Behaviors of Seguinian nonlocal curvature

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Abstract

We study the Seguinian notion nonlocal curvature of a curve and show that to classical signed curvature can be recovered under an appropriate limit, akin to other nonlocal recovery results. It's also shown that Seguinian nonlocal curvature of a curve in a k dimensional subspace is also lives in the same k dimensional subspace, further showing parallels between Seguinian nonlocal and classical signed curvature.

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1 Introduction

1.1 Background

Seguinian nonlocal curvature of curves was introduced by Seguin [2] who defined the $\sigma - \mathit{length}$ for $\sigma \in (0,1)$ of a curve $\mathcal{C}$ relative to an open, bounded set with smooth boundary $\Omega \subset \mathbb{R}^n$ by

$$\mathrm{Len}_\sigma(\mathcal{C}, \Omega) := \int_{\mathcal{D}(\mathcal{C})} r^{1-n-\sigma} \sup_{v \in \mathcal{U} \cap \{u\}^\perp} \chi_\Omega(p + rv) d\mathcal{H}^{2n}(p, u, r).$$

It was shown the corresponding Euler-Lagrange is

$$\left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} \, d\mathcal{H}^{2n-2}(a, b, r), \qquad \text{for all } z \in \mathcal{C},$$

which, since the minimization of classical length occurs when classical curvature vanishes, motivated the Seguinian notion of nonlocal curvature as

$$\kappa_\sigma(z) := \left(\int_{\mathcal{A}^+_{\text{even}}} - \int_{\mathcal{A}^+_{\text{odd}}} \right) \frac{(a \cdot t(z))b - (b \cdot t(z))a}{r^{1+\sigma}} \, d\mathcal{H}^{2n-2}(a, b, r), \qquad \text{for all } z \in \mathcal{C}. \tag{1}$$

This definition is analogous to Abatangelo and Valdinoci's [1] a notion of nonlocal mean curvature of an surface $E \subset \mathbb{R}^n$

$$H_\sigma(z) := \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{E^c}}{|z - x|^{n+\sigma}} dx \qquad \text{for all } z \in \partial E.$$

Abatangelo and Valdinoci showed classical mean curvature can be recovered under an appropriate limit, that is

$$\lim_{\sigma \uparrow 1} (1 - \sigma) H_\sigma(z) = H(z).$$

These two notions of curvature match, up to a multiplicative constant, in $\mathbb{R}^2$. Consequently its known, with appropriate scaling classical signed curvature is recovered from Seguinian nonlocal curvature via the above limit relation in $\mathbb{R}^2$.

1.2 Motivation

The above result in $\mathbb{R}^2$ motivates finding the appropriate scaling factor $\Lambda_{2,\sigma}$ to recover classical curvature. Further, it makes sense to extend this result to $\mathbb{R}^n$ so that the signed curvature of any C^2 curve $\mathcal{C} \subset \mathbb{R}^n$ with non-vanishing signed curvature can be recovered via

$$\lim_{\sigma \uparrow 1} (1 - \sigma) \Lambda_{n,\sigma} \kappa_\sigma(z) = \kappa(z) \qquad \text{for all } z \in \mathcal{C}.$$

Motivated by the relationship between classical curvature of a curve $\mathcal{C}$ and the osculating circle at a point $z \in \mathcal{C}$ $\mathrm{osc}(z)$, we'll show Seguinian curvature abides by

$$\lim_{\sigma \uparrow 1} \left| \kappa_\sigma^{\mathrm{osc}}(z) - \kappa_\sigma^{\mathcal{C}}(z) \right| = 0.$$

Using the calculation of Seguinian curvature of a circle of arbitrary radius we'll be able to define

$$\Lambda_{n,\sigma} := \frac{2^{1/2}\pi^{n-1}}{\sigma R^{\sigma}} \frac{\Gamma(1+\sigma/2)\Gamma((1-\sigma)/2)}{\Gamma((n+1)/2)\Gamma((n+\sigma)/2)\Gamma(1-\sigma/2)}.$$

Together these results will give us the limit relation stated above.

The calculation of the Seguinian curvature of a circle in $\mathbb{R}^n$ will reveal a relationship to the Seguinian curvature of a circle in $\mathbb{R}^2$. This motivates showing a curve in a k dimensional subspace has Seguinian curvature spanned by the same k dimensional subspace.

1.3 Notation

In the following, we always use:

- n to denote the dimension of Euclidean space $\mathbb{R}^n$ with $n \geq 1$
- $\mathcal{C}$ a curve; contextually this will either be a unit circle or arbitrary
- $\lambda : \mathbb{R}^1 \to \mathbb{R}^n$ the parameterization of a curve $\mathcal{C}$
- $\kappa(z)$ the classical curvature of $\mathcal{C}$ at $z \in \mathcal{C}$; $\kappa := \kappa_{\mathcal{C}}(0)$
- $\kappa_\sigma(z)$ the non-local curvature (1) of $\mathcal{C}$ at $z \in \mathcal{C}$; $\kappa_\sigma := \kappa_\sigma(0)$
- $\gamma_v : \mathbb{R} \to \mathcal{M}$ for any manifold $\mathcal{M}$ is a flow such that $\gamma_v'(0) = v$ and $\gamma_(0)$ is set contextually
- $\omega_{k-1} := \frac{2\pi^{k/2}}{\Gamma(k/2)}$ is the surface area of an $k-1$ dimensional unit sphere embedded in k dimensional space
- $\mathcal{H}^k(\cdot)$ the k dimensional Hausdorff measure
- $\mathcal{U}(E) := \{a \in E \mid |a| = 1\}$; $\mathcal{U} := \mathcal{U}(\mathbb{R}^n)$
- $\mathcal{U}_2^{\perp}(E) := \{(a,b) \in \mathcal{U}(E) \times \mathcal{U}(E) \mid a \cdot b = 0\}$; $\mathcal{U}_2^{\perp} := \mathcal{U}_2^{\perp}(\mathbb{R}^n)$
- $E^c := \{x \in \mathbb{R}^n : x \notin E\}$
- $E^{\perp} := \{x \in \mathbb{R}^n \mid \; \forall y \in E \, x \cdot y = 0\}$
- $\chi_E(x) := \begin{cases} 1 : x \in E \\ 0 : x \notin E \end{cases}$
- $\overline{\chi}_E := \chi_E - \chi_{E^c}$
- $\mathcal{D}(p,u,r) := \big\{p + \xi v \mid (u,v) \in \mathcal{U}_2^{\perp}, \xi \in [0,r)\big\}$
- $\mathcal{A}^+(z) := \big\{(a,b,r) \in \mathcal{U}_2^{\perp} \mid (b \cdot t(z)) > 0\big\}$; $\mathcal{A}^+ = \mathcal{A}^+(0)$
- $\mathcal{A}^+_{Even}(z) := \big\{(a,b,r) \in \mathcal{A}^+(z) \mid \mathcal{H}^0(\mathcal{D}(z+ra,b,r) \cap \mathcal{C}) \text{ is even}\big\}$; $\mathcal{A}^+_{Even} := \mathcal{A}^+_{Even}(0)$
- $\mathcal{A}^+_{Odd}(z) := \big\{(a,b,r) \in \mathcal{A}^+(z) \mid \mathcal{H}^0(\mathcal{D}(z+ra,b,r) \cap \mathcal{C}) \text{ is odd}\big\}$; $\mathcal{A}^+_{Odd} := \mathcal{A}^+_{Odd}(0)$
- $(t,n) := (t(z), n(z))$ are the unit (tangent, normal) vectors of $\mathcal{C}$ at z
- $w_z = (w \cdot z)$, i.e. the $z-th$ component of w
- $v^{\perp} = ((v \cdot t)t + (v \cdot n)n)^{\perp} := (v \cdot n)t - (v \cdot t)n$ for any $v \in \mathbb{R}^2$; i.e. a clockwise rotation

- $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt$
- $B(z_1, z_2) := \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt = 2\int_0^{\pi/2} (\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}\, d\theta = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$
- $\mathrm{sgn}(x) := \frac{x}{|x|}$
- $\overline{\mathcal{J}} f := \sqrt{\nabla f^T \nabla f}$ for change of variables
- $\mathcal{J} f := \sqrt{\nabla f \nabla f^T}$ for applying the co-area formula
- $\mathcal{S} := \mathrm{lsp}\{t, n\}$, i.e. the 2D subspace spanned by t, z
- $\mathcal{P}_S := (t \otimes t) + (n \otimes n)$, i.e. the projection onto $\mathcal{S}$
- $\mathcal{T}_p(E)$ is the tangent space of E at p

2 Vanishing non-normal components of Seguinian curvature

Classically it's trivial to show the signed curvature of a curve at a point z is parallel to $n(z)$. Seguinian curvature the same property, which is easy to see in $\mathbb{R}^2$ since

$$\kappa_\sigma(z) \cdot t = \left(\int_{\mathcal{A}^+_{Even}} - \int_{\mathcal{A}^+_{Odd}} \right) \frac{(a \cdot t)(b \cdot t) - (b \cdot t)(a \cdot t)}{r^{1+\sigma}} \, d\mathcal{H}^2(a, b, r) = 0.$$

The following result extends this to $\mathbb{R}^n$.

