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Q.9) Harry is the team lead in a company but is new to Linear Algebra. While working on his project, he arrived at a problem. He got three vectors $v_1, v_2, v_3 \in \mathbb{R}^5$

Let $S = \text{span}\{v_1, v_2, v_3\}$ and $W = \{u_1, u_2, \ldots, u_r\}$

The problem is to find all those u_is that belong to S and if $u_i \in S$, find linear combination of u_i in terms of v_1, v_2, v_3. Explain the method to solve Harry's problem with proper justification. Is the method efficient?

Using the above method solve it

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } W = \left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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Answer:-

In the above problem statement, we see that S is the span of vector v_1, v_2, v_3 in space $\mathbb{R}^5$

This means that S is the set of all vectors that can be derived from all possible linear combinations of vectors v_1, v_2, v_3 for all choices of scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\boxed{\text{Any element in set } S = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3}$$

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We also see that W is a set of vectors $\{u_1, u_2, \ldots, u_r\}$. Now, to determine if a vector $u_i \in S$, we can write a linear system of equations as follows:

$$\boxed{u_i = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3}$$

Since matrixes can be used to compactly represent a linear system of equations $Ax=b$, we can re-write the above equation as follows:

$$\boxed{\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = u_i} \qquad \boxed{\left[\begin{array}{ccc|c} V_1 & V_2 & V_3 & u_i \end{array}\right]}$$

$\boxed{Ax=b}$ Augmented Matrix $[A|b]$

From here, we can use Gaussian elimination to reduce the augmented matrix to the row-echelon form using elementary transformations. And from there, we can perform back-substitute to get the values of $\lambda_1, \lambda_2, \lambda_3$.

And once we have the values of $\lambda_1, \lambda_2, \lambda_3$, we can check their correctness using: $\boxed{u_i = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3}$

Thus in conclusion, this is the method Harry should follow to check if a vector $u_i \in S$

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Now, using this method we will try to see if vectors in set $W \in S$, where:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } W = \left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

CASE 1:

Given that,

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } u_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

Therefore,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{array}\right] \boxed{\text{Swap } R3, R5}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & -3 \end{array}\right] \begin{array}{l} \\ \boxed{R2 - R1} \\ \\ \\ \boxed{R5 - 2R1} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \boxed{R5 + 2R3} \end{array}$$

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If we now convert the row-echelon form of the augmented matrix to a linear system of equations we get :

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 \quad \ldots (1)$$
$$\lambda_2 - \lambda_3 = 0 \quad \ldots (2)$$
$$\lambda_2 + \lambda_3 = 2 \quad \ldots (3)$$
$$\lambda_3 = 0 \quad \ldots (4)$$
$$0\lambda_1 + 0\lambda_2 + 0\lambda_3 = 1 \quad \ldots (5)$$

We do not even need to perform back-substitution here to get the values of $\lambda_1, \lambda_2, \lambda_3$ as the last equation is a false statement. Hence, $u_1 \notin S$ or in other words vector u_1 cannot be represented as linear combination of vectors v_1, v_2, v_3.

CASE 2 :-

Given that,

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Therefore,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{array}\right] \boxed{\text{swap } R3, R5}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & 2 \end{array}\right] \begin{array}{l} \boxed{R2-R1} \\ \\ \\ \\ \boxed{R5-2R1} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \boxed{R5+2R3}$$

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If we now convert the row-echelon of the augmented matrix to a linear system of equations we get.

$$\lambda_1+\lambda_2+\lambda_3=0 \quad \dots(1)$$
$$\lambda_2+\lambda_3=1 \quad \dots(2)$$
$$\lambda_2+\lambda_3=-1 \quad \dots(3)$$
$$\lambda_3=0 \quad \dots(4)$$
$$0\lambda_1+0\lambda_2+0\lambda_3=0 \quad \dots(5)$$

From equation (4), we get $\boxed{\lambda_3=0}$

From equation (3), we get $\boxed{\lambda_2=-1}$, but this does not satisfy equation (2)

Hence, $u_2 \notin S$ or vector u_2 cannot be represented as linear combination of vectors v_1, v_2, v_3.

CASE 3 :

Given that,

$$v_1=\begin{bmatrix}1\\1\\2\\0\\0\end{bmatrix}, v_2=\begin{bmatrix}1\\2\\0\\0\\1\end{bmatrix}, v_3=\begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix} \text{ and } u_3=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$

Therefore,

$$\left[\begin{array}{ccc|c}1&1&1&1\\1&2&0&1\\2&0&0&1\\0&0&1&1\\0&1&1&1\end{array}\right], \left[\begin{array}{ccc|c}1&1&1&1\\1&2&0&1\\0&1&1&1\\0&0&1&1\\2&0&0&1\end{array}\right] \boxed{\text{Swap } R3, R5}$$

$$\left[\begin{array}{ccc|c}1&1&1&1\\0&1&-1&0\\0&1&1&1\\0&0&1&1\\0&-2&-2&-1\end{array}\right]\begin{array}{l}\\ \boxed{R2-R1}\\ \\ \\ \boxed{R5-2R1}\end{array}, \left[\begin{array}{ccc|c}1&1&1&1\\0&1&-1&0\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{array}\right]\begin{array}{l}\\ \\ \\ \\ \boxed{R5+2R3}\end{array}$$

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If we now convert the row-echelon form of the augmented matrix to a linear system of equations we get :

$\lambda_1 + \lambda_2 + \lambda_3 = 1 \quad \dots (1)$

$\lambda_2 - \lambda_3 = 0 \quad \dots (2)$

$\lambda_2 + \lambda_3 = 1 \quad \dots (3)$

$\lambda_3 = 1 \quad \dots (4)$

$0\lambda_1 + 0\lambda_2 + 0\lambda_3 = 1 \quad \dots (5)$

We do not even need to perform back-substitution here to get the values of $\lambda_1, \lambda_2, \lambda_3$ as the last equation is a false statement.

Hence, $u_3 \notin S$ or vector u_3 cannot be represented as linear combination of vectors v_1, v_2, v_3.