# modular bootstrap

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## 1 Introduction

The torus partition function is terms of the character is given by

$$Z(\tau,\bar{\tau}) = \sum_{h,\bar{h}} \rho_{h,\bar{h}} \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{1}$$

$$= \chi_0(\tau)\bar{\chi}_0(\bar{\tau}) + \sum_{h,\bar{h}} \rho_{h,\bar{h}} \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{2}$$

where in the second equation the sum is over $h, \bar{h}$ such they are not simulatenously equal to zero. The character is expresses as

$$\chi_{(c,h)}(\tau) = \mathrm{Tr}_{\mathcal{R}} q^{L_0 - \frac{c}{24}} \tag{3}$$

where $q = e^{-2\pi i\tau}$ and the trace is taken over all the states in highest weight representation $\mathcal{R}$. For maximally degenerate representation a level $l$, the trace would discount all the null states at that level. The character would then have the expression,

$$\chi_{(c,h)}(\tau) = q^{h-\frac{c}{24}} \prod_{\substack{n=1\\ n\neq l}}^{\infty} \left(\frac{1}{1-q^n}\right) \tag{4}$$

$$= \frac{q^{h-\frac{(c-1)}{24}} - (1-q^l)}{\eta(\tau)} \tag{5}$$

$$= \frac{q^{h-\frac{(c-1)}{24}} - q^{h+l-\frac{(c-1)}{24}}}{\eta(\tau)} \tag{6}$$

The vacuum states has a null vector at level $l = 1$, i.e. $L_{-1}|0\rangle = 0$. In that case, the character is expressed as,

$$\chi_0(\tau) = \frac{q^{-\frac{(c-1)}{24}}(1-q)}{\eta(\tau)} \tag{7}$$

More generically, in the absence of null vectors the character takes the form,

$$\chi_h(\tau) = \frac{q^{h-\frac{(c-1)}{24}}}{\eta(\tau)} \tag{8}$$

$$Z(\tau,\bar{\tau}) = \frac{|q|^{-\frac{(c-1)}{12}}|(1-q)|^2}{|\eta(\tau)|^2} + \sum_{h,\bar{h}} \rho_{h,\bar{h}} \frac{q^{h-\frac{(c-1)}{24}}\bar{q}^{\bar{h}-\frac{(c-1)}{24}}}{|\eta(\tau)|^2} \tag{9}$$

Since $\eta(\tau) \to \eta(\frac{-1}{\tau}) = \sqrt{-i\tau}\eta(\tau)$, the expression $|\tau|^{\frac{1}{2}}|\eta(\tau)|^2$ is invariant under $S$ transformation. It is convenient to introduce the reduced partition function,

$$\hat{Z}(\tau,\bar{\tau}) = |\tau|^{\frac{1}{2}}|\eta(\tau)|^2 Z(\tau,\bar{\tau})\,, \tag{10}$$

so that the $S$-invariance can be prescribes as

$$\hat{Z}(\tau,\bar{\tau}) = \hat{Z}(\frac{-1}{\tau},\frac{-1}{\bar{\tau}}) \tag{11}$$

with the reduced characters are given by,

$$\hat{\chi}_0(\tau)\hat{\bar{\chi}}_0(\bar{\tau}) = |\tau|^{\frac{1}{2}}|q|^{-\frac{(c-1)}{12}}|(1-q)|^2\,, \qquad \hat{\chi}_h(\tau)\hat{\bar{\chi}}_{\bar{h}}(\bar{\tau}) = |\tau|^{\frac{1}{2}}q^{h-\frac{(c-1)}{24}}\bar{q}^{\bar{h}-\frac{(c-1)}{24}} \tag{12}$$

Considering a parity invariant partition function i.e.

$$\hat{Z}(\tau,\bar{\tau}) = \hat{Z}(-\bar{\tau},\tau) \tag{13}$$

we can label the degeneracies and characters with the scaling dimension $\Delta = h+\bar{h}$ and spin $s = |h-\bar{h}|$.