Lemma 2.1. *For a curve $\mathcal{C} \in \mathbb{R}^n$ $\kappa_\sigma(z) = (n \otimes n)\kappa_\sigma(z)$.*

Proof. Just as in the 2D case it's trivial to see $(\kappa_\sigma(z) \cdot t) = 0$. Now for $m \in \mathcal{U}\Big(\{n, t\}^\perp\Big)$ consider the transformation $R_m := I - 2(m \otimes m)$ and put $\Phi : \mathcal{A}^+(z) \to \mathcal{A}^+(z)$ given by

$$\Phi(a, b, r) = (R_m a, R_m b, r).$$

It's easy to see Φ is an isometric diffeomorphism of $\mathcal{A}^+(z)$. Fix the integrand of $\kappa_\sigma(z) \cdot m$ by

$$J(a, b, r) := \frac{(a \cdot m)(b \cdot m) - (b \cdot m)(a \cdot m)}{r^{1+\sigma}}$$

and notice

$$\begin{aligned}
r^{1+\sigma} J(R_m a, R_m, b, r) &= (R_m a \cdot t)(R_m b \cdot m) - (R_m b \cdot t)(R_m a \cdot m) \\
&= ((a - 2(a \cdot m)m) \cdot t)((b - 2(b \cdot m)m) \cdot m) - ((b - 2(b \cdot m)m) \cdot t)((a - 2(a \cdot m)m) \cdot m) \\
&= ((a \cdot t) - 2(a \cdot m)(m \cdot t))((b \cdot m) - 2(b \cdot m)) - ((b \cdot t) - 2(b \cdot m)(m \cdot t))((a \cdot m) - 2(a \cdot m)) \\
&= ((a \cdot t))(-(b \cdot m)) - ((b \cdot t))(-(a \cdot m)) \\
&= -(a \cdot t)(b \cdot m) + (b \cdot t)(a \cdot m) = -r^{1+\sigma} J(a, b, r).
\end{aligned}$$

That is, Φ is an isometric diffeomorphism which the integrand is odd over, so that

$$\kappa_\sigma(z) \cdot m = \left(\int_{\mathcal{A}^+_{Even}} - \int_{\mathcal{A}^+_{Odd}} \right) \frac{(a \cdot m)(b \cdot m) - (b \cdot m)(a \cdot m)}{r^{1+\sigma}} \, d\mathcal{H}^2(a, b, r) = 0. \tag{2}$$

$\square$

3 Seguinian curvature of circles

To get a feel for the geometry of Seguinian curvature we calculate $\kappa_\sigma(z)$ of a circle of arbitrary radius. A circle's symmetry allows us to ignore which z we're analyzing, i.e. $\kappa_\sigma := \kappa_\sigma(0) = \kappa_\sigma(z)$, so for the rest of this section we'll simplify notation and focus on κ_σ. Our initial result in $\mathbb{R}^2$ illuminates the fundamental geometries behind the definition, and provides a result we can reuse in $\mathbb{R}^n$.

Theorem 3.1. *For a circle of radius R in $\mathbb{R}^2$*

$$\kappa_\sigma(z) = \frac{2\sqrt{2}}{\sigma R^\sigma} B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) n(z)$$

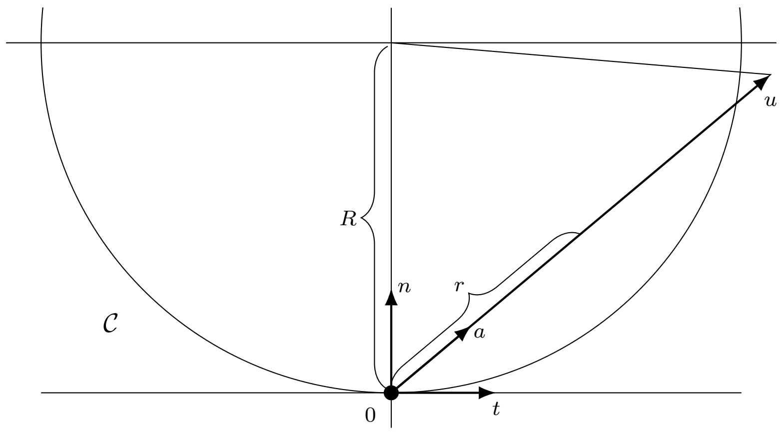
Proof. To begin, notice $\kappa_\sigma(z) = \kappa_\sigma(0) = \kappa_\sigma$ by symmetry and that

$$\kappa_\sigma \cdot t = \left(\int_{\mathcal{A}^+_{Even}} - \int_{\mathcal{A}^+_{Odd}} \right) \frac{(a \cdot t)(b \cdot t) - (b \cdot t)(a \cdot t)}{r^{1+\sigma}} \, d\mathcal{H}^2(a, b, r) = 0,$$

thus we only need to worry about $\kappa_\sigma \cdot n$; i.e. we wish to compute

$$\kappa_\sigma \cdot n = \int_{\mathcal{A}^+} \overline{\chi}_{\mathcal{A}^+_{Even}}(a, b, r) \frac{(a \cdot t)(b \cdot n) - (b \cdot t)(a \cdot n)}{r^{1+\sigma}} \, d\mathcal{H}^2(a, b, r),$$

where $\mathcal{A}^+ = \mathcal{A}^+_{Even} \cup \mathcal{A}^+_{Odd}$ and, for any E $\overline{\chi}_E = \chi_E - \chi_{E^c}$ (i.e. $\overline{\chi}_{\mathcal{A}^+_{Even}} = \chi_{\mathcal{A}^+_{Even}} - \chi_{\mathcal{A}^+_{Odd}}$). The picture below shows the geometric relationship between a given a, b, r with t, n and $u := 2ra$:



For any P on the circle the distance between 0 and P is $2R \sin\theta$ where θ is the angle between $\overrightarrow{0P}$ and t (you can see this by e.g. bisecting the triangle formed by $0, P$ and the center of the circle). Consequently, we only have an odd number of intersections when $(a \cdot n) > 0$ and when $|u| > 2R\sin\theta$, where θ is the angle between u and t. Notably $|u| = 2r$ and since a is a unit vector we have

$$(a \cdot n)^2 + (a \cdot t)^2 = 1, \; (a \cdot t) = |a||t|\cos\theta = \cos\theta \implies (a \cdot n)^2 = 1 - \cos^2\theta = \sin^2\theta \implies (a \cdot n) = |\sin\theta|.$$

Since $(a \cdot n) > 0$ for any (a, b, r) giving odd intersections, we know $\theta \in [0, \pi]$ and thus $\sin\theta = |\sin\theta|$, so $|u| > 2R\sin\theta \iff r > R(a \cdot n)$. Finally putting all this together we can explicitly write

$$\overline{\chi}_{\mathcal{A}^+_{Even}}(a, b, r) = \begin{cases} 1 & (a \cdot n) > 0, \, r < R(a \cdot n), \\ 1 & (a \cdot n) < 0, \\ -1 & \text{otherwise} \end{cases}. \tag{3}$$

The picture also gives us helps us characterize b in terms of a. Since we only care about the b such that $(b \cdot t) > 0$, $(a \cdot n) > 0 \implies b = a^\perp$, i.e. b is a rotated clockwise. Otherwise, b is a counterclockwise rotation so that $b = -a^\perp$, thus

$$b = \operatorname{sgn}(a \cdot n) a^\perp. \tag{4}$$

We're able to use (3), (4) to simplify our integrand i.e.

$$\begin{aligned}\kappa_\sigma\cdot n &= \int_{\mathcal{A}^+}\overline{\chi}_{\mathcal{A}^+_{Even}}(a,b,r)\operatorname{sgn}(a\cdot n)\frac{(a\cdot t)\big(a^\perp\cdot n\big)-\big(a^\perp\cdot t\big)(a\cdot n)}{r^{1+\sigma}}\,d\mathcal{H}^2(a,b,r)\\ &= \int_{\mathcal{A}^+}\overline{\chi}_{\mathcal{A}^+_{Even}}(a,b,r)\operatorname{sgn}(a\cdot n)\frac{-(a\cdot t)(a\cdot t)-(a\cdot n)(a\cdot n)}{r^{1+\sigma}}\,d\mathcal{H}^2(a,b,r)\\ &= -\int_{\mathcal{A}^+}\frac{\overline{\chi}_{\mathcal{A}^+_{Even}}(a,b,r)\operatorname{sgn}(a\cdot n)}{r^{1+\sigma}}\,d\mathcal{H}^2(a,b,r).\end{aligned}$$

