

# Binomial Heaps

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Comp 401: Senior Seminar

3/05/2014

# Outline

1. Binomial Trees
2. Binomial Heaps
3. Implementation Structure
4. Standard Functions
5. Uses of Binomial Heaps

# Outline

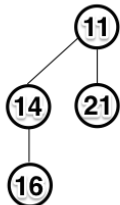
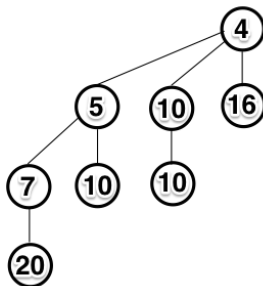
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# Binomial Trees

- A **Binomial Tree** is a specific type of tree that includes the following specifications:
  1. The **order** or **rank** of the binomial tree is the number of children of the root node.
  2. A Binomial Tree of order 0 is a single node.
  3. A Binomial Tree of order  $k$  has  $k$  child nodes, all of which are the roots of binomial trees of orders  $k - 1, k - 2, \dots, 2, 1, 0$  from left to right.

# Binomial Trees: Examples

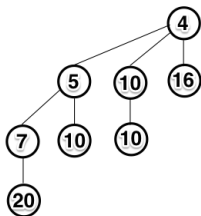
- If a binomial tree has order  $k$ , the orders of the  $k$  child nodes decrease from left to right from  $k - 1$  to 0.

 $k = 2$  $k = 0$  $k = 3$  $k = 1$ 

# Binomial Trees

- If a Binomial Tree has an order  $k$ :
  1. The tree has  $2^k$  nodes.
  2. The height of the tree is  $k$ .
  3. There are  $\binom{k}{d}$  nodes at depth  $d$ .
- $\binom{k}{d} = \frac{k!}{d!(k-d)!}$  is known as the Binomial Coefficient.

Example:  $k = 3, d = 2$



$$\binom{k}{d} = \binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2 * 1} = \frac{6}{2} = 3$$

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# Binomial Heaps

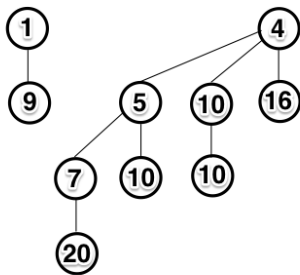
- A **Binomial Heap** is a collection of binomial trees that satisfy the following two binomial heap properties:
  1. The key of any node is greater than or equal to the key of its parent (minimum-heap property).
  2. There cannot be two binomial trees of the same order.



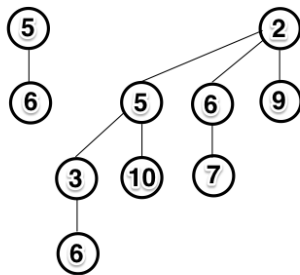
# Binomial Heap: Property #1 (minimum-heap)

- The first property (minimum-heap) ensures that the root is the smallest key in each binomial tree.
- The smallest key of the entire heap is one of the roots.

Min-Heap Property ✓



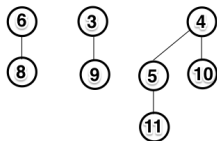
Min-Heap Property ✗



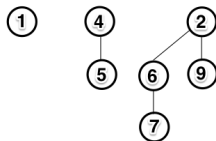
# Binomial Heap: Property #2

- The order of each binomial tree must be unique.

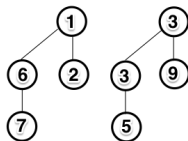
Property #2 ×



Property #2 ✓

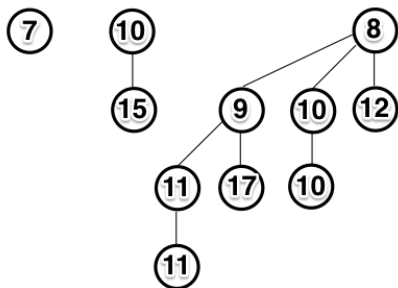


Property #2 ×



## Binomial Heap: Property #2, Cont.

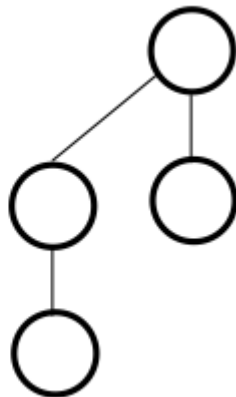
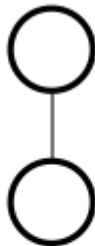
- The second property ensures that if a binomial heap has  $n$  nodes, then it will have at most  $\lfloor \log n \rfloor + 1$  binomial trees.
- The total number of nodes can also be thought of as a binary string, where each binomial tree represents a bit.



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# Implementation Structure



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# Standard Functions: Merge

- Two  $k$  ordered binomial trees are merged into one  $k + 1$  ordered binomial tree.

