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6) Consider a linear system $Ax = b$. Assume that column vectors $a_1, \dots, a_n \in R^n$ are columns of matrix A ie. $A = [a_1\ a_2\ a_3 \dots a_n]$. Let $C = [A/b]$ be the augmented matrix associated with this linear system. Let us consider 2 different scenarios

a) Suppose I interchange column a_i and a_k of the augmented matrix C giving a new matrix C_1 ($i, k <= n$). Now I solve the problem assuming C_1 is my augmented matrix. How are the solutions of augmented matrix C and C_1 related.

b) Suppose I scale the i^{th} column of the augmented matrix C by a giving a new matrix C_2 ($i, k <= n$). Now I solve the problem assuming C_2 is my augmented matrix. How are the solutions of augmented matrix C and C_2 related.

Ans) Since, $Ax = b$ is a linear system of equations, where $[a_1, a_2, a_3 \dots a_n]$ are the column vectors. And $C = [A|b]$ is its augmented matrix.

The general form $Ax = b$ and the augmented matrix $C = [A|b]$ would look like:

General Form:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1i}x_i + \dots + a_{1k}x_k + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2i}x_i + \dots + a_{2k}x_k + \dots + a_{2n}x_n = b_2$$

$\vdots$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{ni}x_i + \dots + a_{nk}x_k + \dots + a_{nn}x_n = b_n$$

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Augmented Matrix :

$$
\left[\begin{array}{cccccccc|c}
\Delta_{11} & \Delta_{12} & \cdots & \Delta_{1i} & \cdots & \Delta_{1k} & \cdots & \Delta_{1n} & b_1 \\
\Delta_{21} & \Delta_{22} & \cdots & \Delta_{2i} & \cdots & \Delta_{2k} & \cdots & \Delta_{2n} & b_2 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots \\
\Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{ni} & \cdots & \Delta_{nk} & \cdots & \Delta_{nn} & b_n
\end{array}\right]
$$

Note: Where the column vector $a_i = [\Delta_{1i}\ \Delta_{2i} \cdots \Delta_{ni}]$

Scenario A

Now, if we interchange the columns a_i and a_k of the augmented matrix C giving a new matrix C_1.

What we have essentially done is re-written our equations as follows :

General Form :

$$\Delta_{11}x_1 + \Delta_{12}x_2 + \cdots + \Delta_{1k}x_i + \cdots + \Delta_{1i}x_k + \cdots + \Delta_{1n}x_n = b_1$$
$$\Delta_{21}x_1 + \Delta_{22}x_2 + \cdots + \Delta_{2k}x_1 + \cdots + \Delta_{2i}x_k + \cdots + \Delta_{2n}x_n = b_2$$
$$\vdots$$
$$\Delta_{n1}x_1 + \Delta_{n2}x_2 + \cdots + \Delta_{nk}x_i + \cdots + \Delta_{ni}x_k + \cdots + \Delta_{nn}x_n = b_n$$

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Augmented Matrix:

$$\left[\begin{array}{cccccccccc|c} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1k} & \cdots & \Delta_{1i} & \cdots & \Delta_{1n} & b_1 \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2k} & \cdots & \Delta_{2i} & \cdots & \Delta_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ \Delta_{n1} & \Delta_{n2} & & \Delta_{nk} & & \Delta_{ni} & & \Delta_{nn} & b_n \end{array}\right]$$

Therefore, we can conclude that:

- Only the i^{th} and k^{th} entries of the solution C will be swapped in the solution C_i.
- All other entries will be the same.

Scenario B

Now, if we scale the i^{th} column of the augmented matrix C by α giving a new matrix C_2

What we have essentially done is re-written our equations as follows:

General Form:

$$\Delta_{11}x_1 + \Delta_{12}x_2 + \cdots + \alpha\Delta_{1i}x_i + \cdots + \Delta_{1n}x_n = b_1$$

$$\Delta_{21}x_1 + \Delta_{22}x_2 + \cdots + \alpha\Delta_{2i}x_i + \cdots + \Delta_{2n}x_n = b_2$$

$$\cdots$$

$$\Delta_{n1}x_1 + \Delta_{n2}x_2 + \cdots + \alpha\Delta_{ni}x_i + \cdots + \Delta_{nn}x_n = b_n$$

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Augmented Matrix:

$$\left[\begin{array}{cccccccc|c} \Delta_{11} & \Delta_{12} & \cdots & \alpha\Delta_{1i} & \cdots & \Delta_{1n} & & & b_1 \\ \Delta_{21} & \Delta_{22} & \cdots & \alpha\Delta_{2i} & \cdots & \Delta_{2n} & & & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & & & \vdots \\ \Delta_{n1} & \Delta_{n2} & & \alpha\Delta_{ni} & & \Delta_{nn} & & & b_n \end{array}\right]$$

Therefore, we can conclude that:

- Only the i^{th} entry of the solution C will be ~~scaled~~ scaled by $1/\alpha$ times the solution of C_2.
- All the other entries will be same.