Now, motivated by the picture above we do a change of variables $u := \phi(a,b,r) = 2ra$. We have

$$\nabla\phi(a,b,r) = \begin{array}{cc} \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) & \\ \left(\begin{array}{c} \sqrt{2}r(b\cdot t) \\ \sqrt{2}r(b\cdot n)\end{array}\right. & \left.\begin{array}{c} 2(a\cdot t) \\ 2(a\cdot n)\end{array}\right) & \begin{array}{c} t \\ n\end{array}\end{array}$$

so

$$\begin{aligned}\big(\overline{\mathcal{J}}\phi(a,b,r)\big)^2 = \big|\nabla\phi^T(a,b,r)\nabla\phi(a,b,r)\big| &= \left|\begin{pmatrix}\sqrt{2}r(b\cdot t) & \sqrt{2}r(b\cdot n)\\ 2(a\cdot t) & 2(a\cdot n)\end{pmatrix}\begin{pmatrix}\sqrt{2}r(b\cdot t) & 2(a\cdot t)\\ \sqrt{2}r(b\cdot n) & 2(a\cdot n)\end{pmatrix}\right|\\ &= \left|\begin{pmatrix}2r^2 & 2\sqrt{2}r((b\cdot t)(a\cdot t)+(b\cdot n)(a\cdot n))\\ 2\sqrt{2}r((b\cdot t)(a\cdot t)+(b\cdot n)(a\cdot n)) & 4\end{pmatrix}\right|\\ &= 8r^2-8((a\cdot t))^2(b\cdot t)^2+(a\cdot n)^2(b\cdot n)^2+2(a\cdot n)(a\cdot t)(b\cdot n)(b\cdot t)\\ &= 8r^2-8\Big((a\cdot t)^2(a\cdot n)^2+(a\cdot n)^2(a\cdot t)^2-2(a\cdot n)^2(a\cdot t)^2\Big)\\ \Longrightarrow\ \overline{\mathcal{J}}\phi &= 2\sqrt{2}r.\end{aligned}$$

Combining this with the fact that

$$\phi\big(\mathcal{A}^+\big)=\mathbb{R}^2,\ \phi^{-1}(u)=\left(\frac{u}{|u|},\operatorname{sgn}(u\cdot n)\frac{u^\perp}{|u|},\frac{|u|}{2}\right),$$

we now have

$$\kappa_\sigma\cdot n = -\int_{\mathbb{R}^2}\frac{\overline{\chi}_{\mathcal{A}^+_{Even}}\Big(\frac{u}{|u|},\operatorname{sgn}(u\cdot n)\frac{u^\perp}{|u|},\frac{|u|}{2}\Big)\operatorname{sgn}(u\cdot n)}{\Big(\frac{|u|}{2}\Big)^{1+\sigma}}\frac{1}{\sqrt{2}|u|}\,d\mathcal{H}^2(u).$$

To further simplify, put $\Pi^+ := \Big\{u\in\mathbb{R}^2 \mid (u\cdot n)<0 \text{ or } \frac{|u|}{2}<R\Big(\frac{u}{|u|}\cdot n\Big)\Big\}$ and notice

$$\overline{\chi}_{\mathcal{A}^+_{Even}}\left(\frac{u}{|u|},\operatorname{sgn}(u\cdot n)\frac{u^\perp}{|u|}\right)=\begin{cases}1 & (u\cdot n)>0,\ \frac{|u|}{2}<R\Big(\frac{u}{|u|}\cdot n\Big),\\ 1 & (u\cdot n)<0,\\ -1 & \text{otherwise}\end{cases} = \overline{\chi}_{\Pi^+}(u).$$

So, further simplifying and leveraging 4.1 we have

$$\kappa_\sigma\cdot n = -2^{\sigma+1/2}\int_{\mathbb{R}^2}\frac{\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u) = -2^{\sigma+1/2}\frac{-2^{1-\sigma}}{\sigma R^\sigma}B\left(\frac{1}{2},\frac{1-\sigma}{2}\right)=\frac{2^{3/2}}{\sigma R^\sigma}B\left(\frac{1}{2},\frac{1-\sigma}{2}\right).$$

$\square$

Now we're prepared to calculate the Seguinian curvature of a circle in $\mathbb{R}^n$, with our strategy as follows:

i. Recognize the similarity with the 2D case and slice the domain along each u intersecting $\mathcal{S} := \operatorname{lsp}\{t, n\}$.

ii. Slice the domain along each radius r.

iii. Show the domain of the inner-most integral with u, r fixed is a pair of spheres.

iv. Evaluate the final integrals.

Theorem 3.2. *For a circle of radius R in $\mathbb{R}^n$*

$$\kappa_\sigma = \frac{\omega_{n-3}^2}{\sigma R^\sigma \sqrt{2}} B\left(\frac{3}{2}, \frac{n-2}{2}\right) B\left(\frac{\sigma+2}{2}, \frac{n-2}{2}\right) B\left(\frac{1}{2}, \frac{1-\sigma}{2}\right) n(z).$$

Proof. First notice that the number of times a disk intersects $\mathcal{C}$ is completely determined by the projection of that disk into $\operatorname{lsp}\{t, n\}$. This motivates using the co-area formula to slice the domain into the line segments $\mathcal{D}(ra, b, r) \cap \mathcal{S}$ where $\mathcal{S} := \operatorname{lsp}\{t, n\}$. To that end, put $\Psi : \mathcal{A}^+ \to \mathbb{R}^2$ so that

$$\mathcal{D}(ra, b, r) \cap \mathcal{S} = \{t\Psi(a, b, r) \mid t \in [0, 1]\}.$$

In order to to determine an exact formula for Ψ, note the following, where $u := \Psi(a, b, r)$:

- There is an isoceles triangle formed by the center of $\mathcal{D}(ra, b, r)$ and the endpoints of $\mathcal{D}(ra, b, r) \cap S$, and thus the component of ra in the direction of u must be half of u's length, i.e.

$$ra \cdot \frac{u}{|u|} = \frac{|u|}{2} \implies 2(ra \cdot u) = |u|^2 = u \cdot u, \tag{5}$$

- Since $\mathcal{D}(ra, b, r) \cap \mathcal{S} \subset \mathcal{D}(ra, b, r)$, $\forall x \in \mathcal{D}(ra, b, r) \cap \mathcal{S}$

$$b \cdot x = 0 \implies b \cdot u = 0. \tag{6}$$

Since $u \in \mathcal{S}$, we can expand via the $\{t, n\}$ basis, take into account (6), and the fact that $b \cdot t > 0$ to see

$$0 = b \cdot u = b_t u_t + b_n u_n \implies u_t = \frac{-b_n u_n}{b_t}. \tag{7}$$

Substituting this back into (5) we have

$$\begin{aligned} 2ra_n u_n - 2ra_t \frac{b_n u_n}{b_t} = u_n^2 + \frac{b_n^2 u_n^2}{b_t^2} &\implies u_n^2 \left(1 + \frac{b_n^2}{b_t^2}\right) + 2ru_n \left(\frac{a_t b_n}{b_t} - a_n\right) = 0 \\ &\implies u_n = 0 \vee u_n \frac{b_t^2 + b_n^2}{b_t^2} + 2r \frac{a_t b_n - a_n b_t}{b_t} = 0. \end{aligned}$$

Notably $u_n \neq 0$ since otherwise (7) would force $u_t = 0$, i.e. $u = 0$, which only occurs for a measure-zero set of $(a, b, r) \in \mathcal{A}^+$. Solving the above equation for u_n we find

$$u_n = -2r \frac{a_t b_n - a_n b_t}{b_t} \frac{b_t^2}{b_t^2 + b_n^2} = 2r \frac{a_n b_t - a_t b_n}{b_t^2 + b_n^2} b_t.$$

Putting $\mathcal{P}_S := (t \otimes t) + (n \otimes n)$ and plugging this back into (7) we find

$$\Psi(a, b, r) = 2r \frac{a_n b_t - a_t b_n}{|\mathcal{P}_S b^\perp|^2} (b_t n - b_n t) = 2r \frac{a_t b_n - a_n b_t}{|\mathcal{P}_S b^\perp|^2} \mathcal{P}_S b^\perp$$

We can further rewrite this by noticing $\mathcal{P}_S b^\perp \cdot a = (b_t t + b_n n)^\perp \cdot a = (b_n t - b_t n)\cdot a = b_n a_t - b_t a_n$, and putting $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$, so that

$$\Psi(a,b,r) = 2r\frac{\mathcal{P}_S b^\perp \cdot a}{\left|\mathcal{P}_S b^\perp\right|^2}\mathcal{P}_S b^\perp = 2r(p(b)\cdot a)p(b) = 2r(p(b)\otimes p(b))a. \tag{8}$$

