1 Introduction

Here I will collect results from geometric measure theory as I explore the subject. I'll be following along in Functions of Bounded Variation and Free Discontinuity Problems by Ambrosio, Fusco & Pallara.

2 Hausdorff Measure

Lemma 2.1 *Let $(X, d_X), (Y, d_Y)$ be metric spaces. For $f : X \to Y$ K-Lipschitz and measurable $E \subset X$ we have*

$$\mathcal{H}^n(f(E)) \leq K^n \, \mathcal{H}^n(E).$$

Proof. Let $E \subset X$ be measurable, $\epsilon > 0, \delta > 0$ be arbitrary. By the definition of $\mathcal{H}^n_\delta$ and the fact that $\operatorname{diam}(f(F)) \leq K \operatorname{diam}(F)$ $\forall\, F \subset X$ (since f is K-Lipschitz), we know

$$\mathcal{H}^n_\delta(E) \geq \sum_h (\operatorname{diam}(E_h))^n - \epsilon \geq \sum_h \left(\frac{\operatorname{diam}(f(E_H))}{K}\right)^n - \epsilon \geq \frac{1}{K^n} \, \mathcal{H}^n_{K\delta}(f(E)) - \epsilon$$

for some countable covering $\{E_h\}$ of E. Since ϵ was arbitrary we find

$$\mathcal{H}^n_{K\delta}(f(E)) \leq K^n \, \mathcal{H}^n_\delta(E).$$

Sending $\delta \to 0$ we find our result. $\square$