MATH 2418: Linear Algebra

Assignment# 6

Due : Tuesday 10/10; 11:59pm Term : `Fall 2023`

[Last Name] [First Name] [Net ID] [Lab Section]

Recommended Problems (do not turn in)

Sec 3.1: 1, 2, 4, 5, 9, 10, 14, 16, 17, 23, 25, 26. **Sec 3.2:** 1, 2, 3, 4, 7, 9, 11, 13, 15, 17, 23, 33, 45. **Sec 3.3:** 1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 20, 26, 29.

1. Which of the following are vector spaces? Justify your answer.

   (a) The set of all polynomials of degree 3.

   (b) The set of all vectors $\mathbf{x} = (x_1, x_2, x_3)$, satisfying $3x_1 + 5x_2 - 9x_3 = 2023$.

   (c) The set of all vectors $\mathbf{x} = (x_1, x_2, x_3)$, satisfying $2024x_1 + x_2 - x_3 = 0$.

   (d) The set of all 3×3 matrices such that $A\mathbf{x} = \mathbf{0}$ has a unique solution.

   (e) The set of all $n \times n$ $(n \in \mathbb{N})$ diagonal matrices.

2. Let U be the set of all vectors (x_1, x_2, x_3) satisfying $3x_1 - x_2 + x_3 = 0$ and V be the set of all vectors (x_1, x_2, x_3) satisfying $2x_1 - x_2 + x_3 = 0$.

   (a) Show that U and V are subspaces of $\mathbb{R}^3$.

   (b) Is the set $U \cup V := \{\mathbf{x} \mid \mathbf{x} \in U \text{ or } \mathbf{x} \in V\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

   (c) Is the set $U \cap V := \{\mathbf{x} \mid \mathbf{x} \in U \text{ and } \mathbf{x} \in V\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

3. Which of the following are spanning sets for $\mathbb{R}^3$? Justify your answer.

   (a) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
   (b) $\{(1, 0, 0), (0, 2, 2), (1, 1, 1)\}$
   (c) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$
   (d) $\{(1, 0, 0), (0, 1, 1), (1, 0, 1)\}$
   (e) $\{(1, 1, 1), (0, 0, 1)\}$

4. Given

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 3 & 6 & 6 & 5 & 5 \\ 4 & 8 & 14 & 16 & 6 \end{bmatrix}$$

(a) Compute the reduced row echelon form R of A;

(b) Which column vectors of R correspond to the free variables? Write each of these vectors as a linear combination of the column vectors corresponding to the pivot variables.

5. For which conditions on $\mathbf{b} = (b_1, b_2, b_3, b_4)$ do there exist solution(s) for the linear system $A\mathbf{x} = \mathbf{b}$?

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 6 & 12 & 12 & 10 \\ 4 & 8 & 14 & 16 \\ 0 & 2 & 6 & 14 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 0 & 6 & 6 & 0 & 12 & 12 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}$

6. (a) Construct a matrix B whose null space consists of all linear combinations of vectors $(3, 0, 6, 9)$ and $(0, 3, 9, -3)$.

   (b) Express matrix B as a sum of two rank one matrices.

7. Find the complete solution $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ to the system $\begin{bmatrix} 6 & 10 & 12 \\ 2 & 4 & 2 \\ 4 & 10 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 14 \\ 4 \\ 16 \end{bmatrix}$.

8. Let

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 6 & 0 & 4 & 12 \\ 0 & 3 & 6 & 15 \\ 6 & 0 & 6 & 24 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 9 \\ k \end{bmatrix}.$$

(a) Find condition on $k \in \mathbb{R}$ such that $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{R}^n$ is solvable.

(b) Find all solutions when condition in a) holds.