With $\mathcal{E}(u) := \Psi^{-1}(\{u\})$, and 4.2 we see

$$\kappa_\sigma = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^2}\frac{1}{|u|}\int_{\mathcal{E}(u)}\overline{\chi}_{\mathcal{A}^+}(a,b,r)\frac{(a\cdot t)b-(b\cdot t)a}{r^{1+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|}{\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}\,d\mathcal{H}^{2n-4}(a,b,r)\,d\mathcal{H}^2(u).$$

As noticed above $\mathcal{C}\in\mathcal{S}$ so $\overline{\chi}_{\mathcal{A}^+}(a,b,r)$ must only depend on where $\mathcal{D}(a,b,r)\cap\mathcal{S}$, i.e. u. Putting[1]

$$\Pi^+ := \left\{u\in\mathcal{S}\mid (u\cdot n)<0 \text{ or } \frac{|u|}{2} < R\left(\frac{u}{|u|}\cdot n\right)\right\}$$

we must have $\overline{\chi}_{\mathcal{A}^+}(a,b,r) = \overline{\chi}_{\Pi^+}(u)$ and so

$$\kappa_\sigma = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^2}\frac{\overline{\chi}_{\Pi^+}(u)}{|u|}\int_{\mathcal{E}(u)}\frac{(a\cdot t)b-(b\cdot t)a}{r^{1+\sigma}}\frac{\left|\mathcal{P}_S b^\perp\right|}{\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}\,d\mathcal{H}^{2n-4}(a,b,r)\,d\mathcal{H}^2(u). \tag{9}$$

From (2.1) we known we can focus on $\kappa_\sigma\cdot n$. It's also true

$$(a\cdot t)(b\cdot n)-(b\cdot t)(a\cdot n) = ((b\cdot n)t-(b\cdot t)n)\cdot a = -\mathcal{P}_S b^\perp\cdot a$$

and since $u = \frac{2r}{|\mathcal{P}_S b^\perp|^2}\big(\mathcal{P}_S b^\perp\cdot a\big)\mathcal{P}_S b^\perp \implies \left|\mathcal{P}_S b^\perp\cdot a\right| = \frac{|u|\left|\mathcal{P}_S b^\perp\right|}{2r}$, $\operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot a\big) = \operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot u\big)$ we have

$$\kappa_\sigma\cdot n = \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\int_{\mathcal{E}(u)}\frac{1}{r^{2+\sigma}}\frac{\operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}\,d\mathcal{H}^{2n-4}(a,b,r)\,d\mathcal{H}^2(u). \tag{10}$$

We can slice the domain of the inner integral along each r, i.e. for $\Phi:\mathcal{E}(u)\to\mathbb{R}^+$ given by

$$\Phi(a,b,r) = r, \tag{11}$$

put $\mathcal{E}(u,r) = \Phi^{-1}(\{r\})$, so that with the co-area calculation in 4.3 we find

$$\begin{aligned}
&\frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\int_{\mathcal{E}(u)}\frac{1}{r^{2+\sigma}}\frac{\operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}\,d\mathcal{H}^{2n-4}(a,b,r)\,d\mathcal{H}^2(u)\\
&\quad= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\int_{\Phi(\mathcal{E}(u))}\int_{\mathcal{E}(u,r)}\frac{1}{r^{3+\sigma}}\frac{\operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2}{\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{1-\frac{|u|^2}{4r^2}}}\,d\mathcal{H}^{2n-5}(a,b)\,dr\,d\mathcal{H}^2(u)\\
&\quad= \frac{-1}{2\sqrt{2}}\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\int_{\Phi(\mathcal{E}(u))}\frac{1}{r^{3+\sigma}\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{1-\frac{|u|^2}{4r^2}}}\int_{\mathcal{E}(u,r)}\operatorname{sgn}\big(\mathcal{P}_S b^\perp\cdot u\big)\left|\mathcal{P}_S b^\perp\right|^2\,d\mathcal{H}^{2n-5}(a,b)\,dr\,d\mathcal{H}^2(u)
\end{aligned}$$

[1] Π^+ is notably the same as in the 2D case. This comes from the inherent geometry of the circle, i.e. this set includes all u who point below the circle (i.e. $(u\cdot n)<0$), or u that lie within the interior of the circle, i.e. $|u|<2R\sin\theta$ where θ is the angle between t and u.

Since the inner most integrand only depends on b it's natural to slice on each b. Put $\Xi : \mathcal{E}(u,r) \to \mathcal{E}_2(u,r)$ where $\Xi(a,b) = b$. With the co-area calculation from 4.4, after putting $\mathcal{E}(u,r,b) = \Xi^{-1}(b)$ and including the result from 4.5, the above integral becomes

$$\frac{-1}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\Phi(\mathcal{E}(u))} \frac{1}{r^{3+\sigma}\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{1-\frac{|u|^2}{4r^2}}} \int_{\mathcal{E}_2(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\big|\mathcal{P}_S b^\perp\big|^2 \frac{\mathcal{H}^{n-3}(\mathcal{E}(u,r,b))}{\Big(2-\frac{|u|^2}{4r^2}\Big)^{-1/2}} \, d\mathcal{H}^{n-2}(b)\, dr\, d\mathcal{H}^2(u)$$

$$= \frac{-\omega_{n-3}}{2\sqrt{2}} \int_{\mathbb{R}^2} \overline{\chi}_{\Pi^+}(u) \int_{\Phi(\mathcal{E}(u))} \frac{\Big(1-\frac{|u|^2}{4r^2}\Big)^{(n-4)/2}}{r^{3+\sigma}} \int_{\mathcal{E}_2(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\big|\mathcal{P}_S b^\perp\big|^2 \, d\mathcal{H}^{n-2}(b)\, dr\, d\mathcal{H}^2(u) \tag{12}$$

To evaluate this second integral we'll need one more application of the co-area formula; we'll split using

$$\zeta : \mathcal{E}_2(u,r) \to [-1,1] \text{ given by } \zeta(b) = b \cdot \frac{u^\perp}{|u^\perp|}.$$

This is a natural transformation since

$$\mathcal{P}_S b^\perp = \left(\left(b \cdot \frac{u}{|u|}\right)\frac{u}{|u|} + \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\frac{u^\perp}{|u^\perp|}\right)^\perp = -\left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\frac{u}{|u^\perp|} = -\zeta(b)\frac{u}{|u^\perp|} \tag{13}$$

Further, since $\mathbb{R}^n = \operatorname{lsp}\Big\{\frac{u}{|u|}, \frac{u^\perp}{|u^\perp|}, f_1, f_2, \dots f_{n-2}\Big\}$ for some orthonormal f_i, it's true that

$$\zeta(b)^2 + (b\cdot f_1)^2 + (b \cdot f_2)^2 + \cdots + (b \cdot f_{n-2})^2 = 1 \implies (b\cdot f_1)^2 + (b \cdot f_2)^2 + \cdots + (b \cdot f_{n-2})^2 = 1 - \zeta(b)^2,$$

and so

$$\begin{aligned} \zeta^{-1}(v) &= \left\{ b \in \mathcal{U}\Big(\{u\}^\perp\Big) \mid v = \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\right\} \\ &= \left\{ b \in \mathcal{U}\Big(\{u\}^\perp\Big) \mid (b\cdot f_1)^2 + (b \cdot f_2)^2 + \cdots + (b \cdot f_{n-2})^2 = 1 - v^2\right\} \\ &= S_{n-3}\Big(\sqrt{1-v^2}\Big) \end{aligned} \tag{14}$$

Combining (13), (14) and 4.6, the inner most integral of (12) becomes

$$\begin{aligned} \int_{\mathcal{E}_2(u,r)} \operatorname{sgn}\big(\mathcal{P}_S b^\perp \cdot u\big)\big|\mathcal{P}_S b^\perp\big|^2 \, d\mathcal{H}^{n-2}(b) &= \int_{\zeta(\mathcal{E}_2(u,r))} \operatorname{sgn}\left(-v\frac{u}{|u^\perp|}\cdot u\right) v^2 \, \frac{\mathcal{H}^{n-3}\big(\zeta^{-1}(v)\big)}{\sqrt{1-v^2}}\, dv \\ &= -\omega_{n-3} \int_{\zeta(\mathcal{E}_2(u,r))} \operatorname{sgn}(v) v^2 \big(1-v^2\big)^{(n-4)/2}\, dv. \end{aligned} \tag{15}$$

