Binomial Heaps

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Comp 401: Senior Seminar

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Outline

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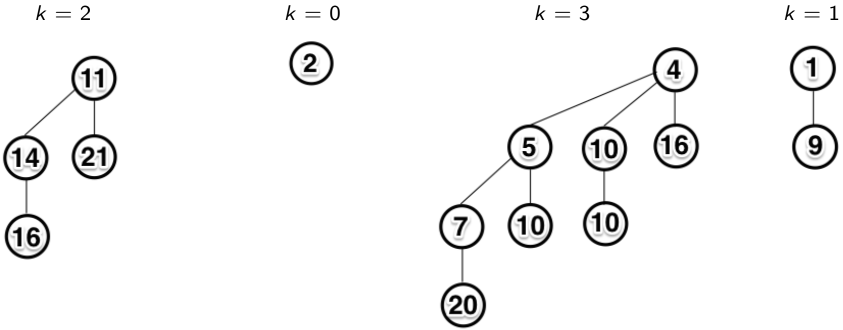
Binomial Trees

- A Binomial Tree is a specific type of tree that includes the following specifications:
  1. The order or rank of the binomial tree is the number of children of the root node.
  2. A Binomial Tree of order 0 is a single node.
  3. A Binomial Tree of order k has k child nodes, all of which are the roots of binomial trees of orders $k-1$, $k-2$, ..., 2, 1, 0 from left to right.

Binomial Trees: Examples

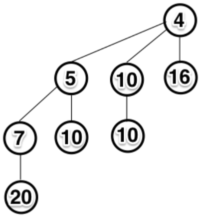
- If a binomial tree has order k, the orders of the k child nodes decrease from left to right from $k-1$ to 0.



Binomial Trees

- If a Binomial Tree has an order k:
  1. The tree has 2^k nodes.
  2. The height of the tree is k.
  3. There are $\binom{k}{d}$ nodes at depth d.
- $\binom{k}{d} = \frac{k!}{d!(k-d)!}$ is known as the Binomial Coefficient.

Example: $k = 3,\ d = 2$



$$\binom{k}{d} = \binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2*1} = \frac{6}{2} = 3$$

Outline

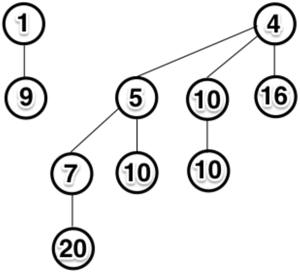
Binomial Heaps

- A Binomial Heap is a collection of binomial trees that satisfy the following two binomial heap properties:
  1. The key of any node is greater than or equal to the key of its parent (minimum-heap property).
  2. There cannot be two binomial trees of the same order.

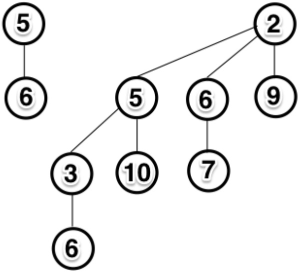
Binomial Heap: Property #1 (minimum-heap)

- The first property (minimum-heap) ensures that the root is the smallest key in each binomial tree.
- The smallest key of the entire heap is one of the roots.

Min-Heap Property ✓



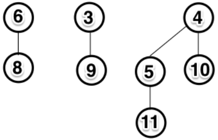
Min-Heap Property ×



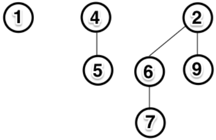
Binomial Heap: Property #2

- The order of each binomial tree must be unique.