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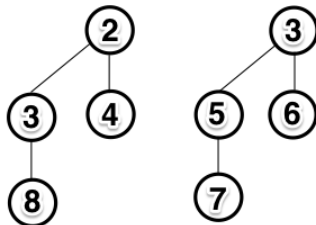
## Merge

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**Input:** Node\* root\_A, Node\* root\_B

```
if root_A.key ≤ root_B.key then
    root_B.right_sibling ← root_A.child
    root_A.child ← root_B
    root_B.parent ← root_A
    return root_A
else
    root_A.right_sibling ← root_B.child
    root_B.child ← root_A
    root_A.parent ← root_B
    return root_B
end if
```

---



# Standard Functions: Merge

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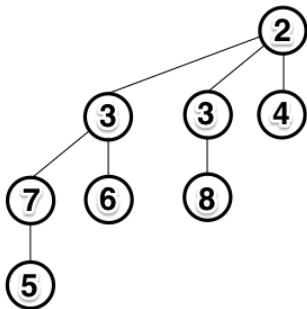
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**Input:** Node\* root\_A, Node\* root\_B

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    root_B.parent ← root_A
    return root_A
else
    root_A.right_sibling ← root_B.child
    root_B.child ← root_A
    root_A.parent ← root_B
    return root_B
end if
```

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# Standard Functions: Join



# Standard Functions: Insert

- Inserts either an integer or an existing node into a Binomial Heap.

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BinomialHeap : Insert ( $O(\log n)$ )

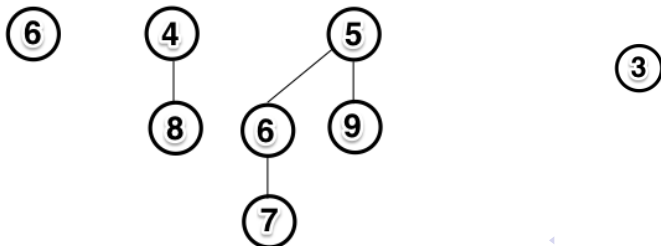
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**Input:** int key

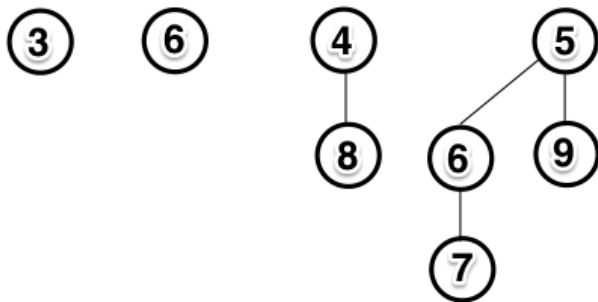
BinomialHeap temp\_heap  $\leftarrow$  new BinomialHeap(key)

this\_heap.join(temp\_heap)

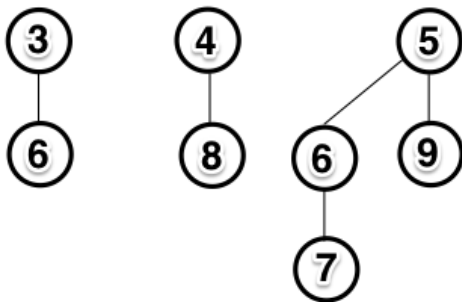
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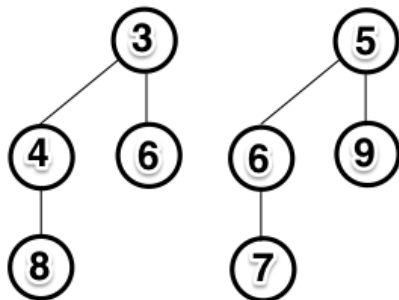
# Standard Functions: Insert



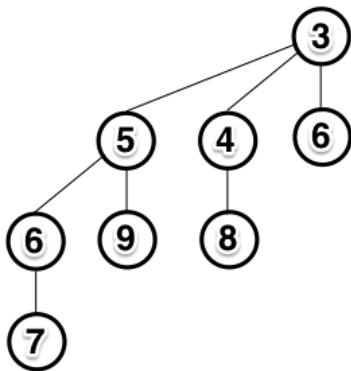
# Standard Functions: Insert



# Standard Functions: Insert



# Standard Functions: Insert



# Standard Functions: DeleteMinimum

- Deletes the node with the smallest key from the Binomial Heap.

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## BinomialHeap : DeleteMinimum

---

**Input:** None

```
Node* deleted_node  $\leftarrow$  this_heap.FindMinimum()  
BinomialHeap temp_heap  $\leftarrow$  new BinomialHeap()  
for each child in delete_node do  
    temp_heap.insert(child)  
end for  
this_heap.join(temp_heap)
```

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# Standard Functions: DeleteMinimum

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## BinomialHeap : DeleteMinimum

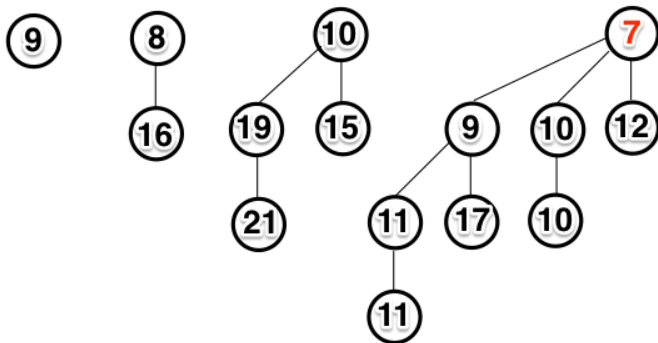
---

**Input:** None

Node\* deleted\_node  $\leftarrow$  this\_heap.FindMinimum()

BinomialHeap temp\_heap  $\leftarrow$  new BinomialHeap() ...

---





# Standard Functions: DeleteMinimum

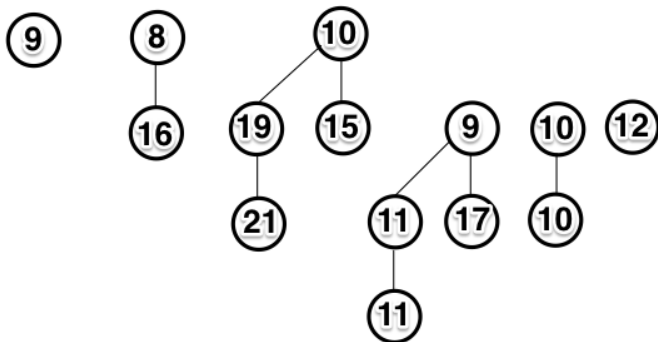
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BinomialHeap : DeleteMinimum, cont.

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```
for each child in delete_node do  
    temp_heap.insert(child)  
end for...
```

---



# Standard Functions: DeleteMinimum, cont.

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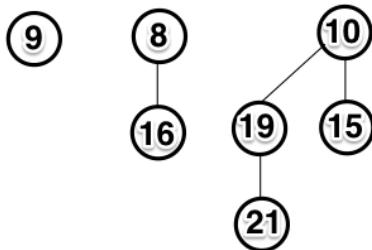
BinomialHeap : DeleteMinimum

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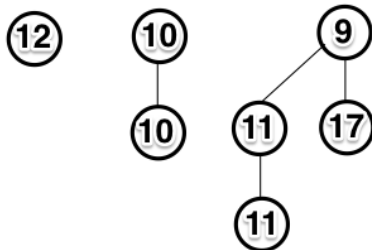
`this_heap.join(temp_heap)`

---

this\_heap



temp\_heap



# Standard Functions: DecreaseKey

- Decreases the key of a given node in the Binomial Heap

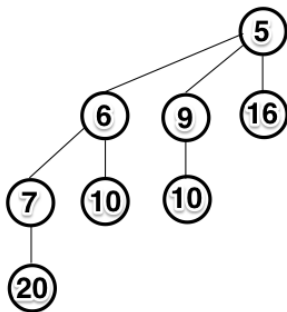
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## BinomialHeap : DecreaseKey

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**Input:** Node\* node, int new\_key  
decrease\_node.key  $\leftarrow$  new\_key  
Node\* temp  $\leftarrow$  decrease\_node  
**while** temp.parent  $\neq$  NULL **do**  
    **if** temp.key < temp.parent.key  
    **then**  
        swap(temp.key, temp.parent.key)  
    **else**  
        break  
    **end if**  
    temp  $\leftarrow$  temp.parent  
**end while**

---



# Standard Functions: DecreaseKey

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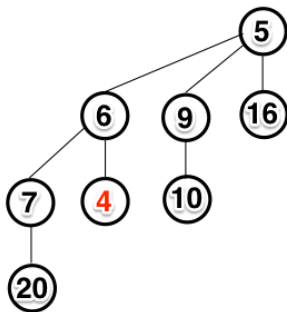
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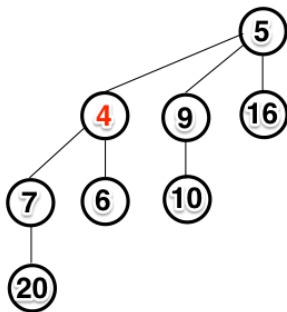
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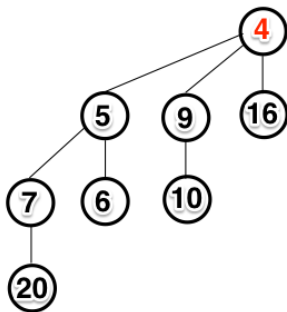
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        swap(temp.key, temp.parent.key)  
    **else**  
        break  
    **end if**  
    temp  $\leftarrow$  temp.parent  
**end while**

---



# Standard Functions: Delete

- Deletes a given node from the Binomial Heap

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BinomialHeap : Delete

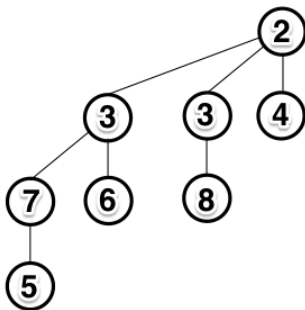
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**Input:** Node\* delete\_node

BinomialHeap.DecreaseKey(delete\_node, -9999)

BinomialHeap.DeleteMinimum()

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# Uses of Binomial Heaps

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# References

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