To understand $\zeta(\mathcal{E}_2(u,r))$ notice $b \in \mathcal{E}_2(u,r) \implies b \cdot t > 0$, so

$$0 < b\cdot t = \left(b \cdot \frac{u}{|u|}\right)\left(t \cdot \frac{u}{|u|}\right) + \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\left(t \cdot \frac{u^\perp}{|u^\perp|}\right) = \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)\left(t \cdot \frac{u^\perp}{|u^\perp|}\right),$$

i.e. $\zeta(\mathcal{E}_2(u,r)) = [-1,0]$ if $\operatorname{sgn}\big(t\cdot u^\perp\big) = -1$ and $[0,1]$ otherwise. Since the integrand in (15) is odd, we can integrate over $[0,1]$ and multiply by $\operatorname{sgn}\big(t \cdot u^\perp\big)$ to take this into account. As we expand (15), note $\operatorname{sgn}\big(t\cdot u^\perp\big) = -\operatorname{sgn}(n \cdot u)$, so that

$$\omega_{n-3}\int_{\zeta(\mathcal{E}_2(u,r))}\operatorname{sgn}(v)v^2\big(1-v^2\big)^{(n-4)/2}\,dv = \operatorname{sgn}(n\cdot u)\omega_{n-3}\int_0^1 v^2\big(1-v^2\big)^{(n-4)/2}\,dv$$
$$= \frac{\operatorname{sgn}(n\cdot u)\omega_{n-3}}{2}\int_0^1 v^{1/2}(1-v)^{(n-4)/2}\,dv$$
$$= \frac{\operatorname{sgn}(n\cdot u)\omega_{n-3}}{2}B\bigg(\frac{3}{2},\frac{n-2}{2}\bigg) \tag{16}$$

Putting (16) back into (12) our calculation becomes

$$\kappa_\sigma\cdot n = \frac{-\omega_{n-3}^2}{4\sqrt{2}}B\bigg(\frac{3}{2},\frac{n-2}{2}\bigg)\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(n\cdot u)\int_{\Phi(\mathcal{E}(u))}\frac{\Big(1-\frac{|u|^2}{4r^2}\Big)^{(n-4)/2}}{r^{3+\sigma}}\,dr\,d\mathcal{H}^2(u). \tag{17}$$

To evaluate the final integrals we recall $\mathcal{E}(u)=\Psi^{-1}(\{u\})$ where Ψ is given in (8). By definition of Φ in (11) we have

$$\Phi(\mathcal{E}(u)) = \Phi\big(\Psi^{-1}(\{u\})\big) = \big\{r \mid 2r(p(b)\cdot a)p(b) = u \text{ for some } (a,b,r)\in\mathcal{A}^+\big\} = \{r \mid r\geq |u|2\},$$

so that the inner integral in (17), after applying the transformation $r\to\frac{|u|}{2}s^{-1/2}$, is

$$\int_{\Phi(\mathcal{E}(u))}\frac{\Big(1-\frac{|u|^2}{4r^2}\Big)^{(n-4)/2}}{r^{3+\sigma}}\,dr = \int_{|u|/2}^{\infty}\bigg(r^2-\frac{|u|^2}{4}\bigg)^{(n-4)/2}r^{1-\sigma-n}\,dr$$
$$= \int_1^0\bigg(\frac{|u|^2}{4}\frac{1}{s}-\frac{|u|^2}{4}\bigg)^{(n-4)/2}\bigg(\frac{|u|}{2}\bigg)^{1-\sigma-n}s^{(n+\sigma-1)/2}\frac{-|u|}{4}s^{-3/2}\,ds$$
$$= \bigg(\frac{|u|}{2}\bigg)^{(n-4)+1-\sigma-n}\frac{|u|}{4}\int_0^1\bigg(\frac{1-s}{s}\bigg)^{(n-4)/2}s^{(n+\sigma-4)/2}\,ds$$
$$= \frac{|u|^{-2-\sigma}}{2^{-1-\sigma}}\int_0^1(1-s)^{(n-4)/2}s^{\sigma/2}\,ds$$
$$= \frac{2^{1+\sigma}}{|u|^{2+\sigma}}B\bigg(\frac{\sigma+2}{2},\frac{n-2}{2}\bigg). \tag{18}$$

Plugging (18) back into (17) our calculation simplifies to

$$\kappa_\sigma\cdot n = \frac{-\omega_{n-3}^2}{4\sqrt{2}}B\bigg(\frac{3}{2},\frac{n-2}{2}\bigg)\int_{\mathbb{R}^2}\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(n\cdot u)\int_{\Phi(\mathcal{E}(u))}\frac{\Big(1-\frac{|u|^2}{4r^2}\Big)^{(n-4)/2}}{r^{3+\sigma}}\,dr\,d\mathcal{H}^2(u)$$
$$= -2^{1+\sigma-5/2}\omega_{n-3}^2B\bigg(\frac{3}{2},\frac{n-2}{2}\bigg)B\bigg(\frac{\sigma+2}{2},\frac{n-2}{2}\bigg)\int_{\mathbb{R}^2}\frac{\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(n\cdot u)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u)$$
$$= -2^{\sigma-3/2}\omega_{n-3}^2B\bigg(\frac{3}{2},\frac{n-2}{2}\bigg)B\bigg(\frac{\sigma+2}{2},\frac{n-2}{2}\bigg)\frac{-2^{1-\sigma}}{\sigma R^\sigma}B\bigg(\frac{1}{2},\frac{1-\sigma}{2}\bigg) \tag{19}$$
$$= \frac{\omega_{n-3}^2}{\sigma R^\sigma\sqrt{2}}B\bigg(\frac{3}{2},\frac{n-2}{2}\bigg)B\bigg(\frac{\sigma+2}{2},\frac{n-2}{2}\bigg)B\bigg(\frac{1}{2},\frac{1-\sigma}{2}\bigg), \tag{20}$$

where in (19) we leveraged 4.1. $\square$

With the main calculation out of the way we're ready to show the first asymptotic relationship in $\mathbb{R}^n$.

Theorem 3.3. *For a circle of radius R*

$$\lim_{\sigma\uparrow 1}(1-\sigma)\Lambda_{n,\sigma}\kappa_\sigma = \frac{1}{R}n$$

where n is the unit normal and

$$\Lambda_{n,\sigma} = TODO. \tag{21}$$

That is, classical curvature of a circle is recovered in the canonical nonlocal limit.

Proof. By (2.1) we know κ_σ only has a normal component so our goal is to show $\kappa_\sigma \cdot n \rightsquigarrow \frac{1}{R}$.
TODO $\square$

4 Appendix

4.1 Supplementary Calculations

Lemma 4.1. *For $\Pi^+ := \left\{u \in \mathbb{R}^2 \mid (u\cdot n) < 0 \text{ or } \frac{|u|}{2} < R\left(\frac{u}{|u|}\cdot n\right)\right\}$ we have*

$$\int_{\mathbb{R}^2} \frac{\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u) = \frac{-2^{1-\sigma}}{\sigma R^\sigma}B\left(\frac{1}{2},\frac{1-\sigma}{2}\right)$$

Proof. Begin by putting

$$I = \int_{\mathbb{R}^2} \frac{\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u).$$

Notably, this integral needs to be taken in a principal value sense, i.e.

$$I = \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^2\setminus B_\epsilon} \frac{\overline{\chi}_{\Pi^+}(u)\operatorname{sgn}(u\cdot n)}{|u|^{2+\sigma}}\,d\mathcal{H}^2(u),$$

where $B_\epsilon = \left\{x \in \mathbb{R}^2 \mid |x| = \epsilon\right\}$. Next, substituting $u = r\cos\theta t + r\sin\theta n$ shows that

$$I = \lim_{\epsilon\downarrow 0}\int_0^{2\pi}\int_\epsilon^\infty \frac{\overline{\chi}_{\Pi^+}(r\cos\theta t + r\sin\theta n)\operatorname{sgn}(\sin\theta)}{r^{2+\sigma}} r\,dr\,d\theta,$$

and further noting

$$\overline{\chi}_{\Pi^+}(r\cos\theta t + r\sin\theta n) = \begin{cases} 1 & \theta\in[0,\pi], \frac{r}{2} < R\sin\theta \\ 1 & \theta\in[\pi,2\pi] \\ 0 & \text{otherwise}\end{cases}$$

$$\operatorname{sgn}(\sin\theta) = \begin{cases} 1 & \theta\in[0,\pi] \\ -1 & \theta\in[\pi,2\pi]\end{cases},$$