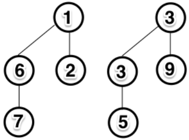
Property #2 ×



Property #2 ✓

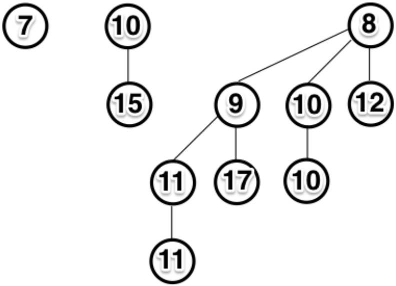


Property #2 ×



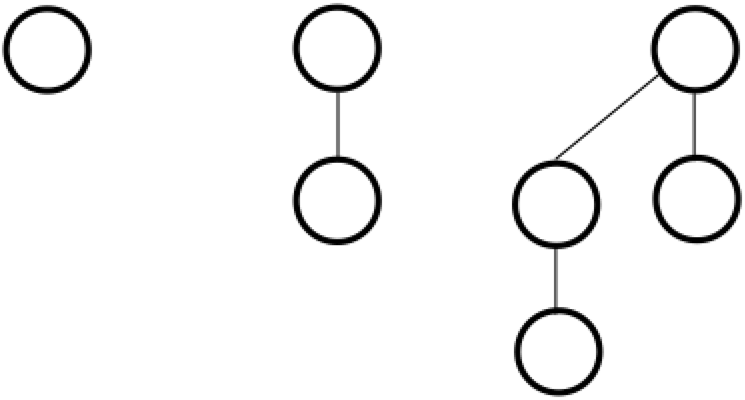
Binomial Heap: Property #2, Cont.

- The second property ensures that if a binomial heap has n nodes, then it will have at most $\lfloor log\ n \rfloor + 1$ binomial trees.
- The total number of nodes can also be thought of as a binary string, where each binomial tree represents a bit.



Outline

Implementation Structure



Outline

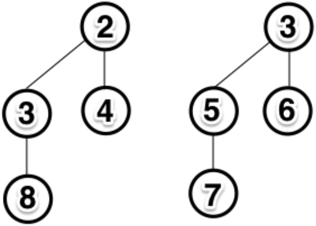
Standard Functions: Merge

- Two k ordered binomial trees are merged into one $k+1$ ordered binomial tree.

Merge

Input: Node* root_A, Node* root_B

```
if root_A.key ≤ root_B.key then
    root_B.right_sibling ← root_A.child
    root_A.child ← root_B
    root_B.parent ← root_A
    return root_A
else
    root_A.right_sibling ← root_B.child
    root_B.child ← root_A
    root_A.parent ← root_B
    return root_B
end if
```



