sphere Volume

We begin by seeing

$$\left(a - \frac{u}{2r}\right)\cdot\left(a - \frac{u}{2r}\right) = 1 + \frac{|u|^2}{4r^2} - 2\frac{(a\cdot u)}{2r} = 1 + \frac{|u|^2}{4r^2} - 2\frac{|u|^2}{4r^2} = 1 - \frac{|u|^2}{4r^2},$$

thus we're able to rewrite our sphere as

$$\mathcal{A}(u,r,b) = \left\{a \in \mathcal{U}\Big(\{b\}^\perp\Big) \mid \left|a - \frac{u}{2r}\right| = \sqrt{1 - \frac{|u^2|}{4r^2}}\right\}.$$

This is an $n-3$ dimensional sphere located at $\frac{u}{2r}$ with radius $\sqrt{1 - \frac{|u|^2}{4r^2}}$ and thus we have

$$\begin{aligned}
\int_{\mathcal{A}(u,r,b)} d\mathcal{H}^{n-3}(a) &= \mathcal{H}^{n-3}(\mathcal{A}(u,r,b)) = S^{n-3}\left(\sqrt{1 - \frac{|u|^2}{4r^2}}\right) \\
&= \frac{2\pi^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)}\left(1 - \frac{\left|u^2\right|}{4r^2}\right)^{(n-3)/2} = \frac{2\pi^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{r^{n-3}}\mathcal{R}^{n-3}(u,r).
\end{aligned}$$

Putting this back into our top-level integral gives us

$$
\begin{aligned}
&\frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{1}{r^{2+\sigma}\mathcal{R}(u,r)}\int_{\mathcal{B}(u)}\left|\mathcal{P}_S b^{\perp}\right|^2\int_{\mathcal{A}(u,r,b)} d\mathcal{H}^{n-3}(a)\,d\mathcal{H}^{n-2}(b)\,dr\,d\mathcal{H}^2(u)\\
&\quad=\frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{1}{r^{2+\sigma}\mathcal{R}(u,r)}\int_{\mathcal{B}(u)}\left|\mathcal{P}_S b^{\perp}\right|^2\frac{2\pi^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{r^{n-3}}\mathcal{R}^{n-3}(u,r)\,d\mathcal{H}^{n-2}(b)\,dr\,d\mathcal{H}^2(u)\\
&\quad=\frac{-\pi^{n/2-1}}{\sqrt{2}\Gamma\left(\frac{n}{2}\right)}\int_{\mathbb{R}^2}\chi(u)\int_{\phi(\mathcal{D}(u))}\frac{\mathcal{R}^{n-4}(u,r)}{r^{\sigma+n-1}}\int_{\mathcal{B}(u)}\left|\mathcal{P}_S b^{\perp}\right|^2 d\mathcal{H}^{n-2}(b)\,dr\,d\mathcal{H}^2(u).
\end{aligned}
$$

4.7 Outer Sphere Volume

Notice $\mathcal{B}(u)=\mathcal{U}\Big(\{u\}^{\perp}\Big)$ and put $\zeta:\mathcal{B}(u)\to[-1,1]$ given by

$$\zeta(b)=b\cdot\frac{u^{\perp}}{|u^{\perp}|}.$$

Since $\mathbb{R}^n=\operatorname{lsp}\Big\{\frac{u}{|u|},\frac{u^{\perp}}{|u^{\perp}|},f_1,f_2,\ldots,f_{n-2}\Big\}$, for some orthonormal $\{f_i\}$, we have

$$\zeta(b)^2+(b\cdot f_1)^2+(b\cdot f_2)^2+\cdots+(b\cdot f_{n-2})^2=1\implies\sqrt{(b\cdot f_1)+(b\cdot f_2)+\cdots+(b\cdot f_{n-2})^2}=\sqrt{1-\zeta(b)^2},$$