means

$$
\begin{aligned}
I &= \lim_{\epsilon\downarrow 0}\int_0^{\pi}\left(\int_{\epsilon}^{2R\sin\theta}\frac{1\cdot 1}{r^{1+\sigma}}\,dr+\int_{2R\sin\theta}^{\infty}\frac{-1\cdot 1}{r^{1+\sigma}}\,dr\right)+\int_{\pi}^{2\pi}\left(\int_{\epsilon}^{\infty}\frac{1\cdot(-1)}{r^{1+\sigma}}\,dr\right)d\theta\\
&= \lim_{\epsilon\downarrow 0}\frac{1}{-\sigma}\int_0^{\pi}\left(r^{-\sigma}\Big|_{\epsilon}^{2R\sin\theta}-r^{-\sigma}\Big|_{2R\sin\theta}^{\infty}\right)d\theta-\frac{1}{-\sigma}\int_{\pi}^{2\pi}r^{-\sigma}\Big|_{\epsilon}^{\infty}\,d\theta\\
&= \frac{-1}{\sigma}\lim_{\epsilon\downarrow 0}\int_0^{\pi}(2R\sin\theta)^{-\sigma}-\epsilon^{-\sigma}+(2R\sin\theta)^{-\sigma}\,d\theta+\int_{\pi}^{2\pi}\epsilon^{-\sigma}\,d\theta\\
&= \frac{-1}{\sigma}\lim_{\epsilon\downarrow 0}2\int_0^{\pi}(2R\sin\theta)^{-\sigma}\,d\theta-\pi\epsilon^{-\sigma}+\pi\epsilon^{-\sigma}\\
&= \frac{-2^{1-\sigma}}{\sigma R^{\sigma}}\int_0^{\pi}(\sin\theta)^{-\sigma}\,d\theta\\
&= \frac{-2^{1-\sigma}}{\sigma R^{\sigma}}\int_{\pi/2}^{-\pi/2}(\cos\theta)^{-\sigma}\,(-d\theta)\ \text{via}\ \theta\to\frac{\pi}{2}-\theta\\
&= \frac{-2^{1-\sigma}}{\sigma R^{\sigma}}2\int_0^{\pi/2}(\cos\theta)^{-\sigma}\,d\theta=\frac{-2^{1-\sigma}}{\sigma R^{\sigma}}B\left(\frac{1}{2},\frac{1-\sigma}{2}\right)
\end{aligned}
$$

□

Lemma 4.2. *For $\Psi:\mathcal{A}^+\to\mathbb{R}^2$ given by $\Psi(a,b,r)=2r(p(b)\otimes p(b))a$, where $p(b)=\frac{\mathcal{P}_S b^{\perp}}{|\mathcal{P}_S b^{\perp}|}$*

$$
\mathcal{J}\Psi(a,b,r)=\frac{\sqrt{2}|u|}{|\mathcal{P}_S b^{\perp}|}\sqrt{2-\frac{|u|^2}{4r^2}}\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}.
$$

Proof. Put $c=\frac{u-(u\cdot a)a}{\sqrt{(u\cdot u)-(u\cdot a)}}$ and $\{e_i\}_{i=1}^{n-3}$ so that $\{a,b,c,e_1,...,e_{n-3}\}$ is an orthonormal basis spanning $\mathbb{R}^n$. We can use this basis to characterize

$$
\mathcal{T}_{(a,b,r)}(\mathcal{A}^+)=\operatorname{lsp}\{(e_i,0,0),(0,e_i,0),(c,0,0),(0,c,0),\epsilon_{a,b},(0,0,1)\mid i=1,...(n-3)\},
$$

where $\epsilon_{a,b}:=\frac{1}{\sqrt{2}}(b,-a,0)$. Additionally, we can use the fact that $p(b),p(b)^{\perp}$ are an orthonormal basis spanning $\mathcal{S}$ to see $\mathcal{T}_{\Psi(a,b,r)}(\mathcal{S})=\operatorname{lsp}\{p(b),p(b)^{\perp}\}$. Next, we compute derivatives along flows as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(e_i,0,0)}(s)\big)\Big|_{s=0} &= 2r(p(b)\otimes p(b))e_i=2r(p(b)\cdot e_i)p(b)=0,\\
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(c,0,0)}(s)\big)\Big|_{s=0} &= 2r(p(b)\otimes p(b))c=2r(p(b)\cdot c)p(b),\\
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(0,0,1)}(s)\big)\Big|_{s=0} &= 2(p(b)\cdot a)p(b).
\end{aligned}
$$

For our last calculations we'll need the following, for $\beta:\mathbb{R}^1\to\mathcal{U}$ such that $\beta(0)=b$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}p(\beta(s))\Big|_{s=0} &= \frac{\mathcal{P}_S\beta'(0)^{\perp}}{|\mathcal{P}_S b^{\perp}|}-\frac{1}{|\mathcal{P}_S b^{\perp}|^3}\mathcal{P}_S b^{\perp}\otimes\big(\mathcal{P}_S\beta'(0)^{\perp}\big)^T\mathcal{P}_S b^{\perp}=\frac{1}{|\mathcal{P}_S b^{\perp}|}(1-(p(b)\otimes p(b)))\mathcal{P}_S\beta'(0)^{\perp}\\
&= \frac{1}{|\mathcal{P}_S b^{\perp}|}\big(p(b)\otimes p(b)+p(b)^{\perp}\otimes p(b)^{\perp}-p(b)\otimes p(b)\big)\mathcal{P}_S\beta'(0)^{\perp}=\frac{(p(b)\cdot\beta'(0))}{|\mathcal{P}_S b^{\perp}|}p(b)^{\perp}.
\end{aligned}
\tag{22}
$$

For $v \in \{c\} \cup \{e_i\}_1^{(n-3)}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(0,v,0)}(s)\big)\Big|_{s=0} &= 2r\bigg(\bigg(\frac{\mathrm{d}}{\mathrm{d}s}p\big(\gamma_{(0,v,0),2}(s)\big)\Big|_{s=0}\bigg)\otimes p + p\otimes\bigg(\frac{\mathrm{d}}{\mathrm{d}s}p\big(\gamma_{(0,v,0),2}(s)\big)\Big|_{s=0}\bigg)\bigg)a \\
&= 2r\bigg(\bigg(\frac{(p(b)\cdot v)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\bigg)\otimes p + p\otimes\bigg(\frac{(p(b)\cdot v)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\bigg)\bigg)a \\
&= 2r\frac{p(b)\cdot v}{|\mathcal{P}_S b^\perp|}\big((p(b)\cdot a)p(b)^\perp + \big(p(b)^\perp\cdot a\big)p(b)\big),
\end{aligned}$$

particularly $\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{(0,e_i,0)}(s)\big)\big|_{s=0} = 0$. Finally,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Psi\big(\gamma_{\epsilon_{a,b}}(s)\big)\Big|_{s=0} &= \sqrt{2}r(p\otimes p)b + 2r\bigg(\bigg(\frac{\mathrm{d}}{\mathrm{d}s}p\big(\gamma_{\epsilon_{a,b}}(s)\big)\Big|_{s=0}\bigg)\odot p\bigg)a \\
&= 2r\Bigg(\Bigg(\frac{\big(p(b)\cdot(-a/\sqrt{2})\big)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\Bigg)\otimes p + p\otimes\Bigg(\frac{\big(p(b)\cdot(-a/\sqrt{2}\big)}{|\mathcal{P}_S b^\perp|}p(b)^\perp\Bigg)\Bigg)a \\
&= -\sqrt{2}r\frac{p(b)\cdot a}{|\mathcal{P}_S b^\perp|}\big((p(b)\cdot a)p(b)^\perp + \big(p(b)^\perp\cdot a\big)p(b)\big)
\end{aligned}$$

Altogether, after putting $p := p(b)$ we get

$$\nabla\Psi(a,b,r) = \frac{2r}{|\mathcal{P}_S b^\perp|}\begin{array}{c} \begin{array}{cccccc} (e_i,0,0) & (0,e_i,0) & (c,0,0) & (0,c,0) & \frac{1}{\sqrt{2}}(b,-a,0) & (0,0,1) \end{array} \\ \begin{pmatrix} 0 & 0 & (p\cdot c)\big|\mathcal{P}_S b^\perp\big| & (p\cdot c)\big(p^\perp\cdot a\big) & -\frac{(p\cdot a)(p^\perp\cdot a)}{\sqrt{2}} & \frac{(p\cdot a)|\mathcal{P}_S b^\perp|}{r} \\ 0 & 0 & 0 & (p\cdot c)(p\cdot a) & -\frac{(p\cdot a)^2}{\sqrt{2}} & 0 \end{pmatrix} \end{array} \begin{array}{c} p \\ p^\perp \end{array}.$$