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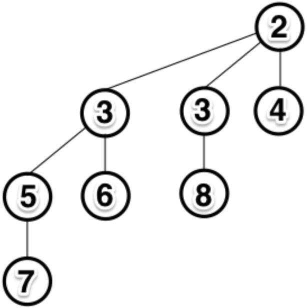
     root_B.child ← root_A
     root_A.parent ← root_B
     return root_B
  end if
```



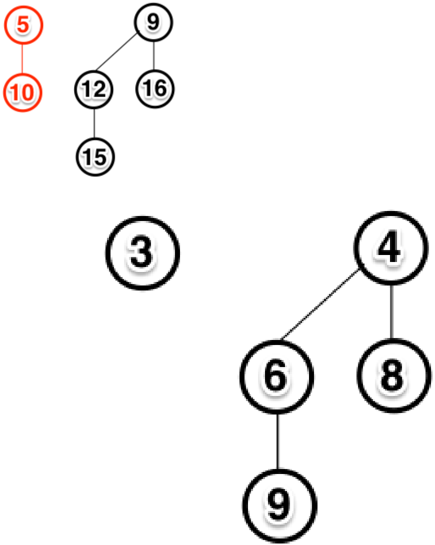
Standard Functions: Join

- Joins one Binomial Heap into another Binomial Heap.

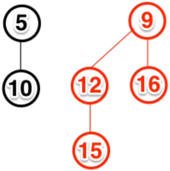
BinomialHeap : Join

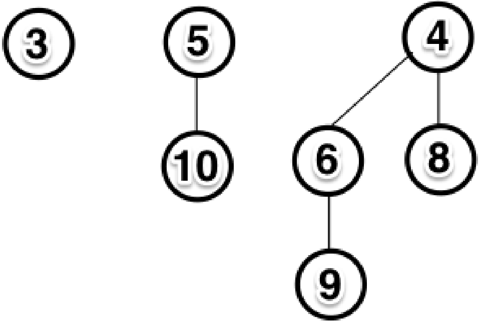
Input: Binomialheap merge_heap
for each *binomialtree* in merge_heap **do**
this_heap.insert(binomialtree.root)
merge binomial trees within this_heap if necessary
end for

Standard Functions: Join



Standard Functions: Join





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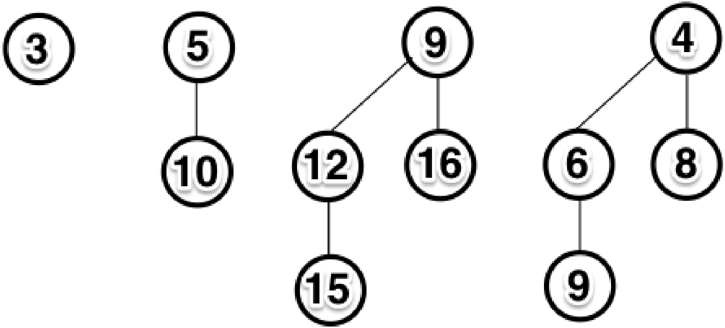

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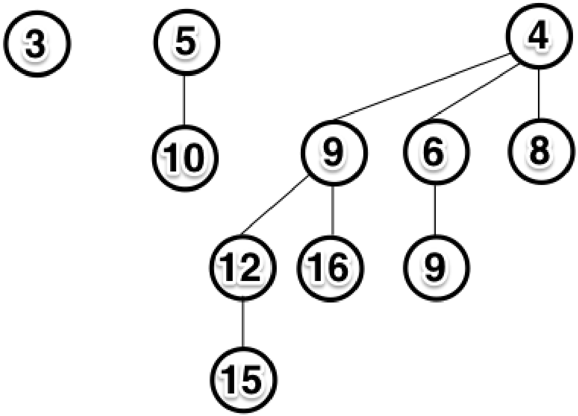
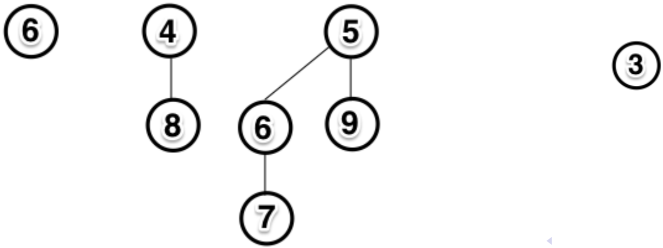


Standard Functions: Insert

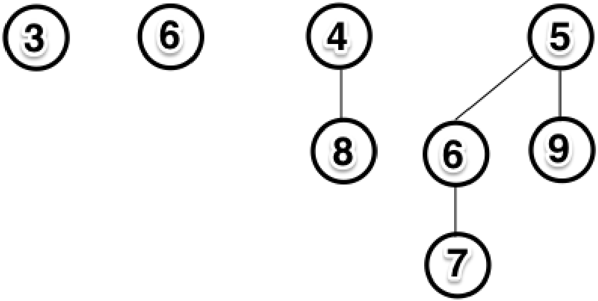
- Inserts either an integer or an existing node into a Binomial Heap.

BinomialHeap : Insert

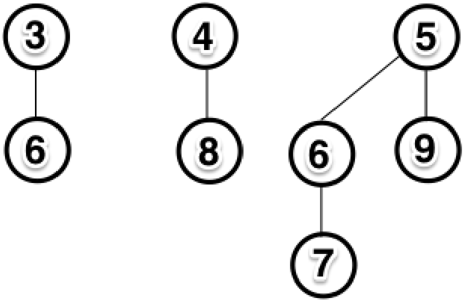
Input: int key
BinomialHeap temp_heap ← new BinomialHeap(key)
this_heap.join(temp_heap)



Standard Functions: Insert



Standard Functions: Insert



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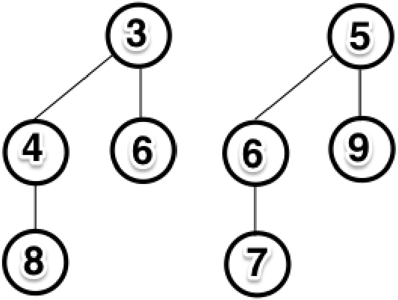

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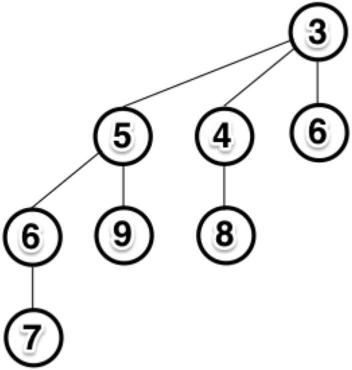


Standard Functions: DeleteMinimum

- Deletes the node with the smallest key from the Binomial Heap.