thus

$$\zeta^{-1}(\{v\})=\left\{b\in\mathcal{B}(u)\mid b\cdot\frac{u^{\perp}}{|u^{\perp}|}=v,\ \sqrt{(b\cdot f_1)+(b\cdot f_2)+\cdots+(b\cdot f_{n-2})^2}=\sqrt{1-q^2}\right\},$$

and so

$$\int_{\zeta^{-1}(v)} d\mathcal{H}^{n-3}(x)=\mathcal{H}^{n-3}\big(\zeta^{-1}(v)\big)=\mathcal{H}^{n-3}\Big(S_{n-3}\Big(\sqrt{1-v^2}\Big)\Big)=\frac{2\pi^{n/2-1}}{\Gamma(n/2-1)}\big(1-v^2\big)^{(n-3)/2}.$$

In order to apply the co-area formula we also need to compute $J\zeta$. To that end, notice $\mathbb{R}^n=\operatorname{lsp}\Big\{\frac{u}{|u|},\frac{u^{\perp}}{|u^{\perp}|},u^*,g_1,\ldots,g_{n-3}\Big\}$ when

$$u^*=\frac{b-\Big(b\cdot\frac{u^{\perp}}{|u^{\perp}|}\Big)\frac{u^{\perp}}{|u|^{\perp}}}{\sqrt{1+\Big(b\cdot\frac{u^{\perp}}{|u^{\perp}|}\Big)^2}},$$

and thus

$$T_b(\mathcal{B}(u))=\operatorname{lsp}\left\{\frac{u^{\perp}}{|u^{\perp}|},g_1,\ldots g_{n-3}\right\}.$$

With this characterization we're able to see

$$\nabla\zeta(b)=\begin{array}{c}\begin{array}{cccc}\frac{u^{\perp}}{|u^{\perp}|} & g_1 & \cdots & g_{n-3}\end{array}\\ \left(\begin{array}{cccc}1 & 0 & \cdots & 0\end{array}\right)\end{array}\ 1\implies J\zeta(b)=1.$$

Lastly, before applying the co-area formula, notice

$$\left|\mathcal{P}_S b^{\perp}\right|=\left|\left(\left(b\cdot\frac{u}{|u|}\right)\frac{u}{|u|}+\left(b\cdot\frac{u^{\perp}}{|u^{\perp}|}\right)\frac{u^{\perp}}{|u^{\perp}|}\right)^{\perp}\right|=\left|b\cdot\frac{u^{\perp}}{|u^{\perp}|}\right|$$

Now we're ready to finally apply the co-area formula to see

$$\begin{aligned}\int_{\mathcal{B}(u)} \left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^{n-2}(b) &= \int_{-1}^{1} |v|^2 \int_{\zeta^{-1}(\{v\})} d\mathcal{H}^{n-3}(x)\, dv \\ &= \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} \int_{-1}^{1} v^2 \big(1-v^2\big)^{(n-3)/2} dv \\ &= \frac{4\pi^{n/2-1}}{\Gamma(n/2-1)} \int_{0}^{1} v^2 \big(1-v^2\big)^{(n-3)/2} dv \\ &= \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} \int_{0}^{1} v^{1/2} (1-v)^{(n-3)/2} dv = \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} B\bigg(\frac{3}{2}, \frac{n-1}{2}\bigg),\end{aligned}$$

where we use evenness of the integrand & the transformation $t^2 \to t$ for the last two lines. Plugging this back into our top-level calculation gives the following:

$$\begin{aligned}&\frac{-\pi^{n/2-1}}{\sqrt{2}\Gamma\big(\frac{n}{2}\big)} \int_{\mathbb{R}^2} \chi(u) \int_{\phi(\mathcal{D}(u))} \frac{\mathcal{R}^{n-4}(u,r)}{r^{\sigma+n-1}} \int_{\mathcal{B}(u)} \left|\mathcal{P}_S b^\perp\right|^2 d\mathcal{H}^{n-2}(b)\, dr\, d\mathcal{H}^2(u). \\ &\quad = \frac{-\pi^{n/2-1}}{\sqrt{2}\Gamma\big(\frac{n}{2}\big)} \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} B\bigg(\frac{3}{2}, \frac{n-1}{2}\bigg) \int_{\mathbb{R}^2} \chi(u) \int_{\phi(\mathcal{D}(u))} \frac{\mathcal{R}^{n-4}(u,r)}{r^{\sigma+n-1}}\, dr\, d\mathcal{H}^2(u)\end{aligned}$$

4.8 Evaluating Integral over Radii

Our computation simplifies to

$$\kappa_\sigma \cdot n = \frac{-\sqrt{2}\pi^{n-2}\big(\frac{n}{2}-1\big)}{\big(\Gamma\big(\frac{n}{2}\big)\big)^2} B\bigg(\frac{3}{2}, \frac{n-1}{2}\bigg) \int_{\mathbb{R}^2} \chi(u) \int_{\frac{|u|}{2}}^{\infty} r^{1-\sigma-n} \left(r^2 - \frac{|u|^2}{4}\right)^{(n-4)/2} dr\, d\mathcal{H}^2(u).$$

Using the transformation $r \to \frac{|u|}{2} s^{-1/2} \implies dr = \frac{-|u|}{4} s^{-3/2} ds$ and $r \to |u|/2 \implies s \to 1,\, r \to \infty \implies s \to 0$ we find

$$\begin{aligned}\int_{\frac{|u|}{2}}^{\infty} r^{1-\sigma-n} \left(r^2 - \frac{|u|^2}{4}\right)^{(n-4)/2} dr &= \left(\frac{|u|}{2}\right)^{1-\sigma-n} \int_0^1 s^{(n+\sigma-1)/2} \left(\frac{|u|^2}{4}\frac{1}{s} - \frac{|u|^2}{4}\right)^{(n-4)/2} \frac{|u|}{4} s^{-3/2} ds \\ &= \frac{|u|^{2-\sigma-n}}{2^{3-\sigma-n}} \left(\frac{|u|}{2}\right)^{n-4} \int_0^1 s^{(n+\sigma-4)/2} \left(\frac{1-s}{s}\right)^{(n-4)/2} ds \\ &= \frac{|u|^{-\sigma-2}}{2^{-\sigma-1}} \int_0^1 s^{\sigma/2} (1-s)^{(n-4)/2} ds \\ &= \frac{2^{1+\sigma}}{|u|^{2+\sigma}} B\bigg(\frac{\sigma+2}{2}, \frac{n-2}{2}\bigg).\end{aligned}$$

Plugging this back into the above integral we find

$$\kappa_\sigma \cdot n = \frac{-2^{3/2+\sigma}\pi^{n-2}\big(\frac{n}{2}-1\big)}{\big(\Gamma\big(\frac{n}{2}\big)\big)^2} B\bigg(\frac{3}{2}, \frac{n-1}{2}\bigg) B\bigg(\frac{\sigma+2}{2}, \frac{n-2}{2}\bigg) \int_{\mathbb{R}^2} \frac{\chi(u)}{|u|^{2+\sigma}}\, d\mathcal{H}^2(u).$$

Just as with the 3D calculation, we're able to convert to polar coordinates and reuse the 2D calculation to find

$$\kappa_\sigma \cdot n = \frac{-2^{3+\sigma}\pi^{n-2}\big(\frac{n}{2}-1\big)}{\sigma\big(\Gamma\big(\frac{n}{2}\big)\big)^2} B\bigg(\frac{3}{2}, \frac{n-1}{2}\bigg) B\bigg(\frac{\sigma+2}{2}, \frac{n-2}{2}\bigg) B\bigg(\frac{1}{2}, \frac{1-\sigma}{2}\bigg).$$