$$\hat{Z}(\tau,\bar{\tau}) = \hat{\chi}_0(\tau)\hat{\bar{\chi}}_0(\bar{\tau}) + \sum_{s\in\mathbb{Z}_+,\,\Delta\in\Omega_s}\rho_{\Delta,s}\left\{\hat{\chi}_{\frac{\Delta+s}{2}}(\tau)\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}(\bar{\tau}) + \hat{\chi}_{\frac{\Delta+s}{2}}(\bar{\tau})\hat{\bar{\chi}}_{\frac{\Delta 0 s}{2}}(\bar{\tau})\right\} \tag{14}$$

## 1.1

We take the starting point to be modular crossing equations (11). To define constraint equations, we need to define differential operators $\mathcal{D}$,

$$\mathcal{D}\left[\hat{Z}(\tau,\bar{\tau}) - \hat{Z}\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}}\right)\right] = 2\mathcal{D}\hat{Z}(\tau,\bar{\tau}) = 0 \tag{15}$$

For simplicity, we take $\tau = i\beta$, $\bar{\tau} = -i\beta$. In this scenario, the character decomposition is to blind to the spin of the operators in the spectrum. This bootstrap problem is refered to as the *spinless bootstrap*. The reduced partition function is given by,

$$\hat{Z}(\beta) = \hat{\chi}_0(\beta) + \sum_{\Delta}\rho_\Delta\hat{\chi}_\Delta(\beta) \tag{16}$$

where,

$$\hat{\chi}_0(\beta) = \beta^{\frac{1}{2}}e^{2\pi\beta\frac{c-1}{12}}(1-e^{-2\pi\beta})^2\,, \qquad \hat{\chi}_\Delta(\beta) = \beta^{\frac{1}{2}}e^{-2\pi\beta(\Delta-\frac{c-1}{12})} \tag{17}$$

The crossing equation is now written as,

$$\hat{Z}\left(\beta\right) = \hat{Z}\left(\frac{1}{\beta}\right) \tag{18}$$

We define

$$V_0 \equiv \hat{\chi}_0(\beta) - \hat{\chi}_0\left(\frac{1}{\beta}\right) = \beta^{\frac{1}{2}}e^{2\pi\beta\frac{c-1}{12}}(1-e^{-2\pi\beta})^2 - \beta^{\frac{-1}{2}}e^{\frac{2\pi}{\beta}\frac{c-1}{12}}(1-e^{\frac{-2\pi}{\beta}})^2 \tag{19}$$

$$V_\Delta \equiv \hat{\chi}_\Delta(\beta) - \hat{\chi}_\Delta\left(\frac{1}{\beta}\right) = \beta^{\frac{1}{2}}e^{-2\pi\beta(\Delta-\frac{c-1}{12})} - \beta^{\frac{-1}{2}}e^{\frac{-2\pi}{\beta}(\Delta-\frac{c-1}{12})} \tag{20}$$

such that the crossing equation takes the form

$$V_0 + \sum_{\Delta}\rho_\Delta V_\Delta = 0 \tag{21}$$

Considering the derivatives $\mathcal{D}^k$,

$$\mathcal{D}^k = \frac{1}{2(2k-1)!}\left[\frac{(\beta+1)^2}{2}\frac{\partial}{\partial\beta}\right]^{2k-1}\left(\frac{1+\beta}{2}\right)\beta^{-\frac{1}{2}} \tag{22}$$

we can get Laguerre polynomials,

$$\begin{aligned}
\mathcal{D}^k V_\Delta &\equiv \mathcal{F}^k_\Delta = L_{2k-1}\left(4\pi x_\Delta\right)e^{-2\pi\Delta} \\
\mathcal{D}^k V_0 &\equiv \mathcal{F}^k_0 = L_{2k-1}(4\pi x_0) + e^{-4\pi}L_{2k-1}(4\pi(x_0+2)) - 2e^{-2\pi}L_{2k-1}(4\pi(x_0+1))
\end{aligned}$$

where $x_\Delta = \Delta - \frac{c-1}{12}$.