```
BinomialHeap : DeleteMinimum
Input: None
  Node* deleted_node ← this_heap.FindMinimum()
  BinomialHeap temp_heap ← new BinomialHeap()
  for each child in delete_node do
    temp_heap.insert(child)
  end for
  this_heap.join(temp_heap)
```

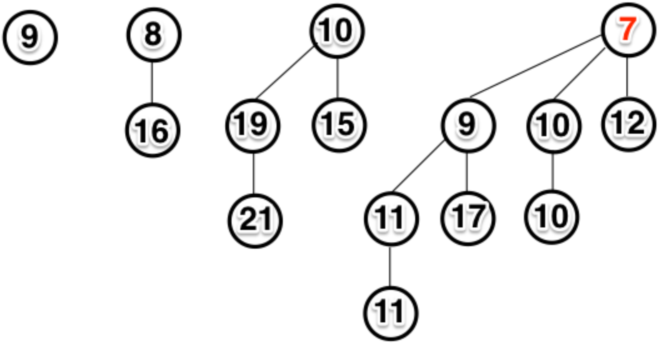
Standard Functions: DeleteMinimum

BinomialHeap : DeleteMinimum

Input: None

Node* deleted_node ← this_heap.FindMinimum()

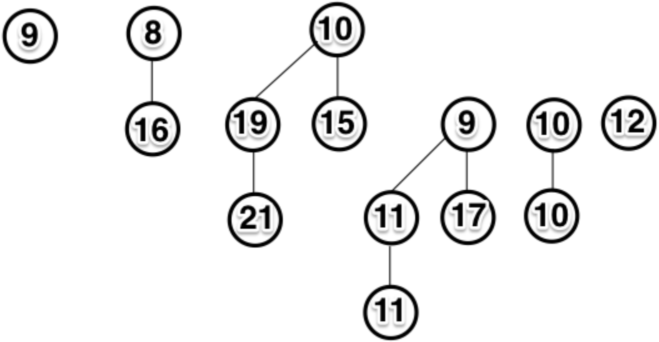
BinomialHeap temp_heap ← new BinomialHeap() ...



Standard Functions: DeleteMinimum

BinomialHeap : DeleteMinimum, cont.

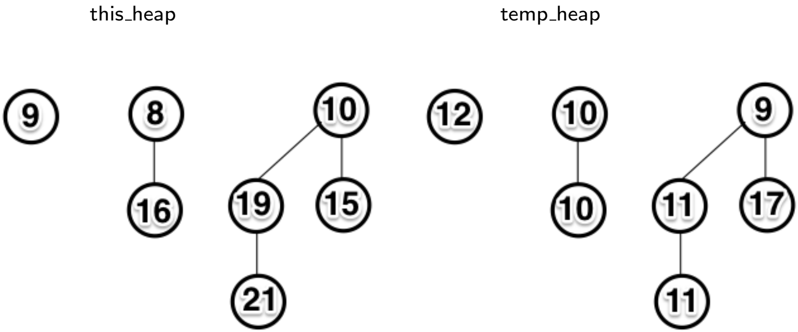
```
for each child in delete_node do
    temp_heap.insert(child)
end for...
```



Standard Functions: DeleteMinimum, cont.

BinomialHeap : DeleteMinimum

this_heap.join(temp_heap)

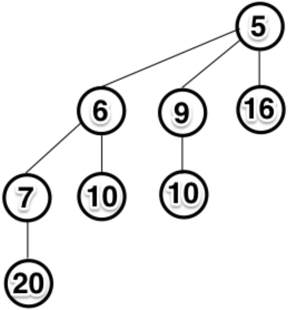


Standard Functions: DecreaseKey

- Decreases the key of a given node in the Binomial Heap

BinomialHeap : DecreaseKey

