Linear Algebra Notes of Gilbert Strang Introduction to Linear Algebra book

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Contents

Chapter 1

Vector Spaces and Subspaces

1.1 Spaces of Vectors

1.2 The Nullspace of A: Solving Ax = 0 and Rx = 0

1.3 The Complete Solution to Ax = b

The "easy" solution x_p is when we take the free variables equal to 0, where the x_{pi} component is 0 if it's at free column or equal to the b_i component if it's at a pivot column. x_p accounts for the particular solution to $Ax = b$ and the nullspace x_n accounts for the general solution to $Ax = 0$. The complete solution is $x = x_p + x_n$.

Chapter 2

Orthogonality

2.1 Orthogonality of Vectors and Subspaces

2.2 Projections onto Lines and Subspaces

We have that for a $\vec{b}$ outside of $C(A)$, its projection onto $C(A)$ is the closest vector to $\vec{b}$ in $C(A)$. The closest vector is denoted as $\vec{p} = A\hat{x}$. Such vector is related to $\vec{b}$ with vectors in the $N(A^T)$ subspace. We have that $\exists e \in N(A^T)$ such $e = \vec{b} - \vec{p}$.

2.2.1 Exercises

Exercise 1. Not completely sure what the extra symmetry condition $P^T = P$ for orthogonal projection means. We have that $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

a) $P^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} row_1 \cdot col_1 & row_1 \cdot col_2 \\ row_2 \cdot col_1 & row_2 \cdot col_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$(I - P)^2 = (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix})^2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

b) A vector $v \in C(P)$ is $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and a $w \in C(I - P)$ is $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, $v \cdot w = 1$ which shows that v and w are not orthogonal $\therefore$ the spaces are not orthogonal.

c) $v^T P^T (I - P^T) w = v^T (P^T - P^T P) w = v^T (0) w = 0$

Exercise 2. We have that $\hat{x} = \frac{a^T b}{a^T a}$. Then, $a^T b = 5$ and $a^T a = 3$. Then, $\hat{x} = \frac{5}{3}$. The projection is then $p = \hat{x} a = \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

2.3 Least Squares Approximations

When solving the impossible solution $A\vec{x} = \vec{b}$, one possible trick to make the equation is by projecting $\vec{b}$ into the $C(A)$ subspace. This projection matrix $P = A(A^TA)^{-}A^T$ reduces the $\vec{e} \in N(A^T)$ to zero of $\vec{b} = \vec{p} + \vec{e}$, where $\vec{p} = A\hat{x}$. Now, we are solving $A\hat{x} = \vec{p}$ which is possible. The solution is $\hat{x} = (A^TA)^{-}A^T\vec{b}$.

The squared error for any $\vec{x} \in \mathbb{R}^n$ is $||\vec{b} - A\vec{x}||^2 = ||\vec{p} - A\vec{x}||^2 + ||\vec{e}||^2$. The minimum of this error is when $\vec{x} = \hat{x} = (A^TA)^{-}A^T\vec{b}$. The meaning of this equation is that it clearly identifies that the error of $||\vec{b} - A\vec{x}||^2$ is clearly minimized when $\vec{p} = A\vec{x}$ and this is satisfied when $\vec{x} = \hat{x} = (A^TA)^{-}A^T\vec{b}$. Substituting for p we have that $||\vec{b} - \vec{p}||^2 = ||\vec{e}||^2$.

One consideration is that the error we refer to the vector that separates $\vec{b}$ and $\vec{p}$ so called $\vec{e} = \vec{b} - \vec{p}$, is slightly different that the error E that we refer to the squared error $E = ||\vec{b} - A\vec{x}||^2$. The error E is only equal to $||\vec{e}||^2$ when $\vec{x} = \hat{x}$.

This method is used when we have too many points to fit a line, in which case the model matrix A with m rows and n columns is tall and thin $(m > n)$. Then, for a desired $\vec{b} \in R^m$ it might be outside of $C(A)$

2.3.1 Exercises

Exercise 3. $\hat{x} = (A^TA)^{-}A^T\vec{b}$. We are given that $\vec{t} = (0, 1, 3, 4)$, so $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$,

I will find A^TA, then $(A^TA)^{-}$ and finally $A^T\vec{b}$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \tag{2.1}$$

$$(A^TA)^{-} = \frac{1}{4 \cdot 26 - 8 \cdot 8} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 & -4 \\ -4 & 2 \end{bmatrix} \tag{2.2}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \tag{2.3}$$

Then, $\hat{x} = (A^TA)^{-}A^T\vec{b} = \frac{1}{4} \begin{bmatrix} 13 & -4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 36 \\ 112 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 \cdot 36 - 4 \cdot 112 \\ -4 \cdot 36 + 2 \cdot 112 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ For $\vec{p} = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \\ 8 \end{bmatrix}$ and $\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 2 \\ 12 \end{bmatrix}$.

2.4 Orthogonal Bases and Gram Schmidt

Chapter 3

Determinants