$$\begin{pmatrix} \mathcal{F}_0^1 \\ \mathcal{F}_0^2 \\ \mathcal{F}_0^3 \\ . \\ . \end{pmatrix} + \sum_\Delta \rho_\Delta \begin{pmatrix} \mathcal{F}_\Delta^1 \\ \mathcal{F}_\Delta^2 \\ \mathcal{F}_\Delta^3 \\ . \\ . \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \end{pmatrix} \tag{23}$$

$$\vec{\mathcal{F}}_0 + \sum_\Delta \rho_\Delta \vec{\mathcal{F}}_\Delta = 0 \tag{24}$$

1. Select the gap $\Delta^g$ such that all $\Delta \geq \Delta^g$.
2. The optimisation problem is phrased as maximising

   $$\vec{\alpha} \cdot \vec{\mathcal{F}}_0 \tag{25}$$

   by varying $\vec{\alpha}$ subject to

   $$\vec{\alpha} \cdot \vec{\mathcal{F}}_\Delta \geq 0 \ . \tag{26}$$

   We can also include a normalisation condition such as $\alpha_1 = 1$.
3. If $\vec{\alpha} \cdot \vec{\mathcal{F}}_0 > 0$ at the resulting $\alpha$, then $\Delta^g$ is rejected and replaced by a new value.

The extremal limit to the optimization problem corresponds to case when $\vec{\alpha} \cdot \vec{\mathcal{F}}_0 \to 0$ — the gap $\Delta^g$ is marginally rejected. If we say that the $\vec{\mathcal{F}_\Delta}$ belong to a vector space $\mathcal{V}$. Then, $\alpha$ lies in a convex set $\mathcal{S} \subset \mathcal{V}^*$, where $\mathcal{V}^*$ is the dual vector space. The set $\mathcal{S}$ is formed by intersection of the (26) and the normalisation condition. Suppose the extremal limit is satisfied by $\vec{\alpha_E}$, then in this situation, there may be some set $\{\Delta_i\}$ for which the constraints saturate on $\mathcal{S}$, that is $\vec{\alpha_E} \cdot \mathcal{F}_{\Delta_i} = 0$, while the rest $\Delta_j \notin \Delta_i$satisfy $\vec{\alpha_E} \cdot \vec{\mathcal{F}}_{\Delta_j} > 0$. We can approximate the neighbourhood of the space $\mathcal{S}$ as $\alpha_E + \kappa_E$, where

$$\kappa_E = \{\vec{\alpha} : \vec{\alpha} \cdot \vec{\mathcal{F}}_{\Delta_i} > 0\} \tag{27}$$

The vector dual to $\kappa_E$ assumes the form,

$$\kappa_E^* = \sum_{\Delta_i} d_{\Delta_i} \mathcal{F}_{\Delta_i} \tag{28}$$

where $d_{\Delta_i} \in \mathbb{R}_+$. For some vector $\vec{v}_E \in \kappa_E$, observe that the maximisation condition implies that

$$\vec{\alpha}_E \cdot \vec{\mathcal{F}}_0 > (\vec{\alpha_E} + \vec{v}_E) \cdot \vec{\mathcal{F}}_0 \tag{29}$$

$$0 > \vec{v}_E \cdot \vec{\mathcal{F}}_0 \ . \tag{30}$$

Therefore, $-\vec{\mathcal{F}}_0$ lies in $\kappa_E^*$. In other words,

$$-\vec{\mathcal{F}}_0 = \sum_{\Delta_i} d_{\Delta_i} \mathcal{F}_{\Delta_i} \ . \tag{31}$$

This is precisely the crossing equation with the degeneracies denoted by $d_{\Delta_i}$.Thus, solving the crossing equations is equivalent to finding an extremal vector $\vec{\alpha}_E$. Moreover, we can state that the solution to the crossing equation is necessarily optimal at a given order of derivatives. We shall refer to solving the crossing equations as the *primal* problem and solving for $\vec{\alpha}$ as the *dual* one.