```
Input: Node* node, int new_key
  decrease_node.key ← new_key
  Node* temp ← decrease_node
  while temp.parent ≠ NULL do
    if temp.key < temp.parent.key then
      swap(temp.key, temp.parent.key)
    else
      exit loop
    end if
    temp ← temp.parent
  end while
```



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Input: Node* node, int new_key
  decrease_node.key $\leftarrow$ new_key
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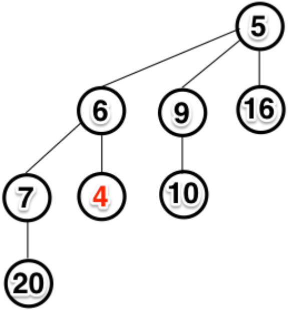
  while temp.parent $\neq$ NULL **do**
    if temp.key $<$ temp.parent.key **then**
      swap(temp.key, temp.parent.key)
    else
      exit loop
    end if
    temp $\leftarrow$ temp.parent
  end while



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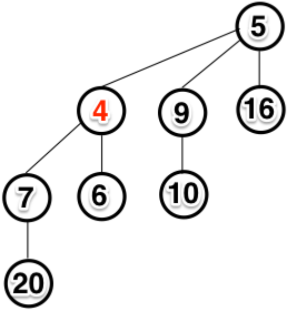

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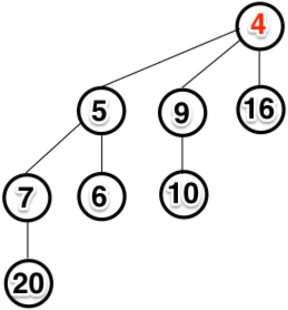
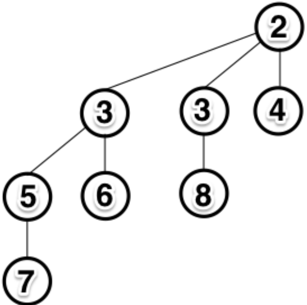


Standard Functions: Delete

- Deletes a given node from the Binomial Heap

BinomialHeap : Delete

Input: Node* delete_node
  BinomialHeap.DecreaseKey(delete_node, -9999)
  BinomialHeap.DeleteMinimum()



Outline

Uses of Binomial Heaps

- Binomial Heaps are used to implement priority queues
- Priority queues can be used in applications that manage network bandwidth or manage events in a discrete event simulation.
- Priority queues can also be used in various algorithms including heapsort, Dijkstra's shortest-path algorithm, Prim's minimum spanning tree algorithm, and Huffman Encoding.

Operation	Linked List	Binary	d-ary	Binomial
FindMinimum	$O(n)$	$O(1)$	$O(1)$	$O(log\ n)$
Insert	$O(log\ n)$	$O(log\ n)$	$O(log_d\ n)$	$O(log\ n)$
Join	$O(1)$	$O(n)$	$O(n)$	$O(log\ n)$
DeleteMinimum	$O(n)$	$O(log\ n)$	$O(d\ log_d\ n)$	$O(log\ n)$
DecreaseKey	$O(1)$	$O(log\ n)$	$O(log_d\ n)$	$O(log\ n)$
Delete	$O(1)$	$O(log\ n)$	$O(d\ log_d\ n)$	$O(log\ n)$

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