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Q3 Consider $n \times n$ elementary matrices where E_{ij} represents the elementary matrix where there is a non-zero value at the ith row & jth column in addition to 1s diagonal. Given a particular elementary matrix E_{ij}, for which other elementary matrices E_{pq} is it the case that $E_{pq}E_{ij} = E_{ij}E_{pq}$?

Answer →

We know that an elementary matrix is a matrix that has:

* 1s on the diagonal and
* a non-zero element at the i^{th} row and j^{th} column.

For example:

$$\begin{array}{c|c} & \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \\ \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5} \end{array}$$

E_{42}

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix}_{5\times3}$$

$E_{42} \times A$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \alpha a_{21}+a_{41} & \alpha a_{22}+a_{42} & \alpha a_{23}+a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix}$$

Matrix multiplication is possible when no. of colums in the first matrix is the same as the number of rows in the second matrix.

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Let us consider E_{ij} as an Elementary matrix, such that:

* it has non-zero element at position α_{ij} and
* the 1 in the i^{th} row it in the i^{th} column

Now when E_{ij} is multiplied with any $A_{n\times m}$ matrix, the following happens:

* the j^{th} row of A is scaled by non-zero value α,
* the scaled row is added to the i^{th} row of A.
* all other elements in A remain unchanged.

Or we can see in other words, it is the same as taking the i^{th} row of A and adding it to the α times j^{th} row of A: $R_i = R_i + \alpha R_j$

By this logic,

if elementary matrix E_{pq} having a non-zero value α_{pq} is multiplied with another elementary matrix E_{ij} having a non-zero value β_{ij} then

if $p = i$ and $q = j$

$E_{pq} = E_{24}$

$\Rightarrow E_{ij} = E_{24}$

$\Rightarrow E_{pq} \times E_{ij} = E_{24} \times E_{24}$ (same as rewriting 2^{nd} row of E_{ij} as $R_{p=2} + \alpha R_{q=4}$

$\Rightarrow E_{ij} E_{pq} = E_{24} \times E_{24}$ (same as re-writing 2^{nd} row of E_{pq} as $R_{i=2} + \beta R_{j=4}$

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Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta+\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha+\beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* if $p = i$ & $q \neq j$

$E_{pq} = E_{24}$

$E_{ij} = E_{23}$

$E_{pq} E_{ij} = E_{24} E_{(23)}$ (Same as rewriting 2^{nd} row of E_{ij} as $R_{p=2} + \alpha R_{q=4}$)

$E_{ij} E_{pq} = E_{23} E_{24}$ (Same as rewriting 2^{nd} row of E_{pq} as $R_{i=2} + \beta R_{j=3}$)

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

if $p \neq i$ & $q = j$

$E_{pq} = E_{43}$

$E_{ij} = E_{23}$

$E_{pq} E_{ij} = E_{43} E_{23}$ (same as rewriting 4^{th} row of E_{ij} as $R_{p=4} + \alpha R_{q=3}$)

$E_{ij} E_{pq} = E_{23} E_{43}$ (same as rewriting 2^{nd} row of E_{pq} as $R_{i=2} + \beta R_{j=3}$)

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{bmatrix}$$

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if $p \neq i$ and $q \neq j$

$E_{pq} = E_{12}$

$E_{ij} = E_{23}$

$E_{pq} E_{ij} = E_{12} E_{23}$ (same as re-writing 1st row of E_{ij} as $R_{p=1} + \alpha R_{q=2}$)

$E_{ij} E_{pq} = E_{23} E_{12}$ (same as re-writing 2nd row of E_{pq} as $R_{i=2} + \beta R_{j=3}$)

Example:

$$\begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha\beta & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, we can conclude that

$E_{pq} E_{ij} = E_{ij} E_{pq}$ when

$p = i$ and $q = j$, or

$p = i$ and $q \neq j$, or

$p \neq i$ and $q = j$

*** —— End —— ***