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7) In a Class, a professor informed students that M is a real 3 by 3 real matrix such that $M^3 = I$. Using the given information, students were asked whether the matrix is invertible and to find the eigen values of M. Is M invertible and find the eigen values of M.

Ans) We know that if a matrix $A \in R^{n \times n}$ is invertible, then $A^{-1}A = I_n = AA^{-1}$

Now, since

$$M^3 = I_3$$

$$\Rightarrow M^2 M = I_3 \quad \text{[Therefore, the matrix } M \text{ is invertible and } M^2 = M^{-1}]$$

$$\Rightarrow M^{-1}M = I_3 \quad [\text{Where } M^2 = M^{-1}]$$

We will next try to find the eigen values of matrix M

We know that A a square matrix is invertible iff it does not have a zero eigen value.

$\therefore$ the matrix M will not have 0 as an eigen value.

Next, let λ and x be the eigenvalue and eigen vectors of matrix M, such that:

$$Mx = \lambda x \quad \text{--- (1)}$$

$$\Rightarrow M^2 M x = M^2 \lambda x \quad \text{(Multiplying } M^2 \text{ on both sides)}$$

$$\Rightarrow M^3 x = \lambda M M x$$

$$\Rightarrow M^3 x = \lambda M \lambda x \quad \text{(from (1))}$$

$$\Rightarrow M^3 x = \lambda^2 M x$$

$$\Rightarrow M^3 x = \lambda^2 \lambda x \quad \text{(from (1))}$$

$$\Rightarrow M^3 x = \lambda^3 x$$

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$\Rightarrow I_3 x = \lambda^3 x$ [since $M^3 = I_3$]

$\Rightarrow x = \lambda^3 x$

$\Rightarrow \lambda^3 x - x = 0$

$\Rightarrow (\lambda^3 - 1) x = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 + \lambda + 1) x = 0$

$\therefore$ the eigenvalues λ of matrix M are

1, $\frac{-1 + i\sqrt{3}}{2}$ and $\frac{-1 - i\sqrt{3}}{2}$

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8) A student from Linear Algebra class received a matrix of the form given below.

$$A = \begin{bmatrix} 0 & \Delta & 1 & 0 \\ \Delta & 0 & \Delta & 0 \\ 1 & \Delta & 0 & \Delta \\ 0 & 0 & \Delta & 1 \end{bmatrix}, \quad \Delta > 1$$

Help him evaluate the column rank, trace and the determinant and then using trace, rank and determinant, what conclusion can be drawn about the signs (+ or -) of eigenvalues of A.

Ans) From the above problem statement, we can see that A is 4 X 4 square matrix.

We know that the trace of a matrix $A \in R^{n \times n}$ is the sum of the diagonals of A.

Therefore

$$tr(A) = \sum_{i=1}^{4} a_{ii} = (0 + 0 + 0 + 1) = 1$$

We will next find the determinant of matrix A:

$$\det(A) = 0 - \Delta \begin{vmatrix} \Delta & \Delta & 0 \\ 1 & 0 & \Delta \\ 0 & \Delta & 1 \end{vmatrix} + 1 \begin{vmatrix} \Delta & 0 & 0 \\ 1 & \Delta & \Delta \\ 0 & 0 & 1 \end{vmatrix} - 0$$

$$= -\Delta\,(\Delta(0 \cdot 1 - \Delta \cdot \Delta) - \Delta(1 \cdot 1 - 0 \cdot \Delta)) + (\Delta(\Delta \cdot 1 - 0 \cdot \Delta))$$

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$$= -\Delta(-\Delta^3 - \Delta) + \Delta^2$$
$$= \Delta^4 + \Delta^2 + \Delta^2$$
$$= \Delta^4 + 2\Delta^2$$

Therefore

$$\det(A) = \Delta^4 + 2\Delta^2$$

We will next find the rank of a matrix A:

Since, $\Delta > 1$ then $\det(A) \neq 0$

Now. we know that a $A \in R^{n \times n}$ matrix has rank n if and only if its determinant is not equal to zero.

Therefore

$rk(A) = 4$

Now using the trace, rank and determinant we will try to draw conclusion about the signs of the eigenvalues of A:

We know that,

λ is an eigenvalue of a matrix $A \in R^{n \times n}$ if and only if λ is a root of the characteristic polynomial $p_A(\lambda)$ of degree n.

Since, A is 4×4 matrix then A must have 4 eigen values λ_1, λ_2, λ_3 and λ_4.

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We also know that, the determinant of a matrix $A \in R^{n \times n}$ is the product of eigenvalues.

Therefore,

$$\det(A) = \prod_{i=1}^{4} \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = \Delta^4 + 2\Delta^2 \neq 0$$

Therefore, we can conclude that either all 4 eigenvalues will have the same sign or at least two of them will have different signs.