To aid in our calculation of $\mathcal{J}\Psi$ put $M := \frac{|\mathcal{P}_S b^\perp|^2}{4r^2}\nabla\Psi(a,b,r)\Psi(a,b,r)^\perp$. We have

$$\begin{aligned}
M_{1,1} &= (p\cdot c)^2\big|\mathcal{P}_S b^\perp\big|^2 + (p\cdot c)^2\big(p^\perp\cdot a\big)^2 + \frac{(p\cdot a)^2\big(p^\perp\cdot a\big)^2}{2} + \frac{(p\cdot a)^2\big|\mathcal{P}_S b^\perp\big|^2}{r^2} \\
&= \frac{\big(p^\perp\cdot a\big)^2}{2}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big) + \frac{\big|\mathcal{P}_S b^\perp\big|^2}{r^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big) \\
M_{1,2} = M_{2,1} &= (p\cdot c)^2(p\cdot a)\big(p^\perp\cdot a\big) + \frac{(p\cdot a)^3\big(p^\perp\cdot a\big)}{2} = \frac{(p\cdot a)\big(p^\perp\cdot a\big)}{2}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big) \\
M_{2,2} &= (p\cdot a)^2(p\cdot c)^2 + \frac{(p\cdot a)^2}{2} = \frac{(p\cdot a)^2}{2}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
M_{1,1}M_{2,2} &= \frac{(p\cdot a)^2\big(p^\perp\cdot a\big)^2}{4}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)^2 + \frac{(p\cdot a)^2\big|\mathcal{P}_S b^\perp\big|^2}{2r^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big) \\
M_{1,2}M_{2,1} &= \frac{(p\cdot a)^2\big(p^\perp\cdot a\big)^2}{4}\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big)^2 \\
\implies |M| &= \frac{(p\cdot a)^2\big|\mathcal{P}_S b^\perp\big|^2}{2r^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big) \\
\implies (\mathcal{J}\Psi(a,b,r))^2 &= \frac{8r^2(p\cdot a)^2}{\big|\mathcal{P}_S b^\perp\big|^2}\Big((p\cdot a)^2 + r^2(p\cdot c)^2\Big)\Big((p\cdot a)^2 + 2(p\cdot c)^2\Big).
\end{aligned}$$

Since $p \in \operatorname{lsp}\{a,b,c\}$ we know $1 = (p\cdot p) = (p\cdot a)^2 + (p\cdot b)^2 + (p\cdot c)^2 = (p\cdot a)^2 + (p\cdot c)^2$, and $u = 2r(p\cdot a)p \implies (p\cdot a) = \frac{|u|}{2r}$ so

$$(p\cdot c)^2 = 1 - (p\cdot a)^2 = 1 - \frac{|u|^2}{4r^2}.$$

Plugging this back in we find

$$\mathcal{J}\Psi(a,b,r) = \frac{\sqrt{2}|u|}{|\mathcal{P}_S b^\perp|^2}\sqrt{2 - \frac{|u|^2}{4r^2}}\sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}$$

□

Lemma 4.3. *Fixing $u \in \mathbb{R}^2$, for Ψ as defined in 4.2, if $\Phi : \Psi^{-1}(u) \to \mathbb{R}^+$ is given by $\Phi(a,b,r) = r$ then*

$$\mathcal{J}\Phi(a,b,r) = \frac{r\sqrt{1-\frac{|u|^2}{4r^2}}}{\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}.$$

Proof. In order to compute $\mathcal{J}\Phi$ we need to characterize $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$, i.e. by finding $2n-4$ orthonormal vectors forming a basis. We can leverage the constraints imposed by $\mathcal{E}(u)$, i.e.

$$u = 2r(p(b)\otimes p(b))a, \tag{23}$$
$$a\cdot a = 1, \tag{24}$$
$$b\cdot b = 1, \tag{25}$$
$$a\cdot b = 0, \tag{26}$$

where $p(b) = \frac{\mathcal{P}_S b^\perp}{|\mathcal{P}_S b^\perp|}$. Put $\boldsymbol{\gamma} : \mathbb{R}^1 \to \mathcal{E}(u)$ such that $\boldsymbol{\gamma}(0) = (a,b,r)$ and $\boldsymbol{\gamma}'(0) = (\delta_a, \delta_b, \delta_r)$ be an arbitrary flow, then (23) combined with (22) shows us

$$0 = \delta_r(p(b)\otimes p(b))a + r\left(\frac{p(b)\cdot\delta_b}{|\mathcal{P}_S b^\perp|}p(b)^\perp \odot p(b)\right)a + r(p(b)\otimes p(b))\delta_a. \tag{27}$$

Next, to simplify notation put $p = p(b)$ so that $\operatorname{lsp}\{p, p^\perp\} = \mathbb{R}^2$ so that (27) shows us

$$\begin{aligned}
0 &= \delta_r(p\cdot a)p + r\left(\frac{p\cdot\delta_b}{|\mathcal{P}_S b^\perp|}p^\perp \odot p\right)a + r(p\cdot\delta_a)p \\
&= (\delta_r(p\cdot a) + r(p\cdot\delta_a))p + r\frac{(p\cdot\delta_b)}{|\mathcal{P}_S b^\perp|}(p^\perp\otimes p + p\otimes p^\perp)a \\
&= \left(\delta_r(p\cdot a) + r(p\cdot\delta_a) + r\frac{(p\cdot\delta_b)}{|\mathcal{P}_S b^\perp|}(p^\perp\cdot a)\right)p + r\frac{(p\cdot\delta_b)}{|\mathcal{P}_S b^\perp|}(p\cdot a)p^\perp.
\end{aligned} \tag{28}$$

From the $p^\perp$ component of (28) we see

$$p\cdot\delta_b = 0, \tag{29}$$

since $p\cdot a = 0$ for only a measure zero set of (a,b). The p component of (28) then simplifies to

$$0 = \delta_r(p\cdot a) + r(p\cdot\delta_a). \tag{30}$$

Lastly, doing the same flow calculation with (24), (25), (26) tells us

$$a \cdot \delta_a = 0, \tag{31}$$
$$b \cdot \delta_b = 0 \tag{32}$$
$$a \cdot \delta_b + \delta_a \cdot b = 0 \tag{33}$$

Put $c = \frac{u-(u\cdot a)a}{\sqrt{(u\cdot u)-(u\cdot a)}}$ and $\{e_i\}_1^{n-3}$ so that $\mathrm{lsp}\Big\{\{a,b,c\} \cup \{e_i\}_1^{n-3}\Big\} = \mathbb{R}^n$. Since $p \in \mathrm{lsp}\{a,b,c\}$ we know $(p \cdot e_i) = 0$ and thus $\{(e_i,0,0)\}_1^{n-3} \cup \{(0,e_i,0)\}_i^{n-3}$ are $2n-6$ orthonormal vectors which span a subspace of $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$.

In order to find two more orthonormal vectors, let's begin by supposing $\delta_b = 0$. In order to find an additional orthoganol basis vector, we want to consider δ_a where $\delta_a \cdot e_i = 0$, and in order to satisfy (31), (33) we must have $\delta_a \cdot a = \delta \cdot b = 0$, so, w.l.o.g. we can consider $\delta_a = c$. Plugging this data into (30) and solving for δ_r gives us

$$\delta_r = \frac{-r(p \cdot \delta_a)}{p \cdot a},$$

so after normalization, we can see an additional orthornomal basis vector spanning $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$ is

$$\begin{aligned}\mu &= \frac{1}{\sqrt{1 + r^2\frac{(p\cdot c)^2}{(p\cdot a)^2}}}\left(c, 0, -r\frac{p \cdot c}{p \cdot a}\right) \\ &= \frac{1}{\sqrt{1 - r^2 + \frac{4r^4}{|u|^2}}}\left(c, 0, \frac{-2r^2\sqrt{1-\frac{|u|^2}{4r^2}}}{|u|}\right),\end{aligned} \tag{34}$$

where we got the second line from the first by noticing $1 = (p \cdot p)^2 = (p \cdot a)^2 + (p \cdot b)^2 + (p \cdot c)^2$ and that $u = 2r(p \cdot a)p \implies |u|2r = (p \cdot a)$. Lastly it's easy to check that

$$\nu = \left(\frac{\sqrt{r^2 - \frac{|u|^2}{4}}}{\sqrt{2r^2 - \frac{|u|^2}{4}}}b, \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}(u - 2ra), 0\right), \tag{35}$$

is a final orthonormal basis vector so that

$$\mathcal{T}_{(a,b,r)}(\mathcal{E}(u)) = \mathrm{lsp}\Big\{\{(e_i,0,0)\}_1^{n-3} \cup \{(0,e_i,0)\}_1^{n-3} \cup \{\mu,\nu\}\Big\}.$$

Next, to compute $\mathcal{J}\Phi$ put $\gamma_v : \mathbb{R}^1 \to \mathcal{E}(u)$ so that $\gamma_v(0) = (a,b,r)$ and $\gamma_v'(0) = v$, then the only non-vanishing derivative is

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\gamma_\mu(s))\Big|_{s=0} = \frac{-2r^2\sqrt{1-\frac{|u|^2}{4r^2}}}{|u|\sqrt{1-r^2+\frac{4r^4}{|u|^2}}} = \frac{-r\sqrt{1-\frac{|u|^2}{4r^2}}}{\sqrt{\frac{|u|^2}{4r^2} - \frac{r^2|u|^2}{4r^2} + r^2}} = \frac{-r\sqrt{1-\frac{|u|^2}{4r^2}}}{\sqrt{r^2 - (r^2-1)\frac{|u|^2}{4r^2}}},$$

so

$$\nabla\Phi(a,b,r) = \frac{-r\sqrt{1-\frac{|u|^2}{4r^2}}}{\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}} \implies \mathcal{J}\Phi(a,b,r) = \frac{r\sqrt{1-\frac{|u|^2}{4r^2}}}{\sqrt{r^2-(r^2-1)\frac{|u|^2}{4r^2}}}.$$

