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Q1 We have studied that during Gaussian Elimination we can write $U = E_m E_{m-1} \cdots E_1 A$ where the matrices E_i are elementary transformations. Is this true for any arbitrary invertible $n \times n$ matrix? If it is true for an arbitrary invertible $n \times n$ matrix, provide a justification. If it is not true for an arbitrary $n \times n$ matrix, explain why and show what modifications you will make to the equation $U = E_m E_{m-1} \cdots E_1 A$ to make it work for any arbitrary invertible $n \times n$ matrix.

We know that we can solve a linear system of equations $Ax = b$ using Gaussian Elimination.

We can do this because what Gaussian Elimination does is, it reduces the linear system of equations $Ax = b$ to a much simpler form $A'x = b'$ (i.e. the row-echelon form) using a series of elementary transformations while keeping the solution set the same. Such the after applying Gaussian Elimination, we can easily use back substitution to find the value of x.

The elementary operations allowed are –

1. Exchange of any two rows.
2. Multiplying any row by a constant
3. Adding a multiple of one row to another

Note –

- All the elementary operations can be represented as an elementary matrix. Where the $n \times n$ elementary matrix E is the result of applying one elementary operation to a $n \times n$ identity matrix
- The elementary matrices are invertible $E^{-1} E = I = E E^{-1}$

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(*) An elementary matrix where only addition of rows is performed in a way such that $R_i \leftarrow R_i - kR_j$ where $i > j$ is a lower triangular matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ * & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & 1 & 0 \\ * & * & \cdots & * & 1 \end{bmatrix}$$

(*) If we multiply two lower triangular matrix the result is also a lower triangular matrix

Now let's assume that we are able to reduce a matix A to row-echelon form (i.e. some upper triangular matrix) U using Gaussian Elimination <u>without performing any row exchanges</u>.

In that case, we can write Gaussian Elimination as a series of elementary matrix multiplication on A such that:

$$E_m E_{m-1} \cdots E_1 A = U$$

$\Rightarrow L^{-1} A = U$ [where $E_m E_{m-1} \cdots E_1 = L^{-1}$]

$\Rightarrow L L^{-1} A = LU$ [multiplying both sides with L]

$\Rightarrow A = LU$ [$LL^{-1} = I$ as elementary matrixes are invertible]

However, if <u>at least one row exchange</u> is needed A can not be factorized to LU

This is because the elementary matrix required to perform a row swap is achived by swapping the columns of a Identity matrix

Example → The elementary matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ will swap row 2 and 3

And these elementary matrices are not lower triangular in nature. As a result when they are multiplied with a lower triangular matrix, the result is no longer lower triangular.

An elementary matrix that represents a row exchange is called a Permutation Matrix P.

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Therefore, if in order to reduce a matrix A to row-echelon form U requires some row exchanges, we can write the Gaussian Elimination as: $A = PLU$

We also know that for a $n \times n$ invertible matrix A (i.e. $\det(A) \neq 0$) a pure LU decomposition exists if and only if all its leading principal minors are non-zero. However, the PLU decomposition always exists.

We can easily prove this with an example:—

let, $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ be a 3×3 invertible matrix

By using the Gauss Jordan Elimination we can easily find A^{-1} as

$[A|I] \rightsquigarrow \cdots \rightsquigarrow [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

check $A^{-1}A = I$

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that A can not be factorized to LU all the leading principle minors are not non-zero.

$|a_{1,1}| = 1$ $\quad |a_{33}| = (4-3) - (4-2) + 2(3-2) = 1$ $\quad$ but $|a_{22}| = 0$

Meaning A can not be reduced to row-echelon from U without row exchange (in this case $R_2 \leftrightarrow R_3$)

Therefore, we can factorize A as PLU

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore, we can conclude that $A = (E_m E_{m-1} \cdots E_1)U$ is not true for any arbitrary invertible $n \times n$ matrix as sometimes elementary row exchanges are required to reduce to row-echelon form. In such cases, we must re-write Gaussian Elimination as $A = (P_M P_{M-1} \cdots P_1)(E_m E_{m-1} \cdots E_1)U$.