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Q.10] Consider system of equations in the matrix form as $Ax=b$, where A is a matrix of order $m \times n$

Case 1.
If $m=5, n=6$, then will the system have solutions for every choice of b? Discuss with explanation.

Case 2.
If $m=6, n=8$ and rank of A is 6, then is it possible to make the system have no solution by changing b? Discuss with explanation.

Case 3.
If $m=10, n=12, b_i=0$ for all i from 1 to 12 then is it possible that all solutions are multiples of one fixed non-zero solution? Discuss with explanation.

Answer →

We know that the rank of a matrix A can be calculated as follows:
If the columns of matrix $A \in R^{m \times n}$ span a vector subspace $U \subseteq R^m$ then $\boxed{rk(A) = dim(U)}$.

where,
- $rk(A)$ is the rank of matrix A
- $dim(U)$ is the number of pivot columns in matrix A found by reducing it to the row-echelon form using Gaussian elimination.

We can extend this concept to find the rank of an augmented matrix $[A|b]$, where $A \in R^{m \times n}$ and $b \in R^{m \times 1}$, by considering the augmented matrix as a matrix of order $\boxed{m \times (n+1)}$ and then by counting the number of pivot columns after reducing it to the row-echelon form

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We also know that a linear system of equations $Ax=b$ is said to be consistent if b can be expressed a linear combination of the pivot columns (i.e, linearly independent columns) of A. And since the rank of a matrix is nothing but the count of pivot columns. We can make some interesting observations:

Here, $Ax=b$ is a linear system of equations and A is a 3×3 matrix, x is a 3×1 matrix and b is a 3×1 matrix.

$rk(A) \neq rk(A|b)$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right]$$

- $rk(A)=2$ and $rk(A|b)=3$
- This means that b cannot be expressed as a linear combination of the pivot columns of A. And hence, no solution exists for $Ax=b$.
- We can also make the same argument just by looking at the equation formed by the last row of [A|b]. The equation is clearly a false statement and hence no solution exists for $Ax=b$.

$rk(A)=rk(A|b)=n$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right]$$

- $rk(A)=3$, $rk(A|b)=3$ and $n=3$
- This means that b can be expressed as a linear combination of the pivot columns of A.
- And by back-substitution we can find values of x_i $(1 \geq i \leq n)$ that satisfy $Ax=b$. and if the values of x_i satisfy $Ax=b$.

$rk(A)=rk(A|b)<n$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- $rk(A)=2$, $rk(A|b)=2$ and $n=3$
- This means that b can be expressed as a linear combination of the pivot columns of A.
- But in this case we cannot use back-substitution we can find values of x_i $(1 \geq i \leq n)$ that satisfy $Ax=b$.

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we can say that the $Ax=b$ has one unique solution.

- As we have n numbers of unknown in matrix x and $>n$ number of equations.
Hence, $Ax=b$ will have infinite number of solutions.

Now using these observations, we will attempt to solve this problem.

CASE 1:

Given that,

- $Ax=b$ is a linear system of equations A is a 5×6 matrix, x is a 6×1 matrix and b is a 5×1 matrix.

This means that rank of A can be:

$rK(A) = 5$

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & b_1 \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & b_2 \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} & b_3 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} & b_4 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} & b_5 \end{array}\right]$$

$1 \geq rK(A) < 5$

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- From the augmented beside, we can see that if $rk(A)$ is 5 then $rk(A|b)$ must also be 5.
- If $rk(A)$ is 5 and $rk(A|b)$ is also 5 and since n is 6, the linear system of equations will have infinite number of solutions.

- Since n is 6, even if $rk(A) = rk(A|b)$, the linear system of equations will have infinite number of solutions.
- If $rk(A) < 5$ and $rk(A|b) = 5$, then the linear system of equations will have 0 solutions.

Thus in conclusion, we see that $Ax = b$, where $m=5$ & $n=6$, will not have solutions for all choices of b as there may be a case where $rk(A) \neq rk(A|b)$.

CASE 2:

Given that,

- $Ax = b$ is a linear system of equations and A is a 6×8 matrix, x is a 8×1 matrix and b is a 6×1 matrix.
- This means that we have $n = 8$ number of unknowns in matrix x.
- $rk(A)$ is 6

This means that:

$$rk(A) = 6 \quad \left[\begin{array}{cccccccc|c} a_{11} & & & & & & \cdots & a_{18} & b_1 \\ 0 & a_{22} & & & & & \cdots & a_{28} & b_2 \\ 0 & 0 & a_{33} & & & & \cdots & a_{38} & b_3 \\ 0 & 0 & 0 & a_{44} & & & \cdots & a_{48} & b_4 \\ 0 & 0 & 0 & 0 & a_{55} & & \cdots & a_{58} & b_5 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & \cdots & a_{68} & b_6 \end{array}\right]$$

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- From the augmented beside, we can see that if $rk(A)$ is 6 then $rk(A|b)$ must also be 6.
- If $rk(A)$ is 6 and $rk(A|b)$ is also 6 and since n is 8, the linear system of equations will have infinite number of solutions.

Thus in conclusion, we see that
- If b cannot be represented as a linear combination of the pivot columns of A, then the linear system of equations will have 0 solutions.
- Otherwise $Ax=b$, where $m=6, n=8$ and $rk(A)=6$, will always have infinite solutions for any given value of b.

CASE 3:

Given that,
- $Ax=b$ is a linear system of equations and A is a 10×12 matrix, x is a 12×1 matrix and b is a 10×1 matrix.
- This means that we have $n=12$ numbers of unknowns in matrix x.
- All the elements of matrix b are zeros.

$$b_i = 0 \, (1 \leq i \geq 12)$$

This means that:

- The rank of matrix A can be at most 10 (i.e $rk(A) \leq 10$).

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- Since all the elements of matrix b are zeros, the rank of A must always be equal to the rank of [A|b] (i.e. $rk(A) = rk(A|b)$). Hence, there is no possibility of having 0 solutions for $Ax=b$.
- And since n is 12, the linear system of equations will have infinite number of solutions.

Thus in conclusion, we see that

- $Ax=b$, where $m=10$, $n=12$ and $b_i=0$ $(1 \leq i \geq 12)$, will have infinite solutions.
- We can hence, find a particular non-zero solution to $Ax=b$ and represent all other solutions as multiples of it.