□

Lemma 4.4. *Fixing $u \in \mathbb{R}^2$, $r \in \mathbb{R}^+$, for Φ as defined in 4.3, if $\Xi : \Phi^{-1}(r) \to \mathcal{E}_2(u,r)$ is given by $\Xi(a,b) = b$ then*

$$\mathcal{J}\Xi(a,b) = \frac{1}{\sqrt{2 - \frac{|u|^2}{4r^2}}}.$$

Proof. In order to characterize $\mathcal{T}_{(a,b)}(\mathcal{E}(u,r))$ we must find $2n-5$ orthornomal basis vectors spanning the space. Similar to 4.3 we can leverage the constraints imposed by $\mathcal{E}(u)$, since $\mathcal{E}(u,r)$ simply introduces a new constraint that r is unchanging. That is, every basis vector of $\mathcal{T}_{(a,b,r)}(\mathcal{E}(u))$ such that r vanishes is a a valid basis vector for $\mathcal{T}_{(a,b)}(\mathcal{E}(u,r))$ when projected onto the first two coordinates, so

$$\mathcal{T}_{(a,b)}(\mathcal{E}(u)) = \{(e_i,0)\}_1^{n-3} \cup \{(0,e_i)\}_1^{n-3} \cup \{\nu\},$$

where ν comes from (35), i.e.

$$\nu = \left(\frac{\sqrt{r^2 - \frac{|u|^2}{4}}}{\sqrt{2r^2 - \frac{|u|^2}{4}}} b, \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}(u - 2ra) \right), \tag{36}$$

and the $\{e_i\}_1^{n-3}$ are defined so that, for $c = \frac{u-(u\cdot a)a}{\sqrt{(u\cdot u)-(u\cdot a)}}$ $\{a,b,c\} \cup \{e_i\}_1^{n-3}$ forms an orthonormal basis spanning $\mathbb{R}^n$.

Next we need to characterize the tangent space of the codomain, i.e. $\mathcal{T}_b(\mathcal{E}_2(u,r))$. Since, for the calculation of $\mathcal{J}\Xi(a,b)$ both a,b are fixed, we can use both a,b for this characterization. Notice

$$\mathcal{E}_2(u,r) = \left\{ b \in \mathcal{U}\Big(\{u\}^\perp\Big) \mid b \cdot t > 0 \right\},$$

so that we need $n-2$ basis vectors respecting the constraints that $b \cdot u = 0, b \cdot b = 1$. Implicitly differentiating these constraints shows us that

$$\delta_b \cdot u = 0, \delta_b \cdot b = 0,$$

and so it becomes clear that every $\delta_b = e_i$ is a valid basis vector for $\mathcal{T}_b(\mathcal{E}_2(u,r))$ for every e_i. It can also be shown that $\frac{\nu_2}{|\nu_2|}$, where ν_2 is the second component of (36), is a valid normalized basis vector orthoganol to every e_i, so that

$$\mathcal{T}_b(\mathcal{E}_2(u,r)) = \operatorname{lsp}\left\{ \{e_i\}_1^{n-3} \cup \left\{ \frac{\nu_2}{|\nu_2|} \right\} \right\}.$$

Now that we've characterized the tangent spaces we can begin to calculate the jacobian factor, i.e. it's easy to see

$$\nabla\xi(a,b) = \begin{array}{c} \begin{array}{ccc} (e_i,0) & (0,e_i) & \nu \end{array} \\ \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & |\nu_2| \end{pmatrix} \end{array} \begin{array}{c} e_i \\ \frac{\nu_2}{|\nu_2|} \end{array} \implies \nabla\xi(a,b)(\nabla\xi(a,b))^T = \begin{array}{c} \begin{array}{cc} \overbrace{}^{n-3} & \overbrace{}^{1} \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & |\nu_2|^2 \end{pmatrix} \end{array} \begin{array}{c} {\scriptstyle n-3} \\ {\scriptstyle 1} \end{array},$$

and so

$$\begin{aligned}\mathcal{J}\Xi(a,b) = |\nu_2| &= \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}\sqrt{|u|^2 - 4r(a\cdot u) + 4r^2} \\ &= \frac{r}{2\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}\sqrt{|u|^2 - 2|u|^2 + 4r^2} \\ &= \frac{r}{\sqrt{r^2 - \frac{|u|^2}{4}}\sqrt{2r^2 - \frac{|u|^2}{4}}}\sqrt{r^2 - \frac{|u|^2}{4}} = \frac{1}{\sqrt{2 - \frac{|u|^2}{4r^2}}}\end{aligned}$$

□

Lemma 4.5. *Fixing $u \in \mathbb{R}^2$, $r \in \mathbb{R}^+$ $b \in \left\{b \in \mathcal{U}\left(\{u\}^\perp\right) \mid b\cdot t > 0\right\}$ then, for Ξ as defined in 4.4*

$$\mathcal{H}^{n-3}\big(\Xi^{-1}(b)\big) = \omega_{n-3}\left(1 - \frac{|u|^2}{4r^2}\right)^{(n-3)/2}.$$

Proof. By definition

$$\Xi^{-1}(b) = \left\{(a,b) \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u\cdot a) = (u\cdot u)\right\} = \left\{a \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u\cdot a) = (u\cdot u)\right\} \times \{b\},$$

so

$$\mathcal{H}^{n-3}\big(\Xi^{-1}(b)\big) = \mathcal{H}^{n-3}\left(\left\{a \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u\cdot a) = (u\cdot u)\right\}\right).$$

Now, for $a \in \mathcal{F}$ for $\mathcal{F} = \left\{a \in \mathcal{U}\left(\{b\}^\perp\right) \mid 2r(u\cdot a) = (u\cdot u)\right\}$

$$\left(a - \frac{u}{2r}\right)\cdot\left(a - \frac{u}{2r}\right) = 1 + \frac{|u|^2}{4r^2} - 2\frac{(a\cdot u)}{2r} = 1 + \frac{|u|^2}{4r^2} - 2\frac{|u|^2}{4r^2} = 1 - \frac{|u|^2}{4r^2},$$

which makes it easy to see

$$\mathcal{F} = \left\{a \in \mathcal{U}\left(\{b\}^\perp\right) : \left|a - \frac{u}{2r}\right|^2 = 1 - \frac{|u|^2}{4r^2}\right\}. \tag{37}$$

Since (37) is an $n-3$ dimensional sphere located at $\frac{u}{2r}$ with radius $\sqrt{1 - \frac{|u|^2}{4r^2}}$ we have

$$\mathcal{H}^{n-3}(\mathcal{F}) = S_{n-3}\left(\sqrt{1 - \frac{|u|^2}{4r^2}}\right) = \omega_{n-3}\left(1 - \frac{|u|^2}{4r^2}\right)^{(n-3)/2}.$$

□

Lemma 4.6. *Fix $u \in \mathbb{R}^2$, $r \in \mathbb{R}^+$, $\mathcal{E}(u,r) = \Phi^{-1}(\{r\})$ for Φ as defined in 4.3, then if $\zeta : \mathcal{E}_2(u,r) \to [-1,1]$ is given by $\zeta(b) = b\cdot\frac{u^\perp}{|u^\perp|}$,*

$$\mathcal{J}\zeta(b) = \sqrt{1 - \left(b\cdot\frac{u^\perp}{|u^\perp|}\right)^2}.$$

Proof. We begin by characterizing $\mathcal{T}_b(\mathcal{E}_2(u,r))$. Put $\{f_i\}_1^{n-3}$

$$u^* = \frac{\frac{u^\perp}{|u^\perp|} - \left(b \cdot \frac{u^\perp}{|u^\perp|}\right) b}{\sqrt{1 - \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)^2}}$$

so that $\left\{\frac{u}{|u|}, b, u^*\right\} \cup \{f_i\}_1^{n-3}$ is an orthonormal basis spanning $\mathbb{R}^n$. We must have $|b| = 1$, $b \cdot u = 1$ since $b \in \mathcal{E}_2(u,r)$, thus if we fix a tangent vector δ_b, we must have $\delta_b \cdot b = 0$, $\delta_b \cdot u = 0$, i.e.

$$\mathcal{T}_b(\mathcal{E}_2(u,r)) = \operatorname{lsp}\{u^*, f_1, f_2, \ldots, f_{n-3}\}.$$

With this, we can compute

$$\nabla\zeta(b) = \begin{array}{cc} \begin{array}{cccc} u^* & g_1 & \cdots & g_{n-3} \end{array} & \\ \begin{pmatrix} \sqrt{1 - \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)^2} & 0 & \cdots & 0 \end{pmatrix} & 1 \end{array} \implies \mathcal{J}\zeta(b) = \sqrt{1 - \left(b \cdot \frac{u^\perp}{|u^\perp|}\right)^2}.$$

□

References

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