In [cite fast conformal bootstrap], they exploit the fact that at an even order $P$ of the derivative, there are exactly $P/2$ roots $\Delta_g = \Delta_1$, $\Delta_2$, ... $\Delta_{P/2}$ along with the fact that $\Delta_2$, ... $\Delta_{P/2}$ are *double* roots. The leads to $P/2 + P/2 - 1 = P - 1$ equations. With the normalisation condition, $\alpha_1 = 1$, imposed, we can use the above equations to calculate the remaining $P - 1$ free $\alpha_k$.

$$\mathcal{L} = \vec{\alpha}\cdot\vec{\mathcal{F}}_0 + \sum_{i=1}^{P/2} b_{\Delta_i}\vec{\alpha}\cdot\vec{\mathcal{F}}_{\Delta_s} + \sum_{j=2}^{P/2} c_{\Delta_j}\vec{\alpha}\cdot\vec{\mathcal{F}}'_{\Delta_d} \tag{32}$$

$$\frac{\delta\mathcal{L}}{\delta b_{\Delta_i}} \equiv \vec{\alpha}\cdot\vec{\mathcal{F}}_{\Delta_i} = 0\,, \qquad \frac{\delta\mathcal{L}}{\delta c_{\Delta_j}} \equiv \vec{\alpha}\cdot\vec{\mathcal{F}'}_{\Delta_j} = 0 \tag{33}$$

$$\frac{\delta\mathcal{L}}{\delta\Delta_j} \equiv b_{\Delta_j}\vec{\alpha}\cdot\vec{\mathcal{F}'}_{\Delta_j} + c_{\Delta_j}\vec{\alpha}\cdot\vec{\mathcal{F}''}_{\Delta_j} \tag{34}$$

$$= c_{\Delta_j}\vec{\alpha}\cdot\vec{\mathcal{F}''}_{\Delta_j} = 0 \tag{35}$$

Assuming that there no higher roots, we can conclude $c_{\Delta_j} = 0$. If we fix $\Delta_1$, we obtain the optimal values of $\Delta$ at the gradient of the lagrangian, which is the crossing equation

$$\frac{\delta\mathcal{L}}{\delta\alpha_k} \equiv \mathcal{F}_0^k + \sum_{i=1}^{P/2} b_{\Delta_i}\mathcal{F}_{\Delta_i}^k = 0 \tag{36}$$

with $k \neq 1$. The dual Lagrangian is given by,

$$\mathcal{G} = \mathcal{F}_0^1 + \sum_{i=1}^{P/2} b_{\Delta_i}\mathcal{F}_{\Delta_i}^1 + \sum_{k=2}^{P}\alpha_k\left(\mathcal{F}_0^k + \sum_{i=1}^{P/2} b_{\Delta_i}\mathcal{F}_{\Delta_i}^k\right) \tag{37}$$

For spinning bootstrap, $\beta = -i\tau$, $\bar{\beta} = i\bar{\tau}$, recalling the generating function of generalised Laguerre polynomial,

$$\sum_n t^n L_n^{(\alpha)}(2\omega)e^{-\omega} = \frac{1}{(1-t)^{\alpha+1}}\exp\left(-\frac{1+t}{1-t}\omega\right)$$

we can define derivatives,

$$\mathcal{D}^j = \frac{1}{\sqrt{2}j!}\left[\frac{(\beta+1)^2}{2}\frac{\partial}{\partial\beta}\right]^j\left(\frac{1+\beta}{2}\right)^{\frac{1}{2}}\beta^{-\frac{1}{4}}$$
$$\bar{\mathcal{D}}^{\bar{j}} = \frac{1}{\sqrt{2}\bar{j}!}\left[\frac{(\bar{\beta}+1)^2}{2}\frac{\partial}{\partial\bar{\beta}}\right]^{\bar{j}}\left(\frac{1+\bar{\beta}}{2}\right)^{\frac{1}{2}}\bar{\beta}^{-\frac{1}{4}}$$

For $j+\bar{j} = $ odd, the derivatives have the effect,

$$\begin{aligned}
&\mathcal{D}^j\bar{\mathcal{D}}^{\bar{j}}\left[\hat{\chi}_{\frac{\Delta+s}{2}}(\tau)\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}(\bar{\tau}) - \hat{\chi}_{\frac{\Delta+s}{2}}\left(-\frac{1}{\tau}\right)\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}\left(-\frac{1}{\bar{\tau}}\right) + (\tau\leftrightarrow\bar{\tau})\right]\\
&\quad = \frac{1}{2}e^{-2\pi(x_{\Delta,s}+\bar{x}_{\Delta,s})}\left(L_j^{-\frac{1}{2}}(4\pi x_{\Delta,s})L_{\bar{j}}^{-\frac{1}{2}}(4\pi\bar{x}_{\Delta,s}) - (-1)^{j+\bar{j}}L_j^{-\frac{1}{2}}(4\pi x_{\Delta,s})L_{\bar{j}}^{-\frac{1}{2}}(4\pi\bar{x}_{\Delta,s}) + (x_{\Delta,s}\leftrightarrow\bar{x}_{\Delta,s})\right)\\
&\quad = e^{-2\pi(x_{\Delta,s}+\bar{x}_{\Delta,s})}\left(L_j^{-\frac{1}{2}}(4\pi x_{\Delta,s})L_{\bar{j}}^{-\frac{1}{2}}(4\pi\bar{x}_{\Delta,s}) + (x_{\Delta,s}\leftrightarrow\bar{x}_{\Delta,s})\right)
\end{aligned}$$

where $x_{\Delta,s} = \frac{\Delta+s}{2} - \frac{c-1}{24}$ and $\bar{x}_{\Delta,s} = \frac{\Delta-s}{2} - \frac{c-1}{24}$.
The action on the vacuum characters is given by

$$\begin{aligned}
&\mathcal{D}^j\bar{\mathcal{D}}^{\bar{j}}\left[\hat{\chi}_0(\tau)\hat{\bar{\chi}}_0(\bar{\tau}) - \hat{\chi}_0\left(-\frac{1}{\tau}\right)\hat{\bar{\chi}}_0\left(-\frac{1}{\bar{\tau}}\right)\right]\\
&\quad = e^{-4\pi x_{0,0}}\left(L_j^{-\frac{1}{2}}(4\pi x_{0,0}) - e^{-2\pi}L_j^{-\frac{1}{2}}(4\pi(x_{0,0}+1))\right)\left(L_{\bar{j}}^{-\frac{1}{2}}(4\pi x_{0,0}) - e^{-2\pi}L_{\bar{j}}^{-\frac{1}{2}}(4\pi(x_{0,0}+1))\right)
\end{aligned}$$

## 1.2

In (14), the $s$ are summed over positive integers and the $\Delta$ lies in a discrete sets $\Omega_s$ corresponding to primaries of spin $s$ with the condition $\Delta > s$ obtained from unitarity. The $S$ transformation has a fixed point at $\tau = i$. The neighbourhood around this point can be parameterized by $\tau = ie^r$ with the fixed point at $r = 0$. The action of $S$ on $r$ is,

$$r \to -r\,.$$

In terms of the new parameter, the modular invariance of the partition function is given by,

$$Z(ie^r, -ie^{\bar{r}}) = Z(ie^{-r}, -ie^{-\bar{r}}) \tag{38}$$

We notes that by taking derivatives at $r = 0$,

$$\left(\frac{\partial}{\partial r}\right)^{N_L}\left(\frac{\partial}{\partial \bar{r}}\right)^{N_R} Z(ie^r, -ie^{\bar{r}})\bigg|_{r=\bar{r}=0} = 0 \qquad \text{for } N_R + N_L \in 2\mathbb{Z}_{\geq 0} + 1 \tag{39}$$

In terms of $\tau$, this translates to

$$\left(\tau\frac{\partial}{\partial \tau}\right)^{N_L}\left(\bar{\tau}\frac{\partial}{\partial \bar{\tau}}\right)^{N_R} Z(\tau, \bar{\tau})\bigg|_{\tau=i,\ \bar{\tau}=-i,} = 0 \qquad \text{for } N_R + N_L \in 2\mathbb{Z}_{\geq 0} + 1 \tag{40}$$

By constraining the partition function to imaginary axis, where $\tau = i\beta$, the above derivative equation implies

$$\left(\beta\frac{\partial}{\partial \beta}\right)^{N} Z(\beta)\bigg|_{\beta=1} = 0 \qquad \text{for } N \in 2\mathbb{Z}_{\geq 0} + 1 \tag{41}$$

The numerical modular bootstrap programme involves listing a set of assumptions derived from analytic considerations, then finding a region in the parameter space that satisfy assumptions. The constraints provided by (39) are central to the approach. Typically, we begin by defining a linear functional,

$$\sum_{m,n} \alpha_{mn}\, \partial_r^m \partial_{\bar{r}}^n|_{r=\bar{r}=0} \qquad \text{with odd } m+n\,. \tag{42}$$

As a result of (39), these functionals have the property that

$$\sum_{m,n} \alpha_{mn}\, \partial_r^m \partial_{\bar{r}}^n|_{r=\bar{r}=0}\, \hat{Z}(\tau, \bar{\tau}) = \alpha\left[\hat{Z}(\tau, \bar{\tau})\right] = 0 \tag{43}$$

where we use $\alpha[.]$ as a short hand notation. Equivalently,

$$\alpha\left[\hat{\chi}_0(\tau)\hat{\bar{\chi}}_0(\bar{\tau})\right] + \sum_{s\in\mathbb{Z}_+,\ \Delta\in\Omega_s} \rho_{\Delta,s}\,\alpha\left[\hat{\chi}_{\frac{\Delta+s}{2}}(\tau)\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}(\bar{\tau}) + \hat{\chi}_{\frac{\Delta+s}{2}}(\bar{\tau})\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}(\bar{\tau})\right] = 0 \tag{44}$$

The derivatives of the characters at different orders can be understood to be components of a vectors $\vec{\mathcal{F}}_{0,0}$ and $\vec{\mathcal{F}}_{\Delta,s}$,

$$\partial_r^m \partial_{\bar{r}}^n|_{r=\bar{r}=0}\, \chi_0(\tau)\hat{\bar{\chi}}_0(\bar{\tau}) = \mathcal{F}_{0,0}^{m,n} \tag{45}$$

and

$$\partial_r^m \partial_{\bar{r}}^n|_{r=\bar{r}=0}\left[\hat{\chi}_{\frac{\Delta+s}{2}}(\tau)\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}(\bar{\tau}) + \hat{\chi}_{\frac{\Delta+s}{2}}(\bar{\tau})\hat{\bar{\chi}}_{\frac{\Delta-s}{2}}(\bar{\tau})\right] = \mathcal{F}_{\Delta,s}^{m,n} \tag{46}$$

Therefore, (44) is equivalent to

$$\vec{\alpha}\cdot\vec{\mathcal{F}}_{0,0} + \sum_{s\in\mathbb{Z}_+,\Delta\in\Omega_s} \rho_{\Delta,s}\vec{\alpha}\cdot\vec{\mathcal{F}}_{\Delta,s} = 0 \tag{47}$$

The coefficients $\alpha_{m,n}$ of the functional correspond to the components of a vector $\vec{\alpha}$. For example, we can describe our bootstrap problem of finding an upper bound on scaling dimension of lowest non-trivial primary in the following manner:

1. Hypothesise that there is a gap $\Delta_s^g \geq s$ such that for all $\Delta \in \Omega_s$, $\Delta \geq \Delta_s^g$.
2. Search for an $\alpha$ such that,
$$\vec{\alpha} \cdot \vec{\mathcal{F}}_{0,0} > 0$$
subject to
$$\vec{\alpha} \cdot \vec{\mathcal{F}}_{\Delta,s} \geq 0\,.$$
3. If such an $\alpha$ exists, then it would be in direct violation of (44), suggesting that the spectrum corresponding to the initial hypothesis is forbidden. We would proved that gap cannot exceed $\Delta_s^g$.

To ensure the problem is numerically tractable, we select a finite number of derivatives odd $n